

ON THE RELATION BETWEEN THE STRATONOVICH AND OGAWA INTEGRALS

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It is shown that for a suitable definition of the nonadapted Stratonovich stochastic integral, the existence of the Ogawa integral implies that the Stratonovich integral exists and the two are equal.

Let $\{W_t, t \in [0, 1]\}$ be a standard Wiener process and let $(\Omega, \mathcal{B}, \mathcal{P})$ be the probability space on the space of continuous functions induced by it. Let $\{X_t, t \in [0, 1]\}$ be a Borel measurable random process such that

$$\int_0^1 X_t^2 dt < \infty \quad \text{a.s.}$$

Consider the following (not necessarily adapted) two types of stochastic integrals.

DEFINITION 1. The (smoothed) Stratonovich integral [12]. The process $X = \{X_t, t \in [0, 1]\}$ is said to be Stratonovich integrable if there exists a random variable denoted $\int_0^1 X_t \circ dW_t$ such that

$$(1) \quad S_\pi = \sum_{i=0}^{n-1} \left(\frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} X_s ds \right) (W_{t_{i+1}} - W_{t_i})$$

converges in probability to $\int_0^1 X_t \circ dW_t$ as $|\pi| = \max_i (t_{i+1} - t_i)$ tends to zero, where π runs over all finite sequences of points $\pi = \{0 = t_0 < \dots < t_n = 1\}$. This definition is equivalent to: For any sequence of refinements $\{\pi^{(m)}, m \geq 1\}$ of $[0, 1]$, such that $|\pi^{(m)}| \rightarrow 0$, the sequence $S_{\pi^{(m)}}$ converges in probability to a limit $\int_0^1 X_t \circ dW_t$ and the limit is independent of the particular sequence.

The equivalence of these two definitions seems to be known; however, we give a proof of it at the end of this note since we have not been able to find a reference for it.

REMARK. There exist other definitions for the Stratonovich integral which are not equivalent to the above definition for which the results of this note may fail to hold. This point will be discussed later.

DEFINITION 2. The Ogawa integral. The process $\dot{X} = \{X_t, t \in [0, 1]\}$ is said to be Ogawa integrable if there exists a random variable denoted $\int_0^1 X_t * dW_t$ such that for any complete orthonormal system $\{e_i, i \geq 1\}$ in $H = L^2(0, 1)$

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we have

$$(2) \quad \sum_{i \geq 1} \left(\int_0^1 X_t e_i(t) dt \right) \int_0^1 e_i(t) W(dt) = \int_0^1 X_t * dW_t,$$

where the series converges in probability.

REMARK. Ogawa defined this integral using a particular orthonormal system; cf. [4].

The purpose of this note is to prove the following.

PROPOSITION. *Suppose that the process X is Ogawa integrable. Then X is Stratonovich integrable and both integrals coincide.*

PROOF. It suffices to show that for any increasing sequence of subdivisions $\{\pi^{(m)}, m \geq 1\}$ such that the mesh of $\pi^{(m)}$ decreases to zero and $\pi^{(m+1)}$ is obtained by refining $\pi^{(m)}$ at one point, the sequence $S_{\pi^{(m)}}$ converges to the Ogawa integral $\int_0^1 X_t * dW_t$ as m tends to infinity. We will introduce a particular complete orthonormal system for which $S_{\pi^{(m)}}$ will be the partial sum of the series (2). Without any loss of generality we may assume that $\pi^{(1)} = \{0, 1\}$. Set $e_1 = 1$. For $m \geq 1$ define e_{m+1} as follows. By our assumption, $\pi^{(m+1)}$ refines $\pi^{(m)}$ at one point only. Assume that this point is in some interval $(d_1^{(m)}, d_2^{(m)})$. Let α satisfying $d_1^{(m)} < \alpha < d_2^{(m)}$ be the point of refinement. Set

$$(3) \quad e_{m+1}(t) = \begin{cases} 0, & \text{if } t \notin (d_1^{(m)}, d_2^{(m)}], \\ \sqrt{\frac{d_2^{(m)} - \alpha}{(\alpha - d_1^{(m)})(d_2^{(m)} - d_1^{(m)})}}, & \text{if } t \in (d_1^{(m)}, \alpha], \\ -\sqrt{\frac{\alpha - d_1^{(m)}}{(d_2^{(m)} - \alpha)(d_2^{(m)} - d_1^{(m)})}}, & \text{if } t \in (\alpha, d_2^{(m)}]. \end{cases}$$

Note that $\int_0^1 e_{m+1}^2(t) dt = 1$. Also note that e_{m+1} is orthogonal to the constants on $[0, 1]$. Moreover since all e_1, \dots, e_m are constant on $(d_1^{(m)}, d_2^{(m)})$, it follows that e_1, e_2, \dots define an orthonormal system. Notice that this system is a generalization of the Haar system and the completeness follows by the same arguments as for the Haar system.

We will compare now the m th Stratonovich approximation $S_{\pi^{(m)}}$ with the m th Ogawa approximation

$$(4) \quad O^m = \sum_{i=1}^m \left(\int_0^1 X_t e_i(t) dt \right) \int_0^1 e_i(t) dW_t.$$

In fact, we will show by induction that they are equal. Note first that for $m = 1$, $S_{\pi^{(1)}} = O^1$. Also note that $S_{\pi^{(m+1)}}$ differs from $S_{\pi^{(m)}}$ only because of changes taking

place in the interval $(d_1^{(m)}, d_2^{(m)}]$. Therefore,

$$\begin{aligned}
 S_{\pi^{(m+1)}} - S_{\pi^{(m)}} &= - \left(\frac{1}{d_2^{(m)} - d_1^{(m)}} \int_{d_1^{(m)}}^{d_2^{(m)}} X_t dt \right) (W(d_2^{(m)}) - W(d_1^{(m)})) \\
 (5) \qquad &+ \left(\frac{1}{\alpha - d_1^{(m)}} \int_{d_1^{(m)}}^{\alpha} X_t dt \right) (W(\alpha) - W(d_1^{(m)})) \\
 &+ \left(\frac{1}{d_2^{(m)} - \alpha} \int_{\alpha}^{d_2^{(m)}} X_t dt \right) (W(d_2^{(m)}) - W(\alpha)).
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 O^{m+1} - O^m &= \left(\int_0^1 e_{m+1}(t) X_t dt \right) \int_0^1 e_{m+1}(t) W(dt) \\
 &= \frac{1}{d_2^{(m)} - d_1^{(m)}} \left(\sqrt{\frac{d_2^{(m)} - \alpha}{\alpha - d_1^{(m)}}} \int_{d_1^{(m)}}^{\alpha} X_t dt - \sqrt{\frac{\alpha - d_1^{(m)}}{d_2^{(m)} - \alpha}} \int_{\alpha}^{d_2^{(m)}} X_t dt \right) \\
 &\quad \times \left(\sqrt{\frac{d_2^{(m)} - \alpha}{\alpha - d_1^{(m)}}} (W(\alpha) - W(d_1^{(m)})) \right. \\
 &\quad \left. - \sqrt{\frac{\alpha - d_1^{(m)}}{d_2^{(m)} - \alpha}} (W(d_2^{(m)}) - W(\alpha)) \right) \\
 (6) \qquad &= \frac{1}{d_2^{(m)} - d_1^{(m)}} \left\{ \left(\frac{d_2^{(m)} - \alpha}{\alpha - d_1^{(m)}} \int_{d_1^{(m)}}^{\alpha} X_t dt \right) (W(\alpha) - W(d_1^{(m)})) \right. \\
 &\quad - \left(\int_{d_1^{(m)}}^{\alpha} X_t dt \right) (W(d_2^{(m)}) - W(\alpha)) \\
 &\quad - \left(\int_{\alpha}^{d_2^{(m)}} X_t dt \right) (W(\alpha) - W(d_1^{(m)})) \\
 &\quad \left. + \frac{\alpha - d_1^{(m)}}{d_2^{(m)} - \alpha} \left(\int_{\alpha}^{d_2^{(m)}} X_t dt \right) (W(d_2^{(m)}) - W(\alpha)) \right\}.
 \end{aligned}$$

Comparing (5) and (6) yields that they are equal which completes the proof. \square

REMARK 1. The example of Rosinski [5] ($X_t = W_{1-t}$) shows that the converse to Proposition 1 is not true.

REMARK 2. If the definition of the Stratonovich integral is changed to read

$$(7) \qquad \int_0^1 X_s \circ dW_s = \lim_{|\pi| \rightarrow 0} \sum_{i=0}^{n-1} \frac{X_{t_i} + X_{t_{i+1}}}{2} (W_{t_{i+1}} - W_{t_i})$$

(cf. [1], page 101), then the result no longer holds since the (nonrandom) integrand

$$X_t = \begin{cases} 1, & t \text{ irrational,} \\ 0, & t \text{ rational,} \end{cases}$$

is Ogawa integrable but not integrable in the sense of (7) or in the sense of

$$(7') \quad \int_0^1 X_s \circ dW_s = \lim_{|\pi| \rightarrow 0} \sum_{i=0}^{n-1} X\left(\frac{t_{i+1} + t_i}{2}\right)(W_{t_{i+1}} - W_{t_i}).$$

Note, however, that if X_t is a continuous semimartingale on the natural filtration of W_t , then the definition

$$(8) \quad \int_0^1 X_s \circ dW_s = \int_0^1 X_s dW_s + \frac{1}{2} \langle X, W \rangle$$

is equivalent to Definition 1. In this case (8) is also equivalent to (7) and (7'). However, if X_t is not adapted to the natural filtration, even if X_t is continuous in t , it is not clear whether in general (7) or (7') coincides with Definition 1; cf. [2] for further information on this point.

REMARK 3. The relation between (7) and the Ogawa integral for adapted integrands has been discussed by Ogawa in [4]; also cf. [3], Proposition 6.2.

REMARK 4. The definitions and results of this note go over directly to the case where $[0, 1]$ is replaced by $[0, 1]^n$ under obvious modifications.

Finally, we turn to the proof that the two definitions of the Stratonovich integral given under Definition 1 are in fact equivalent. Obviously the first version implies the second one. Starting with the second definition, suppose that $S_{\pi_n} \rightarrow \sigma$ in probability for any sequence π_n of refinements with $|\pi_n| \rightarrow 0$ but there exists an $\epsilon > 0$ and a sequence π_n with $|\pi_n| \rightarrow 0$ but $\|S_{\pi_n} - \sigma\|_0 > \epsilon$ for all n where $\|\cdot\|_0$ denotes a pseudonorm corresponding to convergence in probability. Note first that for any fixed subdivision π_0 we have

$$(9) \quad \lim_{|\pi_n| \rightarrow 0} \|S_{\pi_n \vee \pi_0} - S_{\pi_n}\|_0 = 0$$

(where $\pi_n \vee \pi_0$ denotes the subdivision induced by the union of π_n and π_0) for the following reason. The difference between $S_{\pi_n \vee \pi_0}$ and S_{π_n} is a sum of terms of the form $\pm [(W_t - W_s)/(t - s)] \int_s^t X_u du$. The number of terms depends only on π_0 (it is upper bounded by three times the cardinality of π_0) and each of these terms tends to zero in probability as $|\pi_n| \rightarrow 0$ (since $\text{Prob}\{ |[(W_t - W_s)/(t - s)] \int_s^t X_u du| > \delta \}$ tends to zero as $|t - s| \rightarrow 0$), so (9) holds.

Returning to the sequence π_n for which $\|S_{\pi_n} - \sigma\|_0 > \epsilon$, fix an n_0 . By (9) we can find an n_1 (large enough) such that $n_1 > n_0$ and

$$\|S_{\pi_{(n_0)} \vee \pi_{(n_1)}} - S_{\pi_{(n_1)}}\|_0 < \epsilon^2.$$

Next choose n_2 large enough so that

$$\|S_{\pi(n_2) \vee \pi(n_1) \vee \pi(n_0)} - S_{\pi(n_2)}\|_0 < \varepsilon^3$$

and continue recursively. Set $\tilde{\pi}(N) = \pi_{n_0} \vee \pi_{n_1} \vee \dots \vee \pi_{n_N}$. Then after the N th step we have

$$\|S_{\tilde{\pi}(N)} - \sigma\|_0 > \varepsilon - \varepsilon^2 - \varepsilon^3 - \dots - \varepsilon^N.$$

Note that $\tilde{\pi}(N)$ is a system of refinements and $|\tilde{\pi}(N)| \rightarrow 0$ but $\|S_{\tilde{\pi}(N)} - \sigma\|_0 \geq \varepsilon(1 - 2\varepsilon)/(1 - \varepsilon)$, which completes the proof by contradiction.

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