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LARGE DEVIATIONS FOR A CLASS OF ANTICIPATING STOCHASTIC DIFFERENTIAL EQUATIONS

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Consider the family of perturbed stochastic differential equations on $\mathbb{R}^d$,

$$X_t^\varepsilon = X_0^\varepsilon + \sqrt{\varepsilon}\int_0^t \sigma(X_s^\varepsilon)\circ dW_s + \int_0^t b(X_s^\varepsilon)\,ds,$$

$\varepsilon > 0$, defined on the canonical space associated with the standard k-dimensional Wiener process W. We assume that $\{X_0^\varepsilon, \varepsilon > 0\}$ is a family of random vectors not necessarily adapted and that the stochastic integral is a generalized Stratonovich integral. In this paper we prove large deviations estimates for the laws of $\{X_\cdot^\varepsilon, \varepsilon > 0\}$, under some hypotheses on the family of initial conditions $\{X_0^\varepsilon, \varepsilon > 0\}$.

0. Introduction. Consider the Stratonovich differential equation on $\mathbb{R}^d$:

$$X_t = X_0 + \sum_{i=1}^k \int_0^t \sigma_i(s, X_s)\circ dW_s^i + \int_0^t b(s, \omega, X_s)\,ds \tag{0.1}$$

defined on the canonical Wiener space associated with the standard k-dimensional Wiener process $W = \{(W_t^1, \dots, W_t^k), t \in [0,1]\}$. We assume that X_0 is a d-dimensional random vector which may depend on the whole path of W and that the stochastic integral is a generalized Stratonovich integral, as defined in Nualart–Pardoux [8].

Under some regularity conditions on X_0 and the coefficients σ and b, Ocone–Pardoux [9] have proved the existence and uniqueness of solutions of (0.1) when the diffusion coefficient σ may depend on t and the drift coefficient b may depend on both t and ω.

In this paper, we assume that σ and b only depend on the state x, and for $\varepsilon > 0$, we consider the family of perturbed equations

$$X_t^\varepsilon = X_0^\varepsilon + \sum_{i=1}^k \sqrt{\varepsilon}\int_0^t \sigma_i(X_s^\varepsilon)\circ dW_s^i + \int_0^t b(X_s^\varepsilon)\,ds, \tag{0.2}$$

with an arbitrary initial condition X_0^ε. We assume that there exists $x_0 \in \mathbb{R}^d$

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such that for any $\delta > 0$,

$$\lim_{\varepsilon \to 0} \varepsilon \log P\{|X_0^\varepsilon - x_0| > \delta\} = -\infty,$$

and the purpose is to show that the family $\{P^\varepsilon, \varepsilon > 0\}$ of laws of $\{X^\varepsilon, \varepsilon > 0\}$ satisfies a large deviation principle (see Theorem 4.1). We prove that a solution $\{X_t^\varepsilon, t \in [0,1]\}$ of (0.2) can be expressed as the composition of the adapted flow

$$\varphi_t^\varepsilon(x) = x + \sum_{i=1}^k \sqrt{\varepsilon} \int_0^t \sigma_i(\varphi_s^\varepsilon(x)) \circ dW_s^i + \int_0^t b(\varphi_s^\varepsilon(x))\, ds \tag{0.3}$$

and the initial condition X_0^ε, that is, $X_t^\varepsilon = \varphi_t^\varepsilon(X_0^\varepsilon)$.

Due to this fact, it turns out that the key ingredient to reach our goal is a result ensuring the large deviation principle for the flow φ^ε. Then we will easily transfer the large deviation estimates from the flow to X^ε.

Recall that a family $\{P^\varepsilon, \varepsilon > 0\}$ of probabilities on a Polish space E satisfies a large deviation principle with rate function I: $E \to [0, +\infty]$ if I is lower semicontinuous, for every $a > 0$ the set $\{f \in E: I(f) \le a\}$ is compact, and for every open subset G and every closed subset F of E,

$$\liminf_{\varepsilon \to 0} \varepsilon \log P^\varepsilon(G) \ge -\inf\{I(g); g \in G\}, \tag{0.4}$$

$$\limsup_{\varepsilon \to 0} \varepsilon \log P^\varepsilon(F) \le -\inf\{I(g); g \in F\}. \tag{0.5}$$

The paper is organized as follows. In the first section we state uniform Ventzell–Freidlin estimates, which are obtained using Sobolev's inequality. In Section 2 we deduce from these estimates the large deviation principle for the flow φ^ε solution of (0.3), using the ideas of [1] (see also [4] and [10]). In Section 3 we prove that, under weak regularity assumptions on σ and b, $\varphi_t^\varepsilon(X_0^\varepsilon)$ is a solution of (0.2). Section 4 is devoted to prove the large deviation principle of the family of laws of $X_\cdot^\varepsilon = \varphi_\cdot^\varepsilon(X_0^\varepsilon)$. Finally, the last section is an Appendix containing an exponential estimate for a class of Itô processes.

Some remarks on the notation: We make the usual convention on summation over repeated indices. In general C denotes a generic positive constant; its value can change from one expression to another one.

1. Uniform Ventzell–Freidlin estimates. Let $\Omega = \mathscr{C}([0,1]; \mathbb{R}^k)$, P the Wiener measure and $\mathscr{F}$ the completion of the Borel σ-field on Ω with respect to P. We denote by $W_t(\omega) = \omega_t$ the standard Wiener process on $(\Omega, \mathscr{F})$.

Consider the stochastic flow on $\mathbb{R}^d$ defined by

$$\varphi_t^\varepsilon(x) = x + \sqrt{\varepsilon} \int_0^t \sigma_i(\varphi_s^\varepsilon(x)) \circ dW_s^i + \int_0^t b(\varphi_s^\varepsilon(x))\, ds, \tag{1.1}$$

$\varepsilon > 0$, $t \in [0,1]$. During this section we will assume, unless otherwise specified, that the coefficients σ_i, b: $\mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \dots, k$, as well as $m(x) = \frac{1}{2}\sum_{i=1}^k (\partial\sigma_i/\partial x)(x)\sigma_i(x)$, are $\mathscr{C}^2$ functions with bounded partial derivatives up to order 2.

For any integer $m \geq 1$ and $x \in \mathbb{R}^m$, we denote by H_x^m the set of absolutely continuous functions $f \in \mathscr{C}([0,1];\mathbb{R}^m)$ with $f_0 = x$ and $\int_0^1 |\dot{f}_s|^2\,ds < +\infty$. If $x = 0$, we will write H^m instead of H_0^m and set $\tilde{I}(f) = \frac{1}{2}\int_0^1 |\dot{f}_s|^2\,ds$.

Given $f \in H^k$ we consider the function $g(x) \in H_x^d$, which is the solution of the differential equation

$$g_t(x) = x + \int_0^t \{\sigma(g_s(x))\dot{f}_s + b(g_s(x))\}\,ds. \tag{1.2}$$

If A is a compact subset of $\mathbb{R}^d$, then $\sup_{0 \leq t \leq 1} \sup_{x \in A} |g_t(x)| < \infty$. Indeed, the restriction on the growth of the coefficients σ and b entails

$$|g_t(x)| \leq |x| + C\int_0^t (1 + |g_s(x)|)(1 + |\dot{f}_s|)\,ds,$$

and by a generalization of Gronwall's lemma (see Lemma 4.13 in [6]), we get

$$\sup_{0 \leq t \leq 1} \sup_{x \in A} |g_t(x)| \leq C' \exp\left\{C\int_0^1 (1 + |\dot{f}_s|)\,ds\right\} < +\infty$$

for some constants C' and C.

In the sequel we will denote by $\|\cdot\|$ the supremum norm on $\mathscr{C}([0,1];\mathbb{R}^m)$ and by $\|\cdot\|_A$ the supremum norm on $\mathscr{C}([0,1] \times A;\mathbb{R}^m)$. The aim of this section is to prove a result which describes a kind of continuity property of the mapping

$$\Phi\colon \mathscr{C}([0,1];\mathbb{R}^k) \to \mathscr{C}([0,1];\mathscr{C}(\mathbb{R}^d,\mathbb{R}^d))$$

defined by $\Phi(\sqrt{\varepsilon}\,W) = \varphi^\varepsilon$. The precise statement is given below; it is an improvement of Theorem 2.4 in [1] (see also Théorème 4 in [10]). This is what we call uniform Ventzell–Freidlin estimates.

THEOREM 1.1. *Let $f \in H^k$ and $g_t(x)$ be the solution of the ordinary differential equation* (1.2) *and for $\varepsilon > 0$, let $\varphi_t^\varepsilon(x)$ be the adapted flow defined by* (1.1). *Suppose that σ, b and m are of class $\mathscr{C}^2$ with bounded partial derivatives up to order* 2. *Fix positive reals R, λ, η and a compact subset A of $\mathbb{R}^d$. Then there exist $\varepsilon_0 > 0$ and $\alpha > 0$ such that for any ε with $0 < \varepsilon \leq \varepsilon_0$ and any f such that $\tilde{I}(f) \leq \lambda$,*

$$P\{\|\varphi^\varepsilon - g\|_A \geq \eta, \|\sqrt{\varepsilon}\,W - f\| < \alpha\} \leq \exp(-R/\varepsilon). \tag{1.3}$$

We prove Theorem 1.1 by means of Sobolev's inequality. Let p be a positive real number such that $p > (d/2) \vee 1$, A a compact subset of $\mathbb{R}^d$ and v: $A \to \mathbb{R}$ a continuous mapping. There exists a constant C_p such that

$$\sup_{x \in A} |v(x)|^{2p} \leq C_p\left[\int_A \left(|v(x)|^{2p} + \left|\frac{\partial v}{\partial x}(x)\right|^{2p}\right)dx\right]. \tag{1.4}$$

The mappings $x \mapsto \varphi_t^\varepsilon(x)$ and $x \mapsto g_t(x)$ are $\mathscr{C}^1$-diffeomorphisms. Consequently, Sobolev's inequality (1.4) clearly shows that (1.3) can be obtained by estimates of $\sup_{0 \leq t \leq 1} \int_A |\varphi_t^\varepsilon(x) - g_t(x)|^{2p}\,dx$ and $\sup_{0 \leq t \leq 1} \int_A |(\partial\varphi_t^\varepsilon(x)/\partial x) -$

$(\partial g_t(x)/\partial x)|^{2p}\,dx$. Before stating precise results, we prove some technical lemmas.

LEMMA 1.2. *Let $\pi = \{0 = t_0 < \cdots < t_n = 1\}$ be a partition of $[0,1]$ and let $\sigma(s,x,\omega) = \sum_{j=0}^{n-1}\sigma_j(x,\omega)1_{[t_j,t_{j+1})}(s)$ be a family of $\mathbb{R}^N \times \mathbb{R}^k$-valued step processes depending on $x \in \mathbb{R}^d$, such that σ_j is $\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{F}_{t_j}$ measurable and let $I_t(x) = \int_0^t \sigma(s,x,\omega)\,dW_s$ for every $t \in [0,1]$. Fix a compact set $A \subset \mathbb{R}^d$, a real number $p \geq 1$ and, for every $\varepsilon \in (0,1]$, a stopping time τ^ε bounded by 1. Assume that $J_t = \int_A |I_t(x)|^{2p}\,dx$ is finite for any $t \in [0,1]$ and that $M = \sup_{0<\varepsilon\leq 1}\sup_\omega \sup_{0\leq t\leq \tau^\varepsilon(\omega)} \int_A |\sigma(t,x,\omega)|^{2p}\,dx < \infty$. Then for any $\eta > 0$, there exists $\alpha > 0$, depending on η, M, p and n, such that for any $\varepsilon \in (0,1]$, the set*

$$\left\{\sup_{0\leq t\leq \tau^\varepsilon} \varepsilon^p J_t > \eta,\ \|\sqrt{\varepsilon}\,W\| \leq \alpha\right\}$$

is empty.

PROOF. We first majorize $|I_t(x)|$ as follows:

$$\sqrt{\varepsilon}\,|I_t(x)| = \sqrt{\varepsilon}\left|\sum_{j=0}^{n-1}\sigma_j(x,\omega)\big(W_{t_{j+1}\wedge t} - W_{t_j\wedge t}\big)\right| \leq 2\|\sqrt{\varepsilon}\,W\| \sum_{j:\,t_j\leq t}|\sigma_j(x,\omega)|.$$

Consequently, on the set $\{\|\sqrt{\varepsilon}\,W\| \leq \alpha\} \cap \{t \leq \tau^\varepsilon\}$,

$$\begin{aligned}\varepsilon^p J_t = \int_A \left|\sqrt{\varepsilon}\,I_t(x)\right|^{2p} dx &\leq 2^{2p}\alpha^{2p}n^{2p-1}\sum_{j:\,t_j\leq t}\int_A |\sigma_j(x,\omega)|^{2p}\,dx\\ &\leq 2^{2p}\alpha^{2p}n^{2p}M.\end{aligned}$$

It suffices to choose $\alpha < (1/2n)(\eta/M)^{1/2p}$ to obtain the result. □

Let $S\colon \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^k$, $B^\varepsilon\colon [0,1] \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^m$, $\varepsilon \in (0,1]$, be measurable functions satisfying the following conditions.

(H1). There exists a constant $C_2 > 0$ such that for any $x, x' \in \mathbb{R}^N$ and $y, y' \in \mathbb{R}^m$,

$$\begin{aligned}|S(x,y)| &\leq C_2(1+|y|),\\ |S(x,y) - S(x,y')| &\leq C_2|y-y'|,\\ |S(x,y) - S(x',y)| &\leq C_2|y|\,|x-x'|.\end{aligned}$$

(H2). There exists a nonnegative function v such that $i(v) = \int_0^1 v(s)\,ds < \infty$ and a constant $C_3 > 0$ such that for any $s \in [0,1]$, $\varepsilon \in (0,1]$, $x, x' \in \mathbb{R}^N$ and $y, y' \in \mathbb{R}^m$,

$$\begin{aligned}|B^\varepsilon(s,x,y)| &\leq C_3 v(s)(1+|y|),\\ |B^\varepsilon(s,x,y) - B^\varepsilon(s,x,y')| &\leq C_3 v(s)|y-y'|,\\ |B^\varepsilon(s,x,y) - B^\varepsilon(s,x',y)| &\leq C_3 v(s)|y|\,|x-x'|.\end{aligned}$$

Consider a family $\{\Phi_t^\varepsilon(x), t \in [0,1], x \in \mathbb{R}^d, \varepsilon \in (0,1]\}$ of continuous adapted $\mathbb{R}^m$-valued stochastic processes and let $\{\Psi_t^\varepsilon(x), t \in [0,1]\}$ denote the corresponding family of solutions of the stochastic differential system

$$\Psi_t^\varepsilon(x) = \Psi_0^\varepsilon(x) + \sqrt{\varepsilon}\int_0^t S(\Phi_s^\varepsilon(x), \Psi_s^\varepsilon(x))\,dW_s + \int_0^t B^\varepsilon(s, \Phi_s^\varepsilon(x), \Psi_s^\varepsilon(x))\,ds. \tag{1.5}$$

Fix $n \ge 1$, and for $0 \le k \le n$, set $t_k = (k/n)$. For $t \in [t_k, t_{k+1})$, $0 \le k \le n-1$, let

$$\Phi_t^\varepsilon(n,x) = \Phi_{t_k}^\varepsilon(x) \quad \text{and} \quad \Psi_t^\varepsilon(n,x) = \Psi_{t_k}^\varepsilon(x). \tag{1.6}$$

Finally, set

$$\begin{aligned} V_t^\varepsilon(x) &= \sqrt{\varepsilon}\int_0^t S(\Phi_s^\varepsilon(x), \Psi_s^\varepsilon(x))\,dW_s,\\ V_t^\varepsilon(n,x) &= \sqrt{\varepsilon}\int_0^t S(\Phi_s^\varepsilon(x), \Psi_s^\varepsilon(n,x))\,dW_s,\\ \tilde{V}_t^\varepsilon(n,x) &= \sqrt{\varepsilon}\int_0^t S(\Phi_s^\varepsilon(n,x), \Psi_s^\varepsilon(n,x))\,dW_s. \end{aligned} \tag{1.7}$$

We at first present three technical lemmas about the approximations of $\Psi^\varepsilon(x)$ by $\Psi^\varepsilon(n,x)$, $V^\varepsilon(x)$ by $V^\varepsilon(n,x)$ and $V^\varepsilon(n,x)$ by $\tilde{V}^\varepsilon(n,x)$; their proofs depend on a result which is established in the Appendix.

Lemma 1.3. *Let A be a compact subset of $\mathbb{R}^d$ and $p \ge 1$ a real number. For any $\varepsilon \in (0,1]$, let τ^ε be a stopping time bounded by 1 such that*

$$M = \sup_{0<\varepsilon\le 1}\ \sup_w\ \sup_{0\le t\le \tau^\varepsilon} \int_A |\Psi_t^\varepsilon(x)|^{2p}\,dx < \infty.$$

Then for any R, λ, $\delta > 0$, there exists an integer n_0 depending on M, R and δ, such that for any $n \ge n_0$, $\varepsilon \in (0,1]$ and v with $i(v) \le \lambda$,

$$P\left\{\sup_{0\le t\le\tau^\varepsilon} \int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n,x)|^{2p}\,dx \ge \delta\right\} \le \exp\left(-\frac{R}{\varepsilon}\right). \tag{1.8}$$

Proof. For $t \in [t_k, t_{k+1})$, set $V_t^{\varepsilon,k}(x) = \sqrt{\varepsilon}\int_{t_k}^t S(\Phi_s^\varepsilon(x), \Psi_s^\varepsilon(x))\,dW_s$; then

$$\begin{aligned} |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n,x)| &= \left|V_t^{\varepsilon,k}(x) + \int_{t_k}^t B^\varepsilon(s, \Phi_s^\varepsilon(x), \Psi_s^\varepsilon(x))\,ds\right|\\ &\le |V_t^{\varepsilon,k}(x)| + C_3\int_{t_k}^t v(s)(1 + |\Psi_s^\varepsilon(x)|)\,ds. \end{aligned}$$

Hölder's inequality applied to the finite measure $v(s)\,ds$ yields that

$$\begin{aligned} &|\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n,x)|^{2p}\\ &\quad\le C\left\{|V_t^{\varepsilon,k}(x)|^{2p} + \left(\int_{t_k}^t v(s)\,ds\right)^{2p} + \left(\int_{t_k}^t v(s)\,ds\right)^{2p-1}\int_{t_k}^t |\Psi_s^\varepsilon(x)|^{2p} v(s)\,ds\right\}. \end{aligned}$$

Thus, for $t \le \tau^\varepsilon$, $t_k \le t < t_{k+1}$,

$$\int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n,x)|^{2p}\,dx \le C\left\{\int_A |V_t^{\varepsilon,k}(x)|^{2p}\,dx + \left(\int_{t_k}^t v(s)\,ds\right)^{2p}\right\}.$$

Since $v \in L^1([0,1])$, there exists n_1 such that for $n \ge n_1$ and for all $k = 0,1,\dots,n-1$,

$$\int_{t_k}^{t_{k+1}} v(s)\,ds \le \left(\frac{\delta}{2C}\right)^{1/2p}.$$

Therefore, if $\delta' = (\delta/2C)$,

$$P\left\{\sup_{t_k\wedge\tau^\varepsilon \le t < t_{k+1}\wedge\tau^\varepsilon} \int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n,x)|^{2p}\,dx > \delta\right\}$$
$$\le P\left\{\sup_{t_k\wedge\tau^\varepsilon \le t < t_{k+1}\wedge\tau^\varepsilon} \int_A |V_t^{\varepsilon,k}(x)|^{2p}\,dx > \delta'\right\}.$$

Apply Lemma 5.1 (in the Appendix) to the process $Y_t(x) = V_t^{\varepsilon,k}(x)$, $u = t_k$, and the stopping time

$$\tau' = \inf\left\{t \ge t_k : \int_A |V_t^{\varepsilon,k}(x)|^{2p}\,dx > \delta'\right\} \wedge t_{k+1} \wedge \tau^\varepsilon.$$

Then, using the lemma's notation, $T = 1/n$, $Y = \delta'$. Furthermore, for $t_k \le t \le \tau'$,

$$\left(\int_A |\sqrt{\varepsilon}\, S(\Phi_t^\varepsilon(x), \Psi_t^\varepsilon(x))|^{2p}\,dx\right)^{1/p} \le C\varepsilon\left(1 + \int_A |\Psi_t^\varepsilon(x)|^{2p}\,dx\right)^{1/p}$$
$$\le C\varepsilon(1+M)^{1/p} = C'\varepsilon.$$

Hence $M^{(1)} = C'\varepsilon$ and the inequality (5.2) implies that if n_2 is such that $\delta' - (C'/n_2)\delta'^{(p-1)/p}C_4 \ge (\delta'/2)$ for $n \ge n_2$,

$$P\left\{\sup_{t_k\wedge\tau' \le t < t_{k+1}\wedge\tau'} \int_A |V_t^{\varepsilon,k}(x)|^{2p}\,dx > \delta'\right\}$$
$$\le \exp\left(-n\frac{\left(\delta' - (C'/n)\delta'^{(p-1)/p}C_4\varepsilon\right)^2}{2C'\delta'^{(2p-1)/p}C_4\varepsilon}\right).$$

Thus for $n \ge n_1 \vee n_2$,

$$P\left\{\sup_{0\le t\le\tau^\varepsilon} \int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n,x)|^{2p}\,dx > \delta\right\}$$
$$\le \sum_{k=0}^{n-1} P\left\{\sup_{t_k\wedge\tau^\varepsilon \le t < t_{k+1}\wedge\tau^\varepsilon} \int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n,x)|^{2p}\,dx > \delta\right\}$$
$$\le n\exp\left(-n\frac{\delta'^{1/p}}{8C'C_4\varepsilon}\right).$$

Let $n_0 \geq n_1 \vee n_2$ be such that

$$\frac{n_0 \delta'^{1/p}}{8C'C_4} \geq \sup(1, \log n_0 + R).$$

Then for $n \geq n_0$ and $\varepsilon \in (0, 1]$,

$$\varepsilon \log n - n \frac{\delta'^{1/p}}{8C'C_4} \leq -R,$$

which concludes the proof. □

The next lemma compares $V_t^\varepsilon(x)$ and $V_t^\varepsilon(n, x)$.

LEMMA 1.4. *Let A be a compact subset of $\mathbb{R}^d$, $p \geq 1$ a real number and $\{\tau^\varepsilon, 0 < \varepsilon \leq 1\}$ a family of stopping times bounded by* 1. *For any $\eta > 0$, $R > 0$, there exists $\delta > 0$ independent of n such that for every $n \geq 1$ and every $\varepsilon \in (0, 1]$,*

$$P\left\{ \sup_{0 \leq t \leq \tau^\varepsilon} \int_A |V_t^\varepsilon(x) - V_t^\varepsilon(n, x)|^{2p}\, dx \geq \eta, \right.$$

$$\left. \sup_{0 \leq t \leq \tau^\varepsilon} \int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n, x)|^{2p}\, dx \leq \delta \right\} \tag{1.9}$$

$$\leq \exp\left(- \frac{R}{\varepsilon} \right).$$

PROOF. Apply Lemma 5.1 to the Itô process

$$Y_t(x) = V_t^\varepsilon(x) - V_t^\varepsilon(n, x)$$

$$= \sqrt{\varepsilon} \int_0^t [S(\Phi_s^\varepsilon(x), \Psi_s^\varepsilon(x)) - S(\Phi_s^\varepsilon(x), \Psi_s^\varepsilon(n, x))]\, dW_s,$$

$u = 0$, and the stopping time

$$\tau' = \inf\left\{ t > 0 \colon \int_A |V_t^\varepsilon(x) - V_t^\varepsilon(n, x)|^{2p}\, dx \geq \eta \right\}$$

$$\wedge\, \tau^\varepsilon \wedge \inf\left\{ t \geq 0 \colon \int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n, x)|^{2p}\, dx \geq \delta \right\}.$$

Then we may choose $T = 1$, $Y = \eta$. Furthermore, the Lipschitz property of S implies that if $0 \leq t \leq \tau'$,

$$\int_A |S(\Phi_t^\varepsilon(x), \Psi_t^\varepsilon(x)) - S(\Phi_t^\varepsilon(x), \Psi_t^\varepsilon(n, x))|^{2p}\, dx$$

$$\leq C_2^{2p} \int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n, x)|^{2p}\, dx \leq C\delta.$$

Therefore, we may set $M^{(1)} = \varepsilon(C\delta)^{1/p}$ and if δ is such that $\eta - C_4\eta^{(p-1)/p}(C\delta)^{1/p} \geq \eta/2$, the inequality (5.2) yields that

$$P\Big\{\sup_{0\leq t\leq\tau^\varepsilon}\int_A |V_t^\varepsilon(x) - V_t^\varepsilon(n,x)|^{2p}\,dx \geq \eta,$$

$$\sup_{0\leq t\leq\tau^\varepsilon}\int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n,x)|^{2p}\,dx \leq \delta\Big\}$$

$$\leq \exp\left(-\frac{\eta^{1/p}}{8C_4(C\delta)^{1/p}\varepsilon}\right).$$

Suppose furthermore that

$$\delta \leq \frac{\eta}{(8C_4R)^pC};$$

then for every $\varepsilon \in (0,1]$, the exponential inequality (1.9) is obtained. □

The following lemma provides an approximation of $V_t^\varepsilon(n,x)$ by $\tilde{V}_t^\varepsilon(n,x)$.

LEMMA 1.5. *Let A be a compact subset of $\mathbb{R}^d$, $p \geq 1$ a real number and $\{\tau^\varepsilon, 0 < \varepsilon \leq 1\}$ a family of stopping times bounded by 1 such that*

$$M = \sup_{0<\varepsilon\leq 1}\sup_{\omega}\sup_{0\leq t\leq\tau^\varepsilon}\int_A |\Psi_t^\varepsilon(x)|^{2p}\,dx < \infty.$$

Then for any $\eta > 0$, $0 < \delta < 1$ and $R > 0$, there exists $\gamma > 0$ independent of n, such that for every $n \geq 1$ and every $\varepsilon \in (0,1]$,

$$P\Big\{\sup_{0\leq t\leq\tau^\varepsilon}\int_A \left|V_t^\varepsilon(n,x) - \tilde{V}_t^\varepsilon(n,x)\right|^{2p}\,dx \geq \eta,$$

$$\sup_{0\leq t\leq\tau^\varepsilon}\int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n,x)|^{2p}\,dx \leq \delta, \tag{1.10}$$

$$\sup_{0\leq t\leq\tau^\varepsilon}\sup_{x\in A}|\Phi_t^\varepsilon(x) - \Phi_t^\varepsilon(n,x)| \leq \gamma\Big\} \leq \exp\left(-\frac{R}{\varepsilon}\right).$$

PROOF. Apply again Lemma 5.1 to the Itô process

$$Y_t(x) = V_t^\varepsilon(n,x) - \tilde{V}_t^\varepsilon(n,x)$$

$$= \sqrt{\varepsilon}\int_0^t [S(\Phi_s^\varepsilon(x),\Psi_s^\varepsilon(n,x)) - S(\Phi_s^\varepsilon(n,x),\Psi_s^\varepsilon(n,x))]\,dW_s,$$

$u = 0$ and the stopping time

$$\tau' = \tau^\varepsilon \wedge \inf\left\{t \geq 0 \colon \int_A \left|V_t^\varepsilon(n,x) - \tilde{V}_t^\varepsilon(n,x)\right|^{2p} dx \geq \eta\right\}$$
$$\wedge \inf\left\{t \geq 0 \colon \int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n,x)|^{2p}\, dx \geq \delta\right\}$$
$$\wedge \inf\left\{t \geq 0 \colon \sup_{x \in A} |\Phi_t^\varepsilon(x) - \Phi_t^\varepsilon(n,x)| \geq \gamma\right\}.$$

Then we may choose $T = 1$, $Y = \eta$. Furthermore, if $s \leq \tau'$,

$$\int_A |S(\Phi_s^\varepsilon(x), \Psi_s^\varepsilon(n,x)) - S(\Phi_s^\varepsilon(n,x), \Psi_s^\varepsilon(n,x))|^{2p}\, dx$$
$$\leq C_2^{2p} \int_A |\Phi_s^\varepsilon(x) - \Phi_s^\varepsilon(n,x)|^{2p} |\Psi_s^\varepsilon(n,x)|^{2p}\, dx$$
$$\leq C\gamma^{2p}(M + \delta) = (C'\gamma)^{2p},$$

so that we may choose $M^{(1)} = \varepsilon(C'\gamma)^2$. Let $\gamma > 0$ be such that $\eta - \eta^{(p-1)/p} C_4 (C'\gamma)^2 \geq \eta/2$; then (5.2) yields that

$$P\left\{\sup_{0 \leq t \leq \tau^\varepsilon} \int_A \left|V_t^\varepsilon(n,x) - \tilde{V}_t^\varepsilon(n,x)\right|^{2p} dx \geq \eta,\right.$$
$$\sup_{0 \leq t \leq \tau^\varepsilon} \int_A |\Psi_t^\varepsilon(x) - \Psi_t^\varepsilon(n,x)|^{2p}\, dx \leq \delta,$$
$$\left.\sup_{0 \leq t \leq \tau^\varepsilon} \sup_{x \in A} |\Phi_t^\varepsilon(x) - \Phi_t^\varepsilon(n,x)| \leq \gamma\right\} \leq \exp\left(-\frac{\eta^{1/p}}{8C_4\varepsilon(C'\gamma)^2}\right).$$

If furthermore we require that

$$\gamma^2 \leq \frac{\eta^{1/p}}{8C_4(C')^2 R},$$

the inequality (1.10) is fulfilled for any $\varepsilon \in (0,1]$ and any integer $n \geq 1$. □

Theorem 1.1 is a straightforward consequence of the estimates given in the next two propositions and of Sobolev's inequality. In the sequel, p denotes a real number such that $p > (d/2) \vee 1$.

PROPOSITION 1.6. *Assume that $f \in H^k$ and σ, b and m are of class $\mathscr{C}^1$ with bounded partial derivatives. Let $\varphi_t^\varepsilon(x)$ be the solution of* (1.1) *and $g_t(x)$ the solution of* (1.2). *Then given any positive reals R, λ, η and any compact subset A of $\mathbb{R}^d$, there exist $\alpha > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$,*

$$(1.11)\quad P\left\{\left\|\int_A |\varphi_t^\varepsilon(x) - g_t(x)|^{2p}\, dx\right\| \geq \eta, \|\sqrt{\varepsilon}\, W - f\| < \alpha\right\} \leq \exp\left(-\frac{R}{\varepsilon}\right).$$

PROOF. We will follow along the ideas of the proof of Theorem 4 in [10]. For any $\varepsilon > 0$, denote by P^ε the probability on Ω defined by

$$L_\varepsilon = \frac{dP^\varepsilon}{dP} = \exp\left(\frac{1}{\sqrt{\varepsilon}} \int_0^1 \dot{f}_s \, dW_s - \frac{1}{2\varepsilon} \int_0^1 |\dot{f}_s|^2 \, ds \right). \tag{1.12}$$

Then, by Girsanov's theorem, the process $W_t^\varepsilon = W_t - (1/\sqrt{\varepsilon}) f_t$ is a Brownian motion on $(\Omega, \mathscr{F}, P^\varepsilon)$ and

$$\begin{aligned} \varphi_t^\varepsilon(x) &= x + \sqrt{\varepsilon} \int_0^t \sigma(\varphi_s^\varepsilon(x)) \, dW_s^\varepsilon \\ &\quad + \int_0^t \left[b(\varphi_s^\varepsilon(x)) + \varepsilon m(\varphi_s^\varepsilon(x)) + \sigma(\varphi_s^\varepsilon(x)) \dot{f}_s \right] ds. \end{aligned}$$

Let $A^\varepsilon = \{ \| \int_A |\varphi_t^\varepsilon(x) - g_t(x)|^{2p} \, dx \| \geq \eta, \|\sqrt{\varepsilon} W - f\| < \alpha \}$. Then by Schwarz's inequality,

$$P(A^\varepsilon) = \int 1_{A^\varepsilon} L_\varepsilon^{-1} \, dP^\varepsilon \leq \left(P^\varepsilon(A^\varepsilon) \int L_\varepsilon^{-2} \, dP^\varepsilon \right)^{1/2}.$$

The factor $\int L_\varepsilon^{-2} \, dP^\varepsilon$ is bounded by $\exp(C/\varepsilon)$ for some constant C. Indeed,

$$\begin{aligned} \int L_\varepsilon^{-2} \, dP^\varepsilon &= E^{P^\varepsilon}\left[\exp\left(-\frac{2}{\sqrt{\varepsilon}} \int_0^1 \dot{f}_s \, dW_s + \frac{1}{\varepsilon} \int_0^1 |\dot{f}_s|^2 \, ds \right) \right] \\ &= \exp\left(\frac{5}{\varepsilon} \int_0^1 |\dot{f}_s|^2 \, ds \right) \leq \exp\left(\frac{10\lambda}{\varepsilon} \right). \end{aligned}$$

Let $\{\psi_t^\varepsilon(x), t \in [0,1]\}$ be the solution of the stochastic differential system

$$\begin{aligned} \psi_t^\varepsilon(x) &= x + \sqrt{\varepsilon} \int_0^t \sigma(\psi_s^\varepsilon(x)) \, dW_s \\ &\quad + \int_0^t \left[b(\psi_s^\varepsilon(x)) + \varepsilon m(\psi_s^\varepsilon(x)) + \sigma(\psi_s^\varepsilon(x)) \dot{f}_s \right] ds. \end{aligned} \tag{1.13}$$

Then (1.13) is a particular case of (1.5) with $N = m = d$, $\Phi_t^\varepsilon(x) = 0$, $\Psi_0^\varepsilon(x) = x$, $S(x,y) = \sigma(y)$ and $B^\varepsilon(s,x,y) = b(y) + \varepsilon m(y) + \sigma(y)\dot{f}_s$. The assumptions (H1) and (H2) are clearly satisfied with $v(s) = 1 + |\dot{f}_s|$. Therefore, $i(v) \leq 1 + \sqrt{2\lambda}$ if $f \in H^k$ is such that $\tilde{I}(f) \leq \lambda$. The estimate (1.11) on $P(A^\varepsilon)$ reduces to checking that for any $R, \lambda, \eta > 0$, there exists $\alpha > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$,

$$P\left\{ \left\| \int_A |\psi_t^\varepsilon(x) - g_t(x)|^{2p} \, dx \right\| \geq \eta, \|\sqrt{\varepsilon} W\| < \alpha \right\} \leq \exp\left(-\frac{R}{\varepsilon} \right). \tag{1.14}$$

Let $T^\varepsilon = \inf\{t \geq 0: \int_A |\psi_t^\varepsilon(x) - g_t(x)|^{2p} \, dx \geq \eta\} \wedge 1$. Notice that the left-hand side of (1.14) is equal to

$$P\left\{ \sup_{0 \leq t \leq T^\varepsilon} \int_A |\psi_t^\varepsilon(x) - g_t(x)|^{2p} \, dx \geq \eta, \|\sqrt{\varepsilon} W\| < \alpha \right\}.$$

Following the notation introduced in (1.7), we set $V_t^\varepsilon(x) = \sqrt{\varepsilon}\int_0^t \sigma(\psi_s^\varepsilon(x))\,dW_s$. Then

$$|\psi_t^\varepsilon(x) - g_t(x)| \le |V_t^\varepsilon(x)| + \int_0^t \varepsilon |m(\psi_s^\varepsilon(x))|\,ds$$
$$+ \int_0^t \left[|b(\psi_s^\varepsilon(x)) - b(g_s(x))| + |\sigma(\psi_s^\varepsilon(x)) - \sigma(g_s(x))| \|\dot{f}_s\| \right] ds.$$

The Lipschitz and growth conditions on b, σ and m together with $\|g\|_A < \infty$ yield that

$$|\psi_t^\varepsilon(x) - g_t(x)| \le |V_t^\varepsilon(x)| + C\int_0^t |\psi_s^\varepsilon(x) - g_s(x)|\left[1 + |\dot{f}_s|\right] ds + C\varepsilon.$$

By Gronwall's lemma we conclude that for $\tilde{I}(f) \le \lambda$,

$$\begin{aligned} |\psi_t^\varepsilon(x) - g_t(x)| &\le \left[|V_t^\varepsilon(x)| + C\varepsilon\right] \exp\left(C\int_0^1 \left(1 + |\dot{f}_s|\right) ds \right) \\ &\le C|V_t^\varepsilon(x)| + \varepsilon C. \end{aligned} \tag{1.15}$$

Set $C^\varepsilon = \{\sup_{0 \le t \le T^\varepsilon} \int_A |V_t^\varepsilon(x)|^{2p}\,dx \ge \eta, \|\sqrt{\varepsilon}\,W\| < \alpha\}$.

Due to the inequality (1.15), the proof of (1.14) reduces to showing that for any $R, \lambda, \eta > 0$ and $\varepsilon_0 \in (0,1]$, there exists $\alpha > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $f \in H^k$ with $\tilde{I}(f) \le \lambda$,

$$P(C^\varepsilon) \le \exp\left(-\frac{R}{\varepsilon} \right). \tag{1.16}$$

Fix $n \ge 1$ and for $0 \le k \le n$, set $t_k = (k/n)$. Recall that $\psi_t^\varepsilon(n,x) = \psi_{t_k}^\varepsilon(x)$ if $t \in [t_k, t_{k+1})$ and $V_t^\varepsilon(n,x) = \sqrt{\varepsilon}\int_0^t \sigma(\psi_s^\varepsilon(n,x))\,dW_s$. Then we have the following decomposition:

$$C^\varepsilon \subset C_1^\varepsilon \cup C_2^\varepsilon \cup C_3^\varepsilon,$$

where

$$C_1^\varepsilon = \left\{ \sup_{0 \le t \le T^\varepsilon} \int_A |V_t^\varepsilon(n,x)|^{2p}\,dx \ge \frac{\eta}{2^{2p}}, \|\sqrt{\varepsilon}\,W\| < \alpha \right\},$$

$$C_2^\varepsilon = \left\{ \sup_{0 \le t \le T^\varepsilon} \int_A |V_t^\varepsilon(x) - V_t^\varepsilon(n,x)|^{2p}\,dx \ge \frac{\eta}{2^{2p}}, \right.$$
$$\left. \sup_{0 \le t \le T^\varepsilon} \int_A |\psi_t^\varepsilon(x) - \psi_t^\varepsilon(n,x)|^{2p}\,dx \le \delta \right\},$$

$$C_3^\varepsilon = \left\{ \sup_{0 \le t \le T^\varepsilon} \int_A |\psi_t^\varepsilon(x) - \psi_t^\varepsilon(n,x)|^{2p}\,dx > \delta \right\},$$

with $\delta > 0$ and $n \ge 1$ arbitrary.

Lemma 1.4 allows us to choose δ independently of n and $i(v)$ such that for $\varepsilon \in (0,1]$, $P(C_2^\varepsilon) \le \exp(-R/\varepsilon)$. On the other hand, if $t \le T^\varepsilon$, $\int_A |\psi_t^\varepsilon(x)|^{2p}\,dx \le C[\eta + (\|g\|_A)^{2p}]$, which is bounded by a constant depending only on η, A, p, λ

and the Lipschitz coefficients. Thus Lemma 1.3 yields the existence of n_1, depending on δ, λ and R, such that for every $n \geq n_1$, $\varepsilon \in (0,1]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$, $P(C_3^\varepsilon) \leq \exp(-R/\varepsilon)$.

Finally, Lemma 1.2 yields the existence of $\alpha > 0$, depending on n_1, such that $C_1^\varepsilon = \varnothing$ for $n = n_1$. Thus for this choice of α, given any $\varepsilon \in (0, \varepsilon_0] \subset (0,1]$ and $f \in H^k$ such that $\tilde{I}(f) < \lambda$, we have

$$P(C^\varepsilon) \leq 2\exp\left(-\frac{R}{\varepsilon}\right) \leq \exp\left(-\frac{R'}{\varepsilon}\right) \quad \text{if } R' \leq R - \log 2.$$

This completes the proof of the proposition. □

The derivative $\nabla\varphi^\varepsilon(x) = \{(\partial/\partial x)\varphi_t^\varepsilon(x), t \in [0,1]\}$ satisfies the stochastic differential system on $\mathbb{R}^{d\times d}$:

$$\begin{aligned}\nabla\varphi_t^\varepsilon(x) &= Id + \sqrt{\varepsilon}\int_0^t \sigma'(\varphi_s^\varepsilon(x))\nabla\varphi_s^\varepsilon(x)\,dW_s \\ &\quad + \int_0^t \left[\varepsilon m'(\varphi_s^\varepsilon(x)) + b'(\varphi_s^\varepsilon(x))\right]\nabla\varphi_s^\varepsilon(x)\,ds.\end{aligned} \tag{1.17}$$

The derivative $h(x) = \{(\partial/\partial x)g_t(x), t \in [0,1]\}$ is also clearly a solution of the (ordinary) differential system on $\mathbb{R}^{d\times d}$:

$$h_t(x) = Id + \int_0^t \left[b'(g_s(x))h_s(x) + \sigma'(g_s(x))h_s(x)\dot{f}_s\right]ds. \tag{1.18}$$

Since σ' and b' are bounded, Gronwall's lemma implies that for every compact subset A of $\mathbb{R}^d$,

$$\sup_{f:\,\tilde{I}(f)\leq\lambda} \|h\|_A < +\infty. \tag{1.19}$$

The next result is the analogue of Proposition 1.6 for $\nabla\varphi^\varepsilon$.

PROPOSITION 1.7. *Assume that the assumptions of Theorem* 1.1 *are satisfied. Then for any positive numbers R, λ and η and any compact subset A of $\mathbb{R}^d$, there exists $\varepsilon_0 > 0$ and $\alpha > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$,*

$$P\left\{\left\|\int_A |\nabla\varphi_t^\varepsilon(x) - h_t(x)|^{2p}\,dx\right\| \geq \eta, \|\sqrt{\varepsilon}W - f\| < \alpha\right\} \leq \exp\left(-\frac{R}{\varepsilon}\right). \tag{1.20}$$

PROOF. Let $\{\zeta_t^\varepsilon(x), t \in [0,1]\}$ denote the solution of the stochastic differential system on $\mathbb{R}^{d\times d}$:

$$\begin{aligned}\zeta_t^\varepsilon(x) &= Id + \sqrt{\varepsilon}\int_0^t \sigma'(\psi_s^\varepsilon(x))\zeta_s^\varepsilon(x)\,dW_s \\ &\quad + \int_0^t \left[b'(\psi_s^\varepsilon(x))\zeta_s^\varepsilon(x) + \varepsilon m'(\psi_s^\varepsilon(x))\zeta_s^\varepsilon(x) + \sigma'(\psi_s^\varepsilon(x))\zeta_s^\varepsilon(x)\dot{f}_s\right]ds.\end{aligned} \tag{1.21}$$

Then (1.21) is a particular case of (1.5) with $N = d$, $m = d^2$, $S(x,y) = \sigma'(x)y$, $B^\varepsilon(s,x,y) = b'(x)y + \varepsilon m'(x)y + \sigma'(x)y\dot{f}_s$, $\Psi_t^\varepsilon(x) = \zeta_t^\varepsilon(x)$ and $\Phi_t^\varepsilon(x) = \psi_t^\varepsilon(x)$.

The conditions (H1) and (H2) are obviously satisfied with $v(s) = 1 + |\dot{f}_s|$. Let P^ε denote the probability on Ω defined by (1.12), and set $W_t^\varepsilon = W_t - (1/\sqrt{\varepsilon}) f_t$. By Girsanov's theorem the law of $(W, \psi^\varepsilon(x), \zeta^\varepsilon(x))$ under P is the same as the law of $(W^\varepsilon, \varphi^\varepsilon(x), \nabla\varphi^\varepsilon(x))$ under P^ε. Therefore using the argument of the proof of Proposition 1.6, it suffices to show that for any R, λ, $\eta > 0$, there exists $\varepsilon_0 > 0$ and $\alpha > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $f \in H^k$ with $\tilde{I}(f) \le \lambda$,

$$(1.22)\quad P\left\{\left\| \int_A |\zeta_t^\varepsilon(x) - h_t(x)|^{2p}\, dx \right\| \ge \eta, \|\sqrt{\varepsilon}\, W\| < \alpha \right\} \le \exp\left(-\frac{R}{\varepsilon}\right).$$

Let $S^\varepsilon = \inf\{t \ge 0: \int_A |\zeta_t^\varepsilon(x) - h_t(x)|^{2p}\, dx \ge \eta\} \wedge 1$. Notice that

$$\begin{aligned} D^\varepsilon &:= \left\{\left\| \int_A |\zeta_t^\varepsilon(x) - h_t(x)|^{2p}\, dx \right\| \ge \eta, \|\sqrt{\varepsilon}\, W\| < \alpha \right\} \\ &= \left\{ \sup_{0 \le t \le S^\varepsilon} \int_A |\zeta_t^\varepsilon(x) - h_t(x)|^{2p}\, dx \ge \eta, \|\sqrt{\varepsilon}\, W\| < \alpha \right\}, \end{aligned}$$

and that if $t \le S^\varepsilon$, using (1.19), $\int_A |\zeta_t^\varepsilon(x)|^{2p}\, dx \le 2^{2p-1}[\eta + |A|(\|h\|_A)^{2p}] \le C$.

Set, according to (1.7), $V_t^\varepsilon(x) = \sqrt{\varepsilon} \int_0^t \sigma'(\psi_s^\varepsilon(x)) \zeta_s^\varepsilon(x)\, dW_s$.

By definition,

$$\begin{aligned} &|\zeta_t^\varepsilon(x) - h_t(x)| \\ &\quad \le |V_t^\varepsilon(x)| + \int_0^t |\sigma'(\psi_s^\varepsilon(x))\zeta_s^\varepsilon(x) - \sigma'(g_s(x)) h_s(x)| |\dot{f}_s|\, ds \\ &\qquad + \int_0^t \{|b'(\psi_s^\varepsilon(x))\zeta_s^\varepsilon(x) - b'(g_s(x)) h_s(x)| + \varepsilon |m'(\psi_s^\varepsilon(x))\zeta_s^\varepsilon(x)|\}\, ds \\ &\quad \le |V_t^\varepsilon(x)| + \int_0^t \{|\sigma'(\psi_s^\varepsilon(x))| |\zeta_s^\varepsilon(x) - h_s(x)| |\dot{f}_s| \\ &\qquad\qquad + |\sigma'(\psi_s^\varepsilon(x)) - \sigma'(g_s(x))| |h_s(x)| |\dot{f}_s| \\ &\qquad\qquad + |b'(\psi_s^\varepsilon(x))| |\zeta_s^\varepsilon(x) - h_s(x)| + |b'(\psi_s^\varepsilon(x)) - b'(g_s(x))| |h_s(x)| \\ &\qquad\qquad + \varepsilon |m'(\psi_s^\varepsilon(x))| |h_s(x)| + \varepsilon |m'(\psi_s^\varepsilon(x))| |\zeta_s^\varepsilon(x) - h_s(x)|\}\, ds. \end{aligned}$$

Since the partial derivatives of σ, b and m are bounded and Lipschitz functions, we have that

$$\begin{aligned} |\zeta_t^\varepsilon(x) - h_t(x)| \le C\Big[&|V_t^\varepsilon(x)| + \int_0^t |\zeta_s^\varepsilon(x) - h_s(x)| (1 + |\dot{f}_s|)\, ds \\ &+ \int_0^t |\psi_s^\varepsilon(x) - g_s(x)| (1 + |\dot{f}_s|)\, ds + \varepsilon \Big]. \end{aligned}$$

Therefore, Gronwall's lemma yields that for $\tilde{I}(f) \le \lambda$,

$$|\zeta_t^\varepsilon(x) - h_t(x)| \le C\left[|V_t^\varepsilon(x)| + \int_0^t |\psi_s^\varepsilon(x) - g_s(x)| (1 + |\dot{f}_s|)\, ds + \varepsilon \right].$$

This inequality implies that there exists $\varepsilon_1 \in (0, 1]$ and η' which depends on η,

p and λ, such that for any $\varepsilon \in (0, \varepsilon_1]$ and $f \in H^k$ such that $\tilde{I}(f) \leq \lambda$,

$$P(D^\varepsilon) \leq P\left\{ \sup_{0 \leq t \leq S^\varepsilon} \left(\int_A |V_t^\varepsilon(x)|^{2p}\, dx + \int_A |\psi_t^\varepsilon(x) - g_t(x)|^{2p}\, dx \right) > \eta', \right.$$
$$\left. \|\sqrt{\varepsilon}\, W\| < \alpha \right\}.$$

By the proof of Proposition 1.6 (see 1.14), there exists $\varepsilon \in (0,1]$ and $\alpha_2 > 0$ such that for any $\varepsilon \in (0, \varepsilon_2]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$,

$$P\left\{ \sup_{0 \leq t \leq S^\varepsilon} \int_A |\psi_t^\varepsilon(x) - g_t(x)|^{2p}\, dx > \frac{\eta'}{2}, \|\sqrt{\varepsilon}\, W\| < \alpha_2 \right\} \leq \exp\left(-\frac{R}{\varepsilon} \right).$$

Therefore, the proof of the estimate (1.22) on $P(D^\varepsilon)$ reduces to showing the following: Given η', λ, $R > 0$, there exist $\alpha_3 > 0$ and $\varepsilon_3 \in (0,1]$ such that for every $\varepsilon \in (0, \varepsilon_3]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$, if

$$F^\varepsilon = \left\{ \sup_{0 \leq t \leq S^\varepsilon} \int_A |V_t^\varepsilon(x)|^{2p}\, dx > \frac{\eta'}{2}, \right.$$
$$\left. \sup_{0 \leq t \leq S^\varepsilon} \int_A |\psi_t^\varepsilon(x) - g_t(x)|^{2p}\, dx \leq \frac{\eta'}{2}, \|\sqrt{\varepsilon}\, W\| < \alpha_3 \right\},$$

then

$$P(F^\varepsilon) \leq \exp\left(-\frac{R}{\varepsilon} \right). \tag{1.23}$$

Indeed (1.22) will hold for $\varepsilon_0 = \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$ and $\alpha = \alpha_2 \wedge \alpha_3$.

We now prove (1.23). In the sequel we write α instead of α_3 and η instead of $\eta'/2$. For every $n \geq 1$ and $0 \leq k \leq n$, set $t_k = k/n$. Following the notations stated in (1.6), for $0 \leq k \leq n-1$ and $t \in [t_k, t_{k+1})$, set $\psi_t^\varepsilon(n, x) = \psi_{t_k}^\varepsilon(x)$ and $\zeta_t^\varepsilon(n, x) = \zeta_{t_k}^\varepsilon(x)$. Also let $V_t^\varepsilon(n, x) = \sqrt{\varepsilon} \int_0^t \sigma'(\psi_s^\varepsilon(x)) \zeta_s^\varepsilon(n, x)\, dW_s$ and $\tilde{V}_t^\varepsilon(n, x) = \sqrt{\varepsilon} \int_0^t \sigma'(\psi_s^\varepsilon(n, x)) \zeta_s^\varepsilon(n, x)\, dW_s$. As in the proof of Proposition 1.6, set

$$T^\varepsilon = \inf\left\{ t \geq 0 \colon \int_A |\psi_t^\varepsilon(x) - g_t(x)|^{2p}\, dx \geq \eta \right\} \wedge 1.$$

Then for $t \leq T^\varepsilon$, $\int_A |\psi_t^\varepsilon(x)|^{2p}\, dx \leq C[\eta + (\|g\|_A)^{2p}] < \infty$.

Fix $\gamma > 0$; Lemma 1.3 yields the existence of an integer $N_0 \geq 1$ such that for $n \geq N_0$, $\varepsilon \in (0,1]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$,

$$P\left\{ \sup_{0 \leq t \leq T^\varepsilon} \int_A |\psi_t^\varepsilon(x) - \psi_t^\varepsilon(n, x)|^{2p}\, dx \geq \gamma \right\} \leq \exp\left(-\frac{R}{\varepsilon} \right).$$

Furthermore, Proposition 1.6 yields the existence of $\varepsilon_3 \in (0,1]$ and $\alpha_4 > 0$ such that for every $\varepsilon \in (0, \varepsilon_3]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$,

$$P\{T^\varepsilon < 1, \|\sqrt{\varepsilon}\, W\| < \alpha_4\} \leq \exp\left(-\frac{R}{\varepsilon} \right).$$

Therefore, for any $n \geq N_0$, $\varepsilon \in (0, \varepsilon_3]$ and $\tilde{I}(f) \leq \lambda$,

$$(1.24)\quad P\left\{\left\| \int_A |\psi_t^\varepsilon(x) - \psi_t^\varepsilon(n,x)|^{2p}\,dx \right\| \geq \gamma, \|\sqrt{\varepsilon}\,W\| < \alpha_4\right\} \leq 2\exp\left(-\frac{R}{\varepsilon}\right).$$

Lemma 1.3 also yields the existence of an integer N_1, such that for every $n \geq N_1$, $\tilde{I}(f) \leq \lambda$,

$$(1.25)\quad P\left\{ \sup_{0\leq t\leq S^\varepsilon} \int_A |\zeta_t^\varepsilon(x) - \zeta_t^\varepsilon(n,x)|^{2p}\,dx \geq \gamma\right\} \leq \exp\left(-\frac{R}{\varepsilon}\right).$$

Since $2p > d$ and $\zeta_t^\varepsilon - \zeta_t^\varepsilon(n,x) = (\partial/\partial x)[\psi_t^\varepsilon(x) - \psi_t^\varepsilon(n,x)]$, Sobolev's inequality implies the existence of a constant C_p such that

$$\sup_{0\leq t\leq S^\varepsilon}\sup_{x\in A}|\psi_t^\varepsilon(x) - \psi_t^\varepsilon(n,x)| \leq C_p\left\{\sup_{0\leq t\leq S^\varepsilon}\int_A |\psi_t^\varepsilon(x) - \psi_t^\varepsilon(n,x)|^{2p}\,dx + \sup_{0\leq t\leq S^\varepsilon}\int_A |\zeta_t^\varepsilon(x) - \zeta_t^\varepsilon(n,x)|^{2p}\,dx\right\}^{(1/2p)}.$$

Hence from (1.24) and (1.25), there exists a constant $\bar{\gamma}$, which only depends on γ and p, such that for $n \geq N_0 \vee N_1$, $\varepsilon \in (0, \varepsilon_3]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$,

$$(1.26)\quad P\left\{\sup_{0\leq t\leq S^\varepsilon}\sup_{x\in A}|\psi_t^\varepsilon(x) - \psi_t^\varepsilon(n,x)| > \bar{\gamma}, \|\sqrt{\varepsilon}\,W\| < \alpha_4\right\} \leq 3\exp\left(-\frac{R}{\varepsilon}\right).$$

Set $F^\varepsilon \subset F_1^\varepsilon \cup F_2^\varepsilon \cup F_3^\varepsilon \cup F_4^\varepsilon \cup F_5^\varepsilon$, with

$$F_1^\varepsilon = \left\{\sup_{0\leq t\leq S^\varepsilon}\int_A |V_t^\varepsilon(x) - V_t^\varepsilon(n,x)|^{2p}\,dx > \bar{\eta},\ \sup_{0\leq t\leq S^\varepsilon}\int_A |\zeta_t^\varepsilon(x) - \zeta_t^\varepsilon(n,x)|^{2p}\,dx \leq \delta\right\},$$

$$F_2^\varepsilon = \left\{\sup_{0\leq t\leq S^\varepsilon}\int_A \left|V_t^\varepsilon(n,x) - \tilde{V}_t^\varepsilon(n,x)\right|^{2p}\,dx > \bar{\eta},\ \sup_{0\leq t\leq S^\varepsilon}\int_A |\zeta_t^\varepsilon(x) - \zeta_t^\varepsilon(n,x)|^{2p}\,dx \leq \delta,\ \sup_{0\leq t\leq S^\varepsilon}\sup_{x\in A}|\psi_t^\varepsilon(x) - \psi_t^\varepsilon(n,x)|^{2p}\,dx \leq \bar{\gamma}\right\},$$

$$F_3^\varepsilon = \left\{\sup_{0\leq t\leq S^\varepsilon}\int_A \left|\tilde{V}_t^\varepsilon(n,x)\right|^{2p}\,dx > \bar{\eta}, \|\sqrt{\varepsilon}\,W\| < \alpha_3\right\},$$

$$F_4^\varepsilon = \left\{\sup_{0\leq t\leq S^\varepsilon}\int_A |\zeta_t^\varepsilon(x) - \zeta_t^\varepsilon(n,x)|^{2p}\,dx > \delta\right\},$$

$$F_5^\varepsilon = \left\{\sup_{0\leq t\leq S^\varepsilon}\sup_{x\in A}|\psi_t^\varepsilon(x) - \psi_t^\varepsilon(n,x)| > \bar{\gamma}, \|\sqrt{\varepsilon}\,W\| < \alpha_3\right\},$$

where $\bar{\eta}$ depends on η and p, and δ, $\bar{\gamma}$ are arbitrary.

Recall that R, λ and η are fixed. By Lemma 1.4, choose $\delta \in (0,1]$ such that for every $\varepsilon \in (0,1]$, $f \in H^k$ and $n \geq 1$, $P(F_1^\varepsilon) \leq \exp(-R/\varepsilon)$. Apply Lemma 1.5 to obtain $\bar{\gamma} > 0$ such that for every $\varepsilon \in (0,1]$, $f \in H^k$ and $n \geq 1$, $P(F_2^\varepsilon) \leq \exp(-R/\varepsilon)$. By Lemma 1.3, choose an integer $N_2 \geq 1$ such that for $n \geq N_2$, $\varepsilon \in (0,1]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$, $P(F_4^\varepsilon) \leq \exp(-R/\varepsilon)$. Then by (1.26), choose $\alpha_4 > 0$ and $\varepsilon_3 \in (0,1]$ such that for every $n \geq N_0 \vee N_1$, $\varepsilon \in (0, \varepsilon_3]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$, $P(F_5^\varepsilon) \leq 3\exp(-R/\varepsilon)$. Choose an integer $n > N_0 \vee N_1 \vee N_2$, and by Lemma 1.2, choose $\alpha_3 \in (0, \alpha_4]$ such that $F_3^\varepsilon = \varnothing$. Then for $\varepsilon \in (0, \varepsilon_3]$ and $f \in H^k$ with $\tilde{I}(f) \leq \lambda$, $P(F^\varepsilon) \leq 6\exp(-R/\varepsilon) \leq \exp(-\tilde{R}/\varepsilon)$ if $\tilde{R} \leq R - \log 6$; this concludes the proof of (1.23) and of the proposition. □

Proof of Theorem 1.1. Since $p > d/2 \vee 1$, Sobolev's inequality yields the existence of a constant C such that

$$\|\varphi^\varepsilon - g\|_A \leq C\left\{\left\|\int_A |\varphi_t^\varepsilon(x) - g_t(x)|^{2p}\,dx\right\| + \left\|\int_A |\nabla\varphi_t^\varepsilon(x) - h_t(x)|^{2p}\,dx\right\|\right\}^{1/2p}.$$

Therefore the inequalities (1.11) and (1.20) in Propositions 1.6 and 1.7 yield the estimate (1.3) and conclude the proof. □

2. Large deviation principle for the flow. This section contains an application of the uniform Ventzell–Freidlin estimates proved in Theorem 1.1. Our aim is to establish a large deviation principle for the family $\{Q^\varepsilon, \varepsilon > 0\}$ of probabilities on $\mathscr{C}([0,1] \times \mathbb{R}^d; \mathbb{R}^d)$ corresponding to the laws of the stochastic flows

$$\varphi_t^\varepsilon(x) = x + \sqrt{\varepsilon}\int_0^t \sigma_i(\varphi_s^\varepsilon(x)) \circ dW_s^i + \int_0^t b(\varphi_s^\varepsilon(x))\,ds. \tag{2.1}$$

First we will assume that the initial condition x in (2.1) belongs to some arbitrary compact subset A of $\mathbb{R}^d$ and we will prove in this case the large deviation principle on $\mathscr{C}([0,1] \times A; \mathbb{R}^d)$. This result will be obtained as an application of a general statement concerning the transfer of Ventzell–Freidlin estimates (see Proposition 2.1).

Large deviations estimates for the flow on $\mathscr{C}([0,1] \times A; \mathbb{R}^d)$ are all that we need in order to establish the large deviation principle for the anticipating stochastic differential equation (0.2). We will deal with this problem in Section 4.

For any integer $n > 0$, set $A_n = \{x \in \mathbb{R}^d\colon |x| \leq n\}$. The space $\mathscr{C}([0,1] \times \mathbb{R}^d; \mathbb{R}^d)$ is the projective limit of the Polish spaces $\mathscr{C}([0,1] \times A_n; \mathbb{R}^d)$. Then we can finally use the results for large deviations on projective limit spaces (see [2] and [3]) to accomplish the program of this section.

Let (E_i, d_i), $i = 1, 2$, be two Polish spaces and $X_i^\varepsilon\colon \Omega \to E_i$, $\varepsilon > 0$, $i = 1, 2$, be two families of random variables. Assume that $\{X_1^\varepsilon, \varepsilon > 0\}$ satisfies a large deviation principle with rate function $\tilde{I}\colon E_1 \to [0, \infty]$.

Let $F\colon \{\tilde{I} < +\infty\} \to E_2$ be a mapping such that the restriction to the compact sets $\{\tilde{I} \leq a\}$, $a \in [0, \infty)$ is continuous. For any $g \in E_2$, we set $I(g) = \inf\{\tilde{I}(f)\colon F(f) = g\}$. Then we can prove the following result.

PROPOSITION 2.1. *Suppose that for any* $a, R, \eta > 0$, *there exist* $\alpha > 0$ *and* $\varepsilon_0 > 0$ *such that if* $f \in E_1$ *satisfies* $\tilde{I}(f) \le a$,

$$P\{d_2(X_2^\varepsilon, F(f)) \ge \eta, d_1(X_1^\varepsilon, f) < \alpha\} \le \exp\left(-\frac{R}{\varepsilon}\right) \tag{2.2}$$

for any $\varepsilon \le \varepsilon_0$. *Then, the family* $\{X_2^\varepsilon, \varepsilon > 0\}$ *satisfies a large deviation principle with rate function* I.

Proposition 2.1 represents a synthesis of Azencott's results (see [1]) in a general setting well suited to our situation. A similar approach is given in [4]. Its proof relies on the two next estimates:

(i) For any $a, \eta, \delta > 0$, there exists $\varepsilon_0 > 0$ such that if $I(g) \le a$, then for any $\varepsilon \le \varepsilon_0$,

$$\varepsilon \log P\{d_2(X_2^\varepsilon, g) < \eta\} \ge -I(g) - \delta. \tag{2.3}$$

(ii) For any real $a > 0$, set $\Phi(a) = \{g: I(g) \le a\}$. Then, for any $\bar{a}, \eta, \delta > 0$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \le \varepsilon_0$ and $a \le \bar{a}$,

$$\varepsilon \log P\{d_2(X_2^\varepsilon, \Phi(a)) \ge \eta\} \le -a + \delta. \tag{2.4}$$

We will not provide a proof of these estimates since it follows along the same lines as the proofs of Propositions 3.2 and 3.3 in [4].

Proposition 2.1 can be applied in the following situation: Let E_1 be the space $\mathscr{C}([0,1]; \mathbb{R}^k)$ with the distance d_1 given by the supremum norm $\|\cdot\|$. Set $X_1^\varepsilon = \sqrt{\varepsilon}\, W$. Fix an arbitrary compact subset A or $\mathbb{R}^d$ and let $E_2 = \mathscr{C}([0,1] \times A; \mathbb{R}^d)$ endowed with the distance d_2 derived from the supremum norm $\|\cdot\|_A$. Set $X_2^\varepsilon = \varphi^\varepsilon$, the flow defined by (2.1) with $x \in A$.

Let $\tilde{I}(\cdot)$ be the rate function corresponding to the k-dimension Wiener process W, that is,

$$\tilde{I}(f) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{f}_s|^2\, ds, & \text{if } f \in H^k, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.5}$$

For any $f \in H^k$, define $F(f) = g$, where $g_t(x)$ is the solution of the ordinary differential equation (1.2) with $x \in A$. Gronwall's lemma shows the continuity of F: $H^k \to \mathscr{C}([0,1] \times A; \mathbb{R}^d)$ on the compact sets $\{\tilde{I} \le a\}$, $a \in [0, \infty]$. Therefore Theorem 1.1 combined with Proposition 2.1 proves the following result.

PROPOSITION 2.2. *Assume that the coefficients* σ, b *and* $m(\cdot) = \frac{1}{2}\sum_{i=1}^k (\partial\sigma_i/\partial x)(\cdot)\sigma_i(\cdot)$ *are of class* $\mathscr{C}^2$ *with bounded partial derivatives up to the second order. Fix an arbitrary compact subset A of* $\mathbb{R}^d$. *Then, the family* $\{Q_A^\varepsilon, \varepsilon > 0\}$ *of probabilities on* $\mathscr{C}([0,1] \times A; \mathbb{R}^d)$ *corresponding to the laws of the solutions of* (2.1) *with* $x \in A$ *satisfies a large deviation principle with rate function* $I(g) = \inf\{\tilde{I}(f)\colon F(f) = g\}$.

We finally consider the general situation. Set $E_1 = \mathscr{C}([0,1];\mathbb{R}^k)$ as in the preceding case, and let $E_2 = \mathscr{C}([0,1]\times\mathbb{R}^d;\mathbb{R}^d)$ endowed with the metric $d_2(g_1, g_2) = \sum_{n\geq 1} 2^{-n}(\|g_1 - g_2\|_{A_n} \wedge 1)$, which defines the topology of uniform convergence on the compact sets of $[0,1]\times\mathbb{R}^d$.

The mapping F: $H^k \to \mathscr{C}([0,1]\times\mathbb{R}^d, \mathbb{R}^d)$ defined by $F(f) = g$, where g is the solution of (1.2) with $x \in \mathbb{R}^d$, is continuous on the compact sets $\{\tilde{I} \leq a\}$, $a \in [0,\infty)$. Furthermore, condition (2.2) of Proposition 2.1 is satisfied. Indeed, fix $\eta > 0$ and let $N \geq 1$ be such that $\sum_{n\geq N+1} 2^{-n} < \eta/2$. Then

$$P\{d_2(\varphi^\varepsilon, F(f)) \geq \eta,\, d_1(\sqrt{\varepsilon}W, f) < \alpha\}$$

$$\leq P\left\{\sup_{0\leq t\leq 1}\sup_{|x|\leq N}|\varphi_t^\varepsilon(x) - g_t(x)| \geq \eta/2,\ \sup_{0\leq t\leq 1}\left|\sqrt{\varepsilon}W_t - f(t)\right| < \alpha\right\},$$

and the result follows by Theorem 1.1. Therefore Proposition 2.1 yields the large deviation principle for the flow as follows.

THEOREM 2.3. *Assume the hypothesis of Proposition 2.2. Then the family $\{Q^\varepsilon, \varepsilon > 0\}$ of probabilities on $\mathscr{C}([0,1]\times\mathbb{R}^d;\mathbb{R}^d)$ given by the laws of the flows $\{\varphi^\varepsilon, \varepsilon > 0\}$ obeys a large deviation principle with rate function*

$$I(g) = \inf\{\tilde{I}(f)\},$$

where the infimum extends to the functions $f \in H^k$ such that

$$g_t(x) = x + \int_0^t \{\sigma(g_s(x))\dot{f}_s + b(g_s(x))\}\, ds.$$

3. Existence theorem. Let $\pi = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be an arbitrary partition of the interval $[0,1]$ and ν be a nonnecessarily adapted process in $L^2([0,1]\times\Omega;\mathbb{R}^k)$. We define the Riemann sum

$$S^\pi = \sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i}\left(\int_{t_i}^{t_{i+1}} \nu_j(s)\, ds\right)\left(W_{t_{i+1}}^j - W_{t_i}^j\right).$$

The process ν is said to be Stratonovich integrable if the family $\{S^\pi\}$ converges in probability as the norm $|\pi|$ tends to zero. The limit is called the Stratonovich integral of ν and is denoted by $\int_0^1 \nu_s \circ dW_s$. We refer the reader to [8] for conditions ensuring the existence of this integral.

Let σ_i: $\mathbb{R}^d \to \mathbb{R}^d$, $i = 1,\dots,k$, and b: $\mathbb{R}^d \to \mathbb{R}^d$ be sufficiently smooth. Consider the flow on $\mathbb{R}^d$ defined by the adapted equations

$$\begin{aligned}\varphi_t(x) &= x + \int_0^t \sigma_i(\varphi_s(x)) \circ dW_s^i + \int_0^t b(\varphi_s(x))\, ds \\ &= x + \int_0^t \sigma_i(\varphi_s(x))\, dW_s^i + \int_0^t (m + b)(\varphi_s(x))\, ds,\end{aligned} \tag{3.1}$$

where $m(x) = \frac{1}{2}\sum_{i=1}^k (\partial\sigma_i/\partial x)(x)\sigma_i(x)$. The following theorem shows that given an arbitrary initial condition X_0, the process $X_t = \varphi_t(X_0)$ is a solution of (0.1) in the particular case of coefficients σ and b which do not depend on t and ω.

Note that a general result on existence and uniqueness of the solution of (0.1) has been established in [9] as a consequence of a generalized Itô-Ventzell formula. The result needed here is much simpler: The coefficients σ and b only depend on the state x and we do not care about uniqueness, so that we have fewer regularity requirements. Furthermore, the initial condition X_0 need not be regular (in the sense of the calculus of variations on Ω).

THEOREM 3.1. *Let σ, b and m be of class $\mathscr{C}^2$ with bounded partial derivatives up to order 2. Then, for any d-dimensional random vector X_0, the process $X_t = \varphi_t(X_0)$ satisfies the anticipating stochastic differential equation*

$$X_t = X_0 + \int_0^t \sigma_i(X_s) \circ dW_s^i + \int_0^t b(X_s)\, ds. \tag{3.2}$$

PROOF. It suffices to check that the Proposition 7.8 in [8] can be applied to the generalized Stratonovich integral and yields that

$$\int_0^t \sigma_i(\varphi_s(X_0)) \circ dW_s^i = \int_0^t \sigma_i(\varphi_s(x)) \circ dW_s^i \big|_{x=X_0}. \tag{3.3}$$

For $i \in \{1, \dots, k\}$, set $u_i(t, x, \omega) = \sigma_i(\varphi_t(x))$. Then the maps $(t, x, \omega) \in [0,1] \times \mathbb{R}^d \times \Omega \to u_i(t, x, \omega) \in \mathbb{R}^d$ are measurable, and for every $x \in \mathbb{R}^d$, the processes $\{u_i(t, x), t \in [0,1]\}$ are progressively measurable with respect to the filtration $\{\mathscr{F}_t, t \in [0,1]\}$ generated by the Brownian motion $\{W_t, t \in [0,1]\}$. Furthermore, there exists a version of $\varphi_t(x)$, solution of (3.1) such that $\varphi_t(\cdot)$ is $\mathscr{C}^2$ for every t, φ is a.s. jointly continuous in (t, x) and such that for any open bounded subset D of $\mathbb{R}^d$ and any $p \geq 2$,

$$E\int_D \int_0^1 \left\{ |\varphi_t(x)|^p + \left| \frac{\partial}{\partial x} \varphi_t(x) \right|^p + \left| \frac{\partial^2}{\partial x^2} \varphi_t(x) \right|^p \right\} dt\, dx < \infty \tag{3.4}$$

(see, e.g. [5], Chapter II, Theorems 2.2 and 3.3). The hypotheses on the coefficients σ, b and m imply the existence of a constant C_1 such that

$$\sum_{i=1}^k |\sigma_i(x) - \sigma_i(y)| + |b(x) - b(y)| - |m(x) - m(y)| \leq C_1 |x - y|,$$

$$\sum_{i=1}^k |\sigma_i(x)| + |b(x)| + |m(x)| \leq C_1(1 + |x|).$$

Consequently, for any open bounded subset D of $\mathbb{R}^d$ and $p > d$, $p \geq 4$,

$$E\int_D \int_0^1 \left\{ |u_i(t, x)|^p + \left| \frac{\partial}{\partial x} u_i(t, x) \right|^p + \left| \frac{\partial^2}{\partial x^2} u_i(t, x) \right|^p \right\} dt\, dx < \infty.$$

Moreover, by Sobolev's inequality,

$$E\int_0^1 \sup_{x\in D}\left|\frac{\partial}{\partial x}u_i(t,x)\right|^4 dt<\infty.$$

Therefore, assumptions (H1) to (H4) in [8], Proposition 7.8, are satisfied. Let K be a compact subset of $\mathbb{R}^d$. Itô's formula implies that, if $a=\sigma\sigma^*$,

$$\begin{aligned}\sigma_i(\varphi_t(x)) = \sigma_i(x) &+ \int_0^t \nabla_j\sigma_i(\varphi_s(x))\sigma_l^j(\varphi_s(x))\,dW_s^l\\ &+\int_0^t\Big\{\nabla_j\sigma_i(\varphi_s(x))[m+b]^j(\varphi_s(x))\\ &\qquad+\tfrac{1}{2}\nabla_{jl}\sigma_i(\varphi_s(x))a^{jl}(\varphi_s(x))\Big\}\,ds.\end{aligned}$$

The global Lipschitz property of σ implies the existence of a constant C such that $|a(x)|\le C(1+|x|^2)$. For every $r\ge 2$, Burkholder's inequality and Gronwall's lemma yield

$$\sup_{x\in K}E\int_0^1\left\{|\varphi_t(x)|^r+\left|\frac{\partial}{\partial x}\varphi_t(x)\right|^r\right\}dt<\infty.$$

Since σ_i is of class $\mathscr{C}^2$ with bounded partial derivatives of order $k\le 2$, this implies that for any $p>d$, $p\ge 2$, the maps $t\mapsto\sigma_i(\varphi_t(x))$ and $t\mapsto(\partial/\partial x)\sigma_i(\varphi_t(x))$ are continuous from $[0,1]$ into $L^p(\Omega)$ uniformly in $x\in K$, and hence the validity of hypothesis (H5) of Proposition 7.8 ([8]).

Given a subdivision $\pi=\{t_0=0<t_1<\cdots<t_n=1\}$ of $[0,1]$ and given $\alpha\in[0,1]$, set

$$t_j^\alpha=t_j+\alpha(t_{j+1}-t_i),\qquad 0\le j\le n-1,$$

and let

$$\Delta_\pi^\alpha(x)=\sum_{i=1}^k\sum_{j=0}^{n-1}\left[u_i(t_j^\alpha,x)-u_i(t_j,x)\right]\left[W_{t_{j+1}}^i-W_{t_j}^i\right].$$

To conclude the proof of (3.3) it suffices to prove the existence of a measurable map $\beta\colon[0,1]\times\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ such that

$$\Delta_\pi^\alpha(x)\to_{|\pi|\to 0}\alpha\int_0^1\beta(t,x)\,dt \tag{3.5}$$

in $L^2(\Omega)$, uniformly in $x\in K$ and $\alpha\in[0,1]$. This can be shown following the ideas of [13], page 519. Set

$$T_1^\pi(x) = \sum_{i=1}^{k} \sum_{j=0}^{n-1} \int_{t_j}^{t_j^\alpha} \nabla_l \sigma_i(\varphi_s(x)) \sigma_i^l(\varphi_s(x))\, ds - \alpha \int_0^1 2m(\varphi_s(x))\, ds$$

and

$$T_2^\pi(x) = \sum_{i=1}^{k} \sum_{j=0}^{n-1} \int_{t_j}^{t_j^\alpha} \nabla_l \sigma_i(\varphi_s(x)) \sigma_i^l(\varphi_s(x))\, ds - \Delta_\pi^\alpha(x).$$

Then we want to check

$$\lim_{|\pi|\to 0} \sup_{\alpha\in[0,1]} \sup_{x\in K} E\left(|T_1^\pi(x)|^2\right) = 0 \tag{3.6}$$

and

$$\lim_{|\pi|\to 0} \sup_{\alpha\in[0,1]} \sup_{x\in K} E\left(|T_2^\pi(x)|^2\right) = 0. \tag{3.7}$$

For any $x \in K$, $0 \le s < s + \delta \le 1$,

$$E\left\{ \sup_{s\le t\le s+\delta} |\varphi_t(x) - \varphi_s(x)|^4 \right\} \le C\delta E \int_s^{s+\delta} \left(1 + |\varphi_u(x)|^4\right) du. \tag{3.8}$$

For any π, $T_1^\pi(x) = 2[T_3^\pi(x) - T_4^\pi(x)]$, where

$$T_3^\pi(x) = \sum_{j=0}^{n-1} \int_{t_j}^{t_j^\alpha} \left\{ m(\varphi_s(x)) - m\left(\varphi_{t_j}(x)\right) \right\} ds,$$

$$T_4^\pi(x) = \alpha \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left\{ m(\varphi_s(x)) - m\left(\varphi_{t_j}(x)\right) \right\} ds.$$

Let $\pi(t) = t_j$, if $t \in [t_j, t_{j+1})$; then for every $\alpha \in [0,1]$, using (3.8),

$$\begin{aligned} E\left(|T_3^\pi(x)|^4\right) &\le E\left(\int_0^1 \left| m(\varphi_s(x)) - m\left(\varphi_{\pi(s)}(x)\right) \right|^4 ds \right) \\ &\le CE \int_0^1 \left| \varphi_s(x) + \varphi_{\pi(s)}(x) \right|^4 ds \\ &\le C|\pi| \sup_{x\in K} E \int_0^1 \left(1 + |\varphi_u(x)|^4\right) du \\ &\le C|\pi|. \end{aligned}$$

A similar argument yields that $E(|T_4^\pi(x)|^4) \le \alpha C|\pi|$, so that

$$\lim_{|\pi|\to 0} \sup_{\alpha\in[0,1]} \sup_{x\in K} E\left(|T_1^\pi(x)|^4\right) = 0,$$

which concludes the proof of (3.6).

Fix $i = 1, \dots, k$. We have

$$\lim_{|\pi| \to 0} \sup_{\alpha \in [0,1]} \sup_{x \in K} E \left| \sum_{j=0}^{n-1} \left[u_i(t_j^\alpha, x) - u_i(t_j, x) \right] \left[W_{t_{j+1}}^i - W_{t_j^\alpha}^i \right] \right|^2 = 0. \tag{3.9}$$

Indeed, by Schwarz's inequality and the estimate given in (3.8), it follows that

$$\begin{aligned} E & \left| \sum_{j=0}^{n-1} \left[u_i(t_j^\alpha, x) - u_i(t_j, x) \right] \left[W_{t_{j+1}}^i - W_{t_j^\alpha}^i \right] \right|^2 \\ & = E \sum_{j=0}^{n-1} \left[u_i(t_j^\alpha, x) - u_i(t_j, x) \right]^2 \left[W_{t_{j+1}}^i - W_{t_j^\alpha}^i \right]^2 \\ & \le C \left(\sum_{j=0}^{n-1} E \left| \varphi_{t_j^\alpha}(x) - \varphi_{t_j}(x) \right|^4 \right)^{1/2} \left(\sum_{j=0}^{n-1} (t_{j+1} - t_j^\alpha)^2 \right)^{1/2} \\ & \le C|\pi|, \end{aligned}$$

and therefore we obtain (3.9).

Consequently, the proof of (3.7) reduces to showing that, for every $i = 1, \dots, k$,

$$\begin{aligned} \sup_{x \in K} \sup_{\alpha \in [0,1]} E \Bigg\{ \Bigg| & \sum_{j=0}^{n-1} \int_{t_j}^{t_j^\alpha} \nabla_l \sigma_i(\varphi_s(x)) \sigma_i^l(\varphi_s(x)) \, ds \\ & - \sum_{j=0}^{n-1} \left[u_i(t_j^\alpha, x) - u_i(t_j, x) \right] \left[W_{t_j^\alpha}^i - W_{t_j}^i \right] \Bigg|^2 \Bigg\} \to 0 \end{aligned} \tag{3.10}$$

as $|\pi| \to 0$. We have

$$\begin{aligned} A_n := E \Bigg\{ \Bigg| & \sum_{j=0}^{n-1} \Big(\left[u_i(t_j^\alpha, x) - u_i(t_j, x) \right] \left[W_{t_j^\alpha}^i - W_{t_j}^i \right] \\ & - E^{\mathscr{F}_{t_j}} \left(\left[u_i(t_j^\alpha, x) - u_i(t_j, x) \right] \left[W_{t_j^\alpha}^i - W_{t_j}^i \right] \right) \Big) \Bigg|^2 \Bigg\} \\ = E \Bigg\{ & \sum_{j=0}^{n-1} \Big(\left[u_i(t_j^\alpha, x) - u_i(t_j, x) \right] \left[W_{t_j^\alpha}^i - W_{t_j}^i \right] \\ & - E^{\mathscr{F}_{t_j}} \left(\left[u_i(t_j^\alpha, x) - u_i(t_j, x) \right] \left[W_{t_j^\alpha}^i - W_{t_j}^i \right] \right) \Big)^2 \Bigg\} \\ \le & \; 4 \sum_{j=0}^{n-1} E \left(\left[u_i(t_j^\alpha, x) - u_i(t_j, x) \right]^2 \left[W_{t_j^\alpha}^i - W_{t_j}^i \right]^2 \right) \\ \le & \; C|\pi|, \end{aligned}$$

therefore,

$$A_n \to 0 \tag{3.11}$$

as $|\pi| \to 0$, uniformly in $x \in K$ and $\alpha \in [0,1]$.

On the other hand,

$$\begin{aligned}
B_n &:= E\left\{\left|\sum_{j=0}^{n-1}\left(\int_{t_j}^{t_j^\alpha}\nabla_l\sigma_i(\varphi_s(x))\sigma_i^l(\varphi_s(x))\,ds\right.\right.\right.\\
&\qquad\left.\left.\left.-E^{\mathscr{F}_{t_j}}\Big(\big[u_i(t_j^\alpha,x)-u_i(t_j,x)\big]\big[W_{t_j^\alpha}^i-W_{t_j}^i\big]\Big)\right)\right|^2\right\}\\
&= E\sum_{j=0}^{n-1}\left|\int_{t_j}^{t_j^\alpha}\nabla_l\sigma_i(\varphi_s(x))\sigma_i^l(\varphi_s(x))\,ds\right.\\
&\qquad\left.-E^{\mathscr{F}_{t_j}}\Big(\big[u_i(t_j^\alpha,x)-u_i(t_j,x)\big]\big[W_{t_j^\alpha}^i-W_{t_j}^i\big]\Big)\right|^2\\
&\le 2\left\{\sum_{j=0}^{n-1}E\left|\int_{t_j}^{t_j^\alpha}\nabla_l\sigma_i(\varphi_s(x))\sigma_i^l(\varphi_s(x))\,ds\right|^2\right.\\
&\qquad\left.+\sum_{j=0}^{n-1}E\Big(\big[u_i(t_j^\alpha,x)-u_i(t_j,x)\big]^2\big[W_{t_j^\alpha}^i-W_{t_j}^i\big]^2\Big)\right\},
\end{aligned}$$

and, by Schwarz's inequality,

$$\begin{aligned}
\sum_{j=0}^{n-1}E\left|\int_{t_j}^{t_j^\alpha}\nabla_l\sigma_i(\varphi_s(x))\sigma_i^l(\varphi_s(x))\,ds\right|^2 &\le C\sum_{j=0}^{n-1}(t_j^\alpha-t_j)E\int_{t_i}^{t_j^\alpha}\big(1+|\varphi_s(x)|^2\big)\,ds\\
&\le C|\pi|E\int_0^1\big(1+|\varphi_s(x)|^2\big)\,ds\\
&\le C|\pi|.
\end{aligned}$$

Hence

$$B_n \to 0 \tag{3.12}$$

as $|\pi| \to 0$, uniformly in $x \in K$ and $\alpha \in [0,1]$. The convergences (3.11) and (3.12) yield (3.10). Consequently we obtain (3.7) and this concludes the proof of (3.3). □

4. Large deviations for Ocone–Pardoux's anticipating stochastic differential equations. This section deals with the problem which motivated the paper. We prove a large deviation principle for the family $\{X^\varepsilon, \varepsilon > 0\}$ of processes giving the solution of the anticipating stochastic differential equations (0.2). We will assume that the hypotheses of Theorem 3.1 are satisfied. Therefore $X_t^\varepsilon = \varphi_t^\varepsilon(X_0^\varepsilon)$, where φ^ε is the stochastic flow defined by

(1.1).

Let us first quote some well-known results on small perturbations of dynamical systems (see for instance [1]). We consider the $d \times k$ matrix $\sigma(x)$ of the vector fields $\sigma_i(x)$ and the quadratic form defined on $\mathbb{R}^d$ by

$$Q_x(v) = \langle v, \sigma(x)\sigma(x)^* v\rangle.$$

Then the dual quadratic form is defined for $v \in \mathbb{R}^d$ by

$$Q_x^*(v) = \inf\{|w|^2 : w \in \mathbb{R}^k, \sigma(x)w = v\}.$$

Notice that if $a(x) = \sigma(x)\sigma(x)^*$ is invertible, then $Q_x^*(v) = \langle v, a(x)^{-1}v\rangle$.

Fix $x_0 \in \mathbb{R}^d$. The family $\{\varphi^\varepsilon(x_0), \varepsilon > 0\}$ of processes giving the solution of (1.1) with $x = x_0$ satisfies a large deviation principle with rate function $I(\cdot)$ given by

$$I(g) = \begin{cases} +\infty, & \text{if } g \notin H^d_{x_0}, \\ \frac{1}{2}\int_0^1 Q^*_{g_t}[\dot{g}_t - b(g_t)]\,dt, & \text{if } g \in H^d_{x_0} \end{cases} \tag{4.1}$$

(see Théorème 2.13, Chapter III, in [1] and Théorème 7 in [10]). An alternative expression for $I(\cdot)$ is as follows:

$$I(g) = \inf\{\tilde{I}(f) : f \in H^k, g = F_{x_0}(f)\}, \tag{4.2}$$

where $F_{x_0}(f)$ denotes the solution of the ordinary differential equation (1.2) with initial condition $x = x_0$ and $\tilde{I}$ is defined by (2.5). If $I(g)$ is finite, the infimum in (4.2) is attained. We now state the fundamental result of this section.

THEOREM 4.1. *Let σ, b and m be of class $\mathscr{C}^2$ with bounded partial derivatives up to order 2. Moreover, suppose that there exists $x_0 \in \mathbb{R}^d$ such that for any $\delta > 0$,*

$$\limsup_{\varepsilon \to 0} \varepsilon \log P\{|X_0^\varepsilon - x_0| > \delta\} = -\infty. \tag{4.3}$$

Then the family $\{P^\varepsilon, \varepsilon > 0\}$ of laws of $\{X_\cdot^\varepsilon = \varphi_\cdot^\varepsilon(X_0^\varepsilon), \varepsilon > 0\}$ satisfies a large deviation principle with rate function I given by (4.1).

In other words, under hypothesis (4.3), *the perturbed anticipating dynamical system* (0.2) *satisfies the same large deviation principle as the adapted family $\{\varphi^\varepsilon(x_0), \varepsilon > 0\}$. Consequently we have the following estimates: For any open set G and any closed set F of the space $\mathscr{C}([0,1]; \mathbb{R}^d)$,*

$$\liminf_{\varepsilon \to 0} \varepsilon \log P^\varepsilon(G) \geq -\inf\{I(g); g \in G\} \tag{4.4}$$

and

$$\limsup_{\varepsilon \to 0} \varepsilon \log P^\varepsilon(F) \leq -\inf\{I(g); g \in F\}, \tag{4.5}$$

with I defined by (4.1).

PROOF OF THEOREM 4.1. We first prove the lower bound stated in (4.4). Let G be an open subset of $\mathscr{C}([0,1];\mathbb{R}^d)$ and suppose that there exists $g \in G$ such that $I(g) < +\infty$. Notice that, by the definition of I we should have $g_0 = x_0$. For this reason we will write $g(x_0)$ instead of g. Let $\eta > 0$ be such that the ball

$$B_\eta(g) = \{\varphi \in \mathscr{C}([0,1],\mathbb{R}^d): \|\varphi - g\| < \eta\}$$

is included in G. Denote by f an element of H^k such that $g(x_0) = F_{x_0}(f)$ and $I(g) = \tilde{I}(f)$. For this $f \in H^k$, let $g_t(x)$ be the solution of (1.2). The Lipschitz property of σ and b together with Gronwall's lemma implies that

$$\sup_t \sup_{|x-x_0|\le\delta} |g_t(x) - g_t(x_0)| \le \delta \exp\left\{C\int_0^1 \left(1 + |\dot{f}_s|\right) ds\right\}. \tag{4.6}$$

Given $\eta > 0$ we choose $\delta > 0$ such that

$$\sup_t \sup_{|x-x_0|\le\delta} |g_t(x) - g_t(x_0)| \le \frac{\eta}{2}.$$

Then

$$\begin{aligned} P^\varepsilon(G) &= P\{\varphi^\varepsilon(X_0^\varepsilon) \in G\} \ge P\{\|\varphi^\varepsilon(X_0^\varepsilon) - g(x_0)\| < \eta\} \\ &\ge P\{\|\varphi^\varepsilon(X_0^\varepsilon) - g(x_0)\| < \eta, |X_0^\varepsilon - x_0| \le \delta\} \\ &\ge P\left\{\sup_t \sup_{|x-x_0|\le\delta} |\varphi_t^\varepsilon(x) - g_t(x_0)| < \eta, |X_0^\varepsilon - x_0| \le \delta\right\} \qquad (4.7) \\ &\ge P\left\{\sup_t \sup_{|x-x_0|\le\delta} |\varphi_t^\varepsilon(x) - g_t(x)| < \frac{\eta}{2}\right\} - P\{|X_0^\varepsilon - x_0| > \delta\}. \end{aligned}$$

The large deviation estimates for the flow φ^ε given in Proposition 2.2 yield

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \varepsilon \log P\left\{\sup_t \sup_{|x-x_0|\le\delta} |\varphi_t^\varepsilon(x) - g_t(x)| < \frac{\eta}{2}\right\} \\ &\quad\ge -\inf\left\{\tilde{I}(h): g_t(x) = x + \int_0^t \sigma(g_s(x))\dot{h}_s\, ds + \int_0^t b(g_s(x))\, ds\right\} \\ &\quad\ge -\tilde{I}(f) = -I(g(x_0)). \end{aligned}$$

Since $g(x_0) := g$ is an arbitrary function of G verifying $I(g) < \infty$, this last minoration, together with the inequality (4.7) and hypothesis (4.3), provides the desired lower bound.

Let us now establish the upper bound (4.5). Fix a closed set F in $\mathscr{C}([0,1];\mathbb{R}^d)$ and set $\Lambda(F) = \inf\{I(g), g \in F\}$. If $\Lambda(F) = 0$, then (4.5) is obvious. So, we may assume that there exists $a > 0$ such that $a < \Lambda(F)$.

Let $\delta > 0$ be a real number that will be fixed later and $B_\delta(x_0)$ the ball in $\mathbb{R}^d$ centered at x_0 with radius δ. We denote by I' the rate function of the flow $\{\varphi^\varepsilon, \varepsilon > 0\}$ on $\mathscr{C}([0,1] \times B_\delta(x_0);\mathbb{R}^d)$ (see Proposition 2.2).

Consider the compact subsets $K'_a = \{g\colon I'(g) \le a\}$, $K_a = \{\varphi\colon I(\varphi) \le a\}$ of $\mathscr{C}([0,1] \times B_\delta(x_0); \mathbb{R}^d)$ and $\mathscr{C}([0,1]; \mathbb{R}^d)$, respectively.

Let R denote the restriction map defined by

$$R\colon \{g\colon I'(g) < \infty\} \to \{\varphi\colon I(\varphi) < \infty\},$$

$(Rg)_t = g(t, x_0)$. The definitions of I and I' clearly show that $I(R(g)) \le I'(g)$, so that $R(K'_a) \subset K_a$.

For any $\eta > 0$, $\varphi \in \mathscr{C}([0,1]; \mathbb{R}^d)$, $g \in \mathscr{C}([0,1] \times B_\delta(x_0); \mathbb{R}^d)$, set

$$U_\eta(\varphi) = \{\psi \in \mathscr{C}([0,1]; \mathbb{R}^d)\colon \|\psi - \varphi\| < \eta\}$$

and

$$V_\eta(g) = \{h \in \mathscr{C}([0,1] \times B_\delta(x_0); \mathbb{R}^d)\colon \|h - g\|_{B_\delta(x_0)} < \eta\}.$$

Since $K_a \cap F = \varnothing$, for any $\varphi \in K_a$, there exists $\eta_\varphi > 0$ such that $U_{\eta_\varphi}(\varphi) \cap F = \varnothing$.

Let $g \in K'_a$ and $g(x_0) = R(g)$. The sets $V_{(1/2)\eta_{g(x_0)}}(g)$, $g \in K'_a$, give an open covering of the compact set K'_a. Let $V_{(\eta_i/2)}(g^i)$, $g^i \in K'_a$, $i = 1, \dots, r$, denote a finite subcovering, where $\eta_i = \eta_{g_i(x_0)}$. Set $U = \bigcup_{i=1}^r U_{\eta_i}(g^i(x_0))$. We have $U \cap F = \varnothing$.

As in the preceding step, we can choose $\delta > 0$ such that

$$\sup_t \sup_{|x - x_0| \le \delta} \left|g_t^i(x) - g_t^i(x_0)\right| \le \frac{\eta_i}{2} \tag{4.8}$$

for any $i = 1, \dots, r$. Then

$$\begin{aligned} P^\varepsilon(F) &= P\{\varphi^\varepsilon(X_0^\varepsilon) \in F\} \\ &\le P\{\varphi^\varepsilon(X_0^\varepsilon) \in F, |X_0^\varepsilon - x_0| \le \delta\} + P\{|X_0^\varepsilon - x_0| > \delta\} \end{aligned}$$

and

$$\begin{aligned} P\{\varphi^\varepsilon(X_0^\varepsilon) \in F, |X_0^\varepsilon - x_0| \le \delta\} &\le P\{\varphi^\varepsilon(X_0^\varepsilon) \in U^c, |X_0^\varepsilon - x_0| \le \delta\} \\ &\le P\{\varphi^\varepsilon \in V^c\}, \end{aligned}$$

where $V = \bigcup_{i=1}^r V_{(\eta_i/2)}(g^i)$. Since V^c is a closed subset of $\mathscr{C}([0,1] \times B_\delta(x_0); \mathbb{R}^d)$, the large deviation estimates for the flow yield

$$\limsup_{\varepsilon \to 0} \varepsilon \log P\{\varphi^\varepsilon \in V^c\} \le -\inf_{g \in V^c} I(g). \tag{4.9}$$

But $V^c \subset (K'_a)^c$; consequently the right-hand side of (4.9) is bounded by $-a$. From this property and hypothesis (4.3), we deduce that, for any $a < \Lambda(F)$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log P^\varepsilon(F) \le -a.$$

This yields (4.5) and concludes the proof of the theorem. □

REMARK. Using a recent result proved by the authors on composition of large deviation principles, hypothesis (4.3) of Theorem 4.1 can be replaced by a more general statement concerning the validity of a large deviation principle for the family $\{X_0^\varepsilon, \varepsilon > 0\}$. More precisely we have the following result:

THEOREM 4.2 (Proposition 3.1 of [7]). *Let $\{X_0^\varepsilon, \varepsilon > 0\}$ be a family of $\mathbb{R}^d$-valued random variables verifying the following condition*: (i) *There exists a mapping ζ: $H^k \to \mathbb{R}^d$ such that its restriction to the compact sets $\{\tilde{I} \le a\}$, $a \in [0, \infty)$, is continuous, and the pair $(\sqrt{\varepsilon}\, W, X_0^\varepsilon)$ satisfies a large deviation principle on $\mathscr{C}_0([0,1]; \mathbb{R}^k) \times \mathbb{R}^d$ with rate function*

$$I_1(f,g) = \begin{cases} \tilde{I}(f), & \text{if } \tilde{I}(f) < \infty \text{ and } g = \zeta(f), \\ +\infty, & \text{otherwise}. \end{cases}$$

Then $\{X_\cdot^\varepsilon = \varphi_\cdot^\varepsilon(X_0^\varepsilon), \varepsilon > 0\}$ also satisfies a large deviation principle with rate function

$$I(g) = \inf\{\tilde{I}(f): F(f)(\zeta(f)) = g\}$$

for any $g \in \mathscr{C}([0,1]; \mathbb{R}^d)$.

APPENDIX

In this section we establish a technical result which is an application of a well-known inequality for Itô processes (see [11]).

LEMMA 5.1. *Let $\{W_t = (W_t^1, \dots, W_t^k), t \in [0,1]\}$ be the standard k-dimensional Wiener process, $p \ge 1$ a real number and A a compact subset of $\mathbb{R}^N$. Fix $u \in [0,1]$ and consider a family $\{Y_t(x), t \in [u,1]\}$ of N-dimensional Itô processes indexed by $x \in A$, say*

$$Y_t(x) = \int_u^t \sigma_l(s,x,\omega)\, dW_s^l, \tag{5.1}$$

where $\sigma_l(s,x,\omega)$ are measurable in (s,x,ω) and such that $Z_t = \int_A |Y_t(x)|^{2p}\, dx < \infty$ for every $t \in [u,1]$. Let τ be a stopping time, $u \le \tau \le 1$ such that there exists $Y \in [0,\infty)$ with

$$\sup\left\{\int_A |Y_t(x)|^{2p}\, dx; (t,\omega): u \le t \le \tau(\omega)\right\} \le Y < \infty,$$

and let $\sup\{\tau(\omega) - u: \omega \in \Omega\} \le T \le 1$. Assume that there exist a finite constant $M^{(1)}$ such that

$$\sup\left\{\left(\int_A |\sigma(s,x,\omega)|^{2p}\, dx\right)^{1/p}; (t,\omega): u \le t \le \tau(\omega)\right\} \le M^{(1)}.$$

Then there exists a constant C_4 which only depends on p, k and N, such that for every $\eta > C_4 T Y^{(p-1)/p} M^{(1)}$,

$$P\left\{\sup_{u \le t \le \tau} Z_t \ge \eta\right\} \le \exp\left(-\frac{\left[\eta - C_4 T Y^{(p-1)/p} M^{(1)}\right]^2}{2 C_4 T Y^{(2p-1)/p} M^{(1)}}\right). \tag{5.2}$$

PROOF. Itô's formula applied to $f(x) = |x|^{2p}$ yields that

$$|Y_t(x)|^{2p} = \sum_{i=1}^{N} \int_u^t 2p|Y_s(x)|^{2p-2} Y_s^i(x)\sigma_l^i(s,x,\omega)\, dW_s^l$$
$$+ \int_u^t \Bigg\{ \sum_{i=1}^{N} p|Y_s(x)|^{2p-2} a^{i,i}(s,x,\omega)$$
$$+ \sum_{i,j=1}^{N} 2p(p-1)Y_s^i(x)Y_s^j(x)|Y_s(x)|^{2p-4} a^{i,j}(s,x,\omega)\Bigg\}\, ds,$$

where $a = \sigma^*\sigma$. Then Fubini's theorem clearly implies that $\{Z_t, t \in [u,1]\}$ is an Itô process. More precisely,

$$Z_t = \int_u^t C_l(s)\, dW_s^l + \int_u^t D(s)\, ds,$$

where

$$C_l(s) = 2p\int_A |Y_s(x)|^{2p-2} \sum_{i=1}^{N} Y_s^i(x)\sigma_l^i(s,x,\omega)\, dx,$$
$$D(s) = \int_A \Bigg\{ |Y_s(x)|^{2p-2}\left[p\sum_{i=1}^{N} a^{i,i}(s,x,\omega)\right]$$
$$+ |Y_s(x)|^{2p-4} 2p(p-1) \sum_{i,j=1}^{N} Y_s^i(x)Y_s^j(x)a^{i,j}(s,x,\omega)\Bigg\}\, dx.$$

Set

$$\gamma = \sup\left\{ \sum_{l=1}^{k} C_l(s)^2(\omega); (s,\omega) \colon u \le s \le \tau(\omega)\right\},$$
$$\delta = \sup\{|D(s)(\omega)|; (s,\omega) \colon u \le s \le \tau(\omega)\}.$$

Then Lemma 2.19 in Stroock [11] yields that for $\eta > T\delta > 0$,

$$P\Big(\sup_{u \le t \le \tau} Z_t \ge \eta\Big) \le \exp\left(-\frac{(\eta - T\delta)^2}{2T\gamma}\right).$$

We have

$$\left| \sum_{i=1}^{N} Y_s^i(x)\sigma_l^i(s,x,\cdot)\right| \le \left(\sum_{i=1}^{N} |\sigma_l^i(s,x,\cdot)|^2\right)^{1/2} |Y_s(x)|,$$

and Hölder's inequality yields that for $u \le s \le \tau(\omega)$,

$$\sum_{l=1}^{k} C_l(s)^2 \le 4p^2\left(\int_A |Y_s(x)|^{2p}\, dx\right)^{(2p-1)/p} \sum_{l=1}^{k} \left(\int_A \left| \sum_{i=1}^{N} (\sigma_l^i(s,x,\cdot))^2\right|^p dx\right)^{1/p}$$
$$\le C(p,k,N)Y^{(2p-1)/p}M^{(1)}.$$

Similarly, for $u \le s \le \tau(\omega)$,

$$
\begin{aligned}
|D(s)| &\le p\left(\int_A |Y_s(x)|^{2p}\,dx\right)^{(p-1)/p}\left(\int_A\left|\sum_{i=1}^N a^{i,i}(s,x,\omega)\right|^p dx\right)^{1/p} \\
&\quad + 2p(p-1)\left(\int_A |Y_s(x)|^{2p}\,dx\right)^{(p-1)/p} \\
&\quad \times\left(\int_A\left[\sum_{i,j=1}^N |a^{i,j}(s,x,\omega)|^2\right]^{p/2} dx\right)^{1/p} \\
&\le C(p,N)Y^{(p-1)/p}M^{(1)}.
\end{aligned}
$$

Therefore, the estimate (5.2) holds for $C_4 = \sup\{C(p,k,N), C(p,N)\}$. □

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