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ABOUT DIRECT SUMMANDS OF PROJECTIVE MODULES OVER LAURENT POLYNOMIAL RINGS

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ABSTRACT. Suppose A is a local ring and $R = A[X, X^{-1}]$ is a Laurent polynomial ring. We prove that for projective R-modules P and Q with $\operatorname{rank} Q < \operatorname{rank} P$, if Q_f is a direct summand of P_f for a doubly monic polynomial f then Q is also a direct summand of P. We also prove the analogue of the Horrock's theorem for Laurent polynomials rings.

1. INTRODUCTION

In [R] Roy proved that if $R = A[X]$ is a polynomial ring over a local ring A and if P and Q are two projective R-modules with $\operatorname{rank} Q < \operatorname{rank} P$, then Q_f is a direct summand of P_f for some monic polynomial f, implies that Q is a direct summand of P. (All rings are assumed to be noetherian and commutative, and the modules are assumed to be finitely generated.)

In this paper, we shall extend this result of Roy and the consequences to the Laurent polynomial ring case.

We need the following definition for the subsequent discussions.

(1.1) **Definition.** Suppose that $R = A[X, X^{-1}]$ is a Laurent polynomial ring over a commutative ring A. An element f in R is called a *doubly monic Laurent polynomial* if the coefficients of the highest degree and the lowest degree terms are units.

We shall be extending the above theorem of Roy [R] to the Laurent polynomial situation *by replacing "monic polynomial" by "doubly monic Laurent polynomial" in his statement* (see (2.1) below).

As a consequence, we shall be proving the Laurent polynomial version of the Horrock's theorem; that is, if P is a projective module over a Laurent polynomial ring $R = A[X, X^{-1}]$ over a local commutative noetherian ring A and if P_f is free for some doubly monic Laurent polynomial f then P is also free.

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2. The main discussions

The following is the main result of this paper:

(2.1) **Theorem.** *Suppose that* $R = A[X, X^{-1}]$ *is a Laurent polynomial ring over a local noetherian commutative ring* A. *Also suppose that* P *and* Q *are two projective* R-modules *with* $\operatorname{rank} Q < \operatorname{rank} P$. *If* Q_f *is a direct summand of* P_f *for some doubly monic Laurent polynomial* f, *then* Q *is also a direct summand of* P.

Before we go into the proof of (2.1), we discuss the consequences of this theorem.

(2.2) **Theorem.** *Let* $R = A[X, X^{-1}]$ *be as above* (2.1) *and* P, Q *be two projective* R-modules *such that* P_f *is isomorphic to* Q_f *for some doubly monic Laurent polynomial* f. *Then,*

(i) Q *is a direct summand of* $P \oplus L$ *for all nonzero projective* R-modules L.
(ii) P *is isomorphic to* Q *if* P *or* Q *has a rank one direct summand.*
(iii) $P \oplus L$ *is isomorphic to* $Q \oplus L$ *for all rank one projective* R-modules L.
(iv) P *and* Q *have same number of generators.*

Proof of (2.2). We shall only prove (ii), because the rest of the arguments are as in Roy's proof [R, Proposition 3.1].

To Prove (ii), let $P = P' \oplus L$ for some rank one projective R-module L. By (2.1) $Q \approx P' \oplus L'$ for some rank one projective R-module L'. Now it is enough to prove that $L \approx L'$.

Since $P_f \approx Q_f$, it follows that $L_f \approx L'_f$. Hence $(L'L^{-1})_f$ is free. This means that $L'L^{-1}$ is isomorphic to an invertible ideal I of $R = A[X, X^{-1}]$ such that I contains some power of f. Write $J = I \cap A[X]$. As I contains a doubly monic Laurent polynomial, J is also an invertible ideal in $A[X]$. Moreover, since J also contains a monic polynomial, J is a principal ideal. (See [NN] for an elementary proof of this fact.) Now it follows that I is a principal ideal and hence $L \approx L'$. The proof of (ii) is complete.

The following Laurent polynomial analogue of Horrock's theorem is an immediate sequence of (2.2).

(2.3) **Corollary.** *Suppose* $R = A[X, X^{-1}]$ *is as in* (2.1) *and* P *is a projective* R-module. *If* P_f *is free for some doubly monic Laurent polynomial, then* P *is free.*

(2.4) *Remark.* The nonlocal version of (2.3) is also true; that is, if $R = A[X, X^{-1}]$ is a Laurent polynomial ring over a (nonlocal) noetherian commutative ring A and if P is a projective R-module with P_f free for some doubly monic Laurent polynomial f, then P is free. (See [BR, Remark (3.6)] for a proof in the rank one case.)

Proof. Without loss of generality, we can assume that f is a monic polynomial in $A[X]$ with $f(0) = 1$. Now $XA[X] + fA[X] = A[X]$. Since P_f is free, by patching P with a free $A[X]_f$-module, we can get a projective $A[X]$-module Q with $Q_X = P$. Again, by construction Q_f is free. By the Quillen-Suslin theorem it follows that Q is free and hence, P is also free.

3. Proof of Theorem (2.1)

In this section, we shall prove Theorem (2.1). The proof is in the line of Roy's proof [R] with necessary modification for Laurent polynomial rings.

First we have to prove a few lemmas.

(3.1) **Lemma.** *Suppose R is a noetherian commutative ring and P, Q are two projective R-*modules. *Suppose $\phi: Q \to P$ is an R-*linear *map. For an ideal I of R, if ϕ is a split monomorphism* modulo I *then $\phi_{1+I}: Q_{1+I} \to P_{1+I}$ is also a split monomorphism.*

Proof. We need to check that ϕ is a split monomorphism after localizing at maximal ideals containing I. So we can assume that R is local. In this case, it is obvious.

(3.2) **Lemma.** *Suppose $R = A[X, X^{-1}]$ is a Laurent polynomial ring over a local noetherian commutative ring A with maximal ideal* $\mathbf{m}$. *Let P and Q be two projective R-*modules *and $\phi: Q \to P$ be an R-*linear *map. If ϕ is a split monomorphism* modulo $\mathbf{m}$ *and if ϕ_f is a split monomorphism for some doubly monic Laurent polynomial f, then ϕ is also a split monomorphism.*

Proof. By (3.1) $\phi_{1+\mathbf{m}R}$ is a split monomorphism. So, there is an element h in $1 + \mathbf{m}R$ such that ϕ_h is a split monomorphism. Now since f is doubly monic, $Rf + Rh = R$. As ϕ_f is also a split monomorphism, ϕ is a split monomorphism. This completes the proof of (3.2).

(3.3) **Lemma.** *Let $R = A[X, X^{-1}]$ be as above and P, Q be two projective R-*modules. *Let $\phi, \phi': Q \to P$ and $\gamma: P \to Q$ be R-*linear *maps, such that $\gamma\phi' = f\,\mathrm{Id}_Q$ for some doubly monic Laurent polynomial f. For integers $t > 0$, we write $\phi_t = \phi + (X + X^{-1})^t\phi'$. Then for large t, ϕ_t becomes a split monomorphism after inverting a doubly monic polynomial f_t.*

Proof. As in [R], first we assume that Q is free. We also have $\gamma\phi_t = \gamma\phi + (X + X^{-1})^t\gamma\phi' = \gamma\phi + (X + X^{-1})^t f\,\mathrm{Id}_Q$.

Since Q is free, $\gamma\phi_t$ is a matrix. It is also clear that for large integers t, $\det(\gamma\phi_t) = f_t$ is a doubly monic Laurent polynomial. Therefore, ϕ_t becomes a split monomorphism after inverting f_t.

In the general case, find projective R-module Q' such that $Q \oplus Q'$ is free. Define maps $G, G': Q \oplus Q' \to P \oplus Q'$ and $H: P \oplus Q' \to Q \oplus Q'$ by setting $G = \phi \oplus 0$, $G' = \phi' \oplus f\,\mathrm{Id}_{Q'}$, and $H = \gamma \oplus \mathrm{Id}_{Q'}$. It follows from above that for large t, G_t becomes a split monomorphism after inverting a doubly monic polynomial f_t. Hence it also follows that ϕ_t becomes a split monomorphism after inverting f_t. This completes the proof of (3.3).

Now we are ready to prove our main theorem (2.1).

Proof of Theorem (2.1). Since Q_f is isomorphic to a direct summand of P_f, we can find R-linear maps $\phi'\colon Q \to P$ and $\gamma\colon P \to Q$ such that $\gamma\phi' = f\,\mathrm{Id}_Q$ (possibly after replacing f by a power of f).

Bar "$^-$" will denote modulo $\mathbf{m}$, where $\mathbf{m}$ is the maximal ideal of A. So we have $\overline{\gamma}\overline{\phi}' = \overline{f}\,\mathrm{Id}_{\overline{Q}}$. As $\overline{f}$ is doubly monic, $\overline{\phi}'$ is a monomorphism.

Let $\operatorname{rank} P = r$ and $\operatorname{rank} Q = s$. We can find bases $\{\overline{p}_1, \overline{p}_2, \dots, \overline{p}_r\}$ and $\{\overline{q}_1, \overline{q}_2, \dots, \overline{q}_s\}$ for $\overline{P}$ and $\overline{Q}$, respectively, such that $\overline{\phi}'(\overline{q}_i) = \overline{f}_i\overline{p}_i$ for some f_i in R and $i = 1$ to s.

We define an $\overline{R}$-linear map $\overline{\phi}\colon \overline{Q} \to \overline{P}$ such that $\overline{\phi}(\overline{q}_i) = \overline{p}_{i+1}$ for $i = 1$ to s (note that $s < r$). For positive integer t, define $\overline{\phi}'_t = \overline{\phi} + (X + X^{-1})^t\overline{\phi}'$. Since $\{\overline{p}_1, \overline{\phi}'_t(\overline{q}_1), \dots, \overline{\phi}'_t(\overline{q}_s)\}$ is a part of basis for $\overline{P}$, $\overline{\phi}'_t$ is a split monomorphism.

Let $\phi\colon Q \to P$ be a lift of $\overline{\phi}$ and $\phi'_t = \phi + (X + X^{-1})^t\phi'$. Then ϕ'_t is a lift of $\overline{\phi}'_t$. By (3.3), for large t ϕ'_t becomes a split monomorphism after inverting a doubly monic polynomial. Since $\overline{\phi}'_t$ is also a split monomorphism, by (3.2) ϕ'_t is a split monomorphism. So, Q is a direct summand of P. This completes the proof of the theorem.

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