Weak gravitational fields

D. Lerner and J. R. Porter

Citation: Journal of Mathematical Physics **15**, 1413 (1974); doi: 10.1063/1.1666824
View online: http://dx.doi.org/10.1063/1.1666824
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/15/9?ver=pdfcov
Published by the AIP Publishing

Weak gravitational fields

D. Lerner and J. R. Porter

Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15213
(Received 12 February 1973)

We consider the set of C^k bounded tensor fields of type (r,s) on $\mathbf{R}^4$ in the topology of uniform C^k convergence. For each $k \geq 2$, the map sending a metric to its curvature tensor is shown to be analytic at the Minkowski metric. The same is true of the map sending a metric to its Einstein tensor. The well-known linearized theory of gravitation amounts to studying the directional derivatives of these maps. An iterative method for solving the full field equations along an analytic curve of Einstein tensors passing through zero is proposed.

I. INTRODUCTION

A central problem in the general theory of relativity concerns the stability of solutions to Einstein's field equations. Precisely, given a four-manifold M, a stress-energy tensor T, and an exact solution g to the field equations $E(g) = -T$, the problem is to determine all "nearby" solutions and to examine, at least qualitatively, their physical properties. ($E(g) = \{R_{ab} - \frac{1}{2}Rg_{ab}\}dx^a \otimes dx^b$ is the Einstein tensor of the metric g. The map $g \to E(g)$ is called the Einstein map.) There are essentially two approaches to the problem, depending on what one means by the word "nearby."

(a) In the first instance, one considers all metrics g' which are in some sense close to g, computes the energy–momentum tensors $-E(g')$, and examines the physical properties of the resulting space–times (M,g'). One normally requires the introduction of a topology on the set of Lorentz metrics in order to determine whether or not two metrics are close to one another.

(b) In the second instance, one perturbs the energy–momentum tensor T to a nearby T' and attempts to solve the resulting field equations $E(g') = -T'$.

In connection with (a) if one regards all Lorentz metrics on M as being on an equal (mathematical footing, it appears[1] that the only acceptable choice for a topology is the Whitney fine C^k topology. However, it frequently happens that one is *not* concerned with all such metrics, but only those g' which are in some sense close to a *fixed* metric g. In such cases, it is possible to construct a topology which is considerably more tractable than the Whitney topology and at the same time appears to provide a suitable analytic framework within which to attack problem (b).

In this paper, we examine such a topology in the particular case where $M = \mathbb{R}^4$ and the preferred metric is a fixed Minkowski metric η. Section II introduces the necessary mathematical formalism; the set of Lorentz metrics close to η is shown to be an open subset of a Banach space. In Sec. III we show that the curvature map (the map associating with each Lorentz metric its Riemann tensor) is analytic in a neighborhood of η. [The metrics themselves need only be C^k $(k \geq 2)$.] It follows immediately that the Einstein map $g \to E(g)$ is analytic at η. In Sec. IV we briefly discuss the linearized theory of gravitation, which is particularly well-posed in this formalism: The linearized Einstein tensor of the metric $\eta + h$ is simply the derivative of E at η in the direction of h. In Sec. V we discuss an iterative procedure for solving the full field equations along an analytic curve of stress–energy tensors passing through zero.

II. MATHEMATICAL PRELIMINARIES

Fix, once and for all, a global coordinate system (x^a) on $\mathbb{R}^4$ and the Minkowski metric η defined by these coordinates, $\eta_{ab} = \operatorname{diag}\{1,-1,-1,-1\}$. Let $\mathcal{S}_k$ denote the set of C^k twice-covariant symmetric tensor fields on $\mathbb{R}^4$, and for $h \in \mathcal{S}_k$, $x \in \mathbb{R}^4$, put

$$\|h(x)\|_k = \max_{1 \leq a,b,\ldots,\leq 4} \{|h_{ab}(x)|, |h_{cd,e}(x)|, \ldots, |h_{ij,m_1,m_2\cdots m_k}(x)|\}, \tag{1}$$

and set

$$|h|_k = \sup\{\|h(x)\|_k : x \in \mathbb{R}^4\}. \tag{2}$$

Define

$$\mathcal{B}_k \equiv \{h \in \mathcal{S}_k : |h|_k < \infty\}. \tag{3}$$

The $|\cdot|_k$ norm is easily seen to be equivalent to the standard C^k norm; this particular formulation is slightly easier to calculate with. $\mathcal{B}_k$ is a Banach space. Similarly, let $\mathcal{V}_k$ denote the set of four-covariant C^k tensor fields on $\mathbb{R}^4$ having the symmetries of curvature tensors ($R_{[ab][cd]} = R_{abcd}$, $R_{a[bcd]} = 0$). For $R \in \mathcal{V}_k$, define $|R|_k$ as above and let

$$\mathcal{W}_k \equiv \{R \in \mathcal{V}_k : |R|_k < \infty\}. \tag{4}$$

$\mathcal{W}_k$ is a Banach space as well. Notice that $\eta \in \mathcal{B}_k$ and that the ball of radius 1/4 about η consists entirely of Lorentz metrics; it is these which we shall call "close" to η. Thus we are concerned with an open ball in a Banach space. {Notice that the *complete* set of Lorentz metrics contained in $\mathcal{B}_k$ is not an open set; for example, $[1/(1 + r^2)]\eta[r^2 = \sum_a (x^a)^2]$ is not an interior point. This would be a real problem if we were interested in *all* Lorentz metrics.}

III. ANALYTICITY OF THE CURVATURE MAP

Let Ω be the map sending a nondegenerate C^k metric to its C^{k-2} curvature tensor. As mentioned above, the domain of Ω contains an open ball around η in $\mathcal{B}_k$.

Theorem: For any $k \geq 2$, the map Ω: $\mathcal{B}_k \to \mathcal{W}_{k-2}$ is analytic at η. Precisely, for any g in the ball of radius 1/4 about η, write $g = \eta + h$ where $|h|_k < 1/4$; then

$$\Omega(g) = \Omega(\eta + h) = \Omega(\eta) + \sum_{j=1}^{\infty} \frac{1}{j!} D^j\Omega(\eta)\cdot(h, \ldots, h), \quad j \text{ times}, \tag{5}$$

where, as usual,

$$D^j\Omega(\eta)\cdot(h, \ldots, h) = \frac{d}{dt_j} \cdots \frac{d}{dt_1} \left.\{R_{abcd}(\eta + t_1 h_1 + \cdots + t_j h_j)\, dx^a \otimes dx^b \otimes dx^c \otimes dx^d\}\right|_{\substack{t_1 = \cdots = t_j = 0 \\ h_1 = \cdots = h_j = h}} \tag{6}$$

The series on the right converges in norm in the space $\mathcal{W}_{k-2}$.

Proof: We exhibit the power series for $\Omega(\eta + h)$ and show that it converges to $\Omega(\eta + h)$. It is necessary to work in components; all raising and lowering of indices is done with η and the summation convention is employed throughout. We have $g = \eta + h$, where $|h|_k = a < 1/4$. By long division, the components of the inverse matrix to g are

$$g^{cd} = \eta^{cd} - h^{cd} + \sum_{j=1}^{\infty} (-1)^{j+1} h^{ci_1} h_{i_1}{}^{i_2} \cdots h_{i_j}{}^{d}. \qquad (7)$$

This is a series of real-valued functions on $\mathbb{R}^4$; we need to show uniform C^k convergence. Put $b = 4a < 1$, and differentiate the series n times $(0 \leq n \leq k)$. One finds without difficulty that, for any $x \in \mathbb{R}^4$,

$$|(h^{ci_1} h_{i_1}{}^{i_2} h_{i_2}{}^{i_3} \cdots h_{i_j}{}^{d}), a_1 a_2 \cdots a_n(x)| < (j+1)^n b^{j+1}. \qquad (8)$$

Since $\sum_{j=0}^{\infty} (j+1)^n b^{j+1} < \infty$ for $b < 1$ (ratio test), all the series for $g^{cd}, \ldots, g^{cd}, a_1 \cdots a_k$ converge uniformly and absolutely on $\mathbb{R}^4$ (Weierstrass test); and in the notation of Sec. II we have

$$|g^{-1}|_k < |\eta^{-1}|_k + \sum_{j=0}^{\infty} (j+1)^k b^{j+1} = 1 + \sum_{j=0}^{\infty} (j+1)^k b^{j+1} < \infty.$$

So g^{-1} is well defined.

Let $\Gamma^a_{bc}(g)$ be the Christoffel symbols of g with respect to (x^a). Setting $H_{dbc} = \frac{1}{2}\{h_{db,c} + h_{dc,b} - h_{bc,d}\}$, we have

$$\Gamma^a_{bc}(g) = \eta^{ad} H_{dbc} - h^{ad} H_{dbc} + h^{ai} h_i{}^d H_{dbc} - + \cdots \qquad (9)$$

with absolute and uniform C^{k-1} convergence. Thus $\Omega(g) = R_{abcd}(g)\, dx^a \otimes dx^b \otimes dx^c \otimes dx^d$, where

$$R_{abcd}(g) = 2H_{ab[c,d]} + 2H_{as[c}\Gamma^s_{d]b}(g), \qquad (10)$$

and we may expand and regroup in the following way:

$$R_{abcd}(g) = 2H_{ab[c,d]} + 2H_{as[c}H_{|e|d]b}\eta^{se} - 2H_{as[c}H_{|e|d]b}h^{se} + 2H_{as[c}H_{|e|d]b}h^{si}h_i{}^e - \cdots, \qquad (11)$$

where we have convergence in the space $\mathcal{W}_{k-2}$, with $H_{as[c}H_{|e|d]b} = \frac{1}{2}(H_{asc}H_{edb} - H_{asd}H_{ecb})$.

Remark: Because of the absolute and uniform convergence, it follows that the series for $\mathrm{Ric}(g) = R_{cabd}\, g^{-1\,cd} dx^a \otimes dx^b = R_{ab} dx^a \otimes dx^b$ and $R(g) = R_{ab} g^{ab}$ are also convergent. From this it follows immediately that the map $E: \mathcal{B}_k \to \mathcal{B}_{k-2}$ sending a Lorentz metric to its Einstein tensor is also analytic at η in the ball of radius 1/4. Similar remarks apply to the map sending a Lorentz metric to its conformal curvature tensor.

IV. THE LINEARIZED THEORY OF GRAVITATION

The best-known method for obtaining approximate solutions to the field equations is called the linearized theory (see Pirani, Ref. 2, for a fairly complete exposition and references). It has often been remarked that it is not a particularly good method, and in this section we shall see precisely why this is so. The linearized theory proceeds roughly as follows. An energy–momentum tensor T is given; instead of solving the full equations $E(g) = -T$, one replaces E by a linear operator L and considers the simpler equations $L(g) = -T$. $L(g)$ is defined simply by writing $g = \eta + h$, calculating $E(\eta + h)$, and retaining only those terms which are first order in h. The resulting linear system is then solved for h, and one obtains the approximate solution $g = \eta + h$.

Of course, if one now calculates the full Einstein tensor $E(\eta + h)$ for this metric, it will not be equal to $-T$. However, there is a fairly obvious relation between the two quantities, namely

$$DE(\eta)\cdot h = -T. \qquad (12)$$

This should be evident from the remarks in the preceeding section; $DE(\eta)\cdot h$ is just the first term in the power series expansion of $E(\eta + h)$. In words, the linearized Einstein tensor is the derivative of the Einstein map at η in the direction of h. Similarly, the first term $D\Omega(\eta)\cdot h$ in the series (5) or (11) is just the usual linearized curvature tensor of the metric $\eta + h$.

Once it is recast in this formalism, the shortcomings of the linearized theory are readily apparent. The relationship between $\eta + h$ and an exact solution to $E(g) = -T$ is essentially nonexistent. What we have instead is

$$E(\eta + h) + T = \sum_{k=2}^{\infty} \frac{1}{k!} D^k E(\eta)\cdot h^k; \qquad (13)$$

a real solution (if it exists) to $E(g) = -T$ is well approximated by the linearized solution only in the case that the entire power series on the right can be neglected.

V. AN ITERATIVE METHOD FOR SOLVING THE FIELD EQUATIONS

Consider a curve of the form

$$g(t) = \eta + \sum_{i=1}^{\infty} \underset{(i)}{h} \frac{t^i}{i!}, \qquad (14)$$

where, for the sake of definiteness, $|\underset{(i)}{h}|_k < (\frac{1}{4})^{i+1}$. Then for $t \in (-1, 1)$, this defines an analytic curve of metrics passing through η and lying in the ball of radius 1/4 about η in $\mathcal{B}_k$. The image of this curve under the Einstein map will be an analytic curve passing through 0 in $\mathcal{B}_{k-2}$. Setting

$$H(t) = \sum_{i=1}^{\infty} \underset{(i)}{h} \frac{t^i}{i!},$$

we have

$$E(g(t)) = DE(\eta)\cdot H(t) + (1/2!)\, D^2E(\eta)\cdot(H(t), H(t)) + (1/3!)\, D^3E(\eta)\cdot(H(t), H(t), H(t)) + \cdots. \qquad (15)$$

Expanding and regrouping according to powers of t, we have

$$E(g(t)) = \{DE(\eta)\cdot(\underset{(1)}{h})\}t + \{DE(\eta)(\underset{(2)}{h})$$

I

$$+ D^2E(\eta)\cdot(\underset{(1)}{h}, \underset{(1)}{h})\}t^2/2!$$

II

$$+\underbrace{\{DE(\eta)(\underset{(3)}{h}) + 3D^2E(\eta)(\underset{(1)}{h}, \underset{(2)}{h}) + D^3E(\eta)(\underset{(1)}{h}, \underset{(1)}{h}, \underset{(1)}{h})\}}_{\text{III}}$$
$$\times t^3/3! + \cdots. \qquad (16)$$

Now conversely, suppose we are given an analytic curve

$$T(t) = \sum_{i=1}^{\infty} \underset{(i)}{T} \frac{t^i}{i!}$$

of stress–energy tensors with $T(0) = 0$. Then we can try to find a solution curve of the form (14). According to (16), the equations to be solved are then (in order)

$$\text{I: } DE(\eta)\cdot(\underset{(1)}{h}) = -\underset{(1)}{T}, \quad \text{for } \underset{(1)}{h},$$

$$\text{II: } DE(\eta)\cdot(\underset{(2)}{h}) = -\underset{(2)}{T} - D^2E(\eta)(\underset{(1)}{h}, \underset{(1)}{h}), \quad \text{for } \underset{(2)}{h},$$

$$\text{III: } DE(\eta)(\underset{(3)}{h}) = -\underset{(3)}{T} - 3D^2E(\eta)(\underset{(1)}{h}, \underset{(2)}{h})$$
$$- D^3E(\eta)(\underset{(1)}{h}, \underset{(1)}{h}, \underset{(1)}{h}), \quad \text{for } \underset{(3)}{h},$$
$$\cdots \text{etc.} \qquad (17)$$

It should be noted that at each stage of the iteration process, one has only to solve a linear equation, which is, in principle, possible.

[1] D. Lerner, Comm. Math. Phys. (to be published).
[2] F. A. E. Pirani, in *Lectures on General Relativity*, 1964 Brandeis Summer Institute in Theoretical Physics, Vol. 1 (Prentice-Hall, Englewood Cliffs, N. J., 1965).