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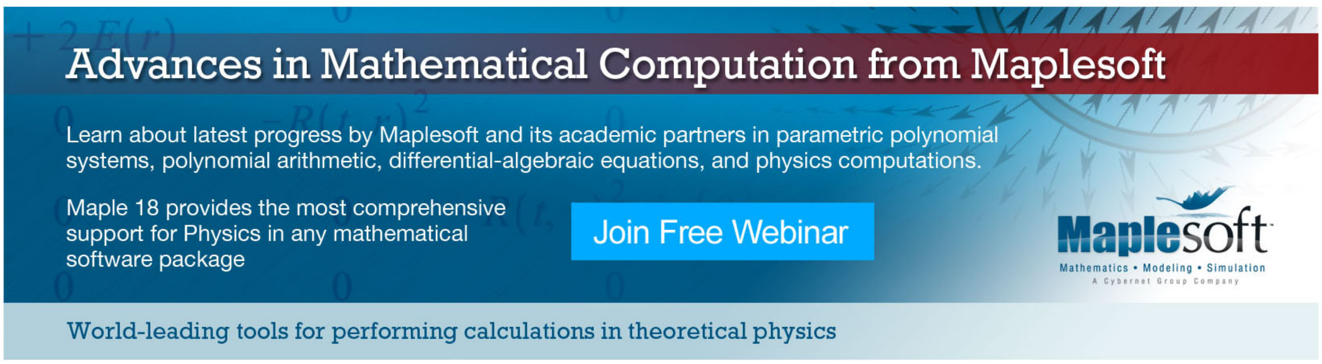

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Asymptotically simple space-time manifolds

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Asymptotic simplicity is shown to be k-stable ($k \geq 3$) at any Minkowski metric on $\mathbb{R}^4$ in both the Whitney fine C^k topology and a coarser topology (in which the C^k twice-convariant symmetric tensors form a Banach manifold whose connected components consist of tensor field asymptotic to one another at null infinity). This result, together with a sequential method for solving the field equations previously proposed by the authors, allows a fairly straightforward proof that a well-known result in the linearized theory holds in the full nonlinear theory as well: There are no nontrivial (i.e., non-Minkowskian) asymptotically simple vacuum metrics on $\mathbb{R}^4$ whose conformal curvature tensors result from prescribing zero initial data on past null infinity.

I. INTRODUCTION

The concept of asymptotically simple space–time manifolds, introduced by Penrose,[1,2] is a fruitful one in the study of asymptotic conditions in general relativity and one which Penrose has used to good advantage.[3] One would like to have more examples of asymptotically simple space–times than the single example now known, namely Minkowski space–time. In this paper it is shown that there are many asymptotically simple space–times; in fact, there is an open neighborhood of any Minkowski metric on $\mathbb{R}^4$ in the Whitney fine C^k topology ($k \geq 3$) on the set of Lorentz metrics on $\mathbb{R}^4$ all of whose elements are asymptotically simple metrics. Using this result and a formulation for weak gravitational fields developed by the authors,[4] we show that a certain linearized solution to the vacuum field equations has an exact counterpart in the full nonlinear theory. Section II deals with definitions and preliminaries; Sec. III gives a proof of the asserted result for asymptotically simple space–times; Sec. IV extends the known linear result to the full theory; and Sec. V gives some concluding remarks and conjectures.

II. MATHEMATICAL PRELIMINARIES

A space–time manifold is a pair (M,g) where M is a four-dimensional C^∞ differentiable manifold and g is a C^3 pseudo-Riemanian metric on M of signature -2 which is time-oriented and possesses no closed timelike C^1 curves. A space–time manifold (M,g) is said to be *asymptotically simple* if there exists a space–time manifold $(\tilde{M},\tilde{g})$ with boundary whose interior is conformal to (M,g) with $\tilde{g} = \Omega^2 g$, $\Omega > 0$, which satisfies:

(1) $\tilde{M}$ is a C^4 differentiable manifold with boundary $\mathscr{I}$ and $\tilde{g}$ is a C^3 pseudo-Riemannian metric on m,

(2) Ω is C^3 on $\tilde{M}$, $\Omega = 0$ on $\mathscr{I}$, and $d\Omega \neq 0$ on $\mathscr{I}$,

(3) Every maximally extended null geodesic in the interior of M^2 intersects $\mathscr{I}$ in precisely two points.[1]

Minkowski space–time $(\mathbb{R}^4, \eta)$ is an example of an asymptotically simple space–time. The manifold $(\tilde{M}, \tilde{\eta})$ whose interior is conformal to $(\mathbb{R}^4, \eta)$ is obtained by constructing the conformally-compactified Minkowski space–time[5,6] and then slicing this manifold apart along the light cone at infinity. The result is represented pictorially in Fig. 1. The points I^+, I^0, and I^- are not points of $\mathscr{I}$.

An exposition on the Whitney topologies applied to problems in general relativity is given by Lerner,[7] and only a brief summary will be given here. Given an arbitrary Riemannian metric μ on a differentiable manifold M, a W_0 (Whitney fine C^0) neighborhood base of a

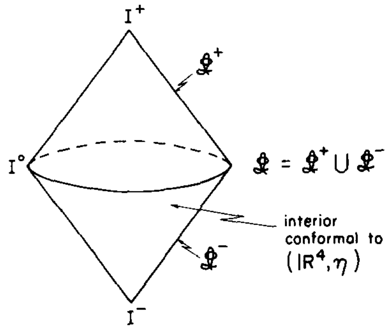

FIG. 1

C^0 tensor field $t \in \Gamma^0(T^r_s(M))$ is given by sets of the form

$$W_\mu(t, \epsilon(x)) = \{s \mid s \in \Gamma^0(T^r_s(M)) \text{ and } \|s - t\|_\mu(x) < \epsilon(x), \forall\ x \in M\}.$$

Here $\epsilon(x)$ is any positive continuous function on M and $\|\cdot\|_\mu(x)$ is the μ norm in $T^r_s M_x$. A W_k (Whitney fine C^k) neighborhood base ($k \geq 0$) of a tensors field $t \in \Gamma^k(T^r_s(M))$ is given by sets of the form

$$W^k_\mu(t, \epsilon(x)) = \{s \mid s \in \Gamma^k(T^r_s(M)) \text{ and } \|s - t\|_\mu(x) < \epsilon(x),$$
$$\|\nabla(s-t)\|_\mu(x) < \epsilon(x), \ldots, \|\nabla^k_s(s-t)\|_\mu(x) < \epsilon(x), \forall\ x \in M\}.$$

where ∇^i_s denotes the totally symmetrized ith covariant derivative with respect to the Riemannian metric μ. This gives a convenient description of the W^k topologies (the Whitney fine C^k topologies); an altogether equivalent formulation, which is manifestly independent of μ and perhaps more physically intuitive as well, is given in Ref. 7.

The set of all C^k Lorentz metrics (pseudo-Riemannian metric tensor fields of signature -2) on M will be denoted by $L^k(M)$. A property P on $\Gamma^k(T^r_s(M))$ is said to be k-stable at a tensor $t \in \Gamma^k(T^r_s(M))$ if there exists an open W^k neighborhood of t all the elements of which possess property P. For example, geodesic completeness is k-stable, $k \geq 2$, on $L^k(M)$ as is time orientability. The purpose of Sec. III is to prove that asymptotic simplicity is k-stable, $k \geq 3$, at η in L^k.

Another topology is used on $L^k(\mathbb{R}^4)$ in Sec. IV which was introduced by Lerner and Porter.[4] Given η and a Minkowski coordinate systems $\{x^i\}$ on $\mathbb{R}^4$ in which $\eta = ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$, for any $t \in \Gamma^k(T^r_s(\mathbb{R}^4))$ set

$$\|t(x)\|_k = \max_{\text{components}} \{|t^{\cdots}_{\cdots}(x)|, |t^{\cdots}_{\cdots,\cdot}(x)|, \ldots, |t^{\cdots}_{\cdots,\underbrace{\cdots}_{k}}(x)|\}$$

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where $t_{\vdots\vdots}(x)$ stands for the components of t in $\{x^i\}$ and $,\cdots$ denotes coordinate derivatives with respect to $\{x^i\}$. Let

$$|t|_k = \sup\{\|t(x)\|_k | x \in \mathbb{R}^4\}.$$

Define

$$B_k^{(r,s)} = \{t \in \Gamma^k(T^r_s) | \, |t|_k < \infty\}.$$

Then $\langle B_k^{(r,s)}, |\cdot|_k \rangle$ is a Banach space; the topology is equivalent to that of uniform C^k convergence.

Given a particular Minkowski metric η on $\mathbb{R}^4$ and the conformal factor Ω mentioned in the definition of asymptotic simplicity, one can define the set of tensor fields of type (r, s) *asymptotic to zero at null infinity:*

$$A_k^{(r,s)} \equiv \{t \in B_k^{(r,s)}: \Omega^2 t \text{ extends to a } C^k \text{ tensor field on } \tilde{M} \text{ with } \Omega^2 t = 0 \text{ on } \mathscr{I}\}.$$

[The conformal factor (Ω^2) used in the definition of $A_k^{(r,s)}$ for certain applications may be changed to another power of Ω. The factor Ω^2 is the correct one to use for metrics asymptotic to η in $B^{(0,2)}$.] $A_k^{(r,s)}$ is a closed subspace of $B_k^{(r,s)}$ and thus a Banach space in its own right. The set $\Gamma^k(T^r_s(\mathbb{R}^4)$ is made into a Banach manifold by taking the sets

$$\{U_\alpha \equiv \alpha + A_k^{(r,s)}: \alpha \in \Gamma^k(T^r_s(\mathbb{R}^4))\}$$

for an (analytic) atlas. Two tensor fields $\alpha, \beta \in \Gamma^k(T^r_s(\mathbb{R}^4))$ lie in the same connected component in this topology iff $\alpha - \beta$ is asymptotic to zero at null infinity. The fields α and β are then said to be asymptotic at infinity. This topology is called the A^k topology on tensor fields. It is clear that the W^k topology is finer than the A^k topology (any open set in the A^k topology is open in the W^k topology as well).

III. STABILITY AND ASYMPTOTIC SIMPLICITY

Theorem 1: Asymptotic simplicity is a k-stable property, $k \geq 3$, in $L^k(\mathbb{R}^4)$ at any Minkowski metric η on $\mathbb{R}^4$.

Proof: It is necessary to exhibit an open W^k neighborhood, $k \geq 3$, of η in $L^k(\mathbb{R}^4)$, the Lorentz metrics in which are all asymptotically simple. First, there exists an open W^k neighborhood U_1 in $\tilde{M} - \mathscr{I}$ of $\tilde{\eta}$, all the Lorentz metrics in which are equal to $\tilde{\eta}$ on $\mathscr{I}$. Since $\tilde{M} - \mathscr{I}$ is differomorphic to $\mathbb{R}^4$ by virtue of the conformal relatedness, an open W^k neighborhood U_1 of η in $\mathbb{R}^4$ is obtained. All the Lorentz metrics in U_1 are asymptotic to the Minkowski metric η. The Lorentz metrics in U_1 satisfy the first two defining properties of asymptotic simplicity (using the same $\tilde{M}$ and Ω as for $(\mathbb{R}^4, \eta)$ and defining $\tilde{g} = \Omega^2 g$ for $g \in U_1$).

The third defining property of asymptotic simplicity states that null geodesics are complete and that, intuitively, they reach "infinity". This is certainly true for Minkowski space and an open W^k neighborhood of η must be exhibited, all the Lorentz metrics in which exhibit this feature. To this end, it is noted that the spray of η is certainly complete and that the completeness of a vector field is a stable property.[7] [The spray of a metric g on M is the map sp: $TM \to TTM$ defined locally as sending $(p, v_p) \to (p, v_p, v_p, -\Gamma(p)(v_p, v_p))$ where $\Gamma(p)$ is the connection of g at p. The spray of a metric is a second-order differential equation on M and its curves give the geodesics of g. Geodesic completeness of g means that the vector field sp(g) is complete.] Thus there exists a W^{k-1} neighborhood of the spray of η in $\Gamma^{k-1}(TTM)$ all the vector fields in which are complete. As the map sending a Lorentz metric to its spray is continuous, there exists an open neighborhood U_2 of η, all the metrics in which are geodesically complete. Let $U_3 = U_1 \cap U_2$.

If a Minkowski coordinate system $\{t = x^0, x^1, x^2, x^3\}$ is chosen for η, so that $ds^2 = dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$, then any null geodesic in $(\mathbb{R}^4, \eta)$ has the property that it crosses each of the $t = $ const hypersurfaces. Thus the null spray of η [in these coordinates $(x^i, v^i) \to (x^i, v^i, v^i, 0)$ where $\eta_{ij} v^i v^j = 0$] is transverse to the hypersurfaces $t \circ \pi = $ const, where π is the projection associated with the tangent bundle, $T\mathbb{R}^4$. This transversal property is stable and so there exists a W^k open neighborhood U_4 of η containing only metrics whose null sprays are transverse to the hypersurfaces $t \circ \pi = $ const. Let $U = U_3 \cap U_4$. The metrics in U satisfy the properties (1) and (2) of asymptotically simplicity, and have complete null sprays transverse to each $t \circ \pi = $ constant hypersurface. The claim is that the metrics will also satisfy property (3). Null geodesics (maximally extended) for a metric g in U cross each $t = $ const hypersurface. Since null geodesics are conformally invariant, the image of a null geodesic of $(\mathbb{R}^4, g)$ under the conformal map is a null geodesic of $(\tilde{M}, \tilde{g})$. Thus a null geodesic (maximally extended) of (M, g) can only fail to have two points on $\mathscr{I}$ if it contains I^+, I^0, or I^-. See Fig. 2. But this is impossible as the past light cone of I^+ is $\mathscr{I}^+$, the light cone of I^0 is $\mathscr{I}$, and the future light cone of I^- is $\mathscr{I}^-$. Thus, for example, the only null geodesics in $(\tilde{M}, \tilde{g})$ containing I^+ are those that lie on $\mathscr{I}^+$ and no null geodesic from $\tilde{M} - \mathscr{I}$ can contain I^+. Thus, we have an open W^k neighborhood U of η in $L(\mathbb{R}^4)$ containing only asymptotically simple space-times.

So there are many asymptotically simple space-times based in $\mathbb{R}^4$; in particular, ones which are not conformally flat.

A slightly stronger version of Theorem 1 can be proved.

Theorem 1': Asymptotic simplicity is stable at η in the A^k topology on the set B_k of C^k Lorentz metrics which are asymptotic to η.

The A^k topology is coarser than the W^k topology and the additional asymptotic condition is essential. The proof proceeds similarly to that above, with one making certain that at each step the W^k open sets can be replaced with A^k open sets all the elements in which are asymptotic to η.

IV. CURVES OF LORENTZ METRICS

Given a curve in $L^k(\mathbb{R}^4)$, $k \geq 2$, of Lorentz metrics at η, g: $t \to g(t)$ such that $g(0) = \eta$, the induced curve of Riemann tensors is denoted by Ω: $t \to \text{Riem}(g(t)) \in \Gamma^{k-2}(T^0_4\mathbb{R}^4)$, the induced curve of Ricci tensors is denoted by Ric: $t \to \text{Ricci}(g(t)) \in \Gamma^{k-2}(T^0_2\mathbb{R}^4)$, and the in-

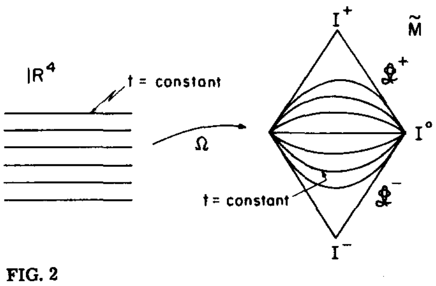

FIG. 2

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duced curve of conformal tensors is denoted by Co: $t \to \text{conf.}(g(t)) \in \Gamma^{k-2}(T^0_4(\mathbb{R}^4))$. If g is in $B_k^{(0,2)}$ for η and Minkowski coordinate system $\{x^i\}$, then the three induced curves Ω, Ric, and Co have images in $B_{k-2}^{(0,4)}$, $B_{k-2}^{(0,2)}$, and $B_{k-2}^{(0,4)}$, respectively.[4] If, in addition, g is analytic at η, then the induced curves are all analytic at zero in the respective spaces.

On the other hand, if an analytic curve is given at 0 in $B_{k-2}^{(0,2)}$, the question arises as to whether there exists an analytic curve in $B_k^{(0,2)}$ of Lorentz metrics at η which induces the given curve in $B_{k-2}^{(0,2)}$ as its curve of Ricci tensors. This, in general, will not be the case; when it is true one can solve the field equations in general relativity by solving sequentially a set of linear partial differential equations. The difficulties are with respect to boundedness of the terms and convergence of the resulting sequence. An additional freedom in the resulting curve is fixed by appropriate initial conditions for g. This additional freedom is a useful adjunct in trying to find such an analytic curve of Lorentz metrics.

Theorem 2: Let $t \to g(t) \in B_k^{(0,2)}$ $(k \geq 3)$ be an analytic curve of vacuum metrics $(\text{Ric}(g(t)) = 0$, all $t)$ on $\mathbb{R}^4$ with $g(0) = \eta$. Suppose that

(a) for all $t, g(t)$ is asymptotic to η,

(b) for each t, the conformal curvature tensor of $g(t)$ results from zero initial data on $\mathscr{I}^-$ in the conformally related $(\tilde{M}, \tilde{\eta})$.

Then $g(t)$ is a curve of Minkowski metrics.

Proof: Given a curve of Lorentz metrics analytic at η, g: $t \to g(t)$, the analyticity at η requires that

$$g(t) = \eta + \sum_{i=1}^{\infty} \underset{(i)}{h} \frac{t^i}{i!}.$$

The requirement that the curve be asymptotic to η is that the curve

$$\sum_{i=1}^{\infty} \underset{(i)}{h} \frac{t^i}{i!}$$

by asymptotic to zero. Tensor fields of any given type in $B_k^{(\cdot,\cdot)}$ asymptotic to zero form a Banach space. There is an open interval about zero for which the curve g has its image in the set of asymptotically simple metrics on $\mathbb{R}^4$ as guaranteed by Theorem 1′; let t be restricted to such an interval. The solution of the equations for the Ricci tensor and conformal tensor for this curve then proceeds sequentially. Since the corresponding maps Ric and Co are analytic at η, the corresponding curves are completely determined by their derivatives for $t = 0$. The zeroth derivative gives conditions automatically satisfied since Minkowski space–time has zero Ricci tensor and zero conformal tensor. The first derivative of the curve of Ricci tensors at $t = 0$ simply gives the vanishing of the linearized Ricci tensor for $\eta + h_{(1)}$,

$$\eta^{ad} \underset{(1)}{h}_{ac,bd} - \eta^{ad} \underset{(1)}{h}_{bc,ad} - \eta^{ad} \underset{(1)}{h}_{ad,bc} + \eta^{ad} \underset{(1)}{h}_{bd,ac} = 0.$$

The first derivative of the curve of conformal tensors at $t = 0$ yields the linearized conformal tensor $C_{(1)abcd}$ of $\eta + h_{(1)}$ which must in the conformally related picture $(\tilde{M}, \tilde{\eta})$ be obtained from zero initial data on the null Cauchy hypersurface $\mathscr{I}^-$. The linearized conformal tensor, $C_{(1)abcd}$, satisfies the linearized Bianchi identities $\eta^{de} C_{(1)abcd,e} = 0$, whose spinor equivalent is $\nabla^{AA'}\psi_{ABCD} = 0$ if ψ_{ABCD} represents $C_{(1)abcd}$. By using the techniques developed by Penrose[1] for handling such zero rest-mass field equations in a conformally invariant manner, a spinor field $\phi_{ABCD} = \Omega^{-1}\psi_{ABCD}$ is obtained on $(\tilde{M}, \tilde{\eta})$ satisfying $\nabla^{AA'}\phi_{ABCD} = 0$. Initial data for ϕ_{ABCD} is given on the null Cauchy hypersurface $\mathscr{I}^-$, namely zero data, and the solution at a point $p \in \tilde{M}$ of the initial value problem for the equation $\nabla^{AA'}\phi_{ABCD} = 0$ can be obtained as a generalized Kirchoff integral over the intersection of the past light cone of p and the initial data surface $\mathscr{I}^-$.[3] For zero initial data, the resulting field ϕ_{ABCD} is zero and thus $\psi_{ABCD} = 0$ or equivalently $C_{(1)abcd} = 0$.

The fact that the linearized Ricci tensor and the linearized conformal tensor of $\eta + h_{(1)}h$ are both zero is equivalent to the vanishing of the linearized Riemann tensor. This fact is best exploited by using the Cartan structure equations in their linearized forms for the determination of $h_{(1)}$. The structure equations are

$$d\theta^a + \omega^a{}_b \wedge \theta^b = 0,$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \tfrac{1}{2}R^a{}_b,$$

where θ^a is a basis for cotangent vector fields, $\omega^a{}_b$ is the connection form whose Riemann curvature form is $R^a{}_b$. In the frame $\theta^a = dx^a$ the metric $\eta + h_{(1)}$ yields the linearized structure equations with linearized connection $\omega^a{}_b$

$$\omega^a{}_b \wedge \theta^b = 0, \tag{1}$$

$$d\omega^a{}_b = 0. \tag{2}$$

Equation (2) says that the 1-form $\omega^a{}_b = \omega^a{}_{bc}dx^c$ is closed and since the manifold is simply connected, $\omega^a{}_b$ is exact,

$$\omega^a{}_{bc} = \mu^a{}_{b,c}. \tag{3}$$

Equation (1) then gives $\mu^a{}_{b,c}dx^b \wedge dx^c = 0$ or that the 1-form $\mu^a = \mu^a{}_b dx^b$ is closed. Again, simple connectivity says that μ^a is exact,

$$\mu^a{}_b = \underset{(1)}{\sigma^a}{}_{,b}.$$

and thus

$$\omega^a{}_{bc} = \underset{(1)}{\sigma^a}{}_{,bc}. \tag{4}$$

The tensor $h_{(1)ab}$ is obtained from the linearized equation for the vanishing of the covariant derivative of the metric, i.e., $Dg_{ab;c}(t)|_{t=0}(h_{(1)ab})$, resulting in

$$\underset{(1)}{h}_{ab,c} = \eta_{ad} \underset{(1)}{\sigma}{}^d{}_{,bc} + \eta_{bd} \underset{(1)}{\sigma}{}^d{}_{,ac}, \tag{5}$$

whose solution (absorbing constants of integration into $\sigma_{(1)}{}^a$) is

$$\underset{(1)}{h}_{ab} = \underset{(1)}{\sigma}_{a,b} + \underset{(1)}{\sigma}_{b,a}, \tag{6}$$

where

$$\underset{(1)}{\sigma}_a = \eta_{ad} \underset{(1)}{\sigma}{}^d.$$

The four functions $\sigma_{(1)a}$ are required to be C^4 and to be appropriately bounded in $B_4^{(0,0)}$.

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A coordinate transformation $x'^a = x^a + t\sigma_{(1)}{}^a$ yields the same curve of metrics which, when expressed in the new coordinate system, has zero linear term:

$$\eta + 0 \cdot t + \sum_{i=2}^{\infty} h'_{(i)} \frac{t^i}{i!}. \tag{7}$$

The next step in the sequential solution is to determine $h'_{(2)}$ from the conditions imposed on the curve of Ricci terms (namely that it be the constant curve 0) and the curve of conformal tensors. The resulting equations for $h'_{(2)}$ are exactly those previously solved for $h_{(1)}$ and the same technique that resulted in Eq. (7) yields a curve of the form

$$\eta + 0 \cdot t + 0 \cdot \frac{t^2}{2!} + \sum_{i=3}^{\infty} h''_{(i)} \frac{t^i}{i!}. \tag{8}$$

Continuing, it is seen that the resulting curve is a curve of Minkowski metrics as required. For each t, the transformation

$$\tilde{x}^a = x^a + \sum_{i=1}^{\infty} \sigma_{(i)}{}^a \frac{t^i}{i!}$$

exhibits $g(t)$ in a Minkowski coordinate system.

V. CONCLUSION

The following questions arise:

(1) Is asymptotic simplicity W^k stable? Theorem 1 is the proof that this is the case at any Minkowski metric on $\mathbb{R}^4$, $(\mathbb{R}^4, \eta)$. The proof utilizes some special properties of Minkowski space and no obvious generalizations of the techniques involved exist. Also note that Theorem 1 gives the stability of weakly asymptotically simple spaces[1] at any weakly asymptotically simple space whose corresponding asymptotically simple space is in the open W^k neighborhood of $(\mathbb{R}^4, \eta)$ exhibited in the theorem.

(2) Is it possible to use the techniques in Sec. IV to generate nontrivial analytic curves of solutions to prescribed field equations? Considerations of this nature are to appear in a forthcoming paper by the authors.

(3) Is it possible to obtain conditions under which the linearized solutions to the field equations actually determine the behavior of solutions in the full theory? Included in this might be a formulation of stable properties in terms of Lyapunov functionals.

[1]R. Penrose, in *Battelle Recontres*, 1967 Lectures in Mathematics and Physics (Benjamin, New York, 1968).
[2]R. Penrose, *An analysis of the Structure of Space–Time* (Princeton U. P., Princeton, N. J., 1966).
[3]E. T. Newman and R. Penrose, Proc. R. Soc. A **305**, 175 (1968).
[4]D. Lerner and J.R. Porter, J. Math. Phys. 15, 1413 (1974).
[5]N. H. Kupier, Ann. Math. **50**, 916 (1949).
[6]H. Rudberg, thesis (Uppsula, 1957).
[7]D. E. Lerner, Comm. Math. Phy. (to be published).

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