Complex *pp* waves and the nonlinear graviton construction

W. D. Curtis

Department of Mathematics, Kansas State University, Manhattan, Kansas 66506

D. E. Lerner

Department of Mathematics, University of Kansas, Lawrence, Kansas 66045

F. R. Miller

Department of Mathematics, Kansas State University, Manhattan, Kansas 66506
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We show how to construct all of the complex *pp* waves using the nonlinear graviton construction of Penrose.

INTRODUCTION

Several years ago, Penrose[1] showed that one could construct generic, half-flat solutions of the complex Einstein equations through the use of deformation theory. His idea is to deform the complex structure of a neighborhood of a projective line in $P_3(C)$. The original undeformed neighborhood contains a four-complex-parameter family of lines which are identified with the points of an open subset of complex Minkowski space. For small deformations, Kodaira's theorems[2,3] guarantee the continued existence of a four-parameter family of "lines" (i.e., compact holomorphic curves) in the deformed space; these are identified as the points of a new complex manifold $\mathcal{G}$. A holomorphic metric is then introduced on $\mathcal{G}$ in a natural way and the resulting complex spacetime is shown to be half-flat, that is, its Ricci tensor vanishes and its conformal curvature tensor is anti-self-dual.

While it is a relatively straightforward matter to construct deformations, the task of finding the four-parameter family of lines in the deformed space is usually very difficult. Because of this, only a few isolated solutions have actually been explicitly constructed, only one of which, to our knowledge, has appeared in print.[4] The purpose of this paper is to show that the simplest half-flat spacetimes, known as complex pp waves or Plebański plane waves,[5] can all be obtained explicitly using the Penrose construction.

1. THE NONLINEAR GRAVITON CONSTRUCTION

In this section we summarize the Penrose construction. For more details, in particular for the proof that $\mathcal{G}$ is half-flat, we refer the reader to Penrose's original article.[1]

Denote a point of $C^4-(0)$ (a twistor) by $Z^\alpha=(\omega^A,\pi_{A'})$, and let $[\omega^A,\pi_{A'}]$ denote the corresponding point in $P_3(C)$. If $x^{AA'}$ is any point in complex Minkowski space, CM, we may associate with it the projective line $L(x)$: $=\{[ix^{AA'}\pi_{A'},\pi_{A'}]\,|\,[\pi_{A'}]\in P_1(C)\}$. If W is a connected open neighborhood of x in CM, the set $PT(W):=\{L(y)\,|\,y\in W\}$ is a connected open neighborhood of $L(x)$ in $P_3(C)$. In this paper we shall consider only the case $W=\text{CM}$ and we set $PT:=PT(\text{CM})$. Notice that PT is just $P_3(C)$ with one projective line removed (namely, all points of the form $[\omega^A,0]$). T is the corresponding set of points in $C^4-(0)$. Then we have:

(a) PT is a holomorphic fiber space over $P_1(C)$ with projection $[\omega^A,\pi_{A'}]\to[\pi_{A'}]$. Similarly we have a fiber space $T\to C^2-(0)$ given by $(\omega^A,\pi_{A'})\to\pi_{A'}$ and the following diagram commutes.

$$\begin{array}{ccc} T & \longrightarrow & PT \\ \downarrow & & \downarrow \\ C^2-(0) & \longrightarrow & P_1(C) \end{array} \qquad (1)$$

(b) The points of CM are in 1—1 correspondence with the global holomorphic cross sections of $PT\to P_1(C)$: Given $x^{AA'}$, define a section by $[\pi_{A'}]\to[ix^{AA'}\pi_{A'}\pi_{A'}]$. Alternately points of CM may be put in correspondence with global holomorphic cross sections of $T\to C^2-(0)$ which are homogeneous of degree 1 in $\pi_{A'}$.

(c) The conformal structure of CM is obtained by observing that points x and y in CM are null-separated iff $L(x)$ and $L(y)$ intersect. If $y=x+\Delta x$ and if $\omega^A(\Delta x,\pi_{A'})$ is the section of $T\to C^2-(0)$ corresponding to Δx, then $\omega^A(\Delta x,\mathring{\pi}_{A'})=0$ for some $\mathring{\pi}_{A'}$ [and hence for $\lambda\mathring{\pi}_{A'}$ for $\lambda\in C-(0)$]. Thus null vectors in CM correspond to global holomorphic sections of $T\to C^2-(0)$ which vanish somewhere. In order to pin down the conformal factor, one makes use of the 2-form $d\omega_A\wedge d\omega^A=\mu$. This will be considered in more detail later.

If we let D (resp. $\hat{D}$) be the subset of $C^2-(0)$ given by $\pi_{0'}\neq0$ (resp. $\pi_{1'}\neq0$), then we get a decomposition of T as the union $U\cup\hat{U}$, where $U=\{(\omega^A,\pi_{A'})\,|\,\pi_{A'}\in D\}$ and $\hat{U}=\{(\hat{\omega}^A,\hat{\pi}_{A'})\,|\,\hat{\pi}_{A'}\in\hat{D}\}$. We may view T as being formed by glueing together U and $\hat{U}$ by the trivial equations $\hat{\omega}^A=\omega^A$, $\hat{\pi}_{A'}=\pi_{A'}$. To deform T, we consider a one-parameter family of patchings of the form

$$\hat{\omega}^A=f^A(\omega,\pi,\lambda),\quad \hat{\pi}_{A'}=\pi_{A'}, \qquad (2)$$

satisfying

$$\begin{aligned} &f^A(\alpha\omega,\alpha\pi,\lambda)=\alpha f^A(\omega,\pi,\lambda),\quad \alpha\in C-(0),\\ &f^A(\omega,\pi,0)=\omega^A. \end{aligned} \qquad (3)$$

Here λ ranges over a neighborhood B of $0\in C$ and the

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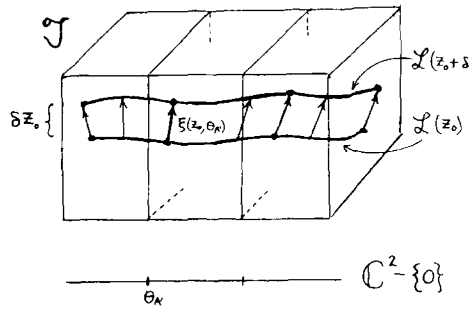

FIG. 1.

functions $\hat{f}^A$ are holomorphic in $C^2\times(D\cap\hat{D})\times B$. For each fixed $\lambda\in B$ the patching (2) gives a fiber space $\mathcal{T}(\lambda)\to C^2-(0)$ (a deformed twistor space). Since the transition functions are homogeneous of degree 1, the identification $(\omega^A, \pi_{A'})\sim(\alpha\omega^A, \alpha\pi_{A'})$, $\alpha\in C-(0)$, is consistent over $D\cap\hat{D}$ and gives rise to a deformed projective twistor space $P\mathcal{T}(\lambda)\to P_1(C)$. An important aspect of these deformed twistor spaces is the existence of a preferred 2-form on the fibers of $\mathcal{T}(\lambda)\to C^2-(0)$; the transformations (2) are required to satisfy $d\hat{\omega}^0\wedge d\hat{\omega}^1 = d\omega^0\wedge d\omega^1+$ (terms involving $d\pi_{A'}$).

Kodaira's stability theorem[2] ensures that for $|\lambda|$ sufficiently small, $P\mathcal{T}(\lambda)\to P_1(C)$ still has a global holomorphic section. His completeness theorem[3] then guarantees the existence of a four-parameter family of global holomorphic sections of $P\mathcal{T}(\lambda)\to P_1(C)$. So for λ fixed and sufficiently small we have an open set $\mathcal{G}\subset C^4$ and for each $z\in\mathcal{G}$ a global holomorphic section, $L_\lambda(z)$, of $P\mathcal{T}(\lambda)\to P_1(C)$, distinct z's giving rise to distinct sections. Each such $L_\lambda(z)$ pulls back to a global holomorphic section of $\mathcal{T}(\lambda)\to C^2-(0)$ which is homogeneous of degree 1. This section, $\mathcal{L}_\lambda(z)$, is represented by a pair $\{\omega^A(z,\pi,\lambda), \hat{\omega}^A(z,\pi,\lambda)\}$ satisfying the transition relation (2). Each function $\omega^A(z,\pi,\lambda)$, $\hat{\omega}^A(z,\pi,\lambda)$ is holomorphic on its domain and homogenneous of degree 1 in $\pi_{A'}$.

Henceforth we shall consider a fixed value of λ and shall omit λ from the notation.

Now let $z_0\in\mathcal{G}$ and let $\delta z=\xi^a(\partial/\partial z^a)(z_0)$ be tangent to $\mathcal{G}$ at z_0. Define $\xi^A(z_0,\pi):=\xi^a(\partial\omega^A/\partial z^a)(z_0,\pi)$ and $\hat{\xi}^A(z_0,\pi):=\xi^a(\partial\hat{\omega}^A/\partial z^a)(z_0,\pi)$. Then

$$\hat{\xi}^A(z_0,\pi)=(\partial\hat{f}^A/\partial\omega^B)(\omega(z_0,\pi),\pi)\xi^B(z_0,\pi). \qquad (4)$$

Thus the pair $\{\xi^A, \hat{\xi}^A\}$ is a section of the normal bundle of $\mathcal{L}(z_0)$ in $\mathcal{T}$. Intuitively the situation is as follows: $\mathcal{L}(z_0)$ is a section of $\mathcal{T}\to C^2-(0)$ which we view as a submanifold of $\mathcal{T}$. We write (to first order) $\mathcal{L}(z_0+\delta z):=\{\omega^A(z_0,\pi)+\xi^A(z_0,\pi);\ \hat{\omega}^A(z_0,\pi)+\hat{\xi}^A(z_0,\pi)\}$.

That is, we have a "nearby" section $\mathcal{L}(z_0+\delta z)$ and the "difference" between the two is a section of the normal bundle of $\mathcal{L}(z_0)$(see Fig. 1). It follows from Kodaira's completeness theorem[3] that the map $\delta z=\xi^a(\partial/\partial z^a)(z_0)\to\xi(z_0,\pi)$ is an isomorphism from $T_{z_0}\mathcal{G}$ onto the space of global holomorphic sections of the normal bundle of $\mathcal{L}(z_0)$ which are homogeneous of degree 1 in $\pi_{A'}$.

The conformal metric on $\mathcal{G}$ is obtained by defining $\xi_z\in T_z$ to be null if and only if $\xi(z,\pi)$ has a zero at some $\mathring{\pi}$ (and hence for any nonzero multiple of $\mathring{\pi}$). See (c) above.

In order to define the actual metric on $\mathcal{G}$ we use the 2-form, $\mu=\epsilon_{AB}\,d\omega^A\wedge d\omega^B=\epsilon_{AB}\,d\hat{\omega}^A\wedge d\hat{\omega}^B$ (mod $d\pi_{A'}$), which is well defined on the fibers of $\mathcal{T}$. If $\xi\in T_z\mathcal{G}$ the relation (4) shows that we may regard $\xi(z,\pi)$ as a vector field along $\mathcal{L}(z)$ which is everywhere tangent to the fibers. Suppose ξ and η are null vectors at z and that $\xi(z,\pi)$, $\eta(z,\pi)$ vanish at $\alpha_{A'}$, $\beta_{A'}$ respectively. Then Penrose defines

$$g_z(\xi,\eta):=(\alpha_{B'}\beta^{B'})\mu(\xi,\eta)/(\alpha_{B'}\pi^{B'})(\beta_{B'}\pi^{B'}). \qquad (5)$$

The right side of (5) is symmetric in (ξ,η). As a function of $\pi_{A'}$, it is homogeneous of degree 0, and holomorphic on $C^2-(0)$. Thus it is constant and so $g_z(\xi,\eta)$ is a well-defined complex number.

2. CONSTRUCTION OF THE COMPLEX pp WAVES

We choose the patching (2) to have the form

$$\hat{\omega}^0=\omega^0,$$
$$\hat{\omega}^1=\omega^1+\lambda h(\omega^0,\pi_{A'}), \qquad (6)$$
$$\hat{\pi}_{A'}=\pi_{A'},$$

where h is homogeneous of degree 1 and holomorphic on $C\times(D\cap\hat{D})$. We set $\lambda=1$ in what follows. We shall write down all global holomorphic sections of the bundle $\mathcal{T}\to C^2-(0)$ obtained using (6) which are also homogeneous of degree 1 in $\pi_{A'}$. Such a section is given by a pair $\{\omega^A(\pi), \hat{\omega}^A(\pi)\}$ satisfying (6) with $\omega^A(\pi)$, $\hat{\omega}^A(\pi)$ homogeneous of degree 1 and holomorphic in D, $\hat{D}$ respectively. Thus $\omega^0(\pi)=\hat{\omega}^0(\pi)$, so these give a holomorphic function on $C^2-(0)$, homogeneous of degree 1. Therefore, there exist u, $\zeta\in C$, constants, such that

$$\hat{\omega}^0(\pi)=\omega^0(\pi)=u\pi_{0'}+\zeta\pi_{1'}. \qquad (7)$$

$\omega^1(\pi)$ and $\hat{\omega}^1(\pi)$ are related as follows:

$$\hat{\omega}^1(\pi)=\omega^1(\pi)+h(\omega^0(\pi),\pi), \quad \pi\in D\cap\hat{D}. \qquad (8)$$

For fixed u and ζ, $h(u\pi_{0'}+\zeta\pi_{1'},\pi_{0'},\pi_{1'})$ is a holomorphic function $D\cap\hat{D}$. To construct a section of our bundle, we must express this function as a difference $\hat{\omega}^1(\pi)-\omega^1(\pi)$ where $\omega^1(\hat{\omega}^1)$ is holomorphic in $D(\hat{D})$ and homogeneous of degree 1. Each distinct way of splitting h will give a pair $\{\hat{\omega}^1,\omega^1\}$ and hence a section. We claim that for each pair (u,ζ) there is a two complex parameter family of splittings. Thus we get a four-parameter family of sections of $\mathcal{T}\to C^2(0)$ as desired.

To see this, let $\xi\neq 0$. Then setting $(\pi_{0'},\pi_{1'})=(1/\xi, 1)$ in (8),

$$\hat{\omega}^1(\xi^{-1},1)=\xi^{-1}\omega^1(1,\xi)+\xi^{-1}h(u+\xi\zeta,1,\xi). \qquad (8')$$

Let $h(\xi):=h(u+\xi\zeta,1,\xi)$ and expand $h(\xi)$ in a Laurent series, $h=\sum_{n=-\infty}^{\infty}b_n(u,\zeta)\xi^n$. Put $\omega^1(1,\xi)=\sum_{n=0}^{\infty}a_n\xi^n$. This series is convergent for all $\xi\in C$, and $\xi^{-1}\omega^1(1,\xi)+\xi^{-1}h(\xi)$ is to be entire in ξ^{-1}. We conclude that in the series expansion of the right side of (8′) all positive powers of ξ disappear. We conclude $a_n = -b_n$ for $n \geq 2$, while $\tilde{\zeta} := a_0$ and $v := a_1$ are free parameters. Thus in the first chart the sections are given by

$$\omega^0(u, v, \zeta, \tilde{\zeta}, \pi_{A'}) = u\pi_{0'} + \zeta\pi_{1'},$$

$$\omega^1(u, v, \zeta, \tilde{\zeta}, \pi_{A'}) = \tilde{\zeta}\pi_{0'} + v\pi_{1'} - \sum_{n=2}^{\infty} b_n(u, \zeta)\frac{(\pi_{1'})^n}{(\pi_{0'})^{n-1}} \tag{9}$$

and in the second chart by

$$\hat{\omega}^0(u, v, \zeta, \tilde{\zeta}, \pi_{A'}) = u\pi_{0'} + \zeta\pi_{1'}$$

$$\hat{\omega}^1(u, v, \zeta, \tilde{\zeta}, \pi_{A'}) = \tilde{\zeta}\pi_{0'} + v\pi_{1'} + \sum_{n=-1}^{\infty} b_{-n}(u, \zeta)\frac{(\pi_{0'})^{n+1}}{(\pi_{1'})^n} \tag{$\hat{9}$}$$

Computing the metric: Consider $(u, v, \zeta, \tilde{\zeta})$ as coordinates of a point in $\mathcal{G}$. Let $(du, dv, d\zeta, d\tilde{\zeta})$ be components of a tangent vector at $(u, v, \zeta, \tilde{\zeta})$. According to the discussion in Sec. 1, we get a section of the normal bundle to the section of $\mathcal{T}$ labelled by $(u, v, \zeta, \tilde{\zeta})$ by writing, in un-hatted coordinates,

$$V^0 = \pi_{0'}\, du + \pi_{1'}\, d\zeta$$

$$V^1 = \pi_{0'}\, d\tilde{\zeta} + \pi_{1'}\, dv - \sum_{n=2}^{\infty}\left(\frac{\partial b_n}{\partial u}\, du + \frac{\partial b_n}{\partial \zeta}\, d\zeta\right)\frac{(\pi_{1'})^n}{(\pi_{0'})^{n-1}}. \tag{10}$$

Assume $d\zeta \neq 0$. Then, for a null vector, $(V^0(\pi_{A'}), V^1(\pi_{A'})) = (0, 0)$ for some $\pi_{A'}$. We must in fact have a zero at $(\pi_{0'}, \pi_{1'}) = (-d\zeta, du)$. But then $V^1 = 0$ gives

$$0 = -d\zeta d\tilde{\zeta} + du dv - \sum_{n=2}^{\infty}\frac{\partial b_n}{\partial u}\frac{(du)^{n+1}}{(d\zeta)^{n-1}}(-1)^{n-1} - \sum_{n=2}^{\infty}\frac{\partial b_n}{\partial \zeta}\frac{(du)^n}{(d\zeta)^{n-2}}(-1)^{n-1},$$

$$0 = du dv - d\zeta d\tilde{\zeta} + \frac{\partial b_2}{\partial \zeta}\, du^2 + \sum_{n=2}^{\infty}(-1)^n\left(\frac{\partial b_n}{\partial u} - \frac{\partial b_{n+1}}{\partial \zeta}\right)\frac{(du)^{n+1}}{(d\zeta)^{n-1}}.$$

But, recalling that $h(u + \xi\zeta, 1, \xi) = \sum_{n=-\infty}^{\infty} b_n(u, \zeta)\xi^n$, we conclude $\partial b_n/\partial u = \partial b_{n+1}/\partial\zeta$. Thus the power series vanishes and the conformal metric is given by

$$ds^2 = \kappa\left(du dv - d\zeta d\tilde{\zeta} + \frac{\partial b_2}{\partial \zeta}\, du^2\right), \tag{11}$$

where κ is an arbitrary nonzero holomorphic function. We now show that the actual metric on $\mathcal{G}$ as defined by Penrose is obtained by taking $\kappa = 2$. Let

$$X := \frac{\partial}{\partial u} - \frac{\partial b_2}{\partial \zeta}\frac{\partial}{\partial v}, \quad Y := \frac{\partial}{\partial v}. \tag{12}$$

X and Y define sections as in (10). For X we have

$$X^0 = \pi_{0'},$$
$$X' = -\pi_{1'}\frac{\partial b_2}{\partial \zeta} - \sum_{n=2}^{\infty}\frac{\partial b_n}{\partial u}(u, \zeta)\frac{(\pi_{1'})^n}{(\pi_{0'})^{n-1}}. \tag{13}$$

For Y we have

$$Y^0 = 0, \quad Y^1 = \pi_{1'}. \tag{14}$$

The section X^A vanishes at $\pi_{A'} = (0, 1)$ while Y^A vanishes at $\pi_{A'} = (1, 0)$. That $X^A(0, 1) = 0$ is not evident in (13), but one must remember "unhatted" coordinates are not valid for $\pi_{0'} = 0$.

Let $x_{A'} = (0, 1)$, $y_{A'} = (1, 0)$. Now $x^{A'} = (1, 0)$, $y^{A'} = (0, -1)$. Then the Penrose inner product of X and Y is

$$g(X, Y) = \frac{x_{A'} y^{A'}\mu(X^A, Y^A)}{(x_{A'}\pi^{A'})(y_{A'}\pi^{A'})},$$

$$x_{A'} y^{A'} = -1, \quad x_{A'}\pi^{A'} = -\pi_{0'}, \quad y_{A'}\pi^{A'} = \pi_{1'}$$

and

$$\mu(X^A(\pi), Y^A(\pi)) = X^0(\pi)Y^1(\pi) - X^1(\pi)Y^0(\pi) = \pi_{0'}\pi_{1'}.$$

So $g(X, Y) = (-1)\pi_{0'}\pi_{1'}/-\pi_{0'}\lambda_{1'} = 1$. On the other hand, if we simply substitute X and Y into (11), to obtain $g(X, Y) = 1$ we find that we must take $\kappa = 2$, as asserted.

Now let $f(u, \zeta)$ be any entire function on C^2. We claim we can choose h so that $\partial b_2(u, \zeta)/\partial\zeta = f(u, \zeta)$. If so, then we will have generated all metrics of the form

$$ds^2 = du dv - d\zeta d\tilde{\zeta} + f(u, \zeta)du^2. \tag{15}$$

It is enough to show we can choose h so as to make $b_2(u, \zeta) = g(u, \zeta)$, where g is a given entire function. Write

$$g(u, \zeta) = \sum_{n, m \geq 0} a_{mn} u^m \zeta^n,$$

where the series converges everywhere. Define

$$\phi(x, y) := \sum_{n, m \geq 0}\frac{a_{mn}}{\binom{m+n}{n}} x^m y^n.$$

Then ϕ is an entire function of x, y. We then write,

$$h(\omega^0, \pi_{0'}, \pi_{1'}) := \phi\left(\frac{\omega^0}{\pi_{0'}}, \frac{\omega^0}{\pi_{1'}}\right)\frac{(\pi_{1'})^2}{\pi_{0'}}.$$

Clearly h is holomorphic on $C \times (\mathcal{D} \cap \hat{\mathcal{D}})$, and h is homogeneous of degree 1:

$$h(u + \zeta\xi, 1, \xi) = \phi\left(\frac{u + \zeta\xi}{1}, \frac{u + \zeta\xi}{\xi}\right)\xi^2 = \sum_{n,m \geq 0}\frac{a_{mn}}{\binom{m+n}{n}}\frac{(u + \zeta\xi)^{m+n}}{\xi^{n-2}},$$

$$\frac{(u + \zeta\xi)^{m+n}}{\xi^{n-2}} = \sum_{k=0}^{m+n}\binom{m+n}{k} u^{m+n-k}\zeta^k\xi^{k-n+2}$$

so when everything is expanded in powers of ξ for fixed u and ζ the coefficient of ξ^2 is $\sum_{n,m \geq 0} a_{mn} u^m \zeta^n = g(u, \zeta)$ as desired. For the metric (15) one can directly show that the Ricci tensor vanishes; whether the Weyl tensor is self-dual or anti-self-dual depends on the choice of complex volume element ϵ_{abcd}. There is on $\mathcal{G}$ a natural choice of ϵ due to the existence of a natural spinor structure on $\mathcal{G}$. In the coordinates $(z^\mu)=(u,v,\zeta,\tilde{\zeta})$ used above, ϵ is specified by $\epsilon_{0123}=-i$. With this choice the space is right-flat, i.e., $*C_{abcd}=-iC_{abcd}$.

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