TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 278, Number 2, August 1983

THE DIVISOR CLASSES OF THE HYPERSURFACE $z^{p^m} = G(x_1,\dots,x_n)$ IN CHARACTERISTIC $p > 0$

BY

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ABSTRACT. In this article we use P. Samuel's purely inseparable descent techniques to study the divisor class groups of normal affine hypersurfaces of the form $z^p = G(x_1,\dots,x_n)$ and develop an inductive procedure for studying those of the form $z^{p^m} = G$. We obtain results concerning the order and type of these groups and apply this theory to some specific examples.

Introduction. In this article we study the divisor class group of normal affine hypersurfaces $F_m \subset \mathbf{A}_k^{n+1}$ defined by equations of the form $z^{p^m} = G(x_1,\dots,x_n)$, where the ground field k is assumed to be algebraically closed of characteristic $p > 0$.

O. Zariski briefly considered surfaces of this type for the case $m = 1$, $n = 2$ in [**ZA**]. Investigations of their geometry have been made by P. Blass [**B1, B2**], who introduced me to this project. P. Samuel in his 1964 Tata notes [**S1**] describes the class group of several of these surfaces, such as $z^p = xy$ and $z^p = x^i + y^j$. Results from Samuel's notes and R. Fossum's book [**FO**] form the foundation of this work, and a brief discussion of these appears in §1.

Facts concerning the order and type of the class group of F_1: $z^p = G(x_1,\dots,x_n)$ appear in §2, together with the calculation of $z^p = H(x_1,\dots,x_n)$, where H is a form of degree not divisible by p.

The case $m > 1$ is attacked in §3. K. Baba in [**BA**] uses higher derivations to study the class group of the hypersurfaces. We develop an alternate, inductive method of attacking $\mathrm{Cl}(F_m$: $z^{p^m} = G)$. Again we collect results about the order and type of these groups, ending this section with some examples.

In §4 the local behavior of $\mathrm{Cl}(F_1$: $z^p = G(x_1, x_2))$ is discussed. In §5 a description of the class group of Krull rings A such that $k[x_1^{p^m},\dots,x_n^{p^m}] \subset A \subset k[x_1,\dots,x_n]$ is given.

0. Notation. 0.1. k-algebraically closed field of characteristic $p > 0$, unless stated otherwise.

0.2. $\mathbf{A}_k^n$-affine n-space over k.

0.3. Surface-irreducible, reduced, two-dimensional, quasiprojective variety over k.

0.4. The notation F: $f(x_1,\dots,x_n) = 0$ means

$$F = \operatorname{Spec} \frac{k[x_1,\dots,x_n]}{(f(x_1,\dots,x_n))} : F \subset \mathbf{A}_k^n.$$

Received by the editors June 16, 1982.

1980 *Mathematics Subject Classification*. Primary 13F15; Secondary 13B10, 14J05, 14C20, 14C22.

0002-9947/82/0000-0706/$06.50

0.5. If A is a Krull ring we denote by $\mathrm{Cl}(A)$ the divisor class group of A.

0.6. If F is a surface we denote by $\mathrm{Cl}(F)$ the divisor class group of the coordinate ring of F.

0.7. For $f(x_1,\ldots,x_n) \in k[x_1,\ldots,x_n]$ we denote by:

deg f—the total degree of f.

$\deg_{x_i} f$—the degree of f in x_i.

$\deg_{x_i,x_j}(f)$—the degree of f in the variables x_i and x_j.

1. Preliminaries. P. Samuel's 1964 Tata notes **[S1]** and R. Fossum's *The divisor class group of a Krull domain* **[FO]** form the framework for this article. What follows is a brief discussion of some results from these works. We begin with Samuel's notes.

DEFINITION. Let A be a domain. A is a *Krull* ring if there exists a family $(v_i)_{i\in I}$ of discrete valuations of $\mathrm{qt}(A)$ such that:

(1) $A = \bigcap_i R_{v_i}$, where R_{v_i} denotes the ring of v_i.

(2) For every $x \neq 0 \in A$, $v_i(x) = 0$, for almost all $i \in I$.

THEOREM 1.1. *A Noetherian integrally closed domain is a Krull ring* (*see* **[S1**, *p*. 5**]**).

DEFINITION. Let A be a domain with quotient field K. A *fractionary ideal* a is an A-submodule of K for which there exists an element $d \in A$ $(d \neq 0)$ such that $da \subset A$. A fractionary ideal is called a *principal ideal* if it is generated by one element. a is said to be *integral* if $a \subset A$. a is said to be *divisorial* if $a \neq (0)$ and if a is an intersection of principal ideals.

DEFINITION. Let $I(A)$ denote the set of nonzero fractionary ideals of the domain A. On $I(A)$ we define an equivalence relation by $a \sim b \Leftrightarrow A\colon a = A\colon b$. The quotient set of $I(A)$ by this equivalence relation is called the set of *divisors of* A, denoted by $D(A)$. For each $a \in I(A)$, we denote by $\bar{a}$ the equivalence class of a in $D(A)$.

DEFINITION. Let A be a Krull domain. The composition law $(a, b) \rightsquigarrow ab$ on $I(A)$ induces a well-defined operation on $D(A)$, thus giving $D(A)$ the structure of an abelian group with identity element $\bar{A}$ (see **[S1**, pp. 1–4**]**). Hereafter we will write this composition law additively. Thus $\bar{a} + \bar{b} = \overline{ab}$ for $\bar{a}, \bar{b} \in D(A)$. Let $F(A)$ denote the subgroup of $D(A)$ generated by the principal divisors (equivalence classes of principal ideals). We denote by $\mathrm{Cl}(A)$ the quotient group $D(A)/F(A)$, called the *divisor class group of* A.

THEOREM 1.2. *Let A be a Krull ring. Then*:

(1) $\mathrm{Cl}(A)$ *is generated by the classes of the height one primes of* A.

(2) A *is factorial if and only if* $\mathrm{Cl}(A) = 0$ (*see* **[S1**, *pp*. 6–7, 18**]**).

Notation. Let $A \subset B$ be rings. Let $p \subset A$ and $q \subset B$ be prime ideals. We write $q \mid p$ if $q \cap A = p$ and we say that q lies over p.

THEOREM 1.3. *Let $A \subset B$ be Krull rings. Suppose that either B is integral over A or that B is a flat A algebra. Then there is a well-defined group homomorphism* ϕ: $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ (*see* **[S1**, *pp*. 19–20**]**).

Let us describe the homomorphism of Theorem 1.3. If q and p are height one primes of B and A with $q \mid p$, we let $e(q:p)$ denote the ramification index of q over p. Then for each height one prime p of A we define $\phi(p) = \Sigma_{q|p} e(q:p)p$, the sum taken over all height one primes in B lying over p. This sum is always finite since B is a Krull ring. We then extend ϕ by linearity. The hypotheses in Theorem 1.3 are needed to guarantee that this map induces a well-defined map on divisor classes.

THEOREM 1.4. *Let A be a Krull ring and S a multiplicatively closed subset in A. Then $S^{-1}A$ is an A-flat Krull ring and*:

(1) ϕ: $\mathrm{Cl}(A) \to \mathrm{Cl}(S^{-1}A)$ *is surjective*; *and*

(2) *if S is generated by prime elements then ϕ is bijective* (*see* [**S1**, *p*. 21]).

REMARK 1.5. In Theorem 1.4, $\ker \phi = H + F(A)/F(A)$, where $H \subset D(A)$ is the subgroup generated by those height one primes p of A such that $p \cap S \neq \varnothing$.

THEOREM 1.6. *Let R be a Krull ring. Then $R[x]$ is a Krull ring and* ϕ: $\mathrm{Cl}(R) \to \mathrm{Cl}(R[x])$ *is bijective* [**S1**, *p*. 22].

Let A be a Noetherian ring and m an ideal contained in the Jacobson radical of A. If we give A the m-adic topology, then (A, m) is called a *Zariski ring*. The completion $\hat{A}$ of A will also be a Zariski ring and is A-flat with $A \subset \hat{A}$.

THEOREM 1.7. *Let (A, m) be a Zariski ring. Then if $\hat{A}$ is a Krull ring, so is A. Also* ϕ: $\mathrm{Cl}(A) \to \mathrm{Cl}(\hat{A})$ *is injective* [**S1**, *p*. 23].

Throughout this article we will concentrate for the most part on the case where $\mathrm{qt}(B)/\mathrm{qt}(A)$ is a purely inseparable extension. The following results are also from Samuel's notes.

We let B be a Krull ring of characteristic $p > 0$. Let Δ be a derivation of $\mathrm{qt}(B)$ such that $\Delta(B) \subset B$. Let $K = \ker(\Delta)$ and $A = B \cap K$. Then A is a Krull ring with B integral over A. Thus we have a map ϕ: $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$.

Set $\mathfrak{L} = \{t^{-1}\Delta t \mid t \in \mathrm{qt}(B) \text{ and } t^{-1}\Delta t \in B\}$. Note that $\mathfrak{L}$ is an additive subgroup of B, called the *group of logarithmic derivatives of* Δ. Set $\mathfrak{L}' = \{u^{-1}\Delta u \mid u \text{ is a unit in } B\}$. Then $\mathfrak{L}'$ is a subgroup of $\mathfrak{L}$.

THEOREM 1.8. (a) *There exists a canonical monomorphism* $\bar{\phi}$: $\ker \phi \to \mathfrak{L}/\mathfrak{L}'$. (b) *If* $[\mathrm{qt}(B): K] = p$ *and* $\Delta(B)$ *is not contained in any height one prime of* B, *then* $\bar{\phi}$ *is an isomorphism* [**S1**, *p*. 62].

THEOREM 1.9. (a) *If* $[\mathrm{qt}(B): K] = p$, *then there exists* $a \in A$ *such that* $\Delta^p = a\Delta$; (b) *an element* $t \in B$ *is in* $\mathfrak{L}$ *if and only if* $\Delta^{p-1}(t) - at + t^p = 0$ [**S1**, *pp*. 63–64].

REMARK 1.10. We take a moment to describe the monomorphism $\bar{\phi}$: $\ker \phi \to \mathfrak{L}/\mathfrak{L}'$. Let $\beta \in \ker \phi \subset \mathrm{Cl}(A)$. Then $\phi(\beta) = tB$ for some $t \in \mathrm{qt}(B)$. The map $\bar{\phi}$ sends β to $t^{-1}\Delta t$.

To see that $t^{-1}\Delta t$ is in B, we write β as a linear combination of height one primes of A, $\beta = n_1 q_1 + \cdots + n_r q_r$, where the q_i are height one primes of A and the n_i are integers. For each i, there is a unique height one prime Q_i in B lying over q_i. By definition $\phi(\beta) = n_1 e_1 Q_1 + \cdots + n_r e_r Q_r$, where e_i denotes the ramification index of

Q_i. $\phi(\beta) = tB$ implies that B: $Q_1^{n_1 e_1} \cdots Q_r^{n_r e_r} = B$: tB. Thus for each height one prime Q of B the ramification index of Q divides $v_Q(t)$ where v_Q is the valuation corresponding to Q. It follows that there exists an $a \in K$ such that $v_Q(t) = v_Q(a)$, i.e. $t = au$ for u a unit in B_Q. Thus $t^{-1}\Delta t = a^{-1}\Delta a + u^{-1}\Delta u = u^{-1}\Delta U$. Since $\Delta(B_Q) \subset B_Q$, we conclude that $t^{-1}\Delta t \in B_Q$ for each height one prime Q of B. Since B is a Krull ring we have that $t^{-1}\Delta t \in B$ (see [**FO**, p. 8]).

These facts are to be found in Fossum's book [**FO**].

THEOREM 1.11. *Let $A = A_0 + A_1 + \cdots$ be a graded Noetherian Krull domain such that A_0 is a field. Let $m = A_1 + \cdots$. Then* $\mathrm{Cl}(A) \to \mathrm{Cl}(A_m)$ *is a bijection* [**FO**, *p*. 42].

THEOREM 1.12. *Let $A = A_0 + A_1 + \cdots$ be a graded Krull domain such that A_0 is a field k. Let k' be an extension field of k. Suppose $A \otimes_k k' = A'$ is a Krull domain. Then A' is a faithfully flat A-module and the induced homomorphism* $\mathrm{Cl}(A) \to \mathrm{Cl}(A')$ *is an injection* [**FO**, *p*. 43].

The next theorem generalizes 1.8.

THEOREM 1.13. *Let $\mathcal{G}$ be a finite group of derivations of a Krull domain B of characteristic $p > 0$. Let A be the fixed subring of $\mathcal{G}$. Let $D_1, \ldots, D_r$ be a basis for $\mathcal{G}$ over $\mathbf{Z}/p\mathbf{Z}$. Then the kernel of the homomorphism* $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ *is isomorphic to a subgroup of V_0/V_0', where V_0 and V_0' are the following subgroups of $L^r, L = \mathrm{qt}(B)$. $V_0 = \{(t^{-1}D_1 t, \ldots, t^{-1}D_r t)$: $t \in \mathrm{qt}(B)$ and $t^{-1}D_i t \in B$ for all $i = 1, \ldots, r\}$ and $V_0' = \{(u^{-1}D_1 u, \ldots, u^{-1}D_r u)$: $u \in B^*\}$ with B^* the units of B (see* [**FO**, *p*. 92]).

REMARK 1.14. The injection in 1.13 is analogous to that of 1.8. If I is a divisorial ideal of A whose class is in the kernel of ϕ: $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$, then B: $(B : IB)$ is a principal ideal, say xB, for some $x \in B$. We then map I to $(x^{-1}D_1 x, \ldots, x^{-1}D_r x)$ in V_0/V_0'.

2. Properties of $\mathrm{Cl}(F$: $z^p = G)$. Throughout this article, unless stated otherwise, k is an algebraically closed field of characteristic $p > 0$. Let $G(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n] \setminus k[x_1^p, \ldots, x_n^p]$ be a polynomial in n variables and $F \subset \mathbf{A}_k^{n+1}$ be the hypersurface defined by the equation $z^p = G(x_1, \ldots, x_n)$.

Since $G \notin k[x_1^p, \ldots, x_n^p]$ if and only if $\partial G/\partial x_i \neq 0$ for some $i = 1, \ldots, n$, we will assume that $\partial G/\partial x_1 \neq 0$.

We will also restrict our attention to hypersurfaces that are normal, or equivalently, to hypersurfaces F: $z^p = G$ for which the greatest common divisor of the n-tuple of polynomials $(\partial G/\partial x_1, \ldots, \partial G/\partial x_n)$ in $k[x_1, \ldots, x_n]$ is 1 (see [**MA**, p. 125]).

Thus we will hereafter assume that G satisfies the condition

$$(*) \qquad \frac{\partial G}{\partial x_1} \neq 0 \quad \text{and} \quad \gcd\left(\frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n}\right) = 1.$$

LEMMA 2.1. *The coordinate ring of F is isomorphic to $A = k[x_1^p, \ldots, x_n^p, G]$.*

PROOF. The coordinate ring of F is $R = k[x_1,\dots,x_n, z]/I$, where I is the ideal in $k[x_1,\dots,x_n, z]$ generated by $z^p - G$. Let Φ: $k[x_1,\dots,x_n, z] \to A$ be the mapping that sends each $\alpha \in k$ to α^p, each x_i to x_i^p, and z to G. This is a surjective homomorphism since k is perfect. Thus the kernel of Φ is a height one prime containing I. Since I is height one, $I = \ker \Phi$. Therefore R is isomorphic to A (note that this isomorphism is not a k-isomorphism).

2.2. For each $i = 1,\dots,n-1$, let D_i: $k(x_1,\dots,x_n) \to k(x_1,\dots,x_n)$ be the k-derivation defined by

$$D_i = \frac{\partial G}{\partial x_{i+1}}\frac{\partial}{\partial x_1} - \frac{\partial G}{\partial x_1}\frac{\partial}{\partial x_{i+1}}.$$

LEMMA 2.3. $\bigcap_{i=1}^{n-1} D_i^{-1}(0) \cap k[x_1,\dots,x_n] = A$.

PROOF. We have that

$$\begin{aligned} k(x_1,\dots,x_n) &\supsetneqq D_1^{-1}(0) \supsetneqq D_2^{-1}(0) \cap D_1^{-1}(0) \\ &\supsetneqq \cdots \supsetneqq D_1^{-1}(0) \cap \cdots \cap D_{n-1}^{-1}(0) \supset \mathrm{qt}(A), \end{aligned}$$

because for each $j = 1,\dots,n-1$, $x_{j+1} \in (D_1^{-1}(0) \cap \cdots \cap D_{j-1}^{-1}(0))$ and $D_j(x_{j+1}) \neq 0$. Since $[k(x_1,\dots,x_n): \mathrm{qt}(A)] = p^{n-1}$, it follows that $\mathrm{qt}(A) = \bigcap_{i=1}^{n-1} D_i^{-1}(0)$. Since A is integrally closed, the result follows.

LEMMA 2.4. *Let* $V = \{(t^{-1}D_1 t,\dots,t^{-1}D_{n-1}t)$: $t \in k(x_1,\dots,x_n)$ *and* $t^{-1}D_i t \in k[x_1,\dots,x_n]\}$. *Then* $\mathrm{Cl}(F)$ *injects into* V.

PROOF. By 1.13 $\mathrm{Cl}(F)$ injects into V/V', where $V' = \{(u^{-1}D_1 u,\dots,u^{-1}D_{n-1}u)$: u is a unit in $k[x_1,\dots,x_n]\}$. Since the units of $k[x_1,\dots,x_n]$ are exactly the elements of k, $V' = 0$.

We can strengthen 2.4 when $n = 2$.

LEMMA 2.5. *If* $n = 2$, *then the injection of* 2.4 *is also surjective*.

PROOF. By 2.1 and 2.3 the coordinate ring of F is isomorphic to $A = k[x_1^p, x_2^p, G]$ and $A = D_1^{-1}(0) \cap k[x_1, x_2]$. Note that $D_1(x_1) = \partial G/\partial x_2$ and $D_1(x_2) = -\partial G/\partial x_1$. Thus the image of D_1 restricted to $k[x_1, x_2]$ is not contained in any height one prime of $k[x_1, x_2]$. By 1.8(b), $\mathrm{Cl}(F) \simeq V$.

LEMMA 2.6. *Let* $t \in k[x_1,\dots,x_n]$ *be a logarithmic derivative of* D_i. *Then* $\deg t \leqslant \deg G - 2$.

PROOF. We have that $t = f^{-1}D_i f$ for some $f \in k(x_1,\dots,x_n)$. There exists $h, g \in k[x_1,\dots,x_n]$ such that $f = g^{-p}h$. Thus $h^{-1}D_i h = t$. We have that $D_i h = h_{x_1}G_{x_{i+1}} - h_{x_{x+1}}G_{x_1}$ is of degree at most $\deg h + \deg G - 2$. This shows that $\deg t \leqslant \deg G - 2$.

PROPOSITION 2.7. $\mathrm{Cl}(F)$ *is a p-group of type* $(p,\dots,p)$ *of order* p^f, *where* $f \leqslant (n-1)g(g-1)/2$, *where* $g = \deg G$.

PROOF. Let $(t_1,\dots,t_{n-1}) \in V$. By (2.6), $\deg t_i \leqslant g - 2$ for each i. We will show that there are at most $p^{g(g-1)/2}$ such t_i for each $i = 1,\dots,n-1$. We begin with t_1.

$$D_1 = \frac{\partial G}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial G}{\partial x_1}\frac{\partial}{\partial x_2}.$$

We have that $k(x_1,\dots,x_n)$ is a purely inseparable extension of $D_1^{-1}(0)$ of degree p. By 1.9 there exists an $a \in k[x_1,\dots,x_n] \cap D_1^{-1}(0)$ such that $D_1^p = aD_1$ and t_1 is a logarithmic derivative of D_1 if and only if

$$D_1^{p-1}(t_1) - at_1 = -t_1^p. \tag{2.7.1}$$

We write $t_1 = \sum_{r+s\leqslant g-2}\alpha_{rs}x_1^r x_2^s$ where $\alpha_{rs} \in k[x_3,\dots,x_n]$. Substituting this expression for t_1 into (2.7.1), we obtain on the left side of this equation a polynomial in x_1 and x_2 whose coefficients are linear expressions in the α_{rs} with coefficients in $k[x_3,\dots,x_n]$. Comparing the coefficients of $x_1^{ep}x_2^{mp}$ on both sides of (2.7.1) we see that for each pair of nonnegative integers (e, m) with $e+m \leqslant g-2$, α_{em} must satisfy an equation of the form

(2.7.2) $L_{em} = \alpha_{em}^p$, where L_{em} is a linear expression in the α_{rs} with coefficients in $k[x_3,\dots,x_n]$.

There are a total of $g(g-1)/2$ such equations.

Let L be an algebraic closure of $k(x_3,\dots,x_n)$. The ring $R = L[\dots,\alpha_{rs},\dots]$ with the relations $L_{rs} = \alpha_{rs}^p$ is a finite-dimensional L-vector space spanned by all monomials in the α_{rs} of degree $\leqslant (p-1)g(g-1)/2$. Thus R' is Artinian and has a finite number of maximal ideals (see [**A–M**, p. 89]).

Therefore, the $g(g-1)/2$ equations in (2.7.2) have a finite number of solutions in L, which by Bezout's theorem [**SH**, p. 198] is at most $p^{g(g-1)/2}$. Hence, the equations in (2.7.2) have at most $p^{g(g-1)/2}$ solutions in $k[x_3,\dots,x_n]$. This implies that there are at most $p^{g(g-1)/2}$ possible t_1's.

Similarly, there are at most $p^{g(g-1)/2}$ possibilities for each t_i, $i = 2,\dots,n-1$, from which it follows that V has order p^f where $f \leqslant (n-1)g(g-1)/2$. Since $V \subset k[x_1,\dots,x_n]$, each element of V has p-torsion. By 2.4, $\mathrm{Cl}(F) \subset V$. The result follows.

The next result, which I proved in [**L2**], is entitled "Ganong's formula". Several useful conversations with R. Ganong [**G1, G2**] led to its discovery. Although Ganong's formula plays a minor role in this article, it is used extensively in [**L1** and **L2**]. For the proof of 2.8 see [**L2**].

THEOREM 2.8 (GANONG'S FORMULA). *Let k be a field of characteristic $p > 0$, $G \in k[x_1, x_2]$ satisfy condition* (*), *D: $k(x_1, x_2) \to k(x_1, x_2)$ be the k-derivation $D = (\partial G/\partial x_2)(\partial/\partial x_1) - (\partial G/\partial x_2)(\partial/\partial x_1)$, and $a \in k[x_1^p, x_2^p, G]$ be such that $D^p = aD$* (*see* 1.9). *Then for all $\alpha \in k(x_1, x_2)$,*

$$D^{p-1}\alpha - a\alpha = -\sum_{i=0}^{p-1} G^i \nabla(G^{p-(i+1)}\alpha),$$

where $\nabla = \partial^{2(p-1)}/(\partial x_1^{p-1}\partial x_2^{p-1})$.

Proposition 2.9 uses Ganong's formula to refine the upper bound of 2.7 in the case $\gcd(\partial G/\partial x_1, \partial G/\partial x_i) = 1$ for each $i = 2,\dots,n$.

PROPOSITION 2.9. *For each* $i = 2,\dots,n$, *let* m_i *be a nonnegative integer such that* $2pm_i \leqslant \deg_{x_1,x_i}(G) \leqslant 2p(m_i+1)$. *Assume that for each* i, $\gcd(\partial G/\partial x_1, \partial G/\partial x_i) = 1$. *Then the order of* $\mathrm{Cl}(F)$ *is* p^f, *where*

$$f \leqslant g(g-1)(n-1)/2 - (p-1)\sum_{i=1}^{n-1} m_i(2m_i - 1).$$

PROOF. For each $i = 1,\dots,n-1$, let D_i be as in 2.2. Let V be as in 2.4. If t is a logarithmic derivative of D_1, then by 2.6, $\deg_{x_1,x_2}(t) \leqslant g-2$. Thus $t = \sum_{r+s\leqslant g-2} c_{rs} x_1^r x_2^s$ for some $c_{rs} \in k[x_3,\dots,x_n]$.

From 1.9 and Ganong's formula we have that

$$\nabla(G^q t) = \begin{cases} 0, & \text{if } 0 \leqslant q \leqslant p-2, \\ t^p, & \text{if } q = p-1, \end{cases} \tag{2.9.1}$$

where $\nabla = \partial^{2(p-1)}/(\partial x_1^{p-1}\partial x_2^{p-1})$.

Since $\nabla(G^{p-1}t) = t^p$ we obtain for each of the c_{rs} an equation of the form

(2.9.2) $\quad l_{rs} = c_{rs}^p$, where l_{rs} is a linear expression in the c_{je} with coefficients in $k[x_3,\dots,x_n]$.

If we regroup terms we can write

$$t = \sum_{0\leqslant u,v\leqslant p-1} \alpha_{uv} x_1^u x_2^v \quad \text{with } \alpha_{uv} = \sum c_{(u+cp)(v+dp)} x_1^{cp} x_2^{dp},$$

where this sum is taken over all pairs (c, d) of nonnegative integers such that $0 \leqslant u + cp + v + dp \leqslant g-2$.

Since $\nabla(G^q t) = 0$ for $q = 0,\dots,p-2$, we obtain the equation

$$L_q\colon \sum_{0\leqslant u,v\leqslant p-1} \alpha_{uv}\nabla(G^q x_1^u x_2^v) = 0 \quad \text{for } 0 \leqslant q \leqslant p-2. \tag{2.9.3}$$

These $(p-1)$ equations (2.9.3) in the α_{uv} with coefficients $\nabla(G^q x_1^u x_2^v)$ are independent over $E = k(x_1^p, x_2^p, x_3,\dots,x_n)$. For suppose that $\beta_q \in E$, $q = 0,\dots,p-2$, such that $\beta_0 L_0 + \cdots + \beta_{p-2}L_{p-2} = 0$.

$$\nabla\left(\left(\beta_0 + \cdots + \beta_{p-2}G^{p-2}\right)x_1^u x_2^v\right) = 0 \quad \text{for all } 0 \leqslant u, v \leqslant p-1, \tag{2.9.4}$$

which implies that $\beta_0 + \beta_1 G + \cdots + \beta_{p-2}G^{p-2} = 0$ and, hence, $\beta_0 = \beta_1 = \cdots = \beta_{p-2} = 0$.

We conclude that $(p-1)$ of the α_{uv}'s are E-linearly dependent on the remaining ones. Note that each α_{uv} involves at least $m_1(2m_1-1)$ of the c_{rs}'s. Thus we have that among the c_{rs}'s there are $(p-1)m_1(2m_1-1)$ of them that are determined by the choice of the remaining $g(g-1)/2 - (p-1)m_1(2m_1-1)$ ones. Each of these remaining ones must satisfy an equation of the form (2.9.2). By the argument used in the proof of 2.7 there are p^{s_1} possibilities for the $g(g-1)/2$-tuple $(c_{00}, c_{10}, c_{01},\dots,c_{0(g-2)})$, where $s_1 \leqslant g(g-1)/2 - (p-1)m_1(2m_1-1)$. Thus there are p^{s_1} possibilities for t.

Similarly, D_i has p^{s_i} logarithmic derivatives for $i = 2,\dots,n-1$, where

$$s_i \leqslant g(g-1)/2 - (p-1)m_i(2m_i-1).$$

It follows that V and, hence, $\mathrm{Cl}(F)$ has order p^s where

$$s \leq (n-1)g(g-1)/2 - (p-1)\sum_{i=1}^{n-1} m_i(2m_i - 1). \quad \text{Q.E.D.}$$

Note that if $\partial G/\partial x_j = 0$ for some $j = 1,\ldots,n-1$ we can replace D_j with $\partial/\partial x_j$ and still have that $\bigcap_{i\neq j} D_i^{-1}(0) \cap (\partial/\partial x_j)^{-1}(0) = A$.

Since $\partial/\partial x_j$ has only 0 as a logarithmic derivative, we have the following results.

LEMMA 2.10. *If G is such that $\partial G/\partial x_j = 0$ for $j = r,\ldots,n$ for some $r > 2$, then $\mathrm{Cl}(F)$ injects into $V' = \{(t^{-1}D_1(t),\ldots,t^{-1}D_{r-1}(t)\colon t \in k(x_1,\ldots,x_n)$ and $t^{-1}D_i(t) \in k[x_1,\ldots,x_n]\}$.*

PROPOSITION 2.11. *With G as in 2.10 the order of $\mathrm{Cl}(F) = p^s$ where $s \leq (r-1) \cdot g(g-1)/2$.*

PROOF. Use 2.10 and the same argument used in the proof of 2.7.

We end this section with some examples.

PROPOSITION 2.12. *Let $h_1,\ldots,h_r$ be distinct homogeneous irreducible polynomials in $k[x_1,\ldots,x_n]$, the sum of whose degrees is g with g not divisible by p. Let $G = h_1 \cdots h_r$ and let F be the hypersurface defined by the equation $z^p = G$. Then F is normal and $\mathrm{Cl}(F)$ has order p^{r-1} generated by the height one primes $GA + h_i^pA$ in $A = k[x_1^p,\ldots,x_n^p, G]$.*

PROOF. By Euler's formula, $\sum_{j=1}^n x_j(\partial G/\partial x_j) = gG$. If h is a factor of $\partial G/\partial x_j$ for $j = 1,\ldots,n$, then h divides G and must be a multiple factor of G. Therefore $\gcd(\partial G/\partial x_1,\ldots,\partial G/\partial x_n) = 1$ and F is normal.

For each pair of positive integers (j,l) with $j \neq l$ and $j, l \leq n$, let $D_{jl}\colon k(x_1,\ldots,x_n) \to k(x_1,\ldots,x_n)$ be the k-derivation defined by

$$D_{jl} = \frac{\partial G}{\partial x_l}\frac{\partial}{\partial x_j} - \frac{\partial G}{\partial x_j}\frac{\partial}{\partial x_l}.$$

Clearly $A \subset (\bigcap D_{jl}^{-1}(0)) \cap k[x_1,\ldots,x_n]$. The reverse inclusion holds by 2.3. Thus $A = (\bigcap D_{jl}^{-1}(0)) \cap k[x_1,\ldots,x_n]$.

Let $\mathcal{H}$ be the $\mathbf{Z}/p\mathbf{Z}$-vector space spanned by the D_{jl}. Let $\overline{D}_1,\ldots,\overline{D}_m$ be a basis for $\mathcal{H}$, and $W = \{(f^{-1}\overline{D}_1 f,\ldots,f^{-1}\overline{D}_m f)\colon f \in k(x_1,\ldots,x_n)$ and $f^{-1}\overline{D}_i(f) \in k[x_1,\ldots,x_n]$ for $i = 1,\ldots,m\}$. By 1.13, $\mathrm{Cl}(F)$ injects into W.

We begin by showing that

(2.12.1) if $(V_1,\ldots,V_m) \in W$, then there exists a homogeneous polynomial $t \in k[x_1,\ldots,x_n]$ such that $t^{-1}\overline{D}_i(t) = v_i$ for $i = 1,\ldots,m$.

Temporarily fix an $i = 1,\ldots,m$. Suppose that $v = u^{-1}\overline{D}_i(u) \in k[x_1,\ldots,x_n]$ where $u \in k(x_1,\ldots,x_n)$. Multiplying u by an element in $k[x_1^p,\ldots,x_n^p]$ we can assume that $u \in k[x_1,\ldots,x_n]$.

Let v_1 and u_1 (v_2 and u_2) be the lowest (highest) degree forms of v and u, respectively.

We have that $\deg(u_1) + g - 2 \leqslant \deg(\overline{D}_i(u)) \leqslant \deg(u_2) + g - 2$. If we compare the forms of lowest and highest degree of both sides of the equality $\overline{D}_i(u) = uv$, we see that $\deg(v_2) + \deg(u_2) \leqslant \deg(u_2) + g - 2$ and $\deg(v_1) + \deg(u_1) \geqslant \deg(u_1) + g - 2$. This implies that $g - 2 \leqslant \deg(v_1) \leqslant \deg(v_2) \leqslant g - 2$ (i.e. v is homogeneous of degree $g - 2$). Thus it must be that $\deg(\overline{D}_i(u)) = \deg(u_2) + g - 2$ and hence $\overline{D}_i(u_2) = u_2 v$.

It follows that if $(v_1,\dots,v_m) \in W$ with $u^{-1}\overline{D}_i(u) = v_i$ for each i, then we can assume that u is a polynomial and the highest degree form of u, say $\tilde{u}$, is such that $\tilde{u}^{-1}\overline{D}_i(\tilde{u}) = v_i$ for each i. This verifies (2.12.1).

Furthermore, if $\tilde{u} = u_1^{e_1}\cdots u_s^{e_s}$ is a prime factorization of $\tilde{u}$, then $e_1u_1^{-1}\overline{D}_i(u_1) + \cdots + e_su_s^{-1}\overline{D}_i(u_s) = v_i$ and each of the elements $e_1u_1^{-1}\overline{D}_i(u),\dots,e_su_s^{-1}\overline{D}_i(u_s) \in k[x_1,\dots,x_n]$, for each i. Thus

(2.12.2) W is generated by all elements of the form $\left(\tilde{u}^{-1}\overline{D}_1(\tilde{u}),\dots,\tilde{u}^{-1}\overline{D}_m(\tilde{u})\right)$ with $\tilde{u}$ an irreducible homogeneous polynomial.

Let $y \in k[x_1,\dots,x_n]$ be irreducible and homogeneous such that y divides $\overline{D}_i(y)$ for each i. Since the $\overline{D}_i$ generate $\mathcal{H}$, y divides $D_{jl}(y)$ for each pair $(j, l), j \neq l \leqslant n$.

Therefore, for each $j = 1,\dots,n$, y divides

$$\begin{aligned}\sum_{l\neq j} x_l D_{jl}(y) &= \sum_{l\neq j} x_l\left(\frac{\partial G}{\partial x_l}\frac{\partial y}{\partial x_j} - \frac{\partial G}{\partial x_j}\frac{\partial y}{\partial x_l}\right)\\ &= \sum_{l=1}^{n} x_l\left(\frac{\partial G}{\partial x_l}\frac{\partial y}{\partial x_j} - \frac{\partial G}{\partial x_j}\frac{\partial y}{\partial x_l}\right)\\ &= gG\frac{\partial y}{\partial x_j} - \deg(y)\frac{\partial G}{\partial x_j}y\end{aligned}$$

by Euler's formula. This implies that y divides $G(\partial y/\partial x_j)$ for each $j = 1,\dots,n$. Since y is irreducible there exists j_0 such that $\partial y/\partial x_j \neq 0$. Then y does not divide $\partial y/\partial x_{j_0}$ and hence y divides G. This fact, together with (2.12.2), implies that

(2.12.3) W is generated by the elements
$$w_q = \left(h_q^{-1}\overline{D}_1(h_q),\dots,h_q^{-1}\overline{D}_m(h_q)\right), \qquad q = 1,\dots,r.$$

Note that $w_1 + \cdots + w_r = (G^{-1}\overline{D}_1(G),\dots,G^{-1}\overline{D}_m(G)) = (0,\dots,0)$. Thus $\{w_1,\dots,w_{r-1}\}$ generate W over $\mathbf{Z}/p\mathbf{Z}$.

If $d_1,\dots,d_{r-1}$ are positive integers such that $d_1w_1 + \cdots + d_{r-1}w_{r-1} = 0$, then $\overline{D}_i(h_1^{d_1}\cdots h_r^{d_r}) = 0$ for each i and thus $h = h_1^{d_1}\cdots h_{r-1}^{d_{r-1}} \in A$. We then have that

$$k(x_1^p,\dots,x_n^p) \subset k(x_1^p,\dots,x_n^p, h) \subset \mathrm{qt}(A).$$

If $k(x_1^p,\dots,x_n^p, h) = \mathrm{qt}(A)$, then there exists $\alpha_0,\dots,\alpha_p \in k[x_1^p,\dots,x_n^p]$ such that $\alpha_0 G = \alpha_1 + \alpha_2 h + \cdots + \alpha_p h^{p-1}$. Since G and h are homogeneous, we may assume that $\alpha_0,\dots,\alpha_p$ are also. Since $\deg(\alpha_i h^{i-1})$ is congruent to $(i-1)\deg h$ modulo p and $\deg(\alpha_0 G) = g \pmod p$, we have that only one of $\alpha_1,\dots,\alpha_p \neq 0$. Thus $\alpha_0 G = \alpha_i h^{i-1}$ for some $i = 1,\dots,p$. But this is clearly impossible.

Thus $k(x_1^p,\dots,x_n^p, h) = k(x_1^p,\dots,x_n^p)$ and $d_1 = \cdots = d_{r-1} = 0 \pmod p$. It follows that $\{w_1,\dots,w_{r-1}\}$ forms a basis for W over $\mathbf{Z}/p\mathbf{Z}$ and the order of W is p^{r-1}.

Finally we note that the nonprincipal height one primes $Q_i = GA + h_i^pA$ in A map to the elements w_i for $i = 1,\dots,r$, under the homomorphism described in 1.14. Therefore $\mathrm{Cl}(F) \simeq W$ and has order p^{r-1} generated by $Q_1,\dots,Q_{r-1}$.

REMARK 2.13. Proposition 2.12 is not valid when p divides g. For we will see in a moment that the hypersurface $z^p = x_1^p \cdots x_p^p + x_0^p$ has nontrivial class group although the polynomial $x_1^p \cdots x_p^p + x_0^p$ is irreducible in $k[x_0, x_1,\dots,x_p]$.

We can use 2.12 to attack some special cases when p divides g.

COROLLARY 2.14. *Let $h_1, h_2,\dots,h_r$ be distinct homogeneous irreducible polynomials in $k[x_1,\dots,x_n]$. Let $G = x_0h_1\cdots h_r$. Then F: $z^p = G \subset \mathbf{A}^{n+2}$ is normal and the order of* $\mathrm{Cl}(F)$ *is p^r.*

PROOF. Let $g = \deg G$. If p does not divide g then the result follows by 2.12.

Suppose then that $g = pm$. Then $\gcd(\partial G/\partial x_1,\dots,\partial G/\partial x_n) = x_0$. Thus $\gcd(\partial G/\partial x_0,\dots,\partial G/\partial x_n) = 1$ and F is normal.

Let $R = k[x_0,\dots,x_n, z]$, $z^p = G$. R is the coordinate ring of F.

We have that $R[1/x_0] \simeq R_1$ where $R_1 = k[x_0',\dots,x_n', z', 1/x_0']$. $(z')^p = h_1'\cdots h_p'$, where $h_j' = h_j(x_1',\dots,x_n')$ for $j = 1,\dots,r$.

The map $R_1 \to R$ is given by $x_0' \to x_0$, $z' \to z/x_0^m$, $x_i' \to x_i/x_0$ for $i = 1,\dots,n$.

By 1.4, $\mathrm{Cl}(R_1) \simeq \mathrm{Cl}(R_2)$, where $R_2 = k[x_0',\dots,x_n', z']$, $(z')^p = h_1'\cdots h_r'$. By 1.6, $\mathrm{Cl}(R_2) \simeq \mathrm{Cl}(R_3)$ where $R_3 = k[x_1',\dots,x_n', z']$, $(z')^p = h_1'\cdots h_r'$. By 2.12, the order of $\mathrm{Cl}(R_3)$ is p^{r-1}. Hence $\mathrm{Cl}(R[1/x_0])$ has order p^{r-1}.

Again by 1.4 we have an exact sequence

$$0 \to \ker\phi \to \mathrm{Cl}(R) \xrightarrow{\phi} \mathrm{Cl}(R[1/x]) \to 0, \tag{2.14.1}$$

where $\ker\phi$ is generated by those height one primes in R that contain x_0. Such a prime ideal would have to contain z also and hence the ideal $x_0R + zR$ which is easily seen to be a nonprincipal height one prime.

Thus $\ker\phi \simeq \mathbf{Z}/p\mathbf{Z}$, from which it follows that $\mathrm{Cl}(R)$ has order p^r.

COROLLARY 2.15. *The divisor class group of the hypersurface $F \subset \mathbf{A}^{np+1}$ defined by the equation $z^p = x_1\cdots x_{np}$ is a direct sum of $np - 1$ copies of $\mathbf{Z}/p\mathbf{Z}$.*

REMARK 2.16. The hypersurface in 2.13 is isomorphic to the hypersurface $z^p = x_1 \cdots x_p$ which has nontrivial class group by 2.15.

3. The hypersurface $z^{p^m} = G$. In this section we study the divisor class group of hypersurfaces of the form $z^{p^m} = G(x_1,\dots,x_n)$. Studies of this type, using higher order derivations, have been conducted by K. Baba **[BA]**. We describe another sort of inductive procedure of obtaining information about $\mathrm{Cl}(F\colon z^{p^m} = G)$.

As always, we let k be an algebraically closed field of characteristic $p > 0$ and $G(x_1,\dots,x_n) \in k[x_1,\dots,x_n]$ satisfy condition $(*)$. For each positive integer m, let $F_m \subset \mathbf{A}_k^{n+1}$ be the hypersurface (necessarily normal) defined by the equation $z^{p^m} = G(x_1,\dots,x_n)$.

LEMMA 3.1. *For each m, the coordinate ring of F_m is isomorphic to* $A_m = k[x_1^{p^m},\ldots,x_n^{p^m}, G]$.

PROOF. Similar to the proof of 2.1.

For each positive integer m, let $B_m = k[x_1^{p^{m+1}},\ldots,x_n^{p^{m+1}}, G^p]$. B_m is clearly isomorphic to A_m and $B_m \subset A_{m+1} \subset A_m$ with A_m integral over B_m. Also $[\mathrm{qt}(A_m) : \mathrm{qt}(A_{m+1})] = p^{m-1}$ and $[\mathrm{qt}(A_{m+1}) : \mathrm{qt}(B_m)] = p$.

By 1.3 there exist group homomorphisms θ_m: $\mathrm{Cl}(B_m) \to \mathrm{Cl}(A_{m+1})$ and ϕ_m: $\mathrm{Cl}(A_{m+1}) \to \mathrm{Cl}(A_m)$. We use derivations to study θ_n and θ_m. We start with θ_m.

Let E_m be the restriction of the derivation $G_{x_1}^{-1}(\partial/\partial x_1)$ on $k(x_1,\ldots,x_n)$ to A_{m+1}.

LEMMA 3.2. *E_m maps A_{m+1} into A_{m+1} and has kernel B_m.*

PROOF. Let $\alpha \in A_{m+1}$. Then $\alpha = \sum_{i=0}^{p-1}\beta_i G^i$ for unique $\beta_i \in B_m$. We have that $E_m(\alpha) = \sum_{i=1}^{p-1} i\beta_i G^{i-1}$.

Thus $E_m(\alpha) \in A_{m+1}$ and $E_m(\alpha) = 0$ if and only if $\beta_1 = \cdots = \beta_{p-1} = 0$, that is, if and only if $\alpha \in B_m$.

PROPOSITION 3.3. *For each positive integer m,* $\mathrm{Cl}(F_m)$ *injects into* $\mathrm{Cl}(F_{m+1})$.

PROOF. With E_m: $A_{m+1} \to A_{m+1}$ as above, let $\mathfrak{L}_{m+1} \subset A_{m+1}$ be the group of logarithmic derivatives of E_m. Let $\mathfrak{L}'_{m+1}$ be the group of logarithmic derivatives of units of A_{m+1}.

Given $\alpha \in A_{m+1}$, $\deg(E_m(\alpha)) \leqslant \deg \alpha - \deg G$. It follows that if $\alpha^{-1}E_m(\alpha) \in A_{m+1}$, then $E_m(\alpha) = 0$. Therefore $\mathfrak{L}_{m+1} = \mathfrak{L}'_{m+1} = 0$. By 1.8, $\ker(\theta_m) = 0$. □

To gain some understanding of ϕ_m: $\mathrm{Cl}(A_{m+1}) \to \mathrm{Cl}(A_m)$ we define derivations D_{mi}: $\mathrm{qt}(A_m) \to \mathrm{qt}(A_m)$ for each $i = 1,\ldots,n-1$.

Given $\alpha \in \mathrm{qt}(A_m)$, there exists unique $\alpha_j \in k[x_1,\ldots,x_n]$ such that

$$\alpha = \sum_{j=0}^{p^m-1} \alpha_j^{p^m} G^j.$$

3.4. Define

$$D_{mi}(\alpha) = \sum_{j=0}^{p^m-1} \left(D_i(\alpha_j)\right)^{p^m} G^j$$

where

$$D_i = \frac{\partial G}{\partial x_{i+1}}\frac{\partial}{\partial x_1} - \frac{\partial G}{\partial x_1}\frac{\partial}{\partial x_{i+1}} \quad \text{for } i = 1,\ldots,n-1.$$

Of course we must show that D_{mi} is indeed a derivation.

LEMMA 3.5. *The mappings D_{mi}, as defined in* 3.4, *are derivations.*

PROOF. Clearly D_{mi} is additive. We show that the multiplicative property holds, that is, $D_{mi}(uv) = uD_{mi}(v) + vD_{mi}(u)$ for all $u, v \in \mathrm{qt}(A_m)$.

Let

$$u = \sum_{j=0}^{p^m-1} \alpha_j^{p^m} G^j \quad \text{and} \quad v = \sum_{j=0}^{p^m-1} \beta_j^{p^m} G^j \in \mathrm{qt}(A_m),$$

where the α_j, $\beta_j \in k(x_1,\dots,x_n)$. We argue by induction on the number of nonzero coefficients appearing in u plus the number of nonzero coefficients appearing in v.

Suppose this sum is 2. Then $u = \alpha^{p^m}G^r$ and $v = \beta^{p^m}G^s$ for some $u, v \in k(x_1,\dots,x_n)$, r, s nonnegative integers.

Then

$$D_{mi}(uv) = D_{mi}\left((\alpha\beta)^{p^m}G^{r+s}\right) = \left(D_i(\alpha\beta)\right)^{p^m}G^{r+s} = (\alpha D_i\beta + \beta D_i\alpha)^{p^m}G^{r+s}$$
$$= \alpha^{p^m}G^r D_{mi}\left(\beta^{p^m}G^s\right) + \beta^{p^m}G^s D_{mi}\left(\alpha^{p^m}G^r\right) = uD_{mi}(v) + vD_{mi}(u).$$

Now assume that the total number of nonzero coefficients appearing in u and v is greater than 2. Let $0 < j_0 < p^m$ be the highest power of G with nonzero coefficient in v. Let this coefficient be γ^{p^m}. Then

$$D_{mi}(uv) = D_{mi}\left(u\left(v - \gamma^{p^m}G^{j_0}\right)\right) + D_{mi}\left(u\gamma^{p^m}G^{j_0}\right),$$

which, by the induction hypothesis,

$$= uD_{mi}\left(v - \gamma^{p^m}G^{j_0}\right) + \left(v - \gamma^{p^m}G^{j_0}\right)D_{mi}(u) + uD_{mi}\left(\gamma^{p^m}G^{j_0}\right) + \gamma^{p^m}G^{j_0}D_{mi}(u)$$
$$= uD_{mi}(v) + vD_{mi}(u).$$

LEMMA 3.6. *Let D_{mi}: $\mathrm{qt}(A_m) \to \mathrm{qt}(A_m)$ be as in* 3.4.

(i) *Then $A_{m+1} = \ker D_{m1} \cap \cdots \cap \ker D_{m(n-1)} \cap A_m$.*

(ii) *Let*

$$V_m = \left\{\left(t^{-1}D_{m1}t,\dots,t^{-1}D_{m(n-1)}t\right) : t \in \mathrm{qt}(A_m) \text{ and } t^{-1}D_{mi}t \in A_m\right\}.$$

Then $\ker \phi_n$ *injects into* V_m.

(iii) *Let $a_i \in D_i^{-1}(0) \cap k[x_1,\dots,x_n]$ be such that $D_{mi}^p = a_iD_i$. Then $D_{mi}^p = a_i^{p^m}D_{mi}$, $i = 1,\dots,n-1$.*

PROOF. (i) Similar to 2.3.

(ii) Similar to 2.4.

(iii) By 1.9, $\exists a_i \in D_i^{-1}(0) \cap k[x_1,\dots,x_n]$ such that $D_i^p = a_iD_i$. Then

$$D_{mi}^p\left(x_1^{p^m}\right) = \left(D_i^p(x_1)\right)^{p^m} = \left(a_iD_i(x_1)\right)^{p^m} = a_i^{p^m}D_{mi}(x_1).$$

PROPOSITION 3.7. *For each $j = 1,2,\dots,n-1$, let*

$$t_j = \alpha_{j0}^{p^m} + \alpha_{j1}^{p^m}G + \cdots + \alpha_{j(p^m-1)}^{p^m}G^{p^m} \in A_m.$$

(a) *If $(t_1,\dots,t_{n-1}) \in V_m$, then $\alpha_{j0} = 0$ if and only if $\alpha_{jr} = 0$ for $r = 0,\dots,p^m - 1$.*

(b) *If $\gcd(G_{x_1}, G_{x_j}) = 1$ in $k[x_1,\dots,x_n]$ for each $j = 1,\dots,n-1$, then $(t_1,\dots,t_{n-1}) \in V_m$ if and only if*

(1) *$\nabla_j(G^q\alpha_{jr}) = 0$ for $0 \leqslant r \leqslant p^m - 1$ and $r \neq 0 \pmod p$, and*

(2) *$\nabla_j(G^q\alpha_{j(sp)}) = \alpha_{j(s+(p-(q+1)p^{m-1}))}^p$ for $s = 0,1,\dots,p^{m-1}-1$, where $\nabla_j = \partial^{2(p-1)}/(\partial x_1^{p-1}\partial x_{j+1}^{p-1})$.*

PROOF. (b) Let $j = 1,\dots,n-1$. By 1.9(b) and 3.6(iii) we have

$$D_{mj}^{p-1}(t_j) - a_j^{p^m}t_j = -t_j^p, \quad \text{where } D_j^p = a_jD_j. \tag{3.7.1}$$

This is equivalent to

$$(3.7.2) \qquad \sum_{r=0}^{p^m-1} \left(D_j^{p-1}\alpha_{jr} - a_j\alpha_{jr}\right)^{p^m} G^r = -\sum_{s=0}^{p^m-1} \alpha_{js}^{p^{m+1}} G^{sp}.$$

Comparing coefficients of G^r in (3.7.2) we obtain

(3.7.3) $(t_1, \ldots, t_{n-1}) \in V_m$ if and only if for each $j = 1, \ldots, n-1$,

(1) $D_j^{p-1}\alpha_{jr} - a_j\alpha_{jr} = 0$ for $r \neq 0 \pmod{p}$, $0 \leqslant r \leqslant p^m - 1$,

and

$$(2) \qquad \sum_{s=0}^{p^{(m-1)}-1} \left(D_j^{p-1}\alpha_{j(sp)} - a_j\alpha_{j(sp)}\right)^{p^m} G^{sp} = -\sum_{s=0}^{p^m-1} \alpha_{js}^{p^{m+1}} G^{sp}.$$

Taking pth roots and comparing coefficients of G^s in (3.7.3)(2), (2) becomes

$$(3.7.4) \qquad \left(D_j^{p-1}\alpha_{j(sp)} - a_j\alpha_{j(sp)}\right)^{p^{m-1}} = -\sum_{i=0}^{p-1} \alpha_{j(s+ip^{(m-1)})}^{p^m} G^{ip^{(m-1)}}$$

$$\text{for } 0 \leqslant s \leqslant p^{(m-1)} - 1,$$

which is equivalent to

$$(3.7.5) \qquad D_j^{p-1}\alpha_{j(sp)} - a_j\alpha_{j(sp)} = -\sum_{i=0}^{p-1} \alpha_{j(s+ip^{(m-1)})}^{p} G^i \quad \text{for } 0 \leqslant s \leqslant p^{(m-1)} - 1.$$

If $\gcd(G_{x_1}, G_{x_j}) = 1$, then we can apply Ganong's formula (2.8) to the left side of (3.7.3)(1) and to the left side of (3.7.5), and comparing coefficients of G^i we obtain (b).

Proof of (a). To prove (a), we proceed by reverse induction on $v(r)$, where $v(r) =$ the highest power of p that divides r.

Note that if $v(r) \geqslant m$ then $\alpha_{jr} = \alpha_0$. Assume then that $v(r) = d < m$. We can write $r = s + cp^{m-1}$ for unique $s = 0, \ldots, p^{(m-1)} - 1$, $c = 0, \ldots, p-1$. Since $v(r) = d$ we have that $s = p^d e$ for some $e = 0, \ldots, p-1$.

By the induction hypothesis $\alpha_{j(sp)} = 0$. By (3.7.5), we see that

$$(3.7.6) \qquad \sum_{i=0}^{p-1} \alpha_{j(s+ip^{(m-1)})}^{p} G^i = 0,$$

which shows that $\alpha_{jr} = 0$.

Theorem 3.8. *For each m,* $\ker\phi_m$ *is a p-group of type $(p, \ldots, p)$ of order p^f where $f \leqslant (n-1)g(g-1)/2$ with $g = \deg G$.*

Proof. For each $j = 1, \ldots, n-1$ let $t_j = \alpha_{j0}^{p^m} + \cdots + \alpha_{j(p^m-1)}^{p^m} G^{p^m-1} \in A_m$. Assume that $(t_1, \ldots, t_{n-1}) \in V_m$.

PROOF. By (3.7.5)

$$D_j^{p-1}\alpha_{j0} - a_j\alpha_{j0} = -\sum_{i=0}^{p-1} \alpha^p_{j(ip^{m-1})}G^i, \quad \text{where } D_j^p = a_jD_j. \tag{3.8.1}$$

Let E_j: $k(x_1,\dots,x_n) \to k(x_1,\dots,x_n)$ be the derivation defined by $E_j = G_{x_j}^{-1}(\partial/\partial x_j)$. Then $E_j(A_1) \subset A_1$ and if $h \in A_1$, then $\deg(E_j(h)) \leqslant \deg h - g$. From (3.8.1) we obtain

(3.8.2)

$$\alpha_{j0}^p = -E_j^{p-1}G^{p-1}\left(D_j^{p-1}\alpha_{j0} - a_j\alpha_{j0}\right) \quad \text{and} \quad p\deg(\alpha_{j0}) \leqslant \deg D_j^{p-1}\alpha_{j0} - a_j\alpha_{j0}.$$

For all $h \in k[x_1,\dots,x_n]$, $\deg(D_jh) \leqslant \deg(h) + g - 2$. Thus

$$\deg(a_j) \leqslant (p-1)(g-2)$$

and

$$\deg D_j^{p-1}\alpha_{j0} - a_j\alpha_{j0} \leqslant \deg(\alpha_{j0}) + (p-1)(g-2). \tag{3.8.3}$$

(3.8.2) and (3.8.3) together imply that

$$\deg(\alpha_{j0}) \leqslant g - 2. \tag{3.8.4}$$

Claim 3.8.5. Let L be an algebraic closure of $k(x_2,\dots,x_{j-1}, x_{j+1},\dots,x_n)$. If $\alpha_1,\dots,\alpha_r \in L[x_1, x_j]$ satisfy (3.8.2) and are $\mathbf{Z}/p\mathbf{Z}$-independent, then $\alpha_1,\dots,\alpha_r$ are L-independent also.

PROOF OF CLAIM. The case $r = 1$ is obvious. We proceed by induction on r. Suppose that $e_1,\dots,e_r \in L$ are such that $e_1\alpha_1 + \cdots + e_r\alpha_r = 0$. From (3.8.2) we obtain $e_1\alpha_1^p + \cdots + e_e\alpha_e^p = 0$. Thus $(e_1^{p-1}e_2 - e_2^p)\alpha_2^p + \cdots + (e_1^{p-1}e_r - e_r^p)\alpha_r^p = 0$.

By the induction hypothesis we have that $e_1^{p-1}e_i - e_i^p = 0$ for $i = 2,\dots,r$. If $e_1 \neq 0$, then $(e_i/e_1)^p = e_i/e_1$, that is $e_i/e_1 \in \mathbf{Z}/p\mathbf{Z}$ for each i. But this contradicts the fact that the α_i are $\mathbf{Z}/p\mathbf{Z}$-independent since $\alpha_1 + (e_2/e_1)\alpha_2 + \cdots + (e_r/e_1)\alpha_r = 0$. Therefore e_1 must equal 0. Using the induction hypothesis again, we have that $e_2 = \cdots = e_r = 0$ also. $\square$

Note that the L-vector space of all polynomials in $L[x_1, x_j]$ of degree $\leqslant g - 2$ has dimension $g(g-1)/2$.

From (3.8.4) and (3.8.5) it follows that the $\mathbf{Z}/p\mathbf{Z}$-vector space of all α_{j0} satisfying (3.8.2) is of dimension at most $g(g-1)/2$.

From 3.7(a) we conclude that V_m, and hence $\ker\phi_m$, has order p^f where $f \leqslant ng(g-1)/2$. Q.E.D.

THEOREM 3.9. *For each m, $\mathrm{Cl}(F_m)$ is a finite p-group of type $(p^{i_1},\dots,p^{i_r})$ where each $i_j \leqslant m$. The order of $\mathrm{Cl}(F_m) \leqslant p^{m(n-1)g(g-k)/2}$, where $g = \deg G$.*

PROOF (BY INDUCTION ON m). For $m = 1$, use 2.7. For each $m \geqslant 1$, we have the exact sequence

$$0 \to \ker\phi_m \to \mathrm{Cl}(F_{m+1}) \xrightarrow{\phi_m} \mathrm{Cl}(F_m) \to 0. \tag{3.9.1}$$

Now just use induction and 3.8.

REMARK 3.10. Using the mappings θ_m and ϕ_m we obtain an inductive procedure for studying $\mathrm{Cl}(F_n)$. For we have the following diagram for each m:

$$\mathrm{Cl}(B_m) \overset{\theta_m}{\hookrightarrow} \mathrm{Cl}(A_{m+1}) \overset{\phi_m}{\to} \mathrm{Cl}(A_m) \tag{3.10.1}$$

$$D_{mj}\colon A \to A_m$$

$$\ker\phi_m \hookrightarrow V_m = \{t^{-1}D_{m1}t,\dots,t^{-1}D_{m(n-1)}t\colon t \in \mathrm{qt}(A_m)$$
$$\& \; t^{-1}D_{mj}t \in A_m, j = 1,2,\dots,n-1\}.$$

We finish this section with two examples.

PROPOSITION 3.11. *Let $h_1,\dots,h_r$ be distinct homogeneous irreducible polynomials in $k[x_1,\dots,x_n]$, the sum of whose degree is g not divisible by p. For each m, let $F_m \subseteq \mathbf{A}^{n+1}$ be the hypersurface defined by $z^{p^m} = G$ where $G = h_1 \cdots h_r$. Then F_m is normal and $\mathrm{Cl}(F_m)$ is a direct sum of $r-1$ copies of $\mathbf{Z}/p^m\mathbf{Z}$, generated by the height one primes $GA_m + h_i^{p^m}A_m$ in A_m, $i = 1,\dots,r-1$.*

PROOF. As in the proof of 2.12, we have that $\gcd(\partial G/\partial x_1,\dots,\partial G/\partial x_n) = 1$. Thus F_m is normal for each m.

Also, as in 2.12, we let $D_{st} = (\partial G/\partial x_t)(\partial/\partial x_s) - (\partial G/\partial x_s)(\partial/\partial x_t)$ for each pair (s,t) with $s \neq t$ and $1 \leqslant s, t \leqslant n-1$. Let $\mathcal{H}$ be the $\mathbf{Z}/p\mathbf{Z}$-vector space spanned by the D_{st}. Let $\bar{D}_1,\dots,\bar{D}_q$ be a basis for $\mathcal{H}$, and $W = \{(f^{-1}\bar{D}_1(f),\dots,f^{-1}\bar{D}_q f)\colon f \in k[x_1,\dots,x_n]$ and $f^{-1}\bar{D}_i f \in k[x_1,\dots,x_n]$ for $i = 1,\dots,q\}$.

For each $m > 0$, let $\bar{D}_{mi}\colon \mathrm{qt}(A_m) \to \mathrm{qt}(A_m)$ be defined by

$$\bar{D}_{mi}\left(\alpha_0^{p^m} + \cdots + \alpha_{p^n-1}^{p^m}G^{p^m-1}\right) = \left(\bar{D}_i\alpha_0\right)^{p^m} + \cdots + \left(\bar{D}_i\alpha_{p^n-1}\right)^{p^m}G^{p^m-1}.$$

$$W_m = \left\{\left(f^{-1}\bar{D}_{m1}f,\dots,f^{-1}\bar{D}_{mq}f\right)\colon f \in \mathrm{qt}(A_m) \text{ and } f^{-1}\bar{D}_{mi}f \in A_m \text{ for } i = 1,\dots,q\right\}.$$

Then as in 3.6(i), $\bigcap_{i=1}^q \ker\bar{D}_{mi} \cap A_m = A_{m+1}$. Since the $\bar{D}_i$ are $\mathbf{Z}/p\mathbf{Z}$ independent, so are the $\bar{D}_{mi}$. By 2.4, $\ker\phi_m$ injects into W_n. We will now demonstrate that W_m has order p^{r-1} and $\ker\phi_m$ surjects onto W_m.

Let $v \in A_m$ be such that $v = f^{-1}\bar{D}_{mi}f$ for some $f \in \mathrm{qt}(A_m)$ and some i. Clearly, we can assume that $f \in A_m$. Let f_1 and v_1 be the lowest degree forms and f_2 and v_2 the highest degree forms of f and v, respectively. Then f_1, v_1, f_2, v_2 all belong to A_m and either $\bar{D}_{mi}(f_2) = 0$ or $\deg\bar{D}_{mi}(f_2) = \deg f_2 + (g-2)p^m$. Similarly, for f_1.

Thus

$$\deg v_2 + \deg f_2 \leqslant \deg f_2 + (g-2)p^m$$

and

$$\deg v_1 + \deg f_1 \geqslant \deg f_1 + (g-2)p^m.$$

It follows that $(g-2)p^m \leqslant \deg v_1 \leqslant \deg v_2 \leqslant (g-2)p^m$. Thus v is homogeneous of degree $(g-2)p^m$. Therefore v can only be of the form $v = u^{p^m}$ for some $u \in k[x_1,\dots,x_n]$ of degree $g-2$. By 1.9(b), v is a logarithmic derivative of $\bar{D}_{mi}$ if and only if $(\bar{D}_{mi})^{p-1}(v) - \bar{a}_{mi}v = -v^p$ where $\bar{a}_{mi}$ is the element of A_m such that

$\bar{D}^p_{mi} = \bar{a}_{mi}\bar{D}_{mi}$. From 3.6(iii), $\bar{a}_{mi} = \bar{a}_i^{p^m}$ where $\bar{a}_i \in k[x_1,\dots,x_n]$ is such that $\bar{D}_i^p = \bar{a}_i\bar{D}_i$. We then have that

(3.11.1) $v \in A_m$ is a logarithmic derivative of $\bar{D}_{mi}$ if and only if $v = u^{p^m}$ where $u \in k[x_1,\dots,x_n]$ and $\bar{D}_i^{p-1}u - \bar{a}_iu = -u^p$ (i.e. u is a logarithmic derivative of $\bar{D}_i$).

Thus

(3.11.2) $(v_1,\dots,v_{n-1}) \in W_m$ if and only if there exists $u_j \in k[x_1,\dots,x_n]$ such that $(u_1,\dots,u_{n-1}) \in W$ with $v_j = u_j^{p^m}$, where W is as in 2.4.

From this fact it follows that the mapping $(u_1,\dots,u_{n-1}) \to (u_1^{p^m},\dots,u_{n-1}^{p^m})$ from W to W_m is an isomorphism.

In 2.12 we showed that W has order p^{n-1}, hence $\ker\phi_m$ has order p^{n-1}. We have that the height one primes $\mathcal{P}_i = GA_m + h_i^{p^m}A_m$ have ramification index 1 over their contractions $Q_i = GA_{m+1} + h_i^{p^{m+1}}A_{m+1}$ in A_{m+1}, and Q_i has ramification index p over their contractions $\mathcal{P}_i' = G^pB_m + h_i^{p^{m+1}}B_m$ in B_m, $i = 1,\dots,r-1$.

Using induction we have that the primes $\mathcal{P}_i$ generate $\mathrm{Cl}(A_m)$ and are each of order p^m, hence the same holds true of the primes $\mathcal{P}_i'$ in B_m.

Since θ_n: $\mathrm{Cl}(B_m) \to \mathrm{Cl}(A_{m+1})$ is injective by 3.3 we see that the elements p^mQ_i are a $\mathbf{Z}/p\mathbf{Z}$-basis for $\ker\phi_m$. Since the ramification indexes $e(\mathcal{P}_i\colon Q_i) = 1$ we have that ϕ_m is surjective. The theorem follows.

PROPOSITION 3.12. *Let $h_1,\dots,h_r$ be distinct irreducible homogeneous polynomials in $k[x_1,\dots,x_n]$. Let $G = x_0h_1\cdots h_r$ and $F_m \subset \mathbf{A}^{n+2}$ be the hypersurface defined by $z^{p^m} = G$. Then $\mathrm{Cl}(F_m)$ is a direct sum of r copies of $\mathbf{Z}/p^m\mathbf{Z}$, generated by the nonprincipal height one primes $Q_i = h_i^{p^m}A_m + GA_m$, $i = 1,\dots,r$.*

PROOF. Let $h = h_1\cdots h_r$ and $R = k[x_0, x_1,\dots,x_n, z]$, $z^{p^m} = G$, which is the coordinate ring of F_m. By 1.4 we have an exact sequence

(3.12.1) $$0 \to H \to \mathrm{Cl}(R) \to \mathrm{Cl}\, R[1/h] \to 0,$$

where H is the subgroup of $\mathrm{Cl}(R)$ generated by those nonprincipal height one primes in R that contain h.

We have that

$$R\left[\frac{1}{h}\right] \simeq k\left[\frac{z^{p^m}}{h}, x_1,\dots,x_n, z, \frac{1}{h}\right] = k\left[x_1,\dots,x_n, z, \frac{1}{h}\right].$$

By 1.4, $k[x_1,\dots,x_n, z, 1/h]$ is factorial. Therefore, $\mathrm{Cl}(R[1/h]) = 0$. From (3.12.1) we see that H is isomorphic to $\mathrm{Cl}(R)$. It follows that $\mathrm{Cl}(A_m)$ is generated by those height one primes in A_m that contain h^{p^m}. Let $Q \subset A_m$ be one such prime. Then there is a unique principal height one prime $fk[x_0,\dots,x_n]$ in $k[x_0,\dots,x_n]$ that lies over Q. f must divide h, thus f must be a k-multiple of h_i for some $i = 1,\dots,r$. Thus $Q = Q_i$ for some $i = 1,\dots,r$, and $\mathrm{Cl}(A_m)$ is generated by the Q_i.

By 3.9, $p^m Q_i = 0$ in $\mathrm{Cl}(A_m)$ for each i. We will now show that the Q_i are $\mathbf{Z}/p^m\mathbf{Z}$-independent. The $m = 1$ case is covered by 2.14. We proceed by induction on m.

We will be done if we show that the elements $p^{m-1}Q_i$ are independent over $\mathbf{Z}/p\mathbf{Z}$. Let $\mathfrak{P}_i' = Q_i \cap B_{m-1} = h_i^{p^m}B_{m-1} + G^p B_{m-1}$. The ramification index of Q_i over $\mathfrak{P}_i$ is p for $i = 1,\ldots,r$. By induction $\{p^{m-2}\mathfrak{P}_i',\ldots,p^{m-2}\mathfrak{P}_r'\}$ are $\mathbf{Z}/p\mathbf{Z}$-independent in $\mathrm{Cl}(B_m)$. Since $\theta_{m-1}(p^{m-2}\mathfrak{P}_i') = p^{m-1}Q_i$ for $i = 1,\ldots,r$ and θ_{m-1} is an injection, the elements $p^{m-1}Q_i$ are independent over $\mathbf{Z}/p\mathbf{Z}$.

REMARK 3.13. Note that if $\mathcal{O}$ is the origin of the surface F_m in 3.12 or 3.13, then by 1.11 the divisor class group of the local ring of F_m at $\mathcal{O}$, $\mathrm{Cl}((F_m)_{\mathcal{O}})$, and $\mathrm{Cl}(F_m)$ are isomorphic.

4. $\mathrm{Cl}(F\colon z^p = G(x_1, x_2))$ for a generic G.

We begin this section by focusing our attention on the case $n = p = 2$. We assume that k is an algebraically closed field of characteristic 2. Let $G(x_1, x_2) \in k[x_1, x_2]$ satisfy condition $(*)$, D be the derivation on $k(x_1, x_2)$ defined by $D = (\partial G/\partial x_2)(\partial/\partial x_1) - (\partial G/\partial x_1)(\partial/\partial x_2)$, $\mathfrak{L} \subset k[x_1, x_2]$ be the group of logarithmic derivatives of D (i.e. $\mathfrak{L} = \{f^{-1}Df \mid f \in k(x, y)$ and $f^{-1}Df \in k[x_1, x_2]\}$), and $F \subset \mathbf{A}_k^3$ be defined by the equation $z^2 = G(x_1, x_2)$.

By 2.5, $\mathrm{Cl}(F) \simeq \mathfrak{L}$.

Observe that $D(G_{x_1}) = G_{x_2}G_{x_1x_1} - G_{x_1}G_{x_1x_2} = G_{x_1}G_{x_1x_2}$. Hence $G_{x_1x_2} = G_{x_1}^{-1}D(G_{x_1}) \in \mathfrak{L}$. Therefore

(4.1) If $G_{x_1x_2} \neq 0$, then $\mathrm{Cl}(F) \neq 0$. Thus for a generic choice of G, the surface F has nontrivial divisor class group.

Note that by 3.3 we have that

(4.2) If $G_{x_1x_2} \neq 0$, then the divisor class group of the surface F_n: $z^{2^n} = G(x_1, x_2)$ is not trivial.

REMARK 4.3. By (4.1), $G_{x_1x_2} \neq 0$ implies that $\mathrm{Cl}(F\colon z^2 = G) \neq 0$. We then should be able to produce a nonprincipal height one prime in $A = k[x^2, y^2, G]$, which is isomorphic to the coordinate ring of F (2.1). This is accomplished with the aid of the next lemma.

LEMMA 4.4. *Let $f \in k[x_1, x_2]$ be such that $f^{-1}Df \in k[x_1, x_2]$. Suppose that $f = g^r h$, where $g \in k[x_1, x_2]$ is irreducible, $h \in k[x_1, x_2]$ is such that* $\gcd(h, g) = 1$, *and r is a positive integer not divisible by p (the characteristic of k). Then $g^{-1}Dg \in k[x_1, x_2]$.*

PROOF. Let $t = f^{-1}Df$. Then $ft = Df = D(g^r h) = rg^{r-1}(Dg)h + g^r Dh$. Thus g divides $rhDg$, which implies that g divides Dg.

We continue the search for the nonprincipal height one prime. Let $G_{x_1} = G_1^{r_1} \cdots G_n^{r_n}$ be a factorization of G_{x_1} into irreducible factors in $k[x_1, x_2]$. Since $G_{x_1x_2} \neq 0$, one of the r_i is not divisible by 2, say r_1. By 4.4, $G_1^{-1}DG_1 \in k[x_1, x_2]$.

Let $I = G_1 k[x_1, x_2] \cap A$. I is clearly a height one prime. To show that I is not principal, first note that $D(G_{x_1}G_{x_2}) = 0$. By 2.3, $G_{x_1}G_{x_2} \in A$. $r_1 = 2s_1 + 1$ for some nonnegative integer s_1. Then

$$G_1^{-2s_1}G_{x_1}G_{x_2} = G_1 G_2^{r_2} \cdots G_n^{r_n} G_{x_2} \in k[x_1, x_2] \cap \mathrm{qt}(A) = A.$$

We conclude that $G_1^{-2s_1}G_{x_1}G_{x_2}$ is an element of I by value 1 in the valuation on $k(x_1, x_2)$ induced by $G_1k[x_1, x_2]$. It follows that the ramification index of $G_1k[x_1, x_2]$ over I is 1. Thus $\bar{\phi}$: $\mathrm{Cl}(A) \to \mathfrak{L}$ maps I to $G_1^{-1}DG_1$. Since $\bar{\phi}$ is well defined, I must be nonprincipal.

REMARK 4.5. Since $\mathrm{Cl}(F\colon z^2 = G(x_1, x_2)) \neq 0$ for a generic G (namely, if $G_{x_1x_2} \neq 0$), we naturally arrive at two questions.

(4.5.1) What is $\mathrm{Cl}(F\colon z^2 = G(x_1, x_2))$ for a generic choice of the coefficients of G?

(4.5.2) Is it also the case that for $p > 2$ the surface $F''z^p = G(x_1, x_2)$ has nontrivial class group?

One approach towards answering these questions is to bound the degree of G and study the corresponding system of equations one obtains via the differential equation of 1.9(b). More explicitly, for a positive integer n, we let $G_n(x_1, x_2)$ be a polynomial in the variables x_1 and x_2 with undetermined coefficients of degree n. Let D be the derivation on $k(x_1, x_2)$ defined by $D = (\partial G_n/\partial x_2)(\partial/\partial x_1) - (\partial G_n/\partial x_1)(\partial/\partial x_2)$. We then try to determine if there is a generic way of choosing the coefficients of G so that the differential equation of 1.9(b) has a fixed number of solutions in $k[x_1, x_2]$.

This approach I used in **[L2]**, demonstrating that for a generic choice of G_n the divisor class group of the surface $F\colon z^p = G_n$ is 0 in the following cases: (i) $p = 3$, $n = 4$, (ii) $p = 3$, $n = 6$, and (iii) $p > 2$, $n = 3$.

Also in **[L2]**, I showed that for a generic G_n, $\mathrm{Cl}(F\colon z^2 = G_n)$ is $\mathbf{Z}/2\mathbf{Z}$ if $n = 5$ or 6, and is a direct sum of four copies of $\mathbf{Z}/2\mathbf{Z}$ if $n = 4$ (see **[L2]** for more details).

In this paper we attempt to shed some light on the local version of these questions. We ask

(4.5.3) Does there exist a group $\mathfrak{M}$ and a generic way of choosing $G(x_1, x_2)$ such that for each singular point $Q \in F\colon z^p = G$, $\mathrm{Cl}(F_Q) \simeq \mathfrak{M}$ (by F_Q we mean the local ring of F at Q)?

PROPOSITION 4.6. *The divisor class group of the ring* $R_n = k[\![x_1^{p^n}, x_2^{p^n}, x_1x_2]\!]$ *is isomorphic to* $\mathbf{Z}/p^n\mathbf{Z}$.

We give two very different and interesting proofs of 4.6. The first of these, which makes use of a proposition of N. Hallier **[HA1]**, involves logarithmic derivatives. The second, a geometric argument, uses results of J. Lipman [**LI2**, pp. 224–240] and P. Blass [**BL1**, pp. 107–121].

The following proposition, whose proof we provide, is due to N. Hallier [**HA1**, p. 2].

PROPOSITION 4.6.1. *Let A be a local Krull domain with maximal ideal m such that A and A/m are of equal characteristic $p > 0$. Let $D\colon A \to A$ be a derivation such that the ideal $I = D(A)\cdot A$ in A is contained in m. Let $a \in A$ be such that $D^p = aD$. If a is a unit in A then each $t \in m$ that is the logarithmic derivative of an element $f \in \mathrm{qt}(A)$ is the logarithmic derivative of a unit u in A.*

PROOF. Replacing f by an element of A^pf we can assume that $f \in A$. If f is a unit in A, we are done. If $f \in m$, then we have by induction that $(af)^{-1}D^nf \in m$ for all positive integers n. Let $u = -1 + (af)^{-1}D^{p-1}(f)$. Then $D(u^{-1})/u^{-1} = f^{-1}Df = t$.

PROPOSITION 4.6.2. *Let A, m, D, I and a be as in* (4.6.1). *Let U be the multiplicative group of units in A. Let $\mathcal{L} = \{f^{-1}Df \mid f \in \mathrm{qt}(A)$ and $f^{-1}Df \in A\}$ be the group of logarithmic derivatives in A and $\mathcal{L}' = \{u^{-1}Du \mid u \in U\}$. Then either $\mathcal{L} = \mathcal{L}'$ or $\mathcal{L}/\mathcal{L}' \simeq \mathbf{Z}/p\mathbf{Z}$.*

PROOF. Let θ: $\mathcal{L} \to A/m$ be the additive group homomorphism mapping each $t \in \mathcal{L}$ to its image $\bar{t}$ in A/m. By (4.6.1), $\ker\theta = \mathcal{L}'$. Thus θ induces an injection $\bar{\theta}$: $\mathcal{L}/\mathcal{L}' \to A/m$.

By 1.9, an element $t \in \mathcal{L} \Leftrightarrow D^{p-1}(t) - at = -t^p$. Thus if $t \in \mathcal{L}$ we have that $\overline{at} = \bar{t}^p$. Since $\bar{a} \neq 0$, the polynomial $x^p - \bar{a}x$ has p distinct roots in an algebraic closure of A/m. It follows that the order of $\bar{\theta}(\mathcal{L}/\mathcal{L}')$ is at most p. Since $\mathcal{L}/\mathcal{L}'$ is a p-group the result follows.

FIRST PROOF OF (4.6). Let D: $k[\![x_1, x_2]\!] \to k[\![x_1, x_2]\!]$ be the derivation defined by $D = x_1(\partial/\partial x_1) - x_2(\partial/\partial x_2)$. Then, as in 3.4, we can use D to define derivations D_n: $R_n \to R_n$ with $\ker D_n = R_{n+1}$ and $D_n^p = D_n$ for each n. If we let ϕ_n: $\mathrm{Cl}(R_{n+1}) \to \mathrm{Cl}(R_n)$ be the homomorphism of 1.3, then by 1.8, $\ker(\phi_n) \simeq \mathcal{L}_n/\mathcal{L}_n'$, where $\mathcal{L}_n = \{f^{-1}D_n(f) \mid f \in \mathrm{qt}(R_n)$ and $f^{-1}D_n f \in R_n\}$ and $\mathcal{L}_n' = \{u^{-1}D_n(u) \mid u$ is a unit in $R_n\}$.

By (4.6.2), the order of $\ker(\phi_n) \leqslant p$. Then by induction we see that the order of $\mathrm{Cl}(R_n) \leqslant p^n$. By 1.7 and 3.12 we have the desired result.

Since the second proof of (4.6) uses techniques not developed in this paper, we give only an outline. For more details see the above-mentioned articles.

SECOND PROOF OF (4.6). Let m be the maximal ideal of R_n. There exists a desingularization f: $X \to \mathrm{Spec}(R_n)$ such that the closed fibre has distinct components $E_1, E_2, \ldots, E_{p^n-1}$ with intersection as follows:

$$\begin{aligned} E_i \cdot E_j &= 1 \quad \text{if } |i-j| = 1,\\ E_i \cdot E_j &= 0 \quad \text{if } 0 \neq |i-j|,\\ E_i^2 &= -2 \end{aligned}$$

(see [**BL1**, pp. 107–121] for the case $n = 1, p > 2$).

The intersection matrix $((E_i\ E_j))$ is given by the $(p^n - 1)$ by $(p^n - 1)$ square matrix

$$\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}$$

with determinant p^n.

From Proposition 17.1 of [**LI2**] and the discussion on p. 225 of [**LI2**] we have that the order of $\mathrm{Cl}(R_n)$ is equal to $(d_1 \cdots d_{p^n-1})^{-1}\det((E_i \cdot E_j))$, where $d_i =$ degree of E_i.

Applying 1.7 and 3.12 we conclude that

$$\frac{1}{d_1 \cdots d_{p^n-1}} \det\big((E_i \cdot E_j)\big) = p^n \quad \text{and} \quad \mathrm{Cl}(R_n) \simeq \mathbf{Z}/p^n\mathbf{Z}.$$

REMARK. 4.7. Suppose that $G(x_1, x_2)$, in addition to satisfying condition $(*)$, is such that the polynomials G_{x_1}, G_{x_2}, and $G_{x_1x_1}G_{x_2x_2} - G^2_{x_1x_2}$ have no points in common (a generic assumption on G).

P. Blass [**BL1**] has shown that under this condition all singularities on F_n: $z^{p^n} = G(x_1, x_2)$ are rational with local equation of the form $z^{p^n} = xy +$ (higher degree terms).

Hereafter we will refer to this additional condition on $G(x_1, x_2)$ as condition (B).

PROPOSITION 4.8. *Let $G \in k[x_1, x_2]$ satisfy condition* (B). *Let Q be a singular point of the surface $F \subset \mathbf{A}^3_k$ defined by the equation $z^{p^n} = G$. Then* $\mathrm{Cl}(F)$ *injects into* $\mathbf{Z}/p^n\mathbf{Z}$.

PROOF. After a linear change of coordinates we may assume that Q is the origin $(x_1 = x_2 = z = 0)$ of F and G has the form $G = x_1x_2 +$ higher degree terms (see Remark 4.7).

Let $A = k[x_1^{p^n}, x_2^{p^n}, G]$ and let m be the maximal ideal of A corresponding to Q.

By 1.7 there exists an injection $\mathrm{Cl}(A_m) \to \mathrm{Cl}(\hat{A})$, where $\hat{A}$ is the completion of A_m at m.

We have that $\hat{A} = k[\![x_1^{p^n}, x_2^{p^n}, G]\!]$. In $k[\![x_1, x_2]\!]$ G factors into a product $G = uv$ where u and v are of the form $u = x_1 +$ higher degree terms, $v = x_2 +$ higher degree terms. Clearly $k[\![x, y]\!] = k[\![u, v]\!]$. Thus $\hat{A} = k[\![u^{p^n}, v^{p^n}, uv]\!]$. By 4.6, $\mathrm{Cl}(\hat{A}) \approx \mathbf{Z}/p^n\mathbf{Z}$.

Thus question (4.5.3), posed at the beginning of this section, can be answered when $p = 2$.

COROLLARY 4.9. *Let k be an algebraically closed field of characteristic* 2, $G \in k[x_1, x_2]$ *satisfy condition* (B), *and Q be a singular point of the surface F: $z^2 = G(x_1, x_2)$. Then* $\mathrm{Cl}(F_Q) \simeq \mathbf{Z}/2\mathbf{Z}$.

PROOF. Note that $G_{x_1x_2} \neq 0$ since G satisfies condition (B) and F has a singularity. Then as in Remark 4.3, there exists an irreducible polynomial $G_1 \in k[x_1, x_2]$ such that G_1 divides G_{x_1} and such that the height one prime $G_1k[x_1, x_2] \cap A = I$ in $A = k[x_1^2, x_2^2, G]$ is nonprincipal. Since Q is a singular point, I is contained in the maximal ideal of A corresponding to Q. Therefore the mapping $\mathrm{Cl}(F) \to \mathrm{Cl}(F_Q)$ of 1.4 is not the zero mapping. By 4.8, $\mathrm{Cl}(F_Q) \subset \mathbf{Z}/2\mathbf{Z}$, from which it follows that $\mathrm{Cl}(F_Q) \approx \mathbf{Z}/2\mathbf{Z}$.

REMARK (4.10). For the case $p > 2$, 4.8 tells us that if Q is a singularity of the surface F: $z^p = G(x_1, x_2)$, with G satisfying condition (B), then $\mathrm{Cl}(F_Q) = 0$ or $\mathbf{Z}/p\mathbf{Z}$. The question as to which, if either, of these groups is $\mathrm{Cl}(F_Q)$ for a generic G is an open one.

5. $\mathrm{Cl}(A)$ **for** A **between** $k[x_1^{p^m}, \ldots, x_n^{p^m}]$ **and** $k[x_1, \ldots, x_n]$. We come to the last topic to be discussed in this article. We show that if A is an integrally closed domain such that $k[x_1^{p^m}, \ldots, x_n^{p^m}] \subset A \subset k[x_1, \ldots, x_n]$, where k is a field of characteristic $p > 0$, then $\mathrm{Cl}(A)$ is a finite p-group of type $(p^{i_1}, \ldots, p^{i_r})$ with each $i_j < mn$. We will use the following fact found in [**JA**, p. 185, Exercise 3].

LEMMA 5.1. *Let P and L be fields such that P is purely inseparable of exponent* 1 *over L and $[P : L] = p^m < \infty$. Then there exists a derivation D of P/L such that $D^{-1}(0) = L$.*

LEMMA 5.2. *Let k be field of characteristic $p > 0$, B an integrally closed finitely-generated k-subalgebra of $k[x_1, \dots, x_n]$ and D a $\mathrm{qt}(B)/k$ derivation such that $[\mathrm{qt}(B) : D^{-1}(0)] = p$. Let $A = D^{-1}(0) \cap B$. Then the homomorphism ϕ: $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ of* 1.3 *has kernel of finite order and type $(p, \dots, p)$.*

PROOF. There exists $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ such that $B = k[f_1, \dots, f_r]$. We can insure by multiplying D by an appropriate element of B that $D(B) \subset B$. Let $\mathfrak{L} \subset B$ be the group of logarithmic derivatives of D. That is, $\mathfrak{L} = \{t^{-1}Dt \mid t \in \mathrm{qt}(B)$ and $t^{-1}Dt \in B\}$. By 1.8(a) there exists an injection $\ker \phi \hookrightarrow \mathfrak{L}$. Let

$$d = \max\{\deg(Df_i) - \deg f_i\}.$$

If $h \in \mathfrak{L}$, then there exists $t \in \mathrm{qt}(B)$ such that $t^{-1}Dt = h$. We can assume $t \in B$, for we can multiply t by a pth power of an element in B to arrange this.

We have that $\deg(Dt) \leqslant \deg t + d$, which implies that $\deg h \leqslant d$. Thus $\mathfrak{L}$ is contained in the k-vector space of polynomials of degree $\leqslant d$, which has dimension $< \infty$.

(5.2.1) If $h_1, \dots, h_s$ are in $\mathfrak{L}$ and are independent over $\mathbf{Z}/p\mathbf{Z}$, then $h_1, \dots, h_s$ are $\bar{k}$-independent ($\bar{k}$ an algebraic closure of k).

We prove (5.2.1) by induction on s. The case $s = 1$ is obvious.

Suppose that $\alpha_1 h_1 + \cdots + \alpha_s h_s = 0$ with $\alpha_i \in k$ and $\{h_1, \dots, h_s\}$ independent over $\mathbf{Z}/p\mathbf{Z}$.

By 1.9, there exists $a \in A$ such that $D^p = aD$. We also have that

$$(5.2.2) \qquad \sum_{i=1}^{s} \alpha_i h_i^p = -\sum_{i=1}^{s} (D^{p-1} - aI)\alpha_i h = -(D^{p-1} - aI)\sum_{i=1}^{s} \alpha_i h = 0,$$

where I is the identity map. Thus $\sum_{i=1}^{s} (\alpha_i)^{1/p} h_i = 0$.

Suppose that $\alpha_s \neq 0$. Then

$$(5.2.3) \qquad \sum_{i=1}^{s-1} \left[(\alpha_s)^{1/p}\alpha_i - \alpha_s(\alpha_i)^{1/p}\right] h_i = (\alpha_s)^{1/p} \sum_{i=1}^{s} \alpha_i h_i - \alpha_s \sum_{i=1}^{s} (\alpha_i)^{1/p} h_i = 0.$$

By induction, $(\alpha_s)^{1/p}\alpha_i - \alpha_s(\alpha_i)^{1/p} = 0$ for $1 \leqslant i \leqslant s - 1$. This implies that $(\alpha_i/\alpha_s)^p = \alpha_i/\alpha_s$ and $\alpha_i/\alpha_s \in \mathbf{Z}/p\mathbf{Z}$ for each i.

Hence $\Sigma\alpha_i h_i = 0$ implies that $\Sigma(\alpha_i/\alpha_s)h_i = 0$, which contradicts the fact that the h_i are $\mathbf{Z}/p\mathbf{Z}$ independent. We conclude that α_s, and hence all α_i, equals 0.

Since $\mathfrak{L} \subset B$, each element of $\mathfrak{L}$ has p-torsion. By (5.2.1) $\mathfrak{L}$ has no more than a finite number of independent elements.

PROPOSITION 5.3. *Let k be a field of characteristic $p > 0$ and let A be an integrally closed domain such that $k[x_1^{p^m}, \dots, x_n^{p^m}] \subsetneq A \subset k[x_1, \dots, x_n]$. Then $\mathrm{Cl}(A)$ is a finite p-group of type $(p^{i_1}, \dots, p^{i_r})$ with each $i_j \leqslant mn - 1$.*

PROOF. $k(x_1, \dots, x_n)$ is a purely inseparable extension of $\mathrm{qt}(A)$ of degree p^s where $s < mn$. There exist fields $k(x_1, \dots, x_n) = L_0 \supset L_1 \supset \cdots \supset L_s = \mathrm{qt}(A)$, with L_i/L_{i+1} a purely inseparable extension of degree p.

For each $i = 0,\ldots,s$, let $A_i = k[x_1,\ldots,x_n] \cap L_i$. Then $A_s = A$ and each A_i is a finite $k[x_1^{p^m},\ldots,x_n^{p^m}]$-module. Thus each A_i is Noetherian and a Krull domain (A_i is an intersection of Krull domains). Hence each A_i is integrally closed (see **[Sl**, p. 5]).

By 5.1, there exist derivations D_i: $L_i \to L_i$ such that $D_i^{-1}(0) = L_{i+1}$.

By 5.2 the homomorphism ϕ_i: $\mathrm{Cl}(A_{i+1}) \to \mathrm{Cl}(A_i)$ has kernel of finite order and of type $(p,\ldots,p)$. Inductively we see that each A_i has class group of finite order and of type $(p^{r_1},\ldots,p^{r_q})$ where each $r_j \leq i$.

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