ON THE EXISTENCE OF MAXIMAL COHEN-MACaulay
MODULES OVER $p$th ROOT EXTENSIONS

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Abstract. Let $S$ be an unramified regular local ring having mixed characteristic $p > 0$ and $R$ the integral closure of $S$ in a $p$th root extension of its quotient field. We show that $R$ admits a finite, birational module $M$ such that $\text{depth}(M) = \text{dim}(R)$. In other words, $R$ admits a maximal Cohen-Macaulay module.

1. Introduction

Let $R$ be a Noetherian local ring. In considering the local homological conjectures over $R$, one may reduce to the situation where $R$ is a finite extension of an unramified regular local ring $S$. Therefore, it is a natural point of departure to assume that $R$ is the integral closure of $S$ in a "well-behaved" algebraic extension of its quotient field. Certainly, when $S$ has mixed characteristic $p > 0$, one ought to consider the case that $R$ is the integral closure of $S$ in an extension of its quotient field obtained by adjoining the $p$th root of an element of $S$. This was done in [Ko] where it was shown that $S$ is a direct summand of $R$, i.e., the Direct Summand Conjecture holds for the extension $S \subseteq R$. In this note we show that a number of the other local homological conjectures hold for such $R$ by showing that $R$ admits a finite, birational module $M$ satisfying $\text{depth}(M) = \text{dim}(R)$ (see [H]). In other words, $R$ admits a maximal Cohen-Macaulay module. Such a module is necessarily free over $S$. Aside from regularity, one of the crucial points in the mixed characteristic case seems to be that $S/pS$ is integrally closed. By contrast, using an example from [HM], Roberts has noted that even if $S$ is a Cohen-Macaulay UFD and $R$ is the integral closure of $S$ in a quadratic extension of quotient fields, $R$ needn’t admit a finite, $S$-free module at all (see [R]). For the example in question, $S$ has mixed characteristic 2, yet $S/2S$ is not integrally closed.

2. Preliminaries

In this section we will establish our notation and present a few preliminary observations. Throughout, $S$ will be a Noetherian normal domain with quotient field $L$. We assume $\text{char}(L) = 0$. Fix $p \in \mathbb{Z}$ to be a prime integer and suppose that either $p$ is a unit in $S$ or that $pS$ is a (proper) prime ideal and $S/pS$ is integrally closed. Let $f \in S$ be an element that is not a $p$th power and select $W$ an indeterminate. Write $F(W) := W^p - f \in S[W]$, a monic irreducible polynomial

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and let $R$ denote the integral closure of $S$ in $K := L(\omega)$, for $\omega$ a root of $F(W)$. Thus $R$ is the integral closure of $S[\omega]$.

Our strategy in this paper is to exploit the fact that $R$ can be realized as $J^{-1}$ for a suitable ideal $J \subseteq S[\omega]$. The study of birational algebras of the form $J^{-1}$ seems to have captured the attention of a number of researchers during the last few years, albeit in notably different contexts (see [EU], [Ka], [KU], [MP] and [V]). Since $J^{-1}$ inherits $S_2$ from $S[\omega]$, this means that in attempting to “construct” $R$, if the candidate is $J^{-1}$ for some $J$, then only the condition $R_1$ must be checked.

The following proposition summarizes some of the conditions relating $R$ to $J^{-1}$ for suitable $J$ that we will call upon in the next section. Parts (i) and (ii) of the proposition were inspired by the main results in [V] and Proposition 3.1 in [KU]. Special cases of part (iii) of the proposition have apparently been known to algebraic geometers for a long while. For some historical comments and fascinating variations, the interested reader should consult [KU].

**Proposition 2.1.** Let $A$ be a Noetherian domain satisfying $S_2$ and assume that $A'$, the integral closure of $A$, is a finite $A$-module.

(i) Suppose $\{P_1, \ldots, P_n\}$ are the height one primes of $A$ for which $A_{P_i}$ is not a DVR. If for each $1 \leq i \leq n$, $\text{rad}(J_i) = P_i$ and $(J_i^{-1})_{P_i} = A_{P_i}$, then $A' = J^{-1}$, for $J := J_1 \cap \cdots \cap J_n$.

(ii) If $A \neq A'$, then $A' = J^{-1}$, for some height one unmixed ideal $J \subseteq A$. Moreover, if $A$ is Gorenstein in codimension one, then $A' = J^{-1}$ for a unique height one unmixed ideal $J$ satisfying $J \cdot J^{-1} = J = (J^{-1})^{-1}$.

(iii) Suppose that $A = B/(F)$ for $F \in B$ a principal prime and $J \subseteq B$ is a grade two ideal arising as the ideal of $n \times n$ minors of an $(n + 1) \times n$ matrix $\phi$. Assume further that $F \in \tilde{J}$ and set $J := J/(F)$. Let $\Delta_1, \ldots, \Delta_{n+1}$ denote the signed minors of $\phi$, write $F := b_1\Delta_1 + \cdots + b_{n+1}\Delta_{n+1}$ and let $\phi'$ denote the $(n+1) \times (n+1)$ matrix obtained by augmenting the column of $b_i$s to $\phi$ (so $F$ is the determinant of $\phi'$). Then $J^{-1}$ can be generated as an $A$-module by $\{\psi_{i,1}/\delta_1, \ldots, \psi_{n+1,n+1}/\delta_{n+1} = 1\}$, where $\psi_{i,i}$ denotes the image in $A$ of the $(i, i)$th cofactor of $\phi'$ and $\delta_i$ denotes the image of $\Delta_i$ in $A$ (which we assume to be non-zero). Moreover, $p.d. B(J) = p.d. B(J^{-1}) = 1$.

**Proof.** To prove (i), note that $J^{-1} = A'_Q$ for all height one primes $Q \subseteq A$. Since $J^{-1}$ and $A'$ are birational and satisfy $S_2$, we obtain $J^{-1} = A'$. For the first statement in (ii), we may, by part (i), consider the case where $A$ is a one-dimensional local ring which is not a DVR. Let $Q$ denote the maximal ideal of $A$. Then $QQ^{-1} \subseteq Q$. Since it always holds that $Q \subseteq QQ^{-1}$, we have $Q = QQ^{-1}$. Therefore $Q^{-1}$ is a finite ring extension properly containing $A$ (since for any ideal $J$, $(JJ^{-1})^{-1}$ is a ring). If $Q^{-1} = A'$, we’re done. If not, then since $Q^{-1}$ inherits $S_2$ from $A$, $Q^{-1}$ contains a height one prime $P$ for which $(Q^{-1})_P$ is not a DVR. Thus $P^{-1}$ is a finite ring extension properly containing $Q^{-1}$. An easy calculation shows that $P^{-1}$, considered over $Q^{-1}$, equals $(QP)^{-1}$, considered over $A$. Iterating this process shows we eventually obtain $A' = J^{-1}$, for some $J \subseteq A$. Now suppose that $A$ is Gorenstein in codimension one. Then $I_Q = (I^{-1})_Q$, for all ideals $I \subseteq A$ and all height one primes $Q \subseteq A$. Therefore, $I = (I^{-1})^{-1}$, for all height one, unmixed ideals $I \subseteq A$. In particular, this holds for $J$. Moreover, if $J^{-1} = A' = K^{-1}$, for $K$ height one and unmixed, then $J = K$. Finally, since $J^{-1}$ is a ring, $(J \cdot J^{-1}) \cdot J^{-1} = J \cdot J^{-1}$, so $J \cdot J^{-1} \subseteq (J^{-1})^{-1} = J$. Thus, $J \cdot J^{-1} = J$, as desired. For (iii), the description of
the generators for \( J^{-1} \) follows either from [MP], Proposition 3.14 or [KU], Lemma 2.5. For the second part of (iii), see [KU], Proposition 3.1.

Returning to our basic set-up, we note that since \( S \) is a normal domain, \( S[\omega] \) satisfies Serre’s condition \( S_2 \). Moreover, since \( \text{char}(S) = 0 \), \( R \) is a finite \( S \)-module. Thus Proposition 2.1 applies. In Section 3 we will identify the ideal \( J \subseteq S[\omega] \) for which \( J^{-1} = R \). In the meantime, we observe that if \( p \) is not a unit in \( S \), then there is a unique height one prime in \( S[\omega] \) containing \( p \). Suppose \( p \mid f \). Then \( P := (\omega, p) \) is clearly the unique height one prime in \( S[\omega] \) containing \( p \). Moreover, \( S[\omega]_p \) is a DVR if and only if \( p^2 \nmid f \). Suppose \( p \nmid f \). If \( f \) is not a \( p \)th power modulo \( pS \), then \( f \) is not a \( p \)th power over the quotient field of \( S/pS \) (since \( S/pS \) is integrally closed) and it follows that \( F(W) \) is irreducible mod \( pS \). Thus \( (p, F(W)) \) is the unique height two prime in \( S[W] \) containing \( F(W) \) and \( p \), so \( pS[\omega] \) is the unique height one prime in \( S[\omega] \) containing \( p \). If \( f \equiv h^p \mod pS \), then \( F(W) \equiv (W - h)^p \mod pS \) and it follows that \( (\omega - h, p)S[\omega] \) is the unique height one prime in \( S[\omega] \) containing \( p \). Thus, in all cases, there exists a unique height one prime in \( S[\omega] \) lying over \( pS \). For the remainder of the paper, we call this prime \( P \). Suppose \( f = h^p + gp \), so \( P = (\omega - h, p)S[\omega] \). Write \( \tilde{P} := (W - h, p)S[W] \) for the preimage of \( P \) in \( S[W] \).

Then
\[
F(W) = W^p - h^p - gp = (W^{p-1} + \cdots + h^{p-1}) \cdot (W - h) - gp.
\]
In \( S[W] \), \( W^{p-1} + \cdots + h^{p-1} \equiv ph^{p-1} \mod (W - h) \), so \( W^{p-1} + \cdots + h^{p-1} \in \tilde{P} \). Thus, \( F(W) \in \tilde{P}^2 \) if and only if \( p \mid g \). In other words, in all cases, \( F_P \) is not principal if and only if \( f = h^p + p^2g \), for some \( h, g \in S \).

3. The main result

In this section we will present our main result, Theorem 3.8. Lemmas 3.2 and 3.3 will enable us to describe the ideal \( J \subseteq S[\omega] \) for which \( R = J^{-1} \). We will then see in the proof of Theorem 3.8 that the module we seek has the form \( I^{-1} \), for some ideal \( I \subseteq J \).

**Lemma 3.1.** Suppose \( p \) is not a unit in \( S \), \( h \in S \backslash pS \) and \( p = 2k + 1 \). Set
\[
C := \sum_{j=1}^{k} (-1)^{j+1} \binom{p}{j} (W \cdot h)^j [WP^{2j} - h^{p-2j}],
\]
\[
C' := C \cdot (p(W - h))^{-1} \text{ and } \tilde{P} := (p, W - h) \cdot S[W]. \text{ Then } C' \not\in \tilde{P}.
\]

**Proof.** Note that since \( p \) divides \( \binom{p}{j} \) for all \( 1 \leq j \leq k \), \( C' \) is a well-defined element of \( S[W] \). Now, \( C' \not\in \tilde{P} \) if and only if the residue class of \( C' \) modulo \( W - h \), as an element of \( S \), does not belong to \( pS \) if and only if \( \sum_{j=1}^{k} (-1)^{j+1} \binom{p}{j} h^{p-1} (p - 2j) \), as an element of \( S \), is not divisible by \( p \). Since
\[
\sum_{j=1}^{k} (-1)^{j+1} \binom{p}{j} h^{p-1}
\]
is divisible by \( p \) and \( h^{p-1} \) is not divisible by \( p \), it is enough to show that
\[
\sum_{j=1}^{k} (-1)^{j+1} \binom{p}{j} \frac{2j}{p}
\]
is not divisible by \( p \), as an element of \( S \). However,
\[
\sum_{j=1}^{k} (-1)^{j+1} \left( \frac{p}{j} \right) \frac{2j}{p} = 2 \sum_{j=1}^{k} (-1)^{j+1} \left( \frac{p-1}{j-1} \right) = (-1)^{k+1} \left( \frac{2k}{k} \right).
\]
Because \( p \) does not divide \( \left( \frac{2k}{k} \right) \) in \( \mathbb{Z} \), \( p \) does not divide \( \left( \frac{2k}{k} \right) \) as an element of \( S \) (since \( pS \neq S \)). Thus \( C' \notin \tilde{P} \), as claimed. \( \square \)

For the next lemma, we borrow the following terminology from [Kap]. We shall say that \( f \in S \) is “square-free” if \( qS_q = fS_q \) for all height one prime ideals \( q \subseteq S \) containing \( f \). Since \( F'(\omega) \cdot R \subseteq S[\omega] \) and \( \omega \cdot F'(\omega) = p \cdot f \), it follows from the discussion in Section 2 that if \( f \) is square-free, then either \( R = S[\omega] \) or \( P \) is the only height one prime for which \( S[\omega]_P \) is not a DVR.

**Lemma 3.2.** Suppose \( f \in S \) is square-free and \( S[\omega] \neq R \) (thus \( p \) is not a unit in \( S \)). Then \( R = P^{-1} \). Moreover, \( R \) is a free \( S \)-module.

**Proof.** We first consider the case \( p > 2 \). Since \( S[\omega] \) is not integrally closed, we have \( f = h^p + p^2 g \), for some \( h \) not divisible by \( p \) and \( q \neq 0 \) in \( S \). Thus, \( P = (\omega - h, p)S[\omega] \).

It follows from the proof and statement of Proposition 2.1 that \( P^{-1} \) is a ring and that \( P^{-1} \) is generated as an \( S[\omega] \)-module by \( \{ 1, \tau \} \), for
\[
\tau = \frac{1}{p} \sum_{j=1}^{p} \omega^{p-j} h^{j-1} = \frac{g \cdot p}{\omega - h}.
\]
Therefore \( P^{-1} = S[\omega, \tau] \). If we show that \( S[\omega, \tau] \) satisfies \( R_1 \), then \( S[\omega, \tau] = R \), since \( P^{-1} \) satisfies \( S_2 \) (as an \( S[\omega] \)-module and as a ring). Since \( f \) is square-free, it suffices to show that \( P^{-1} \) is a DVR for each height one \( Q \subseteq P^{-1} \) containing \( p \). To do this, we find an equation satisfied by \( \tau \) over \( S[\omega] \). On the one hand,
\[
(\omega - h) \cdot \tau = 0 \cdot (w - h) + g \cdot p.
\]
On the other hand,
\[
p \cdot \tau = (\omega - h)^{p-2} \cdot (\omega - h) + c' \cdot p,
\]
where \( c' \) denotes the image in \( S[\omega] \) of the element \( C' \in S[W] \) defined in Lemma 3.1. Therefore, by the standard determinant argument, \( \tau \) satisfies
\[
l(T) := T^2 - c'T - (\omega - h)^{p-2}
\]
over \( S[\omega] \). Now, let \( \pi : S[W, T] \rightarrow S[\omega, \tau] \) denote the canonical map and set \( H := \ker(\pi) \) and let \( Q \subseteq S[\omega, \tau] \) be any height one prime containing \( p \). Then \( Q \) corresponds to a height three prime \( Q' \subseteq S[W, T] \) containing \( p \) and \( H \). Since \( P \subseteq Q \) and \( H \subseteq Q' \), \( W - h \) and \( T^2 - C'T - g(W - h)^{p-2} \) belong to \( Q' \). Therefore, \( Q' = (p, W - h, T) \) or \( Q' = (p, W - h, T - C') \). Suppose \( Q' = (p, W - h, T) \). Then \( Q = (p, \omega - h, \tau)S[\omega, \tau] \). We have
\[
\tau^2 - c' \tau - (\omega - h)^{p-2} = 0 \quad \text{and} \quad p(\tau - c') = (\omega - h)^{p-1}.
\]
By Lemma 3.1, \( c' \notin Q \), so \( \tau - c' \notin Q \), and it follows that \( Q_Q = (\omega - h)_Q \). Now suppose \( Q' = (p, W - h, T - C') \). Then \( Q = (p, \omega - h, \tau - c')S[\omega, \tau] \). Since
\[
\tau^2 - c' \tau - (\omega - h)^{p-2} = 0 \quad \text{and} \quad (\omega - h) \cdot \tau = g \cdot p,
\]
it follows that \( Q_Q = (p)_Q \) (since \( \tau \notin Q \), by Lemma 3.1). Thus, in either case, \( Q_Q \) is principal, so \( R = S[\omega, \tau] = P^{-1} \).
The proof is similar if \( p = 2 \) and \( f = h^2 + 4g \), with \( 2 \nmid h \). One notes that \( P^{-1} = S[\omega, \tau] = S[\tau] \), for \( \tau := \frac{a^2}{2} \omega \) and that \( \tau \) satisfies \( l(T) := T^2 - hT - g \). To show \( R = S[\tau] \), one uses the fact that \( l(T) \) and \( l'(T) \) are relatively prime over the quotient field of \( S/2S \).

To see that \( R \) is a free \( S \)-module, we first note that \( R \) is clearly generated as an \( S \)-module by the set \( \{1, \omega, \ldots, \omega^{p-1}, \tau, \tau\omega, \ldots, \tau\omega^{p-1}\} \). However, \( \tau\omega = pg \cdot 1 + h \cdot \tau \).

This implies that \( \tau\omega^i \) belongs to the \( S \)-module generated by \( \{1, \omega, \ldots, \omega^{p-1}, \tau\} \), for all \( 1 \leq i \leq p - 1 \).

Moreover, since
\[
\omega^{p-1} = -h^{p-1} \cdot 1 - h^{p-2} \cdot \omega - \cdots - h \cdot \omega^{p-2} + p \cdot \tau,
\]
we may displace of \( \omega^{p-1} \) as well. Thus, \( R \) is generated as an \( S \)-module by the set \( \{1, \omega, \ldots, \omega^{p-2}, \tau\} \). Since these elements are clearly linearly independent over \( S \), \( R \) is a free \( S \)-module.

**Lemma 3.3.** Suppose \( f = \lambda \alpha^e \), with \( \alpha \in S \) a prime element, \( \lambda \) a unit in \( S \) and \( 2 \leq e < p \). If \( p \) is not a unit in \( S \), assume \( a = p \). Then there exist integers \( 1 \leq s_1 < s_2 < \cdots < s_{e-1} < p \) satisfying

\[
(i) \quad s_{e-i} \leq p - s_i, \quad 1 \leq i \leq e - 1.
\]

\[
(ii) \quad R = \mathcal{J}^{-1} \text{ for } J := (\omega^{s_{e-i}}, \omega^{s_{e-2}}, \ldots, \omega^{s_{1}}, \alpha^{e-2}, \alpha^{e-1})S[\omega].
\]

**Proof.** We begin by noting that either condition in the hypothesis implies that \( Q := (\omega, a)S[\omega] \) is the only height one prime for which \( S[\omega]_Q \) is not a DVR. Now, since \( p \) and \( e \) are relatively prime, we can find positive integers \( u \) and \( v \) such that \( 1 = u \cdot p + (-v) \cdot e \). If we set \( \tau := \frac{a^{s_{e-i}}}{2} \), then \( \tau^e = \lambda^{-v} \omega \) and \( \tau^{e-1} = \lambda^{-v} a \).

It follows that \( S[\omega, \tau] = \mathcal{S}[\tau] = R \), since either \( p \) is a unit and \( a \) is square-free or \( p \) is not a unit and \( (\tau, p)S[\tau] = \tau S[\tau] \). Thus, \( \{1, \tau, \ldots, \tau^{e-1}\} \) generate \( R \) as an \( S[\omega] \)-module. Since \( u \) and \( e \) are relatively prime, the set \( \{u_j\}_{1 \leq j \leq e-1} \), when reduced \( \mod e \), equals the set \( \{i\}_{1 \leq i \leq e-1} \). This will enable us to replace the generators \( \{1, \tau, \ldots, \tau^{e-1}\} \) by \( \{1, \frac{a}{\sqrt{2}}, \ldots, \frac{a^{e-1}}{\sqrt{2}}\} \). To elaborate, given \( 1 \leq i \leq e - 1 \), there is a unique \( 1 \leq j \leq e - 1 \) such that \( u_j \equiv i \mod e \). Write \( u_j = t_i e + i, t_i \geq 0 \). Then
\[
(1 + ve)j_i = pu_j = t_i e + ip,
\]
so \( vj_i e + j_i = (t_i p)e + ip \). If we write \( ip = s_i e + r \), with \( 0 \leq r < e \), then uniqueness of the \( \omega \)-algorithm gives \( vj_i = t_i p + s_i \) and \( r = j_i \). Thus, \( \tau^{j_i} = \frac{a^{j_i}}{\omega^{s_i}} \frac{a^{e-1}}{\omega^{s_{e-1}}} \) and \( ip = s_i e + j_i \). For \( i = e - 1 \), this yields \( s_{e-1} < p \). Moreover, \( p = (s_{i+1} - s_i) e + (j_{i+1} - j_i) \), so \( s_{i+1} - s_i > 0 \). Similarly, \( ep = (s_{e-1} + s_i) e + (j_{e-1} + j_i) \), so \( s_{e-1} + s_i \leq p \). Thus, \( s_1, \ldots, s_{e-1} \) have the required numerical properties.

We now have \( \{1, \tau, \ldots, \tau^{e-1}\} = \{1, \frac{a}{\sqrt{2}}, \ldots, \frac{a^{e-1}}{\sqrt{2}}\} \). Multiplying by appropriate powers of \( \lambda \) allows us to use \( \{1, \frac{a}{\sqrt{2}}, \ldots, \frac{a^{e-1}}{\sqrt{2}}\} \) as a generating set for \( R \) over \( S[\omega] \). In Proposition 2.1 take \( A := S[\omega], B := S[W], F := F(W) \) and \( J \) the ideal of \( (e - 1) \times (e - 1) \) signed minors of the \( e \times (e - 1) \) matrix
\[
\phi = \begin{pmatrix}
-a & 0 & \cdots & 0 & 0 \\
W^{\alpha e-1} & -a & \cdots & 0 & 0 \\
0 & W^{\alpha e-2} & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & W^{\alpha_2} & -a \\
0 & 0 & \cdots & 0 & W^{\alpha_1}
\end{pmatrix}
\]
with \( \alpha_1 + \alpha_2 + \cdots + \alpha_i = s_i \), for \( 1 \leq i \leq e - 1 \). To obtain \( \phi' \), we augment \( \phi \) by the column whose transpose is \((W^p, 0, \ldots, 0, (-1)^s \lambda a)\) (so \( \det(\phi') = F(W) \)). Then \( J^{-1} \) is generated as an \( S[\omega] \)-module by \( \{1, \frac{\lambda a}{\omega}, \ldots, \frac{\lambda a^{e-1}}{\omega^{e-1}} \} \). Thus, \( R = S[\omega, \tau] = J^{-1} \) for \( J = (\omega^{s-e}, \omega^{s-e-2}a, \ldots, a^{e-1}) \), as desired.

For a proof of the next lemma, see [Ko], Lemma 4.1.

**Lemma 3.4.** In \( S[W] \) consider the ideals \( H := (W^{e_k}, W^{e_{k-1}}a_1, \ldots, W^{e_1}a_{k-1}, a_k) \) and \( K := (W^f, W^{f_{k-1}}b_1, \ldots, W^{f_1}b_{k-1}, b_1) \), where

(i) \( e_k > e_{k-1} > \cdots > e_1 \) and \( f_1 > f_{t-1} > \cdots > f_1 \).
(ii) \( a_1 \mid a_2 \mid \cdots \mid a_k \) and \( b_1 \mid b_2 \mid \cdots \mid b_1 \).
(iii) Each \( a_i \) and \( b_j \) is a product of prime elements.
(iv) For all \( i \) and \( j \), \( a_i \) and \( b_j \) have no prime factor in common.

Then there exist integers \( g_s > \cdots > g_1 \) and products of primes \( c_1 \mid c_2 \mid \cdots \mid c_s \) such that \( H \cap K = (W^{g_s}, W^{g_{s-1}}c_1, \ldots, W^{g_1}c_{s-1}, c_s) \). Moreover, \( H, K \) and \( H \cap K \) are all grade two perfect ideals.

**Lemma 3.5.** Let \( A \) be a domain and \( I \subseteq J \) ideals such that \( J^{-1} \) is a ring. Then \( I^{-1} \) is a \( J^{-1} \)-module if and only if \( I^{-1} = (I \cdot J^{-1})^{-1} \). In particular, if \( x \in J \) and \( x \cdot J^{-1} \subseteq J \), then \( (x \cdot J^{-1})^{-1} \) is a \( J^{-1} \)-module.

**Proof.** We first observe \((I \cdot J^{-1})^{-1}\) is always a \( J^{-1} \)-module. Indeed, \( y \in (I \cdot J^{-1})^{-1} \) implies \( I \cdot J^{-1} y \subseteq R \). Thus \( J^{-1}y = J^{-1}J^{-1}y \subseteq I^{-1} \), so \((I \cdot J^{-1})(J^{-1}y) \subseteq R \) and \( J^{-1}y \subseteq (I \cdot J^{-1})^{-1} \). Therefore, \((I \cdot J^{-1})^{-1}\) is a \( J^{-1} \)-module and the first statement follows easily from this. For the second statement, we note that if \( x \cdot J^{-1} \subseteq J \), then for \( I := x \cdot J^{-1}, I \cdot J^{-1} = x \cdot J^{-1}J^{-1} = x \cdot J^{-1} = I \). Thus, \( I^{-1} = (I \cdot J^{-1})^{-1} \), so \( I^{-1} \) is a \( J^{-1} \)-module by the first statement.

**Remark 3.6.** Proposition 2.2 in [Ko] states that \( R \) is a free \( S \)-module, if \( S \) is an unramified regular local ring and \( p \mid f \). The proof shows that \( R \) is a free \( S \)-module just under the assumption that \( f \) can be written as a product of primes and \( S/pS \) is a domain. In [Ko], Proposition 1.5, it is shown that if \( S \) is a UFD, then there exists a free \( S \)-module \( F \subseteq R \) such that \( pR \) is contained in \( F \). Thus, if \( p \) is a unit in \( S \), then \( R \) is also a free \( S \)-module. Finally, if \( f \) is square-free, \( R \) is a free \( S \)-module by Lemma 3.2. We record these facts in a common setting in the following proposition. For a version of the proposition for \( p^q \)th root extensions, see [Ka], Theorem 4.2.

**Proposition 3.7.** In addition to our standing hypotheses, assume that \( S \) is a UFD. Then \( R \) is a free \( S \)-module in each of the following cases:

(i) \( p \) is a unit in \( S \).
(ii) \( p \) is not a unit and either \( p \mid f \) or \( f \) is square-free.

We are now ready for our theorem.

**Theorem 3.8.** Assume that \( S \) is a regular local ring. Then there exists a finite, birational \( R \)-module \( M \) satisfying \( \text{depth}_S(M) = \dim(R) \). In other words, \( M \) is a maximal Cohen-Macaulay module for \( R \).

**Proof.** By Proposition 3.7, \( R \) is a free \( S \)-module, and therefore Cohen-Macaulay, unless we assume that \( p \) is not a unit in \( S \). Factor \( f \) as a unit \( \lambda \) times prime elements \( a_i \), say \( f = \lambda a_1^{e_1} \cdots a_r^{e_r} \). We may assume that for \( 1 \leq t \leq r \), \( 1 < e_i < p \), if \( 1 \leq i \leq t \) and \( e_i = 1 \), if \( t < i \leq r \). Set \( Q_i := (\omega, a_i)[S[\omega]] \) for \( 1 \leq i \leq t \). For
each $1 \leq i \leq t$ choose $s(i, 1) = \cdots = s(i, e_i - 1)$ satisfying the conclusion of Lemma 3.3 over $S[\omega]_Q$, and set $J_i := (\omega^{s(i, e_i - 1)}, \omega^{s(i, e_i - 2)} a_1, \ldots, \omega^{s(i, 1)} a_i^{e_i - 2}, a_i^{e_i - 1}) S[\omega]$. Thus, $R_{Q_i} = J_i^{-1} Q_i$ for all $i$. We now have two cases to consider. Suppose first that $f$ is not a $p$th power modulo $p^2 S$. We will show that $R$ is Cohen-Macaulay. By our discussion in section two, $Q_1, \ldots, Q_t$ are exactly the height one primes $Q \subseteq S[\omega]$ for which $S[\omega]_Q$ is a DVR, so by Proposition 2.1 and Lemma 3.3, $R = J_i^{-1}$ for $J := J_1 \cap \cdots \cap J_t$. Set $B := S[W]_{(W,N)}$ (for $N$, the maximal ideal of $S$) and use “tilde” to denote pre-images in $B$. By Lemma 3.4, $J \subseteq B$ is a grade two perfect ideal. Therefore, $p.d. B(J) = p.d. B(J^{-1}) = 1$, by Proposition 2.1(iii). Thus, $\text{depth}_B(J^{-1}) = \dim(B) - 1$, so $\text{depth}_S(R) = \dim(R)$, which is what we want.

Suppose that $f$ is a $p$th power modulo $p^2 S$. Write $f = h^p + p^2 g$, for $h, g \in S$, $p \nmid h$. Then $P = (\omega - h, p)$. Moreover, $P$ and $Q_1, \ldots, Q_t$ are the height one primes $Q \subseteq S[\omega]$ for which $S[\omega]_Q$ is a DVR. By Proposition 2.1 and Lemma 3.2, $R = J_i^{-1}$, for $J := J_1 \cap \cdots \cap J_t \cap P$. Now, as in the proof of Lemma 3.3, $J_i^{-1}$ is generated as an $S[\omega]$-module by the set $\{1, \omega a_i, \ldots, \omega a_i^{e_i - 1}\}$, where, for each $i$, $\lambda_i := \prod_{j \neq i} 1 + \omega^a_j$. Thus $K_i = (\omega^{p-s(i, 1)}, \omega^{p-s(i, 2)} a_1, \ldots, a_i^{e_i - 1}) S[\omega]$, for $K_i := a_i^{e_i - 1} J_i^{-1}$ and $1 \leq i \leq t$. By Lemma 3.3, $K_i \subseteq J_i$, so upon setting $I := K_1 \cap \cdots \cap K_t \cap P$, it follows from Lemma 3.5 that $I^{-1}$ is a $J_i^{-1}$-module (since this holds locally for every height one prime in $S[\omega]$). Taking $M := I^{-1}$, we will show that $M$ is the required module. For this, we claim that $I \subseteq B$ is a grade two perfect ideal. If the claim holds, $1 = p.d. B(I) = p.d. B(I^{-1}) = p.d. B(M)$. Thus $\text{depth}_B(M) = \dim(B) - 1$, so $\text{depth}_S(M) = \dim(R)$, which is what we want.

To prove the claim, we set $\tilde{K} := \tilde{K}_1 \cap \cdots \cap \tilde{K}_t$ and consider the short exact sequence

$$0 \longrightarrow B/\tilde{I} \longrightarrow B/\tilde{K} \oplus B/\tilde{P} \longrightarrow B/(\tilde{K} + \tilde{P}) \longrightarrow 0.$$

Since $\tilde{K}$ is a grade two perfect ideal (by Lemma 3.4), the Depth Lemma and the Auslander-Buchsbaum formula imply that $\tilde{I}$ is a grade two perfect ideal, once we show $\text{depth}(B/(\tilde{K} + \tilde{P})) = \dim(B) - 3$. Set $a := a_1^{e_1 - 1} \cdots a_t^{e_t - 1}$. We now argue that $\tilde{K} + \tilde{P} = (a, p, W - h)$. If we can show this, clearly $\text{depth}(B/(\tilde{K} + \tilde{P})) = \dim(B) - 3$ and we will have verified the claim. Take $\tilde{k} \in \tilde{K}$ and consider its image $k$ in $K \subseteq S[\omega]$. Select $Q \subseteq S[\omega]$, a height one prime. If $Q = Q_i$, for some $1 \leq i \leq t$, then $k \in (a_i^{e_i - 1} J_i^{-1}) Q_i = a R_{Q_i}$. If $Q \neq Q_i$ for any $1 \leq i \leq t$, then clearly $k \in a R_{Q_i} = R_Q$. It follows that $k \in a R \cap S[\omega]$. In other words, $k$ is integral over the principal ideal $a S[\omega]$. Therefore, the image of $k$ in $S[\omega]/(\omega - h, p) = S/p S$ is integral over the principal ideal generated by the image of $a$. Since $S/p S$ is integrally closed, the image of $k$ in $S/p S$ is a multiple of the image of $a$. Therefore, $\tilde{k} \in (a, p, W - h) \subseteq S[W]$. It follows that $\tilde{K} \subseteq (a, p, W - h)$. Since $a \in \tilde{K}$, we obtain $\tilde{K} + \tilde{P} = (a, p, W - h)$, which is what we want. This completes the proof of Theorem 3.8.

Remark 3.9. Of course if $S$ is an unramified regular local ring, $S$ fulfills our standing hypotheses, so Theorem 3.8 applies. However, the theorem also applies to certain ramified regular local rings. For instance, take $T$ to be the ring $\mathbb{Z}[X_1, \ldots, X_d]$ localized at $(p, X_1, \ldots, X_d)$ and let $H \in \mathbb{Z}[X_1, \ldots, X_d]$ be any polynomial in $(X_1, \ldots, X_d)^2$ for which $\mathbb{Z}_p[X_1, \ldots, X_d]/(H)$ is an integrally closed domain. If we set $S := T/(p - H)$, then $S$ is a ramified regular local ring and $S/p S$ is an integrally closed domain.
We close with an example where $R$ is not a free $S$-module, yet $R$ admits a finite, birational module which is a free $S$-module. The example is an unramified variation of Koh’s Example (2.4).

**Example 3.10.** Let $S$ be an unramified regular local ring having mixed characteristic 3 and take $x, y \in S$ such that $3, x, y$ form part of a regular system of parameters. Set $a := xy^4 + 9, b := x^4y + 9$ and $f := ab^2$, so $\omega^3 = f = ab^2 = h^3 + 9g$, for $h = x^3y^2$. From Lemmas 3.2 and 3.3 it follows that $R = (Q \cap P)^{-1}$ for $Q := (\omega, b)$ and $P := (\omega - h, 3)$. Set $J := Q \cap P$. We first show that $R = J^{-1}$ is not a free $S$-module. Suppose to the contrary that $J^{-1}$ is free over $S$. As in the proof of Theorem 3.8, set $B := S[W]_{(N,W)}$ and use “tilde” to denote pre-images in $B$. Since $J^{-1}$ is free over $S$, we have $p.d._B(J^{-1}) = 1$, so $J^{-1}$ is a grade one perfect $B$-module.

By [KU, Proposition 3.6], $J$ is a grade one perfect $B$-module, so $J$ is a grade two perfect ideal. On the other hand, $\text{depth}_B(B/J) = 1 + \text{depth}_B(B/(Q+P))$. But, $Q+P = (W, x^4y, x^3y^2, 3)B$, so $B/(Q+P) = S/(3, x^4y, x^3y^2)S$, which is easily seen to have depth equal to $\text{depth}(S) - 3 = \text{depth}(B) - 4$. This is a contradiction, so it must hold that $R$ is not a free $S$-module.

Now, $Q^{-1}$ is generated as an $S[\omega]$-module by $\{1, \omega/h \}$. If we set $K := b \cdot Q^{-1}$, then $K = (\omega^2, b)S[\omega]$. The proof of Theorem 3.8 shows that $M := (K \cap P)^{-1}$ is a finite, birational $R$-module satisfying $\text{depth}_S(M) = \text{dim}(R)$. In other words, $M$ is an $R$-module which is free over $S$. To calculate a basis for $M$, one must calculate $K \cap P$ and then use Proposition 2.1. We leave it to the reader to check that $K \cap P = (\omega^2 - h^2 - 9x^2y^3, b(\omega - h), 3b)$. Therefore, $K \cap P = I_2(\phi)$ for

$$
\phi = \begin{pmatrix} -b & 0 \\ \omega + h & -3 \\ -3x^2y^3 & \omega - h \end{pmatrix}.
$$

The augmented matrix that determines $(K \cap P)^{-1} = M$ is the $3 \times 3$ matrix

$$
\begin{pmatrix} -b & 0 & \omega \\ \omega + h & -3 & x^2y^3 \\ -3x^2y^3 & \omega - h & t \end{pmatrix},
$$

where $t$ is defined by the equation $x^5y^3 = ab + 3t$. By Proposition 2.1, $M$ is generated as an $S[\omega]$-module by the set $\{1, \gamma, \delta\}$, for

$$
\gamma := \frac{-3t - x^2y^3(\omega - h)}{\omega^2 - h^2 - 9x^2y^3}, \quad \delta := \frac{-bt + 3x^2y^3\omega}{b(\omega - h)} = \frac{\omega^2 + \omega h + h^2 + 9x^2y^3}{3b}.
$$

If we show that $\{1, \gamma, \delta\}$ also generate $M$ as an $S$-module, then since they are clearly linearly independent over $S$, they form a basis for $M$ as an $S$-module. To see that $\{1, \gamma, \delta\}$ generate $M$ as an $S$-module, it suffices to show that $\omega, \omega \cdot \gamma$ and $\omega \cdot \delta$ can be expressed as $S$-linear combinations of $\{1, \gamma, \delta\}$. This clearly holds for $\omega$. Using $9x^2y^3 = bx^4y^3 - x^5y^4$, we obtain

$$
\omega \cdot \gamma = \frac{\omega^2}{b} = -x^2y^3 \cdot 1 - h \cdot \gamma + 3 \cdot \delta.
$$

Since $\omega^3 = h^3 + 9g$ and $g = x^5y^3 + bxy^4 + b^2$, we get

$$
\omega \cdot \delta = (3xy^4 + 3b) \cdot 1 + 3x^2y^3 \cdot \gamma + h \cdot \delta,
$$

and the example is complete.
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