

## ON THE EXISTENCE OF MAXIMAL COHEN-MACAULAY MODULES OVER $p$ th ROOT EXTENSIONS

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ABSTRACT. Let  $S$  be an unramified regular local ring having mixed characteristic  $p > 0$  and  $R$  the integral closure of  $S$  in a  $p$ th root extension of its quotient field. We show that  $R$  admits a finite, birational module  $M$  such that  $\text{depth}(M) = \text{dim}(R)$ . In other words,  $R$  admits a maximal Cohen-Macaulay module.

### 1. INTRODUCTION

Let  $R$  be a Noetherian local ring. In considering the local homological conjectures over  $R$ , one may reduce to the situation where  $R$  is a finite extension of an unramified regular local ring  $S$ . Therefore, it is a natural point of departure to assume that  $R$  is the integral closure of  $S$  in a “well-behaved” algebraic extension of its quotient field. Certainly, when  $S$  has mixed characteristic  $p > 0$ , one ought to consider the case that  $R$  is the integral closure of  $S$  in an extension of its quotient field obtained by adjoining the  $p$ th root of an element of  $S$ . This was done in [Ko] where it was shown that  $S$  is a direct summand of  $R$ , i.e., the Direct Summand Conjecture holds for the extension  $S \subseteq R$ . In this note we show that a number of the other local homological conjectures hold for such  $R$  by showing that  $R$  admits a finite, birational module  $M$  satisfying  $\text{depth}(M) = \text{dim}(R)$  (see [H]). In other words,  $R$  admits a maximal Cohen-Macaulay module. Such a module is necessarily free over  $S$ . Aside from regularity, one of the crucial points in the mixed characteristic case seems to be that  $S/pS$  is integrally closed. By contrast, using an example from [HM], Roberts has noted that even if  $S$  is a Cohen-Macaulay UFD and  $R$  is the integral closure of  $S$  in a quadratic extension of quotient fields,  $R$  needn’t admit a finite,  $S$ -free module at all (see [R]). For the example in question,  $S$  has mixed characteristic 2, yet  $S/2S$  is not integrally closed.

### 2. PRELIMINARIES

In this section we will establish our notation and present a few preliminary observations. Throughout,  $S$  will be a Noetherian normal domain with quotient field  $L$ . We assume  $\text{char}(L) = 0$ . Fix  $p \in \mathbb{Z}$  to be a prime integer and suppose that either  $p$  is a unit in  $S$  or that  $pS$  is a (proper) prime ideal and  $S/pS$  is integrally closed. Let  $f \in S$  be an element that is not a  $p$ th power and select  $W$  an indeterminate. Write  $F(W) := W^p - f \in S[W]$ , a monic irreducible polynomial

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and let  $R$  denote the integral closure of  $S$  in  $K := L(\omega)$ , for  $\omega$  a root of  $F(W)$ . Thus  $R$  is the integral closure of  $S[\omega]$ .

Our strategy in this paper is to exploit the fact that  $R$  can be realized as  $J^{-1}$  for a suitable ideal  $J \subseteq S[\omega]$ . The study of birational algebras of the form  $J^{-1}$  seems to have captured the attention of a number of researchers during the last few years, albeit in notably different contexts (see [EU], [Ka], [KU], [MP] and [V]). Since  $J^{-1}$  inherits  $S_2$  from  $S[\omega]$ , this means that in attempting to “construct”  $R$ , if the candidate is  $J^{-1}$  for some  $J$ , then only the condition  $R_1$  must be checked.

The following proposition summarizes some of the conditions relating  $R$  to  $J^{-1}$  for suitable  $J$  that we will call upon in the next section. Parts (i) and (ii) of the proposition were inspired by the main results in [V] and Proposition 3.1 in [KU]. Special cases of part (iii) of the proposition have apparently been known to algebraic geometers for a long while. For some historical comments and fascinating variations, the interested reader should consult [KU].

**Proposition 2.1.** *Let  $A$  be a Noetherian domain satisfying  $S_2$  and assume that  $A'$ , the integral closure of  $A$ , is a finite  $A$ -module.*

- (i) *Suppose  $\{P_1, \dots, P_n\}$  are the height one primes of  $A$  for which  $A_{P_i}$  is not a DVR. If for each  $1 \leq i \leq n$ ,  $\text{rad}(J_i) = P_i$  and  $(J_i^{-1})_{P_i} = A'_{P_i}$ , then  $A' = J^{-1}$ , for  $J := J_1 \cap \dots \cap J_n$ .*
- (ii) *If  $A \neq A'$ , then  $A' = J^{-1}$ , for some height one unmixed ideal  $J \subseteq A$ . Moreover, if  $A$  is Gorenstein in codimension one, then  $A' = J^{-1}$  for a unique height one unmixed ideal  $J$  satisfying  $J \cdot J^{-1} = J = (J^{-1})^{-1}$ .*
- (iii) *Suppose that  $A = B/(F)$  for  $F \in B$  a principal prime and  $\tilde{J} \subseteq B$  is a grade two ideal arising as the ideal of  $n \times n$  minors of an  $(n+1) \times n$  matrix  $\phi$ . Assume further that  $F \in \tilde{J}$  and set  $J := \tilde{J}/(F)$ . Let  $\Delta_1, \dots, \Delta_{n+1}$  denote the signed minors of  $\phi$ , write  $F := b_1 \Delta_1 + \dots + b_{n+1} \Delta_{n+1}$  and let  $\phi'$  denote the  $(n+1) \times (n+1)$  matrix obtained by augmenting the column of  $b'_i$ s to  $\phi$  (so  $F$  is the determinant of  $\phi'$ ). Then  $J^{-1}$  can be generated as an  $A$ -module by  $\{\psi_{1,1}/\delta_1, \dots, \psi_{n+1,n+1}/\delta_{n+1} = 1\}$ , where  $\psi_{i,i}$  denotes the image in  $A$  of the  $(i, i)$ th cofactor of  $\phi'$  and  $\delta_i$  denotes the image of  $\Delta_i$  in  $A$  (which we assume to be non-zero). Moreover,  $p.d._B(J) = p.d._B(J^{-1}) = 1$ .*

*Proof.* To prove (i), note that  $J_Q^{-1} = A'_Q$  for all height one primes  $Q \subseteq A$ . Since  $J^{-1}$  and  $A'$  are birational and satisfy  $S_2$ , we obtain  $J^{-1} = A'$ . For the first statement in (ii), we may, by part (i), consider the case where  $A$  is a one-dimensional local ring which is not a DVR. Let  $Q$  denote the maximal ideal of  $A$ . Then  $QQ^{-1} \subseteq Q$ . Since it always holds that  $Q \subseteq QQ^{-1}$ , we have  $Q = QQ^{-1}$ . Therefore  $Q^{-1}$  is a finite ring extension properly containing  $A$  (since for any ideal  $J$ ,  $(JJ^{-1})^{-1}$  is a ring). If  $Q^{-1} = A'$ , we're done. If not, then since  $Q^{-1}$  inherits  $S_2$  from  $A$ ,  $Q^{-1}$  contains a height one prime  $P$  for which  $(Q^{-1})_P$  is not a DVR. Thus  $P^{-1}$  is a finite ring extension properly containing  $Q^{-1}$ . An easy calculation shows that  $P^{-1}$ , considered over  $Q^{-1}$ , equals  $(QP)^{-1}$ , considered over  $A$ . Iterating this process shows we eventually obtain  $A' = J^{-1}$ , for some  $J \subseteq A$ . Now suppose that  $A$  is Gorenstein in codimension one. Then  $I_Q = ((I^{-1})^{-1})_Q$ , for all ideals  $I \subseteq A$  and all height one primes  $Q \subseteq A$ . Therefore,  $I = (I^{-1})^{-1}$ , for all height one, unmixed ideals  $I \subseteq A$ . In particular, this holds for  $J$ . Moreover, if  $J^{-1} = A' = K^{-1}$ , for  $K$  height one and unmixed, then  $J = K$ . Finally, since  $J^{-1}$  is a ring,  $(J \cdot J^{-1}) \cdot J^{-1} = J \cdot J^{-1}$ , so  $J \cdot J^{-1} \subseteq (J^{-1})^{-1} = J$ . Thus,  $J \cdot J^{-1} = J$ , as desired. For (iii), the description of

the generators for  $J^{-1}$  follows either from [MP], Proposition 3.14 or [KU], Lemma 2.5. For the second part of (iii), see [KU], Proposition 3.1.  $\square$

Returning to our basic set-up, we note that since  $S$  is a normal domain,  $S[\omega]$  satisfies Serre’s condition  $S_2$ . Moreover, since  $\text{char}(S) = 0$ ,  $R$  is a finite  $S$ -module. Thus Proposition 2.1 applies. In Section 3 we will identify the ideal  $J \subseteq S[\omega]$  for which  $J^{-1} = R$ . In the meantime, we observe that if  $p$  is not a unit in  $S$ , then there is a unique height one prime in  $S[\omega]$  containing  $p$ . Suppose  $p \mid f$ . Then  $P := (\omega, p)$  is clearly the unique height one prime in  $S[\omega]$  containing  $p$ . Moreover,  $S[\omega]_P$  is a DVR if and only if  $p^2 \nmid f$ . Suppose  $p \nmid f$ . If  $f$  is not a  $p$ th power modulo  $pS$ , then  $f$  is not a  $p$ th power over the quotient field of  $S/pS$  (since  $S/pS$  is integrally closed) and it follows that  $F(W)$  is irreducible mod  $pS$ . Thus  $(p, F(W))$  is the unique height two prime in  $S[W]$  containing  $F(W)$  and  $p$ , so  $pS[\omega]$  is the unique height one prime in  $S[\omega]$  containing  $p$ . If  $f \equiv h^p \pmod{pS}$ , then  $F(W) \equiv (W - h)^p \pmod{pS}$  and it follows that  $(\omega - h, p)S[\omega]$  is the unique height one prime in  $S[\omega]$  containing  $p$ . Thus, in all cases, there exists a unique height one prime in  $S[\omega]$  lying over  $pS$ . For the remainder of the paper, we call this prime  $P$ . Suppose  $f = h^p + gp$ , so  $P = (\omega - h, p)S[\omega]$ . Write  $\tilde{P} := (W - h, p)S[W]$  for the preimage of  $P$  in  $S[W]$ . Then

$$F(W) = W^p - h^p - gp = (W^{p-1} + \dots + h^{p-1}) \cdot (W - h) - gp.$$

In  $S[W]$ ,  $W^{p-1} + \dots + h^{p-1} \equiv ph^{p-1} \pmod{W - h}$ , so  $W^{p-1} + \dots + h^{p-1} \in \tilde{P}$ . Thus,  $F(W) \in \tilde{P}^2$  if and only if  $p \mid g$ . In other words, in all cases,  $P_P$  is not principal if and only if  $f = h^p + p^2g$ , for some  $h, g \in S$ .

### 3. THE MAIN RESULT

In this section we will present our main result, Theorem 3.8. Lemmas 3.2 and 3.3 will enable us to describe the ideal  $J \subseteq S[\omega]$  for which  $R = J^{-1}$ . We will then see in the proof of Theorem 3.8 that the module we seek has the form  $I^{-1}$ , for some ideal  $I \subseteq J$ .

**Lemma 3.1.** *Suppose  $p$  is not a unit in  $S$ ,  $h \in S \setminus pS$  and  $p = 2k + 1$ . Set*

$$C := \sum_{j=1}^k (-1)^{j+1} \binom{p}{j} (W \cdot h)^j [W^{p-2j} - h^{p-2j}],$$

$C' := C \cdot (p(W - h))^{-1}$  and  $\tilde{P} := (p, W - h) \cdot S[W]$ . Then  $C' \notin \tilde{P}$ .

*Proof.* Note that since  $p$  divides  $\binom{p}{j}$  for all  $1 \leq j \leq k$ ,  $C'$  is a well-defined element of  $S[W]$ . Now,  $C' \notin \tilde{P}$  if and only if the residue class of  $C'$  modulo  $W - h$ , as an element of  $S$ , does not belong to  $pS$  if and only if  $\sum_{j=1}^k (-1)^{j+1} \binom{p}{j} \frac{h^{p-1}}{p} (p - 2j)$ , as an element of  $S$ , is not divisible by  $p$ . Since

$$\sum_{j=1}^k (-1)^{j+1} \binom{p}{j} h^{p-1}$$

is divisible by  $p$  and  $h^{p-1}$  is not divisible by  $p$ , it is enough to show that

$$\sum_{j=1}^k (-1)^{j+1} \binom{p}{j} \frac{2j}{p}$$

is not divisible by  $p$ , as an element of  $S$ . However,

$$\sum_{j=1}^k (-1)^{j+1} \binom{p}{j} \frac{2j}{p} = 2 \cdot \sum_{j=1}^k (-1)^{j+1} \binom{p-1}{j-1} = (-1)^{k+1} \binom{2k}{k}.$$

Because  $p$  does not divide  $\binom{2k}{k}$  in  $\mathbb{Z}$ ,  $p$  does not divide  $\binom{2k}{k}$  as an element of  $S$  (since  $pS \neq S$ ). Thus  $C' \notin \tilde{P}$ , as claimed.  $\square$

For the next lemma, we borrow the following terminology from [Kap]. We shall say that  $f \in S$  is “square-free” if  $qS_q = fS_q$  for all height one prime ideals  $q \subseteq S$  containing  $f$ . Since  $F'(\omega) \cdot R \subseteq S[\omega]$  and  $\omega \cdot F'(\omega) = p \cdot f$ , it follows from the discussion in Section 2 that if  $f$  is square-free, then either  $R = S[\omega]$  or  $P$  is the only height one prime for which  $S[\omega]_P$  is not a DVR.

**Lemma 3.2.** *Suppose  $f \in S$  is square-free and  $S[\omega] \neq R$  (thus  $p$  is not a unit in  $S$ ). Then  $R = P^{-1}$ . Moreover,  $R$  is a free  $S$ -module.*

*Proof.* We first consider the case  $p > 2$ . Since  $S[\omega]$  is not integrally closed, we have  $f = h^p + p^2g$ , for some  $h$  not divisible by  $p$  and  $g \neq 0$  in  $S$ . Thus,  $P = (\omega - h, p)S[\omega]$ . It follows from the proof and statement of Proposition 2.1 that  $P^{-1}$  is a ring and that  $P^{-1}$  is generated as an  $S[\omega]$ -module by  $\{1, \tau\}$ , for

$$\tau = \frac{1}{p} \cdot \sum_{j=1}^p \omega^{p-j} h^{j-1} = \frac{g \cdot p}{\omega - h}.$$

Therefore  $P^{-1} = S[\omega, \tau]$ . If we show that  $S[\omega, \tau]$  satisfies  $R_1$ , then  $S[\omega, \tau] = R$ , since  $P^{-1}$  satisfies  $S_2$  (as an  $S[\omega]$ -module and as a ring). Since  $f$  is square-free, it suffices to show that  $P_Q^{-1}$  is a DVR for each height one  $Q \subseteq P^{-1}$  containing  $p$ . To do this, we find an equation satisfied by  $\tau$  over  $S[\omega]$ . On the one hand,

$$(\omega - h) \cdot \tau = 0 \cdot (\omega - h) + g \cdot p.$$

On the other hand,

$$p \cdot \tau = (\omega - h)^{p-2} \cdot (\omega - h) + c' \cdot p,$$

where  $c'$  denotes the image in  $S[\omega]$  of the element  $C' \in S[W]$  defined in Lemma 3.1. Therefore, by the standard determinant argument,  $\tau$  satisfies

$$l(T) := T^2 - c'T - g(\omega - h)^{p-2}$$

over  $S[\omega]$ . Now, let  $\pi : S[W, T] \rightarrow S[\omega, \tau]$  denote the canonical map and set  $H := \ker(\pi)$  and let  $Q \subseteq S[\omega, \tau]$  be any height one prime containing  $p$ . Then  $Q$  corresponds to a height three prime  $Q' \subseteq S[W, T]$  containing  $p$  and  $H$ . Since  $P \subseteq Q$  and  $H \subseteq Q'$ ,  $W - h$  and  $T^2 - c'T - g(W - h)^{p-2}$  belong to  $Q'$ . Therefore,  $Q' = (p, W - h, T)$  or  $Q' = (p, W - h, T - C')$ . Suppose  $Q' = (p, W - h, T)$ . Then  $Q = (p, \omega - h, \tau)S[\omega, \tau]$ . We have

$$\tau^2 - c'\tau - g(\omega - h)^{p-2} = 0 \quad \text{and} \quad p(\tau - c') = (\omega - h)^{p-1}.$$

By Lemma 3.1,  $c' \notin Q$ , so  $\tau - c' \notin Q$ , and it follows that  $Q_Q = (\omega - h)_Q$ . Now suppose  $Q' = (p, W - h, T - C')$ . Then  $Q = (p, \omega - h, \tau - c')S[\omega, \tau]$ . Since

$$\tau^2 - c'\tau - g(\omega - h)^{p-2} = 0 \quad \text{and} \quad (\omega - h) \cdot \tau = g \cdot p,$$

it follows that  $Q_Q = (p)_Q$  (since  $\tau \notin Q$ , by Lemma 3.1). Thus, in either case,  $Q_Q$  is principal, so  $R = S[\omega, \tau] = P^{-1}$ .

The proof is similar if  $p = 2$  and  $f = h^2 + 4g$ , with  $2 \nmid h$ . One notes that  $P^{-1} = S[\omega, \tau] = S[\tau]$ , for  $\tau := \frac{h+\omega}{2}$  and that  $\tau$  satisfies  $l(T) := T^2 - hT - g$ . To show  $R = S[\tau]$ , one uses the fact that  $l(T)$  and  $l'(T)$  are relatively prime over the quotient field of  $S/2S$ .

To see that  $R$  is a free  $S$ -module, we first note that  $R$  is clearly generated as an  $S$ -module by the set  $\{1, \omega, \dots, \omega^{p-1}, \tau, \tau\omega, \dots, \tau\omega^{p-1}\}$ . However,  $\tau\omega = pg \cdot 1 + h \cdot \tau$ . This implies that  $\tau\omega^i$  belongs to the  $S$ -module generated by  $\{1, \omega, \dots, \omega^{p-1}, \tau\}$ , for all  $1 \leq i \leq p - 1$ . Moreover, since

$$\omega^{p-1} = -h^{p-1} \cdot 1 - h^{p-2} \cdot \omega - \dots - h \cdot \omega^{p-2} + p \cdot \tau,$$

we may dispose of  $\omega^{p-1}$  as well. Thus,  $R$  is generated as an  $S$ -module by the set  $\{1, \omega, \dots, \omega^{p-2}, \tau\}$ . Since these elements are clearly linearly independent over  $S$ ,  $R$  is a free  $S$ -module.  $\square$

**Lemma 3.3.** *Suppose  $f = \lambda a^e$ , with  $a \in S$  a prime element,  $\lambda$  a unit in  $S$  and  $2 \leq e < p$ . If  $p$  is not a unit in  $S$ , assume  $a = p$ . Then there exist integers  $1 \leq s_1 < s_2 < \dots < s_{e-1} < p$  satisfying*

- (i)  $s_{e-i} \leq p - s_i, 1 \leq i \leq e - 1$ .
- (ii)  $R = J^{-1}$  for  $J := (\omega^{s_{e-1}}, \omega^{s_{e-2}}a, \dots, \omega^{s_1}a^{e-2}, a^{e-1})S[\omega]$ .

*Proof.* We begin by noting that either condition in the hypothesis implies that  $Q := (\omega, a)S[\omega]$  is the only height one prime for which  $S[\omega]_Q$  is not a DVR. Now, since  $p$  and  $e$  are relatively prime, we can find positive integers  $u$  and  $v$  such that  $1 = u \cdot p + (-v) \cdot e$ . If we set  $\tau := \frac{a^u}{\omega^v}$ , then  $\tau^e = \lambda^{-u}a$  and  $\tau^p = \lambda^{-v}a$ . It follows that  $S[\omega, \tau] = S[\tau] = R$ , since either  $p$  is a unit and  $a$  is square-free or  $p$  is not a unit and  $(\tau, p)S[\tau] = \tau S[\tau]$ . Thus,  $\{1, \tau, \dots, \tau^{e-1}\}$  generate  $R$  as an  $S[\omega]$ -module. Since  $u$  and  $e$  are relatively prime, the set  $\{uj\}_{1 \leq j \leq e-1}$ , when reduced mod  $e$ , equals the set  $\{i\}_{1 \leq i \leq e-1}$ . This will enable us to replace the generators  $\{1, \tau, \dots, \tau^{e-1}\}$  by  $\{1, \frac{\lambda a}{\omega^{s_1}}, \dots, \frac{\lambda a^{e-1}}{\omega^{s_{e-1}}}\}$ . To elaborate, given  $1 \leq i \leq e - 1$ , there is a unique  $1 \leq j_i \leq e - 1$  such that  $uj_i \equiv i \pmod{e}$ . Write  $uj_i = t_i e + i, t_i \geq 0$ . Then

$$(1 + ve)j_i = pu_j_i = t_i ep + ip,$$

so  $(vj_i)e + j_i = (t_i p)e + ip$ . If we write  $ip = s_i e + r$ , with  $0 \leq r < e$ , then uniqueness of the euclidean algorithm gives  $vj_i = t_i p + s_i$  and  $r = j_i$ . Thus,  $\tau^{j_i} = \frac{a^{uj_i}}{\omega^{vj_i}} = \frac{a^i}{\lambda^{t_i} \omega^{s_i}}$  and  $ip = s_i e + j_i$ . For  $i = e - 1$ , this yields  $s_{e-1} < p$ . Moreover,  $p = (s_{i+1} - s_i)e + (j_{i+1} - j_i)$ , so  $s_{i+1} - s_i > 0$ . Similarly,  $ep = (s_{e-i} + s_i)e + (j_{e-i} + j_i)$ , so  $s_{e-i} + s_i \leq p$ . Thus,  $s_1, \dots, s_{e-1}$  have the required numerical properties.

We now have  $\{1, \tau, \dots, \tau^{e-1}\} = \{1, \frac{a}{\lambda^{t_1} \omega^{s_1}}, \dots, \frac{a^{e-1}}{\lambda^{t_{e-1}} \omega^{s_{e-1}}}\}$ . Multiplying by appropriate powers of  $\lambda$  allows us to use  $\{1, \frac{\lambda a}{\omega^{s_1}}, \dots, \frac{\lambda a^{e-1}}{\omega^{s_{e-1}}}\}$  as a generating set for  $R$  over  $S[\omega]$ . In Proposition 2.1 take  $A := S[\omega], B := S[W], F := F(W)$  and  $\tilde{J}$  the ideal of  $(e - 1) \times (e - 1)$  signed minors of the  $e \times (e - 1)$  matrix

$$\phi = \begin{pmatrix} -a & 0 & \dots & 0 & 0 \\ W^{\alpha_{e-1}} & -a & \dots & 0 & 0 \\ 0 & W^{\alpha_{e-2}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & W^{\alpha_2} & -a \\ 0 & 0 & \dots & 0 & W^{\alpha_1} \end{pmatrix}$$

with  $\alpha_1 + \alpha_2 + \dots + \alpha_i = s_i$ , for  $1 \leq i \leq e - 1$ . To obtain  $\phi'$ , we augment  $\phi$  by the column whose transpose is  $(W^{p-c}, 0, \dots, 0, (-1)^e \lambda a)$  (so  $\det(\phi') = F(W)$ ). Then  $J^{-1}$  is generated as an  $S[\omega]$ -module by  $\{1, \frac{\lambda a}{\omega^{s_1}}, \dots, \frac{\lambda a^{e-1}}{\omega^{s_{e-1}}}\}$ . Thus,  $R = S[\omega, \tau] = J^{-1}$  for  $J = (\omega^{s_{e-1}}, \omega^{s_{e-2}} a, \dots, a^{e-1})$ , as desired.  $\square$

For a proof of the next lemma, see [Ka], Lemma 4.1.

**Lemma 3.4.** *In  $S[W]$  consider the ideals  $H := (W^{e_k}, W^{e_{k-1}} a_1, \dots, W^{e_1} a_{k-1}, a_k)$  and  $K := (W^{f_t}, W^{f_{t-1}} b_1, \dots, W^{f_1} b_{t-1}, b_t)$ , where*

- (i)  $e_k > e_{k-1} > \dots > e_1$  and  $f_t > f_{t-1} > \dots > f_1$ .
- (ii)  $a_1 \mid a_2 \mid \dots \mid a_k$  and  $b_1 \mid b_2 \mid \dots \mid b_t$ .
- (iii) Each  $a_i$  and  $b_j$  is a product of prime elements.
- (iv) For all  $i$  and  $j$ ,  $a_i$  and  $b_j$  have no prime factor in common.

*Then there exist integers  $g_s > \dots > g_1$  and products of primes  $c_1 \mid c_2 \mid \dots \mid c_s$  such that  $H \cap K = (W^{g_s}, W^{g_{s-1}} c_1, \dots, W^{g_1} c_{s-1}, c_s)$ . Moreover,  $H$ ,  $K$  and  $H \cap K$  are all grade two perfect ideals.*

**Lemma 3.5.** *Let  $A$  be a domain and  $I \subseteq J$  ideals such that  $J^{-1}$  is a ring. Then  $I^{-1}$  is a  $J^{-1}$ -module if and only if  $I^{-1} = (I \cdot J^{-1})^{-1}$ . In particular, if  $x \in J$  and  $x \cdot J^{-1} \subseteq J$ , then  $(x \cdot J^{-1})^{-1}$  is a  $J^{-1}$ -module.*

*Proof.* We first observe  $(I \cdot J^{-1})^{-1}$  is always a  $J^{-1}$ -module. Indeed,  $y \in (I \cdot J^{-1})^{-1}$  implies  $I \cdot J^{-1} y \subseteq R$ . Thus  $J^{-1} J^{-1} y = J^{-1} y \subseteq I^{-1}$ , so  $(I \cdot J^{-1})(J^{-1} y) \subseteq R$  and  $J^{-1} y \subseteq (I \cdot J^{-1})^{-1}$ . Therefore,  $(I \cdot J^{-1})^{-1}$  is a  $J^{-1}$ -module and the first statement follows easily from this. For the second statement, we note that if  $x \cdot J^{-1} \subseteq J$ , then for  $I := x \cdot J^{-1}$ ,  $I \cdot J^{-1} = x \cdot J^{-1} J^{-1} = x \cdot J^{-1} = I$ . Thus,  $I^{-1} = (I \cdot J^{-1})^{-1}$ , so  $I^{-1}$  is a  $J^{-1}$ -module by the first statement.  $\square$

*Remark 3.6.* Proposition 2.2 in [Ko] states that  $R$  is a free  $S$ -module, if  $S$  is an unramified regular local ring and  $p \mid f$ . The proof shows that  $R$  is a free  $S$ -module just under the assumption that  $f$  can be written as a product of primes and  $S/pS$  is a domain. In [Ko], Proposition 1.5, it is shown that if  $S$  is a UFD, then there exists a free  $S$ -module  $F \subseteq R$  such that  $pR$  is contained in  $F$ . Thus, if  $p$  is a unit in  $S$ , then  $R$  is also a free  $S$ -module. Finally, if  $f$  is square-free,  $R$  is a free  $S$ -module by Lemma 3.2. We record these facts in a common setting in the following proposition. For a version of the proposition for  $p^n$ th root extensions, see [Ka], Theorem 4.2.

**Proposition 3.7.** *In addition to our standing hypotheses, assume that  $S$  is a UFD. Then  $R$  is a free  $S$ -module in each of the following cases:*

- (i)  $p$  is a unit in  $S$ .
- (ii)  $p$  is not a unit and either  $p \mid f$  or  $f$  is square-free.

We are now ready for our theorem.

**Theorem 3.8.** *Assume that  $S$  is a regular local ring. Then there exists a finite, birational  $R$ -module  $M$  satisfying  $\text{depth}_S(M) = \dim(R)$ . In other words,  $M$  is a maximal Cohen-Macaulay module for  $R$ .*

*Proof.* By Proposition 3.7,  $R$  is a free  $S$ -module, and therefore Cohen-Macaulay, unless we assume that  $p$  is not a unit in  $S$ ,  $p \nmid f$  and  $f$  is not square-free. In particular, we may assume that  $f$  is not a unit in  $S$ . Factor  $f$  as a unit  $\lambda$  times prime elements  $a_i$ , say  $f = \lambda a_1^{e_1} \dots a_r^{e_r}$ . We may assume that for  $1 \leq t \leq r$ ,  $1 < e_i < p$ , if  $1 \leq i \leq t$  and  $e_i = 1$ , if  $t < i \leq r$ . Set  $Q_i := (\omega, a_i)S[\omega]$  for  $1 \leq i \leq t$ . For

each  $1 \leq i \leq t$  choose  $s(i, 1) < \dots < s(i, e_i - 1)$  satisfying the conclusion of Lemma 3.3 over  $S[\omega]_{Q_i}$  and set  $J_i := (\omega^{s(i, e_i - 1)}, \omega^{s(i, e_i - 2)} a_i, \dots, \omega^{s(i, 1)} a_i^{e_i - 2}, a_i^{e_i - 1}) S[\omega]$ . Thus,  $R_{Q_i} = (J_i^{-1})_{Q_i}$  for all  $i$ . We now have two cases to consider. Suppose first that  $f$  is not a  $p$ th power modulo  $p^2 S$ . We will show that  $R$  is Cohen-Macaulay. By our discussion in section two,  $Q_1, \dots, Q_t$  are exactly the height one primes  $Q \subseteq S[\omega]$  for which  $S[\omega]_Q$  is not a DVR, so by Proposition 2.1 and Lemma 3.3,  $R = J^{-1}$  for  $J := J_1 \cap \dots \cap J_t$ . Set  $B := S[W]_{(W, N)}$  (for  $N$ , the maximal ideal of  $S$ ) and use “tilde” to denote pre-images in  $B$ . By Lemma 3.4,  $\tilde{J} \subseteq B$  is a grade two perfect ideal. Therefore,  $p.d._B(J) = p.d._B(J^{-1}) = 1$ , by Proposition 2.1(iii). Thus,  $depth_B(J^{-1}) = dim(B) - 1$ , so  $depth_S(R) = dim(R)$ , which is what we want.

Suppose that  $f$  is a  $p$ th power modulo  $p^2 S$ . Write  $f = h^p + p^2 g$ , for  $h, g \in S$ ,  $p \nmid h$ . Then  $P = (\omega - h, p)$ . Moreover,  $P$  and  $Q_1, \dots, Q_t$  are the height one primes  $Q \subseteq S[\omega]$  for which  $S[\omega]_Q$  is not a DVR. By Proposition 2.1 and Lemma 3.2,  $R = J^{-1}$ , for  $J := J_1 \cap \dots \cap J_t \cap P$ . Now, as in the proof of Lemma 3.3,  $J_i^{-1}$  is generated as an  $S[\omega]$ -module by the set  $\{1, \frac{\lambda_i a_i}{\omega^{s(i, 1)}}, \dots, \frac{\lambda_i a_i^{e_i - 1}}{\omega^{s(i, e_i - 1)}}\}$ , where, for each  $i$ ,  $\lambda_i := \prod_{i \neq j=1}^r \lambda a_j^{e_j}$ . Thus  $K_i = (\omega^{p-s(i, 1)}, \omega^{p-s(i, 2)} a_i, \dots, a_i^{e_i - 1}) S[\omega]$ , for  $K_i := a_i^{e_i - 1} \cdot J_i^{-1}$  and  $1 \leq i \leq t$ . By Lemma 3.3,  $K_i \subseteq J_i$ , so upon setting  $I := K_1 \cap \dots \cap K_t \cap P$ , it follows from Lemma 3.5 that  $I^{-1}$  is a  $J^{-1}$ -module (since this holds locally for every height one prime in  $S[\omega]$ ). Taking  $M := I^{-1}$ , we will show that  $M$  is the required module. For this, we claim that  $\tilde{I} \subseteq B$  is a grade two perfect ideal. If the claim holds,  $1 = p.d._B(I) = p.d._B(I^{-1}) = p.d._B(M)$ . Thus  $depth_B(M) = dim(B) - 1$ , so  $depth_S(M) = dim(R)$ , which is what we want.

To prove the claim, we set  $\tilde{K} := \tilde{K}_1 \cap \dots \cap \tilde{K}_t$  and consider the short exact sequence

$$0 \longrightarrow B/\tilde{I} \longrightarrow B/\tilde{K} \oplus B/\tilde{P} \longrightarrow B/(\tilde{K} + \tilde{P}) \longrightarrow 0.$$

Since  $\tilde{K}$  is a grade two perfect ideal (by Lemma 3.4), the Depth Lemma and the Auslander-Buchsbaum formula imply that  $\tilde{I}$  is a grade two perfect ideal, once we show  $depth(B/(\tilde{K} + \tilde{P})) = dim(B) - 3$ . Set  $a := a_1^{e_1 - 1} \dots a_t^{e_t - 1}$ . We now argue that  $\tilde{K} + \tilde{P} = (a, p, W - h)$ . If we can show this, clearly  $depth(B/(\tilde{K} + \tilde{P})) = dim(B) - 3$  and we will have verified the claim. Take  $\tilde{k} \in \tilde{K}$  and consider its image  $k$  in  $K \subseteq S[\omega]$ . Select  $Q \subseteq S[\omega]$ , a height one prime. If  $Q = Q_i$ , for some  $1 \leq i \leq t$ , then  $k \in (a_i^{e_i - 1} J_i^{-1})_{Q_i} = a R_{Q_i}$ . If  $Q \neq Q_i$  for any  $1 \leq i \leq t$ , then clearly  $k \in a R_Q = R_Q$ . It follows that  $k \in a R \cap S[\omega]$ . In other words,  $k$  is integral over the principal ideal  $a S[\omega]$ . Therefore, the image of  $k$  in  $S[\omega]/(\omega - h, p) = S/pS$  is integral over the principal ideal generated by the image of  $a$ . Since  $S/pS$  is integrally closed, the image of  $k$  in  $S/pS$  is a multiple of the image of  $a$ . Therefore,  $\tilde{k} \in (a, p, W - h)$  in  $S[W]$ . It follows that  $\tilde{K} \subseteq (a, p, W - h)$ . Since  $a \in \tilde{K}$ , we obtain  $\tilde{K} + \tilde{P} = (a, p, W - h)$ , which is what we want. This completes the proof of Theorem 3.8.  $\square$

*Remark 3.9.* Of course if  $S$  is an unramified regular local ring,  $S$  fulfills our standing hypotheses, so Theorem 3.8 applies. However, the theorem also applies to certain ramified regular local rings. For instance, take  $T$  to be the ring  $\mathbb{Z}[X_1, \dots, X_d]$  localized at  $(p, X_1, \dots, X_d)$  and let  $H \in \mathbb{Z}[X_1, \dots, X_d]$  be any polynomial in  $(X_1, \dots, X_d)^2$  for which  $\mathbb{Z}_p[X_1, \dots, X_d]/(\overline{H})$  is an integrally closed domain. If we set  $S := T/(p - H)$ , then  $S$  is a ramified regular local ring and  $S/pS$  is an integrally closed domain.

We close with an example where  $R$  is not a free  $S$ -module, yet  $R$  admits a finite, birational module which is a free  $S$ -module. The example is an unramified variation of Koh’s Example (2.4).

**Example 3.10.** Let  $S$  be an unramified regular local ring having mixed characteristic 3 and take  $x, y \in S$  such that  $3, x, y$  form part of a regular system of parameters. Set  $a := xy^4 + 9$ ,  $b := x^4y + 9$  and  $f := ab^2$ , so  $\omega^3 = f = ab^2 = h^3 + 9g$ , for  $h = x^3y^2$ . From Lemmas 3.2 and 3.3 it follows that  $R = (Q \cap P)^{-1}$  for  $Q := (\omega, b)$  and  $P := (\omega - h, 3)$ . Set  $J := Q \cap P$ . We first show that  $R = J^{-1}$  is not a free  $S$ -module. Suppose to the contrary that  $J^{-1}$  is free over  $S$ . As in the proof of Theorem 3.8, set  $B := S[W]_{(N,W)}$  and use “tilde” to denote pre-images in  $B$ . Since  $J^{-1}$  is free over  $S$ , we have  $p.d._B(J^{-1}) = 1$ , so  $J^{-1}$  is a grade one perfect  $B$ -module. By [KU, Proposition 3.6],  $J$  is a grade one perfect  $B$ -module, so  $\tilde{J}$  is a grade two perfect ideal. On the other hand,  $depth_B(B/\tilde{J}) = 1 + depth_B(B/(\tilde{Q} + \tilde{P}))$ . But,  $\tilde{Q} + \tilde{P} = (W, x^4y, x^3y^2, 3)B$ , so  $B/(\tilde{Q} + \tilde{P}) = S/(3, x^4y, x^3y^2)S$ , which is easily seen to have depth equal to  $depth(S) - 3 = depth(B) - 4$ . This is a contradiction, so it must hold that  $R$  is not a free  $S$ -module.

Now,  $Q^{-1}$  is generated as an  $S[\omega]$ -module by  $\{1, \frac{ab}{\omega}\}$ . If we set  $K := b \cdot Q^{-1}$ , then  $K = (\omega^2, b)S[\omega]$ . The proof of Theorem 3.8 shows that  $M := (K \cap P)^{-1}$  is a finite, birational  $R$ -module satisfying  $depth_S(M) = dim(R)$ . In other words,  $M$  is an  $R$ -module which is free over  $S$ . To calculate a basis for  $M$ , one must calculate  $K \cap P$  and then use Proposition 2.1. We leave it to the reader to check that  $K \cap P = (\omega^2 - h^2 - 9x^2y^3, b(\omega - h), 3b)$ . Therefore,  $K \cap P = I_2(\phi)$  for

$$\phi = \begin{pmatrix} -b & 0 \\ \omega + h & -3 \\ -3x^2y^3 & \omega - h \end{pmatrix}.$$

The augmented matrix that determines  $(K \cap P)^{-1} = M$  is the  $3 \times 3$  matrix

$$\begin{pmatrix} -b & 0 & \omega \\ \omega + h & -3 & x^2y^3 \\ -3x^2y^3 & \omega - h & t \end{pmatrix},$$

where  $t$  is defined by the equation  $x^5y^5 = ab + 3t$ . By Proposition 2.1,  $M$  is generated as an  $S[\omega]$ -module by the set  $\{1, \gamma, \delta\}$ , for

$$\gamma := \frac{-3t - x^2y^3(\omega - h)}{\omega^2 - h^2 - 9x^2y^3} = \frac{\omega}{b}, \quad \delta := \frac{-bt + 3x^2y^3\omega}{b(\omega - h)} = \frac{\omega^2 + \omega h + h^2 + 9x^2y^3}{3b}.$$

If we show that  $\{1, \gamma, \delta\}$  also generate  $M$  as an  $S$ -module, then since they are clearly linearly independent over  $S$ , they form a basis for  $M$  as an  $S$ -module. To see that  $\{1, \gamma, \delta\}$  generate  $M$  as an  $S$ -module, it suffices to show that  $\omega, \omega \cdot \gamma$  and  $\omega \cdot \delta$  can be expressed as  $S$ -linear combinations of  $\{1, \gamma, \delta\}$ . This clearly holds for  $\omega$ . Using  $9x^2y^3 = bx^2y^3 - x^6y^4$ , we obtain

$$\omega \cdot \gamma = \frac{\omega^2}{b} = -x^2y^3 \cdot 1 - h \cdot \gamma + 3 \cdot \delta.$$

Since  $\omega^3 = h^3 + 9g$  and  $g = x^5y^5 + bxy^4 + b^2$ , we get

$$\omega \cdot \delta = (3xy^4 + 3b) \cdot 1 + 3x^2y^3 \cdot \gamma + h \cdot \delta,$$

and the example is complete.



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