

A LINEAR FUNCTION ASSOCIATED TO ASYMPTOTIC PRIME DIVISORS

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ABSTRACT. Let R be a Noetherian standard \mathbb{N}^d -graded ring and M, N finitely generated, \mathbb{N}^d -graded R -modules. Let I_1, \dots, I_s be finitely many homogeneous ideals of R . We show that there exist linear functions $f, g : \mathbb{N}^s \rightarrow \mathbb{N}^d$ such that the associated primes over R_0 of $[\text{Ext}^i(N, M/I_1^{n_1} \cdots I_s^{n_s} M)]_m$ and $[\text{Tor}_i(N, M/I_1^{n_1} \cdots I_s^{n_s} M)]_m$ are stable whenever $m \in \mathbb{N}^d$ satisfies $m \geq f(n_1, \dots, n_s)$ and $m \geq g(n_1, \dots, n_s)$, respectively.

1. INTRODUCTION

In this paper we bring together a couple of themes in the asymptotic theory of ideals. The first of these themes concerns the study of asymptotic prime divisors, i.e., the study of prime ideals associated to a family of ideals or modules indexed by increasing parameters. In [B], Brodmann showed that for a Noetherian ring R , an ideal $I \subseteq R$ and a finitely generated R -module M , the associated primes $\text{Ass}_R(M/I^n M)$ stabilize for n sufficiently large. This result can be derived from a stability result for graded modules: If R is a standard graded ring and M a finitely generated graded R -module, then $\text{Ass}_{R_0}(M_n)$ is stable for n large (see [Mc]). Brodmann's result was extended to finitely many ideals containing regular elements (and $M = R$) in [KMR]. Recently, Sharp and Kingsbury extended this last result to modules and an arbitrary finite collection of ideals (see [KS]). Inspired by this, in [W] the second author initiated a multigraded theory that extends the graded results in [Mc] and recovers cleanly the results from [KMR] and [KS].

The second theme in this paper is that of showing the existence of a linear function derived from properties of powers of ideals. One of the first (and most beautiful) of recent results along these lines is the main result of [Sw]. There, it is shown that if R is a Noetherian ring and $I \subseteq R$ is an ideal, then there exist k independent of n and a primary decomposition of I^n such that the radicals of the components in the decomposition raised to the kn th power belong to the corresponding components. This too was extended to many ideals in [Sh]. The linear functions in [Sw] and [Sh] are derived from properties of powers of the ideal. In [Ko] and [CHT] it is shown that, in the graded case, the Castelnuovo regularity of I^n is eventually given by a linear function. This is an example of a linear function dependent on the powers of a homogeneous ideal, but derived from properties of the

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grading. The linear functions we introduce in this paper are also power dependent and derived from the grading.

Suppose that R is a graded or multigraded ring, I_1, \dots, I_s homogeneous ideals, and M a finitely generated (appropriately) graded R -module. For each n_1, \dots, n_s , $M/I_1^{n_1} \cdots I_s^{n_s} M$ is a graded or multigraded R -module. Thus we can consider the graded pieces of these modules over the zero part of the ring. It follows from the main result in this paper that there exists a linear function h of n_1, \dots, n_s so that the R_0 primes associated to the graded components of $M/I_1^{n_1} \cdots I_s^{n_s} M$ stabilize for values of the grading larger than $h(n_1, \dots, n_s)$. In fact, it is not much harder to prove this result for certain homology modules derived from $I_1^{n_1} \cdots I_s^{n_s}$; so we do this instead. Our main result is facilitated by two new observations of independent interest. The first is that the associated primes of the homology modules derived from $I_1^{n_1} \cdots I_s^{n_s}$ stabilize for n_i sufficiently large. The second observation is the following: In the graded case, if A is a finitely generated graded (or multigraded) module over R , then an explicit value of m can be given beyond which the primes in $\text{Ass}_{R_0}(A_m)$ are stable.

2. NOTATION

Throughout this paper R will denote a Noetherian, commutative ring with identity. *Unless stated otherwise*, all R -modules will be finitely generated. We will deal with rings and modules graded by \mathbb{N}^d , where, by abuse of notation, \mathbb{N} denotes the set of nonnegative integers, and d is any fixed positive integer. For convenience, we may sometimes omit the reference to \mathbb{N}^d in our notation. In particular, if R is an \mathbb{N}^d -graded ring, then it should be assumed that any graded R -module or homogeneous ideal is graded by \mathbb{N}^d as well. We use the term “standard” in the sense of Stanley, i.e., a standard \mathbb{N}^d -graded ring is one that is generated in total degree one.

We will be working with d -tuples from \mathbb{N}^d and s -tuples from \mathbb{N}^s , the latter occurring almost always as superscripts in a power product of s ideals. Rather than burden the reader with excessive notation, we will simplify our notation and use $m \in \mathbb{N}^d$ or $n \in \mathbb{N}^s$ to indicate d -tuples or s -tuples. We will also abbreviate a power product of ideals $I_1^{n_1} \cdots I_s^{n_s}$ in R by I^n , for $n = (n_1, \dots, n_s) \in \mathbb{N}^s$. Thus, for example, if R is an \mathbb{N}^d -graded ring, $I_1, \dots, I_s \subseteq R$ homogeneous ideals, and M a graded R -module, then for $m \in \mathbb{N}^d$ and $n \in \mathbb{N}^s$ we will write $[I^n M]_m$ for the degree m component of the \mathbb{N}^d -graded module $I^n M$. We will use subscripts to denote components of our d -tuples or s -tuples. Thus m_i means the i th component of $m = (m_1, \dots, m_d) \in \mathbb{N}^d$. Superscripts will be used to indicate lists of d -tuples or s -tuples. In particular, we will let $\{e^j\}$ denote the standard basis elements in \mathbb{N}^d or \mathbb{N}^s .

We will often use the partial order on \mathbb{N}^d defined by $n \geq m$ if and only if $n_i \geq m_i$ for all $1 \leq i \leq d$. We will denote by $\sup\{n^1, \dots, n^t\}$ that d -tuple whose i th coordinate is $\max\{(n^1)_i, \dots, (n^t)_i\}$. Finally, for an \mathbb{N}^d -graded ring R , and an \mathbb{N}^d -graded R -module M , we will use the notation $\text{Ass}_R^+(M)$ to denote the set of relevant prime ideals in $\text{Ass}_R(M)$. That is, P belongs to $\text{Ass}_R^+(M)$ if and only if $P \in \text{Ass}_R(M)$ and $R_{e^i} \not\subseteq P$, for all $1 \leq i \leq d$.

3. ASYMPTOTIC STABILITY OF ASSOCIATED PRIMES
OF HOMOLOGY MODULES

In this section we will show that certain homology modules parameterized by $I_1^{n_1} \cdots I_s^{n_s}$ enjoy asymptotic stability of associated primes. R will denote a (not necessarily graded) ring containing the ideals I_1, \dots, I_s . As stated above, for n in \mathbb{N}^s , we will write I^n for $I_1^{n_1} \cdots I_s^{n_s}$. For R -modules $A' \subseteq A, B' \subseteq B, C' \subseteq C$, we will consider homology modules of complexes of the form

$$A/I^n A' \rightarrow B/I^n B' \rightarrow C/I^n C'.$$

For n large, these homology modules can be written as $(U + I^{n-q}V)/I^{n-q}W$ where U, V, W are submodules of a finitely generated R -module T and $W \subseteq V$. Our task will be to show that $Ass_R((U + I^n V)/I^n W)$ is stable for $n \in \mathbb{N}^s, n \gg 0$.

We start by reminding the reader of the following results in the theory of regular ideals.

Lemma 3.1. *Let R, T, W be as above. Suppose that I_1, \dots, I_s, J are ideals of R .*

- (1) *If I_j contains a T -regular element, then there exists $r_j \in \mathbb{N}$ such that for all $n \in \mathbb{N}^s$ satisfying $n_j \geq r_j, (I^{n+e_j} W :_T I_j) = I^n W$.*
- (2) *If J contains a T -regular element, then there exists $k \in \mathbb{N}^s$ such that for all $n \in \mathbb{N}^s, n \geq k, (I^n W :_T J) = I^{n-k}(I^k W :_T J)$.*

Proof. For part (1), if $T = R$, this is Proposition 1.4 of [KMR]. The extension to the module case is given by Lemma 1.3 of [KS]. For the second part, if $x \in J$ is T -regular, then summing $(I^n W :_T x)$ over $n \in \mathbb{N}^s$ gives a finite module over the ring obtained by summing I^n over n . Thus the submodule obtained by summing $(I^n W :_T J)$ is finite as well, and so the result follows by the \mathbb{N}^s -graded form of the Artin-Rees lemma. □

Lemma 3.2. *Let R be a Noetherian ring, I_1, \dots, I_s ideals of R and $U, V \subseteq T$ finitely generated R -modules. Fix a prime ideal P of R such that P contains at least one I_j . Then there exists $n^0 \in \mathbb{N}^s$ such that $(U + I^n V) \cap (0 :_T P) = U \cap (0 :_T P)$ for all $n \geq n^0$.*

Proof. Write $Z := (0 :_T P)$. By the multigraded form of the Artin-Rees lemma, there exists $q \in \mathbb{N}^s$ such that $I^n V \cap (U + Z) = I^{n-q}(I^q V \cap (U + Z))$, for all $n \geq q$. Set $n^0 := q + (1, \dots, 1)$, take $n \geq n^0$ and let $u + v = z \in (U + I^n V) \cap Z$ with $u \in U, v \in I^n V$ and $z \in Z$. Then $-u + z \in (U + Z) \cap I^n V = I^{n-q}(I^q V \cap (U + Z))$. Since $I_j \cdot Z = 0$, it follows that $v = -u + z$ is contained in U . Thus, $u + v$ belongs to U , and therefore to $U \cap Z$, since we also have $z \in Z$. □

For the next lemma, we note that for finite modules $U, V, W \subseteq T$, with $W \subseteq V$, the union over $n \in \mathbb{N}^s$ of the primes in $Ass((U + I^n V)/I^n W)$ is finite since these are included among the primes in $\bigcup_{n \in \mathbb{N}^s} Ass(T/I^n W)$, which is finite (see [KS]). We will also choose $q \in \mathbb{N}^s$ satisfying three conditions. That we can meet the first two conditions follows from Lemma 3.1 and the multigraded form of the Artin-Rees lemma.

Lemma 3.3. *Let R be a Noetherian ring, I_1, \dots, I_s ideals of R and $W, V, U \subseteq T$ finitely generated R -modules, with $W \subseteq V$. Let $q \in \mathbb{N}^s$ satisfy the following conditions:*

- (1) For each $1 \leq j \leq s$, $(I^{n+e^j}W :_T I_j) \subseteq I^nW + N_j$ for all $n \in \mathbb{N}^s$ satisfying $n \geq q$, where N_j denotes the set of elements of T annihilated by some power of I_j .
- (2) For all $1 \leq j \leq s$, $I^nT \cap N_j = 0$, for all $n \in \mathbb{N}^s$, $n \geq q$.
- (3) For any $n \geq q$ and $P \in \text{Ass}((U+I^nV)/I^nW)$, if $I_j \subseteq P$ for some $1 \leq j \leq s$, then the conclusion of Lemma 3.2 holds.

For any $n \geq q$ and $P \in \text{Ass}((U + I^nV)/I^nW)$, if $P = (I^nW : c)$, then for any $1 \leq j \leq s$, either $P = (I^{n+e^j}W : I_jc)$ or $P = (I^{n+e^j}W : a) = (0 : a)$, with $a \in U \cap (0 :_T P)$.

Proof. We first note that if $I_j \not\subseteq P$, then clearly $P = (I^{n+e^j}W : I_jc)$. Suppose then that $I_j \subseteq P$ and $(I^{n+e^j}W : I_jc)$ strictly contains P . Take $r \in (I^{n+e^j}W : I_jc) \setminus P$. Thus, $rc \in (I^{n+e^j}W :_T I_j)$. By (1), we may write $rc = b + a$, with $b \in I^nW$ and $a \in N_j$. Thus, $P = (I^nW : c) = (I^nW : rc) = (I^nW : a)$. Because $a \in N_j$, we have by condition (2) that $P = (I^nW : a) = (0 : a)$. It follows readily from this that $P = (I^{n+e^j}W : a)$. Moreover, $a = rc - b \in (U + I^nV) \cap (0 :_T P)$. Thus by (3), we have $a \in U \cap (0 :_T P)$, which is what we want. □

The following proposition is the main result of this section. Note that if we take α and β in the proposition to be the zero maps, then we recover the results from [KMR] and [KS] cited in the introduction.

Proposition 3.4. *Let R be a Noetherian ring, $I_1, \dots, I_s \subseteq R$ ideals. Consider the complex $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ of finitely generated R -modules. Suppose $A' \subseteq A$, $B' \subseteq B$ and $C' \subseteq C$ are submodules satisfying $\alpha(A') \subseteq B'$ and $\beta(B') \subseteq C'$. For $n \in \mathbb{N}^s$, let $H(n)$ denote the homology of the induced complex*

$$A/I^n A' \xrightarrow{\alpha(n)} B/I^n B' \xrightarrow{\beta(n)} C/I^n C'.$$

Then there exists $k \in \mathbb{N}^s$ such that $\text{Ass}_R(H(n))$ is stable for all $n \in \mathbb{N}^s$, $n \geq k$.

Proof. We first note that there exists $q \in \mathbb{N}^s$ and finitely generated R -modules $U, V, W \subseteq T$ with $W \subseteq V$ such that $H(n)$ is isomorphic to $(U + I^{n-q}V)/I^{n-q}W$ for all $n \in \mathbb{N}^s$, $n \geq q$. Indeed, extending the argument in [T, Proposition 3] to a finite collection of ideals, we take $q \in \mathbb{N}^s$ such that $I^n C' \cap \text{im}(\beta) = I^{n-q}(I^q C' \cap \text{im}(\beta))$ holds for $n \geq q$. It follows that for such n ,

$$H(n) = (\ker(\beta) + I^{n-q}V')/(\text{im}(\alpha) + I^{n-q}W'),$$

where $V' = \beta^{-1}(I^q C')$ and $W' = I^q B'$. For $T := B/\text{im}(\alpha)$, $U := \ker(\beta)/\text{im}(\alpha)$, $V := (V' + \text{im}(\alpha))/\text{im}(\alpha)$ and $W := (W' + \text{im}(\alpha))/\text{im}(\alpha)$, $H(n)$ is isomorphic to $(U + I^{n-q}V)/I^{n-q}W$. Thus, it suffices to take any R -modules $U, V, W \subseteq T$, with $W \subseteq V$, and show that there exists $k \in \mathbb{N}^s$ such that $\text{Ass}_R((U + I^nV)/I^nW)$ is stable for $n \in \mathbb{N}^s$, $n \geq k$.

To start, let P_1, \dots, P_t denote the primes in $\bigcup_{n \in \mathbb{N}^s} \text{Ass}((U + I^nV)/I^nW)$. Take $q \in \mathbb{N}^s$ as in Lemma 3.3. If $(U + I^nV)/I^nW = 0$, for all $n \in \mathbb{N}^s$, $n \geq q$, then clearly $\text{Ass}_R((U + I^nV)/I^nW)$ is independent of n . If not, there exists $p \geq q$ with $(U + I^pV)/I^pW \neq 0$. Increasing q if necessary, we may take $p = q$ and also assume that $(U + I^qV)/I^qW \neq 0$. Now take $P \in \text{Ass}_R((U + I^qV)/I^qW)$; so $P = P_i$, for some $1 \leq i \leq t$. Write $P = (I^qW : c)$, with $c \in U + I^qV$. By Lemma 3.3, for any $1 \leq j \leq s$, either $P = (I^{q+e^j}W : I_jc)$ or $P = (I^{q+e^j}W : a) = (0 : a)$, with

$a \in U \cap (0 :_T P)$. It follows immediately that $P \in \text{Ass}_R((U + I^{q+e^j} V)/I^{q+e^j} W)$ for all j . Continuing, we get $\text{Ass}_R((U + I^n V)/I^n W) \subseteq \text{Ass}_R((U + I^{n^2} V)/I^{n^2} W)$ whenever $q \leq n^1 \leq n^2 \in \mathbb{N}^s$. It follows from this and the fact that the union of the primes in $\text{Ass}_R((U + I^n V)/I^n W)$ is finite that there exists $k \in \mathbb{N}^s$ such that $\text{Ass}_R((U + I^n V)/I^n W)$ is stable for $n \geq k$. \square

As a corollary we have asymptotic stability of the associated primes of Ext and Tor modules derived from $M/I^n M$.

Corollary 3.5. *Let R be a Noetherian ring, I_1, \dots, I_s ideals of R , N, M finitely generated R -modules, and $M' \subseteq M$ a submodule. Fix $i \geq 0$. Then there exists $k \in \mathbb{N}^s$ such that for all $n \in \mathbb{N}^s$, $n \geq k$, the sets $\text{Ass}_R(\text{Ext}_R^i(N, M/I^n M'))$ and $\text{Ass}_R(\text{Tor}_i^R(N, M/I^n M'))$ are stable.*

Proof. Since the proofs for Ext and Tor are essentially the same, we prove the corollary for the Ext modules. But this is clear, since if we take a free resolution of N by finitely generated free R -modules and “Hom” this resolution into $M/I^n M'$, at the i th place in the resulting complex, we have a short complex of the form $A/I^n A' \rightarrow B/I^n B' \rightarrow C/I^n C'$. Hence, the previous proposition applies. \square

Remark 3.6. We cannot expect asymptotic stability of primes associated to the Ext modules if we interchange the arguments N and $M/I^n M'$. Indeed, [Si] provides an example of a local cohomology module $H_J^i(N)$ having infinitely many associated primes. Since the associated primes in a direct limit of modules are contained in the union of the associated primes of the modules comprising the limit, it follows that in this case the union over $t \in \mathbb{N}$ of $\text{Ass}_R(\text{Ext}_R^i(R/J^t, N))$ is not even finite.

4. A THRESHOLD FOR STABILITY OF ASYMPTOTIC PRIMES

Throughout this section R will be a Noetherian standard \mathbb{N}^d -graded ring and A will be a finitely generated graded R -module. In [W], the second author showed that there exists $k \in \mathbb{N}^d$ such that $\text{Ass}_{R_0}(A_m)$ is stable for all $m \in \mathbb{N}^d$, $m \geq k$. In this section we refine the results in [W] in order to give an explicit k . We begin by recalling two lemmas.

Lemma 4.1 ([W, 3.1]). *There exists $l \in \mathbb{N}^d$ such that for each $1 \leq i \leq d$, we have $(0 :_A R_{e^i}) \cap A_m = 0$ whenever $m \in \mathbb{N}^d$ satisfies $m_i \geq l_i$.*

Lemma 4.2 ([W, 3.2]). *Suppose $P \in \text{Ass}_{R_0}(A_m)$ for some $m \in \mathbb{N}^d$, $P = (0 :_{R_0} x)$. Then there exists a homogeneous element $a \in R$ such that $\mathcal{P} := (0 :_R ax)$ is prime (so \mathcal{P} belongs to $\text{Ass}_R(A)$) and $\mathcal{P} \cap R_0 = P$.*

We omit the proofs, but would like to point out that the statements and proofs in this section tacitly allow for the possibility that for some m , and hence all large m , $A_m = 0$ and $\text{Ass}_{R_0}(A_m) = \emptyset$.

Lemma 4.3. *Take $l \in \mathbb{N}^d$ as in Lemma 4.1. Suppose that $m \in \mathbb{N}^d$ satisfies $m_i \geq l_i$ for some $1 \leq i \leq d$. If $P \in \text{Ass}_{R_0}(A_m)$, then $P = \mathcal{P} \cap R_0$ for some $\mathcal{P} \in \text{Ass}_R(M)$ such that $R_{e^i} \not\subseteq \mathcal{P}$. Thus, if $m \geq l$, then every element of $\text{Ass}_{R_0}(A_m)$ is of the form $\mathcal{P} \cap R_0$, for $\mathcal{P} \in \text{Ass}_R^+(A)$.*

Proof. Fix m and, without loss of generality, assume $i = 1$, so that $m_1 \geq l_1$. Take P in $\text{Ass}_{R_0}(A_m)$ and write $P = (0 :_{R_0} x)$, for $x \in A_m \setminus \{0\}$. By Lemma 4.2, we can find homogeneous $a \in R$ such that $\mathcal{P} := (0 :_R ax)$ is a prime ideal and contracts to

P . Since $\deg(ax) \geq m$ and $m_1 \geq l_1$, Lemma 4.1 gives $(0 :_A R_{e^1}) \cap A_{\deg(ax)} = 0$. Thus $ax \notin (0 :_A R_{e^1})$. So $R_{e^1} \not\subseteq \mathcal{P}$ and the first statement holds. The second statement readily follows. \square

Lemma 4.4. *Suppose $\mathcal{P} = (0 :_R x)$, $x \in A_t$, $t \in \mathbb{N}^d$. If $R_{e^i} \not\subseteq \mathcal{P}$, then $\mathcal{P} \cap R_0$ belongs to $Ass_{R_0}(A_{t+\alpha e^i})$ for all $\alpha \in \mathbb{N}$. Thus, if $\mathcal{P} \in Ass_R^+(A)$, then $\mathcal{P} \cap R_0$ belongs to $Ass_{R_0}(A_m)$ for all $m \in \mathbb{N}^d$ satisfying $m \geq t$.*

Proof. We assume, without loss of generality, that $R_{e^1} \not\subseteq \mathcal{P}$. Take $a \in R_{e^1}$, a not in \mathcal{P} . Then $\mathcal{P} = (0 :_R x) = (0 :_R a^\alpha x)$ for all $\alpha \in \mathbb{N}$, which immediately gives what we want. The second statement also now follows at once. \square

We now can give the required $k \in \mathbb{N}^d$. Note that the proposition also describes explicitly the stable set of primes.

Proposition 4.5. *Let R be a Noetherian standard \mathbb{N}^d -graded ring and let A be a finitely generated \mathbb{N}^d -graded R -module. Let $Ass_R^+(A) = \{\mathcal{P}_1, \dots, \mathcal{P}_r\}$ and write $\mathcal{P}_i = (0 :_R c_i)$ with $\deg(c_i) = q^i \in \mathbb{N}^d$, $1 \leq i \leq r$. Take l as in Lemma 4.1. If we set $k := \sup\{q^1, \dots, q^r, l\}$, then for all $m \geq k$, we have $Ass_{R_0}(A_m) = Ass_{R_0}(A_k) = \{\mathcal{P}_1 \cap R_0, \dots, \mathcal{P}_r \cap R_0\}$.*

Proof. Since $k \geq q^i$ for all i , $\mathcal{P}_i \cap R_0$ belongs to $Ass_{R_0}(A_m)$ for all $m \geq k$, by Lemma 4.4. On the other hand, since $k \geq l$, Lemma 4.3 shows that for $m \geq k$, every element of $Ass_{R_0}(A_m)$ is of the form $\mathcal{P}_i \cap R_0$, for some $1 \leq i \leq r$. \square

Remark 4.6. Of course the k in Proposition 4.5 need not be optimal, but we cannot do any better in this generality. On the one hand, if F is a field and $R = A$ is any standard \mathbb{N} -graded F -algebra with $R_n \neq 0$ for all n , then $Ass_{R_0}(A_n)$ is the zero ideal of F for all n , regardless of the degrees of the annihilators of the elements in $Ass_R^+(A)$. On the other hand, let $p \neq q \in \mathbb{Z}$ be primes and let R be a polynomial ring over \mathbb{Z} , with the usual grading and with at least three variables. Take $F, G \in R$ such that F and G are homogeneous, G is irreducible modulo p , F is irreducible modulo q , and $(F, G)R$ is a prime ideal. For example, one could take $R := \mathbb{Z}[S, T, U, V, W, X, Y, Z]$, $F := ST^d + UV^d$ and $G := WX^e + YZ^e$. Then it is not hard to see that for $A := R/(pF, qG)R$, $P := (p, G)R$ and $Q := (q, F)R$, $P = ann(qF)$, $Q = ann(pG)$, $(F, G) = ann(pq)$ and $Ass_R(A) = Ass_R^+(A) = \{P, Q, (F, G)\}$. Thus, since $p\mathbb{Z}$ is not associated to A_n for $n < \deg(F)$ and $q\mathbb{Z}$ is not associated to A_n for $n < \deg(G)$, Proposition 4.5 implies that $Ass_{R_0}(A_n)$ does not stabilize until n reaches k , which is $\max\{\deg(F), \deg(G)\}$. Note also by Proposition 4.5 that the stable value of $Ass_{R_0}(A_n) = \{p\mathbb{Z}, q\mathbb{Z}, (0)\}$.

5. THE MAIN RESULTS

In this section we will prove the results alluded to in the introduction (and stated explicitly in the abstract). We start with a Noetherian standard \mathbb{N}^d -graded ring R and homogeneous ideals $I_1, \dots, I_s \subseteq R$. As in section 3, our main result will be given for the homology modules $H(n)$ derived from complexes of the form $A/I^n A' \rightarrow B/I^n B' \rightarrow C/I^n C'$, where all modules are now \mathbb{N}^d -graded. As before, we will exploit the fact that for large n the $H(n)$ have the form $(U + I^{n-q}V)/I^{n-q}W$, which are also \mathbb{N}^d -graded. By Proposition 4.5, we therefore need to control the degrees of various elements that are annihilated in these modules as $n \in \mathbb{N}^d$ varies. Most of the ingredients for this are in place; we just need to organize the details.

We begin with a lemma recording a couple of facts about the types of functions we are interested in.

Lemma 5.1. *Let R be a Noetherian standard \mathbb{N}^d -graded ring, I_1, \dots, I_s homogeneous ideals and T a finitely generated \mathbb{N}^d -graded R -module.*

- (1) *Let $C \subseteq T$ be a graded submodule and fix $k \in \mathbb{N}^s$. Then there is a linear function $h : \mathbb{N}^s \rightarrow \mathbb{N}^d$ and, for each $n \in \mathbb{N}^s$ with $n \geq k$, a set of homogeneous generators for $I^{n-k}C$ of degrees less than or equal to $h(n)$.*
- (2) *Fix $n^0 \in \mathbb{N}^s$ and set $S := \{n \in \mathbb{N}^s \mid n \geq n^0\}$. For $1 \leq j \leq s$ and $0 \leq i < (n^0)_j$, set $S_{j,i} := \{n \in \mathbb{N}^s \mid n_j = i\}$. Suppose that $g : S \rightarrow \mathbb{N}^d$ and $g_{j,i} : S_{j,i} \rightarrow \mathbb{N}^d$ are given by linear functions. Then there exists a linear function $f : \mathbb{N}^s \rightarrow \mathbb{N}^d$ such that $f(n) \geq g(n)$, if $n \in S$ and $f(n) \geq g_{j,i}(n)$, if $n \in S_{j,i}$.*

Proof. For the first statement, fix homogeneous generating sets for I_1, \dots, I_s and C . Take a^1, \dots, a^s, b in \mathbb{N}^d such that a^j is greater than or equal to the degrees of the generators of I_j and b is greater than or equal to the degrees of the generators of C . Let A be the $s \times d$ matrix over \mathbb{N} whose rows are the a^j , $1 \leq j \leq s$. Then for all $r \in \mathbb{N}^s$, $I^r C$ has a set of generators whose degrees are bounded by $r \cdot A + b$. Thus, for $n \geq k$, $I^{n-k}C$ is generated in degree $h(n) := (n - k) \cdot A + b$ or less.

For the second statement, let $A, A(j, i)$ be $s \times d$ matrices over \mathbb{N} and $b, b^{j,i} \in \mathbb{N}^d$ be such that $g(n) = n \cdot A + b$ for $n \in S$ and $g_{j,i}(n) = n \cdot A(j, i) + b^{j,i}$ for $n \in S_{j,i}$. If we set $C := A + \sum A(j, i)$ and $d := b + \sum b^{j,i}$, then $f(n) := n \cdot C + d$ has the required property. □

Proposition 5.2. *Let R be a Noetherian standard \mathbb{N}^d -graded ring, I_1, \dots, I_s homogeneous ideals, and T a finitely generated \mathbb{N}^d -graded R -module. Let U, V, W be \mathbb{N}^d -graded submodules of T such that $W \subseteq V$. Then there exists $n^0 \in \mathbb{N}^s$ and a linear function $g : \mathbb{N}^s \rightarrow \mathbb{N}^d$ such that for all $n \in \mathbb{N}^s$, $n \geq n^0$,*

$$\text{Ass}_{R_0}([(U + I^n V)/I^n W]_m)$$

is stable for all $m \in \mathbb{N}^d$, $m \geq g(n)$.

Proof. We adopt the notation of Proposition 4.5 with $A(n) := (U + I^n V)/I^n W$. We start by taking $p \in \mathbb{N}^s$ such that $\text{Ass}_R(A(n))$ is stable for $n \in \mathbb{N}^s$, $n \geq p$, which we can do by the proof of Proposition 3.4. In particular, $\text{Ass}_R^+(A(n))$ is independent of n , for $n \geq p$. Let $\{\mathcal{P}_1, \dots, \mathcal{P}_r\}$ be this latter set and write $\mathcal{P}_i := \mathcal{P}_i \cap R_0$, for $1 \leq i \leq r$. Then (with $n \geq p$), $\{\mathcal{P}_1, \dots, \mathcal{P}_r\}$ is the stable set of primes $\text{Ass}_{R_0}([A(n)]_m)$, for $m \gg 0$, by Proposition 4.5.

Now take q satisfying conditions (1)–(3) in Lemma 3.3. Without loss of generality, we may assume $p = q$. Fix $1 \leq i \leq r$ and write $\mathcal{P}_i = (I^p W : c_i)$ with $c_i \in U + I^p V$. Suppose there exists $t^i \in \mathbb{N}^s$ such that $\mathcal{P}_i = (I^{p+t^i} W : a) = (0 : a)$, with $a \in U \cap (0 :_T \mathcal{P}_i)$ homogeneous. Then for $n \in \mathbb{N}^s$, $n \geq p + t^i$, $\mathcal{P}_i = (I^n W : a)$ and the degree of a is independent of n . If no such t^i exists, then it follows from Lemma 3.3 that for all $r^i \in \mathbb{N}^s$, $\mathcal{P}_i = (I^{p+r^i} W : I^{r^i} c_i)$. Thus for $n \in \mathbb{N}^s$, $n \geq p$, $\mathcal{P}_i = (I^n W : c_i(n))$ where $c_i(n)$ is one of the homogeneous generators of $I^{n-p} \cdot c_i$ chosen together with a linear function $h_i : \mathbb{N}^s \rightarrow \mathbb{N}^d$ satisfying $\deg(c_i(n)) \leq h_i(n)$, which can be done by Lemma 5.1.(1). In either case, we have for each i , $k_i \in \mathbb{N}^s$, a linear function $h_i : \mathbb{N}^s \rightarrow \mathbb{N}^d$ and elements $c_i(n) \in U + I^n V$, such that for $n \geq k^i$, $\mathcal{P}_i = (I^n W : c_i(n))$ and $\deg(c_i(n)) \leq h_i(n)$.

We now find a linear function $l : \mathbb{N}^s \rightarrow \mathbb{N}^d$ such that for all i , $[(0 :_{A(n)} R_{e^i})]_m = 0$, whenever $m \in \mathbb{N}^d$ satisfies $m \geq l(n)$ (and $n \in \mathbb{N}^s$ is sufficiently large). Set $R_e := R_{e^1} \cdots R_{e^d}$ and notice that since $(0 :_{A(n)} R_{e^i}) \subseteq (0 :_{A(n)} R_e) \subseteq (0 :_{T/I^n W} R_e)$, it will suffice to find $b \in \mathbb{N}^s$ and a linear function $l : \mathbb{N}^s \rightarrow \mathbb{N}^d$ such that for all $n \geq b$, $[(I^n W :_T R_e)]_m = [I^n W]_m$, for $m \geq l(n)$. Let N denote the elements in T annihilated by a power of R_e and apply Lemma 3.1 to T/N to find $b \in \mathbb{N}^s$ such that for all $n \geq b$, $(I^n W :_T R_e) = I^{n-b}(I^b W :_T R_e) + N \cap (I^n W :_T R_e)$. By Lemma 5.1(1), there exists a linear function $h : \mathbb{N}^s \rightarrow \mathbb{N}^d$ such that $I^{n-b}(I^b W :_T R_e)$ is generated in degrees less than or equal to $h(n)$. Notice that if $x \in I^{n-b}(I^b W :_T R_e)$ and $\deg(x) \geq h(n) + (1, \dots, 1)$, then $x \in I^n W$. Now take $t \geq (1, \dots, 1) \in \mathbb{N}^d$ such that $[N]_m = 0$ for $m \geq t$. Then if we set $l(n) := h(n) + t$, it follows that for all $n \geq b$, $[(I^n W :_T R_e)]_m = [I^n W]_m$ for $m \geq l(n)$. In particular, for any such n , $[(0 :_{A(n)} R_{e^i})]_m = 0$, for all i and all $m \geq l(n)$.

Finally, let $g : \mathbb{N}^s \rightarrow \mathbb{N}^d$ be any linear function satisfying

$$g(n) \geq \sup\{h_1(n), \dots, h_r(n), l(n)\}.$$

Then for all $n \in \mathbb{N}^s$, $n \geq n^0 := \sup_i\{k^i, b\}$ it follows from Proposition 4.5 that $Ass_{R_0}([A(n)]_m)$ is stable for all $m \in \mathbb{N}^d$, $m \geq g(n)$, which is what we want. \square

We may now state and prove the main result in this paper.

Theorem 5.3. *Let R be a Noetherian standard \mathbb{N}^d -graded ring, $I_1, \dots, I_s \subseteq R$ homogeneous ideals and $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ a complex of finitely generated graded R -modules with degree preserving homomorphisms. Suppose $A' \subseteq A$, $B' \subseteq B$ and $C' \subseteq C$ are graded submodules satisfying $\alpha(A') \subseteq B'$ and $\beta(B') \subseteq C'$. For $n \in \mathbb{N}^s$, let $H(n)$ denote the homology of the induced complex*

$$A/I^n A' \xrightarrow{\alpha(n)} B/I^n B' \xrightarrow{\beta(n)} C/I^n C'.$$

Then there exists a linear function $f : \mathbb{N}^s \rightarrow \mathbb{N}^d$ such that for all $n \in \mathbb{N}^s$, $Ass_{R_0}([H(n)]_m)$ is stable for $m \in \mathbb{N}^d$, $m \geq f(n)$.

Proof. As in the proof of the Proposition 3.4, there exists $q \in \mathbb{N}^s$ such that for $n \geq q$, $H(n)$ is isomorphic to a module of the form $(U + I^{n-q}V)/I^{n-q}W$, which in our present case is also \mathbb{N}^d -graded. Thus, by the previous proposition, there is $n^0 \in \mathbb{N}^s$ and a linear function $g : \mathbb{N}^s \rightarrow \mathbb{N}^d$ such that for all $n \in \mathbb{N}^s$, $n \geq n^0$, $Ass_{R_0}([H(n)]_m)$ is stable for $m \in \mathbb{N}^d$, $m \geq g(n)$.

To finish, we must find a linear function $f(n)$ that works for all $n \in \mathbb{N}^s$, not just sufficiently large n . We prove this by induction on s , assuming that we have n^0 and $g(n)$ as above. If $s = 1$, then since $g(n)$ accounts for all but finitely many values of $n \in \mathbb{N}$, we may certainly find a linear function f such that for all $n \in \mathbb{N}$, $Ass_{R_0}([H(n)]_m)$ is stable for $m \in \mathbb{N}^d$, $m \geq f(n)$. Now suppose the result holds for $s - 1$ ideals and all \mathbb{N}^d -graded complexes of the required type. For each $1 \leq j \leq s$ and $0 \leq i < (n^0)_j$, we apply the induction hypothesis with A' , B' and C' replaced by $A'' := I_j^i A'$, $B'' := I_j^i B'$ and $C'' := I_j^i C'$, taking the ideals $I_1, \dots, I_{j-1}, I_{j+1}, \dots, I_s$. For $t \in \mathbb{N}^{s-1}$, let $H_{j,i}(t)$ denote the resulting homology of the complex $A/J_j^t A'' \rightarrow B/J_j^t B'' \rightarrow C/J_j^t C''$, where we are writing J_j^t for the power product $I_1^{t_1} \cdots I_{j-1}^{t_{j-1}} I_{j+1}^{t_{j+1}} \cdots I_s^{t_s}$. Then there exist linear functions $g_{j,i} : \mathbb{N}^{s-1} \rightarrow \mathbb{N}^d$ such that for all $t \in \mathbb{N}^{s-1}$, $Ass_{R_0}([H_{j,i}(t)]_m)$ is stable for $m \geq g_{j,i}(t)$. Set $S := \{n \in \mathbb{N}^s \mid n \geq n^0\}$ and for $1 \leq j \leq s$ and $0 \leq i < (n^0)_j$, set $S_{j,i} :=$

$\{n \in \mathbb{N}^s \mid n_j = i\}$. For each j, i let $g'_{j,i} : S_{j,i} \rightarrow \mathbb{N}^d$ denote the function defined by $g'_{j,i}(n) = g_{j,i}(n_1, \dots, n_{j-1}, n_{j+1}, \dots, n_d)$, for $n \in S_{j,i}$. By Lemma 5.1(2) there exists a linear function $f : \mathbb{N}^s \rightarrow \mathbb{N}^d$ that dominates g restricted to S and the $g'_{j,i}$. It now follows that for all $n \in \mathbb{N}^s$, $\text{Ass}_{R_0}([H(n)]_m)$ is stable for $m \in \mathbb{N}^d$, $m \geq f(n)$. This completes the proof of the theorem. \square

Our promised results follow as an immediate corollary.

Corollary 5.4. *Let R be a Noetherian standard \mathbb{N}^d -graded ring, N, M finitely generated, \mathbb{N}^d -graded R -modules and $M' \subseteq M$ a graded submodule. Let I_1, \dots, I_s be homogeneous ideals of R . Then there exist linear functions $f, g : \mathbb{N}^s \rightarrow \mathbb{N}^d$ such that for all $n \in \mathbb{N}^s$,*

$$\text{Ass}_{R_0}([\text{Ext}^i(N, M/I^n M')]_m) \text{ and } \text{Ass}_{R_0}([\text{Tor}_i(N, M/I^n M')]_m)$$

are stable for $m \in \mathbb{N}^d$, $m \geq f(n)$ and $m \geq g(n)$, respectively.

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