NONLINEAR STABILITY OF PERIODIC TRAVELING WAVE SOLUTIONS OF THE GENERALIZED KORTEWEG–DE VRIES EQUATION[*]

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Abstract. In this paper, we study the orbital stability for a four-parameter family of periodic stationary traveling wave solutions to the generalized Korteweg–de Vries equation $u_t = u_{xxx} + f(u)_x$. In particular, we derive sufficient conditions for such a solution to be orbitally stable in terms of the Hessian of the classical action of the corresponding traveling wave ordinary differential equation restricted to the manifold of periodic traveling wave solutions. We show this condition is equivalent to the solution being spectrally stable with respect to perturbations of the same period in the case when $f(u) = u^2$ (the Korteweg–de Vries equation) and in neighborhoods of the homoclinic and equilibrium solutions if $f(u) = u^{p+1}$ for some $p \geq 1$.

Key words. generalized Korteweg–de Vries equation, periodic waves, orbital stability

AMS subject classifications. 35Q53, 35B10, 35B35

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1. Introduction. This paper concerns the stability analysis of periodic traveling wave solutions of the generalized Korteweg–de Vries (gKdV) equation

$$(1.1) \qquad u_t = u_{xxx} + f(u)_x,$$

where f is a sufficiently smooth nonlinearity satisfying certain convexity assumptions. Probably the most famous equation among this family is given by $f(u) = u^2$, in which case (1.1) corresponds to the Korteweg–de Vries (KdV) equation. The KdV serves as an approximate description of small amplitude waves propagating in a weakly dispersive media. Other choices of the nonlinearity f arise in various applications, such as internal waves and plasmas. Thus, to ensure general application we find it beneficial to consider general nonlinearities in (1.1).

It is well known that (1.1) admits traveling wave solutions of the form

$$(1.2) \qquad u(x,t) = u_c(x + ct), \quad x \in \mathbb{R}, \quad t \in \mathbb{R},$$

for wave speeds $c > 0$. Historically, there has been much interest in the stability of traveling solitary waves of the form (1.2) where the profile u_c decays exponentially to zero as its argument becomes unbounded. Such waves were initially discovered by Russell in the case of the KdV where the traveling wave is termed a soliton. While (1.1) does not in general possess exact "soliton" solutions, which require complete integrability of the partial differential equation (PDE), exponentially decaying traveling wave solutions still exist. Moreover, the stability of such solitary waves is well understood and dates back to the pioneering work of Benjamin (1972), which was then further developed by Bona (1975), Grillakis (1990), Grillakis, Shatah, and Strauss (1987), Bona, Souganidis, and Strauss (1987), Pego and Weinstein (1994, 1992), Weinstein (1985, 1987), and many others. In this theory, it is shown that traveling

solitary waves of (1.1) are orbitally stable if the solitary wave stability index

$$
(1.3) \qquad \frac{\partial}{\partial c} \int_{-\infty}^{\infty} u_c^2 \, dx
$$

is positive, and is orbitally unstable if this index is negative. In the case where (1.1) has a power-law nonlinearity $f(u) = u^{p+1}$, the sign of this stability index is positive if $p < 4$ and is negative if $p > 4$. Moreover, in the work of Pego and Weinstein (1994, 1992) it was shown that the mechanism for this instability is as follows: linearizing the traveling wave PDE

$$
(1.4) \qquad u_t = u_{xxx} + f(u)_x - cu_x,
$$

which is satisfied by the traveling solitary wave profile, about the solution u_c and taking the Fourier transform in time leads to a spectral problem of the form $\partial_x \mathcal{L}[u_c]v = \mu v$ considered on the real Hilbert space $L^2(\mathbb{R})$, where $\mathcal{L}[u_c]$ is a second order self-adjoint differential operator with asymptotically constant coefficients. The authors then make a detailed study of the Evans function $D(\mu)$, which is an analytic function such that if ψ is a solution of (1.4) satisfying $\psi(x) \sim e^{\omega x}$ as $x \to \infty$, then $\psi(x) \sim D(\mu)e^{\omega x}$ as $x \to -\infty$: in essence, $D(\mu)$ plays the role of a transmission coefficient familiar from quantum scattering theory. This approach is motivated by the fact that for $\mathrm{Re}(\mu) > 0$ the vanishing of $D(\mu)$ implies that μ is an L^2 eigenvalue of the linearized operator $\partial_x \mathcal{L}[u_c]$ and conversely. Pego and Weinstein were able to use this machinery to prove that the Evans function satisfies $\lim_{\mu \to +\infty} \mathrm{sign}(D(\mu)) > 0$ as well as the asymptotic relation

$$
D(\mu) = C_1 \left(\frac{\partial}{\partial c} \int_{-\infty}^{\infty} u_c(x)^2 dx \right) \mu^2 + o(|\mu|^2)
$$

in a neighborhood of $\mu = 0$ for some positive constant C_1. Thus, if (1.3) is negative, it follows by the continuity of $D(\mu)$ for $\mu \in \mathbb{R}^+$ that $D(\mu) < 0$ for small positive μ, and hence $D(\mu)$ must have a positive root, thus proving exponential instability of the underlying traveling solitary wave in this case.

In this paper, however, we are concerned with traveling wave solutions of (1.1) of the form (1.2), where this time we require the profile u_c to be a periodic function of its argument. In contrast to the traveling solitary wave theory, relatively little is known concerning the stability of periodic traveling waves of nonlinear dispersive equations such as the gKdV. Existing results usually come in two types: spectral stability with respect to localized or bounded perturbations, and orbital (nonlinear) stability with respect to periodic perturbations. Most spectral stability results seem to rely on a Floquet–Bloch decomposition of the linearized operator and a detailed analysis of the resulting family of spectral problems, or else perturbation techniques which analyze modulational instability (spectrum near the origin).

There is a fairly substantial amount of literature devoted to the stability of the cnoidal solutions of the KdV,

$$
(1.5) \qquad u(x,t) = u_0 + 12k^2\kappa^2 \, \mathrm{cn}^2 \left(\kappa \left(x - x_0 + \left(8k^2\kappa^2 - 4\kappa^2 + u_0 \right) t \right), k \right),
$$

where $k \in [0, 1)$ and κ, x_0, and u_0 are real constants. Such cnoidal waves represent all stationary periodic traveling wave solutions of the KdV and have been studied by McKean (1977) and more recently by Angulo, Bona, and Scialom (2006), Pava and

Natali (2008), and Bottman and Deconinck (2008). The results of McKean (1977) use the complete integrability of the KdV to study the periodic initial value problem in order to show nonlinear stability of the cnoidal solutions to perturbations of the same period. Also using the machinery of complete integrability, Bottman and Deconinck (2008) explicitly compute the spectrum of the linearized operator about the cnoidal wave on the real Hilbert space $L^2(\mathbb{R})$ and show this to be confined to the imaginary axis. In particular, it follows that cnoidal solutions of the KdV are spectrally stable to perturbations of the same period and, more generally, perturbations with periods which are integer multiples of the period of the cnoidal wave. In this paper, we will use the governing PDE, as opposed to properties of the explicit cnoidal wave solution, to upgrade the spectral stability result of Bottman and Deconinck to orbital stability with respect to perturbations of the same period. Finally, Pava and Natali (2008) use a modification of the energy functional techniques of Bona, Souganidis, and Strauss (1987) and Grillakis, Shatah, and Strauss (1987) to study the nonlinear stability problem for periodic solutions of the gKdV. In fact, the authors study a more general version of the governing PDE which we will not consider here. There are very interesting and subtle connections concerning the present work with that of Pava and Natali (2008) which will be made throughout the text. In particular, we will explain how the conditions $(P_0) - (P_3)$ for orbital stability given by Pava and Natali (2008) can be understood from a geometric viewpoint and how these conditions can be extended to the consideration of all periodic traveling waves of the gKdV (not just the "zero mass" solutions).

Returning to the generalized KdV equation (1.1), spectral stability results have recently been obtained by Hărăguş and Kapitula (2008) as well as by Bronski and Johnson (2008). In the former paper, the spectral stability of small amplitude periodic traveling wave solutions of (1.1) with $f(u) = u^{p+1}$ was studied. By using a Floquet–Bloch decomposition of the linearized spectral problem, the authors found that such solutions[1] are spectrally stable if $p \in [1, 2)$ and exhibit a modulational instability if $p > 2$. In particular, they found that such solutions are always spectrally stable to perturbations of the same period: in section 5, we will verify and extend this result through the use of the periodic Evans function. In the work of Bronski and Johnson (2008), a modulational instability analysis of periodic traveling wave solutions of (1.1) was conducted using Floquet theory and developing a perturbation theory appropriate to the Jordan structure of the period map at the origin. As a byproduct of their analysis, a sufficient condition for exponential instability of the underlying periodic traveling wave with respect to periodic perturbations was obtained in terms of the conserved quantities of the gKdV flow. In particular, a stability index was derived in a manner quite similar to the solitary wave theory outlined above such that the negativity of this index implies exponential instability of the periodic traveling wave with respect to perturbations of the same period. The relevant results of this analysis can be found in section 3. It seems natural to consider the role this periodic instability index derived in Bronski and Johnson (2008) plays in the nonlinear stability of the periodic traveling wave. As mentioned above, the analogue of this index controls the nonlinear stability in the solitary wave context. While we show this true in certain cases in sections 5 and 6, this seems not to be the case in general.

Before we state our main theorem, a little notational background is necessary: for more information see sections 2 and 4. The periodic traveling wave solutions of

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[1]In the work of Hărăguş and Kapitula (2008), they consider only small amplitude periodic traveling waves with the integration parameter a (see section 2 for definition) being small.

the gKdV (1.1) form a four-parameter family of solutions of the traveling wave ODE

$$u_{xxx} + f(u)_x - cu_x = 0.$$

One of the defining parameters corresponds to the translation invariance and can be modded out, leaving us with a three-parameter manifold of periodic traveling wave solutions which we define by the parameters $(a, E, c) \in \Omega \subset \mathbb{R}^3$, where c is the wave-speed and a and E arise as constants of integration reducing the traveling wave ODE to quadrature:

$$\frac{u_x^2}{2} = E - \int_{u(0)}^{u} f(s)ds + au + \frac{c}{2}u^2.$$

Notice not every choice of $(a, E, c) \in \mathbb{R}^3$ gives rise to a periodic solution of the traveling wave ODE: the set of all (a, E, c) for which the traveling wave ODE admits periodic solutions is denoted by $\Omega \subset \mathbb{R}^3$ (more will be said on this in the next section). Given such a periodic solution $u(x) = u(x; a, E, c)$ we denote its period by T, its mass by $M := \int_0^T u \, dx$, and its momentum by $P := \int_0^T u^2 dx$, where we consider these as functions of Ω. Moreover, we define the classical action of the traveling wave ODE (in the sense of action angle variables) to be $K := \int_0^T u_x^2 dx$. As we will see in the next section, K serves as a useful generating function for the quantities T, M, and P of the underlying wave. These quantities can be shown to be C^1 in the parameters $(a, E, c) \in \Omega$, and hence we can speak of their partial derivatives (denoted by $T_a := \frac{\partial}{\partial a} T(a, E, c)$, etc.) and gradients. Moreover, we suppose there is a real Hilbert space X such that the Cauchy problem for (1.1) is globally well-posed. With this setup in mind, we now state our main theorem.

THEOREM 1.1. *Let $u(x + c_0 t)$ be a periodic traveling wave solution of* (1.1), *corresponding to an $(a_0, E_0, c_0) \in \Omega$. Moreover, assume the principle minors of the matrix*

(1.6)
$$D_{E,a,c}^2 K(a, E, c) = \begin{pmatrix} T_E & T_a & T_c \\ M_E & M_a & M_c \\ P_E & P_a & P_c \end{pmatrix}$$

satisfy $d_1 = T_E > 0$, $d_2 = T_E M_a - M_E T_a < 0$, and $d_3 = \det(D_{E,a,c}^2 K(a, E, c)) < 0$ at (a_0, E_0, c_0). Then there exist $C_0 > 0$ and $\varepsilon > 0$ such that for all $\phi_0 \in X$ with $\|\phi_0\|_X < \varepsilon$, the solution $\phi(x, t)$ of (1.1) *with initial data $\phi(x, 0) = u(x) + \phi(x)$ satisfies*

$$\inf_{\xi \in \mathbb{R}} \|\phi(\cdot, t) - u(x + c_0 t + \xi)\|_X \leq C_0 \|\phi_0\|_X$$

for all $t > 0$.

The main point in Theorem 1.1 is that the orbital stability of a given periodic traveling wave solution of (1.1) can be determined in terms of the solution itself. Moreover, this yields a geometric characterization of the orbital stability problem for such solutions. Take for example the KdV: in this case we will show that, assuming a particular nondegeneracy condition holds, orbital stability occurs if and only if the underlying solution is spectrally stable to such perturbations. To put this in geometric terms, we will show that the periodic traveling waves have a natural parametrization in terms of three constants arising from the governing ODE. One can then consider the map from this space of constants to the conserved quantities of the gKdV flow, namely, the mass, momentum, and Hamiltonian. If this map is invertible at the corresponding

solution, then the orbital stability will be determined completely in terms of the Jacobian of this map, which is given precisely by the matrix $D^2_{a,E,c}K(a,E,c)$. More will be said on this in later sections.

Remark 1. As we will see in section 5, the condition $T_E > 0$ is satisfied for a large class of nonlinearities by a result of Schaaf (1985). Thus, in such cases one must determine only the sign of d_2 and d_3. Moreover, we will see that $T_E > 0$ is equivalent with the conditions $(P_0) - (P_2)$ presented in Pava and Natali (2008). Also, the notation used for the matrix (1.6) is meant to be suggestive: it is the Hessian matrix of the classical action of the ODE governing traveling wave solutions of (1.1). See section 2 for more details.

The outline for this paper is as follows. Section 2 will be devoted to a study of the basic properties of the periodic traveling wave solution of (1.1). In section 3, we will recall the recent results of Bronski and Johnson (2008) concerning the spectral stability of periodic traveling wave solutions of (1.1) with respect to perturbations of the same period. The resulting instability index will play an important role throughout the rest of the paper. Section 4 is devoted to the proof of Theorem 1.1. Finally, two applications of our theory are described in sections 5 and 6: in section 5, we study the orbital stability of periodic traveling wave solutions of (1.1) in neighborhoods of the homoclinic and equilibrium solutions in the case of a power-law nonlinearity. In section 6, we apply our theory to the KdV and show that periodic traveling wave solutions are orbitally stable if and only if they are spectrally stable to perturbations of the same period as the underlying wave.

2. Properties of the stationary periodic traveling waves. In this section, we recall the basic properties of the periodic traveling wave solutions of (1.1). For each number $c > 0$, a stationary traveling wave solution of (1.1) with wave speed c is a solution of the traveling wave ODE

$$(2.1) \qquad u_{xxx} + f(u)_x - cu_x = 0;$$

i.e., they are solutions of (1.1) which are stationary in the moving coordinate frame defined by $x + ct$. Clearly, such solutions are reducible to quadrature and satisfy

$$(2.2) \qquad u_{xx} + f(u) - cu = a,$$

$$(2.3) \qquad \frac{1}{2}u_x^2 + F(u) - \frac{c}{2}u^2 - au = E,$$

where a and E are real constants of integration and F satisfies $F' = f$, $F(0) = 0$. In order to ensure the existence of periodic orbits of (2.1), we require that the effective potential

$$V(u; a, c) = F(u) - \frac{c}{2}u^2 - au$$

is of class $C^2(\mathbb{R})$ and has a local minimum. Notice this places a restriction on the allowable parameter regime for our problem. This motivates the following definition.

DEFINITION 1. *We define* $\Omega \subset \mathbb{R}^3$ *to be the open set consisting of all triples* (a, E, c) *such that* (2.1) *has at least one periodic solution.*

When $(a, E, c) \in \Omega$, we use the notation $u(x; a, E, c)$ to denote a particular periodic solution;[2] see Figure 2.1. Notice that $\partial\Omega$ corresponds to the solitary wave solutions of the traveling wave ODE.

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[2] In the case where more than one such solution exists for a particular (a, E, c), we can distinguish them by their initial values.
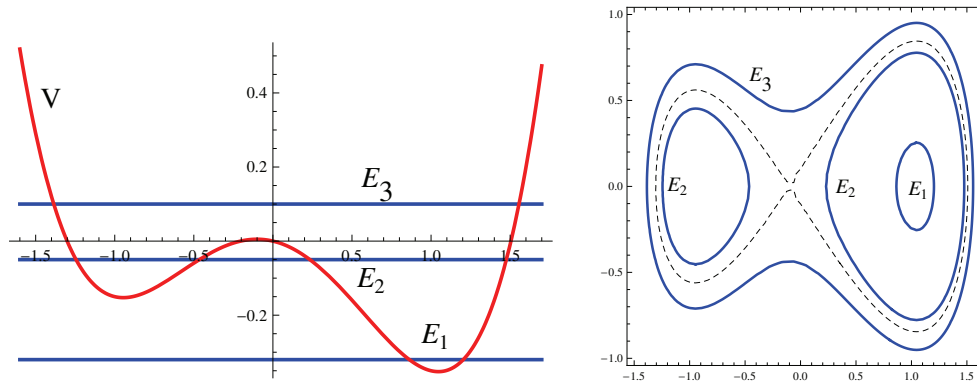
FIG. 2.1. *(Left) A plot of the effective potential energy $V(x; 0.1, 1)$ for the mKdV equation ($f(u) = u^2$), as well as three energy levels $E_1 = -0.32$, $E_2 = -0.05$, and $E_3 = 0.1$. (Right) Plots in phase space of the solutions $u(x; 0.1, E_j, 1)$ corresponding to the three energy levels on the left. Notice that we refer to those solutions corresponding to energy levels E_1 and E_2 as dnoidal type since they are bounded by a homoclinic orbit in phase space (given by the thin dashed line). Moreover, notice that E_2 corresponds to two distinct periodic traveling wave solutions; however, these can be clearly distinguished by their initial values which we have chosen to mod out in our theory.*

Remark 2. The theory presented throughout will apply to all $(a, E, c) \in \Omega$. However, our applications in section 5 will be restricted to the case of dnoidal type solutions, i.e., those solutions which are bounded by a homoclinic or heteroclinic orbit in phase space; such orbits are guaranteed to exist by the requirement that V have a local minima. There are a number of reasons for this restriction: first and foremost is that a result of Schaaf (1985) allows us to identify the sign of the quantity T_E from Theorem 1.1 for a wide class of such solutions; see Lemma 5.1. As we will see in the next section, this information is vital in understanding the number of negative eigenvalues of the second variation of a particular nonlinear functional used throughout this work; see Lemma 4.1 for more details.

Moreover, taking into account the translation invariance of (1.1), it follows that for each $(a, E, c) \in \Omega$ we can construct a one-parameter family of periodic traveling wave solutions of (1.1), namely,

$$u_\xi(x, t) = u(x + ct + \xi; a, E, c),$$

where $\xi \in \mathbb{R}$. Thus, the periodic traveling waves of (1.1) constitute a four-dimensional manifold of solutions. However, outside of the null-direction of the linearized operator which this generates, the added constant of integration does not play an important role in our theory. In particular, we can mod out the continuous symmetry of (1.1) by requiring all periodic traveling wave solutions satisfy the conditions $u_x(0) = 0$ and $V'(u(0)) < 0$. As a result, each periodic solution of (2.1) is an even function of the variable x with a local maximum at $x = 0$.

Remark 3. Notice that a and E are constants of integration arising from integrating (2.1) to quadrature. Moreover, the classical solitary waves corresponding to solutions of (1.1) satisfying $\lim_{x \to \pm\infty} u(x) = 0$ correspond to $a = E = 0$, and hence constitute a codimension two subset of the traveling wave solutions of (1.1). It seems natural then to expect that the stability of periodic traveling waves will involve variations in these extra parameters, just as the solitary wave stability index involves variations in the one (modulo translation) free parameter c.

Throughout this paper, we will always assume that our periodic traveling waves correspond to an (a, E, c) within the open region Ω and that the roots $u_{\pm}$ of $E = V(u; a, c)$ with $V(u; a, c) < E$ for $u \in (u_-, u_+)$ are simple. Moreover, we assume the potential V does not have a local maximum in the open interval (u_-, u_+). It follows that $u_{\pm}$ are C^1 functions of a, E, and c on Ω, and that $u(0) = u_-$. Moreover, given $(a, E, c) \in \Omega$, we define the period of the corresponding solution to be

$$(2.4) \qquad T = T(a, E, c) := 2 \int_{u_-}^{u_+} \frac{du}{\sqrt{2\left(E - V(u; a, c)\right)}}.$$

The above interval can be regularized at the square root branch points u_-, u_+, and in particular we can differentiate the above relation with respect to the parameters a, E, and c within the parameter regime Ω. Similarly, the mass, momentum, and Hamiltonian of the traveling wave are given by

$$(2.5) \qquad M(a, E, c) = \int_0^T u(x)\, dx = 2 \int_{u_-}^{u_+} \frac{u\, du}{\sqrt{2\left(E - V(u; a, c)\right)}},$$

$$(2.6) \qquad P(a, E, c) = \int_0^T u^2(x)\, dx = 2 \int_{u_-}^{u_+} \frac{u^2\, du}{\sqrt{2\left(E - V(u; a, c)\right)}},$$

$$(2.7) \qquad H(a, E, c) = \int_0^T \left(\frac{u_x^2}{2} - F(u)\right) dx = 2 \int_{u_-}^{u_+} \frac{E - V(u; a, c) - F(u)}{\sqrt{2\left(E - V(u; a, c)\right)}}\, du.$$

Notice that these integrals can also be regularized at the branch points $u_{\pm}$ and represent conserved quantities of the gKdV flow restricted to the manifold of periodic traveling wave solutions. In particular, one can differentiate the above expressions with respect to the parameters (a, E, c).

The gradients of the above conserved quantities will play a large role throughout this paper. Notice by the Hamiltonian structure of (2.1), the derivatives of the period, mass, and momentum restricted to a periodic traveling wave $u(\cdot; a, E, c)$ with $(a, E, c) \in \Omega$ satisfy several useful identities. In particular, if we define the classical action

$$K = \oint u_x\, du = \int_0^T u_x^2\, dx = 2 \int_{u_-}^{u_+} \sqrt{2(E - V(u; a, c))}\, du$$

(which is not itself conserved), then this quantity satisfies the following relation:

$$(2.8) \qquad \nabla_{a,E,c} K(a, E, c) = \left\langle M(a, E, c), T(a, E, c), \frac{1}{2} P(a, E, c) \right\rangle,$$

where $\nabla_{a,E,c} = (\partial_a, \partial_E, \partial_c)$. Using the fact that T, M, P, and H are C^1 functions of parameters (a, E, c), the above implies the following relationship between the gradients of the conserved quantities of the gKdV:

$$(2.9) \qquad E \nabla_{a,E,c} T + a \nabla_{a,E,c} M + \frac{c}{2} \nabla_{a,E,c} P + \nabla_{a,E,c} H = 0.$$

See the appendix of Bronski and Johnson (2008) for details of this calculation. The subsequent theory is developed most naturally in terms of the quantities T, M, and P. However, it is possible to restate our results in terms of M, P, and H using the

identity (2.9). This is desirable since these have a natural interpretation as conserved quantities of the PDE (1.1).

We now discuss the parametrization of the periodic solutions of (2.1) in more detail. A major technical necessity throughout this paper is that the constants of motion for the PDE flow defined by (1.1) provide (at least locally) a good parametrization for the periodic traveling wave solutions. In particular, we assume for a given $(a, E, c) \in \Omega$ the conserved quantities (H, M, P) are good local coordinates for the periodic traveling waves near (a, E, c). More precisely, we assume the map $(a, E, c) \to (H(a, E, c), M(a, E, c), P(a, E, c))$ has a unique C^1 inverse in a neighborhood of the point (a, E, c). If we adopt the notation

$$\{f, g\}_{x,y} = \left| \begin{array}{cc} f_x & g_x \\ f_y & g_y \end{array} \right|$$

for 2×2 Jacobians, and $\{f, g, h\}_{x,y,z}$ for the analogous 3×3 Jacobian, it follows this is possible exactly when $\{H, M, P\}_{a,E,c} \neq 0$, which is equivalent to the Jacobian $\{T, M, P\}_{a,E,c}$ being nonzero if $E \neq 0$ by (2.9). Also, we will need to know that two of the quantities T, M, and P provide a local parametrization for the traveling waves with fixed wave speed. By reasoning as above, this happens exactly when the matrix

$$\left(\begin{array}{ccc} T_a & M_a & P_a \\ T_E & M_E & P_E \end{array} \right)$$

has full rank. A sufficient requirement[3] is thus $\{T, M\}_{a,E} \neq 0$.

3. Spectral stability analysis. In this section, we recall the relevant results of Bronski and Johnson (2008) on the spectral stability of periodic traveling wave solutions of the gKdV. Suppose that $u = u(\cdot\;; a, E, c) \in C^3(\mathbb{R}; \mathbb{R})$ is a T-periodic solution of (1.4) with $(a, E, c) \in \Omega$. Linearizing (1.1) about this solution and taking the Laplace transform in time leads to the spectral problem

$$(3.1) \qquad\qquad \partial_x \mathcal{L}[u]v = \mu v$$

considered on $L^2(\mathbb{R}; \mathbb{R})$, where $\mathcal{L}[u] := -\partial_x^2 - f'(u) + c$ is a closed symmetric linear operator with T-periodic coefficients. In particular, since u is bounded, it follows that $\mathcal{L}[u]$ is in fact a self-adjoint operator on $L^2(\mathbb{R})$ with densely defined domain $C^\infty(\mathbb{R})$. Notice that considering (3.1) on $L^2(\mathbb{R})$ corresponds to considering the spectral stability with respect to localized perturbations,[4] and as a result the spectrum $\mathrm{spec}\,(\partial_x \mathcal{L}[u])$ is purely continuous. Moreover, the Hamiltonian nature of (3.1) implies that such a solution is spectrally stable if and only if $\mathrm{spec}\,(\partial_x \mathcal{L}[u]) \subset \mathbb{R}i$.

In order to study the spectrum of the operator $\partial_x \mathcal{L}[u]$ we note that (3.1) can be written as a first order system of the form $\Phi_x = \mathbf{H}(x, \mu)\Phi$. We define the monodromy matrix $\mathbf{M}(\mu)$ to be the corresponding matrix solution with initial condition $\Phi(0) = \mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix. It follows that $\mu \in \mathrm{spec}\,(\partial_x \mathcal{L}[u])$ if and only if there exists a nontrivial bounded function ψ such that $\partial_x \mathcal{L}[u]\psi = \mu\psi$ or, equivalently, if there exists a $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that the periodic Evans function

$$D(\mu, \lambda) = \det\,(\mathbf{M}(\mu) - \lambda\mathbf{I})$$

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[3]Although this is not necessary, we find this condition is needed for the proof of Lemma 4.4. Trivial modifications of the theory of section 3 are needed if $\{T, M\}_{a,E} = 0$ but $\{T, P\}_{a,E} \neq 0$, but we are unable to find such a modification for the orbital stability proof of section 4.

[4]One could also study the spectral stability with respect to uniformly bounded perturbations, but by standard results in Floquet theory the resulting theories are equivalent.

vanishes; see Gardner (1997). In particular, we see that $D(\mu, 1)$ detects spectra which correspond to perturbations which are T-periodic. To study such instabilities, we recall the following result.

LEMMA 3.1. *The function $D(\mu, 1)$ satisfies the following properties:*

1. $D(\mu, 1)$ *is an odd function of μ.*
2. *The limit $\lim_{\mu \to \infty} \text{sign}\,(D(\mu, 1))$ exists and is negative.*
3. *The asymptotic relation*

$$D(\mu, 1) = -\frac{1}{2}\{T, M, P\}_{a,E,c}\, \mu^3 + \mathcal{O}(|\mu|^4)$$

holds in a neighborhood of $\mu = 0$.

For details of the proof, see Proposition 1, Lemma 1, and Theorem 3 of Bronski and Johnson (2008). The main idea is that the integrable structure of the ODE governing the traveling waves (2.1) immediately allows direct computation of the tangent space of the manifold of traveling wave solutions at $\mu = 0$. As such, the calculation is undoubtedly related to the multisymplectic formalism of Bridges (1997). It follows that if $\{T, M, P\}_{a,E,c}$ is negative, then the number of positive roots of $D(\mu, 1)$ is odd and hence one has exponential instability of the underlying periodic traveling wave. Moreover, we will show in Lemma 4.2 that $T_E > 0$ implies $\mathcal{L}[u]$ has exactly one negative eigenvalue. It follows that the linearized operator $\partial_x \mathcal{L}[u]$ has at most one unstable eigenvalue with positive real part, counting multiplicities (see, for example, Theorem 3.1 of Pego and Weinstein (1992)). Since the spectrum of $\partial_x \mathcal{L}[u]$ is symmetric about the real and imaginary axes, it follows that all unstable periodic eigenvalues of the linearized operator must be real. This proves the following extension of Corollary 1 in Bronski and Johnson (2008).

THEOREM 3.2. *Let $u(x; a_0, E_0, c_0)$ be a periodic traveling wave solution of (1.1) with $(a_0, E_0, c_0) \in \Omega$. If T_E is positive and $\{T, M, P\}_{a,E,c}$ is nonzero at (a_0, E_0, c_0), then the solution is spectrally stable to perturbations of the same period if and only if $\{T, M, P\}_{a,E,c}$ is positive at (a_0, E_0, c_0).*

Notice that if $T_E < 0$, the operator $\mathcal{L}[u]$ has two negative eigenvalues by Lemma 4.2 below. Thus, even if $\{T, M, P\}_{a,E,c} > 0$ in this case, there is no way of proving from these methods whether the number of periodic eigenvalues of $\partial_x \mathcal{L}[u]$ with positive real part is equal to zero or two. By drawing a direct analogy with the solitary wave theory one would suspect if $T_E > 0$, then such solutions of (1.1) are nonlinearly stable if and only if $\{T, M, P\}_{a,E,c}$ is positive,[5] i.e., if and only if it is spectrally stable to perturbations of the same period. However, this seems not to be true in general: the sign of the Jacobian $\{T, M\}_{a,E}$ also plays a role in the orbital stability analysis, even though it does not seem to play into the periodic spectral stability theory at all.[6] This possibility of having spectrally stable solutions which are not orbitally stable stands in stark contrast to the solitary wave theory for the gKdV.

4. Orbital stability. In this section, we prove our main theorem on the orbital stability of periodic traveling wave solutions of (1.1). Throughout this section, we assume we have a T-periodic traveling wave solution $u(x; a_0, E_0, c_0)$ of (1.1); i.e., we assume u satisfies

$$(4.1) \qquad \frac{1}{2}u_x^2 + F(u) - \frac{c_0}{2}u^2 - a_0 u = E_0,$$

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[5]In sections 5 and 6, we study cases where this is indeed the case.

[6]One could suspect that $\{T, M\}_{a,E}$ changes sign only when $\{T, M, P\}_{a,E,c}$ does, but this is shown not to be the case in Corollary 4.7 below.

with $(a_0, E_0, c_0) \in \Omega$ and $T = T(a_0, E_0, c_0)$. Moreover, we assume the nonlinearity f present in (1.1) is such that the Cauchy problem for (1.4) is globally well-posed in a real Hilbert space X of real valued T-periodic functions defined on $\mathbb{R}$, which we equip with the standard $L^2([0, T])$ inner product

$$\langle g, h \rangle := \int_0^T g(x)h(x)dx$$

for all $g, h \in X$. Also, we identify the dual space X^* through the usual pairing. In particular, notice that $L^2([0, T])$ is required to be a subspace of X. For example, if $f(u) = u^3/3$, corresponding to the modified KdV equation, then the Cauchy problem for (1.4) is globally well-posed in the space

$$H^s_{\mathrm{per}}([0, T]; \mathbb{R}) = \{g \in H^s([0, T]; \mathbb{R}) : g(x + T) = g(x) \text{ a.e.}\}$$

for all $s \geq \frac{1}{2}$, where we identify the dual space with $H^{-s}_{\mathrm{per}}([0, T]; \mathbb{R})$ through the usual pairing (see Colliander et. al. (2003) for proof). Moreover, due to the structure of the gKdV, we make the natural assumption that the evolution of (1.4) in the space X is invariant under a one-parameter group of isometries G corresponding to spatial translation. Thus, G can be identified with the real line $\mathbb{R}$ acting on the space X through the unitary representation

$$(R_\xi g)(x) = g(x + \xi)$$

for all $g \in X$ and $\xi \in G$. Since the details of our proof work regardless of the form of the nonlinearity f, we make the above additional assumptions on the nonlinearity and make no other references to the exact structure of the space X or f.

In view of the symmetry group G, we now describe precisely what we mean by orbital stability. We define the G-orbit generated by u to be

$$\mathcal{O}_u := \{R_\xi u : \xi \in G\}.$$

Now, suppose we have initial data $\phi_0 \in X$ which is close to the orbit $\mathcal{O}_u$. By orbital stability, we mean that if $\phi(\cdot, t) \in X$ is the unique solution with initial data ϕ_0, then $\phi(\cdot, t)$ is close to the orbit of u for all $t > 0$. More precisely, we introduce a semidistance ρ defined on the space X by

$$\rho(g, h) = \inf_{\xi \in G} \|g - R_\xi h\|_X$$

and use this to define an ε-neighborhood of the orbit $\mathcal{O}_u$ by

$$\mathcal{U}_\varepsilon := \{\phi \in X : \rho(u, \phi) < \varepsilon\}.$$

The main result of this section is the following restatement of Theorem 1.1.

LEMMA 4.1. *Let $u(x) = u(x; a_0, E_0, c_0)$ solve (4.1) with $(a_0, E_0, c_0) \in \Omega$, and suppose the quantities T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are all positive at (a_0, E_0, c_0). Then there exist positive constants C_0, ε_0 such that if $\phi_0 \in X$ satisfies $\rho(\phi_0, u) < \varepsilon$ for some $\varepsilon < \varepsilon_0$, then the solution $\phi(x, t)$ of (1.1) with initial data ϕ_0 satisfies $\rho(\phi(\cdot, t), u) \leq C_0\varepsilon$.*

Remark 4. Notice that Theorem 3.2 implies a periodic solution $u(x; a_0, E_0, c_0)$ of (2.1) is an exponentially unstable solution of (1.1) if $\{T, M, P\}_{a,E,c}$ is negative at

(a_0, E_0, c_0). Thus, the positivity of this Jacobian is a necessary condition for nonlinear stability.

Also, it follows from (2.8) that the Hessian of the classical action $K(a, E, c)$ can be expressed as

$$D_{a,E,c}^2 K(a, E, c) = \begin{pmatrix} M_a & M_E & M_c \\ T_a & T_E & T_c \\ P_a & P_E & P_c \end{pmatrix}.$$

Lemma 4.1 thus states that if $(a_0, E_0, c_0) \in \Omega$, then a sufficient condition for orbital stability is that $D_{a,E,c}^2 K(a, E, c)$ be invertible with precisely one negative eigenvalue. However, this is clearly not sufficient.

We now proceed with the proof of Lemma 4.1, which follows the general methods of Bona, Souganidis, and Strauss (1987) and Grillakis, Shatah, and Strauss (1987). We begin by defining the following functionals on the space X:

$$\mathcal{E}(\phi) := \int_0^T \left(\frac{1}{2}\phi_x(x)^2 - F(\phi(x)) \right) dx,$$

$$\mathcal{M}(\phi) := \int_0^T \phi(x)\, dx,$$

$$\mathcal{P}(\phi) := \frac{1}{2} \int_0^T \phi(x)^2 dx.$$

These functionals represent conserved quantities of the flow generated by (1.1) and correspond to the "energy," "mass," and "momentum," respectively. In particular, if $\phi(x, t)$ is a solution of (1.1) of period T, then the quantities $\mathcal{E}(\phi(\cdot, t))$, $\mathcal{M}(\phi(\cdot, t))$, and $\mathcal{P}(\phi(\cdot, t))$ are constants in time. Also, notice that $\mathcal{E}(u) = H(a_0, E_0, c_0)$, $\mathcal{M}(u) = M(a_0, E_0, c_0)$, and $\mathcal{P}(u) = P(a_0, E_0, c_0)$, where H, M, and P are defined in (2.5)–(2.7).

Remark 5. Throughout the remainder of this paper, the symbols M and P will denote the functionals $\mathcal{M}$ and $\mathcal{P}$, respectively, restricted to the manifold of periodic traveling wave solutions of (1.1) with $(a, E, c) \in \Omega$.

It is easily verified that $\mathcal{E}$, $\mathcal{M}$, and $\mathcal{P}$ are smooth functionals on X, whose first derivatives are smooth maps from X to X^* defined by

$$\mathcal{E}'(\phi) = -\phi_{xx} - f(\phi), \quad \mathcal{M}'(\phi) = 1, \quad \mathcal{P}'(\phi) = \phi.$$

If we now define an augmented energy functional on the space X by

(4.2) $$\mathcal{E}_0(\phi) := \mathcal{E}(\phi) + c_0 \mathcal{P}(\phi) + a_0 \mathcal{M}(\phi) + E_0 T,$$

it follows from (4.1) that $\mathcal{E}_0(u) = 0$ and $\mathcal{E}_0'(u) = 0$. Hence, u is a critical point of the functional $\mathcal{E}_0$.

Remark 6. Notice that the added factor of $E_0 T$ on the right-hand side of (4.2) is not technically needed for our calculation. However, we point out that if we (formally) consider variations in $\mathcal{E}_0$ in the *period*, we obtain

$$\frac{\partial}{\partial T}\mathcal{E}_0(\phi)\big|_{\phi=u} = \frac{1}{2}u_x^2(T) - F(u(T)) + au(T) + E + \left\langle \mathcal{E}_0'(u), \frac{\partial u}{\partial T} \right\rangle$$
$$= 0$$

since $u_x^2(T) = 2E - 2V(u(T); a, c) = 0$ and $\mathcal{E}_0'(u) = 0$. Hence, u is also (formally) a critical point of the modified energy with respect to variations in the period, and the energy level E_0 can be considered as a Lagrange multiplier enforcing fixed period. It would be very interesting to try to make this calculation rigorous and to see if it allows one to extend orbital stability results to include perturbations with period close to the period of the underlying wave; we will make no attempt at such a theory here.

Remark 7. Calculations in a similar vein have been carried out recently by Pava and Natali (2008), Angulo, Bona, and Scialom (2006), and Angulo (2007) in the case where $a = 0$. After dropping the $E_0 T$ factor (see Remark 6), such a periodic traveling profile is a critical point of the energy functional

$$\mathcal{E}(\phi) + c_0 \mathcal{P}(\phi),$$

which is the same as that considered by Bona, Souganidis, and Strauss (1987) and Grillakis, Shatah, and Strauss (1987) in the solitary wave theory. While this is always done for the KdV (due to Galilean invariance), it is not possible for general nonlinearities without restricting your admissible class of traveling wave solutions, i.e., restricting Ω. Thus, when studying nonlinear stability in the periodic setting, the augmented energy given in (4.2) is the appropriate functional to analyze.

To determine the nature of this critical point, we consider its second derivative $\mathcal{E}_0''$, which is a smooth map from X to $\mathcal{L}(X, X^*)$ defined by

$$\mathcal{E}_0''(\phi) = -\phi_{xx} - f'(\phi) + c_0.$$

This formula immediately follows by noticing that the second derivatives of the mass, momentum, and energy functionals are smooth maps from X to $\mathcal{L}(X, X^*)$ given by

$$\mathcal{E}''(\phi) = -\partial_x^2 - f'(\phi), \quad \mathcal{M}''(\phi) = 0, \quad \mathcal{P}''(\phi) = 1.$$

In particular, notice that the second derivative of the augmented energy functional $\mathcal{E}_0$ at the critical point u is precisely linear operator $\mathcal{L}[u]$ arising from linearizing (1.4) with wave speed c_0 about u. It follows from the comments in the previous section that $\mathcal{E}_0''(u)$ is a self-adjoint linear operator on $L_{\text{per}}^2([0, T]; \mathbb{R})$ with compact resolvent. In order to classify u as a critical point of $\mathcal{E}_0$, we must understand the nature of the spectrum of the second variation $\mathcal{L}[u]$: in particular, we need to know the number of negative eigenvalues. This is handled in the following lemma.

LEMMA 4.2. *The spectrum of the operator $\mathcal{L}[u]$ considered on the space* $L_{\text{per}}^2([0, T])$ *satisfies the following trichotomy:*
  (i) *If $T_E > 0$, then $\mathcal{L}[u]$ has exactly one negative eigenvalue, a simple eigenvalue at zero, and the rest of the spectrum is strictly positive and bounded away from zero.*
  (ii) *If $T_E = 0$, then $\mathcal{L}[u]$ has exactly one negative eigenvalue, a double eigenvalue at zero, and the rest of the spectrum is strictly positive and bounded away from zero.*
  (iii) *If $T_E < 0$, then $\mathcal{L}[u]$ has exactly two negative eigenvalues, a simple eigenvalue at zero, and the rest of the spectrum is strictly positive and bounded away from zero.*

Proof. The proof is essentially a consequence of the translation invariance of (1.1) and the Sturm–Liouville oscillation theorem. Indeed, notice that for any $\xi \in G$ the function $R_\xi u$ is a stationary solution of (2.2) with wave speed c_0 and $a = a_0$.

Differentiating this relation with respect to ξ and evaluating at $\xi = 0$ implies that $\mathcal{L}[u]u_x = 0$. Moreover, since u is radially increasing on $[0, T]$ from its local minimum there, u_x is periodic with the same period as u and hence $u_x \in L^2_{\mathrm{per}}([0, T])$. This proves that zero is always a periodic eigenvalue of $\mathcal{L}[u]$ as claimed. To see there is exactly one negative eigenvalue, notice that since u is T-periodic with precisely one local critical point on $(0, T)$, its derivative u_x must have precisely one sign change over its period. By standard Sturm–Liouville theory applied to the periodic problem (see Theorem 2.14 in Magnus and Winkler (1979)), it follows that zero must be either the second or third[7] eigenvalue of $\mathcal{L}[u]$ considered on the space $L^2_{\mathrm{per}}([0, T])$. Our goal is to show that it is the Jacobian T_E which distinguishes between these cases.

Let $\mathbf{m}(\mu)$ be the monodromy matrix corresponding to the second order linear ODE $\mathcal{L}[u]v = \mu v$. Here, we are considering $\mathcal{L}[u]$ as a formal operator without imposing any boundary conditions. From Floquet theory, it is known that the $L^2(\mathbb{R})$ spectrum of $\mathcal{L}[u]$ is characterized by the Floquet discriminant $k(\mu) := \mathrm{tr}(\mathbf{m}(\mu))$. In particular, $\mathcal{L}[u]v = \mu v$ has a nontrivial bounded solution if and only if $k(\mu)$ is real and $|k(\mu)| \leq 2$. As the function $k(\mu)$ is analytic, it follows that the $L^2(\mathbb{R})$ spectrum of $\mathcal{L}[u]$ must be purely continuous, consisting of bands and gaps.[8] The edges of the bands, given by roots of the equation $k(\mu) = \pm 2$, correspond to the periodic and antiperiodic eigenvalues of $\mathcal{L}[u]$. In particular, the solutions of the equation $k(\mu) = 2$ are precisely the eigenvalues of $\mathcal{L}[u]$ considered on $L^2_{\mathrm{per}}([0, T])$. Since $\mathcal{L}[u]u_x = 0$ by the above remarks, it follows that $k(0) = 2$, and hence we must determine if $\mu = 0$ is the second or third root of the equation $k(\mu) = 2$. For this, it is enough to analyze the quantity $k'(0)$. Indeed, the role of this quantity in determining the structure of the spectrum of $\mathcal{L}[u]$ is described by Figure 4.1 and its caption; for more information see the text of Magnus and Winkler (1979). In particular, notice that $\mu = 0$ is the second T-periodic eigenvalue of $\mathcal{L}[u]$ if $k'(0) \geq 0$, and is the third if $k'(0) < 0$.

We now wish to calculate the quantity $k'(0)$ explicitly, showing in particular that $\mathrm{sign}(k'(0)) = \mathrm{sign}(T_E)$. First, notice that the periodic traveling wave solutions of (2.2) are invariant under changes in the energy parameter E associated with the Hamiltonian ODE (2.1). As above, it follows that $\mathcal{L}[u]u_E = 0$. We can thus use u_x and u_E as a basis to compute the matrix $\mathbf{m}(0)$. Differentiating the relation $E = V(u_-; a, c)$ with respect to E and evaluating at (a_0, E_0, c_0) gives $\frac{\partial u_-}{\partial E} V'(u_-; a_0, c_0) = 1$, and hence $\frac{\partial u_-}{\partial E}$ is nonzero at (a_0, E_0, c_0). Defining $y_1(x) = (\frac{du_-}{dE})^{-1} u_E$ and $y_2(x) = -\left(V'(u_-; a_0, c_0)\right)^{-1} u_x(x)$, it follows from direct calculation that

$$y_1(0) = 1, \quad y_2(0) = 0,$$
$$y_1'(0) = 0, \quad y_2'(0) = 1.$$

Calculating $u_E(T)$ by the chain rule, we have

$$\mathbf{m}(0) = \begin{pmatrix} 1 & T_E \\ 0 & 1 \end{pmatrix},$$

where again we have used the fact that $V'(u_-; a, c)\frac{\partial u_-}{\partial E} = 1$. The result now follows by some standard perturbation theory and messy algebra. In particular, using variation of parameters it is possible to express the functions $\{\frac{d}{d\mu} y_j(x)\}_{j=1,2}$ and $\{\frac{d}{d\mu} y_j'(x)\}_{j=1,2}$ in terms of $\{y_j\}_{j=1,2}$ and $\{y_j'\}_{j=1,2}$, respectively. It is then possible to expand $\mathbf{m}(\mu)$

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[7]Clearly, we mean with respect to the natural ordering on $\mathbb{R}$.

[8]A given $\mu \in \mathbb{C}$ is said to be in a band if $-2 \leq k(\mu) \leq 2$ and in a gap otherwise. Notice that since $\mathcal{L}[u]$ is self-adjoint on $L^2(\mathbb{R})$, $k(\mu) \neq 0$ for all μ nonreal.
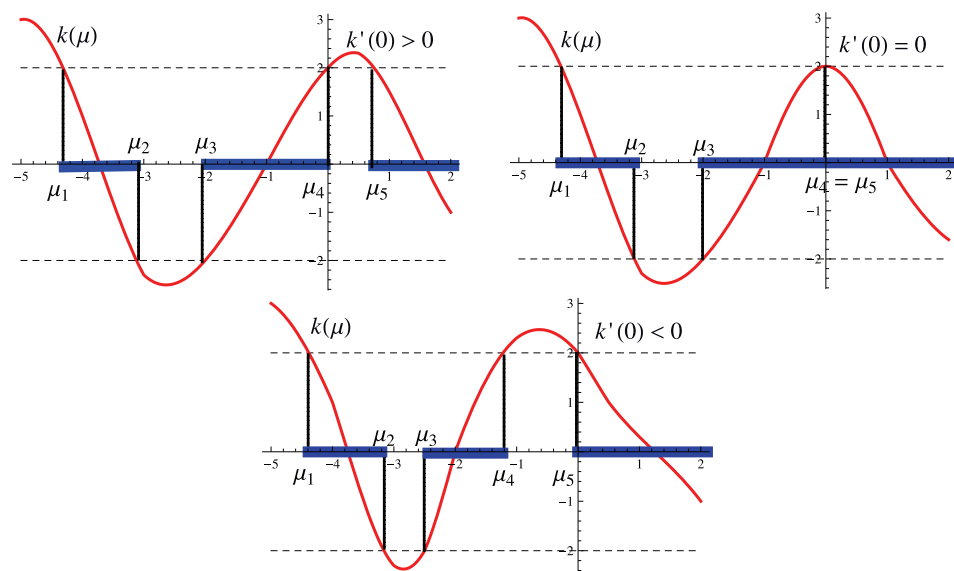
FIG. 4.1. *The above three plots are meant to illustrate the role the quantity $k'(0)$ has in determining the structure of the T-periodic spectrum of $\mathcal{L}[u]$. In each, the horizontal dashed lines correspond to the values ± 2, and the points μ_j correspond to the band edges of $\mathcal{L}[u]$. The points μ_1, μ_4, and μ_5 are T-periodic eigenvalues of L, while the points μ_2 and μ_3 are $2T$-periodic eigenvalues, and the dark lines along the real axis correspond to the $L^2(\mathbb{R})$ spectrum of $\mathcal{L}[u]$. When $k'(0) \neq 0$, zero is a nondegenerate band edge corresponding to a T-periodic eigenvalue of multiplicity one. In particular, notice that whether $\mu = 0$ is a left or right band edge is determined completely by the sign of $k'(0)$. When $k'(0)$ vanishes, the neighboring gap closes and $\mu = 0$ becomes a double point of the spectrum, corresponding to a T-periodic eigenvalue of multiplicity two.*

to first order in a neighborhood of $\mu = 0$, and then using the facts that $\det(\mathbf{m}(0)) = 1$ and $k(0) = 2$ some algebra eventually yields the expression

$$k'(0) = \text{sign}(y_1'(T)) \int_0^T \left(y_2(x)\sqrt{y_1'(T)} + \text{sign}(y_1'(T))\frac{y_1(T) - y_2'(T)}{2\sqrt{y_1'(T)}} y_1(x) \right)^2 dx.$$

Therefore,

$$\text{sign}(k'(0)) = \text{sign}(y_1'(T)) = \text{sign}(T_E)$$

as claimed. This completes the proof. Notice, in particular, that if $T_E = 0$, a second T-periodic solution of $\mathcal{L}[u]v = 0$ is given explicitly by the function $\{u, T\}_{a,E}$. □

Remark 8. The information in Lemma 4.2 is directly related to the nonlinear stability results presented in Pava and Natali (2008) in the case of the gKdV: recall that only the case $a = 0$ is considered. There, the authors construct a curve $(0, E(c), c) \in \Omega$ such that the corresponding periodic solutions are T-periodic. By the implicit function theorem, such a curve (locally) exists if and only if $T_E \neq 0$, and hence property (P_0) in Pava and Natali (2008) is equivalent with the period not having a critical point in the energy at E_0. Moreover, properties (P_1) and (P_2) state that the linear operator $\mathcal{L}[u]$ acting on $L^2_{\text{per}}([0, T])$ has one negative eigenvalue and that the zero eigenvalue is simple. By Lemma 4.2 this is equivalent with the positivity of the Jacobian T_E. Thus, by the result of Schaaf (1985) mentioned earlier, properties $(P_0) - (P_2)$ are satisfied

for a wide class of dnoidal type solutions of the gKdV including all elliptic function solutions of the KdV and mKdV considered in Pava and Natali (2008). Thus, for such solutions one must check only property (P_3). It should be noted, however, that property (P_3) is *not* equivalent with $P_c > 0$ but is rather a reduction of our theory to the case $a = 0$; see Remark 10 below for more details.

In the solitary wave case, the spectrum of the operator $\mathcal{L}[u]$ *always* satisfies (i) in the above trichotomy. However, since the constants a and E are not restricted to be zero in the periodic context, it is not surprising that such a nontrivial trichotomy might exist. Throughout the rest of the paper, unless otherwise stated, we will assume that $T_E > 0$ at (a_0, E_0, c_0), and hence zero is a simple eigenvalue of the operator $\mathcal{L}[u]$ considered on the space $L^2_{\mathrm{per}}(\mathbb{R})$. We will see in the next section that a result of Schaaf (1985) implies that this assumption is valid for a wide class of dnoidal type waves, and hence our results apply to all such solutions. It follows that any $\phi \in X$ can be decomposed as a linear combination of u_x, an element in the positive subspace of $\mathcal{L}[u]$, and χ, where χ is the unique positive eigenfunction of $\mathcal{L}[u]$ with $\|\chi\|_{L^2([0,T])} = 1$ which satisfies $\langle \mathcal{L}[u]\chi, \chi \rangle = -\lambda^2$ for some $\lambda > 0$. It follows that χ is the eigenfunction corresponding to the unique negative eigenvalue $-\lambda^2$ of $\mathcal{L}[u]$.

From Lemma 4.2, we know that u is a degenerate saddle point of the functional $\mathcal{E}_0$ on X, with one unstable direction and one neutral direction. In order to get rid of the unstable direction, we note that the evolution of (1.1) does not occur on the entire space X but on the codimension two submanifold defined by

$$\Sigma_0 := \{\phi \in X : \mathcal{M}(\phi) = M(a_0, E_0, c_0), \ \mathcal{P}(\phi) = P(a_0, E_0, c_0)\}.$$

It is clear that Σ_0 is indeed a smooth submanifold of X in a neighborhood of the group orbit $\mathcal{O}_u$. Moreover, the entire orbit $\mathcal{O}_u$ is contained in Σ_0. The main technical result needed for this section is that the functional $\mathcal{E}_0$ is coercive on Σ_0 with respect to the semidistance ρ, which is the content of the following proposition.

PROPOSITION 4.3. *Let $(a_0, E_0, c_0) \in \Omega$. If each of the quantities T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are positive, then there exist positive constants C_1, δ which depend on (a_0, E_0, c_0) such that*

$$\mathcal{E}_0(\phi) - \mathcal{E}_0(u) \geq C_1 \rho(\phi, u)^2$$

for all $\phi \in \Sigma_0$ such that $\rho(\phi, u) < \delta$.

The proof of Proposition 4.3 is broken down into three lemmas which analyze the quadratic form induced by the self-adjoint operator $\mathcal{L}[u]$. To begin, we define a function ϕ_0 by

$$\phi_0(x) := \{u(x; a, E, c), T(a, E, c), M(a, E, c)\}_{a,E,c}\big|_{(a_0, E_0, c_0)}.$$

It follows from a straightforward calculation that $\phi_0 \in X$ and

$$\mathcal{L}[u]\phi_0 = -\{T, M\}_{E,c} - \{T, M\}_{a,E} u,$$

where the right-hand side is evaluated at (a_0, E_0, c_0). This function plays a large role in the spectral stability theory for periodic traveling wave solutions[9] of (1.1) outlined in section 3. In particular, we have $\partial_x \mathcal{L}[u]\phi_0 = -\{T, M\}_{a,E} u_x$, and hence, assuming $\{T, M\}_{a,E} \neq 0$ at (a_0, E_0, c_0), ϕ_0 is in the generalized periodic null space of

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[9]Actually, the function u_c plays a large role in our analysis via the periodic Evans function. However, since u_c is not in general T-periodic, we work here with its periodic analogue ϕ_0.

the linearized operator $\partial_x\mathcal{L}[u]$. By our assumption that $\{T, M, P\}_{a,E,c} \neq 0$, it follows that $\langle u, \phi_0 \rangle$ is nonzero and hence ϕ_0 does not belong to the set

$$\mathcal{T}_0 = \{\phi \in X : \langle u, \phi \rangle = \langle 1, \phi \rangle = 0\}.$$

Geometrically speaking, $\mathcal{T}_0$ is precisely the tangent space in X to Σ_0. Using the spectral resolution of the operator $\mathcal{L}[u]$, we begin the proof of Proposition 4.3 with the following lemma.

LEMMA 4.4. *Assume that T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are positive. Then $\langle \mathcal{L}[u]\phi, \phi \rangle > 0$ for every $\phi \in \mathcal{T}_0$ which is orthogonal to the periodic null space of $\mathcal{L}[u]$.*

Proof. The proof is essentially found in the work of Bona, Souganidis, and Strauss (1987). By Lemma 4.2 we can write $\phi_0 = \alpha\chi + \beta u_x + p$ and $\phi = A\chi + \widetilde{p}$ for some constants α, β, and A and functions p and $\widetilde{p}$ belonging to the positive subspace of $\mathcal{L}[u]$. By assumption the quantity

$$(4.3) \qquad \langle \mathcal{L}[u]\phi_0, \phi_0 \rangle = -\{T, M\}_{a,E}\{T, M, P\}_{a,E,c}$$

is negative, and hence the above decomposition of ϕ_0 implies that

$$(4.4) \qquad 0 > \left\langle -\lambda^2\alpha\chi + \mathcal{L}[u]p, \alpha\chi + \beta u_x + p \right\rangle = -\lambda^2\alpha^2 + \langle \mathcal{L}[u]p, p \rangle,$$

which gives an upper bound on the positive number $\langle \mathcal{L}[u]p, p \rangle$. Similarly, the assumption that $\phi \in \mathcal{T}_0$ along with the above decomposition of ϕ implies

$$(4.5) \qquad 0 = \langle \mathcal{L}[u]\phi_0, \phi \rangle = -\lambda^2 A\alpha + \langle \mathcal{L}[u]p, \widetilde{p} \rangle.$$

Therefore, a simple application of Cauchy–Schwarz implies

$$\langle \mathcal{L}[u]\phi, \phi \rangle = -\lambda^2 A^2 + \langle \mathcal{L}[u]\widetilde{p}, \widetilde{p} \rangle \geq -\lambda^2 A^2 + \frac{\langle \mathcal{L}[u]\widetilde{p}, p \rangle^2}{\langle \mathcal{L}[u]p, p \rangle} > 0$$

as claimed. $\square$

Remark 9. In the above proof, the positivity of the quantities $\{T, M\}_{a,E}$ and $\{T, M, P\}_{a,E,c}$ was never used; only the product was required to be positive. However, we show in Corollary 4.7 that the former is always positive if the latter is negative.

Also, if $\{T, P\}_{E,c} \neq 0$, then one can repeat the above proof with the function ϕ_0 replaced by $\widetilde{\phi}_0 = \{u, T, P\}_{a,E,c}$. Then (4.3) would be replaced with $\langle \mathcal{L}[u]\widetilde{\phi}_0, \widetilde{\phi}_0 \rangle = \{T, P\}_{E,c}\{T, M, P\}_{a,E,c}$, which we would have to assume to be negative. It follows that $\mathrm{sign}\,(\{T, M\}_{a,E}) = -\mathrm{sign}\,(\{T, P\}_{E,c})$ so long as $\{T, M, P\}_{a,E,c} \neq 0$. In particular, in the case of a power-law nonlinearity, $P_c < 0$ implies $\{T, M\}_{a,E} > 0$ if $\{T, M, P\}_{a,E,c} \neq 0$. It is unknown if $\{T, M, P\}_{a,E,c}$ is negative in this case.

Remark 10. Some comments are now in order concerning the function ϕ_0 and its relation to the work in Pava and Natali (2008), where again the authors consider the $a = 0$ case. Notice that ϕ_0 is essentially constructed to be the periodic version of the function $-\frac{d}{dc}u(x; c)$ from the solitary wave theory in that it is the preimage of the function u under $\mathcal{L}[u]$. Given the restriction $a = 0$ it follows by a direct calculation that the function ϕ_0 can be replaced with the function[10]

$$(4.6) \qquad -\frac{d}{dc}u(x; 0, E(c), c) = \frac{1}{T_E}\{u, T\}_{E,c},$$

---

[10]However, notice that the function in (4.6) does not have zero mean over a period; by the dynamic version of the spectral problem it follows that all nontrivial time dynamics occur on the range of the operator ∂_x on $L^2_{\mathrm{per}}([0, T])$, i.e., the class of mean zero functions.

and hence (4.3) in this case would read

$$\frac{d}{dc}\int_0^{T(0,E(c),c)} u(x;0,E(c),c)^2 dx = 2\left\langle u,\frac{1}{T_E}\{u,T\}_{E,c}\right\rangle$$
$$= \frac{2}{T_E}\{P,T\}_{E,c} < 0.$$

Thus, in the language of the present paper the requirement (P_3) given by Pava and Natali (2008) is equivalent with the product T_E and $\{P,T\}_{E,c}$ being negative. Thus, our results are completely equivalent with the results presented in Pava and Natali (2008) in the case where $a = 0$. The difference in our methods is essentially the following: in Pava and Natali (2008), the authors spend a lot of time constructing the curve $(0,E(c),c) \subset \Omega$ on which the period is constant, but according to our results the construction of this curve is not important; the local tangent vectors give you sufficient information. Moreover, the geometric description presented here allows for, at least for gKdV type equations, an easy verification of properties $(P_0) - (P_2)$ given in Pava and Natali (2008). However, recall here that we consider all of Ω, not just the codimension one subset $\Omega\big|_{a=0}$, and that it is not clear how the Jacobian $\{P,T\}_{E,c}$ relates to the spectral stability of the underlying wave.

Our strategy in proving Proposition 4.3 is to find a particular set of translates of a given $\phi \in \mathcal{U}_\varepsilon$ for which the inequality holds. To this end, we show that for each $\phi \in \mathcal{U}_\varepsilon$ with ε sufficiently small, there exists a set of translates of ϕ which are orthogonal to the periodic null space of $\mathcal{L}[u]$. This is the content of the following lemma.

LEMMA 4.5. *There exist an $\varepsilon > 0$ and a unique C^1 map $\alpha : \mathcal{U}_\varepsilon \to \mathbb{R}$ such that for all $\phi \in \mathcal{U}_\varepsilon$, the function $\phi(\cdot + \alpha(\phi))$ is orthogonal to u_x.*

The proof is presented in the work of Bona, Souganidis, and Strauss (1987) and is an easy result of the implicit function theorem. We now complete the proof of Proposition 4.3 by proving the following lemma.

LEMMA 4.6. *If each of the quantities T_E, $\{T,M\}_{a,E}$, and $\{T,M,P\}_{a,E,c}$ are positive, there exist positive constants $\widetilde{C}$ and ε such that*

$$\mathcal{E}_0(\phi) - \mathcal{E}_0(u) \geq \widetilde{C}\|\phi(\cdot + \alpha(\phi)) - u\|_X^2$$

for all $\phi \in \mathcal{U}_\varepsilon \cap \Sigma_0$.

Proof. Let $\varepsilon > 0$ be small enough such that Lemma 4.5 holds. Fix $\phi \in \mathcal{U}_\varepsilon \cap \Sigma_0$, and write

$$\phi(\cdot + \alpha(\phi)) = (1 + \gamma)u + \left(\beta - \frac{\gamma\langle u\rangle}{T}\right) + y,$$

where $y \in \mathcal{T}_0$. Moreover, define $v = \phi(\cdot + \alpha(\phi)) - u$ and note that by replacing u with $R_\xi u$ if necessary we can assume that $\|v\|_X < \varepsilon$. By Taylor's theorem, we have

$$M(a_0, E_0, c_0) = \mathcal{M}(\phi) = M(a_0, E_0, c_0) + \langle 1, v\rangle + \mathcal{O}\left(\|v\|_X^2\right).$$

Since $\langle 1, v\rangle = \beta T$, it follows that $\beta = \mathcal{O}\left(\|v\|_X^2\right)$. Similarly, we have

$$P(a_0, E_0, c_0) = P(a_0, E_0, c_0) + \langle u, v\rangle + \mathcal{O}\left(\|v\|_X^2\right).$$

Moreover, defining $\langle g\rangle = \int_0^T g(x)dx$ for $g \in L^1_{\mathrm{per}}([0,T];\mathbb{R})$, a direct calculation yields

$$\langle u, v\rangle = \gamma\left(\|u\|_{L^2([0,T])}^2 - \frac{\langle u\rangle^2}{T}\right) + \beta\langle u\rangle.$$

Since $\langle u\rangle^2 < T\|u\|_{L^2([0,T])}^2$ by Jensen's inequality, it follows that $\gamma = \mathcal{O}\left(\|v\|_X^2\right)$.

Now, by Taylor's theorem and the translation invariance of $\mathcal{E}_0$, we have

$$\mathcal{E}_0(\phi) = \mathcal{E}_0\left(\phi(\cdot + \alpha(\phi))\right) = \mathcal{E}_0(u) + \frac{1}{2}\langle \mathcal{L}[u]v, v\rangle + o\left(\|v\|_X^2\right).$$

Hence, by the previous estimates on γ and β, it follows that

$$\mathcal{E}_0(\phi) - \mathcal{E}_0(u) = \frac{1}{2}\langle \mathcal{L}[u]v, v\rangle + o\left(\|v\|_X^2\right)$$
$$= \frac{1}{2}\langle \mathcal{L}[u]y, y\rangle + o\left(\|v\|_X^2\right).$$

Since $y \in \mathcal{T}_0$ and $\langle y, u_x\rangle = 0$ by Lemma 4.5, it follows from Lemma 4.4 that

$$\mathcal{E}_0(\phi) - \mathcal{E}_0(u) \geq \frac{C_1}{2}\|y\|^2 + o\left(\|v\|_X^2\right).$$

Finally, the estimates

$$\|y\|_X = \left\|v - \gamma u - \beta - \frac{\gamma\langle u\rangle}{T}\right\|_X$$
$$\geq \left|\|v\|_X - \left\|\gamma u - \beta - \frac{\gamma\langle u\rangle}{T}\right\|_X\right|$$
$$\geq \|v\|_X - \mathcal{O}\left(\|v\|_X^2\right)$$

prove that $\mathcal{E}_0(\phi) - \mathcal{E}_0(u) \geq \frac{C_1}{4}\|v\|_X^2$ for $\|v\|_X$ sufficiently small. $\square$

Proposition 4.3 now clearly follows by Lemma 4.6 and the definition of the semidistance ρ. It is now straightforward to complete the proof of Lemma 4.1.

Proof of Lemma 4.1. We now deviate from the methods of Bona, Souganidis, and Strauss (1987) and Grillakis, Shatah, and Strauss (1987) and rather follow the direct method of Gallay and Hărăguş (2007). Let $\delta > 0$ be such that Proposition 4.3 holds, and let $\varepsilon \in (0, \delta)$. Assume $\phi_0 \in X$ satisfies $\rho(\phi_0, u) \leq \varepsilon$ for some small $\varepsilon > 0$. By replacing ϕ_0 with $R_\xi\phi_0$ if needed, we may assume that $\|\phi_0 - u\|_X \leq \varepsilon$. Since u is a critical point of the functional $\mathcal{E}_0$, it is clear that we have $\mathcal{E}_0(\phi_0) - \mathcal{E}_0(u) \leq C_1\varepsilon^2$ for some positive constant C_1. Now, notice that if $\phi_0 \in \Sigma_0$, then the unique solution $\phi(\cdot, t)$ of (1.1) with initial data ϕ_0 satisfies $\phi(\cdot, t) \in \Sigma_0$ for all $t > 0$. Thus, Proposition 4.3 implies there exists a $C_2 > 0$ such that $\rho(\phi(\cdot, t), u) \leq C_2\varepsilon$ for all $t > 0$. Thus, $\phi(\cdot, t) \in \mathcal{U}_\varepsilon$ for all $t > 0$, which proves Lemma 4.1 in this case.

If $\phi_0 \notin \Sigma_0$, then we claim we can vary the constants (a, E, c) slightly in order to effectively reduce this case to the previous one. Indeed, notice that since we have assumed $\{T, M, P\}_{a,E,c} \neq 0$ at (a_0, E_0, c_0), it follows that the map

$$(a, E, c) \mapsto (T(u(\,\cdot\,;a, E, c)), M(u(\,\cdot\,;a, E, c)), P(u(\,\cdot\,;a, E, c)))$$

is a diffeomorphism from a neighborhood of (a_0, E_0, c_0) onto a neighborhood of

$$(T(a_0, E_0, c_0), M(a_0, E_0, c_0), P(a_0, E_0, c_0)).$$

In particular, we can find constants a, E, and c with $|a| + |E| + |c| = \mathcal{O}(\varepsilon)$ such that the function

$$\widetilde{u} = \widetilde{u}(\,\cdot\,;a_0 + a, E_0 + E, c_0 + c)$$

solves (1.1), belongs to the space X, and satisfies

$$M(a_0 + a, E_0 + E, c_0 + a) = \mathcal{M}(\phi_0),$$
$$P(a_0 + a, E_0 + E, c_0 + c) = \mathcal{P}(\phi_0).$$

Defining a new augmented energy functional on X by

$$\widetilde{\mathcal{E}}(\phi) = \mathcal{E}_0(\phi) + c\mathcal{P}(\phi) + a\mathcal{M}(\phi) + ET,$$

it follows as before that

$$\widetilde{\mathcal{E}}(\phi(\cdot, t)) - \widetilde{\mathcal{E}}(\widetilde{u}) \geq C_3 \rho(\phi(\cdot, t), \widetilde{u})^2$$

for some $C_3 > 0$ as long as $\rho(\phi(\cdot, t), \widetilde{u})$ is sufficiently small. Since $\widetilde{u}$ is a critical point of the functional $\widetilde{\mathcal{E}}$, we have

$$C_3 \rho(\phi(\cdot, t), \widetilde{u})^2 \leq \widetilde{\mathcal{E}}(\phi_0) - \widetilde{\mathcal{E}}(\widetilde{u}) \leq C_4 \|\phi_0 - \widetilde{u}\|_X^2$$

for some $C_4 > 0$. Moreover, it follows by the triangle inequality that

$$\|\phi_0 - \widetilde{u}\|_X \leq \|\phi_0 - u\|_X + \|u - \widetilde{u}\|_X \leq C_5 \varepsilon$$

for some $C_5 > 0$, and hence there is a $C_6 > 0$ such that

$$\rho(\phi(\cdot, t), u) \leq \rho(\phi(\cdot, t), \widetilde{u}) + \|\widetilde{u} - u\|_X \leq C_6 \varepsilon$$

for all $t > 0$. The proof of Lemma 4.1, and hence Theorem 1.1, is now complete.   □

We would like to point out an interesting artifact of the above proof. Notice that the only stage at which the sign of the quantities $\{T, M, P\}_{a,E,c}$ and $\{T, M\}_{a,E}$ came into play was in the proof of Lemma 4.4, from which we have the following corollary by Theorem 3.2 and (4.3).

COROLLARY 4.7. *On the set Ω, the quantity $\{T, M\}_{a,E}$ is positive whenever T_E is positive and $\{T, M, P\}_{a,E,c}$ is negative.*

Remark 11. Interestingly, in terms of the classical action Corollary 4.7 states that the Hessian of K given in (1.6) cannot be positive definite. We suspect that the underlying reason for this is that the operator $\mathcal{L}[u]$ on $L_{\text{per}}^2([0, T])$ cannot be positive definite due to the translation invariance of (1.1), but we cannot show this as of yet. In any case, it would be interesting to understand the mechanism behind this result as it could possibly illuminate more of the relationship between the dynamics of the gKdV flow and the classical mechanics of the traveling wave ODE.

It follows that we have a geometric theory of the orbital stability of periodic traveling wave solutions of (1.1) to perturbations of the same period as the underlying periodic wave. In the next two sections, we consider specific examples and limiting cases where the signs of these quantities can be easily calculated. First, we consider periodic traveling wave solutions sufficiently close to an equilibrium solution (a local minimum of the effective potential) or to the bounding homoclinic orbit (the separatrix solution). By considering power-law nonlinearities in each of these cases, we give necessary and sufficient[11] conditions for the orbital stability of such solutions. Second, we consider the KdV and prove that all periodic traveling wave solutions are orbitally stable to perturbations of the same periodic as the underlying periodic wave.

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[11]Except in the exceptional case of being near the homoclinic orbit for $p = 4$.

5. Analysis near homoclinic and equilibrium solutions. In this section, we use the theory from section 4 in order to prove general results about the stability of periodic traveling wave solutions of (1.1) in two distinguished limits: as one approaches the solitary wave (i.e., $(a, E, c) \in \Omega$ and consider the limit $T(a, E, c) \to \infty$ for fixed a, c as well as in a neighborhood of the equilibrium solution (i.e., near a nondegenerate local minimum of the effective potential $V(u; a, c)$). Throughout this section, we will consider only power-law nonlinearities.

We begin with considering stability near the solitary wave. Our main result in this limit is that the quantities T_E and $\{T, M\}_{a,E}$ are positive for $(a_0, E_0, c_0) \in \Omega$ with sufficiently large period. Hence, the orbital stability of such a solution in this limit is determined completely by the periodic spectral stability index $\{T, M, P\}_{a,E,c}$, which in turn is controlled by the sign of the solitary wave stability index (1.3). To begin, we point out a result of Schaaf (1985) which gives sufficient conditions for a planar Hamiltonian system to have the period increasing as a function of energy.

LEMMA 5.1. *Assume that G is a C^3 function on $(0, \infty)$ and that G vanishes only at one point x_0 with $G''(x_0) > 0$. Define*

$$A = \{x \in \mathbb{R} : x < x_0 \;\; \text{and} \;\; G(s) < 0 \;\; \text{for all} \;\; s \in (x, x_0)\}.$$

Suppose for each $\alpha \in A$ there exists a periodic solution $x(t) > 0$ of the equation

$$x''(t) + G(x(t)) = 0$$

with initial data $x(0) = \alpha$, $x'(0) = 0$. Let $P(\alpha)$ denote the period of this solution. If G satisfies the two conditions

$$G'(x) > 0, \;\; x \in A \;\; \Rightarrow \;\; 5G''(x)^2 - 3G'(x)G'''(x) > 0;$$
$$G'(x) = 0, \;\; x \in A \;\; \Rightarrow \;\; G(x)G''(x) < 0,$$

then P is differentiable on A and $P'(\alpha) > 0$.

In order to see that Lemma 5.1 applies in our case, we assume the equation is scaled such that $c = 1$, and we define $G(x) := V'(x, a, 1)$ and assume $G(x_a) = 0$, $G'(x_a) > 0$. Define A as above, and notice that $G \in C^3(0, \infty)$. Then for all x such that $G'(x) > 0$, we have $x^p > \frac{1}{p+1}$ and hence

$$\left(5G''^2 - 3G'G'''\right)(x) > 5p^2(p+1)x^{p-2} > 0.$$

Moreover, if $G'(x) = 0$, then $x^p = \frac{1}{p+1}$ and hence

$$G(x)G''(x) = p\left(\frac{1}{p+1} - 1 - a(p+1)x^{p-1}\right).$$

If we assume that a is such that

$$(5.1) \qquad a\left(\frac{1}{p+1}\right)^{(p-1)/p} > \frac{1}{(p+1)^2} - \frac{1}{p+1},$$

then it follows that $G'(x)G''(x) < 0$ when $x^p = \frac{1}{p+1}$. In particular, notice that given any $p > 1$, the inequality (5.1) holds for all $a \geq 0$ and for $a < 0$ sufficiently small. In any case, it follows that we have the following lemma.

LEMMA 5.2. *In the case of a power-law nonlinearity $f(u) = u^{p+1}$, suppose that $(a_0, E_0, 1) \in \Omega$ and assume a_0 and p satisfy (5.1) and that $u(\;\cdot\;; a_0, E_0, 1) > 0$. Then $T_E > 0$ at (a_0, E_0).*

With this in mind, we are able to state our main theorem on the orbital stability of dnoidal type periodic traveling waves of sufficiently large period.

THEOREM 5.3. *In the case of a power-law nonlinearity, i.e., $f(u) = u^{p+1}$ with $p \geq 1$, a dnoidal type periodic traveling wave solution of (1.1) of sufficiently large period and $(a, E, c) \in \Omega$ with $|a|$ sufficiently small is orbitally stable if $p < 4$ and exponentially unstable to perturbations of the same period as the underlying wave if $p > 4$.*

Proof. The proof follows by the work of Bronski and Johnson (2008). In particular, since we are working with a power-law nonlinearity, the periodic traveling wave solutions satisfy the scaling relation

$$(5.2) \qquad u(x; a, E, c) = c^{1/p} u\left(c^{1/2}x; \frac{a}{c^{1+1/p}}, \frac{E}{c^{1+2/p}}, 1\right)$$

from which we get the asymptotic relation

$$\{T, M, P\}_{a,E,c} \sim -T_E M_a \left(\frac{2}{pc} - \frac{1}{2c}\right) P$$

as $\Omega \ni (a, E, c) \to (0, 0, c)$ for a fixed wave speed. Moreover, the quantity M_a is negative for such $(a, E, c) \in \Omega$, again by the work on Bronski and Johnson (2008). Since $\{T, M\}_{a,E} = M_E^2 - T_E M_a$, it follows from Lemma 5.1 and Theorem 3.2 that the solutions $u(x; a, E, c)$ with $(a, E, c) \in \Omega$ of sufficiently large period are orbitally stable if $p < 4$ and exponentially unstable to periodic perturbations if $p > 4$ as claimed. $\qquad \square$

Next, we consider periodic traveling wave solutions near the equilibrium solution. We will use the methods of this paper to prove that such solutions are orbitally stable to periodic perturbations, provided that a is sufficiently small. To begin, we fix a wave speed $c > 0$, assume $|a| \ll 1$, and consider (1.1) with a power-law nonlinearity $f(u) = u^{p+1}$ with $p \geq 1$. Since $T_E > 0$ by Lemma 5.1, it suffices to prove that $\{T, M\}_{a,E}$ and $\{T, M, P\}_{a,E,c}$ are both positive near the equilibrium solution. By continuity, it is enough to evaluate both these indices at the equilibrium and to show they are both positive there. This is the content of the following lemma.

LEMMA 5.4. *Consider (1.1) with a power-law nonlinearity $f(u) = u^{p+1}$ for $p \geq 1$. Then the quantity M_a is negative for all $(a_0, E_0, c_0) \in \Omega$ such that $|a|$ is sufficiently small and the corresponding solution $u(\,\cdot\,; a_0, E_0, c_0)$ is sufficiently close to the equilibrium solution.*[12]

Proof. First, denote the equilibrium solution as $u_{a,c}$ and define $E^*(a, c) = V(u_{a,c}; a, c)$. It follows that $\lim_{E \searrow E^*} T(a, E, c) = \frac{2\pi}{\sqrt{cp}}$ and that the equilibrium solution admits the expansion

$$u_{a,c} = c^{1/p}\left(1 + \frac{a}{p}\right) + \mathcal{O}(a^2).$$

Now, solutions near the equilibrium $u_{a,c}$ can each be written as

$$u(x; a, E, c) = P_{a,E,c}(k_{a,E,c}x),$$

where $k_{a,E,c}T(a, E, c) = 2\pi$ and $P_{a,E,c}$ is a 2π periodic solution of the ODE

$$k_{a,E,c}^2 v'' + v^{p+1} - c^{1+1/p}a = 0$$

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[12]In the case of the KdV ($p = 1$), M_a is negative in a deleted neighborhood of the equilibrium solution.

such that $P_{a,E^*,c} = u_{a,c}$ and $k^2_{a,E^*,c} = (p+1)u^p_{a,c} - c$. Straightforward computations give the expansions

$$P_{a,E,c}(z) = u_{a,c} + \mathcal{O}(\sqrt{E-E^*}\,(1+a^2)),$$
$$k^2_{a,E,c} = cp + (p+1)ca + \mathcal{O}((E-E^*) + a^2).$$

Thus, the mass $M(a, E, c)$ can be expanded as

$$M(a, E, c) = \int_0^{2\pi/k_{a,E,c}} P_{a,E,c}(k_{a,E,c}z)dz$$
$$= \frac{2\pi c^{1/p}}{\sqrt{cp}} \left(1 + \frac{(1-p)a}{2p}\right) + \mathcal{O}(\sqrt{E-E^*} + a^2).$$

It follows that

$$\frac{\partial}{\partial a} M(a, E, c)\big|_{(0,E^*,c)} = \frac{\pi c^{1/p}(1-p)}{p\sqrt{cp}}$$

which is negative for $p > 1$.

The case $p = 1$, which corresponds to the KdV equation, will be discussed in the next section. There we will show that although M_a vanishes *at* the equilibrium solution, it is indeed negative for *nearby* periodic traveling waves with the same wave speed c, i.e., $\frac{\partial^2}{\partial E \partial a} M(a, E, c)\big|_{(0,E^*,c)} < 0$. ☐

Next, we must determine the sign of the periodic spectral stability index. Although it follows from Theorem 4.4 in the work of Hărăguş and Kapitula (2008) that this index must be positive, we present an independent proof based on periodic Evans function methods. To this end, we point out that by the Hamiltonian structure of the linearized operator $\partial_x \mathcal{L}[u]$ we have the identity

$$\{T, M, P\}_{a,E,c} = -\frac{2}{3} \operatorname{tr}\left(\mathbf{M}_{\mu\mu\mu}(0)\right),$$

where $\mathbf{M}(\mu)$ is the corresponding monodromy operator (see Theorem 3 of Bronski and Johnson (2008) for details). Thus, it is sufficient to show that $\operatorname{tr}(\mathbf{M}_{\mu\mu\mu}(0))$ is negative near the equilibrium solution. This is the content of the next lemma.

LEMMA 5.5. *Consider* (1.1), *and suppose* u_0 *is a nondegenerate local minima of the corresponding effective potential* $V(u; a, c)$. *Then* $\operatorname{tr}\left(\mathbf{M}_{\mu\mu\mu}(0)\right) < 0$ *at* u_0.

Proof. The key point is that if we write (3.1) as a first order system of the form $\Phi_x = \mathbf{H}(x, \mu)\Phi$ by the usual procedure, then the matrix $\mathbf{H}(x, \mu)$ reduces to the (spatially) constant matrix

$$\mathbf{H}(\mu) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu & -V''(u_0; a, c) & 0 \end{pmatrix}$$

at the equilibrium solution u_0. Thus, the corresponding monodromy operator at u_0 can be expressed as $\mathbf{M}(\mu) = \exp\left(\mathbf{H}(\mu)T_0\right)$, where $T_0 = \frac{2\pi}{\sqrt{V''(u_0)}}$. Thus, in order to calculate the function $\operatorname{tr}(\mathbf{M}(\mu))$, it is sufficient to calculate the eigenvalues of the constant matrix $\mathbf{H}(\mu)$.

Now, the periodic Evans function corresponding to the constant coefficient system induced by $\mathbf{H}(\mu)$ can be written as

$$D_0(\mu, \lambda) = \det\left(\mathbf{H}(\mu) - \lambda\mathbf{I}\right) = -\lambda^3 - V''(u_0; a, c)\lambda - \mu.$$

In particular, notice that $\frac{\partial}{\partial \lambda} D_0(\mu, \lambda) = -\lambda^2 - V''(u_0; a, c)$. Since $V''(u_0; a, c) > 0$, it follows that the function $D_0(\mu, \ \cdot \ )$ will have precisely one real root for each $\mu \in \mathbb{R}$. This distinguished root is given by the formula

$$\gamma_1(\mu) = \underbrace{\frac{\left(\frac{2}{3}\right)^{1/3} V''(u_0)}{\left(9\mu + \sqrt{3}\sqrt{27\mu^2 + 4V''(u_0)^3}\right)^{1/3}}}_{=:\alpha(\mu)} + \underbrace{\left(-\frac{\left(9\mu + \sqrt{3}\sqrt{27\mu^2 + 4V''(u_0)^3}\right)^{1/3}}{2^{1/3}3^{2/3}}\right)}_{=:\beta(\mu)}.$$

Defining $\omega = \exp(2\pi i/3)$ to be the principle third root of unity, the two complex eigenvalues of $\mathbf{H}(\mu)$ can be written as $\gamma_2(\mu) = \omega\alpha(\mu) + \overline{\omega}\beta(\mu)$ and $\gamma_3(\mu) = \overline{\omega}\alpha(\mu) + \omega\beta(\mu)$, and hence

$$\text{tr}\left(\mathbf{M}(\mu)\right) = \exp\left(\gamma_1(\mu)T_0\right) + \exp\left(\gamma_2(\mu)T_0\right) + \exp\left(\gamma_3(\mu)T_0\right).$$

Now, a straightforward, yet tedious, calculation using the facts that $1 + \omega + \overline{\omega} = 0$ and $\omega^2 = \overline{\omega}$ implies that

$$\text{tr}\left(\mathbf{M}_{\mu\mu\mu}(0)\right) = 9T_0^2\left(\alpha''(0)\beta'(0) + \alpha'(0)\beta''(0)\right) + 3T_0^3\left(\alpha'(0)^3 + \beta'(0)^3\right).$$

Moreover, from the definitions of α and β we have

$$\alpha'(0) = -\frac{1}{2V''(u_0)} = \beta'(0) \ \text{ and } \ \alpha''(0) = \frac{\sqrt{3}}{4V''(u_0)^{5/2}} = -\beta''(0).$$

Therefore, we have the equality

$$\text{tr}\left(\mathbf{M}_{\mu\mu\mu}(0)\right) = -\frac{6\pi^3}{V''(u_0)^{9/2}},$$

which is clearly negative.   □

Therefore, it follows that in the case of a power-nonlinearity and solutions sufficiently close to a nondegenerate minima of the effective potential, each of the quantities T_E, $\{T, M\}_{a,E}$, and $\{T, M, P\}_{a,E,c}$ are positive. Therefore, Theorem 1.1 immediately yields the following result.

THEOREM 5.6. *Consider* (1.1) *with a power-law nonlinearity* $f(u) = u^{p+1}$ *for* $p \geq 1$. *Then the periodic traveling wave solutions with* $(a, E, c) \in \Omega$ *and* $a^2 + (E - E^*)^2$ *sufficiently small are orbitally stable in the sense of Theorem* 1.1.

6. The KdV equation. In this section, we will apply the general theory from section 4 in order to prove that periodic traveling wave solutions of (1.1) with $f(u) = u^2$ and $c > 0$ are orbitally stable with respect to periodic perturbations if and only if they are spectrally stable to such perturbations. Although all solutions of this equation are cnoidal of the form (1.5), our proof does not rely on the explicit form of the solution; instead, we use the form of the traveling wave ODE to reduce such solutions to quadrature and then apply the theory from section 3. To this end, recall from (5.2) that solutions of the KdV equation

(6.1) $u_t = u_{xxx} + (u^2)_x - cu_x$

satisfy the scaling relation $u(x; a, E, c) = c\, u\left(c^{1/2}x; ac^{-1}, Ec^{-3}, 1\right)$. Thus, by scaling we may always assume that $c = 1$ in (6.1); notice that by the transformation $u \mapsto -u$,

once can always consider the KdV equation with positive wave speeds, and there is no loss in generality in assuming that $c = 1$ here. Moreover, we may always assume that $a = 0$ due to the Galilean invariance of the KdV. Therefore, it is sufficient to determine the stability of periodic traveling wave solutions of (6.1) of the form $u(x; 0, E, 1)$. In order to do so, we need the following easily proved lemma.

LEMMA 6.1. *Let μ be a (Borel) probability measure on some interval $I \subset \mathbb{R}$, and let $f, g : I \to \mathbb{R}$ be bounded and measurable functions. Then*

$$(6.2) \qquad \langle fg \rangle - \langle f \rangle \langle g \rangle = \frac{1}{2} \int_{I \times I} (f(x) - f(y)) (g(x) - g(y)) \, d\mu_x d\mu_y,$$

where $\langle f \rangle = \int_I f(x) d\mu$. In particular, if both f and g are strictly increasing or strictly decreasing and if the support of μ is not reduced to a single point, then $\langle fg \rangle > \langle f \rangle \langle g \rangle$.

The proof of this lemma is a trivial result of Fubini's theorem, as one can see by writing the left-hand side of (6.2) as an iterated integral and simplifying the resulting expression. Now, recall from Lemma 5.1 that $T_E > 0$ for periodic traveling wave solutions of (6.1). To conclude orbital stability, we must identify the signs of the Jacobians $\{T, M\}_{a,E}$ and $\{T, M, P\}_{a,E,c}$. The main technical result we need for this section is the following lemma, which uses Lemma 6.1 to guarantee the sign of the quantity in (4.3) is completely determined by the Jacobian $\{T, M, P\}_{a,E,c}$.

LEMMA 6.2. *If $f(u) = u^2$ in (1.1), then $\{T, M\}_{a,E} > 0$ for all $(a_0, E_0, c_0) \in \Omega$ which do not correspond to the unique equilibrium solution.*

Proof. First, notice that $\{T, M\}_{a,E} = M_E^2 - T_E M_a$, and thus by Lemma 5.1 it is enough to prove that $M_a < 0$. Moreover, by scaling it is enough to consider the case $c = 1$ and $a = 0$. It follows for f given as above that we can find functions u_1, u_2, and u_3 which depend smoothly on (a, E, c) within the domain Ω such that

$$3 \left(E - V(u; 0, 1) \right) = (u - u_1)(u - u_2)(u_3 - u).$$

Notice that the assumption that we are not at the equilibrium solution implies that the roots u_i are distinct and moreover that $V'(u_i; 0, 1) \neq 0$. Since $E - V(u_i; 0, 1) = 0$ on Ω, it follows that

$$V'(u_i; 0, 1) \frac{\partial u_i}{\partial a} = u_i.$$

Since we have the relations

$$u_1 < 0, \quad u_2, \ u_3 > 0, \quad \text{and} \quad u_1 + u_2 + u_3 = \frac{3c}{2},$$

it follows that on Ω we have

$$(6.3) \qquad \frac{\partial u_2}{\partial a} < 0, \quad \frac{\partial u_3}{\partial a} > 0, \quad \text{and} \quad \frac{\partial u_2}{\partial a} + \frac{\partial u_3}{\partial a} = -\frac{\partial u_1}{\partial a} > 0.$$

Now, by making the change of variables $u \mapsto s(\theta) = u_2 \cos^2(\theta) + u_2 \sin^2(\theta)$, we have $du = 2\sqrt{(u - u_2)(u_3 - u)}d\theta$, and hence (2.5) yields the expression

$$(6.4) \qquad M(a, E, c) = 2\sqrt{6} \int_0^{\pi/2} \frac{s(\theta) \, d\theta}{\sqrt{s(\theta) - u_1}}.$$

Notice we suppress the dependence of $s(\theta)$ on the parameters (a, E, c). Defining $\sigma(\theta) = \sqrt{s(\theta) - u_1}$, a straightforward computation using (6.3) shows that the derivative of the integrand in (6.4) with respect to the parameter a can be expressed as

$$\frac{\partial}{\partial a} \left(\frac{s(\theta)}{\sqrt{s(\theta) - u_1}} \right) = \frac{\partial u_2}{\partial a} \left(\frac{\cos^2(\theta) - \sin^2(\theta)}{2\sigma(\theta)} \right) - u_1 \frac{\partial u_2}{\partial a} \left(\frac{\cos^2(\theta) - \sin^2(\theta)}{2\sigma(\theta)^3} \right)$$
$$- \left(\frac{\partial u_2}{\partial a} + \frac{\partial u_3}{\partial a} \right) \left(\frac{s(\theta) \cos^2(\theta) - u_1 \left(\cos^2(\theta) - \sin^2(\theta) \right)}{2\sigma(\theta)^3} \right).$$

Since the functions $\cos^2(\theta) - \sin^2(\theta)$ and $\sigma(\theta)^{-1}$ are strictly decreasing on the interval $(0, \pi/2)$, it follows from Lemma 6.1 that

$$\int_0^{\pi/2} \frac{\cos^2(\theta) - \sin^2(\theta)}{\sigma^m(\theta)} \, d\theta > 0$$

for any $m > 0$. Evaluating the above expression at $(a, E, c) = (0, E, 1) \in \Omega$ implies that $s(\theta) > 0$ for all $\theta \in (0, \pi/2)$, and hence (6.3) implies that

$$\int_0^{\pi/2} \frac{\partial}{\partial a} \left(\frac{s(\theta)}{\sqrt{s(\theta) - u_1}} \right) d\theta < 0$$

at $(0, E, 1)$, from which the lemma follows. $\square$

Therefore, our main theorem on the stability of periodic traveling wave solutions of the KdV equation follows by Theorems 1.1 and 3.2 and Lemma 6.2.

THEOREM 6.3. *Let $(a_0, E_0, c_0) \in \Omega$, and assume that $\{T, M, P\}_{a,E,c} \neq 0$ at (a_0, E_0, c_0). The corresponding periodic solution of (2.1) is orbitally stable in the sense of Theorem 1.1 if and only if it is spectrally stable to $T(a_0, E_0, c_0)$ periodic perturbations.*

An interesting corollary of Theorem 6.3 applies to cnoidal wave solutions of the KdV. It was suggested by Benjamin (1974) that such solutions should be stable to perturbations of the same period. As mentioned in the introduction this conjecture has been proved both by using the complete integrability of the KdV by McKean (1977) and by Bottman and Deconinck (2008), and by variational methods by Angulo, Bona, and Scialom (2006). In particular, the work of Bottman and Deconinck (2008) shows that the cnoidal solutions of the KdV are spectrally stable to localized perturbations and are linearly stable to perturbations with the same period as the underlying wave. Clearly then such solutions are spectrally stable with respect to periodic perturbations. Paired with Theorem 6.3, this provides another verification of Benjamin's conjecture in the case where the cnoidal wave has positive wave speed. Moreover, in each of the above papers the authors made use of the exact form of the cnoidal wave solution, and their computations involved tedious elliptic function calculations. In contrast, notice in the proof presented above that we never used the explicit form of the solution; we needed only information about the effective potential and the governing PDE. To summarize, we note the following corollary of the above calculations; as noted above, the restriction on positive wave speed can be relaxed here due to scaling.

COROLLARY 6.4. *The cnoidal wave solutions of (6.1) of the form (1.5) are orbitally stable in the sense of Theorem 1.1.*

7. Concluding remarks. In this paper, we extended the recent results of Bronski and Johnson (2008) on the spectral stability of periodic traveling wave solutions of the gKdV in order to derive sufficient conditions for the orbital stability of the full four-parameter family of periodic traveling wave solutions. By extending the methods of Bona, Souganidis, and Strauss to the periodic case, a new geometric condition was found which could be expressed in terms of a map from the traveling wave parameters to the conserved quantities of the PDE flow restricted to the manifold of periodic traveling waves. Moreover, this paper extended the general results of Pava and Natali (2008) on the gKdV equation with power-law nonlinearity to the case where $a \neq 0$. However, there are still many intriguing questions remaining.

Notice that it is not clear what happens in the case $T_E < 0$; we suspect this is not sufficient to determine orbital instability, even though the linear operator $\mathcal{L}[u]$ acting on $L^2_{\mathrm{per}}([0, T])$ has two negative eigenvalues in this case. In the solitary wave theory, the existence of two negative eigenvalues of the second variation $\mathcal{L}[u]$ indicates instability. However, it is not clear if this is true in the periodic context. Moreover, the role of the Jacobian $\{T, M\}_{a,E}$ is not clear from the above work. As noted previously, the sign of this Jacobian plays no role in the spectral stability theory, but by (4.3) it plays a large role in the nonlinear stability theory. Thus, one is left with the possibility of having a spectrally stable gKdV periodic traveling wave which is not orbitally stable to perturbations of the same period; this stands in stark contrast to the solitary wave case, where orbital stability is equivalent to spectral stability (except possibly on the transition curve). However, it seems quite possible that such a situation arises due to the fact that the solitary waves are a codimension two subset of the family of traveling wave solutions.

Finally, it is not clear how to extend these methods to consider perturbations whose period is, say, an integer multiple of that of the underlying wave. In particular, our method of proof breaks down in Lemma 4.4: if one considers perturbations of period NT with $N > 1$, the operator $\mathcal{L}[u]$ considered on $L^2_{\mathrm{per}}[(0, NT)]$ has more than one negative eigenvalue, and hence the inequality (4.4) no longer has enough information to determine if the quadratic form induced by $\mathcal{L}$ is positive definite on $\mathcal{T}_0$. Thus, the above methods are in need of modification in order to consider such perturbations.

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