A NOTE ON FINITE ABELIAN GERBES
OVER TORIC DELIGNE-MUMFORD STACKS

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Abstract. Any toric Deligne-Mumford stack is a \(\mu\)-gerbe over the underlying
toric orbifold for a finite abelian group \(\mu\). In this paper we give a sufficient
condition so that certain kinds of gerbes over a toric Deligne-Mumford stack
are again toric Deligne-Mumford stacks.

1. Introduction

Let \(\Sigma := (N, \Sigma, \beta)\) be a stacky fan of \(\text{rank}(N) = d\) as defined in [4]. If there
are \(n\) one-dimensional cones in the fan \(\Sigma\), then modelling the construction of toric
varieties [5], [6], the toric Deligne-Mumford stack \(\mathcal{X}(\Sigma) = [Z/G]\) is a quotient stack,
where \(Z = \mathbb{C}^n - V\), the close subvariety \(V \subset \mathbb{C}^n\) is determined by the ideal \(J_\Sigma\)
generated by \(\left\{\prod_{\rho, \sigma} z_\sigma : \sigma \in \Sigma\right\}\) and \(G\) acts on \(Z\) through the map \(\alpha : G \rightarrow (\mathbb{C}^*)^n\)
in the following exact sequence determined by the stacky fan (see [4]):

\[
1 \rightarrow \mu \rightarrow G \xrightarrow{\alpha} (\mathbb{C}^*)^n \rightarrow T \rightarrow 1.
\]

Let \(\overline{\alpha} = \text{Im}(\alpha)\). Then \([Z/\overline{G}]\) is the underlying toric orbifold \(\mathcal{X}(\Sigma_{\text{red}})\). The toric
Deligne-Mumford stack \(\mathcal{X}(\Sigma)\) is a \(\mu\)-gerbe over \(\mathcal{X}(\Sigma_{\text{red}})\).

Let \(\mathcal{X}(\Sigma)\) be a toric Deligne-Mumford stack associated with the stacky fan \(\Sigma\).
Let \(\nu\) be a finite abelian group, and let \(\mathcal{G}\) be a \(\nu\)-gerbe over \(\mathcal{X}(\Sigma)\). We give a
sufficient condition so that \(\mathcal{G}\) is also a toric Deligne-Mumford stack. We have the
following theorem:

**Theorem 1.1.** Let \(\mathcal{X}(\Sigma)\) be a toric Deligne-Mumford stack with stacky fan \(\Sigma\).
Then every \(\nu\)-gerbe \(\mathcal{G}\) over \(\mathcal{X}(\Sigma)\) is induced by a central extension

\[
1 \rightarrow \nu \rightarrow \tilde{G} \rightarrow G \rightarrow 1;
\]

i.e., we have a Cartesian diagram:

\[
\begin{array}{ccc}
\mathcal{G} & \longrightarrow & B\tilde{G} \\
\downarrow & & \downarrow \\
\mathcal{X}(\Sigma) & \longrightarrow & BG.
\end{array}
\]

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In general, the $\nu$-gerbe $G$ is not a toric Deligne-Mumford stack. But if the central extension is abelian, then we have:

**Corollary 1.2.** If the $\nu$-gerbe $G$ is induced from an abelian central extension, then it is a toric Deligne-Mumford stack.

This small note is organized as follows. In Section 2 we construct the new toric Deligne-Mumford stack from an abelian central extension and prove the main results. In Section 3 we give an example of a $\nu$-gerbe over a toric Deligne-Mumford stack.

In this paper, by an orbifold we mean a smooth Deligne-Mumford stack with trivial stabilizers at the generic points.

## 2. The proof of the main results

We refer the reader to [4] for the construction and notation of toric Deligne-Mumford stacks. For the general theory of stacks, see [2].

Let $\Sigma := (N, \Sigma, \beta)$ be a stacky fan. From Proposition 2.2 in [4], we have the following exact sequences:

$$0 \to DG(\beta)^* \to \mathbb{Z}^n \beta \to N \to \text{Coker}(\beta) \to 0,$$

$$0 \to N^* \to \mathbb{Z}^n \beta' \to DG(\beta) \to \text{Coker}(\beta') \to 0,$$

where $\beta'$ is the Gale dual of $\beta$. As a $\mathbb{Z}$-module, $\mathbb{C}^\times$ is divisible, so it is an injective $\mathbb{Z}$-module and hence the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ is exact. We get the exact sequence:

$$1 \to \text{Hom}_{\mathbb{Z}}(\text{Coker}(\beta'), \mathbb{C}^\times) \to \text{Hom}_{\mathbb{Z}}(DG(\beta), \mathbb{C}^\times) \to \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{C}^\times) \to \text{Hom}_{\mathbb{Z}}(N^*, \mathbb{C}^\times) \to 0.$$

Letting $\mu := \text{Hom}_{\mathbb{Z}}(\text{Coker}(\beta'), \mathbb{C}^\times)$, we have the exact sequence (1.1). Let $\Sigma(1) = n$ be the set of one-dimensional cones in $\Sigma$ and $V \subset \mathbb{C}^n$ the closed subvariety defined by the ideal generated by

$$J_\Sigma = \left\langle \prod_{\rho \notin \sigma} z_i : \sigma \in \Sigma \right\rangle.$$

Let $Z := \mathbb{C}^n \setminus V$. From [3], the complex codimension of $V$ in $\mathbb{C}^n$ is at least 2. The toric Deligne-Mumford stack $X(\Sigma) = [Z/G]$ is the quotient stack where the action of $G$ is through the map $\alpha$ in (1.1).

**Lemma 2.1.** If $\text{Codim}_{\mathbb{C}}(V, \mathbb{C}^n) \geq 2$, then $H^1(Z, \nu) = H^2(Z, \nu) = 0$, where $\nu$ is a finite abelian group.

**Proof.** Consider the following exact sequence:

$$0 \to H^0_\nu(\mathbb{C}^n, \nu) \to H^0(\mathbb{C}^n, \nu) \to H^0(Z, \nu) \to$$

$$\to H^1_\nu(\mathbb{C}^n, \nu) \to H^1(\mathbb{C}^n, \nu) \to H^1(Z, \nu) \to$$

$$\to H^2_\nu(\mathbb{C}^n, \nu) \to \cdots.$$

Since $\text{Codim}_{\mathbb{C}}(V, \mathbb{C}^n) \geq 2$, so the real codimension is at least 4 and $H^i_\nu(\mathbb{C}^n, \nu) = 0$ for $i = 1, 2, 3$, so from the exact sequence and $H^i(\mathbb{C}^n, \nu) = 0$ for all $i > 0$ we prove the lemma. □
2.1. The Proof of Theorem 1.1. Consider the following diagram:

\[
\begin{array}{ccc}
Z & \longrightarrow & pt \\
\downarrow & & \downarrow \\
[Z/G] & \longrightarrow & BG
\end{array}
\]

which is Cartesian. Consider the Leray spectral sequence for the fibration \(\pi\):

\[
H^p(BG, R^q\pi_*\nu) = \Rightarrow H^{p+q}([Z/G], \nu).
\]

We compute

\[
H^2([Z/G], \nu) = \bigoplus_{p+q=2} H^p(BG, R^q\pi_*\nu).
\]

First we have that \(R^q\pi_*\nu = [H^q(Z, \nu)/G]\). There are three cases.

1. When \(p = 2, q = 0\), \(R^0\pi_*\nu = \nu\) because \(Z\) is connected, so

\[
H^p(BG, R^q\pi_*\nu) = H^2(BG, \nu).
\]

2. When \(p = 1, q = 1\), \(R^1\pi_*\nu = [H^1(Z, \nu)/G]\), so

\[
H^p(BG, R^q\pi_*\nu) = H^1(BG, H^1(Z, \nu)),
\]

and by Lemma 2.1 \(H^1(Z, \nu) = 0\), so we have \(H^p(BG, R^q\pi_*\nu) = 0\).

3. When \(p = 0, q = 2\), \(R^2\pi_*\nu = [H^2(Z, \nu)/G]\), so

\[
H^p(BG, R^q\pi_*\nu) = H^0(BG, H^2(Z, \nu));
\]

also from Lemma 2.1 \(H^2(Z, \nu) = 0\), and so we have \(H^p(BG, R^q\pi_*\nu) = 0\).

So we get that

\[
H^2([Z/G], \nu) \cong H^2(BG, \nu).
\]

Since for the finite abelian group \(\nu\), the \(\nu\)-gerbes are classified by the second cohomology group with coefficient in the group \(\nu\), and Theorem 1.1 is proved.

2.2. The Proof of Corollary 1.2. Let \(X(\Sigma) = [Z/G]\). The \(\nu\)-gerbe \(G\) over \([Z/G]\) is induced from a \(\nu\)-gerbe \(\widetilde{G}\) over \(BG\) in the following central extension:

\[
1 \longrightarrow \nu \longrightarrow \widetilde{G} \xrightarrow{\varphi} G \longrightarrow 1,
\]

where \(\widetilde{G}\) is an abelian group. So the pullback gerbe over \(Z\) under the map \(Z \longrightarrow [Z/G]\) is trivial. So we have

\[
G = BG \times_{BG} [Z/G] = [Z/\widetilde{G}].
\]

The stack \([Z/\widetilde{G}]\) is this \(\nu\)-gerbe \(G\) over \([Z/G]\). Consider the commutative diagram:

\[
\begin{array}{ccc}
\widetilde{G} & \longrightarrow & G \\
\downarrow \alpha & & \downarrow \\
(C^\times)^n & \xrightarrow{\cong} & (C^\times)^n
\end{array}
\]

where \(\alpha\) is the map in (1.1). So we have the following exact sequences:

\[
1 \longrightarrow \nu \longrightarrow ker(\bar{\alpha}) \longrightarrow \mu \longrightarrow 1
\]

and

\[
1 \longrightarrow ker(\bar{\alpha}) \longrightarrow \widetilde{G} \xrightarrow{\hat{\alpha}} (C^\times)^n \longrightarrow T \longrightarrow 1,
\]
where $T$ is the torus of the simplicial toric variety $X(\Sigma)$. Since the abelian groups $\tilde{G}$, $G$ and $(\mathbb{C}^\times)^n$ are all locally compact topological groups, taking Pontryagin duality and the Gale dual, we have the following diagrams:

\[
\begin{array}{c}
0 \longrightarrow N^* \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \longrightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \longrightarrow \tilde{N}^* \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \longrightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \longrightarrow DG(\tilde{\beta})^* \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \longrightarrow 0,
\end{array}
\]

where $p_\varnothing$ is induced by $\varnothing$ in \(2.1\) under the Pontryagin duality. Suppose $\tilde{\beta} : \mathbb{Z}^n \longrightarrow \tilde{N}$ is given by \{\(b_1, \ldots, b_n\}\}, then $\Sigma := (\tilde{N}, \Sigma, \tilde{\beta})$ is a new stacky fan. The toric Deligne-Mumford stack $X(\Sigma) = [Z/G]$ is the $\nu$-gerbe $G$ over $X(\Sigma)$. □

Remark 2.2. From Proposition 4.6 in [3], any Deligne-Mumford stack is a $\nu$-gerbe over an orbifold for a finite group $\nu$. Our results are the toric case of that general result.

In particular, given a stacky fan $\Sigma = (N, \Sigma, \beta)$, let $\Sigma_{\text{red}} = (\overline{N}, \Sigma, \overline{\beta})$ be the reduced stacky fan, where $\overline{N}$ is the abelian group $N$ modulo torsion, and $\overline{\beta} : \mathbb{Z}^n \longrightarrow \overline{N}$ is given by \{\(\overline{b}_1, \ldots, \overline{b}_n\}\}, which are the images of \{\(b_1, \ldots, b_n\)\} under the natural projection $N \longrightarrow \overline{N}$. Then the toric orbifold $X(\Sigma_{\text{red}}) = [Z/G]$. From [1.1], let $G = Im(\alpha)$. Then we have the following exact sequences:

\[
\begin{align*}
1 & \longrightarrow \overline{G} \longrightarrow (\mathbb{C}^\times)^n \longrightarrow T \longrightarrow 1, \\
1 & \longrightarrow \mu \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1.
\end{align*}
\]

So $G$ is an abelian central extension of $\overline{G}$ by $\mu$. $X(\Sigma)$ is a $\mu$-gerbe over the toric orbifold $X(\Sigma_{\text{red}})$. Any $\mu$-gerbe over the toric orbifold coming from an abelian central extension is a toric Deligne-Mumford stack. This is a special case of the main results and is the toric case of rigidification construction in [1].

Remark 2.3. From the proof of Corollary 1.2 we see that if a $\nu$-gerbe over $X(\Sigma)$ comes from a gerbe over $BG$ and the central extension is abelian, then we can construct a new toric Deligne-Mumford stack.

3. An example

Example 3.1. Let $\Sigma$ be the complete fan of the projective line, $N = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, and $\beta : \mathbb{Z}^2 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ be given by the vectors \{\(b_1 = (1, 0), b_2 = (-1, 1)\)\}. Then $\Sigma = (N, \Sigma, \beta)$ is a stacky fan. We compute that $(\beta)^\vee : \mathbb{Z}^2 \longrightarrow DG(\beta) = \mathbb{Z}$ is given by the matrix $[3,3]$. So we get the following exact sequence:

\[
(3.1) \quad 1 \longrightarrow \mu_3 \longrightarrow \mathbb{C}^\times \longrightarrow (\mathbb{C}^\times)^2 \longrightarrow \mathbb{C}^\times \longrightarrow 1.
\]

The toric Deligne-Mumford stack is $X(\Sigma) = [\mathbb{C}^2 - \{0\}/\mathbb{C}^\times]$, where the action is given by $\lambda(x, y) = (\lambda^3 x, \lambda^3 y)$. So $X(\Sigma)$ is the nontrivial $\mu_3$-gerbe over $\mathbb{P}^1$ coming
from the canonical line bundle over \( \mathbb{P}^1 \). Let \( G \to X(\Sigma) \) be a \( \mu_2 \)-gerbe such that it come from the \( \mu_2 \)-gerbe over \( \mathcal{B}C^\times \) given by the central extension

\[
(3.2) \quad 1 \to \mu_2 \to C^\times \xrightarrow{(\cdot)^2} C^\times \to 1.
\]

From the sequences (3.1) and (3.2), we have

\[
1 \to \mu_3 \otimes \mu_2 \to C^\times \xrightarrow{[6,0]} (C^\times)^2 \to C^\times \to 1.
\]

The Pontryagin dual of \( C^\times \xrightarrow{[6,0]} (C^\times)^2 \) is \( (\tilde{\beta})^\vee : \mathbb{Z}^2 \to \mathbb{Z} \), which is given by the matrix \([6,0]\). Taking the Gale dual we have

\[
\tilde{\beta} : \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}_6,
\]

which is given by the vectors \( \{\overline{b}_1 = (1,0), \overline{b}_2 = (-1,1)\} \). Let \( \overline{\Sigma} = (\overline{N}, \Sigma, \tilde{\beta}) \) be the new stacky fan. Then we have the toric Deligne-Mumford stack \( \mathcal{X}(\overline{\Sigma}) = [C^2 - \{(0)\}/C^\times] \), where the action is given by \( \lambda(x,y) = (\lambda^0 x, \lambda^0 y) \). So \( \mathcal{X}(\overline{\Sigma}) \) is the canonical \( \mu_6 \)-gerbe over \( \mathbb{P}^1 \).

If the \( \mu_2 \)-gerbe over \( \mathcal{B}C^\times \) is given by the central extension

\[
(3.3) \quad 1 \to \mu_2 \to C^\times \times \mu_2 \xrightarrow{\alpha} C^\times \to 1,
\]

where \( \alpha \) is given by the matrix \([1,0]\), then from (3.1) and (3.3), we have

\[
1 \to \mu_3 \otimes \mu_2 \to C^\times \times \mu_2 \xrightarrow{\varphi} (C^\times)^2 \to C^\times \to 1,
\]

where \( \varphi \) is given by the matrix \[
\begin{bmatrix}
3 & 0 \\
3 & 0
\end{bmatrix}.
\]

The Pontryagin dual of \( \varphi \) is: \( (\tilde{\beta})^\vee : \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}_2 \), which is given by the transpose of the above matrix. Taking the Gale dual we get

\[
\tilde{\beta} : \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2,
\]

which is given by the vectors \( \{\overline{b}_1 = (1,0,0), \overline{b}_2 = (-1,1,0)\} \). So \( \overline{\Sigma}' = (\overline{N}', \Sigma, \tilde{\beta}') \) is a stacky fan. The toric Deligne-Mumford stack is \( \mathcal{X}(\overline{\Sigma}') = [C^2 - \{(0)\}/C^\times \times \mu_2] \), where the action is \( (\lambda_1, \lambda_2) \cdot (x,y) = (\lambda_1^3 x, \lambda_2^3 y) \). So \( G' = \mathcal{X}(\overline{\Sigma}') \) is the trivial \( \mu_2 \)-gerbe over \( \mathcal{X}(\Sigma) \) and \( \mathcal{X}(\overline{\Sigma}) \not\cong \mathcal{X}(\overline{\Sigma}') \).

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References


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