

## CONVERGENCE ANALYSIS OF SPECTRAL COLLOCATION METHODS FOR A SINGULAR DIFFERENTIAL EQUATION\*

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**Abstract.** Solutions of partial differential equations with coordinate singularities often have special behavior near the singularities, which forces them to be smooth. Special treatment for these coordinate singularities is necessary in spectral approximations in order to avoid degradation of accuracy and efficiency. It has been observed numerically in the past that, for a scheme to attain high accuracy, it is unnecessary to impose all the pole conditions, the constraints representing the special solution behavior near singularities. In this paper we provide a theoretical justification for this observation. Specifically, we consider an existing approach, which uses a pole condition as the boundary condition at a singularity and solves the reformulated boundary value problem with a commonly used Gauss–Lobatto collocation scheme. Spectral convergence of the Legendre and Chebyshev collocation methods is obtained for a singular differential equation arising from polar and cylindrical geometries.

**Key words.** coordinate singularity, convergence, spectral collocation method

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**1. Introduction.** Physical problems in polar, cylindrical, or spherical geometries often give rise to mathematical models involving singular partial differential equations (PDEs) with smooth solutions. A common feature of these PDEs is that their solutions have special behavior near coordinate singularities, which forces the solutions to be smooth. For the spectral solution of this type of equation, special treatment for the coordinate singularities is needed, since a traditional spectral scheme either does not fully capture the special solution behavior or is ill-suited to fast transform techniques; e.g., see [6, 7, 11].

A number of spectral approaches have been developed in the past in attempts to capture the solution behavior near coordinate singularities. They include those expanding the solution in specially designed basis functions, such as spherical harmonics, parity-modified Fourier series, modified Robert functions, and eigenfunctions of singular Sturm–Liouville problems [6, 9, 11, 20, 21, 22, 27]; approaches using inherent symmetries of the solution [9, 10]; and methods using pole conditions (i.e., compatibility conditions at the center of polar coordinates) as boundary conditions in the collocation context [12] and the Galerkin context [24, 25]. Many of these approaches have been successfully applied to steady state and time dependent problems including the Navier–Stokes equations; e.g., see [13, 14, 20, 23, 26, 28].

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The solution behavior of PDEs near coordinate singularities can be described by an infinite number of pole conditions derived either from the underlying differential equation, by assuming some kind of smoothness of the solution, or more generally from the analyticity of the solution at singularities. Most of the existing methods are developed more or less to accommodate this behavior. However, it is observed numerically first by Orszag [21] and then by many other researchers (e.g., [6]) that it is unnecessary to impose all of the pole conditions in order for a numerical scheme to attain high accuracy. In fact, Huang and Sloan [12], using some of these pole conditions as the boundary conditions at the coordinate singularities, show numerically that the spectral collocation approximation of the Helmholtz equation on the unit disk has a spectral convergence rate.

Three Legendre-type pseudospectral schemes and their convergence analysis have been developed for axisymmetric domains by Bernardi, Dauge, and Maday in their recent book [1]. The basic idea behind these schemes is to incorporate the natural measure  $rdr$  of the coordinate singularity into the quadrature formula defining the spectral approximation. In the radial direction the formula reads as

$$(1.1) \quad \int_0^1 v(r)rdr = \sum_{j=0}^N v(r_j)\omega_j \quad \forall v \in \mathbf{P},$$

where  $\mathbf{P}$  is a polynomial space,  $r_j = (1 + \rho_j)/2$ , and the  $\omega_j$ 's are the corresponding weights. Three sets of  $\rho_j$ 's and  $\mathbf{P}$  are chosen, one for each scheme:

$$\begin{aligned} \text{Method A (Gauss–Radau):} \quad & \mathbf{P} = \mathbf{P}_{2N}, \quad \rho_N = 1, \\ & \rho_j \ (0 \leq j \leq N-1) \text{ are the roots of } L'_{N+1}(\rho); \\ \text{Method B (Gauss–Lobatto):} \quad & \mathbf{P} = \mathbf{P}_{2N-1}, \quad \rho_0 = -1, \quad \rho_N = 1, \\ & \rho_j \ (1 \leq j \leq N-1) \text{ are the roots of } \left( \frac{L_N(\rho) + L_{N+1}(\rho)}{1+\rho} \right)'; \\ \text{Method C (Gauss–Radau):} \quad & \mathbf{P} = \mathbf{P}_{2N-1}, \quad \rho_N = 1, \\ & \rho_j \ (0 \leq j \leq N-1) \text{ are the roots of } L_{N+1}(\rho) - L_N(\rho), \end{aligned}$$

where  $L_N$  is the Legendre polynomial of degree  $N$ . The pseudospectral approximations are then defined through the boundary condition(s) and the Galerkin formulation of the underlying problem in the discrete inner product  $((u, v))_N \equiv \sum_{j=0}^N u(r_j)v(r_j)\omega_j$  induced from the quadrature formula (1.1). It is noted that the nodes used in these schemes are different from those in a traditional (unweighted) spectral collocation method. Moreover, among these three schemes, only Method C is equivalent to a collocation system. Furthermore, the authors of the book suggest that two boundary conditions  $u(0) = 0$  (which is a pole condition, cf. (2.6)) and  $u(1) = g$  be used for a reduced equation (see (2.1)–(2.2)) with  $n \neq 0$ . Thus, only Method B, which uses the Gauss–Lobatto nodes but cannot be interpreted as a collocation scheme, can be applied to the case  $n \neq 0$ .

The objective of this paper is to provide a theoretical justification for the method developed in [12], which uses a pole condition as the boundary condition at the coordinate singularity and solves the reformulated boundary value problem with a Gauss–Lobatto collocation scheme. The method corresponds to the standard quadrature formula

$$(1.2) \quad \int_0^1 v(r)\omega(r)dr = \sum_{j=0}^N w_j v(r_j) \quad \text{for } v \in \mathbf{P}_{2N-1},$$

where  $\omega(r)$  is the weight function and  $r_j = (1 + \rho_j)/2$ . For example,  $\omega(r) = 1$  and  $\rho_j$  ( $0 \leq j \leq N$ ) are the roots of  $(1 - \rho^2)L'_N(\rho)$  for the Legendre collocation method, and  $\omega(r) = (r - r^2)^{-1/2}$  and  $\rho_j = \cos \frac{\pi(N-j)}{N}$  ( $0 \leq j \leq N$ ) for the Chebyshev collocation method. We emphasize that the method of [12], which is no more than a traditional collocation method, is different from those considered by Bernardi, Dauge, and Maday [1]. The method shares with many existing methods the common feature of explicitly using pole conditions, and has been successfully applied to practical problems including the Navier–Stokes equations; e.g., see [13, 14]. Our analysis is given for both the Legendre and Chebyshev schemes. A Chebyshev collocation scheme is often desirable in practical computation because the fast Fourier transformation (FFT) can be utilized. We find that for the current situation with coordinate singularities the corresponding bilinear form lacks the coercive property which is often crucial to the convergence analysis of a Chebyshev scheme. Because of this, the error estimate of the Chebyshev scheme is obtained in the weighted energy norm  $\|\cdot\|_{E_n, \omega}$  with  $\omega(r)$  being the Chebyshev weight function for the reduced equation with  $n > 0$ , but in the unweighted norm  $\|\cdot\|_{E_n}$  for the case  $n = 0$ .

An outline of this paper is as follows. The method of [12] is briefly described in section 2. The convergence analysis of the Legendre and Chebyshev methods is given in section 3. In section 4 we present numerical results to verify the theoretical findings. Finally, section 5 contains conclusions and further comments.

**2. Pole conditions and spectral collocation approximation.** In this section we briefly describe the spectral collocation method of [12] for a model problem

$$(2.1) \quad -\frac{d^2u}{dr^2} - \frac{1}{r} \frac{du}{dr} + \frac{n^2}{r^2}u = f, \quad 0 < r < 1,$$

$$(2.2) \quad u(1) = g,$$

where  $n \geq 0$  is a given integer. This problem is obtained using separation of variables for the Poisson equation on the unit disk.

**2.1. Pole conditions.** Equation (2.1) has a coordinate singularity at  $r = 0$ . Assume that both  $f$  and  $u$  are sufficiently smooth. A Taylor series expansion of  $u$  about  $r = 0$  yields the pole conditions

$$(2.3) \quad O\left(\frac{1}{r^2}\right) : n^2u(0) = 0,$$

$$(2.4) \quad O\left(\frac{1}{r}\right) : (n^2 - 1)\frac{du}{dr}(0) = 0,$$

$$(2.5) \quad O(1) : \left(\frac{n^2}{2} - 2\right)\frac{d^2u}{dr^2}(0) = f(0),$$

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These conditions contain full information about the solution behavior near  $r = 0$ . It was observed first by Orszag [21] and later by many other researchers (see [6]) that it is unnecessary to impose all of these pole conditions in order for a numerical scheme to obtain high accuracy. In fact, using one constraint

$$(2.6) \quad \begin{cases} u(0) = 0 & \text{for } n \neq 0, \\ \frac{du}{dr}(0) = 0 & \text{for } n = 0 \end{cases}$$

as the boundary condition at  $r = 0$ , Huang and Sloan [12] obtain spectrally accurate solutions; also see [24, 25] for the spectral Galerkin approximation. Once a boundary condition has been defined at  $r = 0$ , it is straightforward to apply a traditional spectral collocation scheme to the singular problem (2.1) and (2.2).

**2.2. Legendre and Chebyshev collocation approximations.** Hereafter, the weight functions  $\omega(r) = 1$  and  $\omega(r) = (r-r^2)^{-1/2}$  will be associated with the Legendre and Chebyshev methods, respectively. For simplicity, we use subscript  $\omega$  in common notation for both methods and for those which apply only to the Chebyshev method, and suppress the subscript for the Legendre method.

For a given integer  $N > 0$ , let  $\{\rho_{j,\omega}\}_{j=0}^N$  be a set of Gauss-Lobatto points associated with the weight function  $\omega(r)$ . Define

$$(2.7) \quad r_{j,\omega} = \frac{1 + \rho_{j,\omega}}{2}, \quad j = 0, 1, \dots, N.$$

The solution  $u(r)$  is approximated by

$$(2.8) \quad u^N(r) = \sum_{j=0}^N u_{j,\omega} l_{j,\omega}(r),$$

where  $u_{j,\omega}$  denotes the approximation of  $u(r_{j,\omega})$  and  $l_{j,\omega}(r)$  is the Lagrangian interpolation polynomial

$$(2.9) \quad l_{j,\omega}(r) = \prod_{\substack{i=0 \\ i \neq j}}^N \frac{r - r_{i,\omega}}{r_{j,\omega} - r_{i,\omega}}.$$

A collocation approximation to (2.1), (2.2), and (2.6) is then defined by the collocation equations

$$(2.10) \quad -\frac{d^2 u^N}{dr^2}(r_{j,\omega}) - \frac{1}{r_{j,\omega}} \frac{du^N}{dr}(r_{j,\omega}) + \frac{n^2}{r_{j,\omega}^2} u^N(r_{j,\omega}) = f(r_{j,\omega}),$$

$$j = 1, \dots, N-1,$$

$$(2.11) \quad u^N(1) = g,$$

$$(2.12) \quad \begin{cases} u^N(0) = 0 & \text{for } n \neq 0, \\ \frac{du^N}{dr}(0) = 0 & \text{for } n = 0. \end{cases}$$

Recall that the transformed Gauss-Lobatto quadrature rule satisfies

$$(2.13) \quad \int_0^1 v(r)\omega(r)dr = \sum_{j=0}^N w_{j,\omega} v(r_{j,\omega}) \quad \text{for } v \in \mathbf{P}_{2N-1},$$

where the  $w_{j,\omega}$ 's are the corresponding weights and  $\mathbf{P}_{2N-1}$  is the space of real polynomials (in  $r$ ) of degree no more than  $2N-1$ . The associated interpolation operator  $I^N : C[0, 1] \rightarrow \mathbf{P}_N$  is defined as

$$(2.14) \quad I^N v \in \mathbf{P}_N : \quad (I^N v)(r_{j,\omega}) = v(r_{j,\omega}), \quad j = 0, 1, \dots, N.$$

We use the notation

$$(2.15) \quad \langle u, v \rangle_\omega = \int_0^1 uv\omega dr, \quad \|u\|_\omega = \langle u, u \rangle_\omega^{1/2},$$

$$(2.16) \quad \|u\|_{m,\omega} = \left( \sum_{k=0}^m \left\| \frac{d^k u}{dr^k} \right\|_\omega^2 \right)^{1/2} \quad \text{for } u \in H_\omega^m,$$

$$(2.17) \quad \langle u, v \rangle_{\omega,N} = \sum_{j=0}^N w_{j,\omega} u(r_{j,\omega}) v(r_{j,\omega}), \quad \|u\|_{\omega,N} = \langle u, u \rangle_{\omega,N}^{1/2},$$

where  $H_\omega^m$  ( $m \geq 0$ ) is a (weighted) Sobolev space on  $[0, 1]$ .

For the Legendre collocation scheme, the set of Legendre–Gauss–Lobatto points is defined by  $\rho_0 = -1$ ,  $\rho_N = 1$ , and  $\rho_j$  ( $j = 1, 2, \dots, N-1$ ) being the roots of  $L'_N$ , the first derivative of the Legendre polynomial  $L_N$  of degree  $N$ . We have

$$l_j(r) = \frac{2}{N(N+1)} \frac{r(1-r)\bar{L}'_N(r)}{(r-r_j)\bar{L}_N(r_j)}, \quad w_j = \frac{1}{N(N+1)} \frac{1}{\bar{L}_N^2(r_j)},$$

with  $\bar{L}_N(r)$  being the transformed Legendre polynomial  $L_N(2r-1)$ .

For the Chebyshev approximation, the set of Gauss–Lobatto points is defined by  $\rho_{0,\omega} = -1$ ,  $\rho_{N,\omega} = 1$ , and  $\rho_{j,\omega}$  ( $j = 1, 2, \dots, N-1$ ) being the roots of  $T'_N$ , the derivative of the Chebyshev polynomial  $T_N$  of degree  $N$ . We have

$$l_{j,\omega}(r) = (-1)^{j+1} \frac{2}{c_j N^2} \frac{r(1-r)\bar{T}'_N(r)}{(r-r_{j,\omega})}, \quad w_{j,\omega} = \frac{\pi}{c_j N},$$

where  $\bar{T}_k(r)$  is the transformed Chebyshev polynomial  $T_k(2r-1)$  and

$$c_j = \begin{cases} 2, & j = 0, N, \\ 1, & j = 1, \dots, N-1. \end{cases}$$

### 3. Convergence analysis.

**3.1. Preliminary approximation results.** To start with, we introduce some preliminary results. Hereafter,  $C$  is used to denote the generic constant. We shall assume that  $N \gg m$  (the smoothness order of functions); otherwise, the estimates given below, especially those involving seminorms, will not be true.

**LEMMA 3.1.** *Let  $P^N$  denote the Legendre (or Chebyshev) truncated operator; i.e.,  $P^N v$  is the truncated Legendre (or Chebyshev) series of  $v$ . Then, for  $m \geq 0$  and for  $\omega(r) = 1$  (the Legendre case) or  $\omega(r) = (r-r^2)^{-1/2}$  (the Chebyshev case),*

$$(3.1) \quad \|v - P^N v\|_\omega \leq CN^{-m} \left\| (r-r^2)^{m/2} \frac{d^m v}{dr^m} \right\|_\omega \quad \forall v \in H_\omega^m.$$

**LEMMA 3.2.** *For  $\omega(r) = 1$  (the Legendre case) or  $\omega(r) = (r-r^2)^{-1/2}$  (the Chebyshev case) and for  $m \geq 1$ ,*

$$(3.2) \quad \begin{aligned} & \left\| (r-r^2)^{-1/2} (v - I^N v) \right\|_\omega + N^{-1} \|v - I^N v\|_{1,\omega} \\ & \leq CN^{-m} \left\| (r-r^2)^{\frac{m-1}{2}} \frac{d^m v}{dr^m} \right\|_\omega \quad \forall v \in H_\omega^m. \end{aligned}$$

LEMMA 3.3. Let  $\omega(r) = 1$  (for the Legendre case) or  $\omega(r) = (r - r^2)^{-1/2}$  (for the Chebyshev case). There exists a positive constant  $C$ , independent of  $N$  and  $M$ , such that for all  $\phi \in \mathbf{P}_M$  with  $M$  being any nonnegative integer,

$$(3.3) \quad \|\phi\|_{\omega, N} \leq C \left(1 + \frac{M}{N}\right) \|\phi\|_{\omega},$$

$$(3.4) \quad \|\phi\|_{\omega} \leq \|\phi\|_{\omega, N} \leq C \|\phi\|_{\omega}.$$

The interested reader is referred to [2, 3, 4, 7, 16, 18, 19] for the proofs of these Lemmas. Lemmas 3.1 and 3.2 are the improvements of existing results in terms of the weight and can be obtained by the method in the aforementioned references.

**3.2. Convergence analysis of the Legendre method.** We now proceed to the convergence analysis for the Legendre approximation (2.10)–(2.12). Let  $\phi$  be an arbitrary polynomial in  $\mathbf{P}_N$  satisfying

$$(3.5) \quad \begin{cases} \phi(1) = \phi(0) = 0 & \text{if } n \neq 0, \\ \phi(1) = 0 & \text{if } n = 0. \end{cases}$$

Multiplying (2.10) by  $r_j w_j \phi(r_j)$  and summing over the range of  $j$  from 1 to  $N - 1$ , we have

$$(3.6) \quad \sum_{j=1}^{N-1} w_j \phi(r_j) \left[ -r_j \frac{d^2 u^N}{dr^2}(r_j) - \frac{du^N}{dr}(r_j) + \frac{n^2}{r_j} u^N(r_j) \right] = \sum_{j=1}^{N-1} r_j w_j \phi(r_j) f(r_j).$$

It is not difficult to see from (3.5) that

$$(3.7) \quad \begin{aligned} w_N \phi(r_N) \left[ -r_N \frac{d^2 u^N}{dr^2}(r_N) - \frac{du^N}{dr}(r_N) + \frac{n^2}{r_N} u^N(r_N) \right] \\ = r_N w_N \phi(r_N) f(r_N) \\ = 0. \end{aligned}$$

Noticing that  $u^N(r)/r$  is a polynomial of degree not greater than  $N - 1$ , that  $\phi(r_0) = 0$  (see (3.5)) when  $n \neq 0$ , and that  $(du^N/dr)(r_0) = 0$  when  $n = 0$ , we have

$$(3.8) \quad \begin{aligned} w_0 \phi(r_0) \left[ -r_0 \frac{d^2 u^N}{dr^2}(r_0) - \frac{du^N}{dr}(r_0) + \frac{n^2}{r_0} u^N(r_0) \right] \\ = r_0 w_0 \phi(r_0) f(r_0) \\ = 0. \end{aligned}$$

Thus, (3.6)–(3.8) imply that

$$(3.9) \quad \left\langle -r \frac{d^2 u^N}{dr^2} - \frac{du^N}{dr} + \frac{n^2}{r} u^N, \phi \right\rangle_N = \langle r f, \phi \rangle_N.$$

Since  $r\phi(d^2 u^N/dr^2)$ ,  $\phi(du^N/dr)$ , and  $n^2 u^N \phi/r$  are in  $\mathbf{P}_{2N-1}$ , (2.13) and (3.9) lead to

$$(3.10) \quad \left\langle -r \frac{d^2 u^N}{dr^2} - \frac{du^N}{dr} + \frac{n^2}{r} u^N, \phi \right\rangle = \langle r f, \phi \rangle_N$$

or, taking integration by parts,

$$(3.11) \quad \left\langle \frac{du^N}{dr}, r \frac{d\phi}{dr} \right\rangle + n^2 \left\langle \frac{u^N}{r}, \phi \right\rangle = \langle rf, \phi \rangle_N.$$

Multiplying the continuous equation (2.1) by  $r\phi$  and integrating from  $r = 0$  to 1, we obtain

$$(3.12) \quad \left\langle \frac{du}{dr}, r \frac{d\phi}{dr} \right\rangle + n^2 \left\langle \frac{u}{r}, \phi \right\rangle = \langle rf, \phi \rangle.$$

Then, subtracting (3.11) from (3.12) gives the error equation

$$(3.13) \quad \left\langle \frac{d(u - u^N)}{dr}, r \frac{d\phi}{dr} \right\rangle + n^2 \left\langle \frac{u - u^N}{r}, \phi \right\rangle = \langle rf, \phi \rangle - \langle rf, \phi \rangle_N,$$

which can be written in a simpler form as

$$(3.14) \quad a_{r,n}(u - u^N, \phi) = F_r(\phi) \quad \forall \phi \in \mathbf{P}_N,$$

where

$$(3.15) \quad \begin{aligned} a_{r,n}(u, v) &= \left\langle \frac{du}{dr}, r \frac{dv}{dr} \right\rangle + n^2 \left\langle u, \frac{v}{r} \right\rangle, \\ \|v\|_{E_n} &= a_{r,n}(v, v)^{1/2}, \\ F_r(\phi) &= \langle rf, \phi \rangle - \langle rf, \phi \rangle_N. \end{aligned}$$

We first consider the case  $n \neq 0$ . Recall that we have  $u^N(0) = u(0) = 0$  (cf. (2.6)). Let

$$V_{0g} = \{v \in H^1(I) : v(0) = 0, v(1) = g\}, \quad V_{0g}^N = V_{0g} \cap \mathbf{P}_N.$$

LEMMA 3.4. *Let  $u$  and  $u^N$  be the solutions of the problem (2.1)–(2.2) ( $n \neq 0, u(0) = 0$ ) and the approximation (2.10)–(2.11) ( $u^N(0) = 0$ ), respectively. We have*

$$(3.16) \quad \|u - u^N\|_{E_n} \leq \sup_{\varphi \in V_{00}^N} \frac{F_r(\varphi)}{\|\varphi\|_{E_n}} + 2 \inf_{v \in V_{0g}^N} \|v - u\|_{E_n}.$$

*Proof.* Equation (3.14) can be rewritten as

$$a_{r,n}(v - u^N, \phi) = F_r(\phi) + a_{r,n}(v - u, \phi) \quad \forall \phi \in V_{00}^N \text{ and } v \in V_{0g}^N.$$

Taking  $\phi = v - u^N \in V_{00}^N$  results in

$$\|v - u^N\|_{E_n}^2 = a_{r,n}(v - u^N, \phi) \leq \sup_{\varphi \in V_{00}^N} \frac{F_r(\varphi)}{\|\varphi\|_{E_n}} \|v - u^N\|_{E_n} + a_{r,n}(v - u, \phi).$$

Since

$$\begin{aligned} & a_{r,n}(v - u, \phi) \\ & \leq \|r^{1/2}(v - u)_r\| \cdot \|r^{1/2}\phi_r\| + n^2 \|r^{-1/2}(v - u)\| \cdot \|r^{-1/2}\phi\| \\ & \leq \left( \|r^{1/2}(v - u)_r\|^2 + n^2 \|r^{-1/2}(v - u)\|^2 \right)^{1/2} \left( \|r^{1/2}\phi_r\|^2 + n^2 \|r^{-1/2}\phi\|^2 \right)^{1/2} \\ & \leq C a_{r,n}(v - u, v - u)^{1/2} a_{r,n}(\phi, \phi)^{1/2} \\ & = C \|v - u\|_{E_n} \cdot \|v - u^N\|_{E_n}, \end{aligned}$$

we have

$$\|v - u^N\|_{E_n}^2 \leq C \left( \sup_{\varphi \in V_{00}^N} \frac{F_r(\varphi)}{\|\varphi\|_{E_n}} + \|v - u\|_{E_n} \right) \|v - u^N\|_{E_n}.$$

Then the desired result follows from

$$\|u - u^N\|_{E_n} \leq \|u - v\|_{E_n} + \|v - u^N\|_{E_n}. \quad \square$$

We now use Lemma 3.4 to obtain the estimate of  $\|u - u^N\|_{E_n}$ . For the first term on the right-hand side of (3.16), we have from the Cauchy–Schwarz inequality and Lemmas 3.1–3.3 that, for any  $\phi \in \mathbf{P}_N$ ,

$$\begin{aligned} |F_r(\phi)| &= |\langle rf, \phi \rangle - \langle rf, \phi \rangle_N| \\ &= |\langle rf, \phi \rangle - \langle P^{N-1}(rf), \phi \rangle \\ &\quad + \langle P^{N-1}(rf), \phi \rangle - \langle I^N(rf), \phi \rangle_N| \\ &\leq |\langle rf - P^{N-1}(rf), \phi \rangle| + |\langle P^{N-1}(rf) - I^N(rf), \phi \rangle_N| \\ &\leq \|rf - P^{N-1}(rf)\| \|\phi\| + C \|P^{N-1}(rf) - I^N(rf)\| \|\phi\|_N \\ &\leq \|rf - P^{N-1}(rf)\| \|\phi\| + C (\|rf - P^{N-1}(rf)\| + \|rf - I^N(rf)\|) \|\phi\| \\ (3.17) \quad &= C (\|rf - P^{N-1}(rf)\| + \|rf - I^N(rf)\|) \|\phi\|. \end{aligned}$$

By Lemmas 3.1 and 3.2 (and taking  $v = rf \in H^{\bar{m}}$ ,  $\bar{m} := \max\{m-1, 1\}$ ), we obtain

$$\begin{aligned} |F_r(\phi)| &\leq CN^{1-m} \left\| (r-r^2)^{\frac{\bar{m}-1}{2}} \frac{d^{\bar{m}}(rf)}{dr^{\bar{m}}} \right\| \|\phi\| \\ (3.18) \quad &\leq CN^{1-m} \left\| (r-r^2)^{\frac{\bar{m}-1}{2}} \frac{d^{\bar{m}}(rf)}{dr^{\bar{m}}} \right\| \|\phi\|_{E_n}. \end{aligned}$$

For the second term on the right-hand side of (3.16), taking  $v = I^N u$  and using the definition of the energy norm leads to

$$\begin{aligned} \|v - u\|_{E_n} &\leq |v - u|_1 + \|[r(1-r)]^{-1/2}(v-u)\| \\ (3.19) \quad &\leq CN^{1-m} \left\| (r-r^2)^{\frac{m-1}{2}} \frac{d^m u}{dr^m} \right\|. \end{aligned}$$

Substituting (3.18) and (3.19) into (3.16), we obtain the estimate

$$(3.20) \quad \|u - u^N\|_{E_n} \leq CN^{1-m} \left( \left\| (r-r^2)^{\frac{m-1}{2}} \frac{d^m u}{dr^m} \right\| + \left\| (r-r^2)^{\frac{\bar{m}-1}{2}} \frac{d^{\bar{m}}(rf)}{dr^{\bar{m}}} \right\| \right).$$

We now consider the case  $n = 0$ . Recall again that we have  $\frac{du^N}{dr}(0) = \frac{du}{dr}(0) = 0$ . Define

$$W_g = \{v \in H^1(I) : v(1) = g\}, \quad W_g^N = W_g \cap \mathbf{P}_N.$$

**LEMMA 3.5.** *Let  $u$  and  $u^N$  be the solutions of the problem (2.1)–(2.2) ( $n = 0$ ,  $\frac{du}{dr}(0) = 0$ ) and the approximation (2.10)–(2.11) ( $\frac{du^N}{dr}(0) = 0$ ), respectively. We have*

$$\|u - u^N\|_{E_0} \leq 2 \sup_{\varphi \in W_0^N} \frac{F_r(\varphi)}{\|\varphi\|_{E_0}} + 2 \inf_{v \in W_g^N} \|v - u\|_{E_0}.$$



*Proof.* Equation (3.14) can be written as

$$a_{r,0}(v - u^N, \phi) = F_r(\phi) + a_{r,0}(v - u, \phi) \quad \forall \phi \in W_0^N \text{ and } v \in W_g^N.$$

Taking  $\phi = v - u^N \in W_0^N$ , we have

$$\begin{aligned} \|v - u^N\|_{E_0}^2 &= a_{r,0}(v - u^N, \phi) \\ &\leq \left( \sup_{\varphi \in W_0^N} \frac{F_r(\varphi)}{\|\varphi\|_{E_0}} + \|v - u\|_{E_0} \right) \|v - u^N\|_{E_0}. \end{aligned}$$

Then the conclusion follows.  $\square$

From the Hardy-type inequality

$$\|v\| \leq \left\| \sqrt{r} \frac{dv}{dr} \right\| = \|v\|_{E_0} \quad \forall v \in W_0,$$

Lemma 3.5 leads to the same result as in (3.20). Hence, we have proved the following theorem.

**THEOREM 3.1.** *For any integer  $m \geq 1$ , the Legendre-collocation approximation  $u^N$  defined by the scheme (2.10)–(2.12) for the problem (2.1) and (2.2) satisfies*

$$(3.21) \quad \|u - u^N\|_{E_n} \leq CN^{1-m} \left( \left\| (r - r^2)^{\frac{m-1}{2}} \frac{d^m u}{dr^m} \right\| + \left\| (r - r^2)^{\frac{m-1}{2}} \frac{d^{\bar{m}}(rf)}{dr^{\bar{m}}} \right\| \right),$$

where  $u$  is the exact solution of (2.1) and (2.2) and  $\bar{m} = \max\{m - 1, 1\}$ .

This theorem shows that the Legendre collocation approximation is convergent and the error decays faster than algebraically, provided that the right-hand-side term  $f$  and the solution  $u$  are infinitely differentiable. As shown in [1], the correct regularity requirement for  $u$  and  $f$  should be considered in a weighted Sobolev space

$$(3.22) \quad H_r^s = \left\{ v \left| \sum_{l=0}^s \left\| \sqrt{r} \frac{d^l v}{dr^l} \right\|^2 < \infty \right. \right\}$$

for some integer  $s$ . It is not difficult to see that the terms in the bracket on the right-hand side of (3.21) are bounded for  $u \in H_r^m$  and  $f \in H_r^{\bar{m}}$  with  $m \geq 2$ . In this sense, the result of Theorem 3.1 is optimal.

**3.3. Convergence analysis of the Chebyshev method.** We now consider the convergence of the Chebyshev method (2.10)–(2.12). Let  $\phi$  be the same as in (3.5). As for (3.10), we have

$$(3.23) \quad \left\langle -r \frac{d^2 u^N}{dr^2} - \frac{du^N}{dr} + \frac{n^2}{r} u^N, \phi \right\rangle_{\omega} = \langle rf, \phi \rangle_{\omega, N}$$

or, taking integration by parts,

$$(3.24) \quad \left\langle \frac{du^N}{dr}, r \frac{d(\phi\omega)}{dr} \right\rangle + n^2 \left\langle \frac{u^N}{r}, \phi \right\rangle_{\omega} = \langle rf, \phi \rangle_{\omega, N}.$$

The error equation reads as

$$(3.25) \quad \begin{aligned} &\left\langle \frac{d(u - u^N)}{dr}, r \frac{d(\phi\omega)}{dr} \right\rangle + n^2 \left\langle \frac{u - u^N}{r}, \phi \right\rangle_{\omega} \\ &= \langle rf, \phi \rangle_{\omega} - \langle rf, \phi \rangle_{\omega, N}. \end{aligned}$$

We also write it in a simpler form

$$(3.26) \quad a_{r,n,\omega}(u - u^N, \phi) = F_{r,\omega}(\phi) \quad \forall \phi \in \mathbf{P}_N,$$

where

$$(3.27) \quad \begin{aligned} a_{r,n,\omega}(u, v) &= b_{r,\omega}(u, v) + n^2 \left\langle u, \frac{v}{r} \right\rangle_{\omega}, \\ b_{r,\omega}(u, v) &= \left\langle \frac{du}{dr}, r \frac{d(v\omega)}{dr} \right\rangle, \\ F_{r,\omega}(\phi) &= \langle rf, \phi \rangle_{\omega} - \langle rf, \phi \rangle_{\omega, N}. \end{aligned}$$

It is known that the nonsymmetric bilinear form  $b_{r,\omega}(\cdot, \cdot)$ , without the factor  $r$ , is coercive (see [3, 7, 8, 17, 15]). On the other hand, in the current situation,  $b_{r,\omega}(v, v)$  can become negative for some polynomials subject to the boundary conditions  $v(-1) = v(1) = 0$  or  $\frac{dv}{dr}(-1) = v(1) = 0$ . We have the following Gårding-type inequality.

LEMMA 3.6. *For all  $u, v \in H_{\omega,0}^1$  we have*

$$(3.28) \quad \frac{1}{4} \left\| \sqrt{r} \frac{dv}{dr} \right\|_{\omega}^2 + \frac{3}{8} \left\| \frac{v}{\sqrt{1-r}} \right\|_{\omega}^2 - \frac{1}{8} \left\| \frac{v}{\sqrt{r}} \right\|_{\omega}^2 \leq b_{r,\omega}(v, v) \leq \left\| \sqrt{r} \frac{dv}{dr} \right\|_{\omega}^2,$$

$$(3.29) \quad |b_{r,\omega}(u, v)| \leq 3 \left\| \sqrt{r} \frac{du}{dr} \right\|_{\omega} \left\| \sqrt{r} \frac{dv}{dr} \right\|_{\omega}.$$

*Proof.* For notational simplicity, define

$$(3.30) \quad I_1(v) = \int_0^1 \left( \frac{dv}{dr} \right)^2 r \omega dr.$$

We have from integrating by parts

$$(3.31) \quad \begin{aligned} b_{r,\omega}(v, v) &= I_1(v) - \int_0^1 \frac{dv}{dr} v \frac{1-2r}{2(r-r^2)} r \omega dr \\ &= I_1(v) - \frac{1}{8} \int_0^1 v^2 r (1-2r+4r^2) \omega^5 dr \\ &= I_1(v) - \frac{3}{8} \int_0^1 v^2 r^3 \omega^5 dr - \frac{1}{8} \int_0^1 v^2 r (1-r)^2 \omega^5 dr. \end{aligned}$$

Thus  $b_{r,\omega}(v, v) \leq I_1(v)$ . On the other hand,

$$(3.32) \quad \begin{aligned} 0 &\leq \int_0^1 \left( \frac{dv}{dr} + vr\omega^2 \right)^2 r \omega dr \\ &= I_1(v) + \int_0^1 v^2 r^3 \omega^5 dr + \int_0^1 \frac{d(v^2)}{dr} r^2 \omega^3 dr \\ &= I_1(v) - \frac{1}{2} \int_0^1 v^2 r^2 \omega^5 dr, \end{aligned}$$

which gives

$$(3.33) \quad b_{r,\omega}(v, v) \geq \frac{1}{4} I_1(v) + \frac{3}{8} \int_0^1 v^2 r^2 (1-r) \omega^5 dr - \frac{1}{8} \int_0^1 \frac{v^2}{r} \omega dr.$$

The result (3.28) follows. To prove (3.29), we estimate  $b_{r,\omega}(u, v)$  by

$$(3.34) \quad |b_{r,\omega}(u, v)| \leq \left| \int_0^1 \frac{du}{dr} \frac{dv}{dr} r\omega \, dr \right| + \left| \int_0^1 \frac{du}{dr} v \frac{1-2r}{2(r-r^2)} r\omega \, dr \right| \\ \leq [I_1(u)]^{1/2} [I_1(v)]^{1/2} + I_2(u, v),$$

where

$$(3.35) \quad I_2(u, v) = \left| \int_0^1 \frac{du}{dr} v \frac{1-2r}{2(r-r^2)} r\omega \, dr \right| \leq [I_1(u)]^{1/2} [I_3(v)]^{1/2}$$

with

$$(3.36) \quad I_3(v) = \int_0^1 v^2 \frac{(1-2r)^2}{4(r-r^2)^2} r\omega \, dr = \frac{1}{4} \int_0^1 v^2 r(1-2r)^2 \omega^5 \, dr.$$

On the other hand, from integrating by parts,

$$(3.37) \quad I_2(v, v) = \frac{1}{8} \int_0^1 v^2 (2r^2 + r(1-2r)^2) \omega^5 \, dr \geq \frac{1}{2} I_3(v).$$

Thus we get from (3.37) and (3.35)

$$(3.38) \quad I_3(v) \leq 2I_2(v, v) \leq 2[I_1(v)]^{1/2} [I_3(v)]^{1/2},$$

which gives  $I_3(v) \leq 4I_1(v)$  and

$$(3.39) \quad |b_{r,\omega}(u, v)| \leq [I_1(u)]^{1/2} ([I_1(v)]^{1/2} + [I_3(v)]^{1/2}) \\ \leq 3[I_1(u)]^{1/2} [I_1(v)]^{1/2}. \quad \square$$

We first consider the case  $n \neq 0$ . Let  $V_{0g}$  and  $V_{0g}^N$  be the same as before and

$$\|v\|_{E_n,\omega} = \left( \left\| \sqrt{r} \frac{dv}{dr} \right\|_{\omega}^2 + n^2 \left\| \frac{v}{\sqrt{r}} \right\|_{\omega}^2 \right)^{1/2}.$$

LEMMA 3.7. *Let  $u$  and  $u^N$  be the solutions of the problem (2.1)–(2.2) ( $n \neq 0, u(0) = 0$ ) and the Chebyshev approximation ( $u^N(0) = 0$ ), respectively. We have*

$$(3.40) \quad \|u - u^N\|_{E_n,\omega} \leq C \sup_{\varphi \in V_{00}^N} \frac{F_{r,\omega}(\varphi)}{\|\varphi\|_{E_n,\omega}} + C \inf_{v \in V_{0g}^N} \|v - u\|_{E_n,\omega}.$$

*Proof.* Equation (3.26) can be rewritten as

$$a_{r,n,\omega}(v - u^N, \phi) = F_{r,\omega}(\phi) + a_{r,n,\omega}(v - u, \phi) \quad \forall \phi \in V_{00}^N \text{ and } v \in V_{0g}^N.$$

Taking  $\phi = v - u^N \in V_{00}^N$  and using the inequality (3.28) of Lemma 3.6 yields

$$\|v - u^N\|_{E_n,\omega}^2 \leq C a_{r,n,\omega}(v - u^N, \phi) \leq C \sup_{\varphi \in V_{00}^N} \frac{F_{r,\omega}(\varphi)}{\|\varphi\|_{E_n,\omega}} \|\phi\|_{E_n,\omega} + C a_{r,n,\omega}(v - u, \phi).$$

From (3.29) of Lemma 3.6 we have

$$\begin{aligned}
 & a_{r,n,\omega}(v - u, \phi) \\
 & \leq C \left\| \sqrt{r} \frac{d(v - u)}{dr} \right\|_{\omega} \left\| \sqrt{r} \frac{d\phi}{dr} \right\|_{\omega} + n^2 \left\| \frac{v - u}{\sqrt{r}} \right\|_{\omega} \left\| \frac{\phi}{\sqrt{r}} \right\|_{\omega} \\
 & \leq C \left( \left\| \sqrt{r} \frac{d(v - u)}{dr} \right\|_{\omega}^2 + n^2 \left\| \frac{v - u}{\sqrt{r}} \right\|_{\omega}^2 \right)^{1/2} \left( \left\| \sqrt{r} \frac{d\phi}{dr} \right\|_{\omega}^2 + n^2 \left\| \frac{\phi}{\sqrt{r}} \right\|_{\omega}^2 \right)^{1/2} \\
 & = C \|v - u\|_{E_n, \omega} \cdot \|v - u^N\|_{E_n, \omega},
 \end{aligned}$$

and therefore

$$\|v - u^N\|_{E_n, \omega}^2 \leq C \left( \sup_{w \in V_{00}^N} \frac{F_{r,\omega}(w)}{\|w\|_{E_n, \omega}} + \|v - u\|_{E_n, \omega} \right) \|v - u^N\|_{E_n, \omega}.$$

Then, the desired result follows from the triangular inequality:

$$\|u - u^N\|_{E_n, \omega} \leq \|u - v\|_{E_n, \omega} + \|v - u^N\|_{E_n, \omega}. \quad \square$$

We now use Lemma 3.7 to obtain the estimate of  $\|u - u^N\|_{E_n, \omega}$ . For the first term on the right-hand side of (3.40), we have from the Cauchy–Schwarz inequality and Lemmas 3.1–3.3 that, for any  $\phi \in \mathbf{P}_N$ ,

$$\begin{aligned}
 |F_{r,\omega}(\phi)| & = |\langle rf, \phi \rangle_{\omega} - \langle rf, \phi \rangle_{\omega, N}| \\
 & = |\langle rf, \phi \rangle_{\omega} - \langle P^{N-1}(rf), \phi \rangle_{\omega} + \langle P^{N-1}(rf), \phi \rangle_{\omega} - \langle I^N(rf), \phi \rangle_{\omega, N}| \\
 & = C (\|rf - P^{N-1}(rf)\|_{\omega} + \|rf - I^N(rf)\|_{\omega}) \|\phi\|_{\omega}.
 \end{aligned}$$

Then, by Lemmas 3.1 and 3.2,

$$\begin{aligned}
 |F_{r,\omega}(\phi)| & \leq CN^{1-m} \left\| (r - r^2)^{\frac{m-1}{2}} \frac{d^m(rf)}{dr^m} \right\|_{\omega} \|\phi\|_{\omega} \\
 (3.41) \quad & \leq CN^{1-m} \left\| (r - r^2)^{\frac{m-1}{2}} \frac{d^m(rf)}{dr^m} \right\|_{\omega} \|\phi\|_{E_n, \omega}.
 \end{aligned}$$

For the second term on the right-hand side of (3.16), taking  $v = I^N u$  leads to

$$\begin{aligned}
 \|v - u\|_{E_n, \omega} & \leq \left\| \sqrt{r} \frac{d(v - u)}{dr} \right\|_{\omega} + \left\| \frac{v - u}{\sqrt{r(1 - r)}} \right\|_{\omega} \\
 (3.42) \quad & \leq CN^{1-m} \left\| (r - r^2)^{\frac{m-1}{2}} \frac{d^m u}{dr^m} \right\|_{\omega}.
 \end{aligned}$$

Substituting (3.41) and (3.42) into (3.16), we obtain the estimate

$$\begin{aligned}
 & \|u - u^N\|_{E_n, \omega} \\
 (3.43) \quad & \leq CN^{1-m} \left( \left\| (r - r^2)^{\frac{m-1}{2}} \frac{d^m u}{dr^m} \right\|_{\omega} + \left\| (r - r^2)^{\frac{m-1}{2}} \frac{d^m(rf)}{dr^m} \right\|_{\omega} \right).
 \end{aligned}$$

We now consider the case  $n = 0$ . In this case,  $a_{r,0,\omega}(\cdot, \cdot) = b_{r,\omega}(\cdot, \cdot)$  is not coercive. Thus it does not seem likely to us that an error bound can be obtained in the weighted

energy norm  $\|\cdot\|_{E_0, \omega}$ . For this reason, we conduct the estimation in the energy norm  $\|\cdot\|_{E_0}$  without the Chebyshev weight. We first note that the polar condition  $\frac{du^N}{dr}(0) = 0$  allows us to extend the collocation equations

$$(3.44) \quad -r_{j,\omega} \frac{d^2 u^N}{dr^2}(r_{j,\omega}) - \frac{du^N}{dr}(r_{j,\omega}) = r_{j,\omega} f(r_{j,\omega}), \quad 1 \leq j \leq N-1,$$

to the point  $r = r_{j,\omega} = 0$ . Since the left-hand side of (3.44) is a polynomial of degree  $N-1$ , (3.44) holds for all  $r \in [0, 1]$ , provided that  $f \in \mathbf{P}_{N-2}$ . We introduce an auxiliary interpolation operator  $\tilde{I}^{N-2}: C([0, 1]) \rightarrow \mathbf{P}_{N-2}$  defined by

$$(3.45) \quad \tilde{I}^{N-2}v(r_j) = v(r_j), \quad 1 \leq j \leq N-1.$$

Thus we are able to rewrite (3.44) as

$$(3.46) \quad -r \frac{d^2 u^N}{dr^2}(r) - \frac{du^N}{dr}(r) = r \tilde{I}^{N-2}f(r), \quad 0 \leq r \leq 1.$$

Let  $W_g$  and  $W_g^N$  be the same as before.

LEMMA 3.8. *Let  $u$  and  $u^N$  be the solutions of the problem (2.1)–(2.2) ( $n = 0$ ,  $\frac{du}{dr}(0) = 0$ ) and the Chebyshev collocation approximation ( $\frac{du^N}{dr}(0) = 0$ ), respectively. We have*

$$\|u - u^N\|_{E_0} \leq \sup_{\varphi \in W_0^N} \frac{\tilde{F}_r(\varphi)}{\|\varphi\|_{E_0}} + 2 \inf_{v \in W_g^N} \|v - u\|_{E_0},$$

where  $\tilde{F}_r(\varphi) = \langle rf - r\tilde{I}^{N-2}f, \varphi \rangle$ .

*Proof.* We have from (2.1) and (3.46)

$$a_{r,0}(v - u^N, \phi) = \tilde{F}_r(\phi) + a_{r,0}(v - u, \phi) \quad \forall \phi \in W_0^N \text{ and } v \in W_g^N.$$

Taking  $\phi = v - u^N \in W_0^N$ , we have

$$\begin{aligned} \|v - u^N\|_{E_0}^2 &= a_{r,0}(v - u^N, \phi) \\ &\leq \left( \sup_{\varphi \in W_0^N} \frac{\tilde{F}_r(\varphi)}{\|\varphi\|_{E_0}} + \|v - u\|_{E_0} \right) \|v - u^N\|_{E_0}, \end{aligned}$$

which gives the desired result.  $\square$

We need further to estimate the term  $\tilde{F}_r(\varphi)$ . According to the definition, it is easy to see that  $(r - r^2)\tilde{I}^{N-2}f = I^N((r - r^2)f)$ . Therefore, we have from the Cauchy-Schwarz inequality and Lemma 3.2 that, for any  $\varphi \in W_0^N$ ,

$$(3.47) \quad \begin{aligned} |\tilde{F}_r(\varphi)| &= |\langle (r - r^2)f - I^N((r - r^2)f), (1 - r)^{-1}\varphi \rangle| \\ &\leq \|(r - r^2)f - I^N((r - r^2)f)\| \|(1 - r)^{-1}\varphi\| \\ &\leq CN^{1-m} \left\| (r - r^2)^{\frac{m-1}{2}} \frac{d^m((r - r^2)f)}{dr^m} \right\|_{\omega} \|\varphi\|_{E_0}, \end{aligned}$$

where we have used

$$(3.48) \quad \|(1 - r)^{-1}\varphi\| \leq 2 \left\| \sqrt{r} \frac{d\varphi}{dr} \right\| \quad \forall \varphi \in W_0^N,$$

which can be derived from

$$\begin{aligned}
 0 &\leq \int_0^1 \left( \sqrt{r} \frac{d\varphi}{dr} - \frac{1}{2}(1-r)^{-1}\varphi \right)^2 dr \\
 &= \left\| \sqrt{r} \frac{d\varphi}{dr} \right\|^2 + \frac{1}{4} \|(1-r)^{-1}\varphi\|^2 - \frac{1}{2} \int_0^1 \frac{d(\varphi)^2}{dr} r^{1/2} (1-r)^{-1} dr \\
 (3.49) \quad &\leq \left\| \sqrt{r} \frac{d\varphi}{dr} \right\|^2 - \frac{1}{4} \|(1-r)^{-1}\varphi\|^2.
 \end{aligned}$$

Hence, we have proved the following theorem.

**THEOREM 3.2.** *For any integer  $m \geq 1$ , the Chebyshev-collocation approximation  $u^N$  defined by the scheme (2.10)–(2.12) for the problem (2.1) and (2.2) satisfies*

$$\begin{aligned}
 \|u - u^N\|_{E_n} &\leq CN^{1-m} \left( \left\| (r-r^2)^{\frac{m}{2}-\frac{3}{4}} \frac{d^m u}{dr^m} \right\| + \left\| (r-r^2)^{\frac{m}{2}-\frac{3}{4}} \frac{d^{\bar{m}}(rf)}{dr^{\bar{m}}} \right\| \right. \\
 (3.50) \quad &\quad \left. + \left\| (r-r^2)^{\frac{m}{2}-\frac{3}{4}} \frac{d^{\bar{m}}((r-r^2)f)}{dr^{\bar{m}}} \right\| \right),
 \end{aligned}$$

where  $u$  is the exact solution of (2.1) and (2.2) and  $\bar{m} = \max\{m-1, 1\}$ . For the case  $n \neq 0$ , (3.50) also holds in the stronger norm  $\|\cdot\|_{E_n, \omega}$ .

Thus, we obtain a convergence result similar to that of the Legendre collocation method. As in Theorem 3.1, when  $u \in H_r^m$  and  $f \in H_r^{\bar{m}}$  with  $m \geq 5/2$ , the terms in the bracket on the right-hand side of (3.50) are bounded.

**4. Numerical experiments.** In this section we present some numerical results to demonstrate the accuracy of the Legendre and Chebyshev collocation methods (2.10)–(2.12) for the model problem (2.1)–(2.2).

*Example 1.* The function  $f(r)$  and the Dirichlet boundary condition at  $r = 1$  are chosen such that the exact solution of (2.1) and (2.2) is

$$(4.1) \quad u(r) = r^2 \cos(10\pi r), \quad 0 < r < 1.$$

We note that the energy norm  $\|\cdot\|_{E_n}$  is stronger than the  $L^\infty$  norm for  $n > 0$ . (This is not true for  $n = 0$ .) For this reason, we use the maximum norm to measure the error for the case  $n > 0$ , but use the energy norm for the case  $n = 0$ . These norms are numerically approximated in the computations, viz.,

$$(4.2) \quad E_{0,N} = \left\{ \sum_{j=0}^N r_j \left| \frac{du}{dr}(r_j) - \frac{du^N}{dr}(r_j) \right|^2 w_j \right\}^{1/2} \quad \text{for } n = 0,$$

$$(4.3) \quad E_{1,N} = \max_{0 \leq j \leq N} |u(r_{j,\omega}) - u^N(r_{j,\omega})| \quad \text{for } n = 1,$$

where the Legendre points  $\{r_j\}$  are used in (4.2) and both the Legendre and Chebyshev points  $\{r_{j,\omega}\}$  are used in (4.3). The numerical results for the cases  $n = 0$  and  $n = 1$  are listed in Table 1. The spectral convergence of the methods is clearly shown in the table. One may also notice that the Legendre and Chebyshev collocation methods produce very comparable results.

*Example 2.* The function  $f(r)$  and the boundary condition at  $r = 1$  are chosen such that the problem has a less regular solution

$$(4.4) \quad u(r) = r^{5/2}, \quad 0 < r < 1.$$

TABLE 1

Numerical results obtained with the Legendre (LC) and Chebyshev (CC) collocation methods for Example 1. The convergence order is  $N^{-\text{Order}}$ .

N	n = 0				n = 1			
	LC-method		CC-method		LC-method		CC-method	
	$E_{0,N}$	Order	$E_{0,N}$	Order	$E_{1,N}$	Order	$E_{1,N}$	Order
10	6.9e+01		1.5e+02		1.7e+01		4.1e+01	
20	2.1e+01	1.8	2.4e+01	2.7	3.9e+00	2.2	5.0e+00	3.1
30	1.0e+00	7.4	1.3e+00	7.2	5.3e-02	10.6	6.4e-02	10.7
40	2.0e-03	21.7	2.9e-03	21.2	7.3e-05	22.9	9.2e-05	22.8
50	1.8e-07	41.8	3.5e-07	40.5	6.7e-09	41.7	7.3e-09	42.3
60	1.6e-12	63.9	1.9e-12	66.3	7.8e-14	62.3	1.0e-13	61.3
70	2.2e-13		4.2e-12		1.5e-14		1.2e-13	

TABLE 2

Numerical results obtained with the Legendre collocation method for Example 2. The convergence order is  $N^{-\text{Order}}$ .

N	n = 0		n = 1	
	$E_{0,N}$	Order	$E_{1,N}$	Order
40	3.2e-07		1.7e-08	
80	1.2e-08	4.75	5.6e-10	4.96
120	1.7e-09	4.79	7.4e-11	4.97
160	4.2e-10	4.81	1.8e-11	4.98
200	1.4e-10	4.82	5.8e-12	4.96
240	5.9e-11	4.83	2.4e-12	4.91

The computation is done with the Legendre collocation method for  $n = 0$  and  $n = 1$ . The solution error and the convergence order are listed in Table 2. One can easily see that  $E_{0,N} \approx O(N^{-5})$  and  $E_{1,N} \approx O(N^{-5})$ . That is, the rate of convergence is nearly twice the exponent of  $r$  in (4.4),  $5/2$ . On the other hand, it is not difficult to show that the first term on the right-hand side of (3.21) is bounded for  $m < 5$ , while the second term is bounded for  $\bar{m} < 3$  or  $m < 4$  for the current example. Thus, the right-hand-side terms are bounded for  $m < 4$ . From Theorem 3.1, we have  $\|u - u^N\|_{E_n} \approx O(N^{-3})$ . This indicates that the convergence rate predicted by (3.21) is not sharp, although the estimate is optimal according to the regularity requirement ([1]; also cf. (3.22)). Such an order loss seems typical in the convergence analysis of collocation schemes, especially for problems involving force terms; e.g., see [4] for comparison of typical estimates for the Legendre Galerkin method ((8.7) on p. 274) and the Legendre collocation method ((15.15) on p. 310). It is interesting to note that sharp estimates have been obtained for the  $p$ -version finite element method (which is of Galerkin type); e.g., see Babuska and Suri [5]. Finally, we mention that the Chebyshev collocation method leads to very comparable results.

**5. Conclusions and comments.** In the previous sections we have proved that the Legendre and Chebyshev collocation approximations presented in [12] are convergent and that the error decays faster than algebraically when  $f$  and  $u$  are infinitely differentiable for the singular problem (2.1) and (2.2). Our main results are given in Theorems 3.1 and 3.2.

The key feature of the spectral collocation approximation is that it uses a pole condition as the boundary condition at the singularity and employs a commonly used collocation scheme. Thus, the convergence result provides a theoretical justification for the well-known fact that it is unnecessary to impose all the pole conditions in order for numerical schemes to obtain high accuracy. Because most of the existing spectral

approaches for singular problems use more or less the pole conditions, we expect that our result can also be regarded as a theoretical justification for these methods.

Finally we make a few comments on the method we analyzed. The method has been successfully applied to solving steady-state Navier–Stokes equations in [13, 14]. However, since it uses the Chebyshev or Legendre type of collocation methods in the  $r$  interval  $(0,1)$ , one may suspect that the clustering of grid points near  $r = 0$  leads to a very severe restriction on time steps for time dependent problems. To see this, let us consider the time dependent problem on the unit disk

$$(5.1) \quad u_t = \Delta u + au_x + bu_y.$$

In polar coordinates the equation becomes

$$(5.2) \quad u_t = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + (a \cos \theta + b \sin \theta) \frac{\partial u}{\partial r} + (-a \sin \theta + b \cos \theta) \frac{1}{r} \frac{\partial u}{\partial \theta}.$$

Assume that (5.2) is approximated in  $r$  using the Legendre or Chebyshev collocation, and in  $\theta$  using Fourier collocation. Then it is not difficult to see that for the diffusion term the time step restriction for an explicit integration scheme is

$$(5.3) \quad \Delta t_{\max} \approx \min\{(\Delta r_0)^2, r_1 \Delta r_0\} \approx (r_1)^2 = O\left(\frac{1}{N^4}\right)$$

at  $r \approx 0$  and

$$(5.4) \quad \Delta t_{\max} \approx \min\{(\Delta r_{N-1})^2, r_N \Delta r_{N-1}\} \approx \left(\frac{1}{N^4}\right)$$

at  $r \approx 1$ . Obviously these two time scales are the same. For the convection term we have at  $r \approx 0$

$$(5.5) \quad \Delta t_{\max} \approx \min\{\Delta r_0, r_1\} \approx O\left(\frac{1}{N^2}\right)$$

and at  $r \approx 1$

$$(5.6) \quad \Delta t_{\max} \approx \min\{\Delta r_{N-1}, r_N\} \approx O\left(\frac{1}{N^2}\right).$$

Once again they are the same. Thus, the above simple analysis tells us that the clustering of grid points near the singularity does not result in a time restriction worse than that near the outer boundary. Of course, just like spectral methods applied to nonsingular problems, a restriction  $O(1/N^4)$  on time steps is too severe. Implicit or semi-implicit time integrators should be used. The resultant algebraic systems can be solved using either iterative methods with effective preconditioners [7, 12] or fast direct solvers [24].

For problems in spheric geometries the method can be applied straightforwardly. However, the severe restriction on time steps at the north and south poles could be a potential problem for the method (see discussion in [6, pp. 480–482]). This issue deserves further investigation.

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