# THE CHARACTER OF $\omega_1$ IN FIRST
# COUNTABLE SPACES

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ABSTRACT. We define a cardinal function $\chi(P, Q)$, where $P$ and $Q$ are properties of topological spaces. We show that it is consistent and independent that $\chi(\omega_1, \text{first countable}) = \omega_1$.

In this paper we define and investigate the cardinal function $\chi(P, Q)$, where $P$ and $Q$ are properties of topological spaces. We are particularly interested in whether $\chi(\omega_1, \text{first countable})$ is greater than $\omega_1$. We found this investigation interesting, because it was hard to guess what $\chi(\omega_1, \text{first countable})$ should be (see the paragraph after Example 6); important, because it answers a question arising from the normal Moore space conjecture; and instructive, because it provides an opportunity to use the combinatorial principle $\Diamond^+$ in a topological context.

If $Y$ is a closed subset of a space $X$, define the character of $Y$ in $X$, $\chi(Y, X)$, to be the least cardinal of a set $\mathcal{U}$ of open sets $U$ in $X$, such that for any open set $V$, $Y \subset V \subset X$, there is $U \in \mathcal{U}$ with $Y \subset U \subset V$. The motive for this definition is that if $X'$ is the quotient space of $X$ obtained by shrinking $Y$ to a point $y$, then $\mathcal{U}$ is a neighborhood base for $y$ in $X'$ and $\chi(Y, X)$ is the character of $y$ in $X'$. If $P$ and $Q$ are properties of topological spaces, then $\chi(P, Q)$ is $\sup\{\chi(Y, X): Y \text{ has } P, X \text{ has } Q\}$. This sup need not exist in general, but $\chi(\omega_1, \text{first countable})$ is clearly less than or equal to $2^{\omega_1}$. By abuse of notation, we abbreviate the property of being homeomorphic to an ordinal $\gamma$ as $\gamma$. We also abbreviate first countable as fc.

Let us begin by presenting the application that was the author's motive for this work. One line of attack on the normal Moore space conjecture is to construct absolute examples of normal, not collectionwise normal spaces. After discussing Bing's Example G, Mary Ellen Rudin writes [R], "What we need then is a space of smaller character which is collectionwise Hausdorff as well as normal, but still not collectionwise normal. George has these properties, but has a long way to go to become a Moore space; its character is $c$." In pursuing this line of thought, it is natural to ask the vague question, "Can George be made first countable?" George [F'] is not collectionwise normal

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because there is a closed discrete collection of closed sets homeomorphic to $\omega_1$ that cannot be separated, so a precise version of this question is "Is there an absolute example of a first countable normal space $X$ with a closed discrete collection $\mathcal{Y}$ of closed sets homeomorphic to $\omega_1$ that cannot be separated?"

We answer this question negatively. There is a model $M$ of set theory, obtained by collapsing an inaccessible cardinal to $\omega_2$, in which:

(1) Every normal $T_2$ space of character $\leqslant c$ is collectionwise Hausdorff.

(2) $\chi(\omega_1, \text{fc}) = \omega_1$ (no separation axiom assumed).

Now suppose that $X$ and $\mathcal{Y}$ are as in the precise question. Let $X'$ be the quotient space of $X$ obtained by shrinking each $Y_\gamma \in \mathcal{Y}$ to a point $y_\gamma$. By (2), the character of $X'$ is $\omega_1$; so by (1) the points $y_\gamma$ can be separated in $X'$. By lifting up the collection of open sets, $\mathcal{Y}$ can be separated in $X$.

In §1 we establish notation and discuss combinatorial principles in $L$. §2 is devoted to examples of $\chi(P, Q)$. In §3 we construct, using $\Diamond^+$, a collection-wise normal space demonstrating $\chi(\omega_1, \text{fc}) > \omega_1$. A sketch of the construction of $M$ and the proof of (1) and (2) is in §4. We conclude with a list of questions.

1. We consider an ordinal to be the set of smaller ordinals, equipped with the usual topology. A function is a set of ordered pairs. The interval $(\alpha, \beta]$ is $\{\gamma: \alpha < \gamma \leqslant \beta\}$. We implicitly assume that $\lambda$ is a limit ordinal.

By the Pressing Down Lemma, we mean the fact that if $f$ is defined on the limit ordinals less than $\omega_1$ so that $f(\lambda) < \lambda$, then there is $\beta$ such that for cofinally many $\lambda, f(\lambda) = \beta$.

One result of Jensen's investigations of $L$ has been the abstraction of combinatorial principles and the use of these principles in topology. The history and proofs from $V = L$ of these principles can be found in [D]. Topological applications of $\Diamond$, W, E, and KH can be found in [Ju]. In this section we contrast these principles with $\Diamond^+$.

The Cantor tree (or, the full binary tree on $\omega + 1$) has the property that for $n < \omega$, the cardinality of the $n$th level is $2^n < \omega$, while the cardinality of the $\omega$th level is $c > \omega$. Kurepa's Hypothesis, KH, is the assertion of the existence of an analogue, a tree such that for $\gamma < \omega_1$, the cardinality of the $\gamma$th level is $\omega < \omega_1$, while the cardinality of the $\omega_1$th level is $\omega_2 > \omega_1$.

The principle $\Diamond$ is used to do $2^{\omega_1}$ tasks in $\omega_1$ steps. For example, to construct a Suslin tree we must construct a tree in $\omega_1$ steps while preventing all $2^{\omega_1}$ potential uncountable antichains. However, $\Diamond$ does not suffice to construct a Kurepa tree; we need $\Diamond^+$ to construct $\omega_2$ distinct branches through the tree. To construct a Kurepa tree we use $\Diamond^+$ and a diagonaliza-tion argument to insure that no family of $\omega_1$ branches is the family of all branches.

Principle W asserts the existence of a Kurepa tree plus some control over all countable subsets of branches through the tree.

2. We begin some easy examples of the function $\chi(Y, X)$.

EXAMPLE 1. The character of the unit interval $I$ in the plane $R^2$. Because $I$

is compact and the distance function $d$ is continuous, $d(I, H) = \inf\{d(i, h): i \in I, h \in H\}$ is greater than 0 for every nonempty, closed set $H$ disjoint from $I$. Let $B_n = \{x: d(x, i) < 1/n$ for some $i \in I\}$. Then $\{B_n: n \in \omega\}$ is a basis for $I$ in $R^2$, so $\chi(I, R^2) = \omega$. We have, in fact, shown the well-known fact that $\chi(\text{Compact, Metric}) = \omega$.

EXAMPLE 2. The character of $Z^+$, the positive integers, in the plane $R^2$. We show that $\chi(Z^+, R^2) > \omega$ by contradiction. So assume that $\mathfrak{U} = \{U_n: n \in \omega\}$ is a basis. Let $x_n \in U_n$, $d(x_n, n) < \frac{1}{2}$. Let $V = R^2 - \{x_n: n \in \omega\}$. Now, $V$ is open, $x_n \in U_n$, and $x_n \notin V$, so $\mathfrak{U}$ is not a basis. This contradiction shows that $\chi(Z^+, R^2) > \omega$; a fortiori, it shows $\chi(\omega, \text{fc}) > \omega$. A similar argument shows that if $\gamma$ is a countable ordinal with no last element, then $\chi(\gamma, \text{fc}) > \omega$.

EXAMPLE 3. If $\gamma$ is a countable ordinal with a last element, $\chi(\gamma, \text{fc}) = \omega$. Since $\gamma$ is countable, let $\gamma = \{\gamma_i: i < \omega\}$. Since we are considering only first countable spaces, let $\{B_{i,n}: n \in \omega\}$ be a basis for $\gamma_i$ in $X$. If $s: n \to \omega$, let $V_s = \bigcup\{B_{i, s(i)}: i < n\}$. $\mathcal{V}$, the set of such $V_s$'s, is countable. Now because $\gamma$ is compact, every open set containing (i.e., covering) $\gamma$ has a refinement in $\mathcal{V}$.

EXAMPLE 4. The character of a Lindelöf set in a first countable space is at most $c$. Let $Y \subset X$, $Y$ Lindelöf, $X$ first countable. Now $Y$ as a space itself is first countable Lindelöf, so by Arhangelskiĭ's Theorem [**R**] card $Y \leqslant c$. Now, analogously to Example 3, we need only consider open sets which are the union of a countable number of basic open sets, because $Y$ is Lindelöf. Now because $X$ is first countable and card $Y \leqslant c$, there are only $\omega \cdot c = c$ basic open sets to consider. There are $c^\omega = (2^\omega)^\omega = 2^\omega = c$ countable subsets of $c$, so there is a basis for $Y$ in $X$ of cardinality $\leqslant c$.

EXAMPLE 5. The character of a countable closed, discrete set in a first countable space. Let $d$ be the least cardinal such that there is a family $F = \{f_\gamma: \gamma < d\}$ dominating $^\omega\omega$; i.e. for all $g: \omega \to \omega$ there is $\gamma$ with $f_\gamma(n) > g(n)$ for all $n$. It is clear that every countable set in a first countable space has a basis indexed by such an $F$, i.e. by $d$. The diagonalization argument of Example 2 shows that $d > \omega$. It is known [**H**] that $d < c$ is consistent.

EXAMPLE 6. $\chi(\text{compact, fc}) = c$. Example 4 shows that $\chi(\text{compact, fc}) \leqslant c$; we need a compact $Y$ in a first countable $X$ with $\chi(Y, X) = c$. This example of Alexandroff was suggested to us by Juhász. Let $Y$ and $Z$ be disjoint copies of the unit interval $I$. For $z \in Z$ let $\{z\}$ be open. For $y \in Y$ let the $n$th basic open neighborhood of $y$ be $\{y' \in y: d(y, y') < 1/n\} \cup \{z' \in z: 0 < d(z, z') < 1/n\}$ where $y$ and $z$ correspond to the same point of $I$. Clearly $X$ is first countable, $Y$ is compact. Because every infinite set of $I$ has a limit point, if $U$ is an open set of $X$ containing $Y$, then $X - U$ is finite. Thus $\chi(Y, X) = c$.

Let us now consider $\chi(\omega_1, \text{fc})$. One's first impulse might be that because $\omega_1$ has no last element, $\chi(\omega_1, \text{fc}) > \omega_1$ by arguing as in Example 2. But that argument depends on the existence of a countable sequence without a limit point, and $\omega_1$ is countably compact. So maybe the analogy should rather be with the countably compact countable ordinals, and $\chi(\omega_1, \text{fc}) = \omega_1$.

EXAMPLE 7. $\chi(\omega_1, \text{fc}) \geqslant \omega_1$. Let $X = \omega_1 \times \omega_1$, let $Y = \{\langle \gamma, \gamma \rangle: \gamma \in \omega_1\}$.

Let $U$ be open, $Y \subset U \subset X$. For each $\gamma$ there is $\beta < \gamma$ so that the open rectangle $(\beta, \gamma] \times (\beta, \gamma] \subset U$. By the Pressing Down Lemma, there is $\beta_U$ so that $(\beta_U, \omega_1) \times (\beta_U, \omega_1) \subset U$. If $\{U_i : i \in \omega\}$ is a countable family of such $U$'s, choose $\beta > \sup\{\beta_{U_i} : i \in \omega\}$. Then no $U_i$ is contained in the open set $X - \{\langle \beta, \beta + 1 \rangle\}$, so $\chi(\omega_1, \mathrm{fc}) > \omega$.

3. In this section we show that $\chi(\omega_1, \mathrm{fc}) > \omega_1$ is consistent with the usual axioms of set theory. Specifically, we construct $Y \simeq \omega_1$, $X \supset Y$, $X$ first countable with $\chi(\omega_1, \mathrm{fc}) > \omega_1$ assuming the combinatorial principle $\Diamond^+$.

We define a topological space $X_F$, depending on a set $F$ of functions from $\omega_1$ to $\omega$. The points of $X_F$ are the elements of the set $\omega_1 \times (\omega + 1)$, $Y$ is the set $\omega_1 \times \{\omega\}$. Points of $X - Y$ are isolated. The other points have basic open neighborhoods of the form $V(f, \alpha, \beta) = \{\langle \delta, n \rangle : \alpha < \delta < \beta, f(\delta) < n \leqslant \omega\}$. The set $V(f) = \{\langle \delta, n \rangle : \delta < \omega_1, f(\delta) < n \leqslant \omega\}$ is open if $f \in F$.

Two examples are easily understood. If $F$ is the set of constant functions, then $X_F$ is first countable and $\chi(Y, X) = \omega$. If $F$ is the set of all functions, then a diagonalization arguments shows that $\chi(Y, X) > \omega_1$. But in this case $X$ is not first countable. Our plan is to define $F$ between these two extremes.

We begin by establishing machinery for diagonalization arguments. Call $G$ a *candidate* if $G = \{g_\alpha : \alpha < \omega_1\}$, each $g_\alpha$ is a function from $\omega_1$ to $\omega$, and let $G^* = \{V(g_\alpha) : \alpha < \omega_1\}$. If $\chi(Y, X) > \omega_1$, there is a function $h \colon \omega_1 \to \omega$ and a set of ordinals $\{\gamma_\alpha : \alpha < \omega_1\}$ such that $V(h)$ is open, and for all $\alpha$, $h(\gamma_\alpha) > g_\alpha(\gamma_\alpha)$. Our plan is to place many such $h$'s in $F$.

The following notation will be used in defining such $h$'s. Let $\lambda \leqslant \omega_1$, $K \subset \lambda$, $G = \{g_\alpha : \alpha < \lambda\}$ where each $g_\alpha$ is a function from $\lambda$ to $\omega$. Define:

$$\mathrm{Enum}(K)(\alpha) = \text{the } \alpha\text{th element of } K;$$

$$\mathrm{Diag}(K, G)(\beta) = g_\alpha(\beta) + 1 \quad \text{if } \beta = \mathrm{Enum}(K)(\alpha) \text{ for some } \alpha,$$

$$= 1 \text{ otherwise.}$$

Thus if for some $K$ with $\omega_1$ elements $V(\mathrm{Diag}(K, G))$ is open, then $G^*$ is not a basis.

Because of the form of $\Diamond^+$, we need to code candidates as subsets of $\omega_1$. There is a bijection $\rho \colon \omega_1 \to \omega_1 \times \omega_1 \times \omega$ such that for all limit ordinals $\lambda$, range $\rho \restriction \lambda = \lambda \times \lambda \times \omega$. Define $G^\#$, the subset of $\omega_1$ coding $G$, by $G^\# = \{\rho^{-1}(\alpha, \beta, n) : g_\alpha(\beta) = n\}$.

The combinatorial principle $\Diamond^+$ is the assertion that there is a sequence $\mathcal{W} = \{W_\lambda : \lambda < \omega_1\}$ such that:

(a) $W_\lambda$ is a countable family of subsets of $\lambda$.

(b) For every $S \subset \omega_1$ there is a closed unbounded set $C(S)$ such that for all $\lambda \in C(S)$, $S \cap \lambda \in W_\lambda$ and $C(S) \cap \lambda \in W_\lambda$.

Now define $D(G) = \mathrm{Diag}(C(G^\#), G)$. From $\mathcal{W}$ we can define an auxiliary sequence $\mathcal{W}' = \{W'_\lambda : \lambda < \omega_1\}$ such that:

(a') $W'_\lambda$ is a countable family of functions from $\lambda$ to $\omega$.

(b') For $\lambda \in C(G^\#)$, $D(G) \restriction \lambda \in W'_\lambda$.

(c) For each $n$ and $\lambda$, the constant function $\{\langle \alpha, n \rangle : \alpha < \lambda\} \in W'_\lambda$.

(d) If $\lambda' < \lambda$, $n < \omega$, and $f \in W'_{\lambda'}$ then the function $f \cup \{\langle \lambda', n \rangle\} \cup \{\langle \beta, 1 \rangle : \lambda' < \beta < \lambda\} \in W'_\lambda$.

$\mathcal{W}'$ can be defined by placing in $W'_\lambda$ the countable number of functions required by (b') and (c) and then closing under (d). To show that a countable set of functions can satisfy (b'), note that by (b) we can replace $\lambda \in C(G^\#)$ by $G^\# \cap \lambda \in W_\lambda$, and that by our definitions, $G^\# \cap \lambda = H^\# \cap \lambda$ and $C(G^\#) \cap \lambda = C(H^\#) \cap \lambda$ imply that $D(G) \restriction \lambda = D(H) \restriction \lambda$.

We define $F$ as the set of all functions $f$ from $\omega_1$ to $\omega$ such that for all $\lambda, f \restriction \lambda \in W'_\lambda$. $X$ is first countable by (a') and Hausdorff by (c). To show that $\chi(Y, X) > \omega_1$ it is sufficient to show that $D(G) \in F$ for all candidates $G$, i.e. that $D(G) \restriction \lambda \in W'_\lambda$ for all $\lambda$.

If $\lambda \in C(G^\#)$, then $D(G) \restriction \lambda \in W'_\lambda$ by (b'). If $\lambda \notin C(G^\#)$, then either $D(G) \restriction \lambda$ is constantly 1 or there is (because $C(G^\#)$ is closed) a greatest element $\lambda'$ of $C(G^\#) \cap \lambda$. Then $D(G) \restriction \lambda' \in W'_{\lambda'}$ by (b') and $D(G) \restriction \lambda \in W'_\lambda$ by (d) and the definition of $D$ (which depends on the definition of Diag). Thus, we have shown that $\lozenge^+$ implies $\chi(\omega_1, \text{fc}) > \omega_1$.

We remark here on an easily overlooked point. It is vitally important that $C(G^\#)$ is closed. It is not enough to know that $\{\lambda : G^\# \cap \lambda \in W_\lambda\}$ contains a closed unbounded set; we must know $C(G^\#)$, and that $C(G^\#) \cap \lambda \in W_\lambda$ if $G^\# \cap \lambda \in W_\lambda$. We cannot exclude the possibility that $G^\# \cap \lambda = H^\# \cap \lambda$, but $C(G^\#) \cap \lambda \neq C(H^\#) \cap \lambda$ and, hence, $D(G) \restriction \lambda \neq D(H) \restriction \lambda$.

4. In this section we sketch the proof of (1) and (2). We feel that it is unfortunate that this proof requires the reader to have some knowledge of forcing and models of set theory. (See Question 2.) The model $M$ is the one Silver [**S**] used to show the consistency of not KH. This section is written with the assumption that the reader has at hand a copy of *Trees* [**J**] (probably more widely available than [**S**]).

The construction of $M$ and the proof from the key lemma that $\chi(\omega_1, \text{fc}) = \omega_1$ is exactly the proof of not KH in $M$ as in [**J**, pp. 10–11], so we only state the key lemma and sketch its proof.

First, we code $X$, $Y$ as a subset of $\omega_1$. Let $Q$ be a homeomorphism from $\omega_1$ to $Y$. Let $B(n, \beta)$ be the $n$th basic open neighborhood of $Q(\beta)$ in $X$. For a function $f$ with domain an ordinal $\leqslant \omega_1$, let

$$U(f) = \cup \{B(f(\beta), \beta) : \beta \in \text{dom} f\}.$$

Let $Y_\alpha = \text{range } Q \restriction \alpha + 1$. By Example 3, $\chi(Y_\alpha, X) = \omega$, so there is a family $H = \{h_i^\alpha : i \in \omega, \alpha < \omega_1\}$ of functions such that $\{U(h_i^\alpha) : i \in \omega\}$ is a base for $Y_\alpha$ in $X$. We say that $H$ codes $X$, $Y$.

KEY LEMMA. *If a set $(P, \leqslant)$ of forcing conditions is countably closed and if $H$ coding $X$, $Y$ is in the ground model $V$, then in the extension, $\{U(f) : f \in V, f : \omega_1 \to \omega\}$ is a basis for $Y$ in $X$.*

PROOF. Suppose that in the extension, $k: \omega_1 \to \omega$ and $U(k)$ does not contain any $U(f), f \in V$. Let $p_\phi$ be a condition. We construct by recursion conditions $p_s$ and ordinals $\alpha_s$, for all finite sequences $s$ of natural numbers so that $p_{s\hat{\ }n}$ forces "$U(k)$ does not contain $B(n, \alpha_s)$". This can be done by the assumptions that $(P, \leqslant)$ is countably closed and $U(k)$ does not contain any $U(f)$. Using that $(P, \leqslant)$ is countably closed and a diagonalization argument as in Example 2, we can prove the contradiction that $\{U(h_i^\alpha): i \in \omega\}$ is not a basis for $Y_\alpha$, for $\alpha$ greater than all the $\alpha_s$'s.

The proof of (1) is as in [F]. Because $M$ is $L[G]$, where $G$ is generic over a countably closed notion of forcing and can be coded as a subset of $\omega_2$, GCH and diamond for stationary systems hold in $M$. So (1) holds in $M$.

5. The proofs in §§2 and 3 follow proofs about Kurepa's Hypothesis. So it is natural to ask:

Q1. Is there an implication between KH and $\chi(\omega_1, \mathrm{fc}) > \omega_1$ ?

The formulation of Martin's Axiom lets people who do not know about forcing, let alone iterated forcing, do consistency results. We would like the same for models obtained by collapsing large cardinals to $\omega_2$.

Q2. Formulate and show consistent an axiom which implies not KH, $\chi(\omega_1, \mathrm{fc}) = \omega_1$, not E $(\omega_2)$, Kunen's generalization of Cantor's theorem on the cardinality of first countable compact spaces [Ju], and probably other interesting results. A parameter in the axiom depending on how large the large cardinal was would be of technical interest.

The remaining questions are simply open problems concerning trees of height $\omega_1$. (We learned of Q4 and Q5 from J. Baumgartner.)

Q3. Does $\Diamond^+$ imply there is a Kurepa tree with no Aronszajn subtree [JW]?

Q4. Shelah [Sh] has constructed a "special" special Aronszajn tree solving Countryman's problem. Does every special Aronszajn tree solve Countryman's problem?

Q5. Call two Aronszajn trees $T$, $T^1$ almost isomorphic if there is a cub $C$ such that $\langle \{\tau \in T: \mathrm{ht}(\tau) \in C\}, <_T \rangle \simeq \langle \{\tau \in T^1: \mathrm{ht}(\tau) \in C\}, <_{T'} \rangle$. Does MA + not CH imply that every two Aronszajn trees are almost isomorphic?

The referee asks whether $\chi(\omega_1, \mathrm{fc}) < c$ is consistent, and whether MA + $\neg$ CH has any effect on $\chi(\omega_1, \mathrm{fc})$. The author expects that $\chi(\omega_1, \mathrm{fc}) < c$ is true in Mitchell's model [M], and that MA + $\neg$ CH has no effect on $\chi(\omega_1, \mathrm{fc})$, just as it has no effect on KH [D'].

BIBLIOGRAPHY

[D] K. Devlin, *Aspects of constructibility*, Lecture Notes in Math., vol. 354, Springer-Verlag, Berlin and New York, 1973.

[D'] ———, *Kurepa's hypothesis and the continuum*, Fund. Math. **89** (1975), 23–31. (See Added in Proof, p. 30.)

[F] W. Fleissner, *Normal Moore spaces in the constructible universe*, Proc. Amer. Math. Soc. **46** (1974), 294–298. MR **50** #14682.

[F′]——, *A normal collectionwise Hausdorff, not collectionwise normal space*, General Topology and Appl. **6** (1976), 57–74.

[H] S. Hechler, *On the existence of certain cofinal subsets of $^\omega\omega$*, Proc. Sympos. Pure Math., vol. 13, part 2, Amer. Math. Soc., Providence, R. I., 1974, pp. 155–173. MR **50** #12716.

[J] T. Jech, *Trees*, J. Symbolic Logic **36** (1971), 1–14. MR **44** #1560.

[Ju] I. Juhász, *Consistency results in topology*, Logic Handbook (to appear).

[JW] I. Juhász and W. Weiss, *On a problem of Sikorski*, Fund. Math. (to appear)

[M] W. Mitchell, *Aronszajn trees and the independence of the transfer property*, Ann. Math. Logic **5** (1972), 21–46. MR **47** #1612.

[R] M. E. Rudin, *Lectures on set theoretic topology*, CBMS Regional Conf. Ser. in Math., no. 23, Amer. Math. Soc., Providence, R.I., 1975. MR **51** #4128.

[Sh] S. Shelah, *Decomposing uncountable squares into countably many chains* (to appear).

[S] J. H. Silver, *The independence of Kurepa's conjecture and two-cardinal conjectures in model theory*, Proc. Sympos. Pure Math., vol. 13, part 1, Amer. Math. Soc., Providence, R.I., 1971, pp. 383–390. MR **43** #3112.

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