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LEFT SEPARATED SPACES WITH POINT-COUNTABLE BASES

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ABSTRACT. Theorem 2.2 lists properties equivalent to left separated spaces in the class of T_1 with point-countable bases, with examples preventing plausible additions to this list. For example, X is left iff X is σ-weakly separated or X has a closure preserving cover by countable closed sets, but X is left separated does not imply that X is σ-discrete. Theorem 2.2 is used to show that the following reflection property holds after properly collapsing a supercompact cardinal to ω_2: If X is a not σ-discrete metric space, then X has a not σ-discrete subspace of cardinality less than ω_2. Similar reflection properties are shown true in some models and false in others.

1. Introduction. If P is a property of topological spaces, let $\mathbf{S(P)}$ denote the assertion: "whenever X is a space with property P, then X has a subspace of cardinality $\leq \omega_1$ with property P". In particular, we consider $\mathbf{S_1} \equiv S$ (metric, not σ-discrete). The proof of the consistency of S_1 leads to consideration of $\mathbf{S}_2 \equiv S$ (T_1, point-countable base, not left separated) and $\mathbf{S}_3 \equiv S$ (T_1, point-countable base, not σ-relatively discrete).

§2 starts with an explanation of the topological terminology used in Theorem 2.2, which lists properties equivalent to left separated in the class of T_1 spaces with point countable bases. Examples A and B limit additions to this list.

We will discuss three other refelction properties which can be studied by the same techniques: $\mathbf{S}_4 \equiv S$ (T_1, local density $\leq \omega_1$, countable tightness, not cwH), $\mathbf{S}_5 \equiv S$ (T_1, first countable, not cwH), and $\mathbf{S}_6 \equiv S$ (first countable, not metrizable). A space X is said to be collectionwise Hausdorff (cwH) if every closed discrete subset of X can be simultaneously separated by disjoint open sets. Assuming extra axioms of set theory, there are spaces similar to Examples A and B refuting the S's (§3).

In §4, Axiom R is explained and showns to imply S_1, S_2 and S_4. Further, we discuss decompositions of metric spaces. In §5, we describe iterated proper forcing collapsing a supercompact to ω_2, and show that it makes Axiom R true. This forcing is flexible, and so Theorem 5.1 shows that Axiom R is consistent with CH, with the Proper Forcing Axiom, with MA_{ω_1}, and with $\sim$ CH and $\sim \mathrm{MA}_{\omega_1}$, etc. (assuming, of course, that the existence of a supercompact cardinal is consistent). Thus Axiom R cannot imply S_3, S_5, or S_6 because MA_{ω_1} implies their negations.

§6 contains various remarks which do not fit elsewhere.

Set theoretic notation follows [**Kun**]. In particular, $|X|$ is the cardinality of X, $[X]^{<\kappa} = \{Y \subset X\colon |Y| < \kappa\}$, and ${}^A B = \{f\colon f$ is a function, $\operatorname{dom} f = A$,

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range $f \subset B\}$. ${}^{<\omega}B = \bigcup\{{}^{n}B \colon n \in \omega\}$. For the typist's sake, $[X]^{\omega}$ is often used for $[X]^{<\omega_1}$.

2. Left separated spaces with point countable bases. We call a subspace Y of a space X *relatively discrete* (respectively, *weakly separated*) if there is a family $\{U_y \colon y \in Y\}$ such that U_y is a neighborhood of y and if $y \neq z$ then $y \notin U_z$ and (respectively, or) $z \notin U_y$. Y is *closed, discrete* if Y is relatively discrete and closed in X. Y is *left separated* if there is a well-ordering $<$ of Y so that initial segments are closed in Y. A left separated space is weakly separated; consider the family $\{[y, \infty) \colon y \in Y\}$. When P is one of the above properties, X is σ-P means that X is a countable union of P subspaces.

It is immediate that σ-closed discrete implies σ-relatively discrete implies σ-left separated implies σ-weakly separated. In a metric space, the implications can be reversed. A relatively discrete Y can fail to be closed because of a set L of limit points. L is closed, hence G_δ, i.e., $L = \bigcap\{U_n \colon n \in \omega\}$. Now $Y - U_n$ is closed, discrete, and we conclude that in perfect spaces σ-relatively discrete implies σ-closed discrete. Now let Z be weakly separated by $\{U_z \colon z \in Z\}$ in a semimetric space. For each $n \in \omega$, let $Z_n = \{z \in Z \colon B_n(z) \subset U_z\}$, where $B_n(z) = \{y \in Z \colon d(y, z) < 1/n\}$. If $y, z \in Z_n$, then $(y \notin B_n(z)$ or $z \notin B_n(y))$ because Z is weakly separated and hence $(y \notin B_n(z)$ and $z \notin B_n(y))$ because $d(y, z) = d(z, y)$.

We say that X has *point-countable base* $\mathcal{B}$ if $\mathcal{B}$ is a base for X and, for all $x \in X$, $\{B \in \mathcal{B} \colon x \in B\}$ is countable. By the Nataga-Smirnov-Bing Theorem, metrizable spaces have point-countable bases. A space X is said to have *countable tightness* if whenever $x \in X$, $Y \subset X$ and $x \in \overline{Y}$, then there is $a \in [Y]^{\omega}$ such that $x \in \bar{a}$. Spaces with point countable bases have countable tightness.

We define a game $G(X, \mathcal{B})$ of infinite length played by two players, I and II. Roughly, I aims to play a base for a point of X that II does not guess. More precisely, at turn n, I plays $B_n \in \mathcal{B}$ and II plays a countable subset, a_n, of X. Then they proceed to turn $n+1$. After ω turns, the winner is decided. I wins iff $\{B_n \colon n \in \omega\}$ contains a base for a point $x \in X - \bigcup\{a_n \colon n \in \omega\}$. What is a winning strategy for II? It is a function, s, from the set of I's previous moves to the set of plays allowed II; i.e., $s \colon {}^{<\omega}\mathcal{B} \to [X]^{\omega}$, such that if II plays according to s, then II wins the play of the game; i.e., if $(B_n : n \in \omega)$ contains a base for a point x, then $x \in \bigcup\{s(B_0, \ldots, B_n) \colon n \in \omega\}$. It is possible that II has a simple winning strategy (let us call it a winning tactic, t) which looks only at I's last move and plays only one point. Rephrasing, $t \colon \mathcal{B} \to X$ is such that if $\{B_n : n \in \omega\}$ contains a base for a point x, then there is $n \in \omega$, such that $t(B_n) = x$.

For example, if X is the real line, then I can win by listing a countable base for X. No matter how II plays, $\bigcup\{a_n \colon n \in \omega\}$ is countable, so $X - \{a_n \colon n \in \omega\} \neq \emptyset$, and I's plays contain a base for every x. In the other direction, if X is countable, then II can win by playing X, either all at one turn, or one point at a time. We leave it to the reader to work out that if X is a σ-discrete metric space, then II has a winning strategy.

We say that $\mathcal{H} = \{H_i \colon i \in I\}$, a family of subsets of a space X is *closure preserving* if for all $J \subset I$, $\bigcup\{\overline{H_i} \colon i \in J\} = \overline{\bigcup\{H_i \colon i \in J\}}$. If $\mathcal{H}$ is a family of closed sets, then it is the same to say that arbitrary unions from $\mathcal{H}$ are closed. We can weaken the concept by requiring only that countable increasing unions be closed. Alone, this condition can be very weak, for maybe there are no increasing chains from $\mathcal{H}$. So we add a condition yielding many increasing unions. We will express these ideas in the terminology of proper forcing. (Unfortunately, this leads us to overwork the word "closed".)

Let $\Gamma \subset [X]^\omega$. We say that Γ is *unbounded* if for all $a \in [X]^\omega$ there is $c \in \Gamma$ such that $a \subset c$. We say that Γ is *closed* if whenever $(c_n)_{n\in\omega}$ is an increasing chain from Γ, then $\bigcup\{c_n : n \in \omega\} \in \Gamma$. Γ is a *club* if Γ is closed and unbounded. We say that $\Sigma \subset [X]^\omega$ is *stationary* if for all clubs Γ, $\Sigma \cap \Gamma \neq \emptyset$. If $f : [X]^{<\omega} \to X$ and $Y \subset X$ satisfies $\{f(e) : e \in [Y]^{<\omega}\} \subset Y$, we say that Y is *closed under* f. The following lemma of Kueker [**Kue**; **Ba**$_1$, Theorem 1.4] is basic to proper forcing.

LEMMA 2.1. *If $\Gamma \in [X]^\omega$ is club, then there is a function $f : [X]^{<\omega} \to X$ such that*

$$\{a \in [X]^\omega : a \text{ is closed under } f\} \subset \Gamma.$$

THEOREM 2.2. *Let X be a T_1 space with a point-countable base, $\mathcal{B}$. The following are equivalent.*

(a) *X is left separated in order type $|X|$.*

(a′) *X is σ-weakly separated.*

(b) II *has a winning tactic in $G(X, \mathcal{B})$.*

(b′) II *has a winning strategy in $G(X, \mathcal{B})$.*

(c) *X has a closure preserving cover $\mathcal{H} = \{H_i : i \in I\}$ by countable closed sets.*

(c′) *$\Gamma = \{a \in [X]^\omega : a$ is closed$\}$ contains a closed unbounded subset of $[X]^\omega$.*

PROOF. The implications (a) $\Rightarrow$ (a′), (b) $\Rightarrow$ (b′) and (c) $\Rightarrow$ (c′) are obvious.

(a) $\Rightarrow$ (b) Let $<$ be a well-order showing that X is left separated. Define $t(B)$ to be the $<$-least element of B. If $\{B_n : n \in \omega\}$ includes a base for x, then $x \in B_n \subset [x, \infty)$ for some n, in which case $x = t(B_n)$.

(a′) $\Rightarrow$ (b′) Let $X = \bigcup\{X_n : n \in \omega\}$, where each X_n is weakly separated by $\{U_x : x \in X_n\}$. For $B \in \mathcal{B}$ define $s(B_0, \ldots, B_m) = \{x \in X : x \in B_m \subset U_x\}$. The definition of weakly separated guarantees that for each n, there is at most one point in $s(B_0, \ldots, B_m) \cap X_n$.

(b′) $\Rightarrow$ (c) For $x \in X$, we define H_x containing x such that if $B_0, \ldots, B_n$ each meet H_x, then $s(B_0, \ldots, B_n) \subset H_x$. Explicitly, we define countable sets H_x^k and $\mathcal{B}_x^k$ for $k \in \omega$ by induction on k. Set $H_x^0 = \{x\}$. Set $\mathcal{B}_x^k = \{B \in \mathcal{B} : B \cap H_x^k \neq \emptyset\}$. $\mathcal{B}_x^k$ is countable because H_x^k is countable and $\mathcal{B}$ is point-countable; and set $H_x^{k+1} = \{s(B_0, \ldots, B_n) : B_0, \ldots, B_n \in \mathcal{B}_x^k\}$. Set $H_x = \bigcup\{H_x^k : k \in \omega\}$ and $\mathcal{H} = \{H_x : x \in X\}$.

We have shown that $\mathcal{H}$ covers X and that each $H \in \mathcal{H}$ is countable. We will show that for all $\mathcal{G} \subset \mathcal{H}$, $\bigcup \mathcal{G}$ is closed, so $\mathcal{H}$ is closure preserving. In particular, $H = \bigcup\{H\}$ is closed. Let $\mathcal{G} \subset \mathcal{H}$, set $Y = \bigcup \mathcal{G}$, and let $z \in \overline{Y}$. We will show that $z \in Y$. Let $(B_n : n \in \omega)$ list a base for z. By (b′) there is n so that $z \in s(B_0, \ldots, B_n)$. Let $U = \bigcap\{B_k : k \leq n\}$; because $z \in \overline{Y}$, there is $y \in U \cap Y$. Then $y \in H_x \in \mathcal{G}$. If $y \in H_x^k$, then $z \in H_x^{k+1} \subset Y$.

(c′) $\Rightarrow$ (a) Apply Lemma 2.1 to Γ to obtain $f : [X]^{<\omega} \to X$. Next we claim that if $Y \subset X$ and Y is closed under f, then Y is closed. If Y is countable, this follows from the choice of f; if Y is uncountable and $z \in \overline{Y}$, by countable tightness let $a_0 \in [Y]^\omega$ be such that $z \in \bar{a}_0$. By induction on $j \in \omega$, set $a_{j+1} = a_j \cup \{f(e) : e \in [a_j]^{<\omega}\}$; $a_j \subset Y$ because Y is closed under f. Now $a_\omega = \bigcup\{a_j : j \in \omega\}$ is countable and closed under f; hence $z \in a_\omega \subset Y$.

Now we prove (a) by induction on $|Y|$, $Y \subset X$. If Y is countable, then Y is left separated because it is T_1. Now we assume that $|Y| = \lambda$ and every $Z \in [X]^{<\lambda}$ is left separated. Well order $Y = \{y_\alpha\colon \alpha < \lambda\}$. By induction, define an increasing sequence Z_α of sets closed under f such that $\{y_\beta\colon \beta < \alpha\} \subset Z_\alpha$ and $|Z_\alpha| = |\alpha|$ for infinite α. By induction hypothesis, for each $\alpha < \lambda$, there is a left separation $<_\alpha$ of Z_α. For $y \in Y$, define $\beta(y)$ to be the least α such that $y \in Z_{\beta(y)}$. We define a left separation of Y as follows: $y < z$ if $\beta(y) < \beta(z)$ or $\beta(y) = \beta(z) = \alpha$ and $y <_\alpha z$. the set of predecessors of Y is $\bigcup\{Z_\beta\colon \beta < \alpha\} \cup \{z \in Z_\alpha\colon z <_\alpha y\}$ (where $\beta(y) = \alpha$), a union of two closed sets. □

I thank Fred Galvin for improving this theorem. I explained to him that if X has a point-countable base then the following are equivalent: X is σ-left separated, (b′), and a complicated statement analogous to (c), (c′). The next day Fred suggested assuming T_1, correctly formulated (c), and pointed out Corollary 2.5.

COROLLARY 2.3. *If X is a T_1, left separated space with a point-countable base, then for all $Y \subset X$, $|\overline{Y}| = Y$.*

PROOF. Immediate from Theorem 2.2(c).

A more subtle corollary follows from the following theorem of Fodor [**Fo**; **Wi**, Theorem 3.1.5, p. 69].

THEOREM 2.4. *Let $f\colon X \to [X]^{<\lambda}$. Then X can be written as $\bigcup\{X_\alpha\colon \alpha < \lambda\}$ so that if $x \in X_\alpha$, then $f(x) \cap X_\alpha \subseteq \{x\}$.*

COROLLARY 2.5. *If X is a left separated space with a point countable base, then X can be written as $\bigcup\{X_\alpha\colon \alpha < \omega_1\}$, where each X_α is relatively discrete.*

PROOF. Let $\mathcal{H}$ satisfy Theorem 2.2(c). Define $f\colon X \to [X]^{<\omega_1}$ so that $x \in f(x) \in \mathcal{H}$. Apply Fodor's Theorem. If $x \in X_\alpha$, then $K = \bigcup\{f(y)\colon y \in X_\alpha - \{x\}\}$ is closed, and $(X - K) \cap X_\alpha = \{x\}$. □

We conclude this section with some results about concepts related to those mentioned in Theorem 2.2.

LEMMA 2.6. *If X is a left separated T_1 space with a σ-point-finite base $\mathcal{B} = \bigcup\{\mathcal{B}_n\colon n \in \omega\}$, then X is σ-relatively discrete.*

PROOF. Let $<$ left separate X. For $x \in X$, choose $U_x \in \mathcal{B}$, so that $x \in U_x \subset [x, \infty)$. Set $X_n = \{x \in X\colon U_x \in \mathcal{B}_n\}$. Fix $n \in \omega$. Set $Y_0 = \varnothing$ and

$$Y_{j+1} = \{y \in X_n\colon \text{if } x < y \;\&\; x \in X_n - \bigcup\{Y_k\colon k \leq j\},\ \text{then } y \notin U_x\}.$$

Because $\mathcal{B}_n$ is point-finite, $X_n = \bigcup\{Y_j\colon j < \omega\}$. □

LEMMA 2.7. (a) *If X has a closure preserving cover by finite sets, then X is σ-relatively discrete.*

(b) *If X is σ-relatively discrete, perfect, and collectionwise Hausdorff, then X has a closure preserving cover by finite sets.*

PROOF. (a) As in Corollary 2.5. Theorem 2.4 with $\lambda = \omega$ is due to Erdös and DeBrujn.

(b) By perfect and σ-relatively discrete, $X = \bigcup\{X_n\colon n \in \omega\}$ where each X_n is closed, discrete. Apply collectionwise Hausdorff to obtain disjoint open families $\mathcal{U}_n = \{U_x\colon x \in X_n\}$ with $x \in U_x$ and $U_x \cap (\bigcup\{X_k\colon k \leq n\}) = \{x\}$. We define an open set V_k containing x for $x \in X_n$ by induction on n. For $x \in X_0$, set $V_x = U_x$. For $x \in X_n$, $n > 0$, set

$$F_x = \{y \in X_k : k < n \text{ and } x \in V_y\},$$
$$V_x = U_x \cap \left(\bigcap\{V_y : y \in F_x\}\right),$$
$$H_x = F_x \cup \{x\}.$$

It is routine to check that $y \in H_x$ iff $x \in V_y$. We claim that $\{H_x : x \in X\}$ is closure preserving. Suppose that $z \in \overline{\bigcup\{H_x : x \in X'\}}$; then $y \in H_x \cap V_z$ for some $x \in X'$, $y \in X$. Hence $x \in V_x \subset V_y \subset V_z$ and $z \in H_x$. $\square$

The examples in the following section show that Lemma 2.7 cannot be improved to an equivalence. We may delete from the hypothesis of 2.7(b) neither perfect (Michael line), nor cwH (Cantor tree). The existence of a closure preserving cover by finite sets implies neither perfect (one point compactification of uncountable discrete space) nor cwH (Pixley-Roy).

3. Examples.

EXAMPLE A. Let Λ be the set of countable limit ordinals. For $\delta \in \Lambda$, let $\eta_\delta : \omega \to \delta$ be increasing and cofinal in δ. Define a metric d on Λ by $d(\delta, \varsigma) = 2^{-n}$, where n is least so that $\eta_\delta | n \neq \eta_\varsigma | n$. The conclusion of Corollary 2.3 holds because if $Y \subset \Lambda$, then $\sup \overline{Y} = \sup Y$. The conclusion of Corollary 2.5 holds because each $\{\delta\}$ is relatively discrete. A point-countable base is $\mathcal{B} = \{B_\sigma : \sigma \in^{<\omega} \omega_1\}$, where $B_\sigma = \{\delta \in \Lambda : \eta_\delta | n = \sigma\}$. If $\delta \in B_\sigma$, then $\max \operatorname{range} \sigma < \delta$, so the pressing down lemma yields that Λ is not left separated. In fact, Theorem 2.2(b) is strongly contradicted.

LEMMA 3.1. *Player* I *has a winning strategy in* $G(\Lambda, \mathcal{B})$.

PROOF. It is easy to verify that the following changes in $G(\Lambda, \mathcal{B})$ do not change the existence of a winning strategy: (1) player I may play countably many basic open sets at each turn; (2) at turn $n+1$ player II must play $a_{n+1} \supset a_n$ such that $\sup a_{n+1} > \sup a_n$. (If a_n is enlarged I's task is harder. I can in effect play countably many moves at once using a partition of ω into infinitely many infinite pieces.) Now a strategy for I is to play $\{B_\sigma : \operatorname{range} \sigma \subset \sup a_n\}$ at turn $n+1$. After ω steps, I's plays include a base for $\sup\{a_n : n \in \omega\} \notin \bigcup\{a_n : n \in \omega\}$. $\square$

For subspaces Y of Λ, I has a winning strategy in $G(Y, \mathcal{B})$ iff Y contains a club, II has a winning strategy in $G(Y, \mathcal{B})$ iff Y is nonstationary. Hence the game is undetermined if both Y and $\omega_1 - Y$ are stationary.

The following axiom is true in L, every generic extension of L, and every model of set theory without inner models of many large cardinals (see [**KM**, p. 222]).

AXIOM E. For some regular cardinal $\lambda > \omega_1$, there is $E \subset \{\delta \in \lambda : \operatorname{cf} \lambda = \omega\}$ stationary in λ with $E \cap \beta$ nonstationary in β for all $\beta < \lambda$.

EXAMPLE A1. Assume λ, E satisfy Axiom E. For $\delta \in E$, let $\eta_\delta : \omega \to \delta$ be increasing and cofinal in δ. Define a metric d on E by $d(\delta, \varsigma) = 2^{-n}$, where n is least so that $\eta_\delta | n \neq \eta_\varsigma | n$. Then E is not σ-discrete, but every $Y \in [E]^{<\lambda}$ is σ-discrete (see [**Po**]). Thus Axiom E implies that S_1, S_2 and S_3 fail.

EXAMPLE A2. Assume λ, E satisfy Axiom E; define η_δ as above. Let $Z = E \cup {}^{<\omega}\lambda$. Let ${}^{<\omega}\lambda$ be the set of isolated points. Define the mth basic open set of $\delta \in E$ to be $\{\delta\} \cup \{\eta_\delta | n : m \leq n\}$. Then Z is first countable and locally countable but not cwH. Every $Y \in [Z]^{<\lambda}$ is cwH and metrizable (see [**F**$_3$]). Thus Axiom E implies that S_4, S_5 and S_6 fail.

EXAMPLE B. This space is from Aull [**Au**], who calls it a modification of an example of Miščenko [**Mi**]. I thank Pete Nyikos for calling this example to my attention.

Let $F = \bigcup\{{}^\alpha\omega\colon \alpha \in \omega_1\}$. Basic open sets are indexed by $f \in F$, $n \in \omega$:

$$B(f,n) = \{f\} \cup \{g \in F\colon f \subset g \text{ and } g(\operatorname{dom} f) \geq n\}.$$

We can picture F as a tree where open sets "look up".

(*) If $B(f,n) \cap B(g,m) \neq \varnothing$,
then either $f \subset g$ and $B(f,n) \supset B(g,m)$
or $g \subset f$ and $B(g,m) \subset B(f,n)$.

It is routine to verify that F is a T_1, regular, ultraparacompact, left separated space. (Given an open cover $\mathcal{U}$, define a disjoint refinement by induction on dom f—possible by (*).)

LEMMA 3.2. *F is not σ-relatively discrete.*

PROOF. Towards a contradiction, assume that $\{F_n\colon n \in \omega\}$ and $\theta\colon F \to \omega$ are such that $F = \bigcup\{F_n\colon n \in \omega\}$ and, for each $n \in \omega$, $\{B(f,\theta(f))\colon f \in F_n\}$ is disjoint. By induction on $\alpha < \omega_1$, define $f_\alpha \in {}^\alpha\omega$ so that if $\alpha < \beta$, then $f_\beta(\alpha) = \theta(f_\alpha)$, hence $f_\beta \in B(f_\alpha, \theta(f_\alpha))$. For some $n \in \omega$ and $\alpha < \beta < \omega_1$, $f_\alpha, f_\beta \in F_n$ but $f_\beta \in B(f_\alpha, \theta(f_\alpha))$. Contradiction. □

Hence σ-relatively discrete cannot be added to the list in Theorem 2.2.

Next, we assume that the weak Kurepa Hypothesis fails (∼wKH) and Martin's Axiom (MA_{ω_1}) holds (see [**To$_1$** and **Ba**, Theorem 7.10]).

LEMMA 3.3. ($\mathrm{MA}_{\omega_1}+\sim$ wKH). *Every $Y \in [F]^{\omega_1}$ is σ-closed discrete, and hence metrizable.*

PROOF. If $Y \in [F]^{\omega_1}$, then Y is a tree of height and cardinality ω_1, so Y has at most ω_1 uncountable branches, $\{b_\alpha\colon \alpha < \omega_1\}$ (this is ∼wKH). By induction on $\alpha < \omega_1$ we can define c_α as a final segment of b_α so that $\{c_\alpha\colon \alpha \in \omega_1\}$ is disjoint. We consider $T = Y - \bigcup\{c_\alpha - \{\min c_\alpha\}\colon \alpha < \omega_1\}$. T has no ω_1-branches, so by MA_{ω_1} [**BMR**], T is the union of countably many antichains; i.e., $T = \bigcup\{A_n\colon n \in \omega\}$. Let $Y_n = A_n \cup \bigcup\{c_\alpha\colon \min c_\alpha \in A_n\}$. Then $Y = \bigcup\{Y_n\colon n \in \omega\}$ and if $y \in Y_n$ then $\{f \in Y\colon y \subset f\}$ is totally ordered. Hence each Y_n is relatively discrete.

Fix $\theta\colon Y \to \omega$ so that $\{B(y,\theta(y))\colon y \in Y_n\}$ is disjoint for all $n \in \omega$. For each $n \in \omega$ we define P_n, a poset of finite approximations to a closed subset of Y_n. Let P_n be the set of triples (a, s, η) satisfying

(1) $a \in [Y_n]^{<\omega}$,
(2) $s \in [Y - Y_n]^{<\omega}$,
(3) $\eta\colon s \to \omega$,
(4) $\forall y \in s, \eta(y) \geq \theta(y)$,
(5) $\forall y \in s, B(y, \eta(y)) \cap a = \varnothing$.

We show that P_n is ccc. Aiming for a contradiction, suppose that $W \in [P_n]^{\omega_1}$ is a set of pairwise incompatible elements. By usual counting arguments (Δ-system lemma, discard the root, etc.), we may assume that $W = \{(a_\alpha, s_\alpha, \eta_\alpha)\colon \alpha \in \omega_1\}$ satisfies for some fixed $k \in \omega$, $l \in \omega$ and $m\colon l \to \omega$,

(1) $\{a_\alpha \cup s_\alpha : \alpha \in \omega_1\}$ is disjoint,
(2) $\forall \alpha \in \omega_1$, $a_\alpha = \{a(\alpha, i) : i < k\}$,
(3) $\forall \alpha \in \omega_1$, $s_\alpha = \{s(\alpha, j) : j < l\}$,
(4) $\forall \alpha \in \omega_1 \forall_j < l$, $s(\alpha, j) \in Y_{m(j)}$.

Because W is pairwise incompatible, we can define $\Phi: [\omega_1]^2 \to k \times l \times \{0,1\}$ so that if $\Phi(\alpha, \beta) = (i, j, e)$, then $a(\beta, i) \in B(s(\alpha, j), \eta(s(\alpha, j)))$ and $\alpha < \beta$ if $e = 0$, $\beta < \alpha$ if $e = 1$. Apply Ramsey's Theorem to get $i < k$, $j < l$, $e < 2$ and $\alpha, \beta, \gamma < \omega$ so that

$$a(\gamma, i) \in B(s(\alpha, j), \eta(s(\alpha, j))) \cap B(s(\beta, j), \eta(s(\beta, j))),$$

and $\alpha < \beta < \gamma$ if $e = 0$, $\gamma < \beta < \alpha$ if $e = 1$. This contradicts that $\{B(y, \theta(y)) : y \in Y_{m(j)}\}$ is disjoint.

By MA_{ω_1}, $P = (P_n)^\omega$, the finite support product of ω copies of P_n, has ccc. An element of P can be considered to be a triple of functions, (a, s, η) with domain ω so that $(a(i), s(i), \eta(i)) \in P_n$ and $\{i : (a(i), s(i), \eta(i)) \neq (\emptyset, \emptyset, \emptyset)\}$ is finite for all i. For each $y \in Y_n$, $D_y = \{(a, s, \eta) : (\exists i)(y \in a(i))\}$ is dense. For each $i \in \omega$ and $f \in Y - Y_n$, $D_{if} = \{(a, s, \eta) : f \in s(i)\}$ is dense. By MA_{ω_1}, let G be a filter on P meeting each of the above dense sets. Set $Z_i = \bigcup\{a(i) : (a, s, \eta) \in G\}$. Then each Z_i is closed, discrete and $Y_n = \bigcup\{Z_i : i \in \omega\}$. Because this construction can be done for all $n \in \omega$, Y is σ-closed discrete. Then by $(*)$ and Nagata-Smirnov-Bing, Y is metrizable. □

Thus, $\mathrm{MA}_{\omega_1} + \sim$ wKH implies that S_3 and S_6 fail.

Choose $A \subset \omega_1$ with both A and $\omega_1 - A$ stationary. Let $U(A)$ be the tree of continuous, increasing functions from a countable ordinal to A. Gary Gruenhage suggested using $U(A)$ in place of F in Example B. Because $U(A)$ has no ω_1-branches, we do not need $\sim$wKH before applying MA_{ω_1} and [**BMR**]. This space is essentially the same as the linearly ordered space $M(A)$ of maximal branches through $U(A)$ (see [**To**$_2$, p. 288]). An analog of Example B1 also can be constructed. Thus MA_{ω_1} suffices to refute S_3, S_5 and S_6.

EXAMPLE B1. Assume $\sim$ wKH + MA_{ω_1}. Let F be as in Example B. For each $f \in F$, enumerate $\{h \in F : h \subset f\}$ as $\{h(f, m) : m < \omega\}$ and set $e(f, n) = \{h(f, m) : m \leq n\}$. Let $W = \{(f, g) \in F^2 : f \subset g\}$. Set $X = F \cup W$. Points of W are isolated. The other basic open sets are indexed by $f \in F$ and $n \in \omega$:

$$U(f, n) = \{f\} \cup \{(f, g) \in W : g(\operatorname{dom} f) \geq n\} \cup \{(h, f) \in W : h \notin e(f, n)\}.$$

With this basis X is a Moore space and F is closed discrete. If $A \in [F]^{\omega_1}$, then A is σ-closed discrete in the topology of Example B. It is routine to use that fact to separate A in X.

X is not cwH because F cannot be separated. We follow the proof of Lemma 3.2. Towards a contradiction, assume that $\theta: F \to \omega$ is such that $\{U(f, \theta(f)) : f \in F\}$ is disjoint. By induction on $\alpha < \omega_1$, define $f_\alpha \in {}^\alpha\omega$ so that if $\alpha < \beta$, then $f_\beta(\alpha) \geq \theta(f_\alpha)$; i.e., $(f_\alpha, f_\beta) \in U(f_\alpha, \theta(f_\alpha))$. By the pressing down lemma, there is $n \in \omega$, $e \in [\omega_1]^{<\omega}$ and a stationary set S such that if $\alpha \in S$, then $\theta(f_\alpha) = n$ and $e(f_\alpha, n) = e$. If $\alpha, \beta \in S$, then $(f_\alpha, f_\beta) \in U(f_\alpha, \theta(f_\alpha)) \cap U(f_\beta, \theta(f_\beta))$. Contradiction. □

EXAMPLE C. This example is from [**vD**, Remark 12.6], to which the reader is referred for more details and more information about $\mathfrak{b}$. The point set of C is $({}^\omega\omega) \cup \omega \cup ({}^\omega\omega \times \omega \times \omega)$. Points of ${}^\omega\omega \times \omega \times \omega$ are isolated. The nth neighborhood of $k \in \omega \subset C$ is $\{k\} \cup ({}^\omega\omega \times \{k\} \times (\omega - n))$. The nth neighborhood of $f \in {}^\omega\omega \subset C$ is $\{f\} \cup (\{f\} \times \{(k, f(k)) \colon k \geq n\})$.

Consider subspaces $C(H)$ of the form $H \cup \omega \cup (H \times \omega \times \omega)$, where $H \subset {}^\omega\omega$. If H is bounded, i.e., there is $f \in {}^\omega\omega$ such that for all $h \in H$, $\{n \in \omega \colon f(n) \leq h(n)\}$ is finite, then $C(H)$ is metrizable. If H is unbounded, then $C(H)$ is not cwH. Hence $\mathfrak{b} > \omega_1$ (i.e., every $H \in [{}^\omega\omega]^{\omega_1}$ is bounded) implies that S_5 and S_6 fail.

4. Axiom R and applications. We have seen in §3 that Axiom E implies that all the S_i's fail. Since Axiom E implies the existence of counterexamples, $\sim E$, the negation of Axiom E is potentially weaker than the S_i's. In any case, it is often frustrating to work with $\sim E$ because it is so "linear". (An exception: it follows quickly from Engelking and Lutzer [**EL**] that S (first countable linearly ordered topological space, not paracompact) is equivalent to E.)

The statement that stationary subsets of $[X]^\omega$ reflect is stronger than the statement that stationary subsets of $\{\alpha \in \kappa \colon \operatorname{cf} \alpha = \omega\}$ reflect. Even this seems to be not enough because if X has structure, we may need to reflect to a closed subset of X.

Recall that a space X has *countable tightness* when for all $x \in X$ and $\overline{Y} \in X$, if $x \in \overline{Y}$, then $\exists a \in [Y]^\omega \; x \in \bar{a}$. It is easy to see that in a space of countable tightness, the union of an increasing sequence of closed sets is closed if the sequence has uncountable cofinality. We say that $\Gamma \subset [X]^{<\kappa}$ is *tight* if whenever $\{C_\alpha \colon \alpha < \delta\}$ is an increasing sequence from Γ and $\omega < \operatorname{cf} \delta < \kappa$, then $\bigcup\{C_\alpha \colon \alpha < \delta\} \in \Gamma$.

AXIOM R. If $\Sigma \subset [X]^\omega$ is stationary and $\Gamma \subset [X]^{<\omega_2}$ is tight and unbounded, then there is $Y \in \Gamma$ such that $P(Y) \cap \Sigma$ is stationary in $[Y]^\omega$.

LEMMA 4.1. *Let X be a T_1 space with a point countable base. If every $Y \in [X]^{<\omega_2}$ is left separated, then for all $Y \in [X]^{<\omega_2}$, $|\overline{Y}| = |Y|$. Hence $\Gamma = \{Y \in [X]^{<\omega_2} \colon Y$ is closed$\}$ is tight and unbounded.*

PROOF. For Y finite, use T_1. Let $|Y| = \omega$. Towards a contradiction, assume that $|\overline{Y}| > \omega$, choose $Z \in [X]^{\omega_1}$ with $Y \subset Z \subset \overline{Y}$. Then $|\mathrm{closure}_Z(Y)| = |\overline{Y} \cap Z| = |Z| = \omega_1$. However Z is left separated, so by Corollary 2.3 $|\mathrm{closure}_Z(Y)| = \omega$. Now let $Y = \{y_\alpha \colon \alpha < \omega_1\}$. By countable tightness, $\overline{Y} = \bigcup\{\overline{\{y_\beta \colon \beta < \alpha\}} \colon \alpha < \omega_1\}$. Hence $|\overline{Y}| = \omega_1 \cdot \omega = \omega_1$.

Γ is tight because X has countable tightness. Γ is unbounded because for all $Y \in [X]^{<\omega_2}$, $Y \subset \overline{Y} \in \Gamma$. $\square$

THEOREM 4.2. *Axiom* R *implies S_2 (hence S_1).*

PROOF. Towards a contradiction, assume (a) X is a T_1 not left separated space with a point countable base and (b) every $Y \in [X]^{<\omega_2}$ is left separated. From (a) and Theorem 2.2(c′) we get that $\Sigma = \{a \in [X]^\omega \colon a$ is not closed in $X\}$ is stationary in $[X]^\omega$. From (b) and Lemma 4.1 we get that $\Gamma = \{Y \in [X]^{<\omega_2} \colon Y$ is closed$\}$ is tight and unbounded. Applying Axiom R, we get $Y \in \Gamma$ such that $\Sigma \cap [Y]^\omega = \{a \in [Y]^\omega ; a$ is not closed in $X\} = \{a \in [Y]^\omega \colon a$ is not closed in $Y\}$ is stationary in $[Y]^\omega$. (The second equality is why we wanted Y to be closed.) Hence $Y \in [X]^{<\omega_2}$ is not left separated. Contradiction. $\square$

In order to use the technique of the above proof for other properties, we must first have a characterization of those properties in terms of $[X]^\omega$. We give three more examples. (Forcing and reflection proofs of the consistency of 4.4 and 4.7 are well known.)

Let T be a tree of height ω and B a set of infinite branches through T. We say that B is *nonstationary* if there exists a one-to-one function $f\colon B \to T$ such that for all $b \in B$, $f(b) \in b$. B is *stationary* otherwise. (This terminology is justified by Theorem 4.3.) Set $X = T \cup B$. Say that $Y \subset X$ is B-closed iff (i) for all $b \in B$, $b \subset Y$ implies $b \in Y$ and (ii) $s <_T t$ and $t \in Y$ implies $s \in Y$. The set of B-closed sets is tight.

THEOREM 4.3. *Let T be a tree of height ω and B a set of infinite branches through T. B is stationary if and only if $\{a \in [T \cup B]^\omega\colon a \text{ is closed}\}$ is stationary.*

PROOF. This is Theorem 8.5 of [$\mathbf{Ba_1}$], where the proof is done in detail. The only if direction is similar to (c′) ⇒ (a) of Theorem 1, by induction on cardinality of $X = T \cup B$ and using a function $f\colon [X]^{<\omega} \to X$.

COROLLARY 4.4 (AXIOM R). *If B is a stationary set of branches through a tree T of height ω, then there is a stationary $A \in [B]^{<\omega_2}$.*

PROOF. First prove the analogue of 4.1, then follow the proof of 4.2. □

We can show, again by induction on $|X|$, that B is nonstationary iff there is a function $f'\colon B \to T$ so that $f'(b) \in b$ and $\{\{t \in b\colon f(b) \le t\}\colon b \in B\}$ is disjoint. If we topologize X so that each $t \in T$ is isolated, and a neighborhood of $b \in B$ has the form $\{b\} \cup \{t \in b\colon s \le t\}$ for some $s \in b$, then X is a space similar to Examples B1 and B2. Moreover B is nonstationary iff X is cwH. Generalizing from trees of height ω, Shelah [$\mathbf{Sh_2}$] showed S (locally countable, not cwH) holds after the Levy collapse of a supercompact to ω_2. However, locally countable spaces are very special. Manifolds, which are more geometric, are locally separable and have countable tightness. These ideas lead to the unusual hypotheses of Lemma 4.5. (Also see recent work of Tall, e.g. [$\mathbf{T_2}$].)

LEMMA 4.5. *Let D be a closed discrete subset of a space X and κ a cardinal such that*

(a) $\forall A \in [D]^{<\kappa}$, *$A$ can be separated.*

(b) $\exists \beta < \kappa\, \forall y \in D$, *$y$ has a neighborhood of density $\le \beta$,*

(c) *X has countable tightness at every point of D (i.e.,*

$$(\forall y \in D\, \forall Z \subset X)(y \in \overline{Z} \to \exists a \in [Z]^\omega)(y \in \bar{a})).$$

Then D can be separated iff $\{a \in [X]^\omega\colon \bar{a} \cap D \ne a \cap D\}$ is not stationary.

PROOF. (→) is easy.

(←) Say that $Z \subset X$ is D-closed if $\overline{Z} \cap D = Z \cap D$. By Lemma 1.1, let $f\colon [X]^{<\omega} \to X$ be such that closed under f implies D-closed. By (b), for $y \in D$, choose $Q_y \in [X]^{\le\beta}$ so that $y \in \operatorname{Int}(\overline{Q}_y)$. The proof is by induction on X, again. By (a) D can be separated if $|X| < \kappa$. For $|X| \ge \kappa$, we can define a continuous increasing sequence of sets, X_α, so that $\bigcup\{X_\alpha\colon \alpha < |X|\} = X$, $|X_\alpha| < |X|$, and each X_α is D-closed and Q-closed; (i.e., $y \in X_\alpha$ implies $Q_y \subset X_\alpha$). By inductifon hypothesis, for each α, $D_\alpha = D \cap (X_{\alpha+1} - X_\alpha)$ can be separated in $X_{\alpha+1}$—by $\{U_y : y \in D_\alpha\}$, say. Then

$$\bigcup\{\{\operatorname{Int}(\overline{U}_y) - \overline{X}_\alpha : y \in D_\alpha\}\colon \alpha < |X|\}$$

separates D in X. □

COROLLARY 4.6 (AXIOM R). *If X is ω_1-cwH, has local density ω_1, and has countable tightness, then X is* cwH. *In particular ω_1-cwH manifolds are* cwH.

PROOF. Again, prove the analogue of 4.1 and follow the proof of 4.2. $\square$

A graph $G = (V, E)$ consists of a set V of vertices and a set of edges $E \subset [V]^2$. G is said to have coloring number $\leq \omega$ if there is a well-ordering $<$ of V so that for all $v \in V$, $\{u \in V : \{u, v\} \in E\}$ is finite.

THEOREM 4.7 (AXIOM R). *If $G = (V, E)$ is a graph such that $(Y, E \cap [Y]^2)$ has coloring number $\leq \omega$ for every $Y \in [V]^{<\omega_2}$, then G has coloring number $\leq \omega$.*

PROOF. Say that $Y \subset V$ is E-closed if for all $v \in V$, $\{y \in Y : (y, v) \in E\}$ infinite implies $v \in Y$. Argue as above. $\square$

Below, we will use Axiom R to prove a reflection property which fits well with theorems of Pol [**Po**] and Hansell [**Ha**] on σ-discretely refinable families. Analogous results about σ-discretely decomposible families are left to the reader. These results allow us to replace Axiom SC(ω_2) with Axiom R in Lemma 4.9 of [$\mathbf{F}_1$]. For further discussion and references, see [**Fr**].

Let $\mathcal{E} = (E_\xi : \xi \in I)$ be an indexed family of subsets of a metric space X with σ-discrete base $\mathcal{B} = \bigcup\{B_n : n \in \omega\}$. We will assume that X, I and $\mathcal{B}$ are pairwise disjoint. We say that $\mathcal{E}$ is *σ-discretely refinable* if there exists an indexed family $\mathcal{E}' = (E_{\xi j} : \xi \in I, j \in \omega)$ such that (a) $\forall j \in \omega$, $(E_{\xi j} : \xi \in I)$ is discrete, (b) $\forall_\xi \in I \forall j \in \omega$, $E_{\xi j} \subset E_\xi$, and (c) $\bigcup \mathcal{E}' = \bigcup \mathcal{E}$. We say that $a \subset \mathcal{B} \cup I$ is *$\mathcal{E}$-closed* if whenever $a \cap \mathcal{B}$ contains a base for $x \in \bigcup \mathcal{E}$, then $x \in \bigcup\{E_\xi : \xi \in a \cap I\}$.

LEMMA 4.8. *Let $\mathcal{E}, X, I$ and $\mathcal{B}$ be as above. The following are equivalent.*

(a) *$\mathcal{E}$ is σ-discretely refinable.*

(b) *There is $s : \mathcal{B} \to [I]^\omega$ such that for all $b \in [\mathcal{B}]^\omega$, $b \cup (\bigcup\{s(B) : B \in b\}$ is $\mathcal{E}$-closed.*

(c) *$\{a \in [\mathcal{B} \cup I]^\omega : a$ is $\mathcal{E}$-closed$\}$ contains a club subset $\Gamma \subset [\mathcal{B} \cup I]^\omega$.*

PROOF. The proof of this analogue to 2.2 is routine except for (c)$\Rightarrow$(b). Given Γ, let $f : [\mathcal{B} \cup I]^{<\omega} \to \mathcal{B} \cup I$ be the Kueker function. For $u \in [\mathcal{B}]^{<\omega}$ and $j \in \omega$, define $u_0 = u$, $u_{j+1} = u_j \cup \operatorname{range} f|[u_j]^{<\omega}$ and $u^f = \bigcup\{u_j : j \in \omega\}$. (Compare 2.2(c$'$) $\Rightarrow$(a).)

For $B \in \mathcal{B}$, let

$$U(B) = \{\{B_0, B_1, \ldots, B_m\} \in [\mathcal{B}]^{<\omega} : B_1 \supset B_1 \supset \cdots \supset B_m = B\}$$

be a countable set. Set $s(B) = \bigcup\{u^f : u \in U(B)\}$.

Towards verifying (b), let $\{B_k : k \in \omega\}$ contain a base for $x \in \bigcup \mathcal{E}$. By passing to a subsequence we may assume that $B_0 \supset B_1 \supset \cdots \supset B_m \supset \cdots$. (The modifications in case x is isolated are easy.) For each $m \in \omega$, set $a_m = \{B_0, \ldots, B_m\}^f$; then $\{B_k : k \in \omega\} \subset \bigcup\{a_m : m \in \omega\} \in \Gamma$. Hence for some $m \in \omega$, $x \in a_m \subset s(B_m)$. $\square$

For $Y \subset X$, set $\mathcal{E}|Y = (E_\xi \cap Y : \xi \in I)$ and $I(Y) = \{\xi \in I : E_\xi \cap Y \neq \emptyset\}$. We say that $\mathcal{E}$ is *ω_1-like* if whenever $Y \subset X$ has weight $\leq \omega_1$, $|I(Y)| \leq \omega_1$. If CH holds, then point countable families are ω_1-like. If $\mathcal{C} \subset \mathcal{B}$, set $Y(\mathcal{C}) = \{x \in \bigcup \mathcal{E} : \mathcal{C}$ contains a base for $x\}$.

COROLLARY 4.9 (AXIOM R). *Let $X, \mathcal{E}$ be as above. If $\mathcal{E}$ is ω_1-like and not σ-discretely refinable, then there is $Y \subset X$, weight $Y \leq \omega_1$ such that $\mathcal{E}|Y$ is not σ-discretely refinable.*

PROOF. Set $\Gamma = \{a \in [\mathcal{B} \cup I]^{<\omega_2}\colon A$ is $\mathcal{E}$-closed and $I(Y(A \cap \mathcal{B})) \subset I \cap A\}$. Argue as in 4.2. □

Axiom $\forall S\Diamond_s$ is a consequence of $V = L$, implies CH and is consistent with Axiom R. The proof of the next result follows ideas in [$\mathbf{F_1}$, **Ha**, and **Fr**].

COROLLARY 4.10 (AXIOM R+$\forall S\Diamond_s$). *Let $X, \mathcal{E} = (E_\xi\colon \xi \in I)$ be as above. If $\mathcal{E}$ is point-countable and for all $J \subset I$, $\bigcup\{E_\xi\colon \xi \in J\}$ is Borel in X, then $\mathcal{E}$ is σ-discretely refinable.*

5. Properly collapsing a supercompact cardinal. We will force with complete Boolean algebras, $\mathbf{B}$. A $\mathbf{B}$-name is a function from (already defined) $\mathbf{B}$-names to $\mathbf{B}$. For definition and dicussion of proper forcing, see [$\mathbf{Ba_1}$]. An iterated proper forcing is a sequence $(\mathbf{B}_\alpha\colon \alpha \le \kappa)$ such that for $\alpha < \beta$, $\mathbf{B}_\alpha$ is a complete subalgebra of $\mathbf{B}_\beta$ so that a $\mathbf{B}_\alpha$ name is a $\mathbf{B}_\beta$ name (see, e.g. [$\mathbf{Ba_2}$]). Moreover if $\nu \le \kappa$ is inaccessible, then $\mathbf{B}_\nu = \bigcup\{\mathbf{B}_\alpha\colon \alpha < \nu\}$ is proper and $\mathbf{B}_\kappa = \mathbf{B}_\nu * \dot{\mathbf{A}}_\nu$, where $\mathbf{B}_\nu$ is proper and $\mathbf{A}_\nu \in V^{\mathbf{B}_\nu}$ is proper in $V^{\mathbf{B}_\nu}$.

Let $\mathbf{B}$ have κcc, where κ is regular. If $X \in V$ and $A \in [X]^{<\kappa} \cap V[G]$ (where G is V-generic on $\mathbf{B}$), then A can be represented in V by a name in $\mathcal{N} = \{\dot{A} | \dot{A}\colon \check{Y} \to \mathbf{B}$, where $Y \in [X]^{<\kappa}\}$. If $\Gamma \in V[G]$ and $\Gamma \subset [X]^{<\kappa}$, then Γ can be represented by a $\mathbf{B}$-name, $\dot{\Gamma}\colon \mathcal{N} \to \mathbf{B}$.

THEOREM 5.1. *Let κ be supercompact. Let $(\mathbf{B}_\alpha\colon \alpha \le \kappa)$ be an iterated proper forcing such that for every inaccessible $\nu < \kappa$, $|\mathbf{B}_\nu| < \kappa$ and $\mathbf{B}_\kappa \| - \kappa = \omega_2$. Then $\mathbf{B}_\kappa \| -$ Axiom* R.

PROOF. We may assume that $\mathbf{B}_\kappa \subset V_\kappa$. $\mathbf{B}_\kappa$ is κcc by a Δ-system argument. Assume that in $V[G]$, X, Σ, Γ satisfy the hypothesis of Axiom R. Without loss of generality, $X \in V$. Let $\dot{\Sigma}$ and $\dot{\Gamma}$ be $\mathbf{B}_\kappa$-names representing Σ and Γ of the form discussed above. Let λ be much bigger than everything mentioned so far, and let $j\colon V \to M \supset {}^\lambda V$ be elementary with critical point κ, $j\kappa > \lambda$. For every set S, $j''S = \{js\colon s \in S\} \subset jS$. A careful examination of the form of $\dot{\Sigma}$, $\dot{\Gamma}$ yields the stronger condition (3) below. (The point is that for $\dot{A} \in \mathcal{N}$, $j(\dot{A}) = \{(jx, b)\colon (x, b) \in A\}$. Because $\mathbf{B}_\kappa \subset V_\kappa$, $jb = b$ for $b \in B_\kappa$. There are no "new" elements of $j(\dot{A})$ because $|\dot{A}| < \kappa$.) It is routine to check that $M \models \Psi(\kappa, j''X, j''\dot{\Sigma}, j''\dot{\Gamma}, j\kappa, jX, j\dot{\Sigma}, j\dot{\Gamma})$, where Ψ is the conjunction of

(1) $j''\dot{\Sigma}$ and $j''\dot{\Gamma}$ are $\mathbf{B}_\kappa$-names, $j''X \in [jX]^{<j\kappa}$.

(2) $\mathbf{B}_\kappa \Vdash (j''\dot{\Sigma}$ is a stationary subset of $[j''X]^\omega$, $j''\dot{\Gamma}$ is tight and unbounded in $[j''X]^{<\kappa}$, and $\check{\kappa} = \omega_2)$.

(3) $\mathbf{B}_{j\kappa} \Vdash j''\dot{\Sigma} \subset j\dot{\Sigma}$ and $j''\dot{\Gamma} \subset j\dot{\Gamma}$.

Then $M \models \exists v_1 \exists v_2 \exists v_3 \exists v_4 \Psi(v_1, v_2, v_3, v_4, j\kappa, X, \dot{\Sigma}, j\dot{\Gamma})$. By elementarity

$$V \models \exists v_1 \exists v_2 \exists v_3 \exists v_4 \Psi(v_1, v_2, v_3, v_4, \kappa, jX, \dot{\Sigma}, \dot{\Gamma}).$$

Instantiating, we get $V \models \Psi(\nu, Y, \dot{\Sigma}', \dot{\Gamma}', \kappa, X, \dot{\Sigma}, \dot{\Gamma})$. Let $G_\nu = G \cap \mathbf{B}_\nu$. By (1) we can define $\Sigma' = \mathrm{val}(\dot{\Sigma}', G_\nu)$ and $\Gamma' = \mathrm{val}(\dot{\Gamma}', G_\nu)$. Moreover, by (2) $V[G_\nu] \models \Sigma'$ is stationary in $[Y]^\omega$ and Γ' is tight and unbounded in $[Y]^{<\nu}$. By properness Σ' remains stationary in $V[G]$, and $P(Y) \cap \Sigma \supset \Sigma'$ is stationary. It remains to show that $Y \in \Gamma$.

Because $V[G] \models |Y| = \omega_1$, let $Y = \{x_\alpha\colon \alpha < \omega_1\}$. By induction on $\alpha < \omega_1$, we will define $A_\alpha \in \Gamma' \in V[G_\nu]$ so that $\beta < \alpha$ implies $A_\beta \cup \{x_\beta\} \subset A_\alpha$. Let

$A_0 \in \Gamma'$ be arbitrary. For $\alpha = \beta + 1$, apply ($V[G_\nu] \vDash \Gamma'$ is unbounded in $[Y]^{<\nu}$) to $A_\beta \cup \{x_\beta\}$. For α a limit, $S = \{A_\beta : \beta < \alpha\}$ is probably not in $V[G_\nu]$, but by properness there is $T \in V[G_\nu]$, $T \in [\Gamma']^{<\nu}$, so that $S \subset T$. Because $V[G_\nu] \vDash \nu$ is ω_2, regular, $\bigcup T \in [Y]^{<\nu}$. Apply unbounded to get A_α, $\bigcup T \subset A_\alpha \in \Gamma'$. By construction $\bigcup\{A_\alpha : \alpha < \omega_1\} = Y \in \Gamma$, because Γ is tight. $\square$

6. Digressions. The following three plausible conjectures may summarize and clarify the status of the S's.

1. S_1, S_2 and S_4 are equivalent,
2. S_3, S_5 and S_6 are equivalent; in fact,
3. S_3, S_5 and S_6 are refutable in ZFC.

RE CONJECTURE 3. In [**MS**], Milnor and Shelah construct in ZFC an example from Truss of a family, X, of ω_2 countable sets such that every $Y \in [X]^{<\omega_2}$ has a transversal, but X does not. Here, a transversal is a one-to-one choice function.

In the examples of this paper, reflection properties have the form $S(P_1 \& \sim P_2)$, where P_1 is hereditary. The contrapositives can be stated without negation. For example, S_1: A metric space X is σ-discrete iff every $Y \in [X]^{<\omega_2}$ is σ-discrete. The "only if" can be added because σ-discrete is hereditary. Note that $S(P_1 \& \sim P_2)$ is equivalent to $S(P_1 \& \sim$ hereditarily $P_2)$ because a subset of a small set is small.

We stated the S's as reflection properties and proved some of them by reflection arguments. The contrapositives can be proven by integrating winning tactics t_Y, $Y \in [X]^{<\omega_2}$, into a winning strategy for X (see Theorem 2.2(b), (b$'$)) using a homomorphism $h: P([X]^{<\omega_2}) \to \mathbf{B}$, where $\mathbf{B}$ is a proper complete Boolean algebra. For specific Boolean algebras, a strongly compact cardinal, rather than a supercompact cardinal, suffices [$\mathbf{F_2}$].

A timely letter from Peg Daniels reminded me of another reflection that can be proven in ZFC.

THEOREM 6.1. *Let X be a T_1 space with a point-countable base such that $|X| = \lambda$, singular, and every $Y \in [X]^{<\omega_2}$ is left separated. Then X is left separated.*

PROOF. We follow the proof in [$\mathbf{F}_4$] of the analogous results for locally small spaces and coloring numbers. Let $(\lambda_\alpha)_{\alpha < \operatorname{cf} \lambda}$ be a continuous, increasing sequence of cardinals cofinal in λ with $\lambda_0 \geq \operatorname{cf} \lambda$. By induction on $\beta \leq \omega_1$, we will define continuous increasing sequences $\mathcal{Y}_\beta = \{Y_{\beta\alpha} : \alpha < \operatorname{cf} \lambda\}$ such that

$$(*) \qquad |Y_{\beta\alpha}| = \lambda_\alpha \quad \text{and} \quad \bigcup \mathcal{U}_\beta = X.$$

Let $\mathcal{Y}_0$ be any sequence satisfying $(*)$. If $\beta = \gamma + 1$, by the proof of Lemma 4.1 we can enumerate $\overline{Y}_{\gamma\alpha}$ as $\{x^\gamma_{\alpha\xi} : \xi < \lambda_\alpha\}$ for each α. Set $Y_{\beta\alpha} = \{x^\gamma_{\delta\xi}\} : \xi < \lambda_\alpha$, $\delta < \operatorname{cf} \lambda\}$. $|Y_{\beta\alpha}| = \operatorname{cf} \lambda \cdot \lambda_\alpha = \lambda_\alpha$.

If β is a limit, set $Y_{\beta\alpha} = \bigcup\{Y_{\gamma\alpha} : \gamma < \beta\}$. By countable tightness, $\mathcal{Y}_{\omega_1}$ is a continuous, increasing sequence of closed subsets of X. Thus we can piece together well orderings of $Y_{\omega_1\alpha+1} - Y_{\omega_1\alpha}$ to left separate X (as in 2.2(c$'$) $\to$(a)). $\square$

It is clear that the "right" generalization of countability in Euclidean space to arbitrary metrizable spaces is σ-discreteness. Theorem 2.2 suggests that the "right" further generalization is left separation in spaces with point countable base. In a similar context, Balogh and Junnila [**BJ**] generalized a theorem about σ-discreteness in metric spaces to σ-left separation in T_1 spaces of character $\leq \mathfrak{c}$. They were able to achieve σ-relatively discrete only with additional hypotheses.

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