PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 106, Number 3, July 1989

# STONE-CECH REMAINDERS WHICH MAKE CONTINUOUS IMAGES NORMAL

WILLIAM FLEISSNER AND RONNIE LEVY

(Communicated by Dennis K. Burke)

ABSTRACT. If $f$ is a continuous surjection from a normal space $X$ onto a regular space $Y$, then there are a space $Z$ and a perfect map $bf : Z \to Y$ extending $f$ such that $X \subset Z \subset \beta X$. If $f$ is a continuous surjection from normal $X$ onto Tychonov $Y$ and $\beta X \backslash X$ is sequential, then $Y$ is normal. More generally, if $f$ is a continuous surjection from normal $X$ onto regular $Y$ and $\beta X \backslash X$ has the property that countably compact subsets are closed (this property is called $C$-closed), then $Y$ is normal. There is an example of a normal space $X$ such that $\beta X \backslash X$ is $C$-closed but not sequential. If $X$ is normal and $\beta X \backslash X$ is first countable, then $\beta X \backslash X$ is locally compact.

We began the study of the class of spaces ACRIN (all continuous regular images normal) in [FL], where we showed that if $X$ is a normal space such that $\beta X \backslash X$ is finite, then $X$ is ACRIN. Here, we give more general conditions on $\beta X \backslash X$ which imply that a normal space is ACRIN.

For a Tychonov space $X$, we denote the Stone-Cech compactification of $X$ by $\beta X$. We call a space $Z$ such that $X \subset Z \subset \beta X$ an *intermediate space.* If $X$ and $Y$ are Tychonov spaces and $f : X \to Y$ is a continuous map, we denote by $\beta f : \beta X \to \beta Y$ the Stone extension of $f$.

Let us recall some well-known properties of maps.

**1. Lemma.** *Let $f : X \to Y$ be a continuous surjection.* (a) *If $X$ is normal and $f$ is a closed mapping, then $Y$ is normal.*

(b) *If $X$ is countably compact, then $Y$ is countably compact.*

(c) *If $f$ is perfect and $Y$ is countably compact, then $X$ is countably compact.*

(d) *If $\beta f^{\to}(\beta X \backslash X) = \beta Y \backslash Y$, then $f$ is perfect.*

Let us fix a continuous surjection $f : X \to Y$ from a normal space to a regular space. Let $bf : P \to Y$ be a perfect map extending $f$, where $P$ is an intermediate space. If $Y$ is Tychonov, we will set $P = \beta f^{\leftarrow}(Y)$ and

Received by the editors July 18, 1988 and, in revised form, October 11, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54C05, 54D15.

*Key words and phrases.* Stone-Cech remainder, continuous image, normal, ACRIN, $C$-closed, normality inducing.

The first author gratefully acknowledges support from NSF Grant DMS-8603227 and University of Kansas General Research allocation #3762-XO-0038.

$bf = \beta f|P$. We defer the discussion of how to define $P$ and $bf$ in the case when $Y$ is regular but not Tychonov. From Lemma 1, we see that if $P$ is normal, then $Y$ is normal. Thus, we seek conditions which imply that $P$ is normal.

One such condition is to assert that every intermediate space $Z$ is normal. Barr and Hajek introduced this notion in [BH] and called it normality inducing. Further, they showed that if $X$ is normality inducing, then $X$ is countably compact, and they showed that $X$ is normality inducing if and only if every compact subset of $\beta X\backslash X$ is finite.

A condition which implies that an intermediate space $Z$ is normal is that $Z\backslash X$ is closed in $\beta X\backslash X$. Then $Z$ is normal because it is the union of the normal space $X$ and the compact space $\mathrm{Cl}_{\beta X}(Z\backslash X)$. (See [FL, Lemma 1.1(c)].) Thus, if $X$ is normal and $\beta X\backslash X$ is finite, or more generally, discrete, then $X$ is normality inducing, and hence ACRIN.

Let us assume that $X$ is countably compact. Again from Lemma 1, we see that $P$ is countably compact. Thus, it is not necessary that every intermediate space $Z$ be normal; it is enough to require that countably compact intermediate spaces be normal.

**2. Proposition.** *If $\beta X\backslash X$ is sequential and $Y$ is Tychonov, then $P\backslash X$ is closed in $\beta X\backslash X$. Hence, $Y$ is normal.*

*Proof*. Suppose that $q \in \beta X\backslash X$ and there is a sequence $(p_n)_{n\in\omega}$ in $P\backslash X$ converging to $q$. By the definition of $P$, each $\beta f(p_n)$ is an element of $Y$. Since $Y$ is countably compact, $(\beta f(p_n))_{n\in\omega}$ has a cluster point $y$ in $Y$. By continuity, $(\beta f(p_n))_{n\in\omega}$ converges to $y$ and $y = \beta f(q)$. Therefore, $q \in P$, and $P\backslash X$ is sequentially closed in $\beta X\backslash X$. Because $\beta X\backslash X$ is sequential, $P\backslash X$ is closed in $\beta X\backslash X$. □

A space is called *C-closed* if every countably compact subset is closed. (See [IN].) For example, sequential spaces are $C$-closed. Countable spaces, which need not be sequential, and $P$-spaces, which are sequential if and only if they are discrete, are $C$-closed. If $X$ is normal and not countably compact, then $X$ contains a closed copy of $\omega$. Thus, $\beta X\backslash X$ contains a closed copy of $\beta\omega\backslash\omega$. Therefore, $\beta X\backslash X$ is not $C$-closed—it contains the nonclosed countably compact subset $(\beta\omega\backslash\omega)\backslash\{p\}$ where $p \in \beta\omega\backslash\omega$.

**3. Theorem.** *If $\beta X\backslash X$ is $C$-closed, then every countably compact intermediate space $Z$ is normal. Hence $X$ is ACRIN.*

*Proof*. Towards a contradiction, assume that $H$ and $K$ are disjoint closed subsets of a countably compact intermediate space $Z$ and that $q \in [\mathrm{Cl}_{\beta X}(H) \cap \mathrm{Cl}_{\beta X}(K)]\backslash Z$. If $q$ were an element of $\mathrm{Cl}_{\beta X}(H \cap X) \cap \mathrm{Cl}_{\beta X}(K \cap X)$, then $X$ would not be normal. Assume without loss of generality that $q \notin \mathrm{Cl}_{\beta X}(H\cap X)$. By regularity, there is an open subset $U$ of $\beta X$ with $\mathrm{Cl}_{\beta X}(H \cap X) \subseteq U \subseteq \mathrm{Cl}_{\beta X}(U) \subseteq \beta X\backslash\{q\}$. Then $H\backslash U$ is a countably compact subset of $\beta X\backslash X$

(since it is a closed subset of $Z$), and since $\beta X\backslash X$ is assumed to be $C$-closed, $q \in \mathrm{Cl}_{\beta X\backslash X}(H\backslash X) = H\backslash U \subseteq Z$, contradicting $q \notin Z$. □

If $Y$ is a locally compact space and $\alpha Y = Y \cup \{\infty\}$ is its one-point compactification, then for large enough cardinal $\gamma$, the subspace $(\alpha Y \times \gamma) \cup (\infty, \gamma)$ of $\alpha Y \times (\gamma + 1)$ is a normal space whose Stone-Cech remainder is $Y$. Thus, every locally compact space is a remainder of a normal space. The following result limits the applicability of Theorem 3 to exactly the locally compact spaces if $\beta X\backslash X$ is first countable.

**4. Proposition.** *If $X$ is normal and $\beta X\backslash X$ is first countable, then $\beta X\backslash X$ is locally compact.*

*Proof*. Towards a contradiction, suppose that $p \in \beta X\backslash X$ has a countable nested base in $\beta X\backslash X$ such that for all $n$, $Cl_{\beta X\backslash X}(B_n)$ is not compact. Since $\mathrm{Cl}_{\beta X}(B_n)$ is compact, we may choose distinct $x_n \in \mathrm{Cl}_{\beta X}(B_n) \cap X$. Then $\{x_{2n} : n \in \omega\}$ and $\{x_{2n+1} : n \in \omega\}$ are disjoint closed subsets of $X$, both of whose closures in $\beta X$ contain $p$. This contradicts the normality of $X$. □

There are normal spaces whose Stone-Cech compactifications are $C$-closed, but not locally compact. For example, it is not hard to show that if $E$ is an $\eta_1$-set with the order topology and $X$ is the set of non-$P$-points of the Dedekind compactification of $E$, then $\beta X\backslash X$ is a $P$-space without isolated points, and hence is $C$-closed but not locally compact. Since $X$, a linearly ordered space is normal, it follows from Theorem 3 that every continuous image of $X$ is normal.

We now give alternative definitions of $bf$ and $P$ which require only that $Y$ be regular. The general situation is this: $f$ is a continuous surjection from the normal space $X$ to a regular space $Y$, and $Z$ is a Hausdorff extension of $X$, that is, a Hausdorff space which contains $X$ as a dense subspace. For $p \in Z$, define $\mathscr{N}_p = \{N \cap X : N \text{ is a neighborhood of } p \text{ in } Z\}$. Let $G$ be $\mathrm{Cl}_{Z\times Y}(\mathrm{graph}\, f)$.

**5. Lemma.** *Let $f$, $X$, $Y$, $Z$, and $G$ be as above.* (a) *For $p \in Z\backslash X$, $f \cup \{(p, y)\}$ is continuous if and only if $f[\mathscr{N}_p]$ converges to $y$.*

(b) *If for all $p \in Z$, $f \cup \{(p, y_p)\}$ is a continuous function, then $f \cup \{(p, y_p) : p \in Z\}$ is a continuous function.*

(c) *$(p, y) \in G$ if and only if $f[\mathscr{N}_p]$ adheres to $y$.*

*Further assume that $X$ is normal and $Z$ is $\beta X$.*

(d) *If $f[\mathscr{N}_p]$ adheres to $y$, then $f[\mathscr{N}_p]$ converges to $y$.*

(e) *Hence, $G$ is the graph of a function $bf$ and $bf$ is perfect.*

*Proof*. (a) and (b) are [PW, 4.1(1) and 4.1(n)]; (c) is routine. Proof of (d): Because $X$ is normal and $Z = \beta X$, we may consider points of $Z$ to be ultrafilters of closed subsets of $X$ and basic open sets have the form $N(F) = \{q \in \beta X : F \notin q\}$. We prove the contrapositive. Suppose that $f[\mathscr{N}_p]$ does not converge to $y$. There is a neighborhood $V$ of $y$ which does not contain $f^{\rightarrow}(N)$

for any $N \in \mathcal{N}_p$. Because $Y$ is regular, there is an open $W$ such that $y \in W$ and $\mathrm{Cl}_Y W \subseteq V$. Consider $H = f^{\leftarrow}(\mathrm{Cl}_Y W) = \{x \in X : f(x) \in \mathrm{Cl}_Y W\}$. Then $N(H) \in \mathcal{N}_p$ and $f^{\rightarrow}(N(H)) \cap W = \phi$. Thus, $f[\mathcal{N}_p]$ does not adhere to $y$.

We have arranged things so that the proof of (e) is easy. By (c), (d), and the fact that points of convergence are unique in Hausdorff spaces, $bf$ is a function. Continuity follows from the previous parts. Because $Z$ is compact, the projection onto $Y$ is a closed map; $bf$ is the restriction of projection to the closed set $G$, so it is also closed. Finally, $bf^{\leftarrow}\{y\}$ is $G \cap (Z \times \{y\})$, the intersection of a closed set and a compact set. □

We wish to thank Jack Porter for discussions which transformed our original ad hoc, dot and circle construction into Lemma 5.

We close with some questions.

*Question* 1. If $X$ is normal and $\beta X \backslash X$ is countable, is $\beta X \backslash X$ sequential? If $X$ is normal and $\beta X \backslash X$ is sequential, is $\beta X \backslash X$ locally compact?

*Question* 2. If $X$ is normal and $\beta X \backslash X$ has countable tightness, is $X$ ACRIN? Is there, without extra axioms of set theory, a regular space of countable tightness which is not $C$-closed? (Balogh proved, assuming PFA, that locally compact spaces of countable tightness are $C$-closed. Fedorchuk constructed, assuming $\diamondsuit$, a Tychonov space of countable tightness which is not $C$-closed.)

## References

[BH] D. Barr and D. Hajek, *Normality inducing spaces*, Gen. Top. Appl. **7** (1977), 89–97.

[FL] W. Fleissner and R. Levy, *Ordered spaces all of whose continuous images are normal*, Proc. Amer. Math. Soc. **355** (1989), 231–235.

[IN] M. Ismail and P. Nyikos, *On spaces in which countably compact sets are closed and hereditary properties*, Topology Appl. **11** (1980), 281–290.

[PW] J. Porter and G. Woods, *Extensions and absolutes of Hausdorff spaces*, Springer-Verlag, New York, 1988.

Department of Mathematics, University of Kansas, Lawrence, Kansas 66045

Department of Mathematics, George Mason University, Fairfax, Virginia 22030