

LEMMA ON MEASURABLE CARDINALS

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ABSTRACT. An ordinal is moved by only finitely many measurable cardinals.

We inductively define a class M , analogously to L , Gödel's class of constructible sets. Let $M_0 = \emptyset$; $M_\lambda = \bigcup \{M_\alpha : \alpha < \lambda\}$; and let $M_{\alpha+1}$ be the collection of subsets of M_α definable using parameters by some formula of the infinitary language $L_{\omega_1\omega_1}$ (instead of $L_{\omega\omega}$, the usual finitary language). Let $M = \bigcup \{M_\alpha : \alpha \in \text{OR}\}$; Chang [1] called this class C_{ω_1} . Kunen [2] used the lemma below to show that if there are ω_1 measurable cardinals, then the axiom of choice is false in M . Kunen's proof used the machinery of iterated ultrapowers. The present proof can be read by anyone familiar with the fundamental paper of Scott [3].

A free κ -complete ultrafilter U on κ induces an elementary embedding j of V into the transitive realization N of the ultrapower V^κ/U . We say κ moves δ when there is a U on κ so that $j(\delta) \neq \delta$.

Lemma. *An ordinal is moved by only finitely many measurable cardinals.*

Proof. Suppose $j(\delta) \neq \delta$. Then there is a maximal interval of ordinals moved by j containing δ . It is straightforward to show that this interval is of the form $[\alpha, \beta)$ where $\text{cf}(\alpha) = \kappa$, and $\beta = \sup j^n(y)$ for $y \in [\alpha, \beta)$.

Claim 1. If $\kappa_0 < \kappa_1$ then $j_1(j_0(y)) = j_0(j_1(y))$.

Informally, the idea is that $j_1(j_0(y))$ is N_1 's notion of the j_0 process applied to $j_1(y)$ (N_1 's notion of y), which is $j_0(j_1(y))$ because N_1 's notion is correct.

Let $\phi_i(x, y, U_i)$ be a formula of the language of ZF, obtained in the familiar manner, such that for all x, y , $\phi_i(x, y, U_i)$ iff $j_i(x) = y$ ($i = 0, 1$).

Then for every $y \in \text{OR}$, $\phi_0(y, j_0(y), U_0)$. Clearly

$$\begin{aligned} \phi_0(y, j_0(y), U_0) &\Leftrightarrow \phi_0^{j_1''V}(j_1(y), j_1(j_0(y)), j_1(U_0)) \\ &\Leftrightarrow \phi_0^{N_1}(j_1(y), j_1(j_0(y)), U_0) \Leftrightarrow \phi_0(j_1(y), j_1(j_0(y)), U_0). \end{aligned}$$

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Since $j_0(j_1(\gamma))$ is the unique y such that $\phi_0(j_1(\gamma), y, U_0)$, we get from $\phi_0(j_1(\gamma), j_1(j_0(\gamma)), U_0)$

$$j_1(j_0(\gamma)) = j_0(j_1(\gamma)).$$

Claim 2. If $[\alpha_0, \beta_0]$ and $[\alpha_1, \beta_1]$ intersect, they are nested. Suppose $\alpha_0 < \alpha_1$. Let $\alpha_0 < \gamma < \alpha_1$, $j_1(\gamma) = \gamma$. Then $j_1(j_0^n(\gamma)) = j_0^n(\gamma)$, so no $j_0^n(\gamma) \in [\alpha_1, \beta_1]$. Then either $\beta_0 = \sup j^n(\gamma) < \alpha_1$ or there is n so that $j_0^n(\gamma) < \alpha_1 < \beta_1 < j_0^{n+1}(\gamma)$.

Let $\delta, \{\kappa_i : i \in \omega\}$ be a counterexample to the Lemma. Since $\text{cf}(\alpha_i) \neq \text{cf}(\alpha_j)$ there is an infinite ascending chain $\{\alpha_i : i \in \omega\}$ of ordinals. By Claim 2, there is an infinite descending chain of ordinals. Contradiction.

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