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LINEAR-QUADRATIC FRACTIONAL GAUSSIAN CONTROL*

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Abstract. In this paper a control problem for a linear stochastic system driven by a noise process that is an arbitrary zero mean, square integrable stochastic process with continuous sample paths and a cost functional that is quadratic in the system state and the control is solved. An optimal control is given explicitly as the sum of the well-known linear feedback control for the associated deterministic linear-quadratic control problem and the prediction of the response of a system to the future noise process. The optimal cost is also given. The special case of a noise process that is an arbitrary standard fractional Brownian motion is noted explicitly with an explicit expression for the prediction of the future response of a system to the noise process that is used the optimal control.

Key words. linear-quadratic control with general noise processes, linear-quadratic Gaussian control, control of linear systems with fractional Brownian motions, control of continuous time linear systems

AMS subject classifications. 9N10, 49J15, 60H30, 60G22

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1. Introduction.

The control of a linear stochastic system with a Brownian motion and a cost functional that is quadratic in the state and the control, which is often called the linear-quadratic Gaussian (LQG) control problem, is probably the most well known stochastic control problem for continuous time systems (e.g., [5]). The discrete time LQG control problem was solved in the late 1950s and early 1960s (e.g., [10, 20]) and shortly afterward the continuous time LQG problem was solved (e.g., [21]). These solutions are closely related to the corresponding deterministic linear-quadratic control problem whose solution has its origins in the 19th century from the work of Lagrange and others (cf. [6]). For the continuous time LQG problem an optimal control is a linear feedback control which is identical to an optimal control for the corresponding deterministic linear-quadratic control problem where the Brownian motion is replaced by the zero process. The optimal cost only differs from the deterministic problem's optimal cost by the integral of a function of time. However, the usual methods to solve these two problems are quite distinct. The deterministic linear-quadratic control problem is often solved by a first order nonlinear partial differential equation (Hamilton–Jacobi equation), while the stochastic (or LQG) control problem is often solved by a second order nonlinear partial differential equation (Hamilton–Jacobi–Bellman equation). From the form of these equations or by conjecturing a solution for both of them it follows that both solutions are basically the same quadratic function. However, no intrinsic reasons are provided for why the two optimal controls are the same and why the optimal costs differ only by the integral of a function of time. In this paper this asymmetry of approaches for the deterministic and the stochastic problems is removed by applying a method of completion of squares from deterministic linear control and the use of conditional expectation to

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solve a linear-quadratic control problem for a system with an arbitrary zero mean, square integrable process with continuous sample paths. In a number of previous works some application of the method of completion of squares is used to solve a stochastic control problem (e.g. [7, 18]). The methods of these previous works for continuous time systems have combined the techniques of stochastic calculus for semimartingales with completion of squares to obtain a quadratic expression from which the optimal control is obtained. This approach is not feasible here because of the lack of a suitable stochastic calculus for an arbitrary square integrable process. On an abstract level it can be argued that both the deterministic and the stochastic linear-quadratic control problems are minimization problems in a Hilbert space, so a completion of squares method should be applicable. However, in this abstraction it is not clear that the same explicit method can be used in the minimization for both the deterministic and the stochastic problems.

The determination of an optimal control that is given here only requires computations of linear functionals of the noise process for the stochastic system instead of a nonelementary use of stochastic calculus. The optimal control is obtained from a limit of optimal controls from a sequence of deterministic control problems that use the method of completion of squares. This result implies the existence of a solution of a backward stochastic differential equation for a suitable noise process for this optimization problem. The optimal control is a sum of two terms, one term is the well-known linear feedback term for the case of Brownian motion or the associated deterministic control problem, and the other term is a minimum mean square error prediction of the response of a system to the future noise process. The special case of the noise processes from the family of (standard) fractional Brownian motions that is indexed by the Hurst parameter, $H \in (0, 1)$, is noted. If $H = 1/2$, then the process is a (standard) Brownian motion. If $H \neq 1/2$, then the Gaussian process is neither a semimartingale nor a Markov process. Thus for $H \neq 1/2$ the well-known stochastic calculus for semimartingales or the methods from Markov processes cannot be used.

Only a very limited amount of prior work seems to have been done on control problems, even for a linear system driven only by a fractional Brownian motion. Kleptsyna, LeBreton and Viot [11, 12] obtained an optimal control for a scalar system with a fractional Brownian motion for the Hurst parameter $H \in (1/2, 1)$. Some special cases of a multidimensional system are described in [3] and [4]. Hu and Zhou [8] obtained an optimal control for a scalar bilinear system with a fractional Brownian motion for $H \in (1/2, 1)$ but with the condition that the control is Markovian. In [2] the optimal control of a linear equation in a Hilbert space with a fractional Brownian motion having $H \in (\frac{1}{2}, 1)$ and with a quadratic cost functional is explicitly solved.

The method of verification of an optimal control in this paper is a generalization of the method of completion of squares. For a deterministic control problem a translation of the completion of squares method by an affine term in the system differential equation has been used (e.g., [23]). The method in this paper can be considered as a stochastic generalization of the method for an affine system term for deterministic control. However, the stochastic generalization is more complicated because of the measurability conditions for admissible controls. The authors are unaware of the use of this method by others for stochastic control. For some fractional Brownian motions, a scalar linear-quadratic control problem is solved using a stochastic maximum principle and the solution of a backward stochastic differential equation [11]. The authors are not aware of any solutions to linear-quadratic control problems for general noise processes that are only square integrable with continuous sample paths.

A brief outline of the paper follows. Since systems with fractional Brownian motions are an important application of the results in section 3 the definition and some properties of a fractional Brownian motion are given. In section 3 the control problem is formulated and solved. Initially an optimal control and its associated cost are given for a family of controls that are not required to be adapted to the past of the state process. Then an optimal control and its cost for a family of controls that are adapted to the state process are given. The special case of a fractional Brownian motion as the noise processes is described explicitly because of its importance in modeling physical phenomena and the fact that the prediction used for the optimal control can be given explicitly as a linear functional of the past of the fractional Brownian motion. Finally, in section 4 some concluding remarks are made.

2. Fractional Brownian motion. Fractional Brownian motions are a family of Gaussian processes with continuous sample paths that were defined by Kolmogorov [13] in his study of turbulence [14] and [15]. Hurst [9] implicitly noted the applicability of these processes to model rainfall along the Nile River basin by performing some statistical analysis on the rainfall data that determined the specific fractional Brownian motion. Mandelbrot [16] noted the applicability for economic data and that Hurst's analysis determined a fractional Brownian motion, and Mandelbrot and van Ness [17] developed some of the initial theory for fractional Brownian motions. Since Hurst's initial empirical demonstration of the applicability of a fractional Brownian motion, an empirical justification for this family of processes has been demonstrated in a wide variety of physical phenomena, such as economic data, Internet traffic, turbulence, device noise, medicine, and biology. Thus it seems important to study control problems for systems with a fractional Brownian motion.

Initially the family of real-valued standard fractional Brownian motions is defined.

DEFINITION 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. For each $H \in (0,1)$, a real-valued standard fractional Brownian motion $(B(t), t \geq 0)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian process with continuous sample paths that satisfies*

$$\mathbb{E}[B(t)] = 0, \tag{2.1}$$

$$\mathbb{E}[B(s)B(t)] = (1/2)(t^{2H} + s^{2H} - |t-s|^{2H}) \tag{2.2}$$

for all $s, t \in \mathbb{R}_+$.

An $\mathbb{R}^n$-valued standard fractional Brownian motion with Hurst parameter H is an n-vector of independent real-valued standard fractional Brownian motions each with the Hurst parameter H.

Closely associated with the analysis of fractional Brownian motions are the Riemann–Liouville fractional integrals and derivatives, e.g., [19]. Let V be a Hilbert space. If $\varphi \in L^1([0,T], V)$, then for $\alpha > 0$ the left-sided and the right-sided fractional (Riemann–Liouville) integrals of φ are defined (for almost all $t \in [0,T]$) by

$$\left(I^{\alpha}_{0+}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)\, ds$$

and

$$\left(I^{\alpha}_{T-}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \varphi(s)\, ds,$$

respectively, where $\Gamma(\cdot)$ is the Gamma function. For $\alpha \in (0,1)$ the inverse operators of these fractional integrals are called fractional derivatives and can be given by their respective Weyl representations

$$\left(D_{0+}^{\alpha}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{t^{\alpha}} + \alpha\int_0^t \frac{\psi(t)-\psi(s)}{(t-s)^{\alpha+1}}\,ds\right)$$

and

$$\left(D_{T-}^{\alpha}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{(T-t)^{\alpha}} + \alpha\int_t^T \frac{\psi(s)-\psi(t)}{(s-t)^{\alpha+1}}\,ds\right),$$

where $\psi \in I_{0+}^{\alpha}\left(L^1([0,T],V)\right)$ and $\psi \in I_{T-}^{\alpha}\left(L^1([0,T],V)\right)$, respectively.

Associated with each Gaussian process is a natural Hilbert space. By a factorization of the covariance of a real-valued standard fractional Brownian motion in $L^2([0,T])$ this natural Hilbert space can be described in terms of fractional integrals and derivatives. Let $H \in (0,1)$ be fixed and let $L_H^2([0,T])$ be the Hilbert space where $f, g \in L_H^2$ if $\langle f,f\rangle_H < \infty$ and $\langle g,g\rangle_H < \infty$ and the inner product is given by

$$\langle f,g\rangle_H = \rho(H)\int_0^T u_{\frac{1}{2}-H}^2(r)\left(I_{T-}^{H-\frac{1}{2}}u_{H-\frac{1}{2}}f\right)(r)\left(I_{T-}^{H-\frac{1}{2}}u_{H-\frac{1}{2}}g\right)(r)dr, \tag{2.3}$$

where

$$\rho(H) = \frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)} \tag{2.4}$$

and $u_a(r) = r^a$ for $r \geq 0$ and $a \in \mathbb{R}$. The linear operator I^{α} is understood to be the fractional derivative $D^{-\alpha}$ for $\alpha \in (-1,0)$. If f and g are smooth functions and $H \in (\frac{1}{2},1)$, then

$$\langle f,g\rangle_H = \int_0^T\int_0^T f(s)g(r)\phi_H(s-r)drds, \tag{2.5}$$

where $\phi_H(s) = H(2H-1)|s|^{2H-2}$. If $(B(t), t \geq 0)$ is a real-valued standard fractional Brownian motion with Hurst parameter $H \in (0,1)$ fixed, then

$$\mathbb{E}[B(s)B(t)] = \rho(H)\int_0^T u_{\frac{1}{2}-H}^2(r)\left(I_{T-}^{H-\frac{1}{2}}u_{H-\frac{1}{2}}1_{[0,s]}\right)(r)\left(I_{T-}^{H-\frac{1}{2}}u_{H-\frac{1}{2}}1_{[0,t]}\right)(r)dr. \tag{2.6}$$

It can be shown that the covariance function $\mathbb{E}[B(s)B(t)]$ is an analytic function of H.

This factorization of the covariance of a fractional Brownian motion can be used to solve a prediction problem for a linear functional of a fractional Brownian motion (Wiener-type integral). A result is given in the following proposition [1].

Proposition 2.2. *Let $(B(t), t \geq 0)$ be a real-valued standard fractional Brownian motion with Hurst parameter $H \in (0,1)$ and let $c \in L_H^2$. Then*

$$\mathbb{E}\left[\int_s^t cdB \mid B(r),\ r \in [0,s]\right] = \int_0^s u_{\frac{1}{2}-H}\left(I_{s-}^{-(H-1/2)}\left(I_{t-}^{(H-1/2)}u_{H-1/2}c\right)\right)dB, \tag{2.7}$$

where $u_a(s) = s^a$ for $a > 0, s > 0$ and I^{α} is a fractional integral for $\alpha \in (0,1)$ and a fractional derivative for $\alpha \in (-1,0)$.

3. Control problem. In this section an optimal control problem for a linear stochastic system with an arbitrary zero mean, square integrable process with continuous sample paths and a quadratic cost functional is formulated and solved.

Consider the following controlled linear stochastic system:

$$dX(t) = AX(t)dt + CU(t)dt + dB(t), \tag{3.1}$$

$$X(0) = X_0, \tag{3.2}$$

where $X_0 \in \mathbb{R}^n$ is not random, $X(t) \in \mathbb{R}^n, A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $U(t) \in \mathbb{R}^m$, $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^l)$ denotes the family of linear transformations from $\mathbb{R}^k$ to $\mathbb{R}^l$, $U \in \mathcal{U}$, $(B(t), t \in [0, T])$ is an $\mathbb{R}^n$-valued zero mean, square integrable process with continuous sample paths with $B(0) = 0$ and this process is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $T > 0$ is fixed.

The family of nonadapted admissible controls, $\mathcal{U}$, is

$$\mathcal{U} = \{U : U \text{ is an } \mathbb{R}^m\text{-valued process such that } U \in L^2([0, T]) \text{ a.s.}\}.$$

Let $(\mathcal{F}(t), t \in [0, T])$ be the filtration of $(B(t), t \in [0, T])$. The family of adapted, admissible controls, $\mathcal{U}_a$, is

$$\begin{aligned}\mathcal{U}_a = \{U : U &\text{ is an } \mathbb{R}^m\text{-valued } (\mathcal{F}(t), t \in [0, T])\\ &\text{progressively measurable process such that } U \in L^2([0, T]) \text{ a.s.}\}.\end{aligned}$$

It is elementary to verify that there is one and only one solution to (3.1) for each $U \in \mathcal{U}$ that is obtained by the variation of parameters formula, that is,

$$X(t) = e^{tA}X_0 + \int_0^t e^{A(t-s)}CU(s)ds + \int_0^t e^{A(t-s)}dB(s). \tag{3.3}$$

The (Wiener-type) stochastic integral can be defined by integration by parts, that is,

$$e^{At}\int_0^t e^{-As}dB(s) = e^{At}\left(e^{-At}B(t) + \int_0^t B(s)Ae^{-As}ds\right).$$

The cost functional J is a quadratic functional of X and U that is given by

$$J(U) = \frac{1}{2}E\left[\int_0^T \langle QX(s), X(s)\rangle + \langle RU(s), U(s)\rangle ds\right] + \frac{1}{2}E\langle MX(T), X(T)\rangle, \tag{3.4}$$

where $Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), R \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m), M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $Q > 0$, $R > 0$, and $M \geq 0$ are symmetric linear transformations and $\langle \cdot, \cdot \rangle$ denotes the canonical Euclidean inner product on the Euclidean space of the appropriate dimension.

Initially an optimal control for (3.1) and (3.4) is determined from the family of nonadapted controls, $\mathcal{U}$.

THEOREM 3.1. *For the optimal control problem* (3.1) *and* (3.4) *and the family of admissible, nonadapted controls, $\mathcal{U}$, there is an optimal control U^* that can be expressed as*

$$U^*(t) = -R^{-1}C^T(P(t)X(t) + W(t)), \tag{3.5}$$

where $(P(t), t \in [0, T])$ is the unique symmetric positive definite solution of the Riccati equation

$$\frac{dP}{dt} = -PA - A^T P + PCR^{-1}C^T P - Q, \tag{3.6}$$

$$P(T) = M, \tag{3.7}$$

and $(W(t), t \in [0,T])$ *is the process that satisfies*

$$W(t) = \int_t^T \Phi_P(s,t)P(s)dB(s) \tag{3.8}$$

and Φ_P *is the fundamental solution of the matrix equation*

$$\frac{d\Phi_P(s,t)}{dt} = -(A^T - P(t)CR^{-1}C^T)\Phi_P(s,t), \tag{3.9}$$

$$\Phi_P(s,s) = I. \tag{3.10}$$

Proof. The optimal control problem is solved by constructing a sequence of piecewise linear approximations to the noise process B in (3.1) and using a completion of squares method from deterministic linear control to obtain a sequence of optimal controls for linear systems where the noise process in (3.1) is replaced by a sequence of piecewise linear approximations to the noise process. Then it is shown that this sequence of optimal controls has a limit that is an optimal control for the system (3.1). Let $(B(t), t \in [0,T])$ be the $\mathbb{R}^n$-valued process B in (3.3). For each $n \in \mathbb{N}$, let $T_n = \{t_j^{(n)}, j \in \{0,\ldots,n\}\}$ be a partition of $[0,T]$ such that $0 = t_0^{(n)} < t_1^{(n)}, \ldots < t_n^{(n)} = T$. Assume that $T_{n+1} \supset T_n$ for each $n \in \mathbb{N}$ and that the sequence $(T_n, n \in \mathbb{N})$ becomes dense in $[0,T]$. For example, the sequence $(T_n, n \in \mathbb{N})$ can be the dyadic partitions of $[0,T]$. For each $n \in \mathbb{N}$, let $(B_n(t), t \in [0,T])$ be the piecewise linear process obtained from $(B(t), t \in [0,T])$ and T_n by linear interpolation, that is,

$$B_n(t) = \left[B(t_j^{(n)}) + \frac{B(t_{j+1}^{(n)}) - B(t_j^{(n)})}{t_{j+1}^{(n)} - t_j^{(n)}}(t - t_j^{(n)}) \right] \mathbb{1}_{[t_j^{(n)}, t_{j+1}^{(n)})}(t). \tag{3.11}$$

A nonadapted control problem for the controlled system (3.1) with $(B(t), t \in [0,T])$ replaced by $(B_n(t), t \in [0,T], n \in \mathbb{N})$ given by (3.11) is solved for almost all sample paths of B_n and the solution of (3.1) with B_n replacing B is considered as a translation of the deterministic linear system without B_n.

Let $(B_n(t), t \in [0,T], n \in \mathbb{N})$ be the sequence of processes obtained by (3.11) that converges uniformly almost surely to $(B(t), t \in [0,T])$. Fix $n \in \mathbb{N}$ and consider a sample path of B_n. For this sample path the dependence on $\omega \in \Omega$ is suppressed for notational convenience.

Let $(\phi_n(t), t \in [0,T])$ be the solution of the linear differential equation

$$\frac{d\phi_n}{dt} = -\left[(A^T - P(t)CR^{-1}C^T)\phi_n(t) + P(t)\frac{dB_n}{dt} \right], \qquad \phi_n(T) = 0. \tag{3.12}$$

This differential equation is defined for almost all t and its solution is well defined from the results for linear ordinary differential equations. It follows directly that

$$\phi_n(t) = \int_t^T \Phi_P(s,t)P(s)dB_n(s), \tag{3.13}$$

where Φ_P is the fundamental solution of the matrix equation

$$\frac{d\Phi_P(s,t)}{dt} = -(A^T - P(t)CR^{-1}C^T)\Phi_P(s,t), \tag{3.14}$$

$$\Phi_P(s,s) = I. \tag{3.15}$$

Let $(X_n(t), t \in [0,T])$ be the solution of (3.1) with B replaced by B_n, that is, X_n is the solution of

$$dX_n(t) = (AX_n(t) + CU(t))dt + dB_n(t), \tag{3.16}$$

$$X_n(0) = X_0. \tag{3.17}$$

The dependence of X_n on the control U is suppressed for notational simplicity. The following approach uses completion of squares for the corresponding deterministic control problem (e.g., [23]). By taking the differential of the process $(\langle P(t)X_n(t), X_n(t)\rangle, t \in [0,T])$ and integrating this differential expression it follows that

(3.18)
$$\begin{aligned}\langle P(T)X_n(T), X_n(T)\rangle - \langle P(0)X_0, X_0\rangle = &\int_0^T (\langle (P(t)CR^{-1}C^TP(t) - Q)X_n(t), X_n(t)\rangle \\ &+ 2\langle C^TP(t)X_n(t), U(t)\rangle)dt \\ &+ 2\int_0^T \langle P(t)dB_n(t), X_n(t)\rangle.\end{aligned}$$

Furthermore, compute the differential of $(\langle \phi_n(t), X_n(t)\rangle, t \in [0,T])$ and integrate it to obtain

$$\begin{aligned}-\langle \phi_n(0), X_0\rangle = &\int_0^T (\langle P(t)CR^{-1}C^T\phi_n(t), X_n(t)\rangle + \langle \phi_n(t), CU(t)\rangle)dt \\ &- \int_0^T \langle P(t)dB_n(t), X_n(t)\rangle + \int_0^T \langle \phi_n(t), dB_n(t)\rangle.\end{aligned}$$

Using the above two equalities it follows that

(3.19)
$$\begin{aligned}&J_n^0(U) - \frac{1}{2}\langle P(0)X_0, X_0\rangle - \langle \phi_n(0), X_0\rangle \\ &\quad = \frac{1}{2}\int_0^T [(|R^{-1/2}[RU + C^TPX_n + C^T\phi_n]|^2 - |R^{-1/2}C^T\phi_n|^2)dt + 2\langle \phi_n, dB_n\rangle],\end{aligned}$$

where

$$J_n^0(U) = \frac{1}{2}\int_0^T \langle QX_n(s), X_n(s)\rangle + \langle RU(s), U(s)\rangle ds + \frac{1}{2}\langle MX_n(T), X_n(T)\rangle.$$

Since the arbitrary control U only appears in the first term of (3.19) and this term is quadratic, an optimal control U_n^* is

$$U_n^*(t) = -R^{-1}(C^TP(t)X_n(t) + C^T\phi_n(t)). \tag{3.20}$$

This optimal control is satisfied for almost all $\omega \in \Omega$. This method of minimization is usually called a completion of squares for deterministic linear control.

Since the sequence of processes $(B_n(t), t \in [0,T], n \in \mathbb{N})$ converges uniformly almost surely to the process $(B(t), t \in [0,T])$, it follows that for a fixed control U, the sequence of solutions of (3.16), $(X_n(t), t \in [0,T], n \in \mathbb{N})$, converges uniformly almost surely to the solution of (3.1), $(X(t), t \in [0,T])$. This uniform convergence almost surely follows directly by representing $(X_n(t), t \in [0,T])$ by the variation of parameters formula and performing an integration by parts in (3.3) as follows:

$$\begin{aligned} X_n(t) &= e^{tA}X_0 + \int_0^t e^{A(t-s)}CU(s)ds + \int_0^t e^{A(t-s)}dB_n(s) \\ &= e^{tA}X_0 + \int_0^t e^{A(t-s)}CU(s)ds + B_n(t) + e^{tA}\int_0^t Ae^{-As}B_n(s)ds. \end{aligned} \tag{3.21}$$

Likewise the sequence of solutions of (3.12), $(\phi_n(t), t \in [0,T], n \in \mathbb{N})$, converges uniformly almost surely to $(\phi(t), t \in [0,T])$ by performing integration by parts on the integral expressions for ϕ_n and ϕ in (3.14) and (3.24), respectively, where ϕ is the solution of the stochastic equation

$$d\phi(t) = -(A^T - P(t)CR^{-1}C^T)\phi(t)dt - P(t)dB(t), \tag{3.22}$$

$$\phi(T) = 0 \tag{3.23}$$

that is given by

$$\phi(t) = \int_t^T \Phi_P(s,t)P(s)dB(s). \tag{3.24}$$

Since the sequence $(\phi_n(t), t \in [0,T], n \in \mathbb{N})$ converges uniformly almost surely to ϕ, it follows that $(X_n^{U_n^*}, n \in \mathbb{N})$ converges uniformly almost surely to X^{U^*}, where the superscript denotes the control that is used. This result follows from integration by parts and Gronwall's lemma. Since the sequence $((X_n^{U_n^*}, \phi_n), n \in \mathbb{N})$ converges uniformly almost surely to (X^{U^*}, ϕ) it follows that

(3.25)

$$\begin{aligned} \lim_{n\to\infty} |(J_n^0(U_n^*) - J^0(U^*))| &= \lim_{n\to\infty} \int_0^T [|\langle QX_n, X_n\rangle - \langle QX, X\rangle| + |\langle RU_n^*, U_n^*\rangle \\ &\quad - \langle RU^*, U^*\rangle|dt + |\langle MX_n(T), X_n(T)\rangle - \langle MX(T), X(T)\rangle|] \\ &= 0 \qquad \text{a.s.} \end{aligned}$$

and by contraposition

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}\int_0^T [|\langle QX_n, X_n\rangle - \langle QX, X\rangle| + |\langle RU_n^*, U_n^*\rangle \\ &\quad - \langle RU^*, U^*\rangle|]dt + \mathbb{E}|\langle MX_n(T), X_n(T)\rangle - \langle MX(T), X(T)\rangle| \\ &= \lim_{n\to\infty} \mathbb{E}|(J_n^0(U_n^*) - J^0(U^*))| = 0, \end{aligned} \tag{3.26}$$

where X_n and X are the solutions of (3.16) and (3.1) using U_n^* and U^*, respectively, and U^* satisfies

$$U^*(t) = -R^{-1}(C^TP(t)X(t) + C^T\phi(t)). \tag{3.27}$$

It is shown that U^* is an optimal nonadapted control for (3.1) and (3.4). Assume that there is a subset $\Lambda \in \mathcal{F}$ with $\mathbb{P}(\Lambda) > 0$ and a control $\widetilde{U}$ such that on Λ, $J^0(\widetilde{U}) <$

$J^0(U^*)$. There is a $\Lambda_1 \subset \Lambda$ with $\Lambda_1 \in \mathcal{F}$ and $\mathbb{P}(\Lambda_1) > 0$ and an $\epsilon_0 > 0$ such that $\epsilon_0 < J^0(U^*) - J^0(\widetilde{U})$ and there is an $\epsilon_1 \in (0, \epsilon_0/2)$ and a $\Lambda_2 \subset \Lambda_1$ with $\Lambda_2 \in \mathcal{F}$ and $\mathbb{P}(\Lambda_2) > 0$ such that on Λ_2 there is an $N \in \mathbb{N}$ such that if $n > N$

$$|J_n^0(U_n^*) - J^0(U^*)| < \epsilon_1 \tag{3.28}$$

and

$$|J_n^0(\widetilde{U}) - J^0(\widetilde{U})| < \epsilon_1, \tag{3.29}$$

then on Λ_2

$$\begin{aligned} J_n^0(\widetilde{U}) - J_n^0(U_n^*) &= J_n^0(\widetilde{U}) - J^0(\widetilde{U}) + J^0(\widetilde{U}) - J^0(U^*) + J^0(U^*) - J_n^0(U_n^*) \\ &< \epsilon_1 - \epsilon_0 + \epsilon_1 < 0. \end{aligned}$$

Thus U_n^* is not an optimal control for X_n. So U^* is an optimal control for (3.1) and (3.4). To determine explicitly the optimal cost, additional properties of the noise process are required. □

Since the noise processes in a wide variety of physical phenomena have been empirically identified as fractional Brownian motions, the solution of the control problem (3.1) and (3.4) with an arbitrary standard fractional Brownian motion and the nonadapted family of controls is given explicitly in the following corollary.

COROLLARY 3.2. *Let* $(B(t), t \in [0,T])$ *in* (3.1) *be a standard fractional Brownian motion with a fixed Hurst parameter* $H \in (0,1)$. *For the optimal control problem given by* (3.1) *and* (3.4) *and the family of admissible nonadapted controls* $\mathcal{U}$, *an optimal control* U^* *is given by*

$$\bar{U}^*(t) = R^{-1}C^T(P(t)X(t) + W(t)), \tag{3.30}$$

where $(P(t), t \in [0,T])$ *is the unique positive definite symmetric solution of* (3.6) *and* $(W(t), t \in [0,T])$ *is the process that satisfies*

$$W(t) = \int_t^T \Phi_P(s,t)P(s)dB(s) \tag{3.31}$$

and Φ_P *is the solution of* (3.9). *For* $H \in (\frac{1}{2}, 1)$ *the optimal cost is*

(3.32)

$$\begin{aligned} J(U^*) = &\frac{1}{2}\langle P(0)X_0, X_0\rangle \\ &- \frac{1}{2}\int_0^T \int_t^T \int_t^T \mathrm{tr}(P(r)\Phi_P^T(r,t)CR^{-1}C^T\Phi_P(s,t)P(s))\phi_H(s-r)drdsdt \\ &+ \int_0^T \int_s^T \mathrm{tr}(\Phi_P(s,t)P(s))\phi_H(s-t)dsdt, \end{aligned}$$

where $\phi_H(s) = H(2H-1)|s|^{2H-2}$.

Proof. It suffices to verify the optimal cost (3.32). To compute the optimal cost the expectation of the sequence of costs $(J_n^0, n \in \mathbb{N})$ given by (3.19) is computed. Let $T_n = \{t_0^{(n)}, \dots, t_n^{(n)}\}$ be a partition of $[0,T]$ from the sequence $(T_n, n \in \mathbb{N})$ in the construction of the sequence $(B_n, n \in \mathbb{N})$. Initially consider the sequence $(\mathbb{E}\int_0^T \langle R^{-1}C^T\phi_n, C^T\phi_n\rangle dt, n \in \mathbb{N} >)$. The uniform almost sure convergence of the sequence $(B_n, n \in \mathbb{N})$ to B implies that the sequence $(\phi_n, n \in \mathbb{N})$ converges uniformly almost surely to ϕ. Given $\epsilon > 0$, there is a modulus of continuity $\delta > 0$ on $\Lambda \in \mathcal{F}$ where $\mathbb{P}(\Lambda) > 1 - \epsilon$ such that for $\omega \in \Lambda$ if $|t - s| < \delta$, then

$$|B(t,\omega) - B(s,\omega)| < \epsilon.$$

Thus for $n \geq N$ where N is sufficiently large

$$\begin{aligned}|B_n(t)|1_\Lambda &< |B(t_j^{(n)}|1_\Lambda + \epsilon \\ &< |B(t)| + 2\epsilon.\end{aligned}$$

Thus for $n \geq N$

$$|B_n(t)|^2 1_\Lambda < 2|B(t)|^2 + 4\epsilon. \tag{3.33}$$

Thus the dominated convergence theorem can be applied on Λ to obtain the convergence and by contraposition there is convergence on Ω. Thus it follows that

$$\lim_{n\to\infty}\mathbb{E}\int_0^T \langle R^{-1}C^T\phi_n, C^T\phi_n\rangle dt = \mathbb{E}\int_0^T \langle R^{-1}C^T\phi, C^T\phi\rangle dt. \tag{3.34}$$

Now consider the sequence $(\mathbb{E}\int_0^T \langle \phi_n, dB_n\rangle, n \in \mathbb{N})$. For $n \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}\int_0^T \langle \phi_n, dB_n\rangle &= \mathbb{E}\Sigma_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \langle \phi_n, dB_n\rangle \\ &= \mathbb{E}\Sigma_j \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} \phi_n(s)ds\frac{\Delta B(t_j^{(n)})}{\Delta t_j^{(n)}} \\ &= \mathbb{E}\Sigma_j \phi_n(s_j^{(n)})\Delta B(t_j^{(n)}),\end{aligned} \tag{3.35}$$

where the random variable $s_j^{(n)} \in (t_j^{(n)}, t_{j+1}^{(n)})$ by the mean value theorem for integrals. It is desired to replace $\phi_n(s_j^{(n)})$ by $\phi_n(t_j^{(n)})$. By integration by parts it follows that

$$\begin{aligned}\phi_n(s_j^{(n)}) - \phi_n(t_j^{(n)}) &= \int_{s_j^{(n)}}^T \Phi_P(r, s_j^{(n)})P(r)dB_n(r) \\ &\quad - \int_{t_j^{(n)}}^T \Phi_P(r, t_j^{(n)})P(r)dB_n(r) \\ &= \Phi_P(T, s_j^{(n)})P(T)B_n(T) - P(s_j^{(n)})B_n(s_j^{(n)}) \\ &\quad - \int_{s_j^{(n)}}^T B_n(r)d_r(\Phi_P(r, s_j^{(n)})P(r)) \\ &\quad - \Phi_P(T, t_j^{(n)})P(T)B_n(T) + P(t_j^{(n)})B_n(t_j^{(n)}) \\ &\quad + \int_{t_j^{(n)}}^T B_n(r)d_r(\Phi_P(r, t_j^{(n)})P(r)).\end{aligned} \tag{3.36}$$

Consider the following terms that are obtained for bounding the differences in (3.35):

$$
\begin{aligned}
(3.37)\quad |\phi_n(s_j^{(n)}) - \phi_n(t_j^{(n)})|^2 = \Big| & \Phi_P(T, s_j^{(n)})P(T)B_n(T) - P(s_j^{(n)})B_n(s_j^{(n)}) \\
& - \Phi_P(T, t_j^{(n)})P(T)B_n(T) + P(t_j^{(n)})B_n(t_j^{(n)}) \\
& - \int_{s_j^{(n)}}^T B_n(r)d_r(\Phi_P(r, s_j^{(n)})P(r)) \\
& + \int_{t_j^{(n)}}^T B_n(r)d_r(\Phi_P(r, t_j^{(n)})P(r)) \\
& + \int_{t_j^{(n)}}^T B_n(r)d_r(\Phi_P(r, s_j^{(n)})P(r)) \\
& - \int_{t_j^{(n)}}^T B_n(r)d_r(\Phi_P(r, s_j^{(n)})P(r))\Big|^2 \\
\le\ & 4|\Phi_P(T, s_j^{(n)})P(T)B_n(T) - \Phi_P(T, t_j^{(n)})P(T)B_n(T)|^2 \\
& + 4| - P(s_j^{(n)})B_n(s_j^{(n)}) + P(t_j^{(n)})B_n(t_j^{(n)})|^2 \\
& + 4\left|\int_{t_j^{(n)}}^{s_j^{(n)}} B_n(r)d_r(\Phi_P(r, s_j^{(n)})P(r))\right|^2 \\
& + 4\Big|\int_{t_j^{(n)}}^T B_n(r)d_r(\Phi_P(r, s_j^{(n)})P(r)) \\
& - \int_{t_j^{(n)}}^T B_n(r)d_r(\Phi_P(r, t_j^{(n)})P(r))\Big|^2.
\end{aligned}
$$

It follows directly from the above inequality that

$$(3.38)\qquad \lim_{n\to\infty}\mathbb{E}\Sigma_j|\phi_n(s_j^{(n)}) - \phi_n(t_j^{(n)})|^2 = 0.$$

Since the sequence $(\mathbb{E}\int_0^T\langle\phi_n, dB_n\rangle, n \in \mathbb{N})$ converges by the equalities (3.19) and (3.26), it suffices to evaluate this limit. For notational simplicity this limit is only described for $H \in (\frac{1}{2}, 1)$.

$$
\begin{aligned}
(3.39)\qquad \lim_{n\to\infty}\mathbb{E}\int_0^T\langle\phi_n, dB_n\rangle &= \lim_{n\to\infty}\mathbb{E}\Sigma_j\int_{t_j^{(n)}}^{t_{j+1}^{(n)}}\langle\phi_n, dB_n\rangle \\
&= \lim_{n\to\infty}\mathbb{E}\Sigma_j\phi_n(t_j^{(n)})\Delta B(t_j^{(n)}) \\
&= \int_0^T\int_s^T \mathrm{tr}(\Phi_P(s,t)P(s))\phi_H(s-t)dsdt,
\end{aligned}
$$

where $\phi_H(s) = H(2H-1)|s|^{2H-2}$. The expression for the limit follows from (2.6).

Combining the limiting terms from (3.34) and (3.39) it follows that the optimal cost for $H \in (\frac{1}{2}, 1)$ is

(3.40)

$$\begin{aligned} J(U^*) = &\frac{1}{2}\langle P(0)X_0, X_0\rangle \\ &- \frac{1}{2}\int_0^T \int_t^T \int_t^T \mathrm{tr}(P(r)\Phi_P^T(r,t)CR^{-1}C^T\Phi_P(s,t)P(s))\phi_H(s-r)drdsdt \\ &+ \int_0^T \int_s^T \mathrm{tr}(\Phi_P(s,t)P(s))\phi_H(s-t)dsdt. \qquad \square \end{aligned}$$

Now the control problem (3.1) and (3.4) is considered with the family of adapted controls, $\mathcal{U}_a$. The following result exhibits an optimal control in $\mathcal{U}_a$.

THEOREM 3.3. *For the optimal problem* (3.1) *and* (3.4) *and the family of admissible, adapted controls, $\mathcal{U}_a$, there is an optimal control $\hat{U}^*$ that can be expressed as*

$$\hat{U}^*(t) = -R^{-1}C^T(P(t)X(t) + V(t)), \tag{3.41}$$

where $(P(t), t \in [0,T])$ is the unique symmetric positive definite solution of the Riccati equation

$$\frac{dP}{dt} = -PA - A^TP + PCR^{-1}C^TP - Q, \tag{3.42}$$

$$P(T) = M, \tag{3.43}$$

and $(V(t), t \in [0,T])$ is the process that satisfies

$$V(t) = \mathbb{E}\left[\int_t^T \Phi_P(s,t)P(s)dB(s)|\mathcal{F}(t)\right] \tag{3.44}$$

and Φ_P is the fundamental solution of the matrix equation

$$\frac{d\Phi_P(s,t)}{dt} = -(A^T - P(t)CR^{-1}C^T)\Phi_P(s,t), \tag{3.45}$$

$$\Phi_P(s,s) = I. \tag{3.46}$$

Proof. Consider the control problem where the family of controls is required to be adapted to $(\mathcal{F}(t), t \in [0,T])$, which is the filtration for $(B(t), t \in [0,T])$. It is claimed that the optimal, adapted control is the projection (via conditional expectation) of the optimal control U^*, that is, an optimal adapted control $\widehat{U}^*$ is

$$\begin{aligned} \widehat{U}^*(t) &= \mathbb{E}[U^*(t)|\mathcal{F}(t)] \\ &= -R^{-1}C^T(P(t)X(t) + \mathbb{E}[\phi(t)|\mathcal{F}(t)]). \end{aligned} \tag{3.47}$$

Let $\widetilde{U}^*$ be given by

$$\widetilde{U}^*(t) = -R^{-1}C^T(\phi(t) - \mathbb{E}[\phi(t)|\mathcal{F}(t)]) \tag{3.48}$$

so that

$$U^*(t) = \widehat{U}^*(t) + \widetilde{U}^*(t). \tag{3.49}$$

Note that for each $t \in [0,T]$

$$\mathbb{E}[\widetilde{U}^*(t)|\mathcal{F}(t)] = 0 \qquad \text{a.s.} \tag{3.50}$$

Let $(\bar{U}(t), t \in [0,T])$ be an $(\mathcal{F}(t), t \in [0,T])$ adapted control such that

$$\mathbb{E}J^0(\bar{U}) - \mathbb{E}J^0(\widehat{U}^*) < 0. \tag{3.51}$$

Then there is an elementary control, U_1, such that

$$\mathbb{E}J^0(\widehat{U}^* + U_1) - \mathbb{E}J^0(\widehat{U}^*) < 0, \tag{3.52}$$

where $U_1(t) = \alpha 1_{[t_1,t_2)}(t)$, $t_1, t_2 \in [0,T], t_1 < t_2$, and the $\mathbb{R}^n$-valued random variable α is $\mathcal{F}(t_1)$ measurable. It is shown below that if (3.51) is satisfied, then

$$\mathbb{E}J^0(U^* + U_1) - \mathbb{E}J^0(U^*) < 0, \tag{3.53}$$

which is a contradiction and verifies the optimality of $\widehat{U}^*$ in the family $\mathcal{U}_a$ of $(\mathcal{F}(t), t \in [0,T])$ adapted controls.

Let $(X_1(t), t \in [t_1,T])$ satisfy

$$X_1(t) = \int_{t_1}^{t} e^{A(t-s)} C U_1(s) ds \tag{3.54}$$

and let $(\widetilde{X}^*(t), t \in [t_1,T])$ satisfy

$$\widetilde{X}^*(t) = \int_{t_1}^{t} e^{A(t-s)} B \widetilde{U}^*(s) ds \tag{3.55}$$

and let $(\hat{X}^*(t), t \in [t_1,T])$ satisfy

$$\hat{X}^*(t) = e^{A(t-t_1)} X(t_1) + \int_{t_1}^{T} e^{A(t-s)} C \hat{U}^*(s) ds. \tag{3.56}$$

By the linearity of (3.1), it follows directly that the solutions of (3.1) using the controls $\widehat{U}^* + U_1$, U^*, and $U^* + U_1$ are $\widehat{X}^* + X_1$, $\widehat{X}^* + \widetilde{X}^*$, and $\widehat{X}^* + \widetilde{X}^* + X_1$, respectively. Using the principle of optimality, the integrals in the cost functional (3.3) and the processes are restricted to the interval $[t_1, T]$ for the determination of optimality.

By conditional expectation and (3.50) the following equalities are satisfied for all $t \in [0,T]$:

$$\mathbb{E}\langle R\bar{U}(t), \widetilde{U}^*(t)\rangle = 0, \tag{3.57}$$

$$\mathbb{E}\langle R\hat{U}^*(t), \widetilde{U}^*(t)\rangle = 0. \tag{3.58}$$

Now $\mathbb{E}[J^0(U^*+U_1)-J^0(U^*)]$ is computed.

$$\begin{aligned}
&\mathbb{E}[J^0(U^*+U_1)-J^0(U^*)]\\
&=\frac{1}{2}\mathbb{E}\Bigg[\int_{t_1}^T\langle Q(\widehat{X}^*+X_1),\widehat{X}^*+X_1\rangle dt+2\int_{t_1}^T\langle Q(\widehat{X}^*+X_1),\widetilde{X}^*\rangle dt\\
&\quad+\int_{t_1}^T\langle Q\widetilde{X}^*,\widetilde{X}^*\rangle dt+\int_{t_1}^T\langle R(\widehat{U}^*+U_1),\widehat{U}^*+U_1\rangle dt\\
&\quad+2\int_{t_1}^T\langle R(\widehat{U}^*+U_1),\widetilde{U}^*\rangle dt+\int_{t_1}^T\langle R\widetilde{U}^*,\widetilde{U}^*\rangle dt\\
&\quad+\langle M(\widehat{X}^*+X_1),\widehat{X}^*+X_1\rangle+2\langle M(\widehat{X}^*+X_1),\widetilde{X}^*\rangle+\langle M\widetilde{X}^*,\widetilde{X}^*\rangle\\
&\quad-\int_{t_1}^T\langle Q\widehat{X}^*,\widehat{X}^*\rangle dt-\int_{t_1}^T\langle Q\widetilde{X}^*,\widetilde{X}^*\rangle dt\\
&\quad-2\int_{t_1}^T\langle Q\widehat{X}^*,\widetilde{X}^*\rangle dt-\int_{t_1}^T\langle R\widehat{U}^*,\widehat{U}^*\rangle dt-\int_{t_1}^T\langle R\widetilde{U}^*,\widetilde{U}^*\rangle dt\\
&\quad-2\int_{t_1}^T\langle R\widetilde{U}^*,\widehat{U}^*\rangle dt-\langle M\widehat{X}^*,\widehat{X}^*\rangle-\langle M\widetilde{X}^*,\widetilde{X}^*\rangle-2\langle M\widehat{X}^*,\widetilde{X}^*\rangle\Bigg]\\
&=\mathbb{E}\Bigg[J^0(\widehat{U}^*+U_1)-J^0(\widehat{U}^*)+\int_{t_1}^T\langle QX_1,\widetilde{X}^*\rangle dt\\
&\quad+\int_{t_1}^T\langle RU_1,\widetilde{U}^*\rangle dt+\langle MX_1,\widetilde{X}^*\rangle\Bigg]=\mathbb{E}[J^0(\widehat{U}^*+U_1)-J^0(\widehat{U})].
\end{aligned}$$

The equality $\mathbb{E}\langle RU_1(t),\widetilde{U}(t)\rangle=0$ for all $t\in[t_1,T]$ follows from (3.50). The equality $\mathbb{E}\langle QX_1(t),\widetilde{X}^*(t)\rangle=0$ follows from the fact that X_1 given by (3.54) is $\mathcal{F}(t_1)$ measurable and an approximation of the integral for $\widetilde{X}^*$ by a Riemann sum and the use of conditional expectation and likewise $\mathbb{E}\langle MX_1(T),\tilde{X}^*(T)\rangle=0$. Thus

$$\mathbb{E}[J^0(U^*+U_1)-J^0(U^*)]=\mathbb{E}[J^0(\hat{U}^*+U_1)-J^0(\widehat{U}^*)]<0 \tag{3.59}$$

and U^* is not optimal for the family of nonadapted admissible controls $\mathcal{U}$. This contradiction verifies that $\widehat{U}^*$ is an optimal adapted control. ◻

Let $(B(t),t\in[0,T])$ be a standard fractional Brownian motion process in the stochastic system (3.1). For the control problem given by (3.1) and (3.4) and the family of adapted controls $\mathcal{U}_a$ the following corollary is verified using the proof of the previous theorem and Proposition 2.2.

Corollary 3.4. *For the optimal control problem for the controlled system* (3.1) *with* $(B(t),t\in[0,T])$ *being a standard fractional Brownian motion with* $H\in(0,1)$ *fixed, the cost functional* (3.4)*, and the family of admissible controls,* $\mathcal{U}_a$*, there is an optimal control* $\hat{U}^*$ *that can be expressed as*

$$\hat{U}^*(t)=R^{-1}C^T(P(t)X(t)+K(t)), \tag{3.60}$$

where $(P(t),t\in[0,T])$ *is the unique symmetric positive definite solution of* (3.6) *and the process* $(K(t),t\in[0,T])$ *is given by*

$$\begin{aligned}
K(t)&=\mathbb{E}[W(t)|\mathcal{F}(t)]\\
&=\int_0^t u_{1/2-H}I_{t-}^{1/2-H}(I_{T-}^{H-1/2}u_{H-1/2}1_{[t,T]}\Phi_P(.,t)P)dB
\end{aligned} \tag{3.61}$$

and Φ_P *is the fundamental solution of the matrix equation*

$$\frac{d\Phi_P(s,t)}{dt} = -(A^T - P(t)CR^{-1}C^T)\Phi_P(s,t), \tag{3.62}$$

$$\Phi_P(s,s) = I. \tag{3.63}$$

For $H \in (\frac{1}{2}, 1)$ *the optimal cost is*

$$\begin{aligned} J(U^*) = \frac{1}{2}\langle P(0)X_0, X_0\rangle - \frac{1}{2}\int_0^T \int_0^t \int_0^t u_{\frac{1}{2}-H}(s)u_{\frac{1}{2}-H}(r)\mathrm{tr}(CR^{-1}C^T \\ \times I_{t-}^{\frac{1}{2}-H}(I_{T-}^{H-\frac{1}{2}}U_{H-\frac{1}{2}}1_{[t,T]}\Phi_P(\cdot,t)P)(s) \\ \times I_{t-}^{\frac{1}{2}-H}(I_{T-}^{H-\frac{1}{2}}U_{H-\frac{1}{2}}1_{[t,T]}P\Phi_P^T(\cdot,t))(r)\phi_H(s-r)drdsdt \\ + \int_0^T \int_s^T \mathrm{tr}(\Phi_P(s,t)P(s))\phi_H(s-t)dsdt \end{aligned}$$

and $(W(t), t \in [0,T])$ *is given by* (3.31).

Proof. The integral expression for the conditional expectation in (3.61) follows from Proposition 2.2. To determine the optimal cost for the adapted control it suffices to rewrite the limit of the sequence of expectations of (3.19) as

$$\begin{aligned} J(U) = \frac{1}{2}\langle P(0)X_0, X_0\rangle + \frac{1}{2}\mathbb{E}\int_0^T (|R^{-\frac{1}{2}}[RU + C^TPX + C^T\hat{\phi} + C^T\widetilde{\phi}]|^2 \\ - |R^{\frac{1}{2}}C^T\phi|^2]dt + \lim_{n\to\infty}\mathbb{E}\int_0^T \langle \phi_n, dB_n\rangle. \end{aligned} \tag{3.64}$$

The limit on the right-hand side of (3.64) can be evaluated using the results in [22]. Thus the optimal cost for the control U^* is

$$\begin{aligned} J(U^*) = \frac{1}{2}\langle P(0)X_0, X_0\rangle - \frac{1}{2}\int_0^T \int_0^t \int_0^t u_{\frac{1}{2}-H}(s)u_{\frac{1}{2}-H}(r)\mathrm{tr}(CR^{-1}C^T \\ \times I_{t-}^{\frac{1}{2}-H}(I_{T-}^{H-\frac{1}{2}}U_{H-\frac{1}{2}}1_{[t,T]}\Phi_P(\cdot,t)P)(s) \\ \times I_{t-}^{\frac{1}{2}-H}(I_{T-}^{H-\frac{1}{2}}U_{H-\frac{1}{2}}1_{[t,T]}P\Phi_P^T(\cdot,t))(r)\phi_H(s-r)drdsdt \\ + \int_0^T \int_s^T \mathrm{tr}(\Phi_P(s,t)P(s))\phi_H(s-t)dsdt. \end{aligned} \tag{3.65}$$

□

For the Hurst parameter $H \in (\frac{1}{2}, 1)$ the conditional expectation term can be interpreted as the solution of a backward stochastic differential equation that arises from an application of the stochastic maximum principle. This approach was used in [11] for a scalar system. For a fractional Brownian motion noise process or more generally an arbitrary continuous Gaussian process the optimal system is a linear equation, so it can be shown from the theory of causal linear functional differential equations that the family of sigma algebras generated by the state process is the same as the family of sigma algebras generated by the noise process.

4. Conclusions.

It has been shown that an optimal control can be explicitly determined for the linear-quadratic control problem where the Brownian motion is replaced by an arbitrary zero mean, square integrable process with continuous sample paths. Both a nonadapted and an adapted optimal control are given explicitly. For the case of an arbitrary fractional Brownian motion the results are specialized and made more explicit.

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