LIPSCHITZIAN SOLUTIONS OF PERTURBED NONLINEAR PROGRAMMING PROBLEMS*

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Abstract. We prove that if a second order sufficient condition and a constraint regularity assumption hold, then for sufficiently small perturbations of the constraints and the objective function, the set of local minimizers reduces to a singleton. Moreover, the minimizer and the associated multipliers are Lipschitzian functions of the parameter.

Key words. stability, nonlinear programming, weak convexity

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1. Introduction. This paper deals with the stability of solutions and multipliers of nonlinear programming problems when the data are subjected to small perturbations. In order to formulate the problem, we introduce an open subset U of $\mathbb{R}^n$, a metric space P, functions f and g from $U \times P$ to $\mathbb{R}$ and $\mathbb{R}^m$ and a nonempty closed subset Q of $\mathbb{R}^m$. The problem of interest is then:

$$P(\alpha) \qquad \begin{array}{ll} \text{minimize} & f(x, \alpha), \\ \text{subject to} & g(x, \alpha) \in Q, \qquad x \in U, \end{array}$$

where x is the variable in which the minimization is done and α a perturbation parameter which belongs to P and which remains fixed in the minimization problem.

We are interested in the behavior of local minimizers of $P(\alpha)$ when the parameter α varies. Our main result can be informally stated as follows. Under a set of assumptions dealing with (i) the smoothness of the functions f and g, (ii) the regularity of the constraints at $(\bar{x}, \bar{\alpha})$, (iii) the weak convexity of the set Q and (iv) a strong sufficient second-order condition at $(\bar{x}, \bar{\alpha})$, it is shown that, for small perturbations of the parameter $\bar{\alpha}$, the solution $\bar{x}$ of $P(\bar{\alpha})$ persists and is in Lipschitzian dependence with respect to the parameter. The importance of this Lipschitz property should be appreciated in the light of recent developments of calculus for Lipschitzian mappings (Clarke (1975), Rockafellar (1981)).

The above formulation of problem $P(\alpha)$ allows us to take into account the classical nonlinear programming problem with equality and/or inequality constraints, i.e., $Q = \{0\}^{m_1} \times (-\mathbb{R}_+^{m_2})$ for nonnegative integers m_1 and m_2. The consideration of more general sets Q in $P(\alpha)$ is motivated by the following property of weakly convex sets, a class of sets introduced by Vial (1983) (see also Cornet (1981)), which includes as special cases, convex subsets of $\mathbb{R}^m$ and twice continuously differentiable submanifolds of $\mathbb{R}^m$ with or without a boundary. Let Q be a nonempty closed subset of $\mathbb{R}^m$ and let α be in $\mathbb{R}^m$; then the set of projections of α on Q, denoted $\pi(\alpha) = \{x \in Q \mid \|x - \alpha\| \leq \|x' - \alpha\|$, for all x' in $Q\}$, clearly is the set of solutions of problem $P(\alpha)$ for well chosen mappings f and g. An important property of weakly convex sets is that the mapping π is single-valued and Lipschitzian on a neighborhood of Q. Our main theorem generalizes the known results for problems with equality and/or inequality constraints and also includes the above property of weakly convex sets.

We conclude this section by indicating the link between this paper and the rest of the literature. The standard problem with equality and/or inequality constraints has been studied by Fiacco and McCormick (1968), Robinson (1974), Fiacco (1976). The basic feature of these articles is that, under the strict complementarity slackness assumption, it is proved, using the standard implicit function theorem, that the stationary points (i.e., points that satisfy the first order necessary condition for optimality), and their associated multipliers are differentiable.

The strict complementarity slackness assumption has been removed, in the case of equality and/or inequality constraints by Robinson (1980), Kojima (1980), Jittorntrum (1984) and in the case where Q is a closed convex subset of $\mathbb{R}^m$ by Cornet-Laroque (1986) (see also Cornet-Laroque (1980), Cornet (1981)) and J.-P. Aubin (1981) when the constraints are linear. In these cases, under a somewhat stronger second order sufficient condition, a similar result is shown to hold, namely that the stationary points and associated multipliers are (locally) Lipschitzian mappings of the perturbation. The main tool for these analyses are generalizations of the standard implicit function theorem; for example Robinson (1980) proves a general implicit theorem for "generalized equations" (i.e. of variational inequalities), Cornet-Laroque (1986) use a generalization of the implicit function theorem in the case of Lipschitzian mappings due to Clarke (1976) (see also Auslender (1983)) and J.-P. Aubin (1981) a generalization of it in the case of convex processes.

Our approach in the present paper is different from the previous ones. It is direct in the sense that no implicit function theorem or generalization of it is used. We are able to show the local persistence of a local minimizer of our problem and next the Lipschitzian dependence with respect to the parameter α. The first step owes much to a result of Robinson (1982). Finally we shall mention the work of Levitin (1975) who made an analysis of the Lipschitz dependence of local minimizers, also by a direct approach. However, it contains an apparent error as is pointed out in the paper of Robinson (1982).

Our paper is organized as follows. In § 2, we recall some definitions and state the main result of the paper. We also discuss two noteworthy applications: the first deals with the projection of points on a weakly convex set, and the second deals with the standard nonlinear programming problem. The proof of the main theorem is given in § 3.

2. Statement of the main theorem and some consequences. Let us first introduce some notations and definitions. Let $x = (x_i)$, $y = (y_i)$ be in $\mathbb{R}^q$; we denote $\langle x, y \rangle = \sum_{i=1}^q x_i \cdot y_i$, the scalar product of $\mathbb{R}^q$, and $\|x\| = \langle x, x \rangle^{1/2}$ the Euclidean norm. Let A be a nonempty subset of $\mathbb{R}^q$ and let x be in $\mathbb{R}^q$; we denote $d_A(x) = \inf\{\|a - x\| \mid a \in A\}$, $B(A, \varepsilon) = \{x \in \mathbb{R}^q \mid d_A(x) < \varepsilon\}$ and $\bar{B}(A, \varepsilon) = \{x \in \mathbb{R}^q \mid d_A(x) \leq \varepsilon\}$. Let Q be a subset of $\mathbb{R}^m$ and let x be in $\bar{Q}$; we recall the following definitions of Clarke (1975) of the tangent cone $T_Q(x)$ and the normal cone $N_Q(x)$ to Q at x,

$$T_Q(x) = \{v \in \mathbb{R}^m \mid \text{for all sequences } \{\theta_k\} \subset (0, \infty) \text{ and } \{x_k\} \subset \bar{Q} \text{ such that } \theta_k \to 0,$$

$$x_k \to x, \text{ there exists a sequence } \{v_k\} \to v \text{ such that, for all } k, \, x_k + \theta_k v_k \in \bar{Q}\},$$

$$N_Q(x) = \{\eta \in \mathbb{R}^m \mid \langle \eta, v \rangle \leq 0, \text{ for all } v \in T_Q(x)\}.$$

DEFINITION 1. A subset Q of $\mathbb{R}^m$ is said to be weakly convex, with constant $\rho \geq 0$, at an element y^0 in $\bar{Q}$, if there exists $\varepsilon > 0$ such that, for all y^1, y^2 in $\bar{Q} \cap B(y^0, \varepsilon)$ and for all $\lambda^2 \in N_Q(y^2) \cap \bar{B}(0, 1)$, one has

$$\langle \lambda^2, y^2 - y^1 \rangle \geq -\frac{\rho}{2} \|y^2 - y^1\|^2.$$

It is possible to give the following geometric interpretation of weakly convex sets, of constant $\rho > 0$. Let Q be such a set and let $X = \bar{Q} \cap B(y^0, \varepsilon)$ with $y_0 \in \bar{Q}$. Then for all $x \in X$ and $\eta \in N_Q(x) \cap \bar{B}(0, 1) (= N_X(x) \cap \bar{B}(0, 1))$,

$$X \cap B(x + \rho^{-1}\eta, \rho^{-1}\|\eta\|) = \varnothing.$$

For $\eta \neq 0$, one could view $B(x + \rho^{-1}\eta, \rho^{-1}\|\eta\|)$ as a "supporting ball," very much in a sense analogous to a supporting hyperplane for a convex set. In this terminology, if a set is weakly convex at y^0, one can exhibit a "supporting ball" at each point of the boundary of the set in a neighborhood of y^0. Note that the radius of the "supporting ball" is fixed in the given neighborhood. Clearly, a convex subset Q of $\mathbb{R}^m$ is weakly convex with respect to any constant $\rho \geqq 0$. We refer to Cornet (1981), Vial (1983) for other examples of weakly convex sets (such as C^2 submanifolds in $\mathbb{R}^m$ with or without a boundary) and/or for properties of weakly convex sets.

We posit the following assumptions, which describe the general framework of the paper.

Assumptions A.0

(i) U is an open subset of $\mathbb{R}^n$; P is a metric space endowed with a distance d;

(ii) the functions $f(\cdot, \cdot)$ and $g_i(\cdot, \cdot)$, $i = 1, \cdots, m$, are locally Lipschitzian from $U \times P$ to $\mathbb{R}$;

(iii) for all $\alpha \in P$, the functions $f(\cdot, \alpha)$ and $g_i(\cdot, \alpha)$, $i = 1, \cdots, m$, are twice continuously differentiable from U to $\mathbb{R}$;

(iv) the mappings $\nabla f(\cdot, \cdot)$ and $\nabla g_i(\cdot, \cdot)$, $i = 1, \cdots, m$, of first partial derivatives with respect to the first argument, are locally Lipschitzian from $U \times P$ to $\mathbb{R}^n$;

(v) the mapping $D^2 f(\cdot, \cdot)$ and $D^2 g_i(\cdot, \cdot)$ of second order derivatives with respect to the first argument, are continuous;

(vi) $m = m_1 + m_2$, where m_1 and m_2 are nonnegative integers; C is a nonempty closed subset of $\mathbb{R}^{m_2}$ and $Q = \{0\}^{m_1} \times C$ (with the convention that $Q = \{0\}^m$ if $m_2 = 0$ and $Q = C$ if $m_1 = 0$).

We consider the following perturbed nonlinear programming problem:

$$P(\alpha) \qquad \begin{array}{ll} \text{minimize} & g(x, \alpha), \\ \text{subject to} & g(x, \alpha) \in Q, \\ & x \in U, \end{array}$$

where $g(x, \alpha)$ is the vector in $\mathbb{R}^m$ with coordinates $g_i(x, \alpha)$, $i = 1, \ldots, m$, x is the variable in which the minimization is done, and $\alpha \in P$ is a perturbation term which remains fixed in the minimization problem. Note that it is possible to rewrite the constraints as follows. For all (x, α) in $U \times P$, let $g_E(x, \alpha)$ (resp. $g_I(x, \alpha)$) be the vector in $\mathbb{R}^{m_1}$ (resp. $\mathbb{R}^{m_2}$) with coordinates $g_i(x, \alpha)$, $i = 1, \cdots, m_1$ (resp. $i = m_1 + 1, \cdots, m$). Then x satisfies the constraints of $P(\alpha)$ if and only if:

$$g_E(x, \alpha) = 0 \quad \text{and} \quad g_I(x, \alpha) \in C, \qquad x \in U.$$

With $P(\alpha)$, we associate the following "generalized equation":

$$\nabla f(x, \alpha) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x, \alpha) = 0,$$

(2.1)

$$g(x, \alpha) \in Q \quad \text{and} \quad \lambda = (\lambda_i) \in N_Q(g(x, \alpha)).$$

We shall be concerned with pairs $(x^0, \alpha^0) \in U \times P$ such that x^0 is a local minimizer of $P(\alpha^0)$. If we further assume that:

Assumption A.1. The gradients $\nabla g_i(x^0, \alpha^0)$, $i = 1, \cdots, m$, are linearly independent, then we shall prove later (Lemma 3.1) that there exists $\lambda^0 \in \mathbb{R}^m$ such that $(x^0, \alpha^0, \lambda^0)$ solves (2.1). In other words, (2.1) is the first order necessary condition associated with $P(\alpha)$. For such a triplet we posit the following two assumptions:

Assumption A.2. Q is weakly convex with constant $\rho \geqq 0$ at $g(x^0, \alpha^0)$. (Note that it would be equivalent to replace the above statement by "C is weakly convex with constant $\rho \geqq 0$ at $g_I(x^0, \alpha^0)$.")

Assumption A.3. There exist real numbers $a \geqq 0$ and $c > 0$ such that, for all $h \in \mathbb{R}^n$, one has

$$\left\langle \left[D^2 f(x^0, \alpha^0) + \sum_{i=1}^{m} \lambda_i^0 D^2 g_i(x^0, \alpha^0) \right] h, h \right\rangle$$

$$+ a \langle \nabla f(x^0, \alpha^0), h \rangle^2 + a \sum_{i=1}^{m_1} \langle \nabla g_i(x^0, \alpha^0), h \rangle^2 \geqq c \|h\|^2.$$

Note that the index in the first sum runs from 1 to $m = m_1 + m_2$ and from 1 to m_1 in the second. (A.3) clearly implies the more familiar assumption:

Assumption A.3'. For all $h \in \mathbb{R}^n$, $h \neq 0$, such that $\langle \nabla f(x^0, \alpha^0), h \rangle = 0$ and $\langle \nabla g_i(x^0, \alpha^0), h \rangle = 0$, $i = 1, \cdots, m_1$, one has

$$\left\langle \left[D^2 f(x^0, \alpha^0) + \sum_{i=1}^{m} \lambda_i^0 D^2 g_i(x^0, \alpha^0) \right] h, h \right\rangle > 0.$$

It is an immediate consequence of a lemma of Debreu (1952) that (A.3') implies (A.3). Thus the two assumptions are equivalent.

We can now state the main theorem.

THEOREM 2.1. *Assume* (A.0) *and let* $(x^0, \alpha^0, \lambda^0) \in U \times P \times \mathbb{R}^m$ *satisfy* (A.1), (A.2), (A.3). *Further, assume that the constants* $\rho \geqq 0$ *and* $c > 0$ *satisfy*

Assumption A.4. $c > \lambda_I^0 \| Dg_I(x^0, \alpha^0) \|^2$, *where* $\lambda_I^0 = (\lambda_{m_1}^0 + 1, \cdots, \lambda_m^0)$.

Then, if $(x^0, \alpha^0, \lambda^0)$ *satisfies condition* (2.1), *there exist neighborhoods* U' *of* x^0 *in* U, V' *of* α^0 *in* P *and mappings* $x(\cdot): V' \to U'$, $\lambda(\cdot): V' \to \mathbb{R}^m$ *such that*:

(i) $x(\cdot)$ *and* $\lambda(\cdot)$ *are Lipschitzian*;

(ii) $x(\alpha^0) = x^0$ *and* $\lambda(\alpha^0) = \lambda^0$;

(iii) *for all* α *in* V', $x(\alpha)$ *is the unique minimizer of* $P(\alpha)$ *in* U' *and* $\lambda(\alpha)$ *is the unique Kuhn–Tucker multiplier associated with* $x(\alpha)$ *(i.e.,* $(x(\alpha), \lambda(\alpha))$ *satisfies condition* (2.1)).

The proof of Theorem 2.1 is given in the next section.

Remark 1. Assumption (A.1) cannot be relaxed in the case of equality and/or inequality constraints (i.e., when $Q = \{0\}^{m_1} \times (-\mathbb{R}_+^{m_2})$) by only assuming the Mangasarian–Fromovitz's constraint qualification (see Robinson (1980)).

Remark 2. If C is convex, then (A.4) is trivially satisfied (since convex sets in $\mathbb{R}^{m_2}$ are weakly convex with constant $\rho = 0$). It is worth pointing out that in the case of equality and/or inequality constraints, (A.3) is stronger than the classical sufficient second-order condition of Fiacco and McCormick (1968). However, Theorem 2.1 does not hold if one replaces (A.3) by the classical second order condition as it has been shown by Robinson (1980).

We conclude this section by discussing two noteworthy applications of Theorem 2.1. The first one deals with the projection mapping on a weakly convex subset of $\mathbb{R}^n$.

Let Q be a nonempty closed subset of $\mathbb{R}^n$. For a fixed element α in $\mathbb{R}^n$, we consider the following minimization problem:

$$R(\alpha) \qquad \begin{aligned} &\text{minimize} \quad \tfrac{1}{2}\|x - \alpha\|^2, \\ &\text{subject to} \quad x \in Q, \end{aligned}$$

and we denote by $\pi(\alpha)$ the set of its solutions. Any element of $\pi(\alpha)$ is called a projection of α on Q. The next proposition gives some properties of the (multi-valued) mapping $\alpha \to \pi(\alpha)$ when Q is assumed to be weakly convex.

COROLLARY 2.2. *Let* $(x^0, \alpha^0) \in \mathbb{R}^n \times \mathbb{R}^n$ *be such that* x^0 *is a local minimizer of* $R(\alpha^0)$. *Assume that* Q *is weakly convex with constant* $\rho \geq 0$ *at* x^0 *and that* $1 > \rho \|x^0 - \alpha^0\|$. *Then, there exist neighborhoods* U' *of* x^0, V' *of* α^0 *and a Lipschitzian mapping* $x(\cdot): V' \to U'$ *such that* $x(\alpha^0) = x^0$ *and, for all* $\alpha \in V'$, $x(\alpha)$ *is the unique minimizer of* $R(\alpha)$ *in* U'.

Proof. It is a trivial matter to check that (A.1) and (A.3) are satisfied with $c = 1$ and that the Kuhn–Tucker multiplier λ^0 associated with x^0 satisfies $\lambda^0 = -(x^0 - \alpha^0)$. Thus (A.4) reduces to $1 = c > \rho \|x^0 - \alpha^0\|$. Hence the result.

Remark 3. When $m_1 = 0$, we give here an example showing that the inequality in (A.4) is the best possible. Let $Q = \{x \in \mathbb{R}^m \mid \|x\| \geq 1\}$; clearly, Q is weakly convex at every element x in Q, with constant $\rho = 1$. If $\alpha^0 \neq 0$, let $\mu = \max\{1, \|\alpha^0\|^{-1}\}$, then $x^0 = \mu \alpha^0$ is the unique minimizer of $R(\alpha^0)$. Since the hypotheses of Corollary 2.2 are satisfied at x^0, the conclusion of the corollary holds. However, if $\alpha^0 = 0$, any x^0 such that $\|x^0\| = 1$ is a minimizer of $R(\alpha^0)$. Obviously, $1 = \rho \|x^0 - \alpha^0\|$; hence Assumption (A.4) is violated and one easily sees directly that the conclusion of Corollary 2.2 cannot hold.

The second application deals with standard nonlinear programming. Assume A.0 and assume furthermore that $Q = \{0\}^{m_1} \times (-\mathbb{R}_+^{m_2})$ (i.e., in (vi) of (A.0), $C = -\mathbb{R}_+^{m_2}$). We consider the standard perturbed nonlinear programming problem:

$$S(\alpha) \qquad \begin{aligned} &\text{minimize} \quad f(x, \alpha), \\ &\text{subject to} \quad g_i(x, \alpha) = 0, \ i = 1, \cdots, m_1, \\ &\qquad\qquad\quad g_i(x, \alpha) \leq 0, \ i = m_1 + 1, \cdots, m, \\ &\qquad\qquad\quad x \in U. \end{aligned}$$

With $S(\alpha)$, we associate the first order necessary conditions:

$$\nabla f(x, \alpha) + \sum_{i=1}^{m_1 + m_2} \lambda_i \nabla g_i(x, \alpha) = 0,$$

(2.2) $\qquad g_i(x, \alpha) = 0, \qquad i = 1, \cdots, m_1,$

$$g_i(x, \alpha) \leq 0, \quad \lambda_i \geq 0, \quad \lambda_i g_i(x, \alpha) = 0, \qquad i = m_1 + 1, \cdots, m_1 + m_2.$$

In the case of standard nonlinear programming, Assumptions (A.2) and (A.4) are satisfied, since $Q = \{0\}^{m_1} \times (-\mathbb{R}_+)^{m_2}$ is convex. We give now a consequence of Theorem 2.1, where the Assumptions (A.1) and (A.3) are weakened. First, let us introduce the following notation: for $(x, \alpha) \in U \times P$, satisfying the constraints of $S(\alpha)$, we let $I(x, \alpha) = \{i \in \{m_1 + 1, \cdots, m\} \mid g_i(x, \alpha) = 0\}$ be the set of active inequality constraints.

COROLLARY 2.3. *Assume* (A.0) *and let* $(x^0, \alpha^0, \lambda^0) \in U \times P \times \mathbb{R}^m$ *be such that*:
  (C.1) *the vectors* $\nabla g_i(x^0, \alpha^0)$, $i \in \{1, \cdots, m_1\} \cup I(x^0, \alpha^0)$, *are linearly independent*;
  (C.2) *for all* $h \in \mathbb{R}^n$, $h \neq 0$, *satisfying* $\langle \nabla f(x^0, \alpha^0), h \rangle = 0$ *and* $\langle \nabla g_i(x^0, \alpha^0), h \rangle = 0$, $i \in \{1, \cdots, m_1\} \cup \{i \in I(x^0, \alpha^0) \mid \lambda_i^0 > 0\}$, *then*

$$\left\langle \left[D^2 f(x^0, \alpha^0) + \sum_{i=1}^{m} \lambda_i^0 D^2 g_i(x^0, \alpha^0) \right] h, h \right\rangle > 0.$$

Then, if $(x^0, \alpha^0, \lambda^0)$ *satisfies condition* (2.2), *there exist neighborhoods* U' *of* x^0 *in* U, V' *of* α^0 *in* P *and mappings* $x(\cdot): V' \to U'$, $\lambda(\cdot): V' \to \mathbb{R}^m$ *such that*:

(i) $x(\cdot)$ *and* $\lambda(\cdot)$ *are Lipschitzian*;

(ii) $x(\alpha^0) = x^0$ *and* $\lambda(\alpha^0) = \lambda^0$;

(iii) *for all* $\alpha \in V'$, $x(\alpha)$ *is the unique minimizer of* $S(\alpha)$ *in* U' *and* $\lambda(\alpha)$ *is the unique Kuhn–Tucker multiplier associated with it* (*i.e.*, $(x(\alpha), \lambda(\alpha))$ *satisfies condition* (2.2)).

The proof of Corollary 2.3 is given in the next section.

3. Proofs. We prepare the proofs of Theorem 2.1 by several lemmas. Some of them are more or less known. However the entire proofs are given for the sake of completeness. Our first lemma says that it is sufficient to prove Theorem 2.1 with Assumption (A.3) replaced by the stronger one

Assumption A.3. bis. There exist real numbers $a \geqq 0$, $c > 0$ such that for all $h \in \mathbb{R}^n$ one has

$$\left\langle D^2 f(x^0, \alpha^0) h + \sum_{i=1}^{m} \lambda_i^0 D^2 g_i(x^0, \alpha^0) h, h \right\rangle + a \sum_{i=1}^{m_1} \langle \nabla g_i(x^0, \alpha^0), h \rangle^2 \geqq c \|h\|^2,$$

(i.e. in (A.3.bis) the term $a \langle \nabla f(x^0, \alpha^0), h \rangle^2$ which appears in (A.3) has been removed).

LEMMA 3.0. *If Theorem 2.1 is true when* (A.3) *is replaced by the stronger Assumption* (A.3 *bis*), *then it is also true under Assumption* (A.3).

Proof. Let us assume that the weak form of Theorem 2.1 holds (i.e. with Assumption (A.3) replaced by (A.3bis)). We show that Theorem 2.1 also holds. Let $(x^0, \alpha^0, \lambda^0) \in U \times P \times \mathbb{R}^m$ satisfy Assumptions (A.0), (A.1), (A.2), (A.3), (A.4) together with the necessary conditions (2.1) associated with the problem:

$$\text{minimize } f(x, \alpha)$$

$P(\alpha)$ subject to $g(x, \alpha) \in Q$,

$$x \in U.$$

We now associate to all $\alpha \in P$ the following modified problem:

$$\text{minimize } v$$

subject to $f(x, \alpha) - v = 0$,

$\tilde{P}(\alpha)$

$$g(x, \alpha) \in Q,$$

$$(v, x) \in \mathbb{R} \times U.$$

Clearly (v, x) is a solution of $\tilde{\mathscr{P}}(\alpha)$ if and only if x is a solution of $\mathscr{P}(\alpha)$ and $v = f(x, \alpha)$. Moreover at this solution $[(v, x), (\mu, \lambda)]$ satisfies the necessary conditions (2.1) associated with $\tilde{\mathscr{P}}(\alpha)$ if and only if $\mu = 1$, (x, λ) satisfies the necessary conditions (2.1) associated with $P(\alpha)$ and $v = f(x, \alpha)$.

Hence $((v^0, x^0), (1, \lambda^0))$, with $v^0 = f(x^0, \alpha^0)$ satisfies the necessary conditions (2.1) associated with $\tilde{\mathscr{P}}(\alpha^0)$. From the fact that $(x^0, \alpha^0, \lambda^0)$ satisfies Assumptions (A.0), (A.1), (A.2), (A.3) and (A.4) for problem $\tilde{\mathscr{P}}(\alpha^0)$ one deduces that $((v^0, x^0), (1, \lambda^0))$ satisfies Assumptions (A.0), (A.1), (A.2), (A.3bis) and (A.4) for problem $\tilde{\mathscr{P}}(\alpha^0)$. Applying the weak form of Theorem 2.1 to $((v^0, x^0), (1, \lambda^0))$ and using the above equivalence property between $\mathscr{P}(\alpha)$ and $\tilde{\mathscr{P}}(\alpha)$ one deduces the end of the proof of Lemma 3.0.

In the sequel we shall assume that $(x^0, \alpha^0, \lambda^0)$ satisfies (A.0), (A.1), (A.2), (A.3bis) and (A.4).

LEMMA 3.1. *There exist neighborhoods U_1 of x^0, $U_1 \subset U$, V_1 of α^0 such that, for all $\alpha \in V_1$, if $x \in U_1$ is a local minimizer of $P(\alpha)$, then there exists $\lambda = (\lambda_i) \in \mathbb{R}^m$ such that (x, λ) satisfies the first order necessary conditions at α, i.e.,*

$$\nabla f(x, \alpha) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x, \alpha) = 0,$$

$$g(x, \alpha) \in Q \quad and \quad \lambda \in N_Q(g(x, \alpha)).$$

Proof. Since by (A.1) the gradients $\nabla g_i(x^0, \alpha^0)$, $i \in \{1, \cdots, m\}$ are independent from (A.0), there exist neighborhoods U_1 of x^0, $U_1 \subset U$, V_1 of α^0 such that, for all $(x, \alpha) \in U_1 \times V_1$, the vectors $\nabla g_i(x, \alpha)$, $i \in \{1, \cdots, m\}$, are independent. In the sequel of this proof, since α is fixed, there is no ambiguity denoting $f(x, \alpha)$, $\nabla f(x, \alpha)$, $\cdots$, simply by $f(x)$, $\nabla f(x)$, $\cdots$.

Let $X = \{x \in U_1 | g(x) \in Q\}$. If x is a local minimizer of $P(\alpha)$, from Clarke (1975), for all $v \in T_x(x)$ one has $\langle -\nabla f(x), v \rangle \leq 0$. Hence from Rockafellar (1970, Cor. 16.3.2), it is sufficient to prove that the following inclusion holds:

$$\{u \in \mathbb{R}^n | Dg(x)u \in T_Q(g(x))\} \subset T_X(x).$$

Indeed, let $u \in \mathbb{R}^n$ be such that $Dg(x)u \in T_Q(g(x))$ and let $\{\theta^q\} \subset (0, \infty)$, $\{x^q\} \subset X$ be sequences such that $\theta^q \to 0$ and $x^q \to x$. From the definition of the tangent cone it is sufficient to show that there exists a sequence $\{u^q\} \subset \mathbb{R}^n$, such that $u^q \to u$ and, for all q, $x^q + \theta^q u^q \in X$. Let $v = Dg(x)u$, then $v \in T_Q(g(x))$. Since, for all q, $g(x^q) \in Q$, and $g(x^q) \to g(x)$, there exists a sequence $\{v^q\} \subset \mathbb{R}^m$ such that, for all q, $g(x^q) + \theta^q v^q \in Q$ and $v^q \to v$. We can choose vectors $b_{m+1}, \cdots, b_n$ in $\mathbb{R}^n$ so that the vectors $\nabla g_1(x), \cdots, \nabla g_m(x), b_{m+1}, \cdots, b_n$, form a basis in $\mathbb{R}^n$, and we define the mapping $G: U_1 \to \mathbb{R}^n$ by $G(y) = (g_1(y), \cdots, g_m(y), \langle b_{m+1}, y \rangle, \cdots, \langle b_n, y \rangle)$. Clearly, G is continuously differentiable, and the derivative $DG(x)$ is nonsingular. Hence, by the inverse mapping theorem, there exists q_0 such that, for $q \geq q_0$, there exists $\hat{x}^q \in U_1$ satisfying:

$$g(\hat{x}^q) = g(x^q) + \theta^q v^q,$$

$$\langle b_i, \hat{x}^q \rangle = \langle b_i, x^q \rangle + \theta^q \langle b_i, u \rangle \quad (i = m+1, \cdots, n),$$

and such that $\hat{x}^q \to x$. For $q \geq q_0$, let $u^q = (\hat{x}^q - x^q)/\theta^q$. Recall that, for all q, $g(x^q) + \theta^q v^q \in Q$, hence, for $q \geq q_0$, $g(\hat{x}^q) = g(x^q) + \theta^q v^q \in Q$; thus $\{\hat{x}^q\} \subset X$. Consequently, for $q \geq q_0$, $x^q + \theta^q u^q = \hat{x}^q \in X$. To end the proof of the lemma, it suffices to show that $u^q \to u$. Indeed, from Taylor's theorem, for $q \geq q_0$, one has $g(\hat{x}^q) - g(x^q) = [\int_0^1 Dg(x^q + t(\hat{x}^q - x^q)) \, dt](\hat{x}^q - x^q)$. Dividing by $\theta^q > 0$, one gets $v^q = [\int_0^1 Dg(x^q + t(\hat{x}^q - x^q)) \, dt]u^q$ and one easily deduces that $\lim_{q \to \infty} Dg(x)u^q = \lim_{q \to \infty} v^q$. Recall that $\lim_{q \to \infty} v^q = v = Dg(x)u$; hence, for all $i \leq m$, $\lim_{q \to \infty} \langle \nabla g_i(x), u^q \rangle = \langle \nabla g_i(x), u \rangle$. Furthermore, for all $i \geq m+1$, $\langle b_i, u^q \rangle = \langle b_i, (\hat{x}^q - x^q)/\theta^q \rangle = \langle b_i, u \rangle$. Since the vectors $\{\nabla g_1(x), \cdots, \nabla g_m(x), b_{m+1}, \cdots, b_n\}$ are independent, one deduces that $u^q \to u$. This ends the proof of the lemma.

LEMMA 3.2. *For all $\varepsilon > 0$ and all $c' \in (0, c)$, there exist positive real numbers k_1, k_2, a positive real number δ (independent of ε) and neighborhoods U_2 of x^0, $U_2 \subset U$, V_2 of α^0, such that the two following properties are satisfied.*

(a) *For all (x, α), (y, β), (x^1, α^1), (x^2, α^2) in $U_2 \times V_2$ such that $g(x^1, \alpha^1) \in Q$, $g(x^2, \alpha^2) \in Q$, for all $\lambda \in B(\lambda^0, \delta)$ one has:*

$$\left\langle \left[D^2 f(x, \alpha) + \sum_{i=1}^{m} \lambda_i D^2 g_i(y, \beta) \right] (x^2 - x^1), (x^2 - x^1) \right\rangle$$

$$\geq c' \|x^2 - x^1\|^2 - k_1 \|x^2 - x^1\| d(\alpha^2, \alpha^1) - k_1 d(\alpha^2, \alpha^1)^2.$$

(b) *For all* (x^1, α^1), (x^2, α^2) *in* $U_2 \times V_2$, *such that* $g(x^2, \alpha^2) \in Q$, $g(x^1, \alpha^1) \in Q$, *for all* $\lambda^2 \in N_Q(g(x^2, \alpha^2))$ *one has*:

$$\langle \lambda^2, g(x^2, \alpha^2) - g(x^1, \alpha^1) \rangle$$

$$\geqq \|\lambda_I^2\| \cdot \left[-\frac{\rho}{2} [\|Dg_I(x^0, \alpha^0)\| + \varepsilon]^2 \cdot \|x^2 - x^1\|^2 - k_2\|x^2 - x^1\| \, d(\alpha^2, \alpha^1) - k_2 d(\alpha^2, \alpha^1)^2 \right].$$

Proof. We first claim that, for all $\varepsilon > 0$, there exist a positive real number k' and neighborhoods U' of x^0, $U' \subset U$, V' of α^0 such that, for all (x^1, α^1), (x^2, α^2) in $U' \times V'$, for all $i \in \{1, \cdots, m\}$ one has:

(3.1) $\quad |g_i(x^2, \alpha^2) - g_i(x^1, \alpha^1) - \langle \nabla g_i(x^0, \alpha^0), (x^2 - x^1) \rangle| \leqq \varepsilon \|x^2 - x^1\| + k' d(\alpha^2, \alpha^1).$

Indeed, for all $i \in \{1, \cdots, m\}$, from (A.0), for all $\varepsilon > 0$, there exist open neighborhoods U' of x^0, $U' \subset U$, V' of α^0, such that U' is convex, and, for all $(x, \alpha) \in U' \times V'$, $\|\nabla g_i(x, \alpha) - \nabla g_i(x^0, \alpha^0)\| < \varepsilon$. Furthermore, without any loss of generality, we can assume that there exists a positive real number k' such that g_i is Lipschitzian of constant k' on $U' \times V'$. Hence, from Taylor's theorem,

$$|g_i(x^2, \alpha^2) - g_i(x^1, \alpha^1) - \nabla g_i(x^0, \alpha^0)(x^2 - x^1)|$$

$$\leqq \left| g_i(x^2, \alpha^2) - g_i(x^1, \alpha^2) - \left\langle \left[\int_0^1 \nabla g_i(x^1 + t(x^2 - x^1), \alpha^2) \, dt \right], x^2 - x^1 \right\rangle \right|$$

$$+ |g_i(x^1, \alpha^2) - g_i(x^1, \alpha^1)|$$

$$+ \int_0^1 \|\nabla g_i(x^1 + t(x^2 - x^1), \alpha^2) - \nabla g_i(x^0, \alpha^0)\| \, dt \cdot \|x^2 - x^1\|$$

$$\leqq 0 + k' d(\alpha^2, \alpha^1) + \varepsilon \|x^2 - x^1\|.$$

(a) For all $c' \in (0, c)$, let $c'' \in (c', c)$. From (A.3.bis) and the continuity of the mappings $D^2 f(\cdot, \cdot)$ and $D^2 g_i(\cdot, \cdot)$ ($i = 1, \cdots, m$), there exist neighborhoods U'' of x^0, $U'' \subset U$, V'' of α^0 and a positive real number δ such that, for all (x, α), (y, β) in $U'' \times V''$, for all $\lambda \in B(\lambda^0, \delta)$ and all $h \in \mathbb{R}^n$, one has:

$$\left\langle \left[D^2 f(x, \alpha) + \sum_{i=1}^{m} \lambda_i D^2 g_i(y, \beta) \right] h, h \right\rangle \geqq c'' \|h\|^2 - a \sum_{i=1}^{m_1} \langle \nabla g_i(x^0, \alpha^0), h \rangle^2.$$

Let $\varepsilon > 0$ be such that $c'' - a \cdot m_1 \cdot \varepsilon^2 \geqq c'$, and let U', V' be the neighborhoods associated with ε in the above claim (3.1). Let $U_2 = U' \cap U''$ and $V_2 = V' \cap V''$. Recall that, for all $(x, \alpha) \in U \times V$ such that $g(x, \alpha) \in Q$, from (A.0.vi), for $i \in \{1, \cdots, m_1\}$, $g_i(x, \alpha) = 0$. Hence the end of the proof follows easily from (3.1) and the above inequality.

(b) From the weak convexity Assumption (A.2) and the continuity of the functions g_i ($i = 1, \cdots, m$), there exist open neighborhoods U'' of x^0, $U'' \subset U$, and V'' of α^0 such that, for all (x^1, α^1), (x^2, α^2) in $U'' \times V''$ such that $g(x^k, \alpha^k) \in Q$ ($k = 1, 2$), and for all $\mu^2 \in N_Q(g(x^2, \alpha^2)) \cap \bar{B}(0, 1)$, one has:

$$\langle \mu^2, g(x^2, \alpha^2) - g(x^1, \alpha^1) \rangle \geqq -\frac{\rho}{2} \|g(x^2, \alpha^2) - g(x^1, \alpha^1)\|^2.$$

Take any $\varepsilon > 0$ and let U', V' be the neighborhoods associated with ε in the above claim (3.1). Let $U_2 = U' \cap U''$ and $V = V' \cap V''$. From (3.1) and the above inequality, there exists k_2 such that

$$\langle \mu^2, g(x^2, \alpha^2) - g(x^1, \alpha^1) \rangle$$

$$\geqq -\frac{\rho}{2} [\|Dg_I(x^0, \alpha^0)\| + \varepsilon]^2 \cdot \|x^2 - x^1\|^2 - k_2\|x^2 - x^1\| \cdot d(\alpha^2, \alpha^1) - k_2 d(\alpha^2, \alpha^1)^2.$$

Now letting $\lambda^2 \in N_Q(g(x^2, \alpha^2))$, we let $\mu^2 = (0, \lambda_I^2 / \|\lambda_I^2\|)$ if $\lambda_I^2 \neq 0$ and $\mu^2 = 0$ if $\lambda_I^2 = 0$. Since $Q = \{0\} \times C$, $N_Q(g(x^2, \alpha^2)) = \mathbb{R}^{m_1} \times N_C(g_I(x^2, \alpha^2))$ and since $N_C(g_I(x^2, \alpha^2))$ is a cone, one deduces that $\mu^2 \in N_Q(g(x^2, \alpha^2)) \cap \bar{B}(0, 1)$. Applying the above inequality to μ^2 and noticing that

$$\langle \mu^2, g(x^2, \alpha^2) - g(x^1, \alpha^1) \rangle = (1/\|\lambda_I^2\|)\langle \lambda^2, g(x^2, \alpha^2) - g(x^1, \alpha^1) \rangle,$$

yields the inequality of Lemma 3.2(b).

LEMMA 3.3. *Let us suppose that* (x^0, λ^0) *satisfies the first order necessary condition at* α^0. *Then there exist positive real numbers r and b such that $\bar{B}(x^0, r) \subset U$ and, for all $x \in \bar{B}(x^0, r)$ satisfying $g(x, \alpha^0) \in Q$, one has $f(x, \alpha^0) \geq f(x^0, \alpha^0) + b\|x - x^0\|^2$.*

Proof. For every $\lambda = (\lambda_i) \in \mathbb{R}^m$, let $L(\cdot, \lambda): U \to \mathbb{R}$ be the function defined by $L(x, \lambda) = f(x, \alpha^0) + \sum_{i=1}^m \lambda_i g_i(x, \alpha^0)$. From Taylor's theorem, one has:

$$L(x, \lambda^0) = L(x^0, \lambda^0) + \langle \nabla L(x^0, \lambda^0), x - x^0 \rangle$$

$$+ \int_0^1 (1-t)\langle D^2 L(x^0 + t(x - x^0), \lambda^0)(x - x^0), (x - x^0) \rangle \, dt.$$

Let $\rho > 0$ be the constant of weak convexity defined by (A.2). From (A.4), $c > \rho \|\lambda_I^0\| \cdot \|Dg_I(x^0, \alpha^0)\|^2$. Hence, there exist $c' \in (0, c)$ and $\varepsilon > 0$ such that, if $b = c'/2 - \rho/2\|\lambda_I^0\|[\|Dg_I(x^0, \alpha^0)\| + \varepsilon]^2$, then $b > 0$. Let U_2, V_2 be the neighborhoods of x^0 and α^0 associated with c' and ε in Lemma 3.2, and let r be a positive real number such that $\bar{B}(x^0, r) \subset U_2$. Since (x^0, α^0) satisfies the first order necessary condition (2.1) at α^0, one deduces that $\nabla L(x^0, \lambda^0) = 0$. From Lemma 3.2(a) (taking $\alpha^1 = \alpha^2 = \alpha^0$), for all $x \in \bar{B}(x^0, r)$ one deduces that:

$$L(x, \lambda^0) - L(x^0, \lambda^0) \geq \int_0^1 (1-t)c'\|x - x^0\|^2 \, dt \geq \frac{c'}{2}\|x - x^0\|^2,$$

and, from the definition of L, one gets:

$$f(x, \alpha^0) - f(x^0, \alpha^0) \geq \langle \lambda^0, g(x^0, \alpha^0) - g(x, \alpha^0) \rangle + (c'/2)\|x - x^0\|^2.$$

By Lemma 3.2(b) (taking $\alpha^1 = \alpha^2 = \alpha^0$), for all $x \in \bar{B}(x^0, r) \subset U_2$, one gets:

$$\langle \lambda^0, g(x^0, \alpha^0) - g(x, \alpha^0) \rangle \geq -(\rho/2)\|\lambda_I^0\|[\|Dg_I(x^0, \alpha^0)\| + \varepsilon]^2.$$

Hence, from the two above inequalities and the definition of b, for all $x \in \bar{B}(x^0, r)$, one has $f(x, \alpha^0) - f(x^0, \alpha^0) \geq b\|x - x^0\|^2$. This completes the proof.

LEMMA 3.4. *Let* $(x^0, \lambda^0) \in U \times \mathbb{R}^m$ *satisfy the first order necessary condition (2.1) at* α^0. *Then, for every neighborhood U' of x^0, $U' \subset U$, there exist $r' > 0$ and a neighborhood V' of α^0 such that $B(x^0, r') \subset U'$ and, for all $\alpha \in V'$, there exists a minimizer $x(\alpha)$ of $P(\alpha)$ in $B(x^0, r')$ (i.e., $x(\alpha) \in B(x^0, r')$, $g(x(\alpha), \alpha) \in Q$ and for all $x \in B(x^0, r')$, $g(x, \alpha) \in Q$ one has $f(x(\alpha), \alpha) \leq f(x, \alpha)$).*

The above lemma is related to a previous result of Robinson (1982), proved when Q is a closed convex cone.

Proof. Let U' be a neighborhood of x^0, $U' \subset U$, and let b, r be the positive real numbers defined by Lemma 3.3. There exists a positive real number $r' < r$ and a neighborhood V'_1 of α^0 such that $\bar{B}(x^0, r') \subset U'$ and f is k-Lipschitzian on $\bar{B}(x^0, r') \times V'_1$.

For all $\alpha \in P$, let $\Gamma(\alpha) = \{x \in \bar{B}(x^0, r') | g(x, \alpha) \in Q\}$. We claim that there exists a neighborhood V'_2 of α^0 such that, for all α in V'_2, there exists $x(\alpha) \in \Gamma(\alpha)$ satisfying:

$$f(x(\alpha), \alpha) \leq f(x, \alpha) \quad \text{for all } x \in \Gamma(\alpha).$$

Since f is continuous and, for all α in P, $\Gamma(\alpha)$ is a compact subset of $\mathbb{R}^n$, it is sufficient to prove that, for α in a neighborhood of α^0, $\Gamma(\alpha)$ is nonempty. Indeed, by (A.1) the vectors $\nabla g_i(x^0, \alpha^0)$, $i \in \{1, \cdots, m\}$, are independent. Hence, by the implicit function theorem (Schwartz (1967)), there exists a neighborhood V_2' of α^0 and a continuous mapping $\varphi : V_2' \to \bar{B}(x^0, r)$ such that $\varphi(\alpha^0) = x^0$ and, for all $\alpha \in V_2'$, one has $g(\varphi(\alpha), \alpha) = g(x^0, \alpha^0) \in Q$. This ends the proof of the claim.

We now claim that there exists a neighborhood V' of α^0, $V' \subset V_2'$, such that, for all $\alpha \in V'$, the element $x(\alpha)$ defined before satisfies $\|x(\alpha) - x^0\| < r'$. Clearly, the proof of the claim will end the proof of the lemma. Recall that f is k-Lipschitzian on $\bar{B}(x^0, r') \times V_1'$. Let $V_3' = V_1' \cap B(\alpha^0, r'^2 b / 16k)$ and let $\eta > 0$, $\eta < \min \{r'/2, r'^2 b / 16k\}$. Then, for all $x^1, x^2 \in B(x^0, r')$ satisfying $\|x^2 - x^1\| < \eta$, for all $\alpha \in V_3'$, one has:

$$f(x^1, \alpha) - f(x^2, \alpha^0) \leq k[\eta + d(\alpha, \alpha^0)] < r'^2 b / 8.$$

Furthermore, the multivalued mapping $\alpha \to \Gamma(\alpha) = \{x \in \bar{B}(x^0, r') | g(x, \alpha) \in Q\}$, from V_2' to $\mathbb{R}^n$, is upper semicontinuous, with compact values. Hence, there exists a neighborhood V_4' of α^0, $V_4' \subset V_2'$, such that, for all $\alpha \in V_4'$, $\Gamma(\alpha) \subset B(\Gamma(\alpha^0), \eta)$ and (from the continuity of φ) $\varphi(\alpha) \in B(x^0, \eta)$.

Let $V' = V_1' \cap V_2' \cap V_3' \cap V_4'$. We now show that V' satisfies the conclusion of the lemma. Recall that, for all $\alpha \in V'$, $\varphi(\alpha) \in \Gamma(\alpha)$; thus, from the definition of $x(\alpha)$, one deduces that:

$$f(\varphi(\alpha), \alpha) \geq f(x(\alpha), \alpha).$$

On the other hand, for all $\alpha \in V'$, $x(\alpha) \in \Gamma(\alpha) \subset B(\Gamma(\alpha^0), \eta)$. Hence, there exists $y^0 \in \Gamma(\alpha^0)$ such that $\|y^0 - x(\alpha)\| < \eta$. By Lemma 3.3, one has:

$$f(y^0, \alpha^0) \geq f(x^0, \alpha^0) + b\|y^0 - x^0\|^2.$$

Summing up the two above inequalities, for all $\alpha \in V'$, one gets:

$$b\|y^0 - x^0\|^2 \leq f(\varphi(\alpha), \alpha) - f(x^0, \alpha^0) + f(y^0, \alpha^0) - f(x(\alpha), \alpha).$$

Let us recall that, for $\alpha \in V'$, $\|\varphi(\alpha) - x^0\| < \eta$ and $\|y^0 - x(\alpha)\| < \eta$. Hence, from the above inequality and the inequality defining η, one gets $b\|y^0 - x^0\|^2 \leq r'^2 b / 8 + r'^2 b / 8$. Thus, $\|y^0 - x^0\| < r'/2$, and $\|x(\alpha) - x^0\| \leq \|x(\alpha) - y^0\| + \|y^0 - x^0\| \leq \eta + r'/2 < r'$. This ends the proof of the claim and the proof of the lemma.

In the following, we denote by $Dg(x, \alpha)^*$ the $n \times m$ matrix whose columns are $\nabla g_i(x, \alpha)$ $(i = 1, \cdots, m)$, i.e., the transpose of the $m \times n$ matrix $Dg(x, \alpha)$.

LEMMA 3.5. *Let ε be a positive real number. There exist a positive real number k_3 and neighborhoods U_3 of x^0, $U_3 \subset U$, V_3 of α^0 such that, for all $(x, \alpha) \in U_3 \times V_3$, the matrix $Dg(x, \alpha) \circ Dg(x, \alpha)^*$ is nonsingular and the mapping $\varphi : U_3 \times V_3 \to \mathbb{R}^m$ defined by*

$$\varphi(x, \alpha) = [Dg(x, \alpha) \circ Dg(x, \alpha)^*]^{-1} \circ Dg(x, \alpha) \nabla f(x, \alpha),$$

satisfies the following properties:

$$\|\varphi(x^2, \alpha^2) - \varphi(x^1, \alpha^1)\| \leq k_3[\|x^2 - x^1\| + d(\alpha^2, \alpha^1)] \quad \textit{for all } (x^1, \alpha^1), (x^2, \alpha^2) \textit{ in } U_3 \times V_3;$$

$$\|\varphi(x, \alpha) - \varphi(x^0, \alpha^0)\| \leq \varepsilon, \quad \textit{for all } (x, \alpha) \in U_3 \times V_3.$$

Proof. The proof is a straightforward consequence of the independence Assumption (A.1), using the fact that the mappings $\nabla f(\cdot, \cdot)$, $\nabla g_i(\cdot, \cdot)$ $(i = 1, \cdots, m)$ are locally Lipschitzian and that the mappings $A \to (A \circ A^*)^{-1} \circ A$, defined on the set of $m \times n$ matrices of maximal rank, is infinitely differentiable, and hence locally Lipschitzian.

Proof of Theorem 2.1. Let $c > 0$, $\rho > 0$ be the constants defined by (A.2) and (A.3). From (A.4), $c > \rho \|\lambda_I^0\| \cdot \|Dg_I(x^0, \alpha^0)\|^2$. Hence, there exist $c' \in (0, c)$ and $\varepsilon > 0$ such that

$$(3.2) \qquad c' > \rho[\|\lambda_I^0\| + \varepsilon][\|Dg_I(x^0, \alpha^0)\| + \varepsilon]^2.$$

Let $\delta > 0$ be the constant defined by Lemma 3.2; without any loss of generality, we can suppose that $\varepsilon < \delta$. From (A.0), there exist a positive real number k and neighborhoods U_0 of x^0, $U_0 \subset U$, V_0 of α^0 such that all the mappings $g(\cdot, \cdot)$ $Dg(\cdot, \cdot)$ and $\nabla f(\cdot, \cdot)$ are k-Lipschitzian on $U_0 \times V_0$. Furthermore, let U_1, V_1 (resp. U_2, V_2 and U_3, V_3) be the neighborhoods of x^0 and α^0 associated, in Lemma 3.1 (resp. Lemma 3.2 and Lemma 3.5) with the constants c' and ε as defined above.

Now, by Lemma 3.4, we associate with $U' = U_0 \cap U_1 \cap U_2 \cap U_3$ a positive real number r' and an open neighborhood $\tilde{V}$ of α^0. We take $V' = \tilde{V} \cap V_0 \cap V_1 \cap V_2 \cap V_3$. Let α^1, α^2 be two elements in V'. By Lemma 3.4, there exists a minimizer of $P(\alpha^1)$ (resp. $P(\alpha^2)$) in $B(x^0, r')$ not necessarily unique, that we denote by x^1 (resp. x^2). By Lemma 3.1, let $\lambda^1 \in \mathbb{R}^m$ (resp. $\lambda^2 \in \mathbb{R}^m$) be the Kuhn-Tucker multiplier associated with x^1 (resp. x^2), i.e., such that (x^h, λ^h) $(h = 1, 2)$ satisfies the first order necessary condition at α^h:

$$-\nabla f(x^h, \alpha^h) = Dg(x^h, \alpha^h)*\lambda^h,$$

$$g(x^h, \alpha^h) \in Q \quad \text{and} \quad \lambda^h \in N_Q(g(x^h, \alpha^h)).$$

To end the proof of the theorem, it suffices to show that $\|x^2 - x^1\| \leq Kd(\alpha^2, \alpha^1)$ and $\|\lambda^2 - \lambda^1\| \leq Kd(\alpha^2, \alpha^1)$, where K is a positive real number independent of the choice of α^1, α^2 in V'.

From the first part of the first order necessary condition and Lemma 3.5, for $h = 1, 2$, the matrix $Dg(x^h, \alpha^h) \circ Dg(x^h, \alpha^h)^*$ is invertible and one deduces that:

$$\lambda^h = -[Dg(x^h, \alpha^h) \circ Dg(x^h, \alpha^h)^*]^{-1} Dg(x^h, \alpha^h) \nabla f(x^h, \alpha^h)(=\varphi(x^h, \alpha^h)),$$

$$(3.3) \qquad \|\lambda^2 - \lambda^1\| \leq k_3[\|x^2 - x^1\| + d(\alpha^2, \alpha^1)],$$

$$(3.4) \qquad \|\lambda_I^h - \lambda_I^0\| \leq \lambda^h - \lambda^0\| \leq \varepsilon < \delta.$$

From (3.4) and from Lemma 3.2(b), one deduces that:

$$(3.5) \qquad \begin{aligned} &[\|\lambda_I^0\| + \varepsilon]^{-1} \cdot \langle \lambda^2 - \lambda^1, g(x^2, \alpha^2) - g(x^1, \alpha^1) \rangle \\ &\geq -\rho[\|Dg_I(x^0, \alpha^0)\| + \varepsilon]^2 \|x^2 - x^1\|^2 \\ &\quad -2k_2\|x^2 - x^1\|d(\alpha^2, \alpha^1) - 2k_2 d(\alpha^2, \alpha^1)^2. \end{aligned}$$

Furthermore,

$$(3.6) \qquad \langle \lambda^2 - \lambda^1, g(x^2, \alpha^2) - g(x^1, \alpha^1) \rangle = A + B + C,$$

where

$$A = \langle \lambda^2 - \lambda^1, g(x^2, \alpha^2) - g(x^2, \alpha^1) \rangle,$$

$$B = \langle -\lambda^2, g(x^1, \alpha^1) - g(x^2, \alpha^1) - g(x^1, \alpha^2) + g(x^2, \alpha^2) \rangle,$$

$$C = \langle -\lambda^1, g(x^2, \alpha^1) - g(x^1, \alpha^1) \rangle + \langle -\lambda^2, g(x^1, \alpha^2) - g(x^2, \alpha^2) \rangle,$$

and we consider successively each one of the three terms. From (3.3), using the Cauchy-Schwarz inequality and the fact that g is k-Lipschitzian on $U' \times V'$, one gets:

$$(3.7) \qquad A \leq k\, k_3[\|x^2 - x^1\| + d(\alpha^2, \alpha^1)] \cdot d(\alpha^2, \alpha^1).$$

By (3.4) the multiplier λ^2 is bounded by $\|\lambda^0\| + \varepsilon$; hence, from Taylor's theorem, using the Cauchy-Schwarz inequality, one gets:

$$B \leqq [\|\lambda^0\| + \varepsilon] \|x^2 - x^1\| \left\| \int_0^1 Dg(x^2 + t(x^1 - x^2), \alpha^1) \, dt - \int_0^1 Dg(x^1 + u(x^2 - x^1), \alpha^2) \, du \right\|.$$

Upon performing the change of variable $t = 1 - u$ in the second integral, and using the fact that the derivative $Dg(\cdot, \cdot)$ is k-Lipschitzian on $U' \times V'$, one gets:

$$B \leqq [\|\lambda^0\| + \varepsilon] \|x^2 - x^1\| \cdot \int_0^1 \|Dg(x^2 + t(x^1 - x^2), \alpha^1) - Dg(x^2 + t(x^1 - x^2), \alpha^2)\| \, dt,$$

(3.8)
$$B \leqq k[\|\lambda^0\| + \varepsilon] \|x^2 - x^1\| d(\alpha^2, \alpha^1).$$

From Taylor's theorem, one has:

$$C = \langle -\lambda^1, Dg(x^1, \alpha^1)(x^2 - x^1)\rangle + \langle -\lambda^2, Dg(x^2, \alpha^2)(x^1 - x^2)\rangle$$

$$- \left\langle \left[\int_0^1 (1-t) \sum_{i=1}^m \lambda_i^1 D^2 g_i(x^1 + t(x^2 - x^1), \alpha^1) \, dt \right] (x^2 - x^1), (x^2 - x^1) \right\rangle$$

$$- \left\langle \left[\int_0^1 (1-t) \sum_{i=1}^m \lambda_i^2 D^2 g_i(x^2 + t(x^1 - x^2), \alpha^2) \, dt \right] (x^2 - x^1), (x^2 - x^1) \right\rangle.$$

From the first order necessary condition, Taylor's theorem, and using the fact that $\nabla f(\cdot, \cdot)$ is k-Lipschitzian on $U' \times V'$, one gets:

$$\langle -\lambda^1, Dg(x^1, \alpha^1)(x^2 - x^1)\rangle + \langle -\lambda^2, Dg(x^2, \alpha^2)(x^1 - x^2)\rangle$$

$$= -\langle \nabla f(x^2, \alpha^2) - \nabla f(x^1, \alpha^1), x^2 - x^1 \rangle$$

$$= -\langle \nabla f(x^2, \alpha^2) - \nabla f(x^2, \alpha^1), x^2 - x^1 \rangle - \langle \nabla f(x^2, \alpha^1) - \nabla f(x^1, \alpha^1), x^2 - x^1 \rangle$$

$$\leqq kd(\alpha^2, \alpha^1)\|x^2 - x^1\|$$

$$- \left\langle \left[\int_0^1 D^2 f(x^1 + t(x^2 - x^1), \alpha^1) \, dt \right] (x^2 - x^1), (x^2 - x^1) \right\rangle.$$

One easily gets (by performing the change of variable $t = 1 - u$) that:

$$\int_0^1 t D^2 f(x^1 + t(x^2 - x^1), \alpha^1) \, dt = \int_0^1 (1-u) D^2 f(x^2 + u(x^1 - x^2), \alpha^1) \, du.$$

Hence

$$\int_0^1 D^2 f(x^1 + t(x^2 - x^1), \alpha^1) \, dt = \int_0^1 (1-t) D^2 f(x^1 + t(x^2 - x^1), \alpha^1) \, dt$$

$$+ \int_0^1 (1-u) D^2 f(x^2 + u(x^1 - x^2), \alpha^1) \, du.$$

From (3.4), $\lambda^h \in \bar{B}(\lambda^0, \delta)$ ($h = 1, 2$). Hence, from Lemma 3.2(a) and the above equalities and inequalities

$$C \leqq kd(\alpha^2, \alpha^1)\|x^2 - x^1\|$$

(3.9)
$$- 2 \int_0^1 (1-t)[c'\|x^2 - x^1\|^2 - k_1\|x^2 - x^1\|d(\alpha^2, \alpha^1) - k_1 d(\alpha^2, \alpha^1)^2] \, dt,$$

$$C \leqq -c'\|x^2 - x^1\|^2 + (k + k_1)\|x^2 - x^1\|d(\alpha^2, \alpha^1) + k_1 d(\alpha^2, \alpha^1)^2.$$

Let $b = c' - \rho[\|\lambda_I^0\| + \varepsilon][\|Dg_I(x^0, \alpha^0)\| + \varepsilon]^2$; then, from (3.2) one has $b > 0$. From (3.5), (3.6), (3.7), (3.8), (3.9), there exists a positive real number k' such that:

$$b\|x^2 - x^1\|^2 \le k'd(\alpha^2, \alpha^1)[\|x^2 - x^1\| + d(\alpha^2, \alpha^1)].$$

Let $K = \max\{2, (2k')/b\}$. Then, from the above inequality, one easily deduces that $\|x^2 - x^1\| \le Kd(\alpha^2, \alpha^1)$, and, from (3.3), $\|\lambda^2 - \lambda^1\| \le (k_3 + Kk_3)d(\alpha^2, \alpha^1)$. This ends the proof of the theorem.

We now give the proof of Corollary 2.3.

Proof of Corollary 2.3. Let $(x^0, \alpha^0, \lambda^0)$ satisfy the assumptions of Corollary 2.3 and let:

$$I_+(x^0, \alpha^0) = \{i \in I(x^0, \alpha^0) | \lambda_i^0 > 0\}, \qquad I_0(x^0, \alpha^0) = \{i \in I(x^0, \alpha^0) | \lambda_i^0 = 0\},$$

$$J(x^0, \alpha^0) = \{i \in \{m_1 + 1, \cdots, m\} | g_i(x^0, \alpha^0) < 0\}.$$

For $\alpha \in P$, we consider the following problem:

$$\tilde{S}(\alpha) \qquad \begin{aligned} &\text{minimize } f(x, \alpha) \\ &\text{subject to } g_i(x, \alpha) = 0, \; i \in \{1, \cdots, m_1\} \cup I_+(x^0, \alpha^0), \\ &\qquad g_i(x, \alpha) \le 0, \; i \in I_0(x^0, \alpha^0), \\ &\qquad x \in U. \end{aligned}$$

Clearly $\tilde{S}(\alpha)$ is a problem of type $P(\alpha)$ with $Q = \{0\}^p \times (-\mathbb{R}_+)^q$, where $p = \text{card } I_+(x^0, \alpha^0) + m_1$ and $q = \text{card } I_0(x^0, \alpha^0)$. If we let $\tilde{\lambda}^0$ be the vector in $\mathbb{R}^{p+q}$ with coordinates $\tilde{\lambda}_i^0 = \lambda_i^0$, for $i \in \{1, \cdots, m_1\} \cup I(x^0, \alpha^0)$, then $(x^0, \alpha^0, \tilde{\lambda}^0)$ satisfies the assumptions of Theorem 2.1. Furthermore, $(x^0, \alpha^0, \tilde{\lambda}^0)$ satisfies the necessary condition (2.2) associated with $\tilde{S}(\alpha^0)$, since $(x^0, \alpha^0, \lambda^0)$ satisfies the necessary condition (2.2) associated with $S(\alpha^0)$. Consequently, from Theorem 2.1, there exist open neighborhoods U_1 of x^0 in U, V_1 of α^0 in P and Lipschitzian mappings $x(\cdot): V_1 \to U_1$ and $\tilde{\lambda}(\cdot): V_1 \to \mathbb{R}^{p+q}$ such that $x(\alpha^0) = x^0$, $\tilde{\lambda}(\alpha^0) = \tilde{\lambda}^0$ and, for all $\alpha \in V_1$, the pair $(x(\alpha), \tilde{\lambda}(\alpha))$ satisfies the necessary condition (2.2) associated with problem $\tilde{S}(\alpha)$. We let now $\lambda(\cdot): V_1 \to \mathbb{R}^m$ be the mapping defined by $\lambda_i(\alpha) = \tilde{\lambda}_i(\alpha)$ for $i \in \{1, \cdots, m_1\} \cup I(x^0, \alpha^0)$ and $\lambda_i(\alpha) = 0$ for $i \in J(x^0, \alpha^0)$. Since, for all $i \in I_+(x^0, \alpha^0)$, $\lambda_i(\alpha^0) = \lambda_i^0 > 0$ and, for all $i \in J(x^0, \alpha^0)$, $g_i(x^0, \alpha^0) = g_i(x(\alpha^0), \alpha^0) < 0$, from the continuity of the mappings $\lambda(\cdot), x(\cdot)$ and $g_i(\cdot, \cdot)$, there exists an open neighborhood V_2 of α^0 in V_1 such that, for all α in V_2, for all $i \in I_+(x^0, \alpha^0)$, $\lambda_i(\alpha) > 0$ and, for all $i \in J(x^0, \alpha^0)$, $g_i(x(\alpha), \alpha) < 0$. Consequently, one easily checks that, for all $\alpha \in V_2$, $(x(\alpha), \lambda(\alpha))$ satisfies the necessary condition (2.2) associated with problem $S(\alpha)$.

It now remains to show that there exist neighborhoods V' of α^0 in V_2 and U' of x^0 in U_1 such that, for all $\alpha \in V'$ and all $x \in U'$, $x \ne x(\alpha)$, satisfying $g_i(x, \alpha) = 0$ for $i \in \{1, \cdots, m_1\}$ and, $g_i(x, \alpha) \le 0$, for $i \in \{m_1 + 1, \cdots, m\}$, then one has $f(x, \alpha) > f(x(\alpha), \alpha)$. We prove this assertion by contraposition. Assume that there exist sequences $\{x^k\} \subset U'$ and $\{\alpha^k\} \subset V'$ such that $\{x^k\} \to x^0, \{\alpha^k\} \to \alpha^0$ and, for all k, $x^k \ne x(\alpha^k), g_i(x^k, \alpha^k) = 0$ for $i \in \{1, \cdots, m_1\}$, $g_i(x^k, \alpha^k) \le 0$ for $i \in \{m_1 + 1, \cdots, m\}$ and $f(x^k, \alpha^k) \le f(x(\alpha^k), \alpha^k)$. Without any loss of generality, one can assume that $(x^k - x(\alpha^k))/\|x^k - x(\alpha^k)\|$ converges to some element h in $\mathbb{R}^n$ such that $\|h\| = 1$.

From the mean value theorem and the continuity of $\nabla f(\cdot, \cdot)$ and $\nabla g_i(\cdot, \cdot)$ at (x^0, α^0) one has, for $i = 1, \cdots, m$,

$$\langle \nabla f(x^0, \alpha^0), h \rangle = \lim_{k \to \infty} [f(x^k, \alpha^k) - f(x(\alpha^k), \alpha^k)]/\|x^k - x(\alpha^k)\|,$$

(3.10)

$$\langle \nabla g_i(x^0, \alpha^0), h \rangle = \lim_{k \to \infty} [g_i(x^k, \alpha^k) - g_i(x(\alpha^k), \alpha^k)]/\|x^k - x(\alpha^k)\|.$$

From the very definition of the sequences $\{x^k\}$ and $\{\alpha^k\}$ and recalling that from the first part of the proof, $g_i(x(\alpha^k), \alpha^k) = 0$, $i = \{1, \cdots, m_1\} \cup I_+(x^0, \alpha^0)$, one deduces from (3.10) that

$$
\begin{aligned}
&\langle \nabla f(x^0, \alpha^0), h \rangle \leqq 0, \\
(3.11)\quad &\langle \nabla g_i(x^0, \alpha^0), h \rangle = 0, \qquad i = 1, \cdots, m_1, \\
&\langle \nabla g_i(x^0, \alpha^0), h \rangle \leqq 0, \qquad i \in I_+(x^0, \alpha^0).
\end{aligned}
$$

Since (x^0, λ^0) satisfies the necessary condition (2.2) associated with $S(\alpha^0)$, one gets

$$
(3.12)\qquad \langle \nabla f(x^0, \alpha^0), h \rangle + \sum_{i \in \{1, \cdots, m_1\} \cup I_+(x^0, \alpha^0)} \lambda_i^0 \langle \nabla g_i(x^0, \alpha^0), h \rangle = 0.
$$

From (3.11) and (3.12), since $\lambda_i^0 > 0$, for $i \in I_+(x^0, \alpha^0)$, it follows that the inequalities in (3.11) are in fact equalities. Hence, by (C.2) one gets

$$
(3.13)\qquad \left\langle D^2 f(x^0, \alpha^0) h + \sum_{i=1}^m \lambda_i^0 D^2 g_i(x^0, \alpha^0) h, h \right\rangle > 0.
$$

We end the proof of the corollary by contradicting (3.13). Denote $L_k(x) = f(x, \alpha^k) + \sum_{i=1}^m \lambda_i(\alpha^k) g_i(x, \alpha^k)$. From Taylor's theorem and from the continuity of $D^2 f(\cdot, \cdot)$ and $D^2 g_i(\cdot, \cdot)$ at (x^0, α^0):

$$
\frac{1}{2} \left\langle D^2 f(x^0, \alpha^0) h + \sum_{i=1}^m \lambda_i^0 D^2 g_i(x^0, \alpha^0) h, h \right\rangle
$$

$$
= \lim_{k \to \infty} \int_0^1 (1-t) \left\langle D^2 L_k[x(\alpha^k) + t(x^k - x(\alpha^k))] \frac{(x^k - x(\alpha^k))}{\|x^k - x(\alpha^k)\|}, \frac{x^k - x(\alpha^k)}{\|x^k - x(\alpha^k)\|} \right\rangle dt
$$

$$
= \lim_{k \to \infty} [L_k(x^k) - L_k(x(\alpha^k)) - \langle \nabla L_k(x(\alpha^k)), x^k - x(\alpha^k) \rangle] / \|x^k - x(\alpha^k)\|^2,
$$

where

$$
h = \lim \frac{x^k - x(\alpha^k)}{\|x^k - x(\alpha^k)\|}.
$$

Since $(x(\alpha^k), \lambda(\alpha^k))$ satisfies the necessary conditions (2.2) associated with $S(\alpha^k)$, then $\nabla L_k(x(\alpha^k)) = 0$, $\lambda_i(\alpha^k) g_i(x(\alpha^k), \alpha^k) = 0$, for all $i \in \{1, \cdots, m\}$ and $\lambda_i(\alpha^k) \geqq 0$, for $i \in \{m_1 + 1, \cdots, m\}$. Consequently, $L_k(x(\alpha^k)) = f(x(\alpha^k))$ and $L_k(x^k) = f(x^k) + \sum_{i=1}^m \lambda_i(\alpha^k) g_i(x^k, \alpha^k) \leqq f(x^k)$. Hence, for all k,

$$
L_k(x^k) - L_k(x(\alpha^k)) - \langle \nabla L_k(x(\alpha^k)), x^k - x(\alpha^k) \rangle \leqq 0.
$$

Dividing the above inequality by $\|x^k - x(\alpha^k)\|^2$, passing to the limit, when $k \to \infty$, from the above equalities one deduces that:

$$
\left\langle D^2 f(x^0, \alpha^0) h + \sum_{i=1}^m \lambda_i^0 D^2 g_i(x^0, \alpha^0) h, h \right\rangle \leqq 0
$$

which contradicts (3.13). This ends the proof of the corollary.

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