Simplified realization of two-qubit quantum phase gate with four-level systems in cavity QED

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We propose a method for realizing two-qubit quantum phase gate with 4-level systems in cavity QED. In this proposal, the two logical states of a qubit are represented by the two lowest levels of each system, and two intermediate levels of each system are utilized to facilitate coherent control and manipulation of quantum states of the qubits. The present method does not involve cavity-photon population during the operation. In addition, we show that the gate can be achieved using only two-step operations.

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Recently, a number of schemes have been proposed for realizing quantum computation with up to 6-level systems based on cavity QED or trap assistance [1–10]. These schemes play an important role in building up quantum computers. In this note, we wish to present an alternative method for achieving a two-qubit quantum phase gate with two 4-level systems in cavity QED (a similar model was previously considered in Refs. [9]).

Consider two individual systems I and II each having four levels \(|g\rangle, |g'\rangle, |a\rangle, \text{ and } |b\rangle\) with energy eigenvalues \(E_g\), \(E_{g'}\), \(E_a\), \text{ and } \(E_b\), respectively (Fig. 1). The transition between the two lowest levels \(|g\rangle\) and \(|g'\rangle\) is assumed to be forbidden or very weak. Suppose that the cavity mode is coupled to the \(|a\rangle \rightarrow |b\rangle\) transition of each system and that pulse I (II) is coupled to the \(|g'\rangle \rightarrow |a\rangle\) transition of system I (II) (Fig. 1). The Hamiltonian for the whole system in the Schrödinger picture can then be written as

\[
H = \sum_{l=I,II} \sum_k E_l \sigma_{bl,l} + \sum_k \hbar \omega_c c^{\dagger} c + \sum_{l=I,II} h(\mu c^{\dagger} \sigma_{abl} + \text{H.c.}) + h(\Omega_l e^{i\delta_l} \sigma_{g'bl,l} + \Omega_{bl,l} e^{i\delta_{bl,l}} + \text{H.c.}),
\]

where subscripts I and II represent systems I and II as well as pulses I and II, \(c^{\dagger}\) and \(c\) are the photon creation and annihilation operators of the cavity mode with frequency \(\omega_c\); \(\mu\) is the coupling constant between the cavity mode and the \(|a\rangle \rightarrow |b\rangle\) transition of each system; \(\Omega_l\) is the Rabi frequency of pulse I for the two levels \(|g'\rangle\) and \(|b\rangle\) of system I, while \(\Omega_{bl,l}\) is the Rabi frequency of pulse II for the two levels \(|g'\rangle\) and \(|a\rangle\) of system II; \(\omega_{bl,l}\) and \(\omega_{bl}\) are carrier frequencies of the two pulses I and II, respectively; \(\sigma_{II}=|k\rangle\langle k|=|g,g',a,b\rangle,|g'\rangle\langle g'|\), and \(\sigma_{II}^{\dagger}=|g\rangle\langle g'|\), and \(\sigma_{II}^{\dagger}=|g\rangle\langle g'|\).

Suppose that the cavity mode is largely detuned from the \(|a\rangle \rightarrow |b\rangle\) transition of system I and that pulse II is largely detuned from the \(|g'\rangle \rightarrow |a\rangle\) transition of system II, i.e., \(\Delta_I=\omega_{g'g} - \omega_a \gg \mu\), and pulse I is largely detuned from the \(|g'\rangle \rightarrow |b\rangle\) transition of system I, i.e., \(\Delta_{II}=\omega_{g'g} - \omega_b \gg \Omega_c\), while pulse II is largely detuned from the \(|g'\rangle \rightarrow |a\rangle\) transition of system II, i.e., \(\Delta_{II}=\omega_{g'g} - \omega_{g'g} \gg \Omega_{bl,l}\), where \(\omega_{g'g}=(E_{g'}-E_g)/\hbar, \omega_{g'g}=(E_{g'}-E_g)/\hbar, \text{ and } \omega_{g'g}=(E_{g'}-E_g)/\hbar\) (Fig. 1). Under this condition, the level \(|b\rangle\) for system I and the level \(|a\rangle\) for system II can be adiabatically eliminated [11]. Thus the effective Hamiltonian in the interaction picture can be written as

\[
H_1 = -\hbar \Omega_I^2 \Delta_1^{-1} \sigma_{g'g'g'} + \hbar \Omega_{II}^2 \Delta_{II}^{-1} \sigma_{bbl} + \hbar \Delta_k \mu \sigma_{g'ab} + \hbar(\chi_I e^{i\delta_I} \sigma_{g'bl} + \text{H.c.}) + \hbar(\chi_{II} e^{i\delta_{II}} \sigma_{bbl} + \text{H.c.}),
\]

where \(\delta_I=\Delta_I, \delta_{II}=\Delta_{II}, \chi_I=(\Omega_I/2)(1/\Delta_I+1/\Delta_{II}), \text{ and } \chi_{II}=(\Omega_{II}/2)(1/\Delta_I+1/\Delta_{II})\).

The first two terms in Eq. (2) are ac-Stark shifts of the levels \(|g'\rangle\) and \(|a\rangle\) for system I induced by pulse I and the cavity mode, respectively; while the second two terms in Eq. (2) are ac-Stark shifts of the
levels $|g'\rangle$ and $|b\rangle$ for system II induced by pulse II and the cavity mode, respectively. Finally, the last two terms in Eq. (2) are the familiar Jaynes-Cummings interaction, describing the Raman coupling of the two levels $|g'\rangle$ and $|a\rangle$ for system I as well as the Raman coupling of the two levels $|g'\rangle$ and $|b\rangle$ for system II, respectively.

When $\Delta_1=\Delta_2=\Delta_c \gg \mu^2/\Delta_c , \Omega_{II}^2/\Delta_c , \chi_{II}$; and $\Delta_{II}=\Delta_{II}-\Delta_c \gg \mu^2/\Delta_c , \Omega_{II}^2/\Delta_{II}-\chi_{II}$, there is no energy exchange between the systems (I, II) and the cavity mode. In the following we set $\delta_{II}^=\delta_{II}$ by having $\Delta_{II}^=\Delta_{II}$ which can be readily realized via adjusting the frequencies of the two pulses (Fig. 1). Thus, the energy conserving transitions are between $|ag'I\rangle$ and $|b\rangle$, mediated by $|ag'n+1\rangle$ and $|g'bn-1\rangle$ where the first (second) letter denotes the states $|g'\rangle$ and $|a\rangle$ ($|g'\rangle$ and $|b\rangle$) of system I (system II), and $n$ is the photon number of the cavity mode (Fig. 2). The effective Hamiltonian is then given by [1,2]

$$H_{eff}=-\hbar \frac{\Omega_{II}^2}{\Delta_{II}} \sigma'_{g'g''}^I - \hbar \frac{\mu^2}{\Delta_c} c^I \sigma_{aal}^I + \hbar \frac{\Omega_{II}^2}{\Delta_{II}} \sigma'_{g'g''}^II + \hbar \frac{\mu^2}{\Delta_c} c^I \sigma_{aal}^I + \hbar \frac{\Omega_{II}^2}{\Delta_{II}} \sigma'_{g'g''}^II - \hbar \frac{\mu^2}{\Delta_c} c^I \sigma_{aal}^I + \hbar \lambda (|g'\rangle \langle ab| + |ab\rangle \langle g'|).$$

where $\lambda=\chi_{II}/\delta$, the last two terms in the second line and the two terms in the third line describe the photon-number-dependent Stark shifts induced by the off-resonant Raman coupling, and the last two terms describe the “dipole” coupling between the two systems (I, II) mediated by the cavity mode and the classical pulses. If the cavity is initially in the vacuum state, then the effective Hamiltonian reduces to

$$H_{eff}=-\hbar \left( \frac{\Omega_{II}^2}{\Delta_{II}} \chi_{II} \delta + \frac{\mu^2}{\Delta_c} c^I \sigma'_{g'g''}^I + \hbar \frac{\Omega_{II}^2}{\Delta_{II}} \sigma'_{g'g''}^II - \hbar \frac{\mu^2}{\Delta_c} c^I \sigma_{aal}^I \right) + \hbar \lambda (|g'\rangle \langle ab| + |ab\rangle \langle g'|).$$

Suppose that the two lowest levels $|g\rangle$ and $|g'\rangle$ of each system represent two logical states of a qubit. The time evolution of four logical states for two qubits, under the Hamiltonian (4), are given by

$$|g\rangle|g\rangle_{II} \rightarrow |g\rangle|g\rangle_{II},$$

$$|g\rangle|g'\rangle_{II} \rightarrow e^{-i\epsilon_{aI}t}|g\rangle|g'\rangle_{II},$$

$$|g'\rangle|g\rangle_{II} \rightarrow e^{-i\epsilon_{aII}t}|g'\rangle|g\rangle_{II},$$

$$|g'\rangle|g'\rangle_{II} \rightarrow e^{-i(\epsilon_{aI}+\epsilon_{aII})t/2} \left[ (\cos \lambda^2 + \eta^2) |g'\rangle|g'\rangle_{II} + i \frac{\eta}{\sqrt{\lambda^2 + \eta^2}} \sin \lambda^2 |g'\rangle|g'\rangle_{II} \right] + i \frac{\lambda}{\sqrt{\lambda^2 + \eta^2}} \sin \lambda^2 |g'\rangle|g'\rangle_{II} + i \frac{\lambda}{\sqrt{\lambda^2 + \eta^2}} \sin \lambda^2 |g'\rangle|g'\rangle_{II},$$

where $\epsilon_a=-\Omega_{II}^2/\Delta_{II} - \chi_{II}^2/\delta$, $\epsilon_{aI}=-\Omega_{II}^2/\Delta_{II}$, $\epsilon_{aII}=-\chi_{II}^2/\delta$, and $\eta=(\epsilon_{aI}-\epsilon_{aII})/2$. In the case $\epsilon_{aI}+\epsilon_{aII}=\Omega_{II}^2/\Delta_{II}$, i.e.,

$$\Omega_{II}^2 + \chi_{II}^2/\delta = \Omega_{II}^2 + \chi_{II}^2/\delta,$$

and by setting $\lambda=\pi$, we obtain

$$|g\rangle|g\rangle_{II} \rightarrow |g\rangle|g\rangle_{II},$$

$$|g\rangle|g'\rangle_{II} \rightarrow e^{-i\epsilon_{aI}t}|g\rangle|g'\rangle_{II},$$

$$|g'\rangle|g\rangle_{II} \rightarrow e^{-i\epsilon_{aII}t}|g'\rangle|g\rangle_{II},$$

$$|g'\rangle|g'\rangle_{II} \rightarrow -e^{-i(\epsilon_{aI}+\epsilon_{aII})t/2} |g'\rangle|g'\rangle_{II}.$$

Then we perform the following one-qubit operations:

$$|g'\rangle \rightarrow e^{i\epsilon_{aI}t}|g'\rangle_{II},$$

$$|g'\rangle \rightarrow e^{i\epsilon_{aII}t}|g'\rangle_{II},$$

$$|g'\rangle \rightarrow -e^{-i(\epsilon_{aI}+\epsilon_{aII})t/2} |g'\rangle.$$
metastable \(|g^\prime\rangle\) and \(|a\rangle\) states. And in many solid-state systems that do not have inversion symmetry the level energy diagram would be generic. For solid-state systems such as quantum dots and superconducting quantum interference devices (SQUIDs), the level structure is straightforward to implement by changing external control parameters (e.g., magnetic flux \(\Phi_0\) in the case of SQUID qubits) [7,12]. For instance, an rf SQUID with \(C=90 \text{ fF}, L=100 \text{ pH}, \beta_i=1.12,\) and \(\Phi_0=0.4995 \Phi_0\) [13] has the desired level structure as depicted in Fig. 3.

As described above, the controlled phase operation is implemented by choosing \(n=0\) and carrying out a \(2\pi\) pulse on the \(|g^\prime\rangle|g^\prime\rangle\) and \(|a\rangle|b\rangle\) transition. During this operation, one excites the states \(|a\rangle\) and \(|b\rangle\) with a probability \(\sin^2 \lambda t\) at time \(t\). Thus, decoherence rate would be

\[
\gamma_D = \frac{1}{T} \int_0^T \sin^2 \lambda t (\gamma_a + \gamma_b) dt = \frac{1}{2} (\gamma_a + \gamma_b),
\]

where \(T = \pi/\lambda; \gamma_a\) and \(\gamma_b\) are the decay rates of the levels \(|a\rangle\) and \(|b\rangle\), respectively. We now turn to the experimental matters. First, the typical \(\pi/\lambda\) required for the system-cavity interaction and the decoherence time \(\gamma_D^{-1}\) need to meet \(\pi/\lambda \ll \gamma_D^{-1}\), based on which one can easily find that the following relationship between the decoherence rate \(\gamma_D\) and the coupling strength \(\mu\)

\[
\mu^2 \gg \frac{4 \pi \delta \lambda (\Delta_0 \Delta_{II}) \gamma_D}{\Omega_0 \Omega_{II} (\Delta_c + \Delta_D)(\Delta_c + \Delta_{II})}
\]

should be satisfied. Second, note that the Hamiltonian (2) applies conditional to the adiabatic elimination of the level \(|b\rangle\) for system I and the level \(|a\rangle\) for system II, thus the occupation probability \(P_b\) of the level \(|b\rangle\) for system I and the occupation probability \(P_a\) of the level \(|a\rangle\) for system II, which are approximately given by

\[
P_b \approx \frac{1}{2} \left( \frac{4 \Omega_0^2}{4 \Omega_0^2 + \Delta_D^2} + \frac{4 \mu^2}{4 \mu^2 + \Delta_D^2} \right),
\]

\[
P_a \approx \frac{1}{2} \left( \frac{4 \Omega_{II}^2}{4 \Omega_{II}^2 + \Delta_{II}^2} + \frac{4 \mu^2}{4 \mu^2 + \Delta_{II}^2} \right)
\]

(for \(\Omega_0, \Omega_{II}, \text{ and } \mu\) of similar magnitude), need to be negligibly small in order to reduce the gate error. Lastly, the photon lifetime is given by \(\kappa^{-1} = Q_c/\omega_c\) (\(Q_c\) is the quality factor of the cavity) and the cavity has a probability

\[
P_c \approx \frac{1}{2} \left( \frac{4 \chi_c^2}{4 \chi_c^2 + \delta^2} + \frac{4 \chi_{II}^2}{4 \chi_{II}^2 + \delta^2} \right)
\]

of being excited during the operation, thus the effective decay time of the cavity is \(\kappa^{-1} P_c^{-1}\), which needs to be larger than the system-cavity interaction time \(\pi/\lambda\).

As a quantitative example of this technique, consider a SQUID with the parameters given above and with junction’s damping resistance \(R \approx 1 \text{ GHz}\). Note that SQUIDs with these parameters are available at the present time [14]. With this choice, the decay time of the levels \(|a\rangle\) and \(|b\rangle\) would be \(\gamma_a^{-1} \approx 100 \mu s\) and \(\gamma_b^{-1} \approx 40 \mu s\) (i.e., \(\gamma_0^{-1} \approx 57.1 \mu s\), the \(|a\rangle \rightarrow |b\rangle\) coupling matrix element is \(\phi_{ab} \approx 7.8 \times 10^{-5}\), and the \(|a\rangle \rightarrow |b\rangle\) transition frequency is \(\nu_{ab} \approx 4.9 \text{ GHz}\). Hence, we choose \(\nu_c = \omega_c/(2\pi) = 3.6 \text{ GHz}\) as the cavity-mode frequency. The SQUID-cavity coupling constant for the \(|a\rangle \rightarrow |b\rangle\) transition is given by [15] \(\mu = (1/(L+L_2)\omega_c/2\mu_0\phi_{ab}\Phi_0])B_z(t)\cdot dS\), where \(S\) is any surface bounded by the SQUID ring and \(B_z(t)\) is the magnetic component of the cavity mode in the SQUID loop. For a standing-wave cavity, one has \(B_z(t) = \mu_0/2\sqrt{V}\cos k(z)\), \(|V\), and \(z\) are the wave number, the cavity volume, and the cavity axis, respectively). For a \(10 \times 1 \times 1 \text{ mm}^2\) standing-wave cavity and a SQUID with a \(40 \times 40 \mu\text{m}^2\) loop (located at the cavity-mode antinode), a simple calculation shows \(\mu \approx 4.3 \times 10^8 \text{ s}^{-1}\), i.e., \(\approx 0.05\Delta_c\). By choosing \(\Delta_0 = 20\Omega_0, \Delta_D = 20\Omega_D,\) and \(\Omega_0 = \Omega_{II} = 1.05\mu,\) we have \(\delta \approx \mu \approx 20(\mu_0^2/\Delta_c). 20(\Omega_0^2/\Delta_{II}), 20(\Omega_{II}^2/\Delta_{II}).\) Our calculation shows that (i) the required SQUID-cavity interaction time would be \(t_{c^{-}} = \pi/\lambda \approx 2.8 \mu s\), much shorter than \(\gamma_D^{-1}\); (ii) both \(P_a\) and \(P_b\) are \(< 0.01\); (iii) \(P_c^{-1} \ll 0.01\), thus the effective decay time of the cavity is \(\kappa^{-1} P_c^{-1} \approx 44.2 \mu s\). This is realizable since a superconducting cavity with \(Q_c > 10^4\) was demonstrated by recent experiments [16]. Therefore, within the present cavity QED technique, the implementation of the proposed scheme is possible with a real physical system.

The method can be applied to any physical systems with a four-level configuration as described above. For different systems, there is no difference to the equations. However, the frequency regimes of the cavity mode are likely to be rather different in the physical realizations with different systems, e.g., optical cavities in the case of atoms while microwave cavities in the case of SQUIDs.

The method can be extended to perform logical operations on many qubits—each qubit is embodied by one of many systems described above inside a cavity, due to long-range coherent interaction between systems mediated via the cavity mode. Full parallel operations are possible (e.g., the two-
Quantum-zeno-type effects probably help to avoid the propagation of errors, due to spontaneous emission during the gate performance. If this is to be sure that there is no photon population in the cavity or satisfied. On the other hand, one can check by measurement if setting $|a\rangle$ and $|b\rangle$ during the operation. We point out that in this respect our proposal does not offer any advantage. In principle, the gate errors induced due to the decay of the levels $|a\rangle$ and $|b\rangle$ can be greatly reduced as long as the condition $\pi/\lambda \ll \gamma_0^{-1}$ is well satisfied. On the other hand, one can check by measurement to be sure that there is no photon population in the cavity or spontaneous emission during the gate performance. If this is done continuously and with sufficient efficiency, this will probably help to avoid the propagation of errors, due to quantum-zeno-type effects [4].

Coupling qubits via the cavity/trap-assisted collision without the excitation of the cavity/vibrational mode was previously reported in Refs. [1,2]. And later this idea was applied to realize a two-qubit phase gate with 3-level atoms/ions, by performing operations beyond two-qubit computational sub-space [5,6]. However, our purpose is to show that a two-qubit phase gate is achievable with 4-level systems via only two substeps and without real excitation of the cavity mode. We believe that although our scheme is restricted to the 4-level systems, it is of some interest nonetheless, depending on the systems chosen by experimentalists.

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[13] C. L. and $\beta_L$ are the SQUID’s junction capacitance, loop inductance, and potential shape parameter, respectively. $\Phi_0 = h/2e$ is the flux quantum.