Abstract

This paper presents a new answer to the old question of how to aggregate individual beliefs. We construct a model which allows agents to take arbitrage opportunities against the aggregated belief by making contingent claims against the states, and the aggregator (market maker) regulates the probability of states. When all claims from the agents are mutually covered for every realization of the state, an aggregation of individual beliefs is thus obtained. We prove the existence and uniqueness of the equilibrium aggregation, and also show that the aggregate belief lies in the convex hull of individual beliefs. This model allows us to address some important problems such as how individual agent's attitude toward risk and wealth endowment affect the outcome of the aggregation process, and whether the aggregate belief satisfies the well-known properties like equal treatment.

JEL Classification: D40, D70, D80.

Key words: Aggregation of beliefs; Risk attitudes; Market equlibrium

1 Introduction

How to aggregate individual beliefs is an old problem that has been extensively studied in the literature. Suppose there is a group of individuals (agents) each holding private beliefs about some unknown state of the world. This belief is summarized by a probability distribution $\mathbf{P}_i$ for agent i. Each agent's preferences are characterized by von Neumann and Morgenstern (VNM) expected utility theory. Assuming now there is a risk neutral observer, how would he aggregate those individual beliefs to form his own beliefs?

There has been a vast literature studying the aggregation of beliefs in the Bayesian framework. The existing work is centered around the axiomatic decision analysis, i.e., studying the existence and properties of solutions that satisfy a few important principles (axioms). The most commonly imposed axioms are variations of the so-called *consistency* (or Pareto principle). Loosely speaking, if all agents have similar beliefs over some subset of the states so does the aggregated belief. This body of literature can be roughly summarized along two lines: one is the decision theoretic framework which involves finding the aggregate preferences over individual preferences, i.e., the social welfare function under uncertainty. We refer the readers to, for example, the original work of Harsanyi (1955) and various seminal contributions such as Broome (1987, 1990, 1991) and Mongin (1994). The other line of work is

done in Bayesian statistics which is the so-called *convex pooling* of individual beliefs. One is referred to seminal studies such as Stone (1961) and Genest and Zidek (1986) for a summary of the development afterward. McConway (1981) provides an axiomatic summarization of the logical foundations of linear opinion pool. Mongin (1995) presents a good picture of the interactions of this two lines of work; Rubinstein and Fishburn (1986) nicely unify these two through axiomatic aggregation. There has been more recently work on the aggregation of beliefs, Jouini and Napp (2006 a,b, 2007) deal with aggregations of beliefs in financial markets. Koessler, Noussair and Ziegelmeyer (2012) consider aggregation of beliefs in experiments.

In this paper we attempt to provide a new answer to this old problem. The approach we have taken is not along the line of axiomatic decision analysis, but rather along the line of independent individual decision making process. We build a bridge connecting the axiomatic aggregation with the general equilibrium model by constructing a market-oriented aggregation process. In the model agents are allowed to take arbitrage opportunities against the aggregated belief by making contingent claims, and the aggregator (whom we call the market maker) regulates these arbitrage opportunities that arise from the agents by varying his probabilities assigned to each state. When the aggregation reaches an equilibrium, all the claims from all agents are mutually covered for every realization of the state, and thus an aggregation

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of individual beliefs is obtained. For this reason we name the approach in this paper the *decentralized market approach.*

Even though existing work generates beautiful results on the clarification of the logical relationships among various set of axioms, it does not provide adequate answers to some important problems. For example, because of the different focus of existing studies, they lack a suitable framework for sensitivity analysis, such as how the solution will be affected by changes in some parameters of the model. Another problem is that it is a black-box-approach. The aggregation process is not specified and individual decisions are not postulated in the model. And finally, individual incentives are not addressed in the aggregation process. Individuals are not utility maximizers, thus the individual optimization problem is assumed away.

This paper adopts the decentralized way of aggregation and incorporates individual maximization explicitly into the model. A market is introduced to function as the aggregating device. When the aggregation process reaches an equilibrium, an aggregation is obtained automatically. In our view, such a process that produces an aggregation as output different from the existence theorem (as in the axiomatic framework) because it allows us to do comparative statics. Furthermore, our market-oriented aggregation process is shown to have some well-known properties such as consistency and equal treatment. In the traditional axiomatic aggregation theory, however, all nice properties

of axiomatic aggregation are assumed as axioms, with their consistency being tested through possibility or impossibility theorems.

The rest of the paper is organized as follows. In Section 2 we set up the model with VNM expected utility and prove the existence and uniqueness of the equilibrium aggregation. Section 3 presents the properties of the aggregation. Especially we investigate how agent's attitude toward risk and income endowment affect the outcome of the aggregation process. Section 4 concludes the paper.

2   The model

Suppose we have m agents each holding their beliefs about an unknown state s which takes S possible values, $s \in \{1, 2, ..., S\}$. Let $\mathbf{P}_i$ be a probability distribution which we refer as the belief for agent i, $\mathbf{P}_i = (p_{i1}, p_{i2}, ..., p_{iS})$, and $u_i(x)$ be the von Neumann-Morgenstern expected utility function, $i \in \{1, 2, ..., m\}$.

Assumption 1. The state is unobservable ex ante, but will be observable when market clears.

Assumption 2. Agents are allowed to write claims contingent on the states. Those claims will be cleared when the state becomes known.

Assumption 3. Agents are expected utility maximizers, and each $u_i(x)$ is strictly monotone and concave, with the absolute risk aversion $-\frac{u_i''(x)}{u_i'(x)} \to 0$

as $x \rightarrow \infty$.

Given assumptions 1–3, we intend to derive the market aggregation of individual beliefs. Agents' preferences are defined in the lottery space characterized by state contingent claims $\mathbf{x}_i = (x_{i1}, x_{i2}, ..., x_{iS})$, where x_{is} is the contingent claim for agent i at state s. Agent i is endowed with a income y_i. We denote $\mathbf{Y} = (y_1, .., y_m)$ the endowment vector and $y = \sum_{i=1}^{m} y_i$ aggregate endowment. The aggregate belief is summarized by a probability measure $\mathbf{P}$ over the states, and $\mathbf{P} = (p_1, p_2, ..., p_S)$.

A fair odds line is a line consisting of lotteries with the same expected value given certain belief. When agent i holds private belief $\mathbf{P}_i$ over the states, the *private* fair odds line for agent i can be represented by

$$\mathbf{P}_i \mathbf{x}_i = y_i.$$

Each agent has his own private belief and income. In the VNM expected utility framework it is well known that agent's indifference curve (surface) is tangent to his own fair odds line (plane) $\mathbf{P}_i \mathbf{x}_i = y_i$, along the certainty line .

Since the fairness in the eyes of the aggregator may not be viewed as fair in each individual's eyes, the difference between $\mathbf{P}_i$ and $\mathbf{P}$ generates an angle between the agent's private fair odds line and the aggregator's fair odds line. This provides the agents an opportunity to arbitrage against the aggregator's fair odds line, creating a contingent claim as a function of each realization of

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the state.

This arbitrage opportunity must be fair with respect to the aggregate belief $\mathbf{P}$ itself because the aggregator cannot be expected to lose money permanently. In other words, the arbitrage opportunities provided for the agents are required to have the same expected value as the initial income endowment. This leads to the following constraint for each agent,

$$\mathbf{P}\mathbf{x}_i = y_i, \ \ i = 1, 2, ..., m.$$

If the aggregation of beliefs $\mathbf{P}$ is in equilibrium, then the agents' claims would cover each other at every state of the world. This is how the equilibrium aggregation is generated.

The following lemma characterizes the agents' arbitraging behavior against the aggregate belief $\mathbf{P}$.

Lemma 1 *The aggregate belief $\mathbf{P}$ induces each agent a unique lottery of contingent claims,*

$$\mathbf{x}_i(\mathbf{P}, \mathbf{P}_i, y_i) = \left(x_{i1}(\mathbf{P}, \mathbf{P}_i, y_i), ..., x_{iS}(\mathbf{P}, \mathbf{P}_i, y_i)\right),$$

which is the solution to the following maximization problem

$$\mathbf{x}_i(\mathbf{P}, \mathbf{P}_i, y_i) \ \ = \ \ \arg\max E_{\mathbf{P}_i} u_i\left(\mathbf{x}_i\right)$$

$$s.t. \ \ : \ \ \mathbf{P}\mathbf{x}_i = y_i.$$

When $\mathbf{P}_i = \mathbf{P}$, we have $\mathbf{x}_i(\mathbf{P}, \mathbf{P}_i, y_i) = y_i\mathbf{1} = (y_i, ..., y_i)$.

Proof. Given the concavity of $u_i(\mathbf{x}_i)$, Jensen's inequality implies that $E_{\mathbf{P}_i} u_i(\mathbf{x}_i)$ is a quasi-concave function of the state-contingent consumption, $\mathbf{x}_i = (x_{i1}, ..., x_{iS})$. From the compactness and convexity of budget constraint, we have the following problem for agent i:

$$
\max E_{\mathbf{P}_i} u_i(\mathbf{x}_i) = \sum_{s=1}^{S} p_{is} u_i(x_{is})
$$
$$
s.t. \quad : \quad \mathbf{P}\mathbf{x}_i = y_i, \quad \mathbf{x}_i \geq 0.
$$

It has a unique solution $\mathbf{x}_i(\mathbf{P}, \mathbf{P}_i, y_i) = (x_{i1}, ..., x_{iS})$, that is, each agent has a unique lottery for the expected utility maximization problem. Let λ be the Lagrangian multiplier associated with the budget constraint, the first order conditions are, for $i = 1, ..., m$,

$$
p_{is} \frac{\partial u_i(x_{is})}{\partial x_{is}} = \lambda p_s, \quad s = 1, ..., S
$$

$$
\mathbf{P}\mathbf{x}_i = y_i.
$$

Clearly when $p_{is} = p_s$, we have $\frac{\partial u_i(x_{is})}{\partial x_{is}} = \lambda$ for all s, indicating a unique solution, $(x_{i1}, ... x_{iS}) = (y_i, ..., y_i) = y_i \mathbf{1}$, where $\mathbf{1} = (1, ..., 1)$.

Since the VNM utility function is concave hence the indifference curves for each agent in the lottery space are convex, there exists a unique solution to the agent's maximization problem. ∎

The next proposition establishes the existence and uniqueness of the equilibrium aggregation of the individual beliefs.
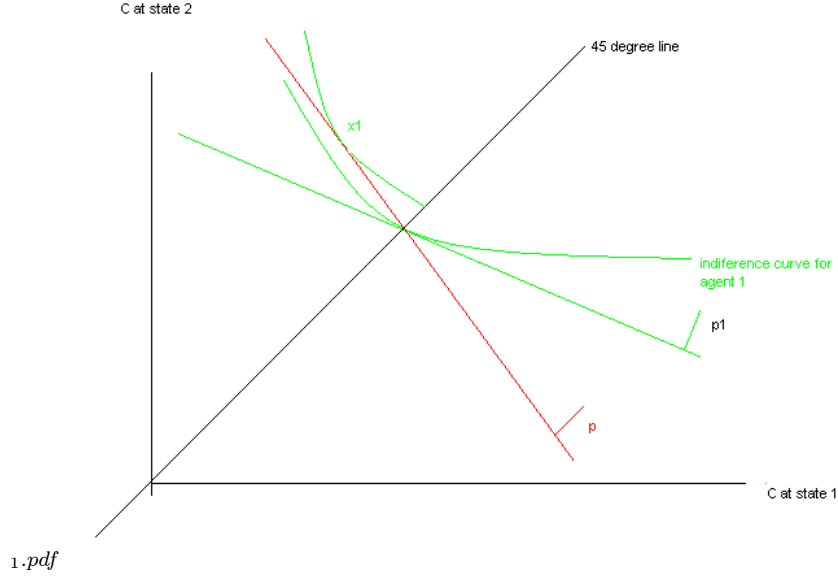
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C at state 2

45 degree line

x1

indiference curve for agent 1

p1

p

C at state 1

₁.pdf

Figure 1: Player 1's optimal choice

Proposition 1 *Given individuals' private belief profile* $(\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_m)$ *and endowment profile,* $(y_1, .., y_m)$, *There is a unique aggregation* $\mathbf{P} = A(\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_m, \ y_1, .., y_m)$, *which clear the contingent claims at every state.*

Proof. The market clearing condition is characterized by the following system of S equations,

$$\sum_{i=1}^{m} x_{is}(\mathbf{P}, \mathbf{P}_i, y_i) = \sum_{i=1}^{m} y_i, \quad s = 1, 2, ..., S.$$

We need to show that the above system has a unique solution for $\mathbf{P}$ given $\mathbf{P}_i$ and y_i, $i = 1, ..., m$. To accomplish this we introduce the following notations.

Let $\mathbf{z}(\mathbf{P}) = (z_1(\mathbf{P}), ... z_S(\mathbf{P}))$, where $z_s(\mathbf{P}) = \sum_{i=1}^{m} x_{is}(\mathbf{P}, \mathbf{P}_i, y_i) - \sum_{i=1}^{m} y_i$,

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be the excess claim under state s and aggregate belief $\mathbf{P}$. An equilibrium aggregation is reached if the excess claim at every state becomes non-positive.

It is clear that from the concavity of the utility function, all the contingent claims made by agents are continuous functions of the aggregate belief $\mathbf{P}$. Individual budget constraint implies that Walras law is satisfied, i.e., $\mathbf{P}\mathbf{z}(\mathbf{P}) = 0$. This implies the existence of some $\mathbf{P}$ such that $\mathbf{z}(\mathbf{P}) \leq 0$. It is summarized in the following lemma.

Lemma 2 *If $\mathbf{z} : \Delta^{S-1} \to R^S$ is a continuous function that satisfies Walras law $\mathbf{P}\mathbf{z}(\mathbf{P}) \equiv 0$, then there is some $\mathbf{P}$ such that $\mathbf{z}(\mathbf{P}) \leq 0$. (see for example Varian pp321).*

■

The uniqueness of the equilibrium aggregation can be obtained by simply applying a theorem in Stella Dafermos (1990) *pp396*.

Lemma 3 *(Stella Dafermos) Assume excess demand $z(P)$ is strictly monotone decreasing in P, that is, $dz \cdot dP < 0$, then there exists at most one Walrasian equilibrium price vector.*

From the first order condition of agent i:

$$p_{is}\frac{\partial u_i(x_{is})}{\partial x_{is}} = \lambda p_s, \quad s = 1, ..., S$$

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and $u"\left(x_{is}\right) < 0$, it is easily that $dx_i \cdot d\mathbf{P} < 0$, thus $dz \cdot d\mathbf{P} = \sum_{i=1}^{m} dx_i \cdot d\mathbf{P} < \mathbf{0}$, that is

$z\left(\mathbf{P}\right)$ is strictly monotone decreasing in $\mathbf{P}$, thus there will be at most one equilibrium.

Thus we have a unique aggregate belief.

The 2×2 case is illustrated in the following disequilibrium pictures. Without loss of generality we assume that $y_1 = y_2$ and the fair odds line for agent 1, $\mathbf{P}_1 \mathbf{x}_1 = y_1$, has a smaller slope than that of agent 2, $\mathbf{P}_2 \mathbf{x}_2 = y_2$. It is clear that when $\mathbf{P} = \mathbf{P}_1$, agent 1 chooses to be in autarky (self-covered), and agent 2 will choose an equilibrium at e_2 as shown in figure 2. Clearly we don't have an equilibrium. Similarly when $\mathbf{P} = \mathbf{P}_2$, agent 2 chooses to be in autarky and agent 1 would choose to be at e_1 as shown in figure 3. Again we don't have an equilibrium.

Now when $\mathbf{P}$ varies continuously from $\mathbf{P}_1$ to $\mathbf{P}_2$, the average demand must cross the 45° line in order to go from below the line to above. When it settles on the 45° line, an equilibrium aggregation of the two beliefs is reached. This is illustrated in figure 4.

The above procedure describes how to generate an aggregation of beliefs. It would be natural to inquire about the welfare properties about this aggregation procedure.
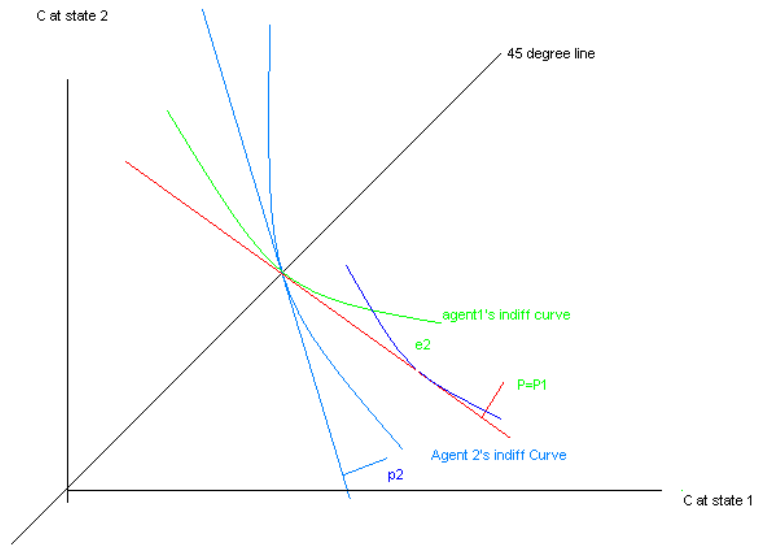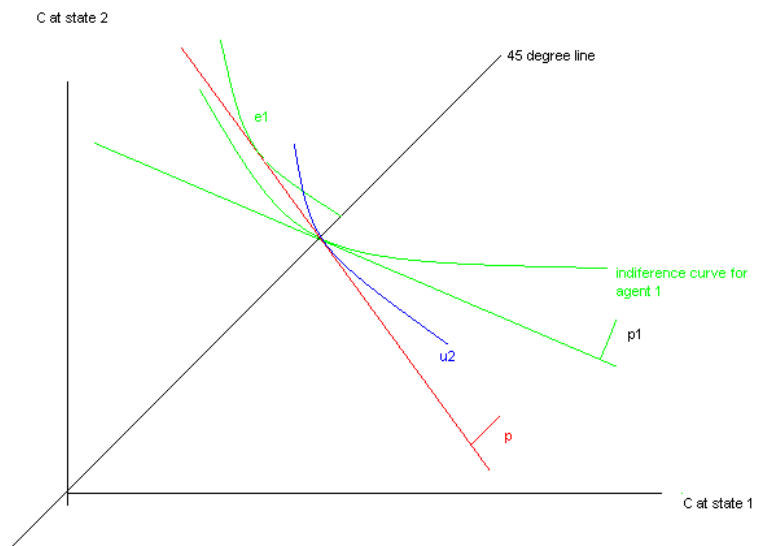
Figure 2: P=P1 is not an equilibrium
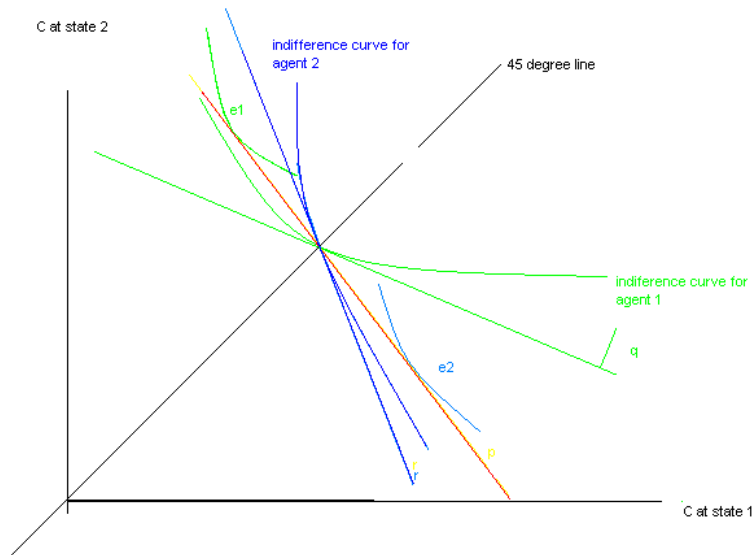

Figure 3: P=P2 is not an equilibrium

Figure 4: Equilibrium is reached

3   Comparative Statics for the Aggregation

We now characterize the properties of the aggregate belief that we derived in the previous section. Specifically, we intend to study factors that affect the aggregation outcome.

How is our aggregation related to the axiomatic aggregation? The next proposition shows that our aggregation satisfies the property of linear pool, or consistency.

Proposition 2 *The aggregation* $\mathbf{P} = A\left(\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_m, \mathbf{Y}\right)$ *lies in the convex hull of the individual beliefs thus it is a convex combination of individual*

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beliefs. There are numbers $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_m) \geq 0$ *depending on the income profile* $\mathbf{Y} = (y_1, .., y_m)$ *such that* $\sum_{i=1}^{m} \alpha_i = 1$, *and* $\mathbf{P} = \sum_{i=1}^{m} \alpha_i \mathbf{P}_i$.

Proof. Suppose the equilibrium aggregation does not lie in the convex hull of the individual beliefs, then $\mathbf{P} \in \Delta^{S-1} \setminus coh\{\mathbf{P}_1, ..., \mathbf{P}_m\}$. The separation theorem of convex sets implies that there exists a hyperplane H separating $\mathbf{P}$ from $coh\{\mathbf{P}_1, ..., \mathbf{P}_m\}$,

$$\mathbf{P} \in H^-, \text{ and } \{\mathbf{P}_1, ..., \mathbf{P}_m\} \subset H^+.$$

Given the budget set for agent i, $\mathbf{P}\mathbf{x}_i = y_i$, each agent moves from endowment $y_i\mathbf{1}$ to some $\mathbf{x}_i$ in the side of the hyperplane H where $\mathbf{P}_i$ lies. That is, $\mathbf{x}_i \in H^+$ for $i = 1, ..., m$. Let h be the normal direction of this separating hyperplane, then

$$\mathbf{x}_i \in H^+ \Leftrightarrow h \cdot (x_i - y_i\mathbf{1}) > 0, \quad i = 1, ..., n$$

Therefore we have the following by summing up above inequalities,

$$\sum_{i=1}^{m} h \cdot (x_i - y_i\mathbf{1}) = h \cdot \left(\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i\mathbf{1} \right) > 0$$

Which violates the market clearing condition:

$$\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i\mathbf{1} = \mathbf{0}$$

Thus P must lie in the convex hull of $\{P_1, P_2, ...P_m\}$.
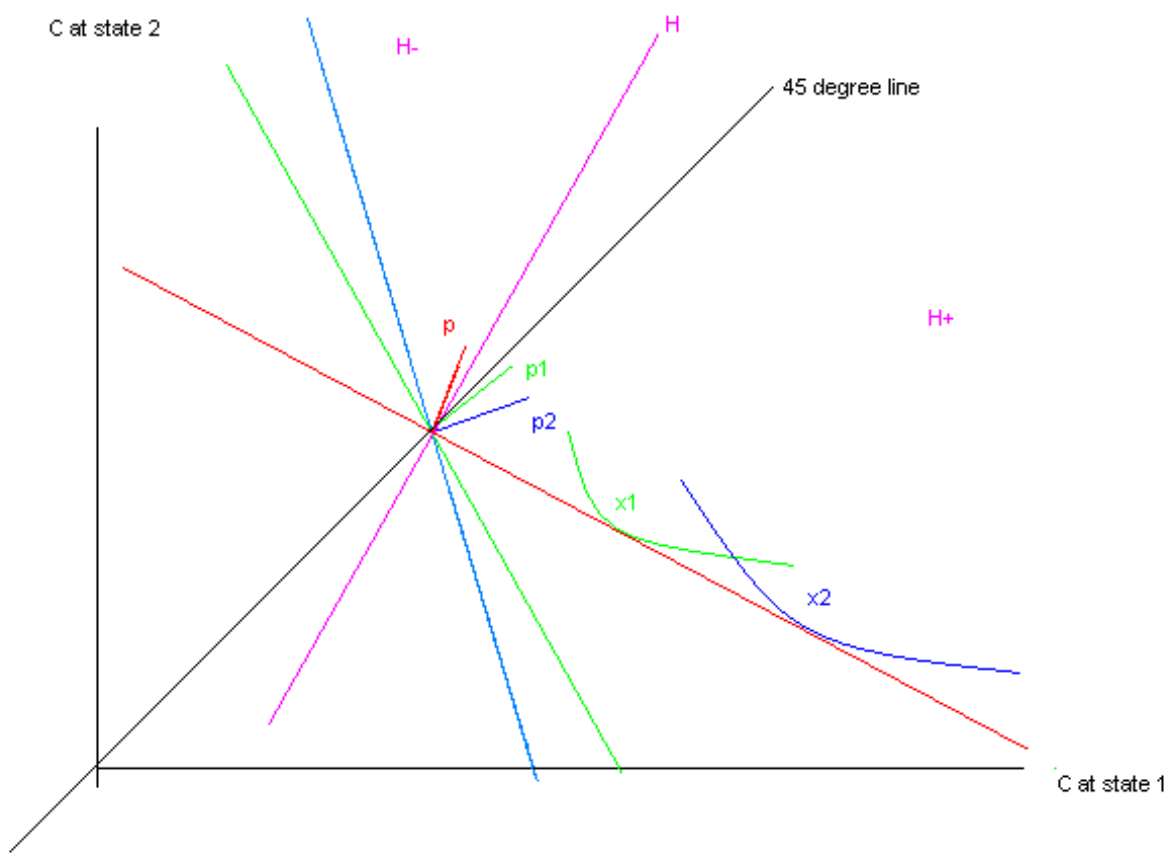
Figure 5: P Lies in the Convex Hull

The case $m = 2$ is illustrated in figure 5. ∎

The next proposition links people's attitude toward risk to the aggregation outcome. It is shown that the contribution of an individual belief to the aggregate belief depends negatively on the degree of risk aversion of that agent.

Proposition 3 *Let* $\mathbf{P} = \sum_{i=1}^{m} \alpha_i P_i$ *be the aggregate belief as defined in proposition 2. When agent $i's$ degree of risk aversion at the endowment point $r_i(y_i) \equiv -\frac{u_i''(y)}{u_i'(y)}|_{y=y_i} \to \infty$, then his weight α_i in the aggregate belief goes to 0, that is,*

$$\alpha_i \to 0, \ as \ r_i \to \infty$$

On the other hand, as the degree of risk aversion $r_i(y_i) \equiv -\frac{u_i''(y)}{u_i'(y)}|_{y=y_i} \to 0$, agent i becomes the dominator (dictator) in the aggregation of the beliefs. That is

$$\alpha_i \to 1, \ as \ r_i \to 0$$

Proof. Without loss of generality, we assume $i = m$ is the agent in consideration. It is obvious that when agent m becomes more and more risk averse, he is becoming asymptotically autarky. That is, his contingent claim $\mathbf{x}_i \to y_i \mathbf{1}$ as the agent's degree of risk aversion goes to infinity. This implies that the market clearing condition with m agents

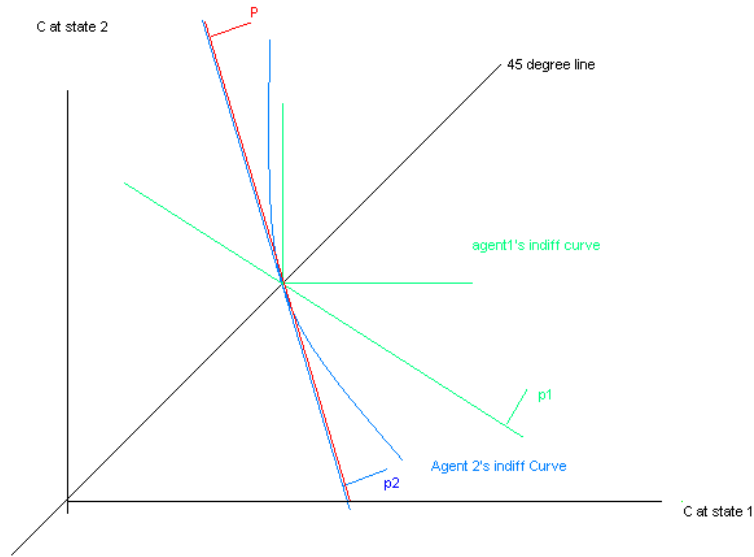$$\sum_{i=1}^{m} x_{is}(\mathbf{P}, \mathbf{P}_i, y_i) = \sum_{i=1}^{m} y_i, \quad s = 1, 2, ..., S,$$

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Figure 6: Weights in Aggregation

is equivalent to the market clearing condition with the rest $(m-1)$ agents:

$$\sum_{i=1}^{m-1} x_{is}(\mathbf{P}, \mathbf{P}_i, y_i) = \sum_{i=1}^{m-1} y_i, \quad s = 1, 2, ..., S.$$

Therefore we have

$$A\left(\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_m\right) = A\left(\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_{m-1}\right).$$

That is the aggregate belief is independent of agent m's belief. Thus α_m must approach to 0 as agent m becomes autarky.

This is illustrated in Figure 6, where agent 1 is infinitely risk averse and has Leontief type of preferences.

On the other hand, if agent m becomes risk neutral when $\mathbf{P} \neq \mathbf{P}_m$, he is

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willing to make claims that are totally concentrating on one state and would have to monopolize that state. This is impossible since all the other risk-averse agents have interior solutions. Therefore the rest $(m-1)$ risk-averse agents will not be able to cover his contingent claims. ■

It was obvious from the proof of the above proposition that the aggregation satisfies the Pareto property. When both agents agree on their beliefs, the aggregate belief equals the individual belief. This property is usually assumed in the axiomatic aggregate framework, see for example Mongin (1995).

What happens to the aggregate belief if agents have the same preferences and endowment but different subjective beliefs? The following proposition tells us that the aggregation puts equal weight on private beliefs. That is, our aggregation satisfies anonymity property.

Proposition 4 *Let* $\mathbf{P} = \sum_{i=1}^{m} \alpha_i \mathbf{P}_i$ *is the aggregate belief as in proposition 2, then If two agents j and k have the same VNM expected utility function up to a positive affine transformation and the same endowment, but different subjective beliefs, then the aggregate belief depends symmetrically on their private beliefs $\mathbf{P}_j$ and $\mathbf{P}_k$, i.e.,*

$$\alpha_j = \alpha_k.$$

Specifically, when all agents have same preferences and same endowment

the aggregate belief is just the arithmetic average of the individual beliefs:

$$\mathbf{P} = \frac{1}{m} \sum_{i=1}^{m} \mathbf{P}_i.$$

Proof. We now prove this equal treatment property. Since agent j and k have the same preferences over uncertainty and same income $y_j = y_k = y$, they would induce the same contingent claims,

$$\mathbf{x}_j(\mathbf{P}, \mathbf{P}_j, y) = \mathbf{x}_k(\mathbf{P}, \mathbf{P}_j, y),$$

or

$$\mathbf{x}_j(\mathbf{P}, \mathbf{P}_k, y) = \mathbf{x}_k(\mathbf{P}, \mathbf{P}_k, y).$$

That means the market clearing condition

$$\sum_{i=1}^{m} \mathbf{x}_i(\mathbf{P}, \mathbf{P}_i, y_i) = \left(\sum_{i=1}^{m} y_i \right) \mathbf{1}$$

is completely symmetric for $\mathbf{P}_j$ and $\mathbf{P}_k$. As a result, the unique solution $\mathbf{P} = \sum_{i=1}^{m} \alpha_i \mathbf{P}_i$ must also be symmetric between $\mathbf{P}_j$ and $\mathbf{P}_k$. That is, α_j must be equal to α_k. ∎

One may also be interested in finding out how the income level influences the aggregation outcome. The next proposition shows agents' endowments positively affect their individual weights in the aggregate belief.

Proposition 5 *If agent i's income endowment y_i goes to zero relative to other agents' income, then agent i's weight α_i on the aggregation also goes to 0.*

Proof. From the first order condition, when $y_i \to 0$, we have $\mathbf{x}_i(\mathbf{P}, \mathbf{P}_i, y_i)$ $\to 0$. Agent i is asymptotically autarky, which implies $\alpha_i \to 0$.

On the other hand when agent i's wealth goes to infinity, assumption 3 implies his degree of risk aversion goes to 0. According to Proposition 3 we must have the aggregate $\mathbf{P} \longrightarrow \mathbf{P}_i$. ∎

We conclude this section with the following examples. The first example demonstrates Proposition 1, 2, 4, and 5. The second one demonstrates Proposition 3.

Example 1: A 2-agent-3-state illustration

Suppose $u_1(x) = u_2(x) = \ln(x)$. The state space $S = \{1, 2, 3\}$. The private beliefs are

$$\mathbf{P}_1 = (\frac{1}{2}, \frac{1}{2}, 0), \quad \mathbf{P}_2 = (\frac{1}{2}, 0, \frac{1}{2}).$$

Two agents agree to the probability of state 1 to be $\frac{1}{2}$, but they disagree to the probability over $\{2, 3\}$.

Given the aggregate belief $\mathbf{P} = (p_1, p_2, p_3)$, where p_s is the probability of state s, agent i maximizes his expected utility:

$$\max E_{\mathbf{P}_i} u_i(\mathbf{x}_i) = \Sigma_{s=1}^{3} p_{is} \ln(x_{is})$$

$$\text{subject to } \mathbf{P}\mathbf{x}_i \leq y_i.$$

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The above problem gives the following contingent claims for agent 1 and 2.

$$x_{11} = \frac{y_1}{2p_1}, \quad x_{12} = \frac{y_1}{2p_2}, \quad x_{13} = 0.$$

$$x_{21} = \frac{y_2}{2p_1}, \quad x_{22} = 0, \quad x_{13} = \frac{y_2}{2p_3}.$$

Imposing market clearing condition:

$$x_{1s} + x_{2s} = y_1 + y_2, \quad s = 1, 2, 3,$$

yields the following equilibrium belief:

$$\mathbf{P} = (p_1, p_2, p_3) = \left(\frac{1}{2}, \frac{y_1}{2(y_1 + y_2)}, \frac{y_2}{2(y_1 + y_2)} \right).$$

One can easily see that the aggregate belief is a convex combination of individual beliefs:

$$\mathbf{P} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2,$$

where $\lambda_i = \frac{y_i}{2(y_1 + y_2)}$. Specifically, when agents have same income, they will have the same weight, and

$$\mathbf{P} = \frac{1}{2}\mathbf{P}_1 + \frac{1}{2}\mathbf{P}_2 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}).$$

Note that the weight distribution across agents in the aggregation process $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \rightarrow (0, 1)$ as $y_1/y_2 \rightarrow 0$.

Example 2: Different degrees of risk aversion

The following example shows how the degree of risk aversion affects the aggregation outcome. Again we consider the 2×3 model. We assume $y_1 = y_2 = y$. The state space $S = \{1, 2, 3\}$. The private beliefs are

$$\mathbf{P}_1 = (\frac{1}{2}, \frac{1}{2}, 0), \quad \mathbf{P}_2 = (\frac{1}{2}, 0, \frac{1}{2}).$$

Suppose $u_1(x) = \frac{x^{1-\alpha}-1}{1-\alpha}$, $u_2(x) = \frac{x^{1-\beta}-1}{1-\beta}$, where $\alpha, \beta \geq 0$, $\alpha, \beta \neq 1$. The risk aversion coefficient is $\frac{\alpha}{x}$ and $\frac{\beta}{x}$, respectively. We set $\alpha = 0$ and $\beta \to \infty$ so that agent 1 is risk neutral and agent 2 is infinitely risk averse. Agent 2's contingent claims are simply (autarky)

$$x_{21} = x_{22} = x_{23} = y.$$

If agent 1's expected utility maximization has interior solutions, then $p_1 = p_2$ and his contingent claims are

$$x_{11} + x_{12} = \frac{y}{p_1}.$$

Summing up the market clearing condition for state 1 and 2 satisfy

$$\frac{y}{p_1} + 2y = 4y.$$

And then

$$p_1 = p_2 = \frac{1}{2}, \ p_3 = 0.$$

Thus we have $\mathbf{P} = \mathbf{P}_1 = \{\frac{1}{2}, \frac{1}{2}, 0\}$, and $\mathbf{P}_2$ doesn't play any role in the aggregation. One can easily show that the case when agent 1 has corner

solutions cannot be an equilibrium since market clearing conditions cannot be satisfied.

4 Conclusion

In this paper, we have constructed an aggregation of individual beliefs through the market where agents are allowed to make contingent claims on every state against the aggregate belief. The aggregation is obtained when the claims made by all the agents are covered by each other at every possible state. We prove that the aggregate belief lies in the convex hull of the individual beliefs and satisfies the properties of consistency and equal treatment. It is also shown that one agent's influence on the aggregate belief is affected by his attitudes toward risk and his wealth level relative to others.

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