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by Daniel Harnett and David Nualart

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Central Limit Theorem for a Stratonovich Integral with Malliavin Calculus

Daniel Harnett, David Nualart*
Department of Mathematics, University of Kansas
405 Snow Hall, Lawrence, Kansas 66045-2142

Abstract

The purpose of this paper is to establish the convergence in law of the sequence of "midpoint" Riemann sums for a stochastic process of the form $f'(W)$, where W is a Gaussian process whose covariance function satisfies some technical conditions. As a consequence we derive a change-of-variable formula in law with a second order correction term which is an Itô integral of $f''(W)$ with respect to a Gaussian martingale independent of W. The proof of the convergence in law is based on the techniques of Malliavin calculus and uses a central limit theorem for q-fold Skorohod integrals, which is a multidimensional extension of a result proved by Nourdin and Nualart in [5]. The results proved in this paper are generalizations of previous work by Swanson [10] and Nourdin and Réveillac [7], who found a similar formula for two particular types of bifractional Brownian motion. We provide two examples of Gaussian processes W that meet the necessary covariance bounds. The first one is the bifractional Brownian motion with parameters $H \le 1/2$, $HK = 1/4$. The second one is a Gaussian process recently studied by Swanson [9] in connection with the fluctuation of empirical quantiles of independent Brownian motion. In the first example the Gaussian martingale is a Brownian motion and in the second case it has a variance equal to t^2.

1 Introduction

The aim of this paper is to obtain a change-of-variable formula in distribution for a class of Gaussian stochastic processes $W = \{W_t, t \ge 0\}$ under certain conditions on the covariance function. These conditions are in the form of upper bounds on the covariance of process increments. For example, the variance on the increment on an interval of length s is bounded by $C\sqrt{s}$, and the covariance between the increments in the intervals $[t-s, t]$ and $[r-s, r]$ is bounded by

$$s^2|t-r|^{-\alpha}(r-s)^{-\beta} + s^2|t-r|^{-\frac{3}{2}},$$

if $0 < 2s \le r < t$ and $|t-r| \ge 2s$, where $1 < \alpha \le \frac{3}{2}$ and $\alpha + \beta = \frac{3}{2}$.

For this process and a suitable function f we study the behavior of the "midpoint" Riemann sum

$$\Phi_n(t) := \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f'(W_{\frac{2j-1}{n}})(W_{\frac{2j}{n}} - W_{\frac{2j-2}{n}}).$$

The limit of this sum as n tends to infinity is the Stratonovich midpoint integral, denoted by $\int_0^t f'(W_s)^\circ dW_s$. We show that the couple of processes $\{(W_t, \Phi_n(t)), t \ge 0\}$ converges in distribution the Skorohod space

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$\mathbb{D}[0,\infty)$ to $\{(W_t, \Phi(t)), t \geq 0\}$, where

$$\Phi(t) = f(W_t) - f(W_0) - \frac{1}{2}\int_0^t f''(W_s)dB_s$$

and $B = \{B_t, t \geq 0\}$ is a Gaussian martingale independent of W with variance $\eta(t)$, depending on the covariance properties of W. This limit theorem can be reformulated by saying that the following Itô formula in distribution holds

$$f(W_t) \stackrel{\mathcal{L}}{=} f(W_0) + \int_0^t f'(W_s)^\circ dW_s + \frac{1}{2}\int_0^t f''(W_s)dB_s. \tag{1}$$

The above mentioned convergence is proven by showing the convergence in law of a $d-$dimensional vector $(\Phi_n(t_1), \dots, \Phi_n(t_d))$ and a tightness argument. To show the convergence in law of the finite dimensional distributions, we show first, using the techniques of Malliavin calculus, that $\Phi_n(t)$ is asymptotically equivalent to a sequence of iterated Skorohod integrals involving $f''(W_t)$. We then apply our $d-$dimensional version of the central limit theorem for multiple Skorohod integrals proved by Nourdin and Nualart in [5].

Recent papers by Swanson [10], Nourdin and Réveillac [7], and Burdzy and Swanson [2] presented results comparable to (1) for a specific stochastic process. In [10], a change-of-variable form was found for a process equivalent to the bifractional Brownian motion with parameters $H = K = 1/2$, arising as the solution to the one-dimensional stochastic heat equation with an additive space-time white noise. This result was proven mostly by martingale methods. In [7], the authors used Malliavin calculus prove a change-of-variable formula for fractional Brownian motion with Hurst parameter $H = 1/4$. More recently, [6] studied the case of fractional Brownian motion with $H = 1/6$. In that paper, weak convergence was proven in $\mathbf{D}[0,\infty)$, and the Riemann sums are based on the trapezoidal approximation.

It happens that the conditions on the process W are satisfied by a bifractional Brownian motion with parameters $H \leq 1/2,\ HK = 1/4$. In this case $\eta(t) = Ct$ and the process B is a Brownian motion. This includes both cases studied in [7] and [10], and extends to a larger class of processes. For another example, we consider a class of Gaussian processes with twice-differentiable covariance function of the form

$$\mathbb{E}\left[W_r W_t\right] = r\phi\left(\frac{t}{r}\right), \quad t \geq r,$$

where ϕ is a bounded function on $[1,\infty)$ such that

$$\phi'(x) = \frac{\kappa}{\sqrt{x-1}} + \frac{\psi(x)}{\sqrt{x}},$$

and ψ is bounded, differentiable and $|\psi'(x)| \leq C(x-1)^{-\frac{1}{2}}$, with the additional condition that $|\phi''(x)| \leq Cx^{-\frac{1}{2}}(x-1)^{-\frac{3}{2}}$. This class of Gaussian processes includes the processes arising as the limit of normalized α-quantiles of a system of independent Brownian motions studied by Swanson in [9]. In particular, if $\alpha = \frac{1}{2}$, we obtain

$$\phi(x) = \sqrt{x}\ \arctan\left(\frac{1}{\sqrt{x-1}}\right).$$

It is surprising to remark that in this case $\eta(t) = Ct^2$. This is related to the fact that the variance of the increments of W on the interval $[t-s, t]$ behaves as $C\sqrt{s}$, when s is small, although the variance of $W(t)$ behaves as Ct.

The outline of this paper is as follows: In Section 2, we introduce the basic environment, and recall some aspects of Malliavin calculus that will be used. In Section 3, a multi-dimensional version of a central limit theorem that appears in [5] is given. In Section 4, the theorem is applied to prove convergence of $\Phi_n(t)$. Section 5 discusses two examples of suitable process families. Finally, Section 6 contains proofs of three of the longer lemmas from Section 4. Most of the notation in this paper follows that of [5].

2 Preliminaries and notation

Let $W = \{W(t), t \geq 0\}$ be a centered Gaussian process defined on a probability space $(\Omega, \mathcal{F}, P)$ with continuous covariance function

$$\mathbb{E}[W(t)W(s)] = R(t,s).$$

We will always assume that $\mathcal{F}$ is the $\sigma-$algebra generated by W. Let $\mathcal{E}$ denote the set of step functions on $[0,T]$ for $T > 0$; and let $\mathfrak{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathfrak{H}} = R(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \mapsto W(t)$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian space spanned by W. We denote this isometry by $h \mapsto W(h)$. In this way, $\{W(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process. For integers $q \geq 1$, let $\mathfrak{H}^{\otimes q}$ denote the q^{th} tensor product of $\mathfrak{H}$. We use $\mathfrak{H}^{\odot q}$ to denote the symmetric tensor product.

For integers $q \geq 1$, let $\mathcal{H}_q$ be the q^{th} Wiener chaos of W, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(W(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where $H_q(x)$ is the q^{th} Hermite polynomial, defined as

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}.$$

For $q \geq 1$, it is known that the map

$$I_q(h^{\otimes q}) = H_q(W(h)) \tag{2}$$

provides an isometry between the symmetric product space $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\frac{1}{\sqrt{q!}}\|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and $\mathcal{H}_q$. By convention, $\mathcal{H}_0 = \mathbb{R}$ and $I_0(x) = x$.

2.1 Elements of Malliavin Calculus

Following is a brief description of some identities that will be used in the paper. The reader may refer to [5] for a brief survey, or to [8] for detailed coverage of this topic. Let $\mathcal{S}$ be the set of all smooth and cylindrical random variables of the form $F = g(W(\phi_1), \ldots, W(\phi_n))$, where $n \geq 1$; $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function with compact support, and $\phi_i \in \mathfrak{H}$. The Malliavin derivative of F with respect to W is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial w_i}(W(\phi_1), \ldots, W(\phi_n))\phi_i.$$

In particular, $DW(h) = h$. By iteration, for any integer $q > 1$ we can define the q^{th} derivative D^qF, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot q})$. For example, if $F = g(W(t))$, then $D^2F = g''(W(t))\mathbf{1}_{[0,t]}^{\otimes 2}$.

For any integer $q \geq 1$ and real number $p \geq 1$, let $\mathbb{D}^{q,p}$ denote the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{q,p}}$ defined as

$$\|F\|_{\mathbb{D}^{q,p}}^p = \mathbb{E}\left[|F|^p\right] + \sum_{i=1}^{q} \mathbb{E}\left[\|D^iF\|_{\mathfrak{H}^{\otimes i}}^p\right].$$

We denote by δ the Skorohod integral, which is defined as the adjoint of the operator D. This operator is also referred to as the divergence operator in [8]. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of δ, Dom δ, if and only if,

$$\left|\mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}}\right]\right| \leq c_u\sqrt{E[F^2]}$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant which depends only on u. If $u \in \mathrm{Dom}\ \delta$, then the random variable $\delta(u) \in L^2(\Omega)$ is defined for all $F \in \mathbb{D}^{1,2}$ by the duality relationship,

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathfrak{H}}\right].$$

This is sometimes called the Malliavin integration by parts formula. We iteratively define the multiple Skorohod integral for $q \geq 1$ as $\delta(\delta^{q-1}(u))$, with $\delta^0(u) = u$. For this definition we have,

$$\mathbb{E}\left[F\delta^q(u)\right] = \mathbb{E}\left[\langle D^q F, u\rangle_{\mathfrak{H}^{\otimes q}}\right],$$

where $u \in \mathrm{Dom}\ \delta^q$ and $F \in \mathbb{D}^{q,2}$. Moreover, if $h \in \mathfrak{H}^{\odot q}$, then we have $\delta^q(h) = I_q(h)$.

For $f, g \in \mathfrak{H}^{\otimes p}$, the following integral multiplication formula holds:

$$\delta^p(f)\delta^p(g) = \sum_{r=0}^{p} r!\binom{p}{r}\delta^{2p-2r}(f \otimes_r g), \tag{3}$$

where $\otimes_r$ is the contraction operator (see, *e.g.,* [5], Sec. 2).

We will use the Meyer inequality for the Skorohod integral, (see, for example Prop. 1.5.7 of [8]). Let $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes k})$ denote the corresponding Sobolev space of $\mathfrak{H}^{\otimes k}$-valued random variables. Then for $p \geq 1$ and integers $k \geq q \geq 1$, we have,

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leq c_{k,p}\|u\|_{\mathbb{D}^{k,p}}(\mathfrak{H}^{\otimes q}) \tag{4}$$

for all $u \in \mathbb{D}^{k,p}(\mathfrak{H}^{\otimes k})$ and some constant $c_{k,p}$.

The following three results will be used in the proof of Theorem 4.3. The reader may refer to [5] and [8] for details.

Lemma 2.1. *Let $q \geq 1$ be an integer.*

1. *Assume $F \in \mathbb{D}^{q,2}$, u is a symmetric element of* $\mathrm{Dom}\ \delta^q$*, and $\left\langle D^r F, \delta^j(u)\right\rangle_{\mathfrak{H}^{\otimes r}} \in L^2(\Omega, \mathfrak{H}^{\otimes q-r-j})$ for all $0 \leq r+j \leq q$. Then $\langle D^r F, u\rangle_{\mathfrak{H}^{\otimes r}} \in$* $\mathrm{Dom}\ \delta^r$ *and*

$$F\delta^q(u) = \sum_{r=0}^{q}\binom{q}{r}\delta^{q-r}\left(\langle D^r F, u\rangle_{\mathfrak{H}^{\otimes r}}\right).$$

2. *Suppose that u is a symmetric element of $\mathbb{D}^{j+k,2}(\mathfrak{H}^{\otimes j})$. Then we have,*

$$D^k\delta^j(u) = \sum_{i=0}^{j\wedge k}\binom{k}{i}\binom{j}{i} i!\,\delta^{j-i}\left(D^{k-i}u\right).$$

3. *Let u, v be symmetric functions in $\mathbb{D}^{2q,2}(\mathfrak{H}^{\otimes q})$. Then*

$$\mathbb{E}[\delta^q(u)\delta^q(v)] = \sum_{i=0}^{q}\binom{q}{i}^2\mathbb{E}\left[\left\langle D^{q-i}u, D^{q-i}v\right\rangle_{\mathfrak{H}^{\otimes(2q-i)}}\right].$$

In particular,

$$\|\delta^q(u)\|^2_{L^2(\Omega)} = \mathbb{E}\left[\delta^q(u)^2\right] \leq \sum_{i=0}^{q}\binom{q}{i}^2\mathbb{E}\left[\left\|D^{q-i}u\right\|^2_{\mathfrak{H}^{\otimes(2q-i)}}\right].$$

Proof of 1. This is proved in [5] (see Lemma 2.1). It follows by induction from the relation $F\delta(u) = \delta(Fu) + \langle DF, u\rangle_{\mathfrak{H}}$ (see [8], Prop. 1.3.3).

Proof of 2. This follows from repeated application of the relation $D\delta(u) = u + \delta(Du)$, (see [8], Prop. 1.3.2).

Proof of 3. This follows from repeated application of the duality property. (see [5], eq. (2.12)). □

3 A central limit theorem for multiple Skorohod integrals

Let $X = \{X(h), h \in \mathfrak{H}\}$ be an isonormal Gaussian process associated with a real-separable Hilbert space $\mathfrak{H}$, defined on a probability space $(\Omega, \mathcal{F}, P)$. We assume that $\mathcal{F}$ is generated by X. The purpose of this section is to prove a multi-dimensional version of a theorem proved in [5] (see Theorem 3.1). We begin by defining the notion of stable convergence.

Definition 3.1. Assume F_n is a sequence of $d-$dimensional random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, and F is a $d-$dimensional random variable defined on $(\Omega, \mathcal{G}, P)$, where $\mathcal{F} \subset \mathcal{G}$. We say that F_n *converges stably* to F as $n \to \infty$, if, for any continuous and bounded function $f : \mathbb{R}^d \to \mathbb{R}$ and $\mathbb{R}$-valued, $\mathcal{F}-$measurable random variable Z, we have

$$\lim_{n\to\infty} \mathbb{E}\left(f(F_n)Z\right) = \mathbb{E}\left(f(F)Z\right).$$

Theorem 3.2. *Let $q \geq 1$ be an integer, and suppose that F_n is a sequence of random variables in $\mathbb{R}^d$ of the form $F_n = \delta^q(u_n) = \left(\delta^q(u_n^1), \ldots, \delta^q(u_n^d)\right)$, for a sequence of $\mathbb{R}^d-$valued symmetric functions u_n in $\mathbb{D}^{2q,2q}(\mathfrak{H}^{\otimes q})$. Suppose that the sequence F_n is bounded in $L^1(\Omega, \mathfrak{H})$ and that:*

(a) $\left\langle u_n^j, \bigotimes_{\ell=1}^m (D^{a_\ell} F_n^{j_\ell}) \otimes h\right\rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in $L^1(\Omega)$ for all integers $1 \leq j, j_\ell \leq d$, all integers $1 \leq a_1, \ldots, a_m, r \leq q-1$ such that $a_1 + \cdots + a_m + r = q$; and all $h \in \mathfrak{H}^{\otimes r}$.

(b) For each $1 \leq i, j \leq d$, $\left\langle u_n^i, D^q F_n^j\right\rangle_{\mathfrak{H}^{\otimes q}}$ converges in $L^1(\Omega, \mathfrak{H})$ to a random variable s_{ij}, such that the matrix $\Sigma := \left(s_{ij}\right)_{d\times d}$ is nonnegative definite (that is, $\lambda^T \Sigma \lambda \geq 0$ for all nonzero $\lambda \in \mathbb{R}^d$).

Then F_n converges stably to a random variable in $\mathbb{R}^d$ with conditional Gaussian law $\mathcal{N}(0, \Sigma)$ given X.

Remark 3.3. Conditions (a) and (b) mean that for $q \geq 1$, some combinations of lower-order derivative products are negligible. For example, for $q = 2$, then the following scalar products will converge to zero in $L^1(\Omega, \mathfrak{H})$:

- $\left\langle u_n^i, h_1 \otimes h_2\right\rangle_{\mathfrak{H}^{\otimes 2}}$ for all $h_1, h_2 \in \mathfrak{H}$.
- $\left\langle u_n^i, DF_n^j \otimes h\right\rangle_{\mathfrak{H}^{\otimes 2}}$ for all $h \in \mathfrak{H}$ and all j (including $i = j$).
- $\left\langle u_n^i, DF_n^j \otimes DF_n^k\right\rangle_{\mathfrak{H}^{\otimes 2}}$ for all $1 \leq k, j \leq d$.

Only the q^{th}-order derivative products converge to a nontrivial random variable. Usually (see Section 6), the term $\left\langle u_n^i, D^q F_n^j\right\rangle_{\mathfrak{H}^{\otimes q}}$ has the same asymptotic behavior as $\left\langle u_n^i, u_n^j\right\rangle_{\mathfrak{H}^{\otimes q}}$.

Remark 3.4. It suffices to impose condition (a) for $h \in \mathcal{S}_0$, where $\mathcal{S}_0$ is a total subset of $\mathfrak{H}^{\otimes r}$.

Proof of Theorem 3.2.
As in the 1-dimensional case considered in [5], we will use the conditional characteristic function. Given any $h_1, \ldots h_m \in \mathfrak{H}$, we want to show that the sequence

$$\xi_n = \left(F_n^1, \ldots, F_n^d, X(h_1), \ldots, X(h_m)\right)$$

converges in distribution to a vector $\left(F_\infty^1, \ldots F_\infty^d, X(h_1), \ldots, X(h_m)\right)$, where, for any vector $\lambda \in \mathbb{R}^d$, F_∞ satisfies

$$\mathbb{E}\left(e^{i\lambda\cdot F_\infty} | X(h_1), \ldots, X(h_m)\right) = \exp\left(-\frac{1}{2}\lambda^T \Sigma \lambda\right), \tag{5}$$

where $\lambda \cdot F_n = \sum_{j=1}^d \lambda_j F_n^j$ denotes the usual scalar product in $\mathbb{R}^d$, and we use this notation to avoid confusion with the scalar product in $\mathfrak{H}$.

Dropping to a subsequence if necessary, we may assume that ξ_n converges in distribution to a limit $\left(F_\infty^1, \ldots F_\infty^d, X(h_1), \ldots X(h_m)\right)$. Let $Y := g\left(X(h_1), \ldots, X(h_m)\right)$, where $g \in \mathcal{C}_b^\infty(\mathbb{R}^m)$, and consider $\phi_n(\lambda) = \phi(\lambda, \xi_n) := \mathbb{E}\left(e^{i\lambda\cdot F_n} Y\right)$ for $\lambda \in \mathbb{R}^d$. The convergence in law of ξ_n implies that:

$$\lim_{n\to\infty} \frac{\partial \phi_n}{\partial \lambda_j} = \lim_{n\to\infty} i\mathbb{E}\left(F_n^j e^{i\lambda\cdot F_n} Y\right) = i\mathbb{E}\left(F_\infty^j e^{i\lambda\cdot F_\infty} Y\right) \tag{6}$$

On the other hand, using the duality property of the Skorohod integral and the Malliavin derivative:

$$\begin{aligned}\frac{\partial \phi_n}{\partial \lambda_j} &= i\mathbb{E}\left(\delta^q(u_n^j) e^{i\lambda\cdot F_n} Y\right) = i\mathbb{E}\left(\left\langle u_n^j, D^q\left(e^{i\lambda\cdot F^n} Y\right)\right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\ &= i\sum_{a=0}^{q} \binom{q}{a} \mathbb{E}\left(\left\langle u_n^j, D^a\left(e^{i\lambda\cdot F_n}\right) \widetilde{\otimes}\, D^{q-a} Y\right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\ &= i\left\{\mathbb{E}\left\langle u_n^j, Y D^q e^{i\lambda\cdot F_n}\right\rangle_{\mathfrak{H}^{\otimes q}} + \sum_{a=0}^{q-1}\binom{q}{a}\mathbb{E}\left\langle u_n^j, D^a e^{i\lambda\cdot F_n} \widetilde{\otimes}\, D^{q-a} Y\right\rangle_{\mathfrak{H}^{\otimes q}}\right\}\end{aligned} \tag{7}$$

By condition (a), we have that $\left\langle u_n^j, D^a e^{i\lambda\cdot F_n} \widetilde{\otimes}\, D^{q-a} Y\right\rangle_{\mathfrak{H}^{\otimes q}}$ converges to zero in $L^1(\Omega)$ when $a < q$, so the sum term vanishes as $n \to \infty$, and this leaves

$$\begin{aligned}\lim_{n\to\infty} i\mathbb{E}\left\langle u_n^j, Y D^q e^{i\lambda\cdot F_n}\right\rangle_{\mathfrak{H}^{\otimes q}} &= \lim_{n\to\infty} i\sum_{k=1}^{d} \mathbb{E}\left(i\lambda_k e^{i\lambda\cdot F_n}\left\langle u_n^j, Y D^q F_n^k\right\rangle_{\mathfrak{H}^{\otimes q}}\right) \\ &= -\sum_{k=1}^{d}\mathbb{E}\left(\lambda_k e^{i\lambda\cdot F_\infty} s_{kj} Y\right)\end{aligned}$$

because the lower-order derivatives in $D^q e^{i\lambda\cdot F_n}$ also vanish by condition (a). Combining this with (6), we obtain:

$$i\mathbb{E}\left(F_\infty^j e^{i\lambda\cdot F_\infty} Y\right) = -\sum_{k=1}^{d} \lambda_k \mathbb{E}\left(e^{i\lambda\cdot F_\infty} s_{kj} Y\right).$$

This leads to the PDE system:

$$\frac{\partial}{\partial \lambda_j}\mathbb{E}\left(e^{i\lambda\cdot F_\infty} | X(h_1), \ldots, X(h_m)\right) = -\sum_{k=1}^{d} \lambda_k s_{kj} \mathbb{E}\left(e^{i\lambda\cdot F_\infty} | X(h_1), \ldots, X(h_m)\right)$$

which has solution (5). □

4 Central limit theorem for the Stratonovich integral

Suppose that $W = \{W_t, t \geq 0\}$ is a centered Gaussian process, as in Section 2, that meets conditions (i) through (v), below, for any $T > 0$, where the constants C_i may depend on T.

(i) For any $0 < s \leq t \leq T$, there is a constant C_1 such that

$$\mathbb{E}\left[(W_t - W_{t-s})^2\right] \leq C_1 s^{\frac{1}{2}}.$$

(ii) For any $s > 0$ and $2s \leq r, t \leq T$ with $|t - r| \geq 2s$,

$$|\mathbb{E}\left[(W_t - W_{t-s})(W_r - W_{r-s})\right]| \leq C_1 s^2 |t-r|^{-\alpha}(t \wedge r - s)^{-\beta} + s^2|t-r|^{-\frac{3}{2}};$$

for positive constants α, β, γ, such that $1 < \alpha \leq \frac{3}{2}$ and $\alpha + \beta = \frac{3}{2}$.

(iii) For $0 < t \le T$ and $0 < s \le r \le T$,

$$|\mathbb{E}\left[W_t(W_{r+s} - 2W_r + W_{r-s})\right]| \le \begin{cases} C_2 s^{\frac{1}{2}} & if\ r < 2s\ or\ |t-r| < 2s \\ C_2 s^2 \left((r-s)^{-\frac{3}{2}} + |t-r|^{-\frac{3}{2}}\right) & if\ r \ge 2s\ and\ |t-r| \ge 2s \end{cases}$$

for some positive constant C_2.

(iv) For any $0 < s \le t \le T - s$

$$|\mathbb{E}\left[W_t(W_{t+s} - W_{t-s})\right]| \le \begin{cases} C_3 s^{\frac{1}{2}} & if\ t < 2s \\ C_3 s (t-s)^{-\frac{1}{2}} & if\ t \ge 2s \end{cases}$$

and for each $0 < s \le r \le T$,

$$|\mathbb{E}\left[W_r(W_{t+s} - W_{t-s})\right]| \le \begin{cases} C_3 s^{\frac{1}{2}} & if\ t < 2s\ or\ |t-r| < 2s \\ C_3 s (t-s)^{-\frac{1}{2}} + C_3 s |t-r|^{-\frac{1}{2}} & if\ t \ge 2s\ and\ |t-r| \ge 2s \end{cases}$$

for some positive constant C_3. In addition, for $t > 2s$,

$$|\mathbb{E}\left[W_s(W_t - W_{t-s})\right]| \le C_3 s^{\frac{1}{2}+\gamma}(t-2s)^{-\gamma}$$

for some $\gamma > 0$.

(v) Consider a uniform partition of $[0, \infty)$ with increment length $1/n$. Define for integers $j, k \ge 0$ and $n \ge 1$:

$$\beta_n(j,k) = \mathbb{E}\left[\left(W_{\frac{j+1}{n}} - W_{\frac{j}{n}}\right)\left(W_{\frac{k+1}{n}} - W_{\frac{k}{n}}\right)\right].$$

Next, define

$$\eta_n^+(t) = \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1, 2k-1)^2 + \beta_n(2j-2, 2k-2)^2;$$

$$\eta_n^-(t) = \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-2, 2k-1)^2 + \beta_n(2j-1, 2k-2)^2.$$

Then for each $t \ge 0$,

$$\lim_{n\to\infty} \eta_n^+(t) = \eta^+(t) \quad \text{and} \quad \lim_{n\to\infty} \eta_n^-(t) = \eta^-(t)$$

both exist, where $\eta^+(t), \eta^-(t)$ are nonnegative and nondecreasing functions.

Consider a real-valued function $f \in \mathcal{C}^9(\mathbb{R})$, such that f and all its derivatives up to order 9 have at most exponential growth, that is

$$\left|f^{(k)}(x)\right| < K_1 \exp(K_2|x|^\alpha),\ x \in \mathbb{R},\ \alpha < 2$$

for $k = 0, \ldots, 9$, and positive constants K_1, K_2. We will refer to this as Condition (0).

In the following, the term C represents a generic positive constant, which may change from line to line. The constant C may depend on T and the constants in conditions (0) and (i) - (v) listed above.

The results of the next lemma follow from conditions (i) and (ii).

Lemma 4.1. *Using the notation described above, for integers $0 \le a < b$ and integers $r, n \ge 1$, we have the estimate,*

$$\sum_{j,k=a}^{b} |\beta_n(j,k)|^r \le C(b-a+1)n^{-\frac{r}{2}}.$$

Proof. Suppose first that $r = 1$. Let $I = \{(j,k) : a \le j,k \le b, |k-j| \ge 2, j \wedge k \ge 2\}$, and $J = \{(j,k) : a \le j,k \le b, (j,k) \notin I\}$. Consider the decomposition

$$\sum_{j,k=a}^{b} |\beta_n(j,k)| = \sum_{(j,k)\in I} |\beta_n(j,k)| + \sum_{(j,k)\in J} |\beta_n(j,k)|\,.$$

Then by condition (ii), the first sum is bounded by

$$\sum_{(j,k)\in I} n^{-\frac{1}{2}}|j-k|^{-\alpha} \le Cn^{-\frac{1}{2}}(b-a+1),$$

and the second sum, using condition (i) and Cauchy-Schwarz, is bounded by $Cn^{-\frac{1}{2}}(b-a+1)$. For the case $r > 1$, condition (i) implies $|\beta_n(j,k)| \le C_1 n^{-\frac{1}{2}}$ for all j,k. It follows that we can write,

$$\sum_{j,k=a}^{b} |\beta_n(j,k)|^r \le C_1 n^{-\frac{r-1}{2}} \sum_{j,k=a}^{b} |\beta_n(j,k)| \le C(b-a+1)n^{-\frac{r}{2}}.$$

□

Corollary 4.2. *Using the notation of Lemma 4.1, for each integer $r \ge 1$,*

$$\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left(|\beta_n(2j-1,2k-1)|^r + |\beta_n(2j-1,2k-2)|^r + |\beta_n(2j-2,2k-1)|^r + |\beta_n(2j-2,2k-2)|^r\right) \le C\left\lfloor \frac{nt}{2} \right\rfloor n^{-\frac{r}{2}}.$$

Proof. Note that

$$\begin{aligned}
&\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left(|\beta_n(2j-1,2k-1)|^r + |\beta_n(2j-1,2k-2)|^r + |\beta_n(2j-2,2k-1)|^r + |\beta_n(2j-2,2k-2)|^r\right) \\
&\quad = \sum_{j,k=0}^{2\lfloor \frac{nt}{2} \rfloor - 1} |\beta_n(j,k)|^r\,.
\end{aligned}$$

□

Consider a uniform partition of $[0,\infty)$ with increment length $1/n$. The Stratonovich midpoint integral of $f'(W)$ will be defined as the limit in probability of the sequence (see [10]):

$$\Phi_n(t) := \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f'(W_{\frac{2j-1}{n}})(W_{\frac{2j}{n}} - W_{\frac{2j-2}{n}}). \tag{8}$$

We introduce the following notation, as used in [5]: $\varepsilon_t := \mathbf{1}_{[0,t]}$; and $\partial_{\frac{j}{n}} := \mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}$.

The following is the major result of this section.

Theorem 4.3. *Let f be a real function satisfying condition (0), and let $W = \{W_t, t \geq 0\}$ be a Gaussian process satisfying conditions (i) through (v). Then:*

$$(W_t, \Phi_n(t)) \xrightarrow{\mathcal{L}} \left(W_t, f(W_t) - f(W_0) - \frac{1}{2} \int_0^t f''(W_s)\, dB_s \right)$$

as $n \to \infty$ in the Skorohod space $\mathbf{D}[0,\infty)$, where $\eta(t) = \eta^+(t) - \eta^-(t)$ for the functions defined in condition (v); and $B = \{B_t, t \geq 0\}$ is scaled Brownian motion, independent of W, and with variance $\mathbb{E}\left[B_t^2\right] = \eta(t)$.

The rest of this section consists of the proof of Theorem 4.3, and is presented in a series of lemmas. The proofs of Lemmas 4.4, 4.5, and 4.9, which are rather technical, are deferred to Section 6. We begin with an expansion of $f(W_t)$, following the methodology used in [10]. Consider the telescoping series

$$f(W_t) = f(W_0) + \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \left[f(W_{\frac{2j}{n}}) - f(W_{\frac{2j-2}{n}}) \right] + f(W_t) - f(W_{\frac{2}{n}\lfloor \frac{nt}{2} \rfloor}),$$

where the sum is zero by convention if $\lfloor \frac{nt}{2} \rfloor = 0$. Using a Taylor series expansion of order 2, we obtain

$$\begin{aligned} \Phi_n(t) &= f(W_t) - f(W_0) - \frac{1}{2} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f''(W_{\frac{2j-1}{n}}) \left(\Delta W_{\frac{2j}{n}}^2 - \Delta W_{\frac{2j-1}{n}}^2 \right) \\ &\quad - \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} R_0(W_{\frac{2j}{n}}) + \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} R_1(W_{\frac{2j-2}{n}}) - \left(f(W_t) - f(W_{\frac{2}{n}\lfloor \frac{nt}{2} \rfloor}) \right), \end{aligned}$$

where R_0, R_1 represent the third-order remainder terms in the Taylor expansion, and can be expressed in integral form as:

$$R_0(W_{\frac{2j}{n}}) = \frac{1}{2} \int_{W_{\frac{2j-1}{n}}}^{W_{\frac{2j}{n}}} (W_{\frac{2j}{n}} - u)^2 f^{(3)}(u)\, du; \quad \text{and} \tag{9}$$

$$R_1(W_{\frac{2j-2}{n}}) = -\frac{1}{2} \int_{W_{\frac{2j-2}{n}}}^{W_{\frac{2j-1}{n}}} (W_{\frac{2j-2}{n}} - u)^2 f^{(3)}(u)\, du. \tag{10}$$

By condition (0) we have for any $T > 0$ that

$$\lim_{n\to\infty} \mathbb{E} \sup_{0 \leq t \leq T} \left| f(W_t) - f(W_{\frac{2}{n}\lfloor \frac{nt}{2} \rfloor}) \right| = 0,$$

so this term vanishes uniformly on compacts in probability (ucp), and may be neglected. Therefore, it is sufficient to work with the term

$$\Delta_n(t) := f(W_t) - f(W_0) - \frac{1}{2} \Psi_n(t) + R_n(t), \tag{11}$$

where

$$\Psi_n(t) = \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f''(W_{\frac{2j-1}{n}}) \left(\Delta W_{\frac{2j}{n}}^2 - \Delta W_{\frac{2j-1}{n}}^2 \right); \quad \text{and}$$

$$R_n(t) = \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \left(R_1(W_{\frac{2j-2}{n}}) - R_0(W_{\frac{2j}{n}}) \right).$$

We will first decompose the term $\Psi_n(t)$, using a Skorohod integral representation. Using (2) and the second Hermite polynomial, one can write $\Delta W^2(h) = 2H_2(W(h)) + 1 = \delta^2(h^{\otimes 2}) + 1$ for any $h \in \mathfrak{H}$ with $\|h\|_{\mathfrak{H}} = 1$. It follows that,

$$\Psi_n(t) = \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f''(W_{\frac{2j-1}{n}})\delta^2\left(\partial_{\frac{2j-1}{n}}^{\otimes 2} - \partial_{\frac{2j-2}{n}}^{\otimes 2}\right).$$

From Lemma 2.1, we have for random variables u, F

$$F\delta^2(u) = \delta^2(Fu) + 2\delta\left(\langle DF, u\rangle_{\mathfrak{H}}\right) + \left\langle D^2F, u\right\rangle_{\mathfrak{H}^{\otimes 2}},$$

so we can write:

$$\begin{aligned}
\Psi_n(t) &= \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \delta^2\left(f''(W_{\frac{2j-1}{n}})\left(\partial_{\frac{2j-1}{n}}^{\otimes 2} - \partial_{\frac{2j-2}{n}}^{\otimes 2}\right)\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} 2\delta\left(f^{(3)}(W_{\frac{2j-1}{n}})\left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}}^{\otimes 2} - \partial_{\frac{2j-2}{n}}^{\otimes 2}\right\rangle_{\mathfrak{H}}\right) \\
&\quad + \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} f^{(4)}(W_{\frac{2j-1}{n}})\left(\left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}}\right\rangle_{\mathfrak{H}}^2 - \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}}^2\right) \\
&:= F_n(t) + B_n(t) + C_n(t).
\end{aligned}$$

Hence, we have $\Delta_n(t) = f(W_t) - f(W_0) - \frac{1}{2}\left(F_n(t) + B_n(t) + C_n(t)\right) + R_n(t)$. In the next two lemmas, we show that the terms $B_n(t), C_n(t)$, and $R_n(t)$ converge to zero in probability as $n \to \infty$. The proofs of these lemmas are deferred to Section 6.

Lemma 4.4. *Let $0 \le r < t \le T$. Using the notation defined above,*

$$\mathbb{E}\left[(R_n(t) - R_n(r))^2\right] \le C\left(\left\lfloor \frac{nt}{2} \right\rfloor - \left\lfloor \frac{nr}{2} \right\rfloor\right) n^{-\frac{3}{2}}$$

for some positive constant C, which may depend on T. It follows that for any $0 \le t \le T$, $R_n(t)$ converges to zero in probability as $n \to \infty$.

Lemma 4.5. *Let $0 \le r < t \le T$. Using the above notation, there exist constants C_B, C_C such that*

$$\mathbb{E}\left[(B_n(t) - B_n(r))^2\right] \le C_B\left(\left\lfloor \frac{nt}{2} \right\rfloor - \left\lfloor \frac{nr}{2} \right\rfloor\right) n^{-\frac{3}{2}}; \text{ and}$$
$$\mathbb{E}\left[(C_n(t) - C_n(r))^2\right] \le C_C\left(\left\lfloor \frac{nt}{2} \right\rfloor - \left\lfloor \frac{nr}{2} \right\rfloor\right) n^{-\frac{3}{2}}.$$

It follows that for any $0 \le t \le T$, $B_n(t)$ and $C_n(t)$ converge to zero in probability as $n \to \infty$.

Corollary 4.6. *Let $Z_n(t) := R_n(t) - \frac{1}{2}B_n(t) - \frac{1}{2}C_n(t)$. Then given $0 \le t_1 < t < t_2 \le T$, there exists a positive constant C such that*

$$\mathbb{E}\left[|Z_n(t) - Z_n(t_1)|\,|Z_n(t_2) - Z_n(t)|\right] \le C(t_2 - t_1)^{\frac{3}{2}}.$$

Proof. By lemmas (4.4) and (4.5),

$$\begin{aligned}\mathbb{E}\left[(Z_n(t_2)-Z_n(t_1))^2\right] &\le 3\mathbb{E}\left[(R_n(t_2)-R_n(t_1))^2\right]+2\mathbb{E}\left[(B_n(t_2)-B_n(t_1))^2\right]\\ &\quad+2\mathbb{E}\left[(C_n(t_2)-C_n(t_1))^2\right]\\ &\le C\left(\left\lfloor\frac{nt_2}{2}\right\rfloor-\left\lfloor\frac{nt_1}{2}\right\rfloor\right)n^{-\frac{3}{2}}.\end{aligned}$$

Then by Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}\left[|Z_n(t)-Z_n(t_1)|\,|Z_n(t_2)-Z_n(t)|\right] &\le \left(\mathbb{E}\left[(Z_n(t)-Z_n(t_1))^2\right]\;\mathbb{E}\left[(Z_n(t)-Z_n(t_1))^2\right]\right)^{\frac{1}{2}}\\ &\le C\left(\left\lfloor\frac{nt_2}{2}\right\rfloor-\left\lfloor\frac{nt_1}{2}\right\rfloor\right)^{\frac{3}{2}}n^{-\frac{3}{2}}.\end{aligned}$$

This estimate implies the required bound $C(t_2-t_1)^{\frac{3}{2}}$, see, for example [1], p. 156. □

Next, we will develop a comparable estimate for differences of the form $F_n(t)-F_n(r)$. In order to prove this estimate, we need a technical lemma which will be used here and also in Section 6.

Lemma 4.7. *Suppose a,b are nonnegative integers such that $a+b\le 9$. For fixed $T>0$ and interval $[t_1,t_2]\subset[0,T]$, let*

$$g_a=\sum_{\ell=\lfloor\frac{nt_1}{2}\rfloor+1}^{\lfloor\frac{nt_2}{2}\rfloor} f^{(a)}(W_{\frac{2\ell-1}{n}})\left(\partial^{\otimes 2}_{\frac{2\ell-1}{n}}-\partial^{\otimes 2}_{\frac{2\ell-2}{n}}\right).$$

Then we have for $1\le p<\infty$

$$\mathbb{E}\left[\|D^b g_a\|^p_{\mathfrak{H}^{\otimes 2+b}}\right]\le C\left(\left\lfloor\frac{nt_2}{2}\right\rfloor-\left\lfloor\frac{nt_1}{2}\right\rfloor\right)^{\frac{p}{2}}n^{-\frac{p}{2}}.$$

Proof. We may assume $t_1=0$ with $t_2\le T$. For each b we can write

$$\begin{aligned}&\mathbb{E}\left[\left(\|D^b g_a\|^2_{\mathfrak{H}^{\otimes 2+b}}\right)^{\frac{p}{2}}\right]\\ &=\mathbb{E}\left[\left(\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor} f^{(a+b)}(W_{\frac{2\ell-1}{n}})f^{(a+b)}(W_{\frac{2m-1}{n}})\left\langle\varepsilon^{\otimes b}_{\frac{2\ell-1}{n}},\varepsilon^{\otimes b}_{\frac{2m-1}{n}}\right\rangle_{\mathfrak{H}^{\otimes b}}\left\langle\partial^{\otimes 2}_{\frac{2\ell-1}{n}}-\partial^{\otimes 2}_{\frac{2\ell-2}{n}},\partial^{\otimes 2}_{\frac{2m-1}{n}}-\partial^{\otimes 2}_{\frac{2m-2}{n}}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right)^{\frac{p}{2}}\right]\\ &\le\mathbb{E}\left[\sup_{0\le s\le t}\left|f^{(a+b)}(W_s)\right|^p\right]\left(\sup_{\ell,m}\left|\left\langle\varepsilon_{\frac{2\ell-1}{n}},\varepsilon_{\frac{2m-1}{n}}\right\rangle_{\mathfrak{H}}\right|^b\right)^{\frac{p}{2}}\left(\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor}\left|\left\langle\partial^{\otimes 2}_{\frac{2\ell-1}{n}}-\partial^{\otimes 2}_{\frac{2\ell-2}{n}},\partial^{\otimes 2}_{\frac{2m-1}{n}}-\partial^{\otimes 2}_{\frac{2m-2}{n}}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|\right)^{\frac{p}{2}}.\end{aligned}$$

Recall that condition (0) holds for f and its first 9 derivatives, so the first two terms are bounded. For the last term, note that by Corollary 4.2 with $r=2$,

$$\begin{aligned}&\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor}\left|\left\langle\partial^{\otimes 2}_{\frac{2\ell-1}{n}}-\partial^{\otimes 2}_{\frac{2\ell-2}{n}},\partial^{\otimes 2}_{\frac{2m-1}{n}}-\partial^{\otimes 2}_{\frac{2m-2}{n}}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|\\ &\quad=\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor}\left|\beta_n(2\ell-1,2m-1)^2-\beta_n(2\ell-1,2m-2)^2-\beta_n(2\ell-2,2m-1)^2+\beta_n(2\ell-2,2m-2)^2\right|\\ &\quad\le C\left\lfloor\frac{nt_2}{2}\right\rfloor n^{-1}.\end{aligned}$$

□

Lemma 4.8. *For $0 \le s < t \le T$, write*

$$F_n(t) - F_n(s) = \sum_{j=\lfloor \frac{ns}{2} \rfloor + 1}^{\lfloor \frac{nt}{2} \rfloor} \delta^2 \left(f''(W_{\frac{2j-1}{n}}) \left(\partial_{\frac{2j-1}{2}}^{\otimes 2} - \partial_{\frac{2j-2}{n}}^{\otimes 2} \right) \right)$$

Then given $0 \le t_1 < t < t_2 \le T$, there exists a positive constant C such that

$$\mathbb{E}\left[|F_n(t) - F_n(t_1)|^2 |F_n(t_2) - F_n(t)|^2 \right] \le C(t_2 - t_1)^2. \tag{12}$$

Proof. First, for each $n \ge 1$, we want to show that there is a C such that,

$$\mathbb{E}\left[(F_n(t_2) - F_n(t_1))^4 \right] \le C \left(\left\lfloor \frac{nt_2}{2} \right\rfloor - \left\lfloor \frac{nt_1}{2} \right\rfloor \right)^2 n^{-2}.$$

By the Meyer inequality (4) there exists a constant $c_{2,4}$ such that

$$\mathbb{E}\left| \left(\delta^2(u_n) \right)^4 \right| \le c_{2,4} \|u_n\|^4_{\mathbb{D}^{2,4}(\mathfrak{H}^{\otimes 2})},$$

where in this case,

$$u_n = \sum_{j=\lfloor \frac{nt_1}{2} \rfloor + 1}^{\lfloor \frac{nt_2}{2} \rfloor} f''(W_{\frac{2j-1}{n}}) \left(\partial_{\frac{2j-1}{n}}^{\otimes 2} - \partial_{\frac{2j-2}{n}}^{\otimes 2} \right)$$

and

$$\|u_n\|^4_{\mathbb{D}^{2,4}(\mathfrak{H}^{\otimes 2})} = \mathbb{E}\|u_n\|^4_{\mathfrak{H}^{\otimes 2}} + \mathbb{E}\|Du_n\|^4_{\mathfrak{H}^{\otimes 3}} + \mathbb{E}\|D^2 u_n\|^4_{\mathfrak{H}^{\otimes 4}}.$$

From Lemma 4.7 we have $\mathbb{E}\|u_n\|^4_{\mathfrak{H}^{\otimes 2}}$, $\mathbb{E}\|Du_n\|^4_{\mathfrak{H}^{\otimes 3}}$, $\mathbb{E}\|D^2 u_n\|^4_{\mathfrak{H}^{\otimes 4}} \le C \left(\lfloor \frac{nt_2}{2} \rfloor - \lfloor \frac{nt_1}{2} \rfloor \right)^2 n^{-2}$, and so it follows that,

$$\mathbb{E}\left[\left(\delta^2(u_n) \right)^4 \right] \le C \left(\left\lfloor \frac{nt_2}{2} \right\rfloor - \left\lfloor \frac{nt_1}{2} \right\rfloor \right)^2 n^{-2}.$$

From this result, given $0 \le t_1 < t < t_2$, it follows from the Hölder inequality that

$$\begin{aligned} \mathbb{E}\left[|F_n(t) - F_n(t_1)|^2 |F_n(t_2) - F_n(t)|^2 \right] &\le \left(\mathbb{E}\left[|F_n(t) - F_n(t_1)|^4 \right] \right)^{\frac{1}{2}} \left(\mathbb{E}\left[|F_n(t_2) - F_n(t)|^4 \right] \right)^{\frac{1}{2}} \\ &\le C \left(\left\lfloor \frac{nt_2}{2} \right\rfloor - \left\lfloor \frac{nt_1}{2} \right\rfloor \right)^2 n^{-2}. \end{aligned}$$

As in Corollary 4.6, this implies the required bound $C(t_2 - t_1)^2$. □

By Corollary 4.6 and Lemma 4.8, it follows that $\Delta_n(t) = f(W_t) - f(W_0) - \frac{1}{2}F_n(t) + Z_n(t)$ is tight, since both sequential parts $F_n(t), Z_n(t)$ are tight. Further, we have that $Z_n(t)$ tends to zero in probability, and $F_n(t)$ is in a form suitable for Theorem 3.2. In the next lemma, we show that the conditions of Theorem 3.2 are satisfied by $F_n(t)$ evaluated at a finite set of points.

Lemma 4.9. *Fix $0 = t_0 < t_1 < t_2 < \cdots < t_d$. Set $F_n^i = F_n(t_i) - F_n(t_{i-1})$ for $i = 1, \ldots d$, and let $F_n = (F_n^i, \ldots, F_n^d)$. Then under conditions (0), and (i) - (v), F_n satisfies conditions (a) and (b) of Theorem 3.2, and so given W, F_n converges stably as $n \to \infty$ to a random variable $\xi = (\xi_1, \ldots, \xi_d)$ with distribution $\mathcal{N}(0, \Sigma)$, where Σ is a diagonal $d \times d$ matrix with entries:*

$$s_i^2 = \int_{t_{i-1}}^{t_i} f''(W_s)^2 \eta(ds),$$

where $\eta(t) = \eta^+(n) - \eta^-(t)$ is as defined in condition (v).

Remark 4.10. As we will see later, $\eta(t)$ is continuous, nonnegative, and nondecreasing.

It follows from the structure of Σ that F_n converges stably to a d-dimensional vector with independent components of the form

$$F_\infty^i = \zeta_i \sqrt{\int_{t_{i-1}}^{t_i} f''(W_s)^2 \eta(ds)},$$

where each $\zeta_i \sim \mathcal{N}(0,1)$. Thus, we may conclude that for each i,

$$F_n^i \xrightarrow{\mathcal{L}} \int_{t_{i-1}}^{t_i} f''(W_s)\, dB_s$$

for a Brownian motion $B = \{B_t, t \geq 0\}$ that is independent of W_t, with $\mathbb{E}\left[B_t^2\right] = \eta(t)$.

Proof of Theorem 4.3 To prove Theorem 4.3, it is enough to show that for any finite set of times $0 = t_0 < t_1 < t_2 < \cdots < t_d$ we have

$$(\Delta_n(t_1), \Delta_n(t_2) - \Delta_n(t_1), \ldots, \Delta_n(t_d) - \Delta_n(t_{d-1})) \xrightarrow{\mathcal{L}} (\Delta(t_1), \Delta(t_2) - \Delta(t_1), \ldots, \Delta(t_d) - \Delta(t_{d-1}))$$

as $n \to \infty$; and that $\Delta_n(t)$ satisfies the tightness condition

$$\mathbb{E}\left[|\Delta_n(t) - \Delta_n(t_1)|^\gamma \; |\Delta_n(t_2) - \Delta_n(t)|^\gamma\right] \leq C(t_2 - t_1)^\alpha \tag{13}$$

for $0 \leq t_1 < t < t_2 < \infty$, $\gamma > 0$, and $\alpha > 1$.

For $\Delta_n(t) = f(W_t) - f(W_0) - \frac{1}{2}F_n(t) + Z_n(t)$, we have shown in Lemmas 4.4 and 4.5 that

$$Z_n(t) = R_n(t) - \frac{1}{2}\left(B_n(t) + C_n(t)\right) \xrightarrow{\mathcal{P}} 0$$

for each $0 \leq t \leq T$, and hence $Z_n(t_i) - Z_n(t_{i-1}) \xrightarrow{\mathcal{P}} 0$ for each t_i, $1 \leq i \leq d$. By Lemma 4.9, the pair (W, F_n) converges in law to (W, F_∞), where F_∞ is a $d-$dimensional random vector with conditional Gaussian law and whose covariance matrix is diagonal with entries

$$s_i^2 = \int_{t_{i-1}}^{t_i} f''(W_s)^2 \eta(ds).$$

It follows that, conditioned on W, each component may be expressed as an independent Gaussian random variable, equivalent in law to

$$\int_{t_{i-1}}^{t_i} f''(W_s)\, dB_s,$$

where $B = \{B_t, t \geq 0\}$ is a Brownian motion independent of W with $\mathbb{E}\left[B_t^2\right] = \eta(t)$. Finally, tightness follows from Lemma 4.8 and Corollary 4.6. Theorem 4.3 is proved. □

5 Examples

5.1 Bifractional Brownian Motion

The bifractional Brownian motion is a generalization of fractional Brownian motion, first introduced by Houdré and Villa [3]. It is defined as a centered Gaussian process $B^{H,K} = \{B^{H,K}(t), t \geq 0\}$,with covariance defined by,

$$\mathbb{E}[B_t^{H,K} B_s^{H,K}] = \frac{1}{2^K}\left(t^{2H} + s^{2H}\right)^K + \frac{1}{2^K}|t-s|^{2HK},$$

where $H \in (0,1)$, $K \in (0,1]$ (Note that the case $K=1$ corresponds to fractional Brownian motion with Hurst parameter H). The reader may refer to [4] and its references for further discussion of properties.

In this section, we show that the results of Section 4 are valid for bifractional Brownian motion with parameter values H, K such that $H \le 1/2$ and $2HK = 1/2$. In particular, this includes the end point cases $H = 1/4,\ K = 1$ studied in [7], and $H = 1/2,\ K = 1/2$ studied in [10].

Proposition 5.1. *Let $\left\{B_t^{H,K}, t \ge 0\right\}$ denote a bifractional Brownian motion. The covariance conditions (i) - (iv) of Section 4 are satisfied for values of $0 < H \le 1/2$ and $0 < K \le 1$ such that $2HK = 1/2$.*

Proof. Condition (i).

$$\begin{aligned}\mathbb{E}\left[\left(B_t^{H,K} - B_{t-s}^{H,K}\right)^2\right] &= t^{2HK} + \frac{2}{2^K}(t-s)^{2HK} - \left[t^{2H} + (t-s)^{2H}\right]^K - \frac{2}{2^K}s^{2HK} \\ &\le \left[\left|\sqrt{t} - \frac{1}{2^K}\left(t^{2H} + (t-s)^{2H}\right)^K\right| + \left|\sqrt{t-s} - \frac{1}{2^K}\left(t^{2H} + (t-s)^{2H}\right)^K\right| + \frac{1}{2^K}s^{\frac{1}{2}}\right] \\ &\le Cs^{\frac{1}{2}},\end{aligned}$$

where we used the inequality $a^m - b^m \le (a-b)^m$ for $a > b > 0$ and $m < 1$.

Condition (ii).

$$\begin{aligned}&\mathbb{E}\left[(B_t^{H,K} - B_{t-s}^{H,K})(B_r^{H,K} - B_{r-s}^{H,K})\right] \\ &= \frac{1}{2^K}\left(\left[t^{2H} + r^{2H}\right]^K - \left[t^{2H} + (r-s)^{2H}\right]^K - \left[(t-s)^{2H} + r^{2H}\right]^K + \left[(t-s)^{2H} + (r-s)^{2H}\right]^K\right) \\ &\quad + \frac{1}{2^K}\left(|t-r+s|^{2HK} - 2|t-r|^{2HK} + |t-r-s|^{2HK}\right).\end{aligned}$$

This can be interpreted as the sum of a position term, $\frac{1}{2^K}\varphi(t,r,s)$, and a distance term, $\frac{1}{2^K}\psi(t-r,s)$, where

$$\varphi(t,r,s) = \left[t^{2H} + r^{2H}\right]^K - \left[t^{2H} + (r-s)^{2H}\right]^K - \left[(t-s)^{2H} + r^{2H}\right]^K + \left[(t-s)^{2H} + (r-s)^{2H}\right]^K; \text{ and}$$

$$\psi(t-r,s) = |t-r+s|^{2HK} - 2|t-r|^{2HK} + |t-r-s|^{2HK}.$$

We begin with the position term. Note that if $K=1$, then $\varphi(t,r,s) = 0$, so we may assume $K < 1$ and $H > \frac{1}{4}$. Assume $0 < s \le r \le t$, and let $p := t - r$. By Fundamental Theorem of Calculus, we can write $\varphi(t, t-p, s)$ as

$$\begin{aligned}&2HK\int_0^s \left[t^{2H} + (t-p-\xi)^{2H}\right]^{K-1}(t-p-\xi)^{2H-1} - \left[(t-s)^{2H} + (t-p-\xi)^{2H}\right]^{K-1}(t-p-\xi)^{2H-1}d\xi \\ &= \int_0^s\int_0^s 4H^2K(1-K)\left[(t-\eta)^{2H} + (t-p-\xi)^{2H}\right]^{K-2}(t-\eta)^{2H-1}(t-p-\xi)^{2H-1}\, d\xi\, d\eta \\ &\le 4H^2K(1-K)s^2\left[(t-r)^{2H} + (r-s)^{2H}\right]^{K-2}(t-r)^{2H-1}(r-s)^{2H-1} \\ &\le Cs^2\ (t-r)^{2HK-2H-1}(r-s)^{2H-1}.\end{aligned}$$

This implies condition (ii) for the position term taking $\alpha = \frac{1}{2} + 2H > 1$ and $\beta = 1 - 2H$.

Next, consider the distance term $\psi(t-r,s)$. Without loss of generality, assume $r<t$. Again using an integral representation, we have

$$\begin{aligned}\psi(t-r,s) &= |t-r+s|^{2HK} - 2|t-r|^{2HK} + |t-r-s|^{2HK} \\ &= \int_0^s 2HK\left[(t-r+\xi)^{2HK-1} - (t-r-\xi)^{2HK-1}\right]\,d\xi \\ &= \int_0^s \int_{-\xi}^{\xi} 2HK(2HK-1)\left[t-r+\eta\right]^{2HK-2} d\eta\, d\xi \\ &\le Cs^2(t-r-s)^{2HK-2} \le Cs^2|t-r|^{-\frac{3}{2}},\end{aligned}$$

since $|t-r|\ge 2s$ implies $(t-r-s)^{-\frac{3}{2}} \le 2^{\frac{3}{2}}|t-r|^{-\frac{3}{2}}$.

Condition (iii).

$$\begin{aligned}&\left|\mathbb{E}\left[B_t^{H,K}(B_{r+s}^{H,K} - 2B_r^{H,K} + B_{r-s}^{H,K})\right]\right| \\ &\quad = \frac{1}{2^K}|[t^{2H}+(r+s)^{2H}]^K - 2[t^{2H}+r^{2H}]^K + [t^{2H}+(r-s)^{2H}]^K \\ &\qquad - \frac{1}{2^K}\left[|t-r+s|^{2HK} - 2|t-r|^{2HK} + |t-r-s|^{2HK}\right]|.\end{aligned}$$

Take first the term, $\varphi(t,r,s)$. If $r<2s$, then

$$|[t^{2H}+(r+s)^{2H}]^K - 2[t^{2H}+r^{2H}]^K + [t^{2H}+(r-s)^{2H}]^K| \le Cs^{2HK} = Cs^{\frac{1}{2}},$$

based on the inequality $a^K - b^K \le (a-b)^K$ for $a>b>0$ and $K<1$. Hence, we will assume $r\ge 2s$. If $K=1$, then $H=\frac{1}{4}$, and we have

$$\begin{aligned}\left|\sqrt{r+s} - 2\sqrt{r} + \sqrt{r-s}\right| &= \left|\int_0^s \frac{1}{2\sqrt{r+x}}\,dx - \int_0^s \frac{1}{2\sqrt{r-s+x}}\,dx\right| \\ &= \frac{1}{4}\int_0^s\int_0^s \frac{1}{(r-s+x+y)^{\frac{3}{2}}}\,dy\,dx \\ &\le \frac{1}{4}s^2(r-s)^{-\frac{3}{2}};\end{aligned}$$

and if $K<1$,

$$\begin{aligned}&|\varphi(t,r,s)| \\ &= \left|\int_0^s 2HK[t^{2H}+(r+x)^{2H}]^{K-1}(r+x)^{2H-1}dx - \int_0^s 2HK[t^{2H}+(r-s+x)^{2H}]^{K-1}(r-s+x)^{2H-1}dx\right| \\ &\le \left|\int_0^s\int_0^s 4H^2K(K-1)[t^{2H}+(r-s+x+y)^{2H}]^{K-2}(r-s+x+y)^{4H-2}\,dy\,dx\right| \\ &\quad + \left|\int_0^s\int_0^s 2H(2H-1)K[t^{2H}+(r-s+x+y)^{2H}]^{K-1}(r-s+x+y)^{2H-2}\,dy\,dx\right| \\ &\le 4H^2K(1-K)s^2(r-s)^{2HK-2} + 2H(1-2H)Ks^2(r-s)^{2HK-2} \le Cs^2(r-s)^{-\frac{3}{2}}.\end{aligned}$$

This bound for $\varphi(t,r,s)$ also holds in the case $|t-r|<2s$, so the bound of $Cs^{\frac{1}{2}}$ is valid for this case. Next for the second term. Note that if $|t-r|<2s$, then

$$\left|\frac{1}{2^K}\left(|t-r+s|^{2HK} - 2|t-r|^{2HK} + |t-r-s|^{2HK}\right)\right| \le 2(3s)^{2HK} \le Cs^{\frac{1}{2}}.$$

If $|t-r| \geq 2s$, then we have

$$\begin{aligned}\left|\sqrt{|t-r|+s} - 2\sqrt{|t-r|} + \sqrt{|t-r|-s}\right| &= \left|\int_0^s \frac{1}{2\sqrt{|t-r|+x}}dx - \int_0^s \frac{1}{2\sqrt{|t-r|-s+x}}dx\right| \\ &= \int_0^s \int_0^s \frac{1}{(|t-r|-s+x+y} \ dy \ dx \\ &\leq \frac{s^2}{4(|t-r|-s)^{\frac{3}{2}}} \leq \frac{s^2}{2|t-r|^{\frac{3}{2}}},\end{aligned}$$

using the inequality $\frac{1}{|t-r|-s} \leq \frac{2}{|t-r|}$ for $|t-r| \geq 2s$. This bound for $\psi(t-r,s)$ holds even in the case $r < 2s$, so the bound of $Cs^{\frac{1}{2}}$ when $r < 2s$ is verified as well.

Condition (iv).
For the first part, we have for all $t \geq s$,

$$\left|\mathbb{E}\left[B_t^{H,K}\left(B_{t+s}^{H,K} - B_{t-s}^{H,K}\right)\right]\right| = \left|\frac{1}{2^K}\left[t^{2H} + (t+s)^{2H}\right]^K - \frac{1}{2^K}\left[t^{2H} + (t-s)^{2H}\right]^K\right|.$$

This is bounded by $Cs^{\frac{1}{2}}$ if $t < 2s$. On the other hand, if $t \geq 2s$,

$$\begin{aligned}\left|\frac{1}{2^K}\left[t^{2H} + (t+s)^{2H}\right]^K - \frac{1}{2^K}\left[t^{2H} + (t-s)^{2H}\right]^K\right| &= \left|\frac{1}{2^K}\int_{-s}^s 2HK\left[t^{2H} + (t+x)^{2H}\right]^{K-1}(t+x)^{2H-1} \ dx\right| \\ &\leq Cs(t-s)^{2HK-1} \ = \ Cs(t-s)^{-\frac{1}{2}}.\end{aligned}$$

For $0 < s \leq r \leq T$ with $t \geq 2s$ and $|t-r| \geq 2s$,

$$\begin{aligned}\left|\mathbb{E}\left[B_r^{H,K}\left(B_{t+s}^{H,K} - B_{t-s}^{H,K}\right)\right]\right| &\leq \left|\frac{1}{2^K}\left[r^{2H} + (t+s)^{2H}\right]^K - \frac{1}{2^K}\left[r^{2H} + (t-s)^{2H}\right]^K\right| \\ &\quad + \left|\frac{1}{2^K}|r-t+s|^{2HK} - \frac{1}{2^K}|r-t-s|^{2HK}\right| \\ &\leq Cs(t-s)^{-\frac{1}{2}} + Cs|r-t|^{-\frac{1}{2}}.\end{aligned}$$

If $t < 2s$ or $|t-r| < 2s$, then we have an upper bound of $Cs^{\frac{1}{2}}$ by condition (i) and Cauchy-Schwarz.

For the third bound, if $t > 2s$,

$$\begin{aligned}\left|\mathbb{E}\left[B_s^{H,K}\left(B_t^{H,K} - B_{t-s}^{H,K}\right)\right]\right| &\leq \left|\frac{1}{2^K}\left[s^{2H} + t^{2H}\right]^K - \frac{1}{2^K}\left[s^{2H} + (t-s)^{2H}\right]^K\right| \\ &\quad + \left|\frac{1}{2^K}(t-s)^{2HK} - \frac{1}{2^K}(t-2s)^{2HK}\right| \\ &\leq \frac{2}{2^K}\int_0^s HK\left[s^{2H} + (t-s+x)^{2H}\right]^K (t-s+x)^{2H-1} \ dx \\ &\quad + \frac{1}{2^{K+1}}\int_0^s (t-2s+x)^{-\frac{1}{2}}dx \\ &\leq Cs(t-2s)^{-\frac{1}{2}} = Cs^{\frac{1}{2}+\gamma}(t-2s)^{-\gamma}\end{aligned}$$

for $\gamma = \frac{1}{2}$.

$\square$

Proposition 5.2. *Let $B^{H,K}$ be a bifractional Brownian motion with parameters $H \le 1/2$ and $HK = 1/4$. Then Condition (v) of Section 4 holds, with the functions $\eta^+(t) = 2C_K^+ t$ and $\eta^-(t) = 2C_K^- t$, where*

$$C_K^+ = \frac{1}{4^K}\left(2 + \sum_{m=1}^{\infty}\left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2\right),$$

$$C_K^- = \frac{(2-\sqrt{2})^2}{2^{2K+1}} + \frac{1}{4^K}\sum_{m=1}^{\infty}\left(\sqrt{2m+2} - 2\sqrt{2m+1} + \sqrt{2m}\right)^2.$$

Proof. As in Prop. 5.1, we use the decomposition,

$$\beta_n(j,k) = \frac{1}{2^K}\varphi\left(\frac{j}{n}, \frac{k}{n}, \frac{1}{n}\right) + \frac{1}{2^K}\psi\left(\frac{j-k}{n}, \frac{1}{n}\right) = 2^{-K} n^{-\frac{1}{2}}\varphi(j,k,1) + 2^{-K} n^{-\frac{1}{2}}\psi(j-k,1).$$

The first task is to show that

$$\lim_{n\to\infty} \sum_{j,k=1}^{\lfloor nt \rfloor} n^{-1}\varphi(j,k,1)^2 \;=\; 0. \tag{14}$$

Proof of (14). We consider two cases, based on the value of H. First, assume $H < \frac{1}{2}$. Then

$$\begin{aligned}
\varphi(j,k,1) &= \left[(j+1)^{2H} + (k+1)^{2H}\right]^K - \left[(j+1)^{2H} + k^{2H}\right]^K \\
&\quad - \left[j^{2H} + (k+1)^{2H}\right]^K + \left[j^{2H} + k^{2H}\right]^K \\
&= \int_0^1 2HK\left[(j+1)^{2H} + (k+x)^{2H}\right]^{K-1}(k+x)^{2H-1}dx \\
&\quad - \int_0^1 2HK\left[j^{2H} + (k+x)^{2H}\right]^{K-1}(k+x)^{2H-1}dx \\
&= \int_0^1\int_0^1 4H^2K(1-K)\left[(j+y)^{2H} + (k+x)^{2H}\right]^{K-2}(k+x)^{2H-1}(j+y)^{2H-1}\,dy\,dx \\
&\le Ck^{2HK-2H-1}j^{2H-1} = Ck^{-\frac{1}{2}-2H}j^{2H-1}.
\end{aligned}$$

With this bound, it follows that

$$\begin{aligned}
\frac{1}{n}\sum_{j,k=1}^{\lfloor nt \rfloor}\varphi(j,k,1)^2 &\le \frac{C}{n}\sum_{j=1}^{\lfloor nt \rfloor} j^{4H-2}\sum_{k=1}^{\infty} k^{-1-4H} \\
&\le \frac{C}{n}\lfloor nt \rfloor^{4H-1} \le Ctn^{4H-2},
\end{aligned}$$

which tends to zero as $n \to \infty$ because $H < \frac{1}{2}$.

Next, the case $H = \frac{1}{2}$. Note that this implies $K = \frac{1}{2}$, and we have

$$|\varphi(j,k,1)| = \left|\sqrt{j+k+2} - 2\sqrt{j+k+1} + \sqrt{j+k}\right| \le C(j+k)^{-\frac{3}{2}}.$$

So with this bound,

$$\begin{aligned}
\sum_{j,k=1}^{\lfloor nt \rfloor} n^{-1}\varphi(j,k,1)^2 &\le \frac{C}{n}\sum_{j,k=1}^{\lfloor nt \rfloor}(j+k)^{-3} \\
&\le \frac{C}{n}\sum_{j=1}^{\lfloor nt \rfloor}\sum_{m=j+1}^{\infty} m^{-3} \le \frac{C}{n}\sum_{j=1}^{\lfloor nt \rfloor} j^{-2}
\end{aligned}$$

which tends to zero as $n \to \infty$ because j^{-2} is summable. Hence, (14) is proved.

From (14), it follows that to investigate the limit behavior of $\eta_n^+(t), \eta_n^-(t)$, it is enough to consider

$$\frac{1}{n}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \psi(2j-2k,1)^2 + \psi(2j-2k,1)^2 = \frac{2}{n}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \psi(2j-2k,1)^2; \text{ and}$$

$$\frac{1}{n}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \psi(2j-2k+1,1)^2 + \psi(2j-2k-1,1)^2 = \frac{2}{n}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \psi(2j-2k+1,1)^2;$$

since the sums of $\psi(2j-2k+1,1)^2$ and $\psi(2j-2k-1,1)^2$ are equal by symmetry. We start with

$$\begin{aligned}
&\frac{1}{n}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \psi(2j-2k,1)^2 \\
&= \frac{1}{4^K n}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left(\sqrt{|2j-2k+1|} - 2\sqrt{|2j-2k|} + \sqrt{|2j-2k-1|}\right)^2 \\
&= \frac{1}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} 4 + \frac{2}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{k=1}^{j-1} \left(\sqrt{2j-2k+1} - 2\sqrt{2j-2k} + \sqrt{2j-2k-1}\right)^2 \\
&= \frac{4\lfloor \frac{nt}{2} \rfloor}{4^K n} + \frac{2}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{m=1}^{j-1} \left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2 \\
&= \frac{4\lfloor \frac{nt}{2} \rfloor}{4^K n} + \frac{2}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{m=1}^{\infty} \left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2 \\
&\quad - \frac{2}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{m=j}^{\infty} \left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2,
\end{aligned}$$

where the last term tends to zero since

$$\sum_{m=j}^{\infty} \left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2 \leq \sum_{m=j}^{\infty} (2m-1)^{-3} \leq C(2j-1)^{-2},$$

and,

$$\frac{C}{n}\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} (2j-1)^{-2} \longrightarrow 0$$

as $n \to \infty$ .We therefore conclude that,

$$\begin{aligned}
\eta^+(t) &= \lim_{n\to\infty} \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left(\beta_n(2j-1,2k-1)^2 + \beta_n(2j-2,2k-2)^2\right) \\
&= \lim_{n\to\infty} \frac{2}{n}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \psi(2j-2k,1)^2 = 2C_{K,1}t,
\end{aligned}$$

where

$$C_{K,1} = \frac{1}{4^K}\left(2 + \sum_{m=1}^{\infty}\left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2\right).$$

For the other term,

$$\begin{aligned}&\frac{1}{n}\sum_{j,k=1}^{\lfloor \frac{nt}{2}\rfloor}\psi(2j-2k+1,1)^2\\ &\quad= \frac{1}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}(2-\sqrt{2})^2 + \frac{2}{4^K n}\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\sum_{k=1}^{j-1}\left(\sqrt{2j-2k+2} - 2\sqrt{2j-2k+1} - \sqrt{2j-2k}\right)^2.\end{aligned}$$

Hence, by a similar computation,

$$\eta^-(t) = \lim_{n\to\infty}\sum_{j,k=1}^{\lfloor \frac{nt}{2}\rfloor}\beta_n(2j-1,2k-2)^2 + \beta_n(2j-2,2k-1)^2 = 2C_{K,2}t,$$

where

$$C_{K,2} = \frac{(2-\sqrt{2})^2}{2^{2K+1}} + \frac{1}{4^K}\sum_{m=1}^{\infty}\left(\sqrt{2m+2} - 2\sqrt{2m+1} + \sqrt{2m}\right)^2.$$

then we take $C_K = 2(C_{K,1} - C_{K,2})$. □

As a concluding remark, it is easy to show that $C_{K,1} > C_{K,2}$, and in general we have $\eta^+(t) \geq \eta^-(t)$.

5.2 A Gaussian process with differentiable covariance function

Consider the following class of Gaussian processes. Let $\{F_t, t \geq 0\}$ be a mean-zero Gaussian process with covariance defined by,

$$\mathbb{E}\left[F_r F_t\right] = r\phi\left(\frac{t}{r}\right), \quad t \geq r \tag{15}$$

where $\phi : [1,\infty) \to \mathbb{R}$ is twice-differentiable on $(1,\infty)$ and satisfies the following:

(ϕ.1) $\|\phi\|_\infty := \sup_{x\geq 1}|\phi(x)| = c_{\phi,0} < \infty$.

(ϕ.2) For $1 < x < \infty$,

$$|\phi'(x)| \leq \frac{c_{\phi,1}}{\sqrt{x-1}}.$$

(ϕ.3) For $1 < x < \infty$,

$$|\phi''(x)| \leq c_{\phi,2}x^{-\frac{1}{2}}(x-1)^{-\frac{3}{2}}.$$

where $c_{\phi,j}$, $j = 0,1,2$ are nonnegative constants.

Proposition 5.3. *The process $\{F_t, t \geq 0\}$ described above satisfies Conditions (i) - (iv) of Theorem 4.1.*

Proof. Condition (i). By Condition (ϕ.2),

$$\begin{aligned}
\mathbb{E}\left[(F_t - F_{t-s})^2\right] &= t\phi(1) + (t-s)\phi(1) - 2(t-s)\phi\left(1 + \frac{s}{t-s}\right) \\
&\leq 2(t-s)\left|\phi\left(1+\frac{s}{t-s}\right) - \phi(1)\right| + s\,|\phi(1)| \\
&\leq 2(t-s)\left|\int_1^{1+\frac{s}{t-s}} \phi'(x)\ dx\right| + s\|\phi\|_\infty \\
&\leq 2(t-s)\int_1^{1+\frac{s}{t-s}} \frac{c_{\phi,1}}{\sqrt{x-1}} dx + s\|\phi\|_\infty \\
&\leq Cs^{\frac{1}{2}}\sqrt{t-s} + s\|\phi\|_\infty \\
&\leq Cs^{\frac{1}{2}},
\end{aligned}$$

where the constant C depends on $\max\left\{\sqrt{T}, \|\phi\|_\infty\right\}$.

Condition (ii). For $2s \leq r \leq t-2s$ we have by the Mean Value Theorem,

$$\begin{aligned}
|\mathbb{E}\left[F_tF_r - F_{t-s}F_r - F_tF_{r-s} + F_{t-s}F_{r-s}\right]| &= \left|r\ \left[\phi\left(\frac{t}{r}\right) - \phi\left(\frac{t-s}{r}\right)\right] - (r-s)\ \left[\phi\left(\frac{t}{r-s}\right) - \phi\left(\frac{t-s}{r-s}\right)\right]\right| \\
&\leq s \sup_{\left[\frac{t-s}{r}, \frac{t}{r-s}\right]} |\phi''(x)| \left(\frac{t}{r-s} - \frac{t-s}{r}\right) \\
&\leq c_{\phi,2}s\left(\frac{t-s}{r}\right)^{-\frac{1}{2}}\left(\frac{t-s}{r} - 1\right)^{-\frac{3}{2}}\ \left(\frac{ts}{r(r-s)}\right) \\
&\leq \frac{C\sqrt{T}\ s^2}{(t-r)^{\frac{3}{2}}} = C\sqrt{T}\ s^2|t-r|^{-\frac{3}{2}}.
\end{aligned}$$

Condition (iii). By symmetry we can assume $r \leq t$. Consider the following cases: First, suppose $2s \leq r \leq t-2s$. Then we have

$$\begin{aligned}
|\mathbb{E}\left[F_t(F_{r+s} - 2F_r + F_{r-s})\right]| &= \left|(r+s)\phi\left(\frac{t}{r+s}\right) - 2r\phi\left(\frac{t}{r}\right) + (r-s)\phi\left(\frac{t}{r-s}\right)\right| \\
&= \left|(r+s)\left[\phi\left(\frac{t}{r+s}\right) - \phi\left(\frac{t}{r}\right)\right] - (r-s)\left[\phi\left(\frac{t}{r}\right) - \phi\left(\frac{t}{r-s}\right)\right]\right| \\
&\leq \frac{st}{r} \sup_{\left[\frac{t}{r+s}, \frac{t}{r-s}\right]} |\phi''(x)| \left(\frac{t}{r-s} - \frac{t}{r+s}\right) \\
&\leq \frac{2s^2t^2c_{\phi,2}}{r(r-s)(r+s)}\left(\frac{r+s}{t}\right)^{\frac{1}{2}}\left(\frac{r+s}{t-r-s}\right)^{\frac{3}{2}} \\
&\leq \frac{Cs^2t^{\frac{3}{2}}}{r(t-r)^{\frac{3}{2}}}.
\end{aligned}$$

There are two possibilities, depending on the value of r. If $r \geq \frac{t}{2}$, then $\frac{t}{r} \leq 2$, and we have a bound of

$$Cs^2\left(\frac{t}{r}\right)\left(\frac{\sqrt{T}}{(t-r)^{\frac{3}{2}}}\right) \leq 2C\sqrt{T}\ s^2|t-r|^{-\frac{3}{2}}.$$

on the other hand, if $r < \frac{t}{2}$, then $\frac{t}{t-r} \leq 2$ and the bound is

$$Cs^2\left(\frac{t}{t-r}\right)\left(\frac{\sqrt{T}}{r\sqrt{t-r}}\right) \leq 2C\sqrt{T}\ s^2\left[(r-s)^{-\frac{3}{2}} + |t-r|^{-\frac{3}{2}}\right].$$

For the case $|t-r| < 2s$, assume that $t = r + ks$ for some $0 \leq k < 2$. Then

$$\begin{aligned}
&|\mathbb{E}\left[F_t\left(F_{r+s} - 2F_r + F_{r-s}\right)\right]| \\
&= \left|(t\wedge(r_s))\,\phi\left(\frac{t\vee(r+s)}{t\wedge(r+s)}\right) - 2r\phi\left(\frac{t}{r}\right) + (r-s)\phi\left(\frac{t}{r-s}\right)\right| \\
&= \left|(t\wedge(r_s))\,\phi\left(\frac{t\vee(r+s)}{t\wedge(r+s)}\right) - (r+s)\phi(1) - 2r\phi\left(\frac{t}{r}\right) + 2r\phi(1) + (r-s)\phi\left(\frac{t}{r-s}\right) - (r-s)\phi(1)\right| \\
&\leq 3(r+s)\left|\phi\left(1 + \frac{(k+1)s}{r-s}\right) - \phi(1)\right| \ \leq\ 3(r+s)\left|\int_1^{1+\frac{(k+1)s}{r-s}} \phi'(x)\ dx\right| \\
&\leq 3(r+s)\int_1^{1+\frac{(k+1)s}{r-s}} \frac{c_{\phi,1}}{\sqrt{x-1}}dx\ \leq\ C\sqrt{T}s^{\frac{1}{2}}.
\end{aligned}$$

For the last case, note that if $t \wedge r < 2s$, then we have an upper bound of $8sc_{\phi,0} \leq Cs^{\frac{1}{2}}$, since $\mathbb{E}\left[F_sF_t\right] \leq s\|\phi\|_\infty$.

Condition (iv). Take first the bound for $\mathbb{E}\left[F_t(F_{t+s} - F_{t-s})\right]$. Note that if $t < 2s$, then an upper bound of $Cs^{\frac{1}{2}}$ is clear, so we will assume $t \geq 2s$. We have

$$\begin{aligned}
|\mathbb{E}\left[F_tF_{t+s} - F_tF_{t-s}\right]| &= \left|t\phi\left(\frac{t+s}{t}\right) - (t-s)\phi\left(\frac{t}{t-s}\right)\right| \\
&\leq (t-s)\sup_{\left[\frac{t+s}{t},\frac{t}{t-s}\right]}|\phi'(x)|\left|\frac{t+s}{t} - \frac{t}{t-s}\right| + s\left|\phi\left(\frac{t+s}{t}\right)\right| \\
&\leq c_{\phi,1}\ \frac{s^2}{t}\sqrt{\frac{t}{t+s}}\sqrt{\frac{t}{s}} + c_{\phi,0}\ s\frac{\sqrt{T}}{\sqrt{t-s}} \\
&\leq Cs\sqrt{T}\ (t-s)^{-\frac{1}{2}}.
\end{aligned}$$

For the case $r \neq t$, first assume $r \leq t - 2s$. By condition (ϕ.2),

$$\begin{aligned}
|\mathbb{E}\left[F_rF_{t+s} - F_rF_{t-s}\right]| &= \left|r\phi\left(\frac{t+s}{r}\right) - r\phi\left(\frac{t-s}{r}\right)\right| \leq 2s\sup_{\left[\frac{t-s}{r},\frac{t+s}{r}\right]}|\phi'(x)| \\
&\leq \frac{2s\sqrt{r}\ c_{\phi,1}}{\sqrt{t-r-s}} \leq \frac{C\sqrt{T}\ s}{\sqrt{t-r}}.
\end{aligned}$$

If $r \geq t + 2s$, then

$$\begin{aligned}
|\mathbb{E}\left[F_rF_{t+s} - F_rF_{t-s}\right]| &= \left|(t+s)\phi\left(\frac{r}{t+s}\right) - (t-s)\phi\left(\frac{r}{t-s}\right)\right| \\
&\leq t\int_0^{2s}\left|\phi'\left(\frac{r}{t-s+x}\right)\right|\ dx + 2s\|\phi\|_\infty \\
&\leq \frac{2stc_{\phi,1}\sqrt{t+s}}{\sqrt{r-t}} + \frac{2sc_{\phi,0}\sqrt{T}}{\sqrt{t-s}} \\
&\leq Cs(r-t)^{-\frac{1}{2}} + Cs(t-s)^{-\frac{1}{2}}.
\end{aligned}$$

For the case $t < 2s$ or $|r-t| < 2s$, the bound follows from condition (i) and Cauchy-Schwarz.

For the third part of condition (iv), we have for $t > 2s$,

$$\begin{aligned}\mathbb{E}\left[F_sF_t - F_sF_{t-s}\right] &= s\phi\left(\frac{t}{s}\right) - s\phi\left(\frac{t-s}{s}\right)\\ &\le s \sup_{\left[\frac{t-s}{s},\frac{t}{s}\right]} |\phi(x)| \left(\frac{t}{s} - \frac{t-s}{s}\right)\\ &\le \frac{c_{\phi,1}s}{\sqrt{\frac{t-s}{s}-1}}\\ &\le Cs^{\frac{3}{2}}(t-2s)^{-\frac{1}{2}}\\ &= Cs^{\frac{1}{2}+\gamma}(t-2s)^{-\gamma}\end{aligned}$$

where $\gamma = \frac{1}{2}$. □

Proposition 5.4. *Suppose $\phi(x)$ satisfies condition $(\phi.1)$, and in addition $\phi(x)$ satisfies:*

$$(\phi.4): \quad \phi'(x) = \frac{\kappa}{\sqrt{x-1}} + \frac{\psi(x)}{\sqrt{x}},$$

where $\kappa \in \mathbb{R}$ and $\psi : (1,\infty) \to \mathbb{R}$ is a bounded differentiable function satisfying $|\psi'(1+x)| \le C_\psi x^{-\frac{1}{2}}$ for some positive constant C_ψ. Then Condition (v) of Theorem 4.1 is satisfied, with $\eta^+(t) = C_\beta^+ t^2$, and $\eta^-(t) = C_\beta^- t^2$ for positive constants C_β^+, C_β^-.

Remark 5.5. Observe that condition $(\phi.4)$ implies $(\phi.2)$ but not $(\phi.3)$.

Proof. We want to show

$$\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor} \beta_n(2j-1,2k-1)^2 \longrightarrow C_{\beta,1}t^2; \tag{16}$$

$$\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor} \beta_n(2j-2,2k-2)^2 \longrightarrow C_{\beta,2}t^2; \quad \text{and} \tag{17}$$

$$\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor} \beta_n(2j-1,2k-2)^2 \longrightarrow C_{\beta,3}t^2; \tag{18}$$

so that $C_\beta^+ = C_{\beta,1} + C_{\beta,2}$, and $C_\beta^- = 2C_{\beta,3}$. We will show computations for (16), with the others being similar. As in Prop. 5.2,

$$\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor} \beta_n(2j-1,2k-1)^2 = \sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor} \beta_n(2j-1,2j-1)^2 + 2\sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor}\sum_{k=1}^{j-1} \beta_n(2j-1,2k-1)^2,$$

so it is enough to show

$$\lim_{n\to\infty} \sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor}\sum_{k=1}^{j-1} \beta_n(2j-1,2k-1)^2 = C_1t^2; \text{ and} \tag{19}$$

$$\lim_{n\to\infty} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1,2j-1)^2 = C_2 t^2. \tag{20}$$

Proof of (19). For $1 \le k \le j-1$ we have

$$\begin{aligned}\beta_n(2j-1,2k-1) &= \frac{2k}{n}\left(\phi\left(\frac{2j}{2k}\right) - \phi\left(\frac{2j-1}{2k}\right)\right) - \frac{2k-1}{n}\left(\phi\left(\frac{2j}{2k-1}\right) - \phi\left(\frac{2j-1}{2k-1}\right)\right) \\ &= \frac{2k}{n}\int_{\frac{2j-1}{2k}}^{\frac{2j}{2k}} \phi'(x)\, dx - \frac{2k-1}{n}\int_{\frac{2j-1}{2k-1}}^{\frac{2j}{2k-1}} \phi'(x)\, dx.\end{aligned}$$

Using the change of index $j = k+m$ and a change of variable for the two integrals, this becomes,

$$\beta_n(2j-1,2k-1) = \frac{1}{n}\int_{2m-1}^{2m} \phi'\left(1+\frac{y}{2k}\right)\, dy - \frac{1}{n}\int_{2m}^{2m+1} \phi'\left(1+\frac{y}{2k-1}\right)\, dy. \tag{21}$$

With the decomposition of (ϕ.4), we will address (21) in two parts. Using the first term, we have

$$\begin{aligned}&\frac{\kappa}{n}\int_{2m-1}^{2m}\sqrt{\frac{2k}{y}}\, dy - \frac{\kappa}{n}\int_{2m}^{2m+1}\sqrt{\frac{2k-1}{y}}\, dy \\ &= \frac{2\kappa}{n}\left[\sqrt{2k}\left(\sqrt{2m}-\sqrt{2m-1}\right) - \sqrt{2k-1}\left(\sqrt{2m+1}-\sqrt{2m}\right)\right].\end{aligned}$$

We are interested in the sum,

$$\sum_{k=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{m=1}^{\lfloor \frac{nt}{2} \rfloor - k} \frac{4\kappa^2}{n^2}\left[\sqrt{2k}\left(\sqrt{2m}-\sqrt{2m-1}\right) - \sqrt{2k-1}\left(\sqrt{2m+1}-\sqrt{2m}\right)\right]^2. \tag{22}$$

We can write

$$\begin{aligned}&\sqrt{2k}\left(\sqrt{2m}-\sqrt{2m-1}\right) - \sqrt{2k-1}\left(\sqrt{2m+1}-\sqrt{2m}\right) \\ &= -\sqrt{2k-1}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right) + \left(\sqrt{2k}-\sqrt{2k-1}\right)\left(\sqrt{2m}-\sqrt{2m-1}\right).\end{aligned}$$

Observe that

$$\left[\left(\sqrt{2k}-\sqrt{2k-1}\right)\left(\sqrt{2m}-\sqrt{2m-1}\right)\right]^2 \le \frac{1}{(2k-1)(2m-1)},$$

and so

$$\frac{4\kappa^2}{n^2}\sum_{k=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{m=1}^{\lfloor \frac{nt}{2} \rfloor - k} \frac{1}{(2k-1)(2m-1)} \le \frac{4\kappa^2}{n^2}\left(\sum_{k=1}^{\lfloor \frac{nt}{2} \rfloor}\frac{1}{2k-1}\right)^2 \le \frac{C\log(nt)^2}{n^2}.$$

Therefore the contribution of this term is zero, and it follows by Cauchy-Schwarz that the only significant term is

$$\begin{aligned}&\frac{4\kappa^2}{n^2}\sum_{k=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{m=1}^{\lfloor \frac{nt}{2} \rfloor - k}(2k-1)\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2 \\ &= 4\kappa^2\sum_{m=1}^{\lfloor \frac{nt}{2} \rfloor}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2\sum_{k=1}^{\lfloor \frac{nt}{2} \rfloor - m}\frac{2k-1}{n^2} \\ &= 4\kappa^2\sum_{m=1}^{\lfloor \frac{nt}{2} \rfloor}\left(\sqrt{2m+1}-2\sqrt{2m}+\sqrt{2m-1}\right)^2\frac{\left(\lfloor \frac{nt}{2} \rfloor - m\right)^2}{n^2},\end{aligned}$$

which converges as $n \to \infty$ to

$$\kappa^2 t^2 \sum_{m=1}^{\infty} \left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2, \ \kappa = \frac{1}{2} K e^{-\frac{\beta^2}{2}}.$$

Next, we consider the term $\frac{1}{\sqrt{x}}\psi(x)$. The contribution of this term to (21) is

$$\frac{1}{n}\int_{2m-1}^{2m} \sqrt{\frac{2k}{2k+y}}\ \psi\left(1+\frac{y}{2k}\right)\ dy - \frac{1}{n}\int_{2m}^{2m+1} \sqrt{\frac{2k-1}{2k-1+y}}\ \psi\left(1+\frac{y}{2k-1}\right)\ dy. \tag{23}$$

We can bound (23) by

$$\begin{aligned}
&\frac{1}{n}\left|\int_{2m-1}^{2m} \sqrt{\frac{2k}{2k+y}}\ \psi\left(1+\frac{y}{2k}\right)\ dy - \int_{2m}^{2m+1} \sqrt{\frac{2k-1}{2k-1+y}}\ \psi\left(1+\frac{y}{2k-1}\right)\ dy\right| \\
&\quad \le \frac{1}{n}\left[\sup_{(1,\infty)} |\psi(x)| \frac{\sqrt{2k}-\sqrt{2k-1}}{\sqrt{2k+2m-1}} + \sqrt{\frac{2k}{2k+2m-1}}\left|\int_{2m-1}^{2m} \psi\left(1+\frac{y}{2k}\right)\ dy - \int_{2m}^{2m+1} \psi\left(1+\frac{y}{2k-1}\right)\ dy\right|\right] \\
&\quad = \frac{1}{n}\left(A_{k,m} + B_{k,m}\right).
\end{aligned}$$

Since $|\psi(x)|$ is bounded, we have

$$A_{k,m} \le \frac{C}{\sqrt{2k-1}\sqrt{2k+2m-1}} \le \frac{C}{\sqrt{2k-1}\ \sqrt{2m-1}}\ . \tag{24}$$

For $B_{k,m}$ using that $|\psi'(x+1)| \le C x^{-\frac{1}{2}}$,

$$\begin{aligned}
&\left|\int_{2m-1}^{2m} \psi\left(1+\frac{y}{2k}\right)\ dy - \int_{2m}^{2m+1} \psi\left(1+\frac{y}{2k-1}\right)\ dy\right| \\
&= \left|\int_{2m-1}^{2m} \psi\left(1+\frac{u}{2k}\right) - \psi\left(1+\frac{u+1}{2k-1}\right)\ du\right| \\
&\le \int_{2m-1}^{2m} \left|\int_{\frac{u+1}{2k-1}}^{\frac{u}{2k}} \psi'(1+v)dv\right| du \\
&\le C\int_{2m-1}^{2m}\int_{\frac{u}{2k}}^{\frac{u+1}{2k-1}} v^{-\frac{1}{2}} dv\ du \le \frac{C}{\sqrt{2k-1}}\left(\sqrt{2m+1}-\sqrt{2m}\right) \\
&\le \frac{C}{\sqrt{2k-1}\sqrt{2m-1}}
\end{aligned}$$

so that

$$B_{k,m} \le \sqrt{\frac{2k}{2k+2m-1}} \cdot \frac{C}{\sqrt{2k-1}\sqrt{2m-1}} \le \frac{C}{\sqrt{2k-1}\sqrt{2m-1}}. \tag{25}$$

Hence, from (24) and (25), we obtain

$$\begin{aligned}
&\sum_{k=1}^{\lfloor \frac{nt}{2} \rfloor} \sum_{m=1}^{\lfloor \frac{nt}{2} \rfloor - k} \frac{C}{n^2}\left(\frac{1}{\sqrt{2k-1}\sqrt{2m-1}}\right)^2 \\
&\quad \le \frac{C}{n^2}\sum_{k,m=1}^{\lfloor \frac{nt}{2} \rfloor} \frac{1}{(2m-1)(2k-1)}\ \le\ \frac{C\log(n)^2}{n^2}
\end{aligned}$$

so the portion represented by (23) tends to zero as $n \to \infty$. Since this term is not significant, it follows by Cauchy-Schwarz that the behavior of

$$\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \sum_{k=1}^{j-1} \beta_n(2j-1, 2k-1)^2$$

is dominated by eq. (22), and we have the result (19), with

$$C_1 = \frac{\kappa^2}{4} \sum_{m=1}^{\infty} \left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1} \right)^2 .$$

Proof of (20). For each j,

$$\begin{aligned}
\beta_n(2j-1, 2j-1)^2 &= \left(\frac{2j}{n}\phi(1) - 2\frac{2j-1}{n}\phi\left(\frac{2j}{2j-1}\right) + \frac{2j-1}{n}\phi(1) \right)^2 \\
&= \frac{1}{n^2}\left[\phi(1) + (4j-2)\left(\phi(1) - \phi\left(1 + \frac{1}{2j-1}\right)\right)\right]^2 \\
&= \frac{\phi(1)^2}{n^2} + \frac{4(2j-1)\phi(1)}{n^2}\left(\phi(1) - \phi\left(1 + \frac{1}{2j-1}\right)\right) + \frac{4(2j-1)^2}{n^2}\left(\phi(1) - \phi\left(1 + \frac{1}{2j-1}\right)\right)^2 .
\end{aligned}$$

Since $\left|\phi(1) - \phi\left(1 + \frac{1}{2j-1}\right)\right| \le \frac{c_{\phi,3}}{\sqrt{2j-1}}$ by (ϕ.3), we see that

$$\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \left[\frac{\phi(1)^2}{n^2} + \frac{4(2j-1)\phi(1)}{n^2}\left|\phi(1) - \phi\left(1 + \frac{1}{2j-1}\right)\right|\right] \le Cn^{-\frac{1}{2}};$$

which implies only the last term is significant in the limit. Again we use (31) to obtain:

$$\begin{aligned}
\phi(1) - \phi\left(1 + \frac{1}{2j-1}\right) &= -\int_1^{1+\frac{1}{2j-1}} \phi'(x)\, dx \\
&= \kappa \int_1^{1+\frac{1}{2j-1}} \frac{1}{\sqrt{x-1}}\, dx - \int_1^{1+\frac{1}{2j-1}} \frac{1}{\sqrt{x}}\psi(x)\, dx \\
&= \frac{2\kappa}{\sqrt{2j-1}} + O\left(\frac{1}{2j-1}\right);
\end{aligned}$$

hence

$$\frac{4(2j-1)^2}{n^2}\left(\phi(1) - \phi\left(1 + \frac{1}{2j}\right)\right)^2 = \frac{16\kappa^2(2j-1)^2}{n^2(2j-1)} + O\left(\frac{j^{\frac{1}{2}}}{n^2}\right),$$

and taking $n \to \infty$,

$$\lim_{n\to\infty} \sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \frac{16\kappa^2(2j-1)}{n^2} + O\left(\frac{j^{\frac{1}{2}}}{n^2}\right) = 4\kappa^2 t^2,$$

which gives (20). Thus (16) is proved with $C_{\beta,1} = 4\kappa^2 + \frac{\kappa^2}{2}\sum_{m=1}^{\infty}\left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2$, $\kappa = \frac{1}{2}Ke^{-\frac{\beta^2}{2}}$.

By similar computations,

$$C_{\beta,2} = 4\kappa^2 + \frac{\kappa^2}{2}\sum_{m=1}^{\infty}\left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2; \text{ and}$$

$$C_{\beta,3} = 4\kappa^2 + \frac{\kappa^2}{2}\sum_{m=1}^{\infty}\left(\sqrt{2m+2} - 2\sqrt{2m+1} + \sqrt{2m}\right)^2;$$

and so

$$C_\beta^+ = C_{\beta,1} + C_{\beta,2} = 8\kappa^2 + \kappa^2\sum_{m=1}^{\infty}\left(\sqrt{2m+1} - 2\sqrt{2m} + \sqrt{2m-1}\right)^2,$$

$$C_\beta^- = 2C_{\beta,3} = 8\kappa^2 + \kappa^2\sum_{m=1}^{\infty}\left(\sqrt{2m+2} - 2\sqrt{2m+1} + \sqrt{2m}\right)^2.$$

Note that $C_\beta^+ \geq C_\beta^-$, and it follows that $\eta(t) = \eta^+(t) - \eta^-(t)$ is nonnegative. □

For a particular example, we consider the following family of Gaussian processes. Let $\{B_t, t \geq 0\}$ be a standard Brownian motion. Then let $\{F_t, t \geq 0\}$ be a mean-zero Gaussian process with covariance given by

$$\mathbb{E}[F_r F_t] = \frac{\mathbb{P}\left(B_r \leq q(r), B_t \leq q(t)\right) - \alpha^2}{u(q(r); r)u(q(t); t)}, \tag{26}$$

where $u(x;t) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$ is the Gaussian density with mean zero and variance $t > 0$; and

$$\alpha = \mathbb{P}\left(B_r \leq q(r)\right) = \mathbb{P}\left(B_t \leq q(t)\right).$$

This family of processes was studied by Jason Swanson in a recent paper [9], and they appear in the limit of normalized empirical quantiles of a system of independent Brownian motions.

Assume $0 \leq r < t \leq T$ and fix $0 < \alpha < 1$. We have

$$\alpha = \mathbb{P}(B_t \leq q(t)) = \Phi\left(\frac{q(t)}{\sqrt{t}}\right);$$

where $\Phi(x)$ is the cumulative distribution function for a $\mathcal{N}(0,1)$ random variable. It follows that we can write

$$q(r) = \sqrt{r}\Phi^{-1}(\alpha); \quad q(t) = \sqrt{t}\Phi^{-1}(\alpha),$$

and

$$\left[u(q(r); r)\ u(q(t); t)\right]^{-1} = 2\pi\sqrt{rt}e^{-\Phi^{-1}(\alpha)^2} = 2\pi K\sqrt{rt} \tag{27}$$

where we denote $K = e^{-\Phi^{-1}(\alpha)^2}$. Next, let $\beta := \Phi^{-1}(\alpha)$; and let η, ξ denote two independent $\mathcal{N}(0,1)$ random variables. One can write

$$\begin{aligned}\mathbb{P}\left(B_r \leq q(r), B_t - B_r + B_r \leq q(t)\right) - \alpha^2 &= \mathbb{P}\left(\sqrt{r}\xi \leq \sqrt{r}\beta; \sqrt{\frac{t-r}{r}}\eta + \xi \leq \frac{\sqrt{t}\beta}{\sqrt{r}}\right) - \alpha^2 \\ &= \mathbb{P}\left(\xi \leq \beta; \sqrt{\frac{t}{r} - 1}\ \eta + \xi \leq \sqrt{\frac{t}{r}}\beta\right) - \alpha^2. \end{aligned}\tag{28}$$

Combining (27) and (28), we have for $0 \leq r \leq t \leq T$

$$\mathbb{E}\left[F_r F_t\right] = 2\pi K r\sqrt{\frac{t}{r}}\left[\mathbb{P}\left(\xi \leq \beta; \sqrt{\frac{t}{r} - 1}\ \eta + \xi \leq \sqrt{\frac{t}{r}}\beta\right) - \alpha^2\right],$$

so that the covariance of this process has the form $r\phi\left(\frac{t}{r}\right)$, with

$$\phi(x) = K\sqrt{x}\int_{-\infty}^{\beta}\left(\int_{-\infty}^{\sqrt{\frac{x}{x-1}}\beta-\frac{\xi}{\sqrt{x-1}}} e^{-\frac{\eta^2+\xi^2}{2}}d\eta - \int_{-\infty}^{\beta} e^{-\frac{\eta^2+\xi^2}{2}}d\eta\right)\, d\xi$$

$$= 2\pi K\sqrt{x}\int_{-\infty}^{\beta}\left[\Phi\left(\sqrt{\frac{x}{x-1}}\,\beta - \frac{\xi}{\sqrt{x-1}}\right) - \Phi(\beta)\right]\,\frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}d\xi. \tag{29}$$

Remark 5.6. Consider the case $\alpha = 1/2$. Then $K = 1$, $\beta = 0$, and by transformation of variables it can be shown that (29) has the closed form

$$\phi(x) = \sqrt{x}\arctan\left(\frac{1}{\sqrt{x-1}}\right).$$

Proposition 5.7. *Let $\beta \in \mathbb{R}$, $K > 0$, and let $\Phi(z)$ denote the $\mathcal{N}(0,1)$ distribution function. The function*

$$\phi(x) = \begin{cases} 2\pi\Phi(\beta)\left(1-\Phi(\beta)\right), & x = 1 \\ 2\pi K\sqrt{x}\int_{-\infty}^{\beta}\left[\Phi\left(\sqrt{\frac{x}{x-1}}\,\beta - \frac{\xi}{\sqrt{x-1}}\right) - \Phi(\beta)\right]\,\frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}d\xi, & x > 1 \end{cases}$$

satisfies conditions (ϕ.1), (ϕ.2), (ϕ.3) and (ϕ.4).

Proof. First, note that since $\xi \le \beta$, this means that as $x \searrow 1$ we have $\sqrt{\frac{x}{x-1}}\,\beta - \frac{\xi}{\sqrt{x-1}} \longrightarrow +\infty$, so it follows that $\phi(1^+) = \phi(1)$ and that ϕ is continuous on $[1,\infty)$ and twice-differentiable on $(1,\infty)$. Hence, to show condition (ϕ.1) it is enough to investigate large values of x. For this, consider the behavior of the term

$$\left|\Phi\left(\sqrt{\frac{x}{x-1}}\,\beta - \frac{\xi}{\sqrt{x-1}}\right) - \Phi(\beta)\right|; \quad x > 1.$$

By Mean Value Theorem, this is bounded above by

$$\|\Phi'\|_\infty\left[|\beta|\left(\sqrt{\frac{x}{x-1}}-1\right) + \frac{|\xi|}{\sqrt{x-1}}\right] \le \frac{|\beta|}{\sqrt{x-1}}\left(\sqrt{x}-\sqrt{x-1}\right) + \frac{|\xi|}{\sqrt{x-1}}$$

$$\le \frac{|\beta|}{x-1} + \frac{|\xi|}{\sqrt{x-1}}. \tag{30}$$

From this estimate, it follows that $\|\phi\|_\infty < \infty$, since for $x \ge 2$ we have

$$|\phi(x)| \le 2\pi K\sqrt{x}\int_{-\infty}^{\beta}\left[\frac{|\beta|}{x-1} + \frac{|\xi|}{\sqrt{x-1}}\right]\frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}d\xi$$

$$\le \frac{2\pi K\sqrt{x}|\beta|}{x-1} + 2\pi K\sqrt{\frac{x}{x-1}}\mathbb{E}\left[|\xi|\right] \le C.$$

For condition (ϕ.2),

$$\begin{aligned}
\phi'(x) &= \frac{\phi(x)}{2x} + 2\pi K\sqrt{x}\int_{-\infty}^{\beta}\Phi'\left(\sqrt{\frac{x}{x-1}}\beta - \frac{\xi}{\sqrt{x-1}}\right)\frac{d}{dx}\left(\sqrt{\frac{x}{x-1}}\beta - \frac{\xi}{\sqrt{x-1}}\right)\frac{e^{-\frac{\xi^2}{2}}}{\sqrt{2\pi}}d\xi \\
&= \frac{\phi(x)}{x} + K\sqrt{x}\int_{-\infty}^{\beta}\exp\left(-\frac{1}{2}\left(\sqrt{\frac{x}{x-1}}\beta - \frac{\xi}{\sqrt{x-1}}\right)^2 - \frac{\xi^2}{2}\right)\left(\frac{\beta(x-1) - x\beta + \xi\sqrt{x}}{2\sqrt{x}(x-1)^{\frac{3}{2}}}\right)d\xi \\
&= \frac{\phi(x)}{x} + K\sqrt{x}\int_{-\infty}^{\beta}\exp\left(-\frac{\beta^2}{2} - \frac{(\xi\sqrt{x}-\beta)^2}{2(x-1)}\right)\left(\frac{\xi\sqrt{x}-\beta}{2\sqrt{x}(x-1)^{\frac{3}{2}}}\right)d\xi \\
&= \frac{\phi(x)}{x} + \frac{Ke^{-\frac{\beta^2}{2}}}{2(x-1)}\int_{-\infty}^{\beta\frac{\sqrt{x}-1}{\sqrt{x-1}}} ue^{-\frac{u^2}{2}}\sqrt{\frac{x-1}{x}}\,du \\
&= \frac{\phi(x)}{x} - \frac{Ke^{-\frac{\beta^2}{2}}e^{-\frac{\beta^2}{2}\left(\frac{\sqrt{x}-1}{\sqrt{x-1}}\right)^2}}{2\sqrt{x(x-1)}}.
\end{aligned}$$

Thus (ϕ.2) is satisfied, since

$$\left|\frac{\phi(x)}{x}\right| \le \frac{\|\phi\|_\infty}{\sqrt{x(x-1)}} \le \frac{c_{\phi,0}}{\sqrt{x(x-1)}},$$

and $K\exp\left(-\frac{\beta^2}{2}\left(1+\left(\frac{\sqrt{x}-1}{\sqrt{x-1}}\right)^2\right)\right)$ is bounded. Next, we have

$$\begin{aligned}
\phi''(x) &= \frac{\phi'(x)}{x} - \frac{\phi(x)}{x^2} + \frac{Ke^{-\frac{\beta^2}{2}\left(1+\left(\frac{\sqrt{x}-1}{\sqrt{x-1}}\right)^2\right)}}{2\sqrt{x(x-1)}}\left[\frac{1}{2x} + \frac{1}{2(x-1)} + \beta^2\left(\frac{\sqrt{x}-1}{\sqrt{x-1}}\right)\left(\frac{1}{2\sqrt{x(x-1)}} - \frac{\sqrt{x}-1}{2(x-1)^{\frac{3}{2}}}\right)\right] \\
&\le Cx^{-\frac{1}{2}}(x-1)^{-\frac{3}{2}},
\end{aligned}$$

hence (ϕ.3) is satisfied when (ϕ.2) is used for the first term.

To show (ϕ.4), let $\kappa = \frac{1}{2}Ke^{-\frac{\beta^2}{2}}$, and from the above expression for $\phi'(x)$ we can write

$$\frac{\kappa e^{-\frac{\beta^2}{2}\left(\frac{\sqrt{x}-1}{\sqrt{x-1}}\right)^2}}{\sqrt{x}\sqrt{x-1}} = \frac{\kappa}{\sqrt{x-1}} - \kappa\left(\frac{\sqrt{x} - e^{-\frac{\beta^2}{2}\left(\frac{\sqrt{x}-1}{\sqrt{x-1}}\right)^2}}{\sqrt{x(x-1)}}\right)$$

so that

$$\phi'(x) = -\frac{\kappa}{\sqrt{x-1}} + \frac{\psi(x)}{\sqrt{x}}, \tag{31}$$

where

$$\psi(x) = \frac{\phi(x)}{2\sqrt{x}} - \kappa\left(\frac{\sqrt{x} - e^{-\frac{\beta^2}{2}\left(\frac{\sqrt{x}-1}{\sqrt{x-1}}\right)^2}}{\sqrt{x-1}}\right). \tag{32}$$

Note that $\psi(x)$ is bounded, since $|\phi(x)|$ is bounded and

$$\lim_{x\to 1^+}\frac{\sqrt{x} - e^{-\frac{\beta^2}{2}\left(\frac{\sqrt{x}-1}{\sqrt{x-1}}\right)^2}}{\sqrt{x-1}} = 0.$$

Moreover, it follows from (ϕ.2) and (32) that $|\psi'(x+1)| \le Cx^{-\frac{1}{2}}$. □

6 Proof of the technical Lemmas

We begin with two technical lemmas. The first is a version of Corollary 4.2 with disjoint intervals.

Lemma 6.1. *For* $0 \le t_0 < t_1 \le t_2 < t_3 \le T$,

$$\lim_{n\to\infty} \sum_{j=\lfloor \frac{nt_0}{2} \rfloor+1}^{\lfloor \frac{nt_1}{2} \rfloor} \sum_{k=\lfloor \frac{nt_2}{2} \rfloor+1}^{\lfloor \frac{nt_3}{2} \rfloor} \left| \left\langle \partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}}, \partial^{\otimes 2}_{\frac{2k-1}{n}} - \partial^{\otimes 2}_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}^{\otimes 2}} \right| = 0.$$

Proof. We may assume $t_0 = 0$ and $t_1 = t_2$. Observe that

$$\left\langle \partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}}, \partial^{\otimes 2}_{\frac{2k-1}{n}} - \partial^{\otimes 2}_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}^{\otimes 2}}$$
$$= \beta_n(2j-1, 2k-1)^2 - \beta_n(2j-1, 2k-2)^2 - \beta_n(2j-2, 2k-1)^2 + \beta_n(2j-2, 2k-2)^2.$$

Therefore, it is enough to show that,

$$\sum_{j=0}^{\lfloor nt_2 \rfloor} \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(j,k)^2 \le Cn^{-\varepsilon} \tag{33}$$

for some $\varepsilon > 0$. We can decompose the sum in (33) as:

$$\sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(0,k)^2 + \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(\lfloor nt_2 \rfloor, k)^2 + \sum_{j=1}^{\lfloor nt_2 \rfloor-1} \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(j,k)^2.$$

By condition (iv), for some $\gamma > 0$ we have

$$\sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(0,k)^2 \le \sup_{1\le j\le \lfloor nt_3 \rfloor} |\beta_n(0,k)| \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} |\beta_n(0,k)|$$
$$\le Cn^{-1} \sum_{k=\lfloor nt_2 \rfloor+2}^{\lfloor nt_3 \rfloor} (k-1)^{-\gamma} + Cn^{-1} \le Cn^{-\gamma}.$$

By condition (ii), for some $1 < \alpha \le \frac{3}{2}$,

$$\sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(\lfloor nt_2 \rfloor, k)^2 \le \beta_n(\lfloor nt_2 \rfloor, \lfloor nt_2 \rfloor + 1)^2 + Cn^{-1} \sum_{k=\lfloor nt_2 \rfloor+2}^{\lfloor nt_3 \rfloor} \beta_n(\lfloor nt_2 \rfloor, k)$$
$$\le Cn^{-1} + Cn^{-1} \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} (k - \lfloor nt_2 \rfloor)^{-\alpha} \le Cn^{-1},$$

and again by condition (ii), for $\beta = \frac{3}{2} - \alpha$,

$$\sum_{j=1}^{\lfloor nt_2 \rfloor-1} \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \beta_n(j,k)^2 \le Cn^{-1} \sum_{j=1}^{\lfloor nt_2 \rfloor-1} \sum_{k=\lfloor nt_2 \rfloor+1}^{\lfloor nt_3 \rfloor} \left[(k - \lfloor nt_2 \rfloor)^{-\alpha} j^{-\beta} + (k-j)^{-\frac{3}{2}} \right]$$
$$\le Cn^{-1} \left(\sum_{k=1}^{\lfloor nt_3 \rfloor} k^{-\alpha} \right) \left(\sum_{j=1}^{\lfloor nt_2 \rfloor} j^{-\beta} \right) + Cn^{-1} \sum_{j=1}^{\lfloor nt_2 \rfloor} (\lfloor nt_2 \rfloor - j)^{-\frac{1}{2}}$$
$$\le Cn^{-\beta} + Cn^{-\frac{1}{2}};$$

hence the sum is bounded by $Cn^{-\varepsilon}$ for $\varepsilon = \min\left\{\beta, \gamma, \frac{1}{2}\right\}$. □

Lemma 6.2. *For $0 \le t \le T$ and integer $j \ge 1$,*

$$\left|\left\langle \varepsilon_t, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| \le Cn^{-\frac{1}{2}}$$

for a positive constant C which depends on T.

Proof. By conditions (i) and (ii), we have for $j \ge 1$ and $t > 0$,

$$\begin{aligned}\left|\left\langle \varepsilon_t, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| &\le \sum_{k=0}^{\lfloor nt \rfloor -1} \left|\left\langle \partial_{\frac{k}{n}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| + \left|\left\langle \varepsilon_t - \varepsilon_{\lfloor nt \rfloor}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}}\right| \\ &\le C \sum_{k=0}^{\infty} n^{-\frac{1}{2}} \left(|j-k|^{-\alpha} \wedge 1\right) + O(n^{-\frac{1}{2}}) \le Cn^{-\frac{1}{2}}. \qquad (34)\end{aligned}$$

□

6.1 Proof of Lemma 4.4

By the Lagrange theorem for the Taylor expansion remainder, the terms $R_0(W_{\frac{2j}{n}}), R_1(W_{\frac{2j-2}{n}})$ can be expressed in integral form:

$$R_0(W_{\frac{2j}{n}}) = \frac{1}{2}\int_{W_{\frac{2j-1}{n}}}^{W_{\frac{2j}{n}}} (W_{\frac{2j}{n}} - u)^2 f^{(3)}(u)\, du; \quad \text{and}$$

$$R_1(W_{\frac{2j-2}{n}}) = -\frac{1}{2}\int_{W_{\frac{2j-2}{n}}}^{W_{\frac{2j-1}{n}}} (W_{\frac{2j-2}{n}} - u)^2 f^{(3)}(u)\, du.$$

After a change of variables, we obtain

$$R_0(W_{\frac{2j}{n}}) = \frac{1}{2}(W_{\frac{2j}{n}} - W_{\frac{2j-1}{n}})^3 \int_0^1 v^2 f^{(3)}(vW_{\frac{2j-1}{n}} + (1-v)W_{\frac{2j}{n}})\, dv;$$

and

$$R_1(W_{\frac{2j-2}{n}}) = \frac{1}{2}(W_{\frac{2j-2}{n}} - W_{\frac{2j-1}{n}})^3 \int_0^1 v^2 f^{(3)}(vW_{\frac{2j-1}{n}} + (1-v)W_{\frac{2j-2}{n}})\, dv.$$

Define

$$G_0(2j) = \frac{1}{2}\int_0^1 v^2 f^{(3)}(vW_{\frac{2j-1}{n}} + (1-v)W_{\frac{2j}{n}})\, dv;$$

and

$$G_1(2j-2) = \frac{1}{2}\int_0^1 v^2 f^{(3)}(vW_{\frac{2j-1}{n}} + (1-v)W_{\frac{2j-2}{n}})\, dv.$$

We may assume $r = 0$. Define $\Delta W_{\frac{\ell}{n}} = W_{\frac{\ell+1}{n}} - W_{\frac{\ell}{n}}$. We want to show that

$$\mathbb{E}\left[\left(\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} \left\{G_0(2j)\Delta W^3_{\frac{2j-1}{n}} + G_1(2j-2)\Delta W^3_{\frac{2j-2}{n}}\right\}\right)^2\right] \le C\left\lfloor \frac{nt}{2} \right\rfloor n^{-\frac{3}{2}}. \qquad (35)$$

This part of the proof was inspired by a computation in [7] (see Lemma 4.2). Consider the Hermite polynomial identity $x^3 = H_3(x) + 3H_1(x)$. We use the map $\delta^q(h^{\otimes q}) = q!H_q(W(h))$ (see (2) in Sec.

2), for $h \in \mathfrak{H}$ with $\|h\|_{\mathfrak{H}} = 1$. For each j, let $w_j := \|\Delta W_{\frac{j}{n}}\|_{\mathfrak{H}}$, and note that condition (i) implies $w_j \leq Cn^{-\frac{1}{4}}$ for all j. Then

$$\frac{\Delta W^3_{\frac{j}{n}}}{w_j^3} = H_3\left(\frac{\Delta W_{\frac{j}{n}}}{w_j}\right) + 3H_1\left(\frac{\Delta W_{\frac{j}{n}}}{w_j}\right) = \delta^3\left(\frac{\partial^{\otimes 3}_{\frac{j}{n}}}{w_j^3}\right) + 3\delta\left(\frac{\partial_{\frac{j}{n}}}{w_j}\right)$$

so that

$$\Delta W^3_{\frac{j}{n}} = \frac{1}{2}\delta^3(\partial^{\otimes 3}_{\frac{j}{n}}) + w_j^2\delta(\partial_{\frac{j}{n}}).$$

It follows that we can write,

$$\begin{aligned} &G_0(2j)\Delta W^3_{\frac{2j-1}{n}} - G_1(2j-2)\Delta W^3_{\frac{2j-2}{n}} \\ &\quad = G_0(2j)\delta^3(\partial^{\otimes 3}_{\frac{2j-1}{n}}) - G_1(2j-2)\delta^3(\partial^{\otimes 3}_{\frac{2j-2}{n}}) \\ &\quad + 3w_{2j}^2 G_0(2j)\delta(\partial_{\frac{2j-1}{n}}) - 3w_{2j-1}^2 G_1(2j-2)\delta(\partial_{\frac{2j-2}{n}}). \end{aligned}$$

It is enough to verify the individual inequalities

$$\mathbb{E}\left[\left|\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} G_0(2j)\delta^3(\partial^{\otimes 3}_{\frac{2j-1}{n}})\right|^2\right] \leq C\left\lfloor \frac{nt}{2}\right\rfloor n^{-\frac{3}{2}}, \tag{36}$$

$$\mathbb{E}\left[\left|\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} G_1(2j-2)\delta^3(\partial^{\otimes 3}_{\frac{2j-2}{n}})\right|^2\right] \leq C\left\lfloor \frac{nt}{2}\right\rfloor n^{-\frac{3}{2}}, \tag{37}$$

$$\mathbb{E}\left[\left|\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} w_{2j}^2 G_0(2j)\delta(\partial_{\frac{2j-1}{n}})\right|^2\right] \leq C\left\lfloor \frac{nt}{2}\right\rfloor n^{-\frac{3}{2}}, \tag{38}$$

and

$$\mathbb{E}\left[\left|\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} w_{2j-1}^2 G_1(2j-2)\delta(\partial_{\frac{2j-2}{n}})\right|^2\right] \leq C\left\lfloor \frac{nt}{2}\right\rfloor n^{-\frac{3}{2}}. \tag{39}$$

We will show (36) and (38), with (37) and (39) essentially similar.

Proof of (36). Using (3) and the duality property,

$$\mathbb{E}\left[\left(\sum_{j=1}^{\lfloor \frac{nt}{2} \rfloor} G_0(2j)\delta^3(\partial^{\otimes 3}_{\frac{2j-1}{n}})\right)^2\right]$$

$$\begin{aligned} &= \mathbb{E}\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor}\left[G_0(2j)G_0(2k)\left(\sum_{r=0}^{3}\delta^{6-2r}(\partial^{\otimes 3-r}_{\frac{2j-1}{n}} \otimes \partial^{\otimes 3-r}_{\frac{2k-1}{n}})\left\langle \partial_{\frac{2j-1}{n}}, \partial_{\frac{2k-1}{n}}\right\rangle_{\mathfrak{H}}^r\right)\right] \\ &\leq \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor}\sum_{r=0}^{3}\left|\left\langle \partial_{\frac{2j-1}{n}}, \partial_{\frac{2k-1}{n}}\right\rangle_{\mathfrak{H}}^r\right| \mathbb{E}\left[\left|\left\langle D^{6-2r}\left(G_0(2j)G_0(2k)\right), \partial^{\otimes 3-r}_{\frac{2j-1}{n}} \otimes \partial^{\otimes 3-r}_{\frac{2k-1}{n}}\right\rangle_{\mathfrak{H}^{\otimes 6-2r}}\right|\right]. \end{aligned}$$

For integers $r \geq 0$, we have

$$D^r G_0(2j) = D^r \int_0^1 \frac{1}{2} v^2 f^{(3)} \left(vW_{\frac{2j-1}{n}} + (1-v) W_{\frac{2j}{n}} \right) dv$$

$$= \frac{1}{2} \int_0^1 v^2 f^{(3+r)} \left(vW_{\frac{2j-1}{n}} + (1-v) W_{\frac{2j}{n}} \right) \left(v\varepsilon_{\frac{2j-1}{n}}^{\otimes r} + (1-v) \varepsilon_{\frac{2j}{n}}^{\otimes r} \right) dv. \tag{40}$$

By product rule and (40) we have

$$\mathbb{E}\left[\left| \left\langle D^{6-2r} \left(G_0(2j) G_0(2k) \right), \partial_{\frac{2j-1}{n}}^{\otimes 3-r} \otimes \partial_{\frac{2k-1}{n}}^{\otimes 3-r} \right\rangle_{\mathfrak{H}^{\otimes 6-2r}} \right| \right] \tag{41}$$

$$\leq C \sum_{a+b=6-2r} \mathbb{E}\left[\sup_{0 \leq v,w \leq 1} \left| f^{(a)}(vW_{\frac{2j-1}{n}} + (1-v) W_{\frac{2j-2}{n}}) f^{(b)}(wW_{\frac{2k-1}{n}} + (1-w) W_{\frac{2k-2}{n}}) \right| \right]$$
$$\times \int_0^1 \int_0^1 \left| \left\langle \left(v\varepsilon_{\frac{2j-1}{n}}^{\otimes a} + (1-v) \varepsilon_{\frac{2j}{n}}^{\otimes a} \right) \otimes \left(w\varepsilon_{\frac{2k-1}{n}}^{\otimes b} + (1-w) \varepsilon_{\frac{2k}{n}}^{\otimes b} \right), \partial_{\frac{2j-1}{n}}^{\otimes 3-r} \otimes \partial_{\frac{2k-1}{n}}^{\otimes 3-r} \right\rangle_{\mathfrak{H}^{\otimes 6-2r}} \right| dv\, dw.$$

Notice that by condition (0), $\mathbb{E}\left[\sup \left| f^{(3+r)}(\xi) \right|^p \right] < \infty$, where the supremum is taken over the random variables $\{\xi = vW_{s_1} + (1-v) W_{s_2}, 0 \leq v \leq 1, 0 \leq s_1, s_2 \leq T\}$. From Lemma 6.2, for integers a and b with $a + b = 6 - 2r$, we have the following estimate

$$\int_0^1 \int_0^1 \left| \left\langle \left(v\varepsilon_{\frac{2j-1}{n}}^{\otimes a} + (1-v) \varepsilon_{\frac{2j}{n}}^{\otimes a} \right) \otimes \left(w\varepsilon_{\frac{2k-1}{n}}^{\otimes b} + (1-w) \varepsilon_{\frac{2k}{n}}^{\otimes b} \right), \partial_{\frac{2j-1}{n}}^{\otimes 3-r} \otimes \partial_{\frac{2k-1}{n}}^{\otimes 3-r} \right\rangle_{\mathfrak{H}^{\otimes 6-2r}} \right| dv\, dw \leq C n^{-(3-r)}. \tag{42}$$

It follows that if $r \neq 0$, then by Lemma 4.1, Equation (41), and Equation (42)

$$C \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left| \left\langle \partial_{\frac{2j-1}{n}}, \partial_{\frac{2k-1}{n}} \right\rangle_{\mathfrak{H}}^r \right| \mathbb{E}\left[\left| \left\langle D^{6-r} \left(G_0(2j) G_0(2k) \right), \partial_{\frac{2j-1}{n}}^{\otimes 3-r} \otimes \partial_{\frac{2k-1}{n}}^{\otimes 3-r} \right\rangle_{\mathfrak{H}^{\otimes 6-2r}} \right| \right]$$
$$\leq C n^{r-3} \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \left| \left\langle \partial_{\frac{2j-1}{n}}, \partial_{\frac{2k-1}{n}} \right\rangle_{\mathfrak{H}}^r \right|$$
$$\leq C \left\lfloor \frac{nt}{2} \right\rfloor n^{\frac{r}{2}-3},$$

which satisfies (35) because $\frac{r}{2} - 3 \leq -\frac{3}{2}$. On the other hand, if $r = 0$, then

$$\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} C n^{-3} \leq C \left\lfloor \frac{nt}{2} \right\rfloor n^{-2},$$

and we are done with (36).

Proof of (38). Proceeding along the same lines as above,

$$\mathbb{E}\left[\left(\sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor} w_{2j}^2 G_0(2j)\delta\left(\partial_{\frac{2j-1}{n}}\right)\right)^2\right] = \mathbb{E}\left[\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor} w_{2j}^2 w_{2k}^2 G_0(2j)G_0(2k)\left\{\delta^2\left(\partial_{\frac{2j-1}{n}}\otimes\partial_{\frac{2k-1}{n}}\right) + \left\langle\partial_{\frac{2j-1}{n}},\partial_{\frac{2k-1}{n}}\right\rangle_{\mathfrak{H}}\right\}\right]$$
$$\leq Cn^{-1}\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor}\mathbb{E}\left[\mathbb{E}\sup_{0\leq\ell\leq\lfloor\frac{nt}{2}\rfloor}|G_0(\ell)|^2\left|\left\langle\partial_{\frac{2j-1}{n}},\partial_{\frac{2k-1}{n}}\right\rangle_{\mathfrak{H}}\right|\right]$$
$$+ Cn^{-1}\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor}\mathbb{E}\left[\sum_{a+b=2}\mathbb{E}\left|\left\langle D^aG_0(2j)D^bG_0(2k),\delta^2\left(\partial_{\frac{2j-1}{n}}\otimes\partial_{\frac{2k-1}{n}}\right)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|\right].$$

By Lemma 4.1 we have an estimate for the second term:

$$Cn^{-1}\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor}\left|\left\langle\partial_{\frac{2j-1}{n}},\partial_{\frac{2k-1}{n}}\right\rangle_{\mathfrak{H}}\right| \leq C\left\lfloor\frac{nt}{2}\right\rfloor n^{-\frac{3}{2}}.$$

Then the first term has the same estimate as (41) when $r=2$, which proves (38) and the lemma.

6.2 Proof of Lemma 4.5

As in Lemma 4.4, we may assume $r=0$. Start with $B_n(t)$. Define

$$\gamma_n(t) := \sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor} f^{(3)}(W_{\frac{2j-1}{n}})\left\langle\varepsilon_{\frac{2j-1}{n}},\partial_{\frac{2j-1}{n}}^{\otimes 2}-\partial_{\frac{2j-2}{n}}^{\otimes 2}\right\rangle_{\mathfrak{H}}$$
$$= \sum_{j=1}^{\lfloor\frac{nt}{2}\rfloor} f^{(3)}(W_{\frac{2j-1}{n}})\left\langle\varepsilon_{\frac{2j-1}{n}},\partial_{\frac{2j-1}{n}}-\partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}}\left(\partial_{\frac{2j-1}{n}}-\partial_{\frac{2j-2}{n}}\right),$$

so that $B_n(t)=2\delta(\gamma_n(t))$. By Lemma 2.1, we have $\|\delta(\gamma_n(t))\|_{L^2(\Omega)}^2 \leq \mathbb{E}\|\gamma_n(t)\|_{\mathfrak{H}}^2 + \mathbb{E}\|D\gamma_n(t)\|_{\mathfrak{H}^{\otimes 2}}^2$. We can write

$$\|\gamma_n(t)\|_{\mathfrak{H}}^2 = \sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor} f^{(3)}(W_{\frac{2j-1}{n}})f^{(3)}(W_{\frac{2k-1}{n}})\left\langle\varepsilon_{\frac{2j-1}{n}},\partial_{\frac{2j-1}{n}}-\partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}}$$
$$\times\left\langle\varepsilon_{\frac{2k-1}{n}},\partial_{\frac{2k-1}{n}}-\partial_{\frac{2k-2}{n}}\right\rangle_{\mathfrak{H}}\left\langle\partial_{\frac{2j-1}{n}}-\partial_{\frac{2j-2}{n}},\partial_{\frac{2k-1}{n}}-\partial_{\frac{2k-2}{n}}\right\rangle_{\mathfrak{H}}$$
$$\leq \sup_{0\leq s\leq t}\left|f^{(3)}(W_s)\right|^2 \sup_{1\leq j\leq\lfloor nt\rfloor}\left\langle\varepsilon_{\frac{2j-1}{n}},\partial_{\frac{2j-1}{n}}-\partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}}^2 \sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor}\left|\left\langle\partial_{\frac{2j-1}{n}}-\partial_{\frac{2j-2}{n}},\partial_{\frac{2k-1}{n}}-\partial_{\frac{2k-2}{n}}\right\rangle_{\mathfrak{H}}\right|.$$

Note that $\mathbb{E}\left[\sup_{0\leq s\leq t}|f^{(3)}(W_s)|^2\right]=C$ by condition (0), and by Lemma 6.2, $\left|\left\langle\varepsilon_t,\partial_{\frac{2j-1}{n}}-\partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}}\right| \leq C_2 n^{-\frac{1}{2}}$ for all j,t. By Corollary 4.2 we know,

$$\sum_{j,k=1}^{\lfloor\frac{nt}{2}\rfloor}\left|\left\langle\partial_{\frac{2j-1}{n}}-\partial_{\frac{2j-2}{n}},\partial_{\frac{2k-1}{n}}-\partial_{\frac{2k-2}{n}}\right\rangle_{\mathfrak{H}}\right| \leq C\left\lfloor\frac{nt}{2}\right\rfloor n^{-\frac{1}{2}}.$$

Hence, it follows that $\mathbb{E}\|\gamma_n(t)\|_{\mathfrak{H}}^2 \le C\left\lfloor \frac{nt}{2}\right\rfloor n^{-1}n^{-\frac{1}{2}} \le C\left\lfloor \frac{nt}{2}\right\rfloor n^{-\frac{3}{2}}$. Next,

$$D\gamma_n(t) = \sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor} f^{(4)}(W_{\frac{2j-1}{n}})\left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}} \left(\varepsilon_{\frac{2j-1}{n}} \otimes \left(\partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}\right)\right)$$

and this implies

$$\begin{aligned}
\|D\gamma_n(t)\|_{\mathfrak{H}^{\otimes 2}}^2 \le \sup_{0\le s\le t}\left|f^{(4)}(W_s)\right|^2 & \left|\sum_{j,k=1}^{\lfloor \frac{nt}{2}\rfloor}\left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}}\left\langle \varepsilon_{\frac{2k-1}{n}}, \partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}}\right\rangle_{\mathfrak{H}}\right| \\
& \times \left|\left\langle \varepsilon_{\frac{2j-1}{n}} \otimes \left(\partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}\right), \varepsilon_{\frac{2k-1}{n}} \otimes \left(\partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}}\right)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \\
\le \sup_{0\le s\le t}\left|f^{(4)}(W_t)\right|^2 & \left(\sup_j \left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}}^2\right) \\
\times \sup_{0\le s,r\le t}\left|\langle \varepsilon_s, \varepsilon_r\rangle_{\mathfrak{H}}\right| & \sum_{j,k=1}^{\lfloor \frac{nt}{2}\rfloor}\left|\left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}}\right\rangle_{\mathfrak{H}}\right|.
\end{aligned}$$

By condition (0), $\mathbb{E}\left[\sup_{0\le s\le t}|f^{(4)}(W_s)|\right]$ is bounded, and $\sup_{0\le s,r\le t}|\langle \varepsilon_r, \varepsilon_s\rangle_{\mathfrak{H}}|$ is bounded. Hence, it can be seen that $\mathbb{E}\|D\gamma_n(t)\|_{\mathfrak{H}^{\otimes 2}}^2$ gives the same estimate as $\gamma_n(t)$.

For $C_n(t)$, using condition (0) and the identity $a^2 - b^2 = (a-b)(a+b)$, we can write

$$\mathbb{E}\left[C_n(t)^2\right] \le \mathbb{E}\left[\sup_{0\le s\le t}|f^{(4)}(W_s)|^2\right]\left(\sup_{1\le j\le \frac{nt}{2}}\left|\left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}}\right|\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\left|\left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} + \partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}}\right|\right)^2$$

By Lemma 6.2, $\left|\left\langle \varepsilon_{\frac{2j-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}\right\rangle_{\mathfrak{H}}\right| \le C_2 n^{-\frac{1}{2}}$, and by condition (iv),

$$\sum_{j=1}^{\lfloor \frac{nt}{2}\rfloor}\left|\left\langle \varepsilon_{\frac{2j-1}{n}}, \mathbf{1}_{[\frac{2j-2}{n}, \frac{2j}{n})}\right\rangle_{\mathfrak{H}}\right| \le Cn^{-\frac{1}{2}} + Cn^{-\frac{1}{2}}\sum_{j=2}^{\lfloor \frac{nt}{2}\rfloor}(2j-2)^{-\frac{1}{2}} \le C\left\lfloor \frac{nt}{2}\right\rfloor^{\frac{1}{2}} n^{-\frac{1}{2}}.$$

Hence it follows that $\mathbb{E}\left[C_n(t)^2\right] \le C\left\lfloor \frac{nt}{2}\right\rfloor n^{-2}$ for some constant C, and the lemma is proved.

6.3 Proof of Lemma 4.9

For $i = 1, \ldots, d$, set

$$u_n^i = \sum_{j=\lfloor \frac{nt_{i-1}}{2}\rfloor + 1}^{\lfloor \frac{nt_i}{2}\rfloor} f''(W_{\frac{2j-1}{n}})\left(\partial_{\frac{2j-1}{n}}^{\otimes 2} - \partial_{\frac{2j-2}{n}}^{\otimes 2}\right),$$

and recall that $F_n^i = \delta^2(u_n^i)$. As in Remark 3.3, we want to show:
Condition (a). For each $1 \le i \le d$, the following converge to zero in $L^1(\Omega)$:

(a.1) $\langle u_n^i, h_1 \otimes h_2\rangle_{\mathfrak{H}^{\otimes 2}}$ for all $h_1, h_2 \in \mathfrak{H}$ of the form ε_τ (see Remark 3.4).

(a.2) $\langle u_n^i, DF_n^j \otimes h\rangle_{\mathfrak{H}^{\otimes 2}}$ for each $1 \le j \le d$ and $h \in \mathfrak{H}$.

(a.3) $\langle u_n^i, DF_n^j \otimes DF_n^k\rangle_{\mathfrak{H}^{\otimes 2}}$ for each $1 \le j, k \le d$.

Condition (b).

(b.1) $\left\langle u_n^i, D^2 F_n^j \right\rangle_{\mathfrak{H}^{\otimes 2}} \longrightarrow 0$ in L^1 if $i \neq j$.
(b.2) $\left\langle u_n^i, D^2 F_n^i \right\rangle_{\mathfrak{H}^{\otimes 2}}$ converges in L^1 to a random variable of the form

$$F_\infty^j = c \int_{t_{i-1}}^{t_i} f''(W_s)^2 \eta(ds).$$

The proofs of (a.1) and (a.2) are essentially the same as given in [5] (see Theorem 5.2) but the proof of (a.3) is new.

Proof of (a.1). We may assume $i = 1$. Let $h_1 \otimes h_2 = \varepsilon_s \otimes \varepsilon_\tau \in \mathfrak{H}^{\otimes 2}$ for some values $s, \tau \in [0, t]$. Then

$$\left\langle u_n^1, h_1 \otimes h_2 \right\rangle_{\mathfrak{H}^{\otimes 2}} = \sum_{j=1}^{\lfloor \frac{nt_1}{2} \rfloor} f''(W_{\frac{2j-1}{n}}) \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_\tau \right\rangle_{\mathfrak{H}};$$

so that

$$\left| \left\langle u_n^1, h_1 \otimes h_2 \right\rangle_{\mathfrak{H}^{\otimes 2}} \right| \leq \sup_{0 \leq s \leq t} |f''(W_s)| \sup_{1 \leq j \leq \lfloor \frac{nt_1}{2} \rfloor} \sup_{0 \leq s \leq t_1} \left| \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \right| \sum_{j=1}^{\lfloor \frac{nt_1}{2} \rfloor} \left| \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_\tau \right\rangle_{\mathfrak{H}} \right|.$$

It follows from condition (iii) that for fixed $\tau \geq 0$

$$\begin{aligned} \sum_{j=1}^{\lfloor \frac{nt_1}{2} \rfloor} \left| \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_\tau \right\rangle_{\mathfrak{H}} \right| &= \sum_{j=1}^{\lfloor \frac{nt_1}{2} \rfloor} \left| \mathbb{E} \left[W_\tau (W_{\frac{2j}{n}} - 2W_{\frac{2j-1}{n}} + W_{\frac{2j-2}{n}}) \right] \right| \\ &\leq Cn^{-\frac{1}{2}} + Cn^{-\frac{1}{2}} \sum_{j=2}^{\lfloor \frac{nt_1}{2} \rfloor} \left((2j-2)^{-\frac{3}{2}} + |\tau - 2j|^{-\frac{3}{2}} \wedge 1 \right) \\ &\leq Cn^{-\frac{1}{2}} \end{aligned} \tag{43}$$

and Lemma 6.2 implies,

$$\sup_{1 \leq j \leq \lfloor \frac{nt_1}{2} \rfloor} \sup_{0 \leq s \leq t} \left| \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \varepsilon_s \right\rangle_{\mathfrak{H}} \right| \leq Cn^{-\frac{1}{2}}$$

so that

$$\mathbb{E} \left(\left| \left\langle u_n^1, h_1 \otimes h_2 \right\rangle_{\mathfrak{H}^{\otimes 2}} \right| \right) \leq Ct_1 n^{-1} \longrightarrow 0.$$

Proof of (a.2). As in (a.1), assume $i = 1$. Using the same technique as in (a.1), we can write $DF_n^j \otimes h$ as $DF_n^j \otimes \varepsilon_\tau$ for some $\tau \in [0, T]$. By Lemma 2.1, $DF_n^j = D\delta^2(u_n^j) = \delta^2(Du_n^j) + \delta(u_n^j)$, which gives

$$\left\langle u_n^1, DF_n^j \otimes \varepsilon_\tau \right\rangle_{\mathfrak{H}^{\otimes 2}} = \left\langle u_n^1, \delta^2(Du_n^j) \otimes \varepsilon_\tau \right\rangle_{\mathfrak{H}^{\otimes 2}} + \left\langle u_n^1, \delta(u_n^j) \otimes \varepsilon_\tau \right\rangle_{\mathfrak{H}^{\otimes 2}}.$$

For the first term,

$$\begin{aligned} \mathbb{E} \left| \left\langle u_n^1, \delta^2(Du_n^j) \otimes \varepsilon_\tau \right\rangle_{\mathfrak{H}^{\otimes 2}} \right| &= \sum_{\ell=1}^{\lfloor \frac{nt_1}{2} \rfloor} \mathbb{E} \left| f''(W_{\frac{2\ell-1}{n}}) \left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \delta^2(Du_n^j) \right\rangle_{\mathfrak{H}} \left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \varepsilon_\tau \right\rangle_{\mathfrak{H}} \right| \\ &\leq 2\mathbb{E} \left[\sup_{0 \leq s \leq t_1} |f''(W_s)| \right] \mathbb{E} \left[\sup_{1 \leq \ell \leq \lfloor \frac{nt_1}{2} \rfloor} \left| \left\langle \partial_{\frac{\ell}{n}}, \delta^2(Du_n^j) \right\rangle_{\mathfrak{H}} \right| \right] \sum_{\ell=1}^{\lfloor \frac{nt_1}{2} \rfloor} \left| \left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \varepsilon_\tau \right\rangle_{\mathfrak{H}} \right|. \end{aligned}$$

By (43), the sum has estimate $Cn^{-\frac{1}{2}}$, and for the second term we can take

$$\left|\left\langle \partial_{\frac{\ell}{n}}, \delta^2(Du_n^j)\right\rangle_{\mathfrak{H}}\right| \leq \sup_{\ell} \|\partial_{\frac{\ell}{n}}\|_{\mathfrak{H}} \, \|\delta^2(Du_n^j)\|_{\mathfrak{H}}.$$

It follows from condition (i) that $\|\partial_{\frac{\ell}{n}}\|_{\mathfrak{H}} \leq Cn^{-\frac{1}{4}}$. This leaves the $\|\delta^2(Du_n^j)\|_{\mathfrak{H}}$ term. By the Meyer inequality for a process taking values in $\mathfrak{H}$,

$$\mathbb{E}\left[\|\delta^2(Du_n^j)\|_{\mathfrak{H}}^2\right] \leq c_1\mathbb{E}\|Du_n^j\|_{\mathfrak{H}^{\otimes 3}}^2 + c_2\mathbb{E}\|D^2u_n^j\|_{\mathfrak{H}^{\otimes 4}}^2 + c_3\mathbb{E}\|D^3u_n^j\|_{\mathfrak{H}^{\otimes 5}}^2, \tag{44}$$

so that by Lemma 4.7, $\mathbb{E}\left[\|\delta^2(Du)\|_{\mathfrak{H}}^2\right] \leq C$, and we have

$$\mathbb{E}\left|\left\langle u_n^1, \delta^2(Du_n^j) \otimes \varepsilon_\tau\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \leq Cn^{-\frac{3}{4}}.$$

Then similarly,

$$\left|\left\langle u_n^1, \delta(u_n^j) \otimes \varepsilon_t\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \leq 2\left[\sup_{0\leq s\leq t_1} |f''(W_s)| \sup_{\ell} \left|\left\langle \partial_{\frac{\ell}{n}}, \delta(u_n^j)\right\rangle_{\mathfrak{H}}\right| \sum_{\ell} \left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \varepsilon_t\right\rangle_{\mathfrak{H}}\right|\right].$$

Similar to the above case, for each $1 \leq \ell \leq \left\lfloor \frac{nt_1}{2} \right\rfloor$,

$$\begin{aligned}\mathbb{E}\left[\left|\left\langle \partial_{\frac{\ell}{n}}, \delta(u_n^j)\right\rangle_{\mathfrak{H}}\right|\right] &\leq \mathbb{E}\left[\|\partial_{\frac{\ell}{n}}\|_{\mathfrak{H}} \|\delta(u_n^j)\|_{\mathfrak{H}}\right] \\ &\leq Cn^{-\frac{1}{4}}\left(\mathbb{E}\|u_n^j\|_{\mathfrak{H}^{\otimes 2}} + \mathbb{E}\|Du_n^j\|_{\mathfrak{H}^{\otimes 3}}\right) \leq Cn^{-\frac{1}{4}},\end{aligned}$$

hence with (43) we have

$$\mathbb{E}\left[\left|\left\langle u_n^1, \delta(u_n^j) \otimes \varepsilon_\tau\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|\right] \leq Cn^{-\frac{3}{4}}.$$

Proof of (a.3). For this term we consider the product $\left\langle u_n^i, DF_n^j \otimes DF_n^k\right\rangle_{\mathfrak{H}^{\otimes 2}}$. Lemma 6.1 shows that scalar products of this kind are small in absolute value when the time intervals are disjoint, therefore it is enough to consider the worst case $\left\langle u_n^1, DF_n^1 \otimes DF_n^1\right\rangle_{\mathfrak{H}^{\otimes 2}}$, and assume $t_1 = t$. We have

$$\begin{aligned}\mathbb{E}\left[\left|\left\langle u_n^1, DF_n^1 \otimes DF_n^1\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|\right] &\leq \sum_{\ell=1}^{\left\lfloor \frac{nt}{2}\right\rfloor} \left|\mathbb{E}\left[\left\langle f''(W_{\frac{2\ell-1}{n}})\left(\partial_{\frac{2\ell-1}{n}}^{\otimes 2} - \partial_{\frac{2\ell-2}{n}}^{\otimes 2}\right), DF_n^1 \otimes DF_n^1\right\rangle_{\mathfrak{H}^{\otimes 2}}\right]\right| \\ &\leq C\sum_{\ell=1}^{\left\lfloor \frac{nt}{2}\right\rfloor} \mathbb{E}\left[\left|\left\langle \partial_{\frac{2\ell-1}{n}}, DF_n^1\right\rangle_{\mathfrak{H}}^2 - \left\langle \partial_{\frac{2\ell-2}{n}}, DF_n^1\right\rangle_{\mathfrak{H}}^2\right|\right] \\ &\leq C\sum_{\ell=1}^{\left\lfloor \frac{nt}{2}\right\rfloor} \mathbb{E}\left[\left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, DF_n^1\right\rangle_{\mathfrak{H}}\right| \;\; \left|\left\langle \mathbf{1}_{[\frac{2\ell-2}{n}, \frac{2\ell}{n}]}, DF_n^1\right\rangle_{\mathfrak{H}}\right|\right].\end{aligned}$$

Using the decomposition $DF_n^1 = \delta^2(Du_n^1) + \delta(u_n^1)$, the above summand expands into four terms:

$$\begin{aligned}&(1) \quad \left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \delta^2(Du_n^1)\right\rangle_{\mathfrak{H}}\right| \;\; \left|\left\langle \mathbf{1}_{[\frac{2\ell-2}{n}, \frac{2\ell}{n}]}, \delta^2(Du_n^1)\right\rangle_{\mathfrak{H}}\right| \\ &(2) \quad \left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \delta^2(Du_n^1)\right\rangle_{\mathfrak{H}}\right| \;\; \left|\left\langle \mathbf{1}_{[\frac{2\ell-2}{n}, \frac{2\ell}{n}]}, \delta(u_n^1)\right\rangle_{\mathfrak{H}}\right| \\ &(3) \quad \left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \delta(u_n^1)\right\rangle_{\mathfrak{H}}\right| \;\; \left|\left\langle \mathbf{1}_{[\frac{2\ell-2}{n}, \frac{2\ell}{n}]}, \delta^2(Du_n^1)\right\rangle_{\mathfrak{H}}\right| \\ &(4) \quad \left|\left\langle \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}, \delta(u_n^1)\right\rangle_{\mathfrak{H}}\right| \;\; \left|\left\langle \mathbf{1}_{[\frac{2\ell-2}{n}, \frac{2\ell}{n}]}, \delta(u_n^1)\right\rangle_{\mathfrak{H}}\right|.\end{aligned}$$

We will show computations for the terms (1) and (4) only, with the others similar. For (1) we have

$$
\begin{aligned}
& C\sum_{\ell=1}^{\lfloor\frac{nt}{2}\rfloor}\mathbb{E}\left[\left|\left\langle\partial_{\frac{2\ell-1}{n}}-\partial_{\frac{2\ell-2}{n}},\delta^2(Du_n^1)\right\rangle_{\mathfrak{H}}\right|\ \left|\left\langle\mathbf{1}_{[\frac{2\ell-2}{n},\frac{2\ell}{n}]},\delta^2(Du_n^1)\right\rangle_{\mathfrak{H}}\right|\right] \\
&= C\sum_{\ell,m,m'=1}^{\lfloor\frac{nt}{2}\rfloor}\mathbb{E}\left|\left\langle\partial_{\frac{2\ell-1}{n}}-\partial_{\frac{2\ell-2}{n}},\delta^2\left(f^{(3)}(W_{\frac{2m-1}{n}})\varepsilon_{\frac{2m-1}{n}}\left(\partial^{\otimes 2}_{\frac{2m-1}{n}}-\partial^{\otimes 2}_{\frac{2m-2}{n}}\right)\right)\right\rangle_{\mathfrak{H}}\right| \\
&\quad\times\left|\left\langle\mathbf{1}_{[\frac{2\ell-2}{n},\frac{2\ell}{n}]},\delta^2\left(f^{(3)}(W_{\frac{2m'-1}{n}})\varepsilon_{\frac{2m'-1}{n}}\left(\partial^{\otimes 2}_{\frac{2m'-1}{n}}-\partial^{\otimes 2}_{\frac{2m'-2}{n}}\right)\right)\right\rangle_{\mathfrak{H}}\right| \\
&\le C\sup_{1\le k\le\lfloor\frac{nt}{2}\rfloor}\left(\mathbb{E}\left[\left\|\delta^2\left(f^{(3)}(W_{\frac{2k-1}{n}})\left(\partial^{\otimes 2}_{\frac{2k-1}{n}}-\partial^{\otimes 2}_{\frac{2k-2}{n}}\right)\right)\right\|_{\mathfrak{H}^{\otimes 2}}\right]\right)^2 \\
&\quad\times\sum_{\ell,m,m'=1}^{\lfloor\frac{nt}{2}\rfloor}\left|\left\langle\partial_{\frac{2\ell-1}{n}}-\partial_{\frac{2\ell-2}{n}},\varepsilon_{\frac{2m-1}{n}}\right\rangle_{\mathfrak{H}}\right|\ \left|\left\langle\mathbf{1}_{[\frac{2\ell-2}{n},\frac{2\ell}{n}]},\varepsilon_{\frac{2m'-1}{n}}\right\rangle_{\mathfrak{H}}\right|.
\end{aligned}
$$

By Lemmas 2.1 and 4.7, the Skorohod integral term is bounded by $Cn^{-\frac{1}{2}}$, and we use conditions (iii) and (iv) for the scalar products to obtain an estimate of the form

$$
Cn^{-2}\sum_{\ell,m,m'=1}^{\lfloor\frac{nt}{2}\rfloor}\left((2m-1)^{-\frac{3}{2}}+|2\ell-2m|^{-\frac{3}{2}}\right)\ \left((2\ell-2)^{-\frac{1}{2}}+|2\ell-2m'|^{-\frac{1}{2}}\right)\le Cn^{-\frac{1}{2}}.
$$

For term (4), we have by a computation similar to the proof of Lemma 4.7,

$$
\mathbb{E}\left[\left\|\delta\left(f^{(3)}(W_{\frac{2k-1}{n}})\left(\partial_{\frac{2k-1}{n}}-\partial_{\frac{2k-2}{n}}\right)\right)\right\|_{\mathfrak{H}}\right]\le Cn^{-\frac{1}{4}},
$$

and by conditions (i) and (ii) we have

$$
\begin{aligned}
& Cn^{-\frac{3}{2}}\sum_{\ell,m,m'=1}^{\lfloor\frac{nt}{2}\rfloor}\left|\left\langle\partial_{\frac{2\ell-1}{n}}-\partial_{\frac{2\ell-2}{n}},\partial_{\frac{2m-1}{n}}-\partial_{\frac{2m-2}{n}}\right\rangle_{\mathfrak{H}}\right|\ \left|\left\langle\mathbf{1}_{[\frac{2\ell-2}{n},\frac{2\ell}{n}]},\partial_{\frac{2m'-1}{n}}-\partial_{\frac{2m'-2}{n}}\right\rangle_{\mathfrak{H}}\right| \\
&\le Cn^{-\frac{3}{2}}\sum_{\ell,m,m'=1}^{\lfloor\frac{nt}{2}\rfloor}\left(|2\ell-2m|^{-\alpha}\right)\ \left(|2\ell-2m'|^{-\alpha}\right) \\
&\le Cn^{-\frac{1}{2}}.
\end{aligned}
$$

Proof of (b.1). By Lemma 2.1, we can expand D^2F_n as follows:

$$
\left\langle u_n^i,D^2F_n^j\right\rangle_{\mathfrak{H}^{\otimes 2}}=\left\langle u_n^i,\delta^2(D^2u_n^j)\right\rangle_{\mathfrak{H}^{\otimes 2}}+4\left\langle u_n^i,\delta(Du_n^j)\right\rangle_{\mathfrak{H}^{\otimes 2}}+2\left\langle u_n^i,u_n^j\right\rangle_{\mathfrak{H}^{\otimes 2}} \tag{45}
$$

Without loss of generality, we may assume that u_n^i is defined on $[0,t_1]$ and F_n^j is defined on $[t_1,t_2]$ for $t_1<t_2$, so that the sums are over

$$
u_n^i=\sum_{\ell=1}^{\lfloor\frac{nt_1}{2}\rfloor}f''(W_{\frac{2\ell-1}{n}})\left(\partial^{\otimes 2}_{\frac{2\ell-1}{n}}-\partial^{\otimes 2}_{\frac{2\ell-2}{n}}\right);\quad\text{and}\quad u_n^j=\sum_{m=\lfloor\frac{nt_1}{2}\rfloor+1}^{\lfloor\frac{nt_2}{2}\rfloor}f''(W_{\frac{2m-1}{n}})\left(\partial^{\otimes 2}_{\frac{2m-1}{n}}-\partial^{\otimes 2}_{\frac{2m-2}{n}}\right).
$$

First term

$$\mathbb{E}\left|\left\langle u_n^i, \delta^2(D^2u_n^j)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|$$

$$= \mathbb{E}\left|\left\langle \sum_{\ell=1}^{\lfloor\frac{nt_1}{2}\rfloor} f''(W_{\frac{2\ell-1}{n}})\left(\partial^{\otimes 2}_{\frac{2\ell-1}{n}} - \partial^{\otimes 2}_{\frac{2\ell-2}{n}}\right), \delta^2\left(\sum_{m=\lfloor\frac{nt_1}{2}\rfloor+1}^{\lfloor\frac{nt_2}{2}\rfloor} f^{(4)}(W_{\frac{2m-1}{n}})\varepsilon^{\otimes 2}_{\frac{2m-1}{n}} \otimes \left(\partial^{\otimes 2}_{\frac{2m-1}{n}} - \partial^{\otimes 2}_{\frac{2m-2}{n}}\right)\right)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|$$

$$\leq \mathbb{E}\left[\sup_{0\leq s\leq t}|f''(W_s)|\right]\mathbb{E}\left[\sum_\ell\sum_m\left|\left\langle \varepsilon^{\otimes 2}_{\frac{2m-1}{n}}, \partial^{\otimes 2}_{\frac{2\ell-1}{n}} - \partial^{\otimes 2}_{\frac{2\ell-2}{n}}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \left|\delta^2\left(f^{(4)}(W_{\frac{2m-1}{n}})\left(\partial^{\otimes 2}_{\frac{2m-1}{n}} - \partial^{\otimes 2}_{\frac{2m-2}{n}}\right)\right)\right|\right]$$

$$\leq \mathbb{E}\left[\sup_{0\leq s\leq t}|f''(W_s)|\right]\sup_m\|\delta^2(g_4)\|_{L^2(\Omega)}\sum_{\ell=1}^{\lfloor\frac{nt_2}{2}\rfloor}\sum_{m=1}^{\lfloor\frac{nt_2}{2}\rfloor}\left[\left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-1}{n}}\right\rangle^2_{\mathfrak{H}} - \left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-2}{n}}\right\rangle^2_{\mathfrak{H}}\right]$$

First we need an estimate for the $\delta^2(g_4)$ term, where in the notation of Lemma 4.7,

$$g_4 := f^{(4)}(W_{\frac{2m-1}{n}})\left(\partial^{\otimes 2}_{\frac{2m-1}{n}} - \partial^{\otimes 2}_{\frac{2m-2}{n}}\right).$$

By Lemma 2.1, $\|\delta^2(g_4)\|_{L^2(\Omega)} \leq c_1\mathbb{E}\|g_4\|_{\mathfrak{H}^{\otimes 2}} + c_2\mathbb{E}\|Dg_4\|_{\mathfrak{H}^{\otimes 3}} + c_3\mathbb{E}\|D^2g_4\|_{\mathfrak{H}^{\otimes 4}}$, and so $\|\delta^2(g_4)\|_{L^2(\Omega)} \leq Cn^{-\frac{1}{2}}$ for each $\lfloor\frac{nt_1}{2}\rfloor < m \leq \lfloor\frac{nt_2}{2}\rfloor$. We can write,

$$\mathbb{E}\left|\left\langle u_n^i, \delta^2(D^2u_n^i)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \leq Cn^{-\frac{1}{2}}\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor}\left|\left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-1}{n}}\right\rangle^2_{\mathfrak{H}} - \left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-2}{n}}\right\rangle^2_{\mathfrak{H}}\right|$$

We need an estimate for the double sum. We have by condition (iii),

$$\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor}\left[\left|\left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-1}{n}}\right\rangle^2_{\mathfrak{H}} - \left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-2}{n}}\right\rangle^2_{\mathfrak{H}}\right|\right] \leq \sup_{\ell,m}\left|\left\langle \varepsilon_{\frac{2m-1}{n}}, \mathbf{1}_{[\frac{2\ell-2}{n},\frac{2\ell}{n}]}\right\rangle_{\mathfrak{H}}\right|\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor}\left|\left\langle \varepsilon_{\frac{2m-1}{n}}, \partial_{\frac{2\ell-1}{n}} - \partial_{\frac{2\ell-2}{n}}\right\rangle_{\mathfrak{H}}\right|$$

$$\leq Cn^{-\frac{1}{2}}\sum_{\ell,m=1}^{\lfloor\frac{nt_2}{2}\rfloor}C_2n^{-\frac{1}{2}}\left[\left(|\ell-m|^{-\frac{3}{2}} + (\ell-1)^{-\frac{3}{2}}\right)\wedge 1\right]$$

$$\leq Cn^{-1}\sum_{\ell=1}^{\lfloor\frac{nt_2}{2}\rfloor}\sum_{p=1}^{\infty}p^{-\frac{3}{2}} \leq C$$

This provides an upper bound for the double sum, hence the first term of (45) is $O(n^{-\frac{1}{2}})$. Note that in the above estimate the double sum is taken over $1 \leq \ell, m \leq \lfloor\frac{nt_2}{2}\rfloor$. It follows that this estimate also holds for the case $i = j$, that is, $\mathbb{E}\left|\left\langle u_n^i, \delta^2(D^2u_n^i)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| \leq Cn^{-\frac{1}{2}}$.

Second Term

Using $t_1 < t_2$ as above,

$$\mathbb{E}\left|\left\langle u_n^i, \delta(Du_n^j)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right| = \mathbb{E}\left|\left\langle \sum_{j=1}^{\lfloor\frac{nt_1}{2}\rfloor} f''(W_{\frac{2j-1}{n}})\left(\partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}}\right), \delta\left(\sum_{k=\lfloor\frac{nt_1}{2}\rfloor}^{\lfloor\frac{nt_2}{2}\rfloor} f^{(3)}(W_{\frac{2k-1}{n}})\varepsilon_{\frac{2k-1}{n}} \otimes \left(\partial^{\otimes 2}_{\frac{2k-1}{n}} - \partial^{\otimes 2}_{\frac{2k-2}{n}}\right)\right)\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|$$

$$= \mathbb{E}\left|\sum_j \sum_k f''(W_{\frac{2j-1}{n}}) \left\langle \varepsilon_{\frac{2k-1}{n}}, \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}} \right\rangle_{\mathfrak{H}} \left\langle \partial_{\frac{2j-1}{n}} - \partial_{\frac{2j-2}{n}}, \partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}} \delta\left(f^{(3)}(W_{\frac{2k-1}{n}}) \left(\partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}}\right)\right)\right|$$

$$\leq C\,\mathbb{E}\left[\sup_{0\leq s\leq t} |f''(W_s)|\right] \left(\sup_{s,j}\left|\left\langle \varepsilon_s, \partial_{\frac{j}{n}}\right\rangle_{\mathfrak{H}}\right|\right) \left(\sup_k \|\delta(g_3)\|_{L^2(\Omega)}\right) \sum_{j=0}^{\lfloor nt_2\rfloor} \sum_{k=0}^{\lfloor nt_2\rfloor} \left|\left\langle \partial_{\frac{j}{n}}, \partial_{\frac{k}{n}}\right\rangle_{\mathfrak{H}}\right|,$$

where in this case, g_3 corresponds to the term including $f^{(3)}(W_t)$. It follows from Lemma 6.2 that $\sup|\langle \varepsilon_s, \partial_{k/n}\rangle_{\mathfrak{H}}| \leq Cn^{-\frac{1}{2}}$; and the double sum is bounded by $Cn^{\frac{1}{2}}$ by Corollary 4.2. This leaves an estimate for $\|\delta(g_3)\|_{L^2(\Omega)}$. By Lemma 2.1, $\|\delta(g_3)\|_{L^2(\Omega)} \leq c_1\|g_3\|_{\mathfrak{H}} + c_2\|Dg_3\|_{\mathfrak{H}^{\otimes 2}}$. For this case,

$$\|g_3\|_{\mathfrak{H}}^2 \leq \mathbb{E}\left[\sup_{0\leq s\leq t} |f^{(3)}(W_s)|^2\right] \left\|\partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}}\right\|_{\mathfrak{H}}^2 \leq Cn^{-\frac{1}{2}},$$

hence $\|g_3\|_{\mathfrak{H}} \leq Cn^{-\frac{1}{4}}$. Similarly,

$$\|Dg_3\|_{\mathfrak{H}^{\otimes 2}} \leq \mathbb{E}\left[\sup_{0\leq s\leq t} |f^{(4)}(W_s)|\right] \sup_{0\leq s\leq t} \|\varepsilon_s\|_{\mathfrak{H}} \left\|\partial_{\frac{2k-1}{n}} - \partial_{\frac{2k-2}{n}}\right\|_{\mathfrak{H}} \leq Cn^{-\frac{1}{4}},$$

hence the second term is $O(n^{-\frac{1}{4}})$. As in the first term, the double sum estimate shows that this result also holds for $\langle u_n^i, \delta(DF_n^i)\rangle_{\mathfrak{H}^{\otimes 2}}$.

Third Term

We can write

$$\left|\langle u_n^i, u_n^j\rangle_{\mathfrak{H}^{\otimes 2}}\right| \leq \sup_{0\leq s\leq t} |f''(W_s)|^2 \sum_{\ell=1}^{\lfloor \frac{nt_1}{2}\rfloor} \sum_{m=\lfloor\frac{nt_1}{2}\rfloor+1}^{\lfloor \frac{nt_2}{2}\rfloor} \left|\left\langle \partial^{\otimes 2}_{\frac{2\ell-1}{n}} - \partial^{\otimes 2}_{\frac{2\ell-2}{n}}, \partial^{\otimes 2}_{\frac{2m-1}{n}} - \partial^{\otimes 2}_{\frac{2m-2}{n}}\right\rangle_{\mathfrak{H}^{\otimes 2}}\right|$$

and it follows from Lemma 6.1 that $\mathbb{E}\left|\langle u_n^i, u_n^j\rangle_{\mathfrak{H}^{\otimes 2}}\right| \leq Cn^{-\varepsilon}$, for some $\varepsilon > 0$.

Proof of (b.2). As in case (b.1), this has the expansion (45). From remarks in the proof of (b.1), the first two terms have the same estimate as the $i \neq j$ case, hence only the term $\langle u_n^i, u_n^i\rangle_{\mathfrak{H}^{\otimes 2}}$ is significant.

Third Term

Assume for the summation terms that the indices run over $\lfloor \frac{nt_{i-1}}{2}\rfloor + 1 \leq j,k \leq \lfloor\frac{nt_i}{2}\rfloor$. We have

$$\langle u_n^i, u_n^i\rangle_{\mathfrak{H}^{\otimes 2}} = \sum_{j,k} f''(W_{\frac{2j-1}{n}}) f''(W_{\frac{2k-1}{n}}) \left\langle \partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}}, \partial^{\otimes 2}_{\frac{2k-1}{n}} - \partial^{\otimes 2}_{\frac{2k-2}{n}}\right\rangle_{\mathfrak{H}^{\otimes 2}}.$$

Expanding the product, observe that,

$$\left\langle \partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}}, \partial^{\otimes 2}_{\frac{2k-1}{n}} - \partial^{\otimes 2}_{\frac{2k-2}{n}}\right\rangle_{\mathfrak{H}^{\otimes 2}} = \beta_n(2j-1,2k-1)^2 - \beta_n(2j-1,2k-2)^2 - \beta_n(2j-2,2k-1)^2 + \beta_n(2j-2,2k-2)^2,$$

where $\beta_n(\ell,m)$ is as defined for condition (v). For each n, define discrete measures on $\{1,2,\dots\}^{\otimes 2}$ by

$$\mu_n^+ := \sum_{j,k=1}^{\infty} \beta_n(2j-1,2k-1)^2 + \beta_n(2j-2,2k-2)^2\delta_{jk};$$

$$\mu_n^- := \sum_{j,k=1}^{\infty} \beta_n(2j-1,2k-2)^2 + \beta_n(2j-2,2k-1)^2\delta_{jk}.$$

where in this case δ_{jk} denotes the Kronecker delta. In the following, we show only η_n^+, with η_n^- being similar. It follows from condition (v) that for each $t > 0$,

$$\mu^+\left([0,t]^2\right) := \lim_{n\to\infty} \mu_n\left(\left\lfloor \frac{nt}{2} \right\rfloor, \left\lfloor \frac{nt}{2} \right\rfloor\right)$$
$$= \lim_n \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1,2k-1)^2 + \beta_n(2j-2,2k-2)^2 = \eta^+(t).$$

Moreover, if $0 < s < t$ then

$$\mu_n\left(\left\lfloor \frac{ns}{2} \right\rfloor, \left\lfloor \frac{nt}{2} \right\rfloor\right) = \mu_n\left(\left\lfloor \frac{ns}{2} \right\rfloor, \left\lfloor \frac{ns}{2} \right\rfloor\right) + \sum_{j=1}^{\lfloor \frac{ns}{2} \rfloor} \sum_{k=\lfloor \frac{ns}{2} \rfloor+1}^{\lfloor \frac{nt}{2} \rfloor} \beta_n(2j-1,2k-1)^2 + \beta_n(2j-2,2k-2)^2$$

which converges to $\mu^+([0,s]^2)$ because the disjoint sum vanishes by Lemma 6.1. Hence, we can conclude that μ_n converges weakly to the measure given by $\mu^+([0,s]\times[0,t]) = \eta^+(s\wedge t)$. It follows by continuity of $f''(W_t)$ and Portmanteau Theorem that

$$\sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} f''(W_{\frac{2j-1}{n}})f''(W_{\frac{2k-1}{n}})\left(\beta_n(2j-1,2k-1)^2 + \beta_n(2j-2,2k-2)^2\right)$$
$$= \int_{\mathbb{R}^2} f''(W_s)f''(W_u)\mathbf{1}_{s<t}\mathbf{1}_{u<t}\mu_n^+(ds,du)$$

converges to

$$\int_0^t f''(W_s)^2\eta^+(ds).$$

Combining this result with a similar integral defined for μ^-, we have for $t > 0$,

$$\lim_{n\to\infty} \sum_{j,k=1}^{\lfloor \frac{nt}{2} \rfloor} f''(W_{\frac{2j-1}{n}})f''(W_{\frac{2k-1}{n}})\left\langle \partial^{\otimes 2}_{\frac{2j-1}{n}} - \partial^{\otimes 2}_{\frac{2j-2}{n}}, \partial^{\otimes 2}_{\frac{2k-1}{n}} - \partial^{\otimes 2}_{\frac{2k-2}{n}} \right\rangle_{\mathfrak{H}^{\otimes 2}}$$
$$= \int_0^t f''(W_s)\ \mu^+(ds) - \int_0^t f''(W_s)\ \mu^-(ds) = \int_0^t f''(W_s)\ \eta(ds)$$

where we define $\eta(t) = \eta^+(t) - \eta^-(t)$. It follows that on the subinterval $[t_{i-1}, t_i]$ we have the result

$$\left\langle u_n^i, u_n^i \right\rangle_{\mathfrak{H}^{\otimes 2}} \longrightarrow \int_{t_{i-1}}^{t_i} f''(W_s)^2\eta(ds)\, ds$$

in $L^1(\Omega)$ as $n \to \infty$. $\square$

References

[1] P. Billingsley, *Convergence of Probability Measures*, Wiley, 1968.

[2] K. Burdzy and J. Swanson (2010), "A change of variable formula with Itô correction term," *Ann. Prob.* 38: 1817-1869.

[3] C. Houdré and J. Villa (2003), "An example of infinite dimensional quasi-helix," *Contemp. Math.* 336: 3-39.

[4] P. Lei and D. Nualart (2008), “A decomposition of the bifractional Brownian motion and some applications,” *Statistics and Probability Letters* 10.1016.

[5] I. Nourdin and D. Nualart (2010), “Central limit theorems for multiple Skorokhod integrals,” *J. Theor. Prob.* 23: 39-64.

[6] I. Nourdin, A. Réveillac and J. Swanson (2010), “The weak Stratonovich integral with respect to fractional Brownian motion with Hurst parameter 1/6,” arXiv: 1006.4238v1.

[7] I. Nourdin and A. Rèveillac (2009), “Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: the critical case H = 1/4,” *Ann. Prob.* 37: 2200-2230.

[8] D. Nualart, *The Malliavin Calculus and Related Topics*, 2^{nd} Ed. Springer, 2006.

[9] J. Swanson (2011), “Fluctuations of the empirical quantiles of independent Brownian motions,” *Stochastic Process Appl.*, 121(3):479-514.

[10] J. Swanson (2007), “Variations of the solution to a stochastic heat equation,” *Ann. Prob.* 35: 2122-2159.