Comments

Pollak and Wachter on the Household Production Function Approach

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In "The Relevance of the Household Production Function and Its Implications for the Allocation of Time," Pollak and Wachter (1975) have provided a valuable analytical interpretation of the theory underlying the "new home economics." In addition, they have provided insights into serious potential abuses of the household production function approach. However, their identification of one such potential abuse led them to terminate their analysis prematurely with the rejection of the entire shadow-price concept on which much of the new home economics is based. Their conclusion is unwarranted.

Pollak and Wachter maintain that joint production is inherently important in household technology, and they argue that joint production breaks the link between the existing household production function approach and the neoclassical theory on which that approach is based. They also argue that joint production results in the confounding of tastes and technology within shadow prices. But such "confounding" could pose a fundamental theoretical problem only if the postulated "confounding" can be translated into an identification problem. I shall equate a particular theoretical structural model with the household production function approach. I shall demonstrate that all functions in that structural form do have known neoclassical properties, and I shall discuss the identification of the structure. I shall derive household structure

and prove its identification when tastes are Bergson and technology is Hybrid Diewert. The structure will be shown to be overidentified, and joint production will be shown to increase the number of overidentifying restrictions. I argue that in the general case joint production commonly tends to assist in identification without introducing any nonneoclassical theoretical complications.

Having rejected commodity shadow prices, Pollak and Wachter recommend an alternative. I shall equate that alternative with a reduced-form approach not having capabilities comparable to those of the household production function approach.

I. Introduction

Following Pollak and Wachter, let $X = (x_1, \ldots, x_n)'$ be a vector of goods, and let $Z = (z_1, \ldots, z_m)'$ be a vector of "commodities" generated from goods by the household's production process. Let U be the household's utility function, which we shall assume is defined over commodity vectors, and let $P = (p_1, \ldots, p_n)'$ be the vector of goods prices. Pollak and Wachter have shown that a cost function $C(P, Z)$ exists such that the household maximizes $U(Z)$ subject to the constraint $C(P, Z) = \mu$, where μ is total expenditure available.[1] The solution is a system of commodity demand equations $Z = f(P, \mu)$. Translating U back into the goods space, we can also derive goods demand functions $X = h(P, \mu)$.

Assuming that household production is characterized by constant returns to scale, Pollak and Wachter have shown that the household can be shown equivalently to solve for that value Z^* which will

$$\text{maximize } U(Z) \text{ subject to } \pi'Z = \mu, \tag{1}$$

where $\pi = (\pi_1, \ldots, \pi_m)'$ is the gradient of $C(P, Z)$ with respect to Z and π therefore is a function of (P, Z). Then $\pi_i(P, Z)$ is defined to be the shadow price of the ith commodity.

As Pollak and Wachter have observed, much of the appeal of the commodity shadow-price approach lies in its ability to use functions having known neoclassical properties. However, Pollak and Wachter maintain that, if the constraint $\pi(P, Z)'Z = \mu$ is nonlinear in Z, then the link with conventional theory is broken, since commodity demand functions derived from (1) would "correspond to those in a model in which consumers are monopsonists or are offered tie-in sales" (p. 258). But the commodity shadow prices $\pi(P, Z)$ do depend upon Z whenever household production exhibits jointness, which Pollak and Wachter maintain is inherently characteristic of household production processes. Hence

[1] Pollack and Wachter have considered the additional inclusion of time in the model. The theoretical issues I shall discuss will be presented in such a manner as to be unchanged by such complications.

Pollak and Wachter conclude that in the usual case the household production function approach must model a nonneoclassical decision problem for which "there are virtually no substantive results" (p. 258). [2] On these grounds, which I shall show to be specious, they immediately reject the use of commodity shadow prices as arguments of commodity demand functions. [3] When Z is not measurable, they recommend the estimation of h. Otherwise they recommend estimation of f (perhaps preceded by a prior-stage estimation of technology).

Pollak and Wachter have observed correctly that their function f does not have the known properties of conventional neoclassical demand functions. In addition, f and h depend upon both technology and preferences in a manner that provides little information about either. But as Pollak and Wachter have explained, the primary objective of the new home economics is to avoid "confounding tastes and technology" (p. 260). I shall show that the use of commodity shadow prices permits us to isolate sources of taste and technological change while using *only* functions having known conventional neoclassical properties.

II. Basic Constructs

Shadow prices are usually defined in terms of the normal to a separating hyperplane constructed at a solution point. That construction is dependent upon the location of the solution point, which need not be solely supply or technology determined. Now recall that $Z^* = f(P, \mu)$ is the household's solution value for Z. Define π^* by $\pi^* = \pi(P, Z^*)$, and let us instruct the household to reselect Z conditionally upon π^* to [4]

$$\text{maximize } U(Z) \text{ subject to } \pi^{*\prime}Z = \mu. \tag{2}$$

The constraint in problem (2) is the hyperplane contemplated by the shadow-price approach. The solution for Z (in terms of $[\pi^*, \mu]$) can be denoted implicitly by [5] $G(Z, \pi^*, \mu) = 0$. By comparing the first-order

[2] Observe carefully that the issue they raise is the availability of theoretical knowledge of the properties of the functions used in the household production function approach. This mathematical question about nonstochastic function properties is independent of the separate statistical question of the endogeneity of any random variables. E.g., the endogeneity of the random variable $\pi(P, Z)$ follows trivially from the direct functional dependency of $\pi(P, Z)$ upon the endogenous random variable Z. Yet this endogeneity is irrelevant to Pollak and Wachter's contention of the presumed unavailability of known theoretical properties of the function π or of any other function in household structure.

[3] In order to address these issues, I shall not exclude joint production from this paper.

[4] To permit our prior computation of π^*, I assume that the household already has solved its full decision problem (1). Nevertheless, problem (2) can be defined formally, despite the seeming redundancy of its objectives.

[5] Formally we could use the conventional notation $Z = g(\pi^*, \mu)$, but I use the implicit function notation G to emphasize the fact that an explicit closed-form solution for g need not always exist. Practical and relevant examples are provided in Barnett (1977; 1978, chap. 7).

conditions for the solution to problem (2) with those for the solution to the household's actual decision problem (1), we can see immediately that $G(Z^*, \pi^*, \mu) = 0$. We now can say that the household acts *as if* it were solving problem (2), and we see that G has all of the known conventional properties of neoclassical (implicit) demand functions.[6] The question now is whether this merely definitional construct, G, can be incorporated into the household structural model in such a manner that all functions in the structure have known neoclassical properties and such that each function depends either solely upon tastes or solely upon technology.

III. The Issue

Substitute $\pi(P, Z)$ for π^* in G to get

$$G[Z, \pi(P, Z), \mu] = 0. \tag{3}$$

We shall require the constant-commodity-consumption goods demand functions F_i, $i = 1, \ldots, n$, which determine the cost-minimizing goods consumption quantities at given (Z, P). By Shepherd's Lemma we know that $F_i(Z, P) = [\partial C(P, Z)]/\partial p_i$. Defining F by $F = (F_1, \ldots, F_n)'$, we have[7]

$$X = F(Z, P). \tag{4}$$

Adjoining (4) to (3), we acquire a complete system of $n + m$ simultaneous equations in the $n + m$ endogenous variables (X, Z) and the $n + 1$ exogenous variables[8] (P, μ). I shall call this complete system (with

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[6] The function G should not be confused with the composite function defined by the substitution of the function $\pi(P, Z)$ for the value of the argument π^* in G. The fact that the function π depends upon Z is not relevant to the properties of the function G. The function G neither knows nor cares where the commodity shadow prices came from.

[7] By the homogeneity of F in P and by Euler's Theorem, we know that the cost function can be determined from F. Hence (4) fully defines the technology.

[8] Asymptotically efficient estimators of this system are available from full information maximum likelihood (FIML) estimation. See Barnett (1976; 1978, chap. 4). Consistent but not asymptotically efficient estimators are available at lower computing cost through nonlinear two-stage least squares (2SLS) or nonlinear three-stage least squares (3SLS). See, e.g., Amemiya (1974, 1975, 1977) and Gallant (1977). These latter estimators are also robust to specification and data errors. Relevant computer programs are contained in the Eisenpress (IBM), TSP (Harvard), and TROLL packages. Pollack and Wachter appear to advocate (or perhaps to impute to the household production function approach) a two-stage approach in which technology (perhaps [4]) is estimated separately in a first stage. This two-step estimator is not consistent (since the system is not block recursive) and has no known desirable properties. In fact it has no known properties (or available standard errors) at all. I have not explicitly introduced an error structure into ([3], [4]), but a conventional additive error commonly would be a convenient choice. Observe that we can estimate the full system without deleting an arbitrary equation, since the usual disturbance covariance matrix singularity problem does not arise. Although the budget constraint $\sum P_k X_k = \mu$ does create a linear dependency between the equations of (4), it does not generate singularity of the covariance matrix of the joint error vector

any appropriate error structure) the household structural form.[9] It utilizes only the functions G and (π, F) which each relate solely to preferences or to technology, respectively.[10] Furthermore, if we have theories of taste or technological change, we can incorporate them individually into the specification of G or of (π, F), respectively. Observe that all of the functions in ([3], [4]) have known conventional neoclassical properties. It is this structural form which I shall identify with the commodity shadow-price approach.[11]

We shall solve the system of equations (3, 4) for (Z, X) in terms of (P, μ). The solution is

$$Z = f(P, \mu), \tag{5'}$$

$$X = h(P, \mu). \tag{5''}$$

These are precisely the two equation systems that Pollak and Wachter have suggested that we estimate, the first when Z is measurable and the second otherwise. We now see that the two models recommended to us by Pollak and Wachter consist of the two sets of equations defining the household's theoretical (exclusive of an error structure) reduced form.

The source of Pollak and Wachter's objections to the commodity shadow-price concept now becomes clear. For forecasting purposes, the reduced form places solely "explanatory" (predetermined) variables on the right-hand side and permits direct interpretation of cause and effect relationships. In a structural form, the right-hand side can depend upon endogenous variables, and, in our structural form ([3], [4]), $\pi(P, Z)$ does depend upon the endogenous variables Z. It is the imputation of explanatory power to commodity shadow prices that Pollak and

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added to ([3], [4]). This can be seen by applying to ([3], [4]) the usual singularity deriva-tion (for conventional demand systems), while recalling that Z is random on the right-hand side of (4).

[9] To specify technology, I could have used the production or cost function rather than my "factor" demand functions F. But whether alone or adjoined to (3), the resulting system would be incomplete (having an unequal number of endogenous variables and equations). An incomplete system does not define the joint distribution of the endogenous variables and therefore cannot define any model. An analogous use of factor demand equations to complete a system has been considered in a production context by Hall (1973).

[10] Recall that G lies in a one-to-one correspondence with preferences, while F lies in a one-to-one correspondence with technology. But $\pi(P, Z)$ has Z as an argument, and Z depends upon preferences as well as technology. However, the *function* π itself depends solely upon the cost function.

[11] Pollak and Wachter maintain that the commodity shadow-price approach dictates the use of a two-stage estimation procedure. In the first stage, commodity prices are estimated from a specification depending solely upon technology. This procedure permits viewing commodity shadow prices as household "supply" determined. In the second stage, household commodity demand is estimated conditionally upon commodity shadow prices. My presentation of the commodity shadow-price approach postulates no such two-state process. I assume that problem (1) is the one and only problem that the household actually solves, and the household solves that joint production and consumption decision in one step. Problem (2) is only a mathematical construct.

Wachter have warned us against convincingly. But to use the shadow-price function, π, in the construction of ([3], [4]), we have no need to impute explanatory power to the value of the variables[12] $\pi(P, Z)$.

Each function in the reduced form can carry joint information both about preferences and technology. If we wish to investigate properties of the household's structure or to consider household structural change, we must use a structural parameterization permitting the unscrambling of tastes from technology.[13] To investigate technological change, we can explore shifts in the parameters of the function π and the function F. I do not explore variations in the value of $\pi(P, Z)$, since such variations depend upon preferences as well as technology. In brief, the merits of Pollak and Wachter's approach are precisely those of a reduced-form system, while those of the shadow-price approach are precisely those of a structural form. But the advantages of structural-form estimation are well known.[14]

When Z is measurable, Pollak and Wachter advocate estimating (5′) (perhaps preceded by the estimation of technology). The most general approach to modeling f would involve parameterizing f directly.[15] Each parameter of such a direct reduced-form parameterization could carry information about both preferences and technology, and untangling the

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[12] Different households may have identical technologies and be "given" identical (P, μ) without having identical shadow prices. In such a case, I should conclude that shadow prices are different as the result of differing tastes. To view shadow prices as explanatory would reverse the direction of causation.

[13] E.g., without a structural parameterization habit formation could not be incorporated into the model without confounding tastes and technology. Prior estimation of technology would not help. Of course in the exceptional borderline case of exact identification, structural-form parameters for a fixed structure can be computed from reduced-form parameters. But structural change can be investigated only in terms of changes in structural-form parameters. Furthermore, even if the structural form were exactly identified, nonlinearities that typically exist in both the structural and reduced form would severely complicate solution for the structural-form parameters from the reduced form.

[14] Furthermore, reduced-form forecasts are easily computed numerically from an identified structural form, so that the objectives of the reduced form can be served by the structural form itself. No need exists ever to estimate directly or to solve analytically for the inherently less informative reduced form. Also observe that the household's structural form ([3], [4]) is well designed for deriving refutable theoretical results. The properties of all the functions in ([3], [4]) are restricted by neoclassical demand and production theory, regardless of whether or not joint production exists. Those restrictions imply restrictions upon the response of Z to variations in tastes, technology, and (P, μ). But such theoretical results are very weak unless further assumptions are made about tastes and technology. In this context it is frequently productive to exclude corner solutions and inferior commodities. Excluding joint production is neither necessary nor desirable. In contrast, observe that Pollak and Wachter's function f itself does not possess conventional neoclassical demand properties, although its actual properties can be deduced from ([3], [4]).

[15] The selection of such a direct reduced-form parameterization preferably should be guided by a duality theory, although in practice we may be satisfied with a parameterization of f which only approximates the underlying theory.

two sources (whether or not technology is itself estimated in a prior stage) would rarely be feasible. This is truly a reduced-form approach. Alternatively, we could structurally parameterize (5′) by selecting parameterizations of preferences and technology (rather than directly of f) and then deriving (5′) in terms of those original structural parameters. But this approach would be of little practical value, since the resulting system would be derivable only in pathological cases,[16] and even in those rare cases the resulting system would be far more difficult to estimate than our model.[17]

IV. Identification of the Structural Form

The household production function approach can be viewed as predicated upon the ability of system ([3], [4]) to unscramble tastes from technology. The dependence of shadow prices both upon tastes and technology (which troubles Pollak and Wachter) could pose a fundamental methodological problem only if that joint dependency resulted in an identification problem; furthermore, my own advocacy of ([3], [4]) as the household's structural model would be unsupportable if it were unidentified. To dispel in advance any truly serious potential doubts about the use of shadow prices, I shall disprove nonidentification by counterexample.

Consider a two-good, two-commodity household. I shall assume that the household's commodity demand functions are of the Bergson form $z_i = \beta_i(\mu/\pi_i)$, with $\beta_i > 0$ for $i = 1, 2$. I shall assume that the house-

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[16] To complete the system, suppose that we were to adjoin (4) to (5′). Then suppose that we were to attempt to derive the resulting system from a prior parameterization of tastes and technology. We could first derive ([3], [4]), which indexes the equivalence class of structural forms consisting of all elementary transformations of ([3], [4]). To pass from the shadow-price approach, defined (in the wide sense) by this equivalence class, to the Pollak and Wachter equations (5′), we must be able to solve the structural model (3) explicitly for a closed-form representation of the reduced form equations (5′). As is true in general for nonlinear structures, this rarely is possible. As a simple example, consider Hybrid-Diewert technology and CES commodity preferences in the two-good, two-commodity case, with the same notation introduced in Section IV below. Let $x = \sqrt{z_2}$ and let β be an arbitrary integer exceeding 4. Apply the binomial expansion to terms of order β, and collect all terms onto the right-hand side. To separate z_1 and z_2, we must be able to solve this polynomial for x. But the polynomial is full and of order $\beta + 1$, and it is well known in Galois theory that the general polynomial of degree exceeding 4 is not solvable (except in terms of Fuchsian functions, which are not empirically implementable). Hence we see that to parameterize ([4], [5′]) structurally we must back up to an implicit representation, i.e., into an element of the equivalence class defining the shadow-price approach itself.

[17] The sole such case of which I am aware (permitting both joint technology and a closed-form representation of [5′]) is the case of Cobb-Douglas preferences and Hybrid-Diewert technology. In that case, ([4], [5′]) is both nonlinear in the variables and deeply nonlinear in its parameters, while our system ([3], [4]) is nonlinear in the variables but fully linear in all of its parameters.

hold has a Hybrid-Diewert joint cost function.[18] Using a_{ijkl}, $i, j, k, l = 1, 2$, to denote parameters, we have

$$C(z_1, z_2, p_1, p_2) = a_{1111}z_1p_1 + a_{1122}z_2p_1 + a_{2211}z_1p_2$$
$$+ a_{2222}z_2p_2 + 2a_{1211}z_1\sqrt{p_1p_2} + 2a_{1222}z_2\sqrt{p_1p_2}$$
$$+ 2a_{1112}p_1\sqrt{z_1z_2} + 2a_{2212}p_2\sqrt{z_1z_2}$$
$$+ 4a_{1212}\sqrt{z_1z_2}\sqrt{p_1p_2}.$$

We can then derive the household structural form[19] ([3], [4]). The rank (sufficient) condition for identification can be shown to be satisfied, and the system thereby is identified.[20] The order condition is satisfied for an equation if the number of exclusion restrictions on the equation exceeds the number of equations in the system by more than one. The number of exclusion restrictions on each equation is 14, and the number of equations in the system is four. The system is considerably over-identified.

I now shall explore the effect of nonjointness on identification. To impose nonjointness, we set $a_{1112} = a_{1212} = a_{2212} = 0$. Eleven exclusion restrictions remain in each of two equations of ([3], [4]), while 13 remain in each of the other two equations. We have lost three overidentifying restrictions in each of two equations and one overidentifying restriction in each of the other two equations. Joint production helps in identification.

The results acquired from my counterexample reflect general properties of the theoretical structural form ([3], [4]) rather than properties specific to the chosen specification. The large number of exclusion restrictions results from the nonlinearity in the variables inherent to ([3], [4]) and from the fundamental difference between the structures of[21] (3) and (4).

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[18] See Hall (1973) for the definition of the Hybrid-Diewert cost function.

[19] Detailed verification of all of my results on this counterexample is contained in a prior draft of this paper, available upon request.

[20] Since the structure in this case is nonlinear only in its variables (rather than its parameters), the rank and order conditions for identification are available immediately from Fisher (1966, chap. 5).

[21] It is well known that nonlinear functions of exogenous variables can be viewed as new exogenous variables. Although the exogenous variables P may appear in all of the equations of ([3], [4]), some functions of the elements of P certainly will be missing from some equations. The result is exclusion restrictions on those equations whenever the functions do exist elsewhere in the system. Fisher (1966) has proved that terms involving endogenous variables can generate exclusion restrictions in a similar manner, and ([3], [4]) is inherently nonlinear in both its endogenous and exogenous variables. The fundamental difference between the structure of (3) and (4) assures the existence of many such exclusion restrictions in the combined system ([3], [4]). Furthermore, observe that μ occurs in (3) but not in (4). Each occurrence of μ alone or in an interaction with an endogenous or exogenous variable of (3) provides an exclusion restriction on (4). Finally, in my example I have ignored the 15 available cross-equation parameter restrictions. Such

Joint production does not hinder identification. In fact, it is known in general that such interactions and the nonlinearities which result do not hinder (and commonly assist in) identification.[22]

V. Conclusion

The household structural form that I have identified with the household production function approach does not and need not contain commodity shadow prices as predetermined or supply-determined variables to which causality can be imputed but contains them, rather, as functions of both the exogenous variables P and the endogenous variables Z. The household's structural form contains only functions having conventional neoclassical properties, with each function related solely and identifiably either to preferences or to technology. Causality can be imputed to explainable taste and technological change and to variations in the exogenous variables (P, μ). The existence of joint production poses no problems in the modeling of household structure.

Pollak and Wachter also discuss the formidable problems involved in defining and measuring commodity-consumption quantities. Those issues are independent of Pollak and Wachter's more fundamental critique of the theory underlying the household production function approach, and I have abstracted from such measurement problems. Nevertheless, measurement problems cannot be ignored in practice, and they undoubtedly exclude many household decisions from the domain of attractive applications of the household production function approach.[23]

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restrictions cannot hinder identification and usually help considerably. See Koopmans, Rubin, and Leipnik (1950, p. 94) and Wegge (1965). Hence our results are very conservative. Also observe that only system (4) depends upon tastes. Hence, if an identification problem should arise in a particular application (although I have been unable to construct such an example), further restrictions can be introduced into the system by postulating systematic taste or technological change. The literature on the new home economics is rich in household characteristics that can be appropriate choices as shift variables in tastes and technology.

[22] A rigorous proof in the case of nonlinearity in the variables is available in Fisher (1966, pp. 148–51).

[23] Furthermore, I agree with Pollak and Wachter that some applications of that approach have (unnecessarily) imputed causation to shadow prices. But I do not believe that an approach should be rejected by identifying it definitionally with its abuses. The solution to the abuse of an approach is the proper use of the approach.

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