The CAPM Risk Adjustment Needed for Exact Aggregation over Financial Assets

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> "The economic statistics that the government issues every week should come with a warning sticker: User beware. In the midst of the greatest information explosion in history, the government is pumping out a stream of statistics that are nothing but myths and misinformation."
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> Michael J. Mandel, "The Real Truth about the Economy: Are Government Statistics so much Pulp Fiction? Take a Look," *Business Week*, cover story, November 7, 1994, pp. 110-118.

1. Introduction

In the case of perfect certainty, the Divisia index exactly tracks any aggregator function. This follows from the fact that the Divisia line integral is directly derivable from the first order conditions for optimizing behavior. This result is especially well known in the case of consumer behavior, in which the Divisia index is derived directly from the total differential of the demand function, after substitution of the first order conditions for maximizing utility subject to a budget constraint. However, the exact tracking property of the Divisia index also applies to the demand for monetary services by firms and the supply of produced monetary services by financial intermediaries. See Barnett (1987).

Risk aversion is another story. The first order conditions in the case of risk aversion are Euler equations. Since those are not the first order conditions used in deriving the Divisia index under perfect certainty, the tracking ability of the unadjusted Divisia index is compromised. The degree to which the tracking ability degrades is a function of the degree of risk aversion and the amount of risk. In principle this problem could be solved by estimating the Euler equations by generalized method of moments and producing the estimated exact rational expectations monetary aggregator function. This inference procedure is in accordance with the one widely advocated as the solution to the Lucas Critique and more recently also advocated as the solution to what Chrystal and MacDonald (1994, p. 76) have called the Barnett Critique.[1] However estimation of aggregator functions, while in strict accordance with the principles of microeconomic aggregation theory, produces results that depend upon the parametric specification of the aggregator function and the choice of econometric estimator for estimating the parameters of the aggregator.

Index number thoery exists precisely for the purpose of permitting specification-free, nonparametric tracking ability. The Divisia index is such a parameter free index number and hence depends only upon data. While the Divisia index number is known to permit exact tracking for any economic aggregator function under perfect certainty (see Hulten (1973)), that index has never been extended to a statistical index number that will track exactly under risk aversion. In fact to our knowledge, no nonparametric, statistical index numbers have ever previously been derived directly from Euler equations in a manner that retains tracking ability under risk. In this paper, we derive a statistical index number directly from the Euler equations.[2] The resulting index number turns out to be an extension of the original Divisia index derived by Francois Divisia (1925) under perfect certainty, such that our extended Divisia index remains exact under risk aversion and reduces to the usual Divisia index in the special case of perfect certainty. The derivation is analogous to that for the usual Divisia index, but our extended Divisia index is derived from the Euler equations that are the correct first order conditions produced from rational behavior of economic agents under risk.

If additional assumptions are imposed, we find that the resulting generalized Divisia index has a direct connection with the capital asset pricing model (CAPM) in finance. In a sense our theory is a simultaneous generalization of both the CAPM and of economic index number theory, since our theory contains both as nested special cases. In particular, CAPM deals with a two dimensional tradeoff between expected return and risk, while the Divisia index deals with the two dimensional tradeoff between investment return and liquidity. Our generalized theory includes the three dimensional tradeoff among mean return, risk, and liquidity. The two well known special cases are based upon two-dimensional sections, which are orthogonal to each other, through the relevant three dimensional space.

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[1]According to the Barnett critique, the appearance of structural shift can be produced from an inconsistency between the aggregator function tracked by the index number used to produce the data and the aggregator function that is implied by the structural model within which the index number is used. The use of simple sum monetary aggregates as variables within models is an extreme example. See Chrystal and MacDonald (1994).

[2]Many of the results in this paper are based upon those in the recent working paper by Barnett and Liu (1995).

A particularly productive area of possible application of this new index number is monetary aggregation, since money market assets are characterized by substantially different degrees of each of the three characteristics: mean rate of return, risk, and liquidity, expecially when the collection of money market assets include those subject to prepayment penalties, such as Series EE bonds and nonnegotiable certificates of deposit, and those subject to regulated low rates of return, such as currency. When central banks first pruduced monetary aggregates, all of the components over which they aggregated yielded no interest. Hence there was perfect certainty about the rate of return on each component. In addition, since that rate of return was exactly zero for each component, the user costs were know to be the same for each component. Under those circumstances, it is well known in aggregation theory that the correct method of aggregation is simple summation. But monetary assets no longer yield the same rates of return and cannot be viewed as perfect substitutes. In addition, the interest yield is not a monetary service, so that the interest yield's capitalized value, while embedded in the value of the stock of such assets, is not part of the economic monetary stock. The capitalized value of the monetary service flow, net of that interest yield, is the economy's economic monetary stock. Furthermore, since interest is not paid in advance, there is some degree of uncertaintly about that rate of return, which is needed to compute the foregone interest (user cost) of any interest-yielding monetary asset. These observations indicate that the ability to track a nonlinear aggregator function under risk is needed to be able to measure the economy's monetary service flow.

In the case of the current monetary aggregates, the component assets yield rates of return having low variance and low correlation with consumption. As a result, the ordinary Divisia index produced from perfect certaintly first order conditions may be adequate to track the service flows of those collections of assets. But there is growing research interest in the possibility of incorporating into monetary aggregates some assets that have substantial risk, such as common stock and bond funds. See, e.g., Barnett and Zhou (1994) and Feldstein and Stock (1994). With such potential component assets, the perfect certainty first order conditions are not suitable and hence the ordinary Divisia monetary aggregates may not track well.

1.2. Money in the Utility Function

A large and growing literature seeks to explain why rate-of-return-dominated money exists (i.e., has positive value) in equilibrium. This issue is important and merits much research. Nevertheless, it is well known in general equilibrium theory that if money has positive value in equilibrium, then a derived utility function containing money exists such that behavior can be described by maximizing that derived utility subject to a budget constraint. See, e.g., Quirk and Saposnik (1968), Arrow and Hahn (1971), Feenstra (1986), Phlips and Spinnewyn (1982), and Samuelson (1948). The same result is available for firms. See, e.g., Fischer (1974). In fact the resulting derived utility or production function has indeed been derived for many of the explicit mechanisms producing positive value for money in equilibrium.

The converse of the theorem is especially challenging in its implications. If a model containing an explicit motive for holding money cannot generate a derived utility function from its original utility function and constraints, then money cannot have positive value in equilibrium. Only two possible conclusions can then be reached. Either the model fails to produce a positive equilibrium quantity of money, and money is driven out of existence in equilibrium as a result of its rate dominance, or the asset being modeled as "money" is in fact not money and need not exist in equilibrium.

Empirically that well known theorem and its converse imply that behavior under explicit motives for holding money induces rational (transitive, consistent) behavior within the space of goods and monetary assets, and indeed many tests of the axioms of revealed preference have accepted those axioms in that space. See, e.g., Swofford and Whitney (1987). However, knowledge of the induced preference preordering over that Cartesian product space is not sufficient to produce a unique inverse mapping back to the original decision. Hence, the properties of the derived utility function do not uniquely reveal the explicit motive for holding money. The nonuniqueness of the inverse mapping is a constructive reason for interest in models containing an explicit motive for holdng money. But for some purposes we do not need to know that motive.[3]

Macroeconomists, monetary economists, and central bankers have many reasons for wanting an explanation of the fact that positive quantities of money are held in equililbrium, despite the fact that many monetary assets are rate dominated as investments. However, in aggregation theory we have no need for such an

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[3]Manufacturers of roller skates may need to know why people buy roller skates, despite the fact that roller skates are dominated by subway trains as means of transportation. Hence manufacturers of roller skates may wish to know about the basic-wants (elementary consumption characteristics) and preferences over those characteristics applicable to roller skate and subway train service consumption. However, we know that since roller skates have positive value in equilibrium, a preference preordering exists over the space of subway trains and roller skates, and we know that the two goods are most certainly not perfect substitutes. That fact alone is sufficient to permit us to deduce that simple sum aggregation over subway trains and roller skates makes no sense whatsoever, and any empirical research using such a simple sum aggregate over those two goods is contaminated by that data. We need no information about the preference preordering over the space of elementary consumption characteristics to deduce that result in microeconomic aggregation theory.

explanation. Monetary assets are indeed held in positive quantities, and not all monetary assets yield the same investment rate of return. Within the derived utility function over the Cartesian product space of goods and monetary assets, we know that monetary assets having different own rates of return are most certainly not perfect substitutes. That fact alone is sufficient to permit us to deduce that simple sum aggregation over currency and any interest yielding monetary asset makes no sense whatsoever, and any empirical research using such a simple sum aggregate over such assets is contaminated by that data. We need no further information about the explicit motive for holding money to deduce that result in microeconomic aggregation theory.

In this paper, we display the derivation of the CAPM-extended Divisia monetary index based upon the derived utility function containing money. However, it should be observed that exactly the same result would be produced from any explicit motive for holding money, such as a model having transactions technology constraint. Even the case of perfect substitution between components is a nested special case of the index derived below.

2. Consumer Demand for Monetary Assets

2.1 The Decision

In this section we formulate a representative consumer's stochastic decision problem over consumer goods and monetary assets. The consumer's decisions are made in discrete time over an infinite planning horizon for the time intervals, t, t+1, ..., s, ..., where t is the current time period. The variables used in defining the consumer's decision are as follows: $\mathbf{x}_s$ = n dimensional vector of real consumption of goods and services during period s, $\mathbf{p}_s$ = n dimensional vector of goods and services prices and of durable goods rental prices during period s, $\mathbf{a}_s$ = k dimensional vector of real balances of monetary assets during period s, ρ_s = k dimensional vector of nominal holding period yields of monetary assets, A_s = holdings of the benchmark asset during period s, R_s = the one-period holding yield on the benchmark asset during period s, I_s= the sum of all other sources of income during period s, and $p_s^* = p_s^*(\mathbf{p}_s)$ = the true cost of living index.

Define Y to be the consumer's survival set, assumed to be a compact subset of the n+k+2 dimensional nonnegative orthant. The consumer's consumption possibility set, S(s) for period s is S(s) = { $(\mathbf{a}_s, \mathbf{x}_s, A_s) \in Y$:

$$\sum_{i=1}^{n} p_{is}x_{is} = \sum_{i=1}^{k} [(1+\rho_{i,s-1})\, p_{s-1}^*\, a_{i,s-1} - p_s^*\, a_{is}] + (1+R_{s-1})p_{s-1}^*\, A_{s-1} - p_s^*\, A_s + I_s\}.$$

The benchmark asset A_s provides no services other than its yield R_s. As a result, the benchmark asset does not enter the consumer's contemporaneous utility function. The asset is held only as a means of accumulating wealth. The consumer's subjective rate of time preference, ξ , is assumed to be constant.[4] The single period utility function, $u(\mathbf{a}_t, \mathbf{x}_t)$, is assumed to be increasing and strictly quasiconcave.

The consumer's decision problem is the following.[5]

Problem 1: Choose the deterministic point $(\mathbf{a}_t,\mathbf{x}_t,A_t)$ and the stochastic process $(\mathbf{a}_s,\mathbf{x}_s,A_s)$, s = t+1, ..., ∞, to maximize

$$u(\mathbf{a}_t, \mathbf{x}_t) + E_t \left[\sum_{s=t+1}^{\infty} \left(\frac{1}{1+\xi}\right)^{s-t} u(\mathbf{a}_s, \mathbf{x}_s)\right] \tag{2.1}$$

subject to $(\mathbf{a}_s, \mathbf{x}_s, A_s) \in S(s)$ for s ³ t, and also subject to the transversality condition

$$\lim_{s\to\infty} E_t \left(\frac{1}{1+\xi}\right)^{s-t} A_s = 0.$$

The transversality condition rules out perpetual borrowing at the benchmark rate, R_t.

2.2 Existence of a Monetary Aggregate for the Consumer

In order to ensure the existence of a monetary aggregate for the consumer, we partition the vector of monetary asset quantities, $\mathbf{a}_s$, such that $\mathbf{a}_s = (\mathbf{m}_s, \mathbf{h}_s)$. We correspondingly partition the vector of interest rates of those assets, ρ_s, such that $\rho_s=(\mathbf{r}_s,\mathbf{i}_s)$. We then assume that the utility function, u, is blockwise weakly separable in $\mathbf{m}_s$ and in $\mathbf{x}_s$ for some such partition of $\mathbf{a}_s$ and blockwise strongly separable in $\mathbf{h}_s$.[6] Hence there exist a monetary aggregator ("category utility") function, M, a consumer goods aggregator function, X, and utility functions, F and H, such that

$$u(\mathbf{m}_s,\mathbf{h}_s, \mathbf{x}_s) = F(M(\mathbf{m}_s), X(\mathbf{x}_s)) + H(\mathbf{h}_s), \tag{2.2}$$

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[4]Although money may not exist in the elementary utility function, there exists a derived utility function that contains money, so long as money has positive value in equilibrium. See, e.g., Arrow and Hahn (1971), Phlips and Spinnewyn (1982), Quirk and Saposnik (1968), Samuelson (1984), and Feenstra (1986). We implicitly are using that derived utility function.

[5]We do not consider aggregation over economic agents in this paper. But some results that may be relevant can be found in Blackorby and Schworm (1984).

[6]The strong separability assumption is largely for expository convenience. Weak separability would be sufficient. It is also possible that some of our results could be aquired from duality theory, using results such as those in Blackorby and Russell (1994) and Blackorby, Davidson, and Schworm (1991). However, the applicability of duality theory under risk is not currently highly developed, and we have based all of our results on functional structure imposed directly on the primal decision.

and we define the implied utility function $V(\mathbf{m}_s, c_s)$ by $V(\mathbf{m}_s, c_s)=F(M(\mathbf{m}_s), c_s)$, where aggregate consumptions of goods is defined by $c_s=X(\mathbf{x}_s)$. Then it follows that the exact monetary aggregate is

$$M_s = M(\mathbf{m}_s). \tag{2.3}$$

We define the dimension of $\mathbf{m}_s$ to be k_1, and the dimension of $\mathbf{h}_s$ to be k_2, so that $k=k_1+k_2$. The fact that blockwise weak separability is a necessary condition for exact aggregation is well known in the perfect certainty case.[7] In fact, if the resulting aggregator function also is linearly homogeneous, the theory of two stage budgeting can be used to prove that the consumer behaves as if the exact aggregate were an elementary good. Since two stage budgeting theory is not applicable under risk, we provide in the appendix an aggregation theorem proving that $M(\mathbf{m}_s)$ can be treated as a quantity aggregate, in a well defined sense, even under risk.

The Euler equations which will be of the most use to us below are those for monetary assets. Those Euler equations are:

$$E_s\left[\frac{\partial V}{\partial m_{is}} - \rho \frac{p_s^*(R_s - r_{is})}{p_{s+1}^*} \frac{\partial V}{\partial c_{s+1}}\right] = 0 \tag{2.4a}$$

for $s \geq t$ and $i = 1, \ldots, k_1$, where $\rho=\frac{1}{1+\xi}$ and where p_s^* is the exact price aggregate that is dual to the consumer goods quantity aggregate c_s.[8] Similarly we can acquire the Euler equation for the consumer goods aggregate c_s, rather than for each of its components. The resulting Euler equation for c_s is

$$E_s\left[\frac{\partial V}{\partial c_s} - \rho \frac{p_s^*(1+R_s)}{p_{s+1}^*} \frac{\partial V}{\partial c_{s+1}}\right] = 0 \tag{2.4b}$$

3. The Perfect Certainty Case

In the perfect certainty case, nonparametric index number theory is highly developed and is applicable to monetary aggregation. In the perfect certainty case, Barnett (1978,1980) proved that the contemporaneous real user cost of the services of m_{it} is π_{it}, where

$$\pi_{it} = \frac{R_t - r_{it}}{1+R_t} \tag{3.1}$$

The corresponding nominal user cost is $p_t^*\pi_{it}$. It can be shown that the solution value of the exact monetary aggregate $M(\mathbf{m}_t)$ can be tracked without error in continuous time (see, e.g., Barnett (1980)) by the Divisia index:

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[7]Regarding the highly developed theory of aggregation over goods under perfect certainty, see, e.g., Fisher and Shell (1972) and Blackorby, Primont, and Russell (1978)..

[8]Assuming that X is linearly homogeneous, the exact price aggregator function is the unit cost function.

$$d \log M_t = \sum_{i=1}^{k_1} s_{it} \; d \log m_{it} , \tag{3.2}$$

where the user cost evaluated expenditure shares are $s_{it} = \pi_{it} m_{it} / \sum_{j=1}^{k_1} \pi_{jt} m_{jt}$. The flawless tracking ability of the index in the perfect certainty case holds regardless of the form of the unknown aggregator function, M. However, under risk the ability of equation (3.2) to track $M(\mathbf{m}_t)$ is compromised.

4. The New Generalized Divisia Index

4.1 The User Cost of Money Under Risk Aversion

We now return to the Euler equations for optimal behavior of consumers under risk. Those Euler equations are displayed in equation (2.4a) for monetary assets and equation (2.4b) for consumer goods. Our objective is to find the formula for the user cost of monetary services in a form that is applicable to our model of decision under risk. The following definition for the contemporaneous user cost simply states that the real user cost price of a monetary asset is the marginal rate of substitution between those assets and consumer goods.[9]

Definition 1: The contemporaneous risk adjusted real user cost price of the services of monetary asset i is Π_{it}, defined such that $\Pi_{it} = \dfrac{\partial V}{\partial m_{it}} \Big/ \dfrac{\partial V}{\partial c_t}$.

No expectations opertors appear in that definition, since the marginal utilities at t are known with certainty in period t. Nevertheless, formula (3.1), which applies under perfect certainty, cannot be correct under risk, since the interest rates in equation (3.1) are not known contemporaneously, so the right hand side of equation (3.1) is stochastic, while Definition 1 defines Π_{it} to be deterministic. In this section we derive the correct deterministic formula for the user cost defined by Definition 1.[10]

For notational convenience, we sometimes convert the nominal rates of return, r_{it} and R_t, to real total rates of return, $1 + r^*_{it}$ and $1 + R^*_t$, such that

$$1 + r^*_{it} = \frac{p^*_t (1 + r_{it})}{p^*_{t+1}} \text{ and } 1 + R^*_t = \frac{p^*_t (1 + R_t)}{p^*_{t+1}} , \tag{4.1}$$

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[9]The nominal user cost is the real user cost multiplied by the price aggregate for consumer goods. Hence Definition 1 could be restated to be that the ratio of the nominal user cost of a monetary asset to the price aggregate for consumer goods equals the marginal rate of substitution between that monetary asset and consumer goods.

[10]Note that equation (4.1) remains correct for forward period s, so long as any expectations used in the definition of the user cost are defined relative to information available in that same period s, as is the case in the Euler equations (2.4a) and (2.4b). However, we shall produce our derivation for contemporaneous period t.

where r^*_{it} and R^*_t defined in that manner are called the real rates of excess return. Under this change of variables and observing that current period marginal utilities are known with certainty, Euler equations (2.4a) and (2.4b) become:

$$\frac{\partial V}{\partial m_{it}} - \rho E_t\left[\left(R^*_t - r^*_{it}\right)\frac{\partial V}{\partial c_{t+1}}\right] = 0 \tag{4.2}$$

and

$$\frac{\partial V}{\partial c_t} - \rho E_t\left[\left(1 + R^*_t\right)\frac{\partial V}{\partial c_{t+1}}\right] = 0 \tag{4.3}$$

We now can prove our user cost theorem under risk.

Theorem 1: The risk adjusted user cost of the services of monetary asset i under risk is $\Pi_{it} = \pi_{it} + \psi_{it}$, where

$$\pi_{it} = \frac{E_t R_t - E_t r_{it}}{1 + E_t R_t} \tag{4.4}$$

and

$$\psi_{it} = \rho\left(1 - \pi_{it}\right)\frac{\mathrm{Cov}\left(R^*_t, \frac{\partial V}{\partial c_{t+1}}\right)}{\frac{\partial V}{\partial c_t}} - \rho\frac{\mathrm{Cov}\left(r^*_{it}, \frac{\partial V}{\partial c_{t+1}}\right)}{\frac{\partial V}{\partial c_t}}. \tag{4.5}$$

Proof: Equation (4.2) can be rewritten for current period t to be

$$\frac{\partial V}{\partial m_{it}} = \rho E_t\left[\left(R^*_t - r^*_{it}\right)\frac{\partial V}{\partial c_{t+1}}\right]. \tag{4.6}$$

If the marginal utility and the interest rates in the expectation on the right hand side of (4.6) were uncorrelated, we could write the expectation of the product as the product of the expectations. But under our assumption of weak separability in monetary assets, $\mathbf{m}_t$, the utility function V can be written in the form $V(\mathbf{m}_t,c_t) = F(M(\mathbf{m}_t),c_t)$, where the consumer is risk neutral if and only if F is linear in $M_t=M(\mathbf{m}_t)$ and in c_t. Hence under risk neutrality, V must be linear in c_t, so that the marginal utility of consumption must be a constant. But without risk neutrality and the resulting constancy of the marginal utility of consumption, we have no reason to expect the interest rates and marginal utility on the right hand side of (5.3) to be uncorrelated. The result is that (4.6) becomes

$$\frac{\partial V}{\partial m_{it}} = \rho E_t\left[\frac{\partial V}{\partial c_{t+1}}\right](E_t R_t^* - E_t r_{it}^*) + \rho Cov(R_t^*, \frac{\partial V}{\partial c_{t+1}}) - \rho Cov(r_{it}^*, \frac{\partial V}{\partial c_{t+1}}) \qquad (4.7)$$

where the covariances would become zero, if we were to assume risk neutrality. Similarly, without risk neutrality, equation (4.3) becomes

$$\frac{\partial V}{\partial c_t} = \rho E_t\left[\frac{\partial V}{\partial c_{t+1}}\right] + \rho E_t\left[R_t^*\right] E_t\left[\frac{\partial V}{\partial c_{t+1}}\right] + \rho Cov(R_t^*, \frac{\partial V}{\partial c_{t+1}}). \qquad (4.8)$$

By eliminating $\rho E_t\left[\frac{\partial V}{\partial c_{t+1}}\right]$ between equations (4.7) and (4.8), we get

$$\frac{\partial V}{\partial m_{it}} = (\pi_{it} + \psi_{it})\frac{\partial V}{\partial c_t}, \qquad (4.9)$$

where

$$\pi_{it} = \frac{E_t R_t^* - E_t r_{it}^*}{1 + E_t R_t^*} \qquad (4.10)$$

and

$$\psi_{it} = \rho(1 - \pi_{it})\frac{Cov(R_t^*, \frac{\partial V}{\partial c_{t+1}})}{\frac{\partial V}{\partial c_t}} - \rho\frac{Cov(r_{it}^*, \frac{\partial V}{\partial c_{t+1}})}{\frac{\partial V}{\partial c_t}}. \qquad (4.11)$$

Using equation (4.1) to convert the real rates in equation (4.10) back to nominal rates, equation (4.10) becomes (4.4), while equation (4.11) is immediately identical to equation (4.5). Solving equation (4.9) for $\Pi_{it} = \pi_{it} + \psi_{it}$, Theorem 1 follows from Definition 1. **Q.E.D.**

Under risk neutrality, the covariances in (4.11) would all be zero, since the utility function would be linear in consumption. Hence the user cost would reduce to π_{it}, as defined in equation (4.4). The following corollary is immediate.

Corollary 1 to Theorem 1: Under risk neutrality, the user cost formula is the same as equation (3.1) in the perfect certainty case, but with all interest rates replaced by their expectations.

However, under risk aversion the utility function is strictly concave in consumption, so that marginal utility is inversely related to consumption. In principle, it is possible for the interest rate on a slightly risky investment to reduce the risk in the consumer's consumption stream if that interest rate and consumption are negatively correlated. Because of the inverse relationship between consumption and marginal utility, we conclude that risk is decreased by an investment if the rate of return is positively correlated with marginal utility. For monetary assets, with little or no principle risk and low volatility, the riskiness of the asset is likely to contribute relatively little to the riskiness of the household's consumption stream, and hence the sign of the covariance between the asset's rate of return and of the consumption stream is not easy to predict a priori. But with a very risky asset, such as common stock, it is far more likely that holding such a risky investment will increase risk, rather than decrease it. That occurs if the rate of return on the asset is positively correlated with consumption and thereby negatively correlated with marginal utility. This phenomenon is central to the consumption based capital asset pricing model (CCAPM).

Consider the interpretation of equation (4.5), which defines the adjustment for risk under risk aversion. Suppose we normalize relative to $\frac{\partial V}{\partial c_t}$, so that we need not consider the denominator of equation (4.5). Now consider first the second term on the right hand side of equation (4.5). Suppose that the own rate of return on monetary asset i is positively correlated with the marginal utility of consumption of goods, so that holding that monetary asset decreases risk. Since holding the asset decreases the consumer's consumption risk, we should expect that the risk-adjusted user cost price $\Pi_{it}=\pi_{it} + \psi_{it}$ that the consumer would have to "pay" to hold that asset would be decreased as that positive covariance increases, and that is precisely what the second term of equation (4.5) would do in that case. Conversely, if the covariance between the own rate and the marginal utility of consumption of goods is negative, so that holding the asset increases the risk of the consumer's consumption stream, the second term in equation (4.5) introduces a positive term into the risk-adjusted user cost $\Pi_{it}=\pi_{it} + \psi_{it}$ to reflect the increased cost of holding the asset as that covariance increases the consumer's risk. If the central bank were to introduce common stock or bond funds into monetary aggregates or other assets having substantial principal risk, we should expect to find the latter case would apply to those assets.

Now consider the first term on the right hand side of equation (4.5). The benchmark rate is the interest rate foregone by not holding the benchmark asset. If the benchmark rate decreases consumption risk through a positive covariance between the benchmark rate and the marginal utility of consumption of goods, then the opportunity cost of foregoing the benchmark asset by holding monetary asset i instead, is increased. Hence we should expect that such a positive covariance should increase the risk adjusted user cost Π_{it}, as indeed is the effect of the first term of equation (4.5). Conversely if that covariance is negative, so that holding the benchmark asset increases the consumer's risk, then foregoing the benchmark asset in favor of monetary asset i decreases risk and hence results in a subtraction from the risk adjusted user cost, Π_{it}, of holding asset i.[11]

4.2 The Generalized Divisia Index Under Risk Aversion

The ordinary Divisia index was derived by Francois Divisia from the first order conditons for rational consumer behavior under perfect certainty. In the case of risk aversion, the first order conditions are Euler equations, and we have found that those Euler equations for monetary assets demanded by consumers can be put into the form (4.2), which we now use to derive a generalized Divisia index, as follows.[12]

Theorem 2: In the share equations $s_{it} = \pi_{it}m_{it} / \Sigma_{j=1}^{k1} \pi_{jt}\, m_{jt}$, replace the unadjusted user costs ${}^{1}_{it}$, defined by (3.1), by the risk adjusted user costs Π_{it}, defined by Definition 1, to produce the adjusted shares $S_{it} = \Pi_{it}m_{it} / \Sigma_{j=1}^{k1} \Pi_{jt}\, m_{jt}$.

Under our weak separability assumption, $V(\mathbf{m}_t,c_t) = F(M(\mathbf{m}_t),c_t)$, and our assumption that the monetary aggregator function M is linearly homogeneous, the following generalized Divisia index is true under risk:

$$d \log M_t = \sum S_{it} d \log m_{it} \qquad (4.12)$$

Proof: Under our weak separability assumption, $V(\mathbf{m}_t,c_t) = F(M(\mathbf{m}_t),c_t)$, we have that

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[11]Note that the magnitude of the adjustment from the first term of (4.5) depends upon the size of the unadjusted user cost π_{it} of monetary asset i . The unadjusted user cost defined by (4.4) would equal 1.0 in the limit as the benchmark rate increases towards infinity, while the own rates on the component assets are held down at low levels by regulation. In that case the first term of equation (4.5) would produce no risk adjustment, although the second term still would. This special case is far from likely and should not be expected to be encountered with actual data. In another special case, the unadjusted user cost would equal zero, if the expected rate of return on monetary asset i equals that on the benchmark asset. In that case, the unadjusted user cost π_{it} would drop out of the adjustment term ψ_{it} defined by equation (4.5).

[12]Our generalized Divisia index should not be confused with Stahl's (1983) generalized Divisia index. Stahl's index under perfect certainty provides a parametric extension to the case of affine homothetic aggregator function, in contrast with the Caves, Christensen, and Diewert (1982a,b) nonparametric approximation to the Malmquist index aggregator, when the utility function is completely nonhomothetic. In our proofs under risk, we limit ourselves to the case of homothetic category utility function as aggregator, but the extension to nonhomotheticity could be produced by either Stahl's method or by Caves, Christensen, and Diewert's method.

$$\frac{\partial V}{\partial m_{it}} = \frac{\partial F}{\partial M_t}\frac{\partial M_t}{\partial m_{it}} \quad . \tag{4.13}$$

Substituting (4.10) into (4.13), we acquire

$$\frac{\partial M_t}{\partial m_{it}} = (\pi_{it} + \psi_{it})\frac{\frac{\partial V}{\partial c_t}}{\frac{\partial F}{\partial M_t}} \; . \tag{4.14}$$

Since the total differential of M_t=M($\mathbf{m}_t$) is,

$$dM_t = \sum_i \frac{\partial M}{\partial m_{it}} dm_{it} \tag{4.15}$$

we can substitute (4.14) into (4.15) to get

$$dM_t = \frac{\frac{\partial V}{\partial c_t}}{\frac{\partial F}{\partial M_t}} \sum (\pi_{it} + \psi_{it}) dm_{it} \, . \tag{4.16}$$

Using the linear homogeneity of M, we have from Euler's theorem for homogeneous functions that

$$M_t = \sum_i \frac{\partial M}{\partial m_{it}} m_{it} \; . \tag{4.17}$$

Substituting (4.14) into (4.17), we acquire

$$M_t = \frac{\frac{\partial V}{\partial c_t}}{\frac{\partial F}{\partial M_t}} \sum (\pi_{it} + \psi_{it}) m_{it} \, . \tag{4.18}$$

Dividing (4.16) by (4.18), we get equation (4.12). **Q.E.D.**

Hence we see that the exact tracking of the Divisia monetary index is not compromised by risk aversion, so long as the adjusted user costs $\pi_{it} + \psi_{it}$ are used in computing the index. As we have observed, the adjusted user costs reduce to the usual user costs in the case of perfect certainty and our generalized Divisia index (4.12) reduces to the usual Divsia index (3.2). Similarly the risk neutral case is acquired as the special case with ψ_{it}=0, so that equation (4.10) serves as the user cost . In short, our generalized Divisia index (4.12) is a true generalization in the sense that the risk neutral and perfect certainty cases are strictly nested special cases. Formally that conclusion is the following:

Corollary 1 to Theorem 2: Under risk neutrality, the generalized Divisia index (4.12) reduces to (3.2), where the user costs in the formula are defined by (4.10). Under perfect certainty, the user costs reduce to (3.1).

The need for the generalization can be explained as follows. The consumer has a three dimensional decision, in terms of asset characteristics. The monetary assets having nonzero own rates of return produce investment returns, contribute to risk, and provide liquidity services. Our objective is to track the nested utility function, $M(\mathbf{m}_t)$, which measures solely liquidity and is the true economic monetary aggregate. To do so, we must remove the other two motives: investment yield and risk aversion. While those two motives are relevant to savings and intertemporal substitution, we seek to track the liquidity flow alone. The ordinary Divisia monetary aggregate removes the investment motive and would track the liquidity services, if there were no risk. The generalized Divisia index removes both the investment motive and and the aversion-to-risk motive to extract the liquidity service flow, when the data is produced by consumer's who in fact are making decisions that involve a three way tradeoff among mean investment return, risk aversion, and liquidity service consumption.[13]

5. The CCAPM Special Case

As a means of illustrating the nature of the risk adjustment ψ_{it}, we consider a special case, based upon the usual assumptions in CAPM theory of either quadratic utility or Gaussian stochastic processes. Direct empirical use of theorems 1 and 2, without any CAPM simplifications, would require availability of prior econometric estimates of the parameters of the utility function V and of the subjective rate of time discount. Under the usual CAPM assumptions, we show in this section that empirical use of theorems 1 and 2 would require prior estimation of only one property of the utility function: the degree of risk aversion, on which a large body of published information is available.

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[13]The intent of the Divisia index is to track the category utility function, $M(\mathbf{m}_t)$, which reflects the liquidity services received from holding the components of the aggregate. It is not M, but rather the outer function, F, in the consumer's utility function that reflects the consumer's degree of risk aversion, and it is the integration in the resulting expected utility function that brings in the probability distributions defining the amount of risk in the economic environment of the consumer. But the market opportunity cost evaluated at the margin by the unadjusted user cost π_{it} of asset i reflects both the liquidity premium embedded in the expected rate of return and also the risk premium demanded by risk averse consumers, when an asset's own rate of return is correlated with the consumer's consumption stream. Hence the unadjusted user costs cannot be used to eliminate the marginal utilities in the total differential (4.15). The adjustment ψ_{it} to the opportunity cost in $\pi_{it} + \psi_{it}$ can be viewed as extracting the risk premium at the margin and leaving only the liquidity premium, as is needed to track the marginal utilities in the total differential (4.15) that produces the Divisia index's tracking capability.

Consider first the following case of utility that is quadratic in consumption of goods, conditionally on the level of monetary asset service consumption:

Assumption 1: Let V have the form

$$V(\mathbf{m}_t, c_t) = F(M(\mathbf{m}_t), c_t) = A(M(\mathbf{m}_t))c_t - \frac{1}{2}B(M(\mathbf{m}_t))c_t^2, \qquad (5.1)$$

where A is a positive, increasing, concave function and B is a nonnegative, decreasing, convex function.[14]

The alternative assumption is Guassianity, as follows:

Assumption 2: Let (r_{it}^*, c_{t+1}) be a bivariate Gaussian process for each asset i = 1, . . . , k_1.

We also make the following conventional CAPM assumption:[15]

Assumption 3: The benchmark rate process is deterministic or already risk-adjusted, so that R_s^* is a risk free rate for all s^3t.

Under this assumption, it follows that $Cov(R_t^*, \frac{\partial V}{\partial c_{t+1}})$ equals zero.

We define $H_{t+1}=H(M_{t+1},c_{t+1})$ to be the well known Arrow-Pratt measure of absolute risk aversion,

$$H(M_{t+1},c_{t+1}) = \frac{-E_t[V'']}{E_t[V']}, \qquad (5.2)$$

where $V' = \partial V(\mathbf{m}_{t+1},c_{t+1})/\partial c_{t+1}$ and $V'' = \partial^2 V(\mathbf{m}_{t+1},c_{t+1})/\partial c_{t+1}^2$. In this definition, risk aversion is measured relative to consumption risk, conditionally upon the level of monetary services produced by $M_{t+1}=M(\mathbf{m}_t)$. Under

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[14]In CAPM applications, it also is necessary to assume that all observations are to the left of the quadratic maximum.

[15]It amounts to the assumption that the risk premium already has been extracted from the benchmark rate. In practice, this assumption is harmless, since the risk premia adjustments below are applied to all component assets before the benchmark rate is computed---usually as an upper envelope of the component rates. If other asset paths are also included among those used to produce the upper envelope, then this assumption requires that the same risk premia adjustments also be applied to those paths before the upper envelope is generated. Since the risk premia already have been extracted at the time that the envelope is produced, the benchmark rate automatically is risk adjusted.

risk aversion, H_{t+1} is positive and increases as the degree of absolute risk aversion increases. The following lemma is central to our Theorem 3.

Lemma 1: Under Assumption 3 and either Assumption 1 or Assumption 2, the user cost risk adjustment, ψ_{it}, defined by equation (4.5) reduces to

$$\psi_{it} = \frac{1}{1+R_t^*} H_{t+1} \mathrm{Cov}(r_{it}^*, c_{t+1}) \tag{5.3}$$

Proof: Assuming that R_s^* is a risk free rate for all s³t, equation (4.3) simplifies to

$$\frac{\partial V}{\partial c_t} = \rho(1+R_t^*) E_t\left[\frac{\partial V}{\partial c_{t+1}}\right], \tag{5.4}$$

and the risk adjustment term (4.5) simplifies to

$$\psi_{it} = -\ \rho \frac{\mathrm{Cov}(r_{it}^*, \frac{\partial V}{\partial c_{t+1}})}{\frac{\partial V}{\partial c_t}}. \tag{5.5}$$

Substituting (5.4) into (5.5), we acquire

$$\psi_{it} = -\frac{1}{1+R_t^*} \frac{\mathrm{Cov}(r_{it}^*, \frac{\partial V}{\partial c_{t+1}})}{E_t\left[\frac{\partial V}{\partial c_{t+1}}\right]}. \qquad \text{(5.6}$$

Consider first the case in which we accept Assumption 1. Substituting the quadratic specification (5.1) into (5.6), we get

$$\psi_{it} = \frac{1}{1+R_t^*}\left[\frac{-EV''}{EV'}\right] \mathrm{Cov}(r_{it}^*, c_{t+1}), \tag{5.7}$$

which under our definition of H_{t+1} is identical to (5.3).

Now consider the alternative possibility of accepting Assumption 2 instead of Assumption 1. Applying Stein's lemma for bivariate normal distributions to equation (5.6), we again acquire (5.7) and thereby (5.3).[16]

Q.E.D.

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[16]For Stein's lemma, see Stein (1973). Alternatively see Ingersoll (1987, p. 13, eq. 62)) or Rubinstein (1976).

Observe that equation (5.3) provides a CCAPM (consumption CAPM) type result, since the risk adjustment term ψ_{it} is very much like the risk premium on a risky asset in CCAPM. In CCAPM, as in our model, compensation for risk is proportional to the covariance of the asset's return with consumption through the factor $Cov(r_{it}, c_{t+1})$ in (5.3) and also to the degree of risk aversion H_{t+1} in (5.3).[17]

In effect, what the adjustment does for very risky rates is to remove the the risk premium from $E_t r_{it}$ so that the adjusted user cost becomes positive. To see this more clearly, define $Z_t = H_{t+1} c_t$, where Z_t is a modified (time shifted) Arrow-Pratt relative risk aversion measure. Our theorem now follows immediately.

Theorem 3: Under the assumptions of Lemma 1, we have

$$\Pi_{it} = \frac{E_t R_t^* - \left(E_t r_{it}^* - \phi_{it}\right)}{1 + E_t R_t^*}. \tag{5.8}$$

where

$$\phi_{it} = Z_t Cov\left(r_{it}^*, \frac{c_{t+1}}{c_t} \right) \tag{5.9}$$

Proof: Substitute (5.3) and (4.4) into $\Pi_{it} = \pi_{it} + \psi_{it}$ and substitute $Z_t = H_{t+1} c_t$. **Q.E.D.**

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[17]If own interest rates are positively correlated with consumption, (5.3) is positive, since H_{t+1} would be positive under risk aversion. Alternatively, if the asset's return is not sufficiently risky to dominate the direction of the net shocks to consumption from risk, the opposite could happen. The asset's rate of return could correlate negatively with the consumption stream in a manner tending to decrease the household's consumption risk, and hence (5.3) would be negative. In the CCAPM theory of finance, beta of a very risky asset is usually positive, where beta is defined to be $\beta_{ic} = Cov(r_{it}^*, c_{t+1}) / Var(c_{t+1})$. The subscript c in β_{ic} designates "consumption based" beta, and the lack of a time subscript in the notation β_{ic} results from the assumption of stationarity of the interest rate and consumption bivariate process. Clearly the usual finance view of positive β_{ic} can hold if and only if $Cov(r_{it}, c_{t+1})$ is positive.

This conclusion about the sign of the adjustment term, ψ_{it}, in the adjusted user cost $\pi_{it} + \psi_{it}$ of very risky assets is especially revealing, when the benchmark rate is defined to be riskless, as we have just done. Consider the definition of the unadjusted user cost in equation (4.4). Since we now are assuming that the benchmark rate is defined to be the maximum available rate of return on a risk free asset, we can conclude that the benchmark asset has no embedded liquidity premium and cannot be less than the own rate of return on any risk free monetary asset i. Hence (4.4) is nonnegative if monetary asset i is risk free. But suppose that consumers are risk averse and that monetary asset i is not risk free. Then $E_t r_{it}$ will contain a risk premium, despite the fact that R_t does not (or its risk premium has been removed). Hence the unadjusted user cost (4.4) could be negative. But as we have just observed, ψ_{it} in this case will be positive, and we would expect that in fact it will be sufficiently positive to offset the possible negativity of the unadjusted user cost to produce positive value of the adjusted user cost $\pi_{it} + \psi_{it}$.

As is evident from this theorem, the risk premium adjustment is ϕ_{it}, where c_{t+1}/c_t is a measure of the consumption growth rate. Hence the risk adjustment depends upon relative risk aversion and the covariance between the consumption growth path c_{t+1}/c_t and the real rates of excess return r^*_{it}. We see that the adjusted user cost $\Pi_{it} = \pi_{it} + \psi_{it}$ can be written in the same form as the unadjusted user cost (4.4), if the benchmark rate is defined to be risk free and if the risk premium adjustment ϕ_{it} is subtracted out of the expected value of the real rates of excess return r^*_{it}. As we have observed, that adjustment should be expected to decrease the expected own rate of return, if the asset is very risky and thereby contibutes positively to consumption risk.

6. The Magnitude of the Adjustment

In accordance with the large and growing literature on the equity premium puzzle, we should expect that ϕ_{it} is small for most i. In fact our initial computations of that risk premium adjustment term for the components of the existing monetary aggregates produces very small correction terms, such that the usual Divisia monetary aggregate and the extended Divisia monetary aggregate are nearly identical to within the roundoff error of the data. There recently has been growing interest in the inclusion of common stock and bond mutual fund in monetary aggregates, but no such change has yet been made officially by the Federal Reserve. In addtion, our confidence in the availble data on those additional potential components is limited at the present time. Since the rates of return on those assets are subject to much more risk than the rates of return on the existing components of the monetary aggregates, the gain in moving from the usual Divisia index to the extended Divisia index could be greater, if those assets are incorporated into monetary aggregates. Nevertheless, our initial computations with data including those risky assets produce small, although no longer trivial, differences between the extended and unextended Divisia monetary aggregates. The equity premium puzzle issues with CCAPM do not go away with incorporation of assets with substantial rate of return risk.

One possible explanation of the surprisingly small risk adjustment terms, even with risky assets, may be aggregation over economic agents. In some sense, the risk adjustments may tend to cancel each other out across economic agents, and our initial computations, mentioned above, are produced from data aggregated over economic agents. To explore this possibility further, we now turn to the use of simulated data generated from a single modeled economic agent. Generation of that data requires numerical solution of that consumer's Euler equations.

Having generated the simulated data, we compare the exact aggregator function implied by the weakly separable structure of the utility function with the unextended Divisia monetary aggregate to isolate the effect of the needed risk adjustment, since the correctly computed adjusted Divisia aggregate should track the exact aggregator function accurately.

The loss in tracking ability of the unextended Divisia index under risk has been investigated previously by Barnett, Kirova, and Pasupathy (1995), but with an aggregator function that was estimated using actual data aggregated over computers. Hence there was no control in that study for the possible decrease in the risk adjustment that may have been produced by aggregation over consumers or for possible specification error in the estimated parametric model. In particular, there is no way to know the degree to which the tracking error of the unadjusted Divisia index was produced by the missing risk adjustment or alternatively by the specification error in the estimated utility function of the representative consumer. By the use of simulated data below, we know the exact aggregator function for the simulated single consumer and hence we control for the possible downward bias that may have been produced by earlier studies using macroeconomic data and we eliminate any possible specification error effect, since our simulated data is exactly consistent with rational behavior of our modeled consumer.

7. Generation of the Simulated Data

7.1 Introduction

To show how well the unadjusted Divisia monetary aggregate tracks the `true' theoretical aggregate for one consumer, we numerically solve for the individual monetary assets from the Euler equations associated with a dynamic optimization problem. We use the calculated rational expectation equilibrium to determine both the Divisia and the theoretical monetary aggregate, and we compare how well the unadjusted Divisia index tracks the movement of the theoretical aggregate. The magnitude of the effect of the missing risk adjustment in the unadjusted Divisia index is measured by the gap between our two simulated aggregates.

Although there are a number of numerical methods that solve for the rational expectation equilibrium of such problems, for this paper we have selected the approach advocated by Den Haan and Marcet (1990).[18] The Den Haan and

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[18]See Taylor and Uhligh (1990) for a list and comparison of the known methodologies to solving dynamic optimization problems.

Marcet approach has been shown to provide good results for a number of complex optimization problems.[19] Furthermore, the complexity of the dynamic programming problem found in this paper is an important factor in choosing the Den Haan and Marcet algorithm, since this solution method does not require discretizing the state space---a formidable task for any computer, when the number of state variables is as plentiful as in our model.

7.2 The Parameterized Model of Preferences

In this section we formulate a discrete time period optimization problem for a consumer, under the assumption that monetary assets are weakly separable from consumer goods, so that an exact aggregate exists over consumer goods and another exact aggregate exists over monetary assets. Since our model of tastes will be parameterized, the model can be viewed as a special case of the model that was used above to derive the extended Divisia index, which therefore can be assumed to track the exact aggregate over monetary assets accurately. In our model, there are three component monetary assets within the exact monetary aggregate. We make that choice so that we can use the empirical results found by Barnett and Yue (1991) to set the unknown parameters of the model. In addition we have tried to use the same notation that Barnett and Yue (1991) used in an attempt to keep some consistency in that literature.

We assume that the consumer's intertemporal utility, (2.1), has the form

$$F\Big(M(\mathbf{m}_t), c_t\Big) + E_t\left[\sum_{s=t+1}^{\infty} \rho^{s-t} F\Big(M(\mathbf{m}_s), c_s\Big)\right] \tag{7.1}$$

where F is the constant relative risk aversion utility function

$$F(M_s, c_s) = \frac{1}{\sigma}\left[c_s^{\beta} M_s^{1-\beta}\right]^{\sigma}$$

with $\sigma \in (-\infty, 0) \cup (0,1)$, $\rho \in (0,1)$, and $\beta \in (0,1)$, and where c_s and $M_s = M(\mathbf{m}_s)$ are respectively the consumption good and monetary asset quantity aggregates respectively.[20] The three dimensional vector $\mathbf{m}_s$ contains the three component assets. As $\sigma \to 0$, the consumer's utility function becomes $F(M_s, c_s) = \log(c_s^{\beta} M_s^{1-\beta})$.

We assume that the exact monetary aggregator function, $M(\mathbf{m}_s)$, is the CES function $M(\mathbf{m}_s) = \left(\sum_{i=1}^{3} \delta_i m_{is}^{\alpha}\right)^{1/\alpha}$ with $\sum_{i=1}^{3} \delta_i = 1$ and $\alpha \in (0,1]$. The simple sum and Cobb-Douglas aggregates are nested special cases, since the monetary aggregator function becomes Cobb-Douglas, if $\alpha \to 0$, and simple sum if $\alpha = 1$.

7.3 The Euler Equations

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[19]See Den Haan and Marcet (1989), Marshall (1992), and Bansal et.al. (1992).

[20]In much of this literature, a finite period planning horizon is assumed, as in Jensen (1995). But the resulting exact aggregate is the same in the finite and infinite planning cases. See Barnett (1995) regarding the equivalency.

Following Barnett and Yue (1991) and Poterba and Rotemberg (1987), the Euler equations from the consumer's decision in Problem 1 with the above specification for the objective function are

$$c_s^{\sigma\beta-1} M_s^{\sigma(\beta-1)} = E_s\left[\left(\rho \frac{p_s^*}{p_{s+1}^*}(1+R_s)\right)^{-1} c_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)}\right]$$

$$c_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{1s}^{1-\alpha} = E_s\left[\left(\frac{1}{\delta_1}\frac{\beta\rho}{1-\beta}\frac{p_s^* R_s}{p_{s+1}^*}\right)^{-1} c_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)}\right]$$

$$c_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{2s}^{1-\alpha} = E_s\left[\left(\frac{1}{\delta_2}\frac{\beta\rho}{1-\beta}\frac{p_s^*(R_s - r_{2s})}{p_{s+1}^*}\right)^{-1} c_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)}\right]$$

$$c_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{3s}^{1-\alpha} = E_s\left[\left(\frac{1}{\delta_3}\frac{\beta\rho}{1-\beta}\frac{p_s^*(R_s - r_{3s})}{p_{s+1}^*}\right)^{-1} c_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)}\right]$$

In the dynamic programming Problem 1, the consumer faces the exogenous stochastic state vector $\phi_s = \left(R_{s-1}, r_{2,s-1}, r_{3,s-1}, p_s^* / p_{s-1}^*, I_s / I_{s-1}\right)'$, along with the endogenously evolving state vector $\mathbf{m}_{s-1}$. Together they define the state vector $\sigma_s' = (\mathbf{m}_{s-1}', \phi_s')$, while the consumer's control vector for the optimization problem is $\mathbf{z}_s' = (\mathbf{m}_s', c_s)$.

We assume that the stochastic process $\{\phi_s\}$ behaves as two independent Markovian processes such that

$$\phi_{1s} = \mathbf{a}_1 + \Lambda_1 \phi_{1,s-1} + \mathbf{u}_{1s}$$

and

$$\ln(\phi_{2s}) = \mathbf{a}_2 + \Lambda_2 \ln(\phi_{2,s-1}) + \mathbf{u}_{2s}$$

where $\phi_{1s} = (R_{s-1}, r_{2,s-1}, r_{3,s-1})'$ and $\phi_{2s} = \left(p_s^* / p_{s-1}^*, I_s / I_{s-1}\right)'$.[21] The disturbances $\mathbf{u}_{1s}$ and $\mathbf{u}_{2s}$ are both distributed *i.i.d.* $N(\mathbf{0}, \Omega_i)$ for, $i = 1,2$, with Λ_1 and Ω_1 being 3×3 matrices, while Λ_2 and Ω_2 are 2 X 2 matrices. The processes' intercept terms $\mathbf{a}_1$ and $\mathbf{a}_2$ are 3×1 and 2×1 vectors, respectively.

Each Markovian process is a first-order vector autoregressive process. The natural logarithmic transformation of ϕ_{2s} ensures that the process is stationary. Since ϕ_{1s} is comprised only of interest rates, there is no reason to transform this process. Regarding the method of computing the benchmark asset's rate of return, the sources of exogenous data, and the algorithms used in solving the system, see Jensen (1995).

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[21]A more complex stochastic processes for the exogenous state process could be used if the economist so desires. We here define the log of a vector to be the vcctor of the logarithms.

If we define the economy's parameter vector as $\lambda = \left(\rho, \sigma, \alpha, \beta, \delta_1, \delta_2, \delta_3, \mathbf{a}_1', \mathbf{a}_2', \text{vec}(\boldsymbol{\Omega}_1)', \text{vec}(\boldsymbol{\Omega}_2)'\right)$, then for each value of λ, the above optimization problem defines a nonlinear mapping from the exogenous state process $\{\phi_s\}$ to $\{\mathbf{z}_s\}$. The theoretical monetary aggregate, $\{M_s\}$, and the Divisa monetary aggregate

$$Q_s = Q_{s-1} \prod_{i=1}^{3} \left(\frac{m_{is}}{m_{i,s-1}} \right)^{s_{is}^*}$$

can then be calculated from $\{\phi_s\}$ and $\{\mathbf{z}_s\}$, where

$$s_{is}^* = \frac{1}{2}\left(s_{is} + s_{i,s-1}\right)$$

$$s_{is} = \frac{\pi_{is} m_{is}}{\sum_{j=1}^{3} \pi_{js} m_{js}},$$

and the user costs π_{is} are as in equation (3.1).

7.4 Solving the Euler Equations

To explore the implications of risk aversion and of substitutability between component assets for the tracking ability of the unadjusted Divisia index, we use the solution for $\{\mathbf{z}_s\}$ provided in Jensen (1995) for various settings of the degree of risk aversion and of substitutability. Based upon the simulation results in his paper, we generate the plots of the unadjusted Divisia index and of the exact parametric aggregate. Since the procedures used by Jensen (1995) are provided in detail in his paper, we outline the approach only briefly.

A rapidly growing literature exists on numerical methods for solving dynamic optimization problems.[22] Of those currently algorithms currently available the Parameterized Expectation Approach (PEA) of Den Haan and Marcet (1990) has performed well in head-to-head tests with other algorithms. In addition, the PEA approach has been applied to a number of different economic areas, including growth models (Den Haan and Marcet (1990)), asset markets with heterogenous agents (Ketterer and Marcet (1989), Marcet and Singleton (1990)), and monetary economies (Den Haan (1990a, 1990b), Marshall (1992), Bansal et al (1994, 1995)).

The PEA method approximates the expectation operators found in the Euler equations by parameterizing them with a basis function that spans the set of expectation operators in Hilbert space. The basis function is a globally flexible functional form having arguments that are the state variables σ_s.[23] In this paper we have chosen the first-order polynomial

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[22]See Taylor and Uhlig (1990) for an excellet source on the available methods and a comparison of their performance record.
[23]See Barnett and Jonas (1982), Barnett and Yue (1988), and Gallant (1982) for the properties and examples of globally flexible functions.

function, as in Jensen (1995). One then iterates over the parameters of these flexible functional forms until a convergence criterion is met.

Intuitively, each iteration of the PEA method can be viewed as nonlinear least-square learning behavior by the consumer.[24] Once learning is no longer occurring, the agent selects that stochastic solution $\{z_s\}$ that satisfies the Euler equations, given the learned prediction of the expectation operators. Hence, it can be argued that the numerical solution found with the PEA approach is an equilibrium for an agent restricted to the learning associated with a specific globally flexible functional form.

In applying the PEA approach, Jensen (1991) used the convergence criterion recommended by Bansal et. al. (1992), except for cases in which the degree of risk aversion was set at high levels. In those cases, Jensen (1995) used the convergence criterion advocated by Marshall (1992) to determine the stopping point for the PEA algorithm.[25] Marshall argues that the PEA algorithm converges to an approximate equilibrium, if the coefficient vectors, $\mathbf{v}_i$ for i=1,2,3,4, are the optimal prediction vectors.

To determine whether the numerical solution is close to the `true' solution, Jensen (1995) employed the Den Haan and Marcet test statistic (DHM-stat) [see Den Haan and Marcet (1994) and Taylor and Uhlig (1990)]. The DHM-stat provides a test of the theoretical martingale property $E\left[\nu_s \otimes \mathbf{h}_{sij}\right] = 0$, where ν_s is the 4 by 1 residual vector from the Euler equations and $\mathbf{h}_{sij} = \sigma^i_{s-j}$. If Hansen's (1982) regularity conditions hold and the approximation is an exact solution satisfying the above martingale property, then the DHM-stat $TBA^{-1}B$ will be distributed as χ^2 with 20 degrees of freedom, where $B = (1/T)\sum_{s=t}^{t+T}\left[\nu_s \otimes \mathbf{h}_{sij}\right]$ and $A = (1/T)\sum_{s=t}^{t+T}\left[\nu_s \otimes \mathbf{h}_{sij}\right]\left[\nu_s \otimes \mathbf{h}_{sij}\right]'$.

To ensure that our solution reflects the empirical world, Jensen (1995) set the parameters, λ , in the Euler equations equal to the generalized method of moment estimates of those parameters found by Barnett and Yue (1991). The other parameters associated with the two Markovian processes were set equal to the estimated parameters for the two VAR(1), using monthly data from January 1960 to December 1990.

The size of the simulation is equal to 100, i.e. T=100. Jensen chose a small simulation size, because of the explosive nature of the PEA approach for large simulations when the solutions to the Euler equations are not stationary

[24]See Marcet and Sargent (1989a, 1989b) for examples of a linear least-square learning model that have a locally stable equilibrium.
[25]We initially used the convergence criterion found in Eq. (14) but the PEA algorithm failed to converge within 40,000 iterations.

processes.[26] We know from Corollary 1 to Theorem 1 that the unadjusted Divisia index will track well under risk neutrality. Hence our interest is in other cases. Based upon Jensen's (1995) solutions of the Euler equations conditionally upon Barnett and Yue's (1991) parameter estimates, we plot both the unadjusted Divisia monetary aggregate and the exact nested aggregator function in Figures 1-14.

Clearly the tracking ability of the unadjusted Divisia index is not independent of the degree of risk aversion, and at any setting of the degree of risk aversion, the tracking ability depends upon the substitutability among the components of the aggregator function. Hence the risk-adjusted Divisia index and the associated risk-adjusted user costs are likely to be useful, in some non-risk-neutral cases, although in many cases with moderate risk aversion, the gain may be slight. It also appears that aggregation over economic agents is a relevant factor, since we find greater loss in tracking ability with simulated data from one consumer than with the economy's aggregate per capita data.[27] In addition, it is interesting to observe from the data that under risk aversion, the unadjusted Divisia index is more volatile than the exact aggregate. Since the risk adjusted Divisia index, (4.12), can be expected to track the exact aggregate very accurately (in fact perfectly in continuous time), we find that the risk adjustment to the user costs tends to smooth the volatility of the Divisia monetary aggregate.

For details about the parameter estimation and data, see Barnett and Yue (1991), and for details of the solution for the endogenous variables conditionally upon the exogenous processes, see Jensen (1995). At this point, we are not comfortable about reaching stronger conclusions in this regard, since competing approaches to solving rational expectations models may be preferable in some of the cases considered above, and hence we cannot be certain of the degree to which our conclusions have been affected by inaccuracies in the solution method used. It would be useful to repeat these experiments with the alternative solution methods proposed by Coleman (1990, 1991), Baxter (1991), or Baxter, Crucini, and Rouwenhorst (1990).

8. Velocity Function Behavior under Risk

8.1 Introduction

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[26]Marcet and Marshall (1992) provide a set of conditions and an alternative PEA approach that overcomes the problems associated with unstable solutions. They advocate taking a larger number of samples of random disturbances to generate the exogeneous state variables and use the sample average of the polynomial's coefficient estimates, v_i's, as the coefficients for approximating the expectation operators.

[27]Regarding the results with aggregate per capita data, see Barnett and Liu (1996) and Barnett, Kirova, and Pasupathy (1995).

Having established the relationship between interest rate risk and asset user costs along with the extended aggregation theory for aggregation over assets, when the relevant user costs are the risk adjusted user costs, we now turn to the exploration of the implications of these theoretical results for the behavior of the velocity function under interest rate risk.

Explanation for the behaviour of money velocity go back to as early as the seventeenth century (see Humphrey (1993) for a review). But instability of the money velocity function since the late 1970's in the U.S. has called for a reexamination of the traditional views. See, e.g., Stone and Thornton (1987). One line of research focuses on the correct measurement of money snd challenges the traditional practice of ignoring the aggregation problem in monetary economics research and policy design.[28] For example, Barnett *er al* (1984) found that the coefficients in a conventional demand-for-money equation using Divisia monetary aggregation are more stable than those in the same equation with simple sum monetary aggregation on quarterly data from 1959:1 to 1982:4. Another line of research focuses on the effects of institutional change on money velocity. See, e.g., Bordor and Jonung (1981, 1987, and 1989). Another hypothesis, proposed by Friedman (1983), attributes the several substantial declines of M1 velocity since 1981 to the increased money growth variability following the change of Federal Reserve operating procedures in October 1979. However, empirical tests of his variability hypothesis have not provided uniform evidence.[29] More recently, some research has investigated whether the observed variability of money velocirty can be explained in monetary general equilibrium models since Lucas (1978), Svensson (1985), and Lucas and Stockey (1987). With the cash-in-advance specification, simulation results for money velocity in general equilibrium models have been generally disappoionted. See, e.g., Hordrick *et al* (1991) and Giovannini and Labadie (1991).[30]

In the rest of this paper we explore the determinants of money velocity and the causes of the instability of a traditional money velocity fucntion in a monetary general equilibrium model. One difference between our model and the previous ones in the monetary economics literature is that monetary assets in our model pay interest.

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[28]For the measurement of money, see Barnett (1980, 1987), Barnett *et al* (1991), Belongia (1995), and Serletis (1995).
[29] See Belongia (1985), Fisher and Serletis (1989), Hall and Noble (1987), Mehera (1989), and Thornton (1995) for empirical evidence of the variability hypothesis.
[30]Hordrick *et al* (1991) and Giovannini and Labadie (1991) find an almost-constant money velocity in their simulation results. A constant velocity level is apparently counterfactual. It is, however, frequently assumed in a traditional quantity theory of money. Recently, a constant growth rate of velocity has been assumed in testing the quantity theory by Bullard (1994).

Another feature of our model is that the principle of monetary aggregation theory are imposed. When nominal interest rates are certain, it is found that the unadjusted user cost (3.1) is the relevant determinant of money velocity. Under risk aversion the adjusted user cost defined in Section 4.1 is the determinant of money velcoity. We find that the uncertainty of nomal interest rates and the degree of risk aversion play an important role in determining the stability of the money velocity fucntion. If the covariances change between interest rate and consumption growth rate or between interest rate and the real money growth rate, the model predicts that the usual money velocity function, defined without a CAPM risk adjustment internal to the function, will shift. Hence the coefficients of the usual money velocity fucntion may change, if time-varying risk is present, as for example could be generated by an ARCH type process.[31] In fact, anything that causes covariances to change between interest rates and the consumption growth rate or between interest rates and the real money growth rate will contribute to a shift of the usual money velocity function. These causes could include financial innovations or money growth variability, [32] as previously investigated in the literature. In this sense, our study provides a general and coherent theoretical explanation for the instability of money velocity and nests many earlier explanations as special cases. In addition, our approach derives the money velocity function that internalizes the risk correction and hence remains stable, even if the interest rate processes exhibit stochastic volatility (i.e., variations in variance over time).

We simulate the process of the slope coefficient of a simple traditional money velocity function based on U.S. quarterly data from 1960.1 to 1992.4 and find that the theoretical model's explanatory power is moderately high, especially during the 1973-1976 and 1979-1982 periods. The Swamy and Tinsley (1980) random coefficient model for money velocity is estimated to compare the behavior of the estimated stochastic coefficients with the simulated coefficients from the theoretical model. There are some important similarities between the estimated stochastic coefficient process and the simulated stochastic coefficient process.

The remainder of Section 8 is organized as follows. Section 8.2 develops a monetary general equilibrium model in which monetary assets provide monetary services as well as interest income. Section 8.3 derives the theoretical results for money velocity under the assumption that nominal interest rates are known. It is shown that Latane's equation (Latane (1954)) is a special case of the model developed in this section. Section 8.4 generalizes the result

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[31]Time-varying coefficient models of money velocity are increasingly used in the literature. See, e.g., Dueker (1993,1995).

[32] For the effect of financial innovation on the economy, see recent work by Thornton (1994).

when the assumption of certain nominal interest rates is relaxed. The effect of risk aversion and interest rate uncertainty is investigated. Section 8.5 presents the results from model simulation and from estimation of a random coefficient model. Section 8.6 provides concluding remarks regarding the behavior of velocity.

8.2 Assumptions and Theoretical Specifications

In this section we outline an infinite-horizon, representative-agent model with a set of monetary assets which pay interest. Suppose that there exist k monetary assets. Monetary asset i pays nominal return rate r_{it} at the end of time period t. Money supply is assumed to be exogenous and serves as a moving endowment point in the consumer's budget constraint. There exists one nominal bond with holding period yield R_t, paid at the end of period t. We assume that

$$R_t \geq \max\{r_{it}, i = 1, ...,k\}, \ \forall t,$$

so that monetary assets are dominated in holding period returns by the nomial bond, which is assumed to yield no monetary services. Hence the nominal bond in this model is the benchmark asset. The price for the bond in period t is P_{bt}. There exists one equity asset, which is the exogenous endowment asset and yields resource flow $d_t > 0$. The price of one unit of the equity is P_{st}, and dividend d_t per unit is paid before the share is sold. The only consumption good is the resource flow d_t whcih is perishable. The price of that consumption goods is p_t^* in period t. There are finitely many idntical consumers with utility fucntion $V(c_t, \mathbf{m}_t)$, which is continuous, increasing, and concave in all its arguments, where c_t is the demand for consumption goods, and $\mathbf{m}_t = (m_{1t}, m_{2t}, ..., m_{kt})'$ is a vector of real monetary assets held during period t.

The exogenous supply of monetary asset i in period t is X_{it}, and let $\sum_{i=1}^{k} X_{it} = X_t$ be the simple aum aggregate of money supply. The representative consumer is assumed to maximize the infinite lifetime expected utility (2.1) in the form (7.1), which can be written as

$$T = E_t \sum_{t=0}^{\infty} \rho^t V(c_t, \mathbf{m}_t). \tag{8.1}$$

The budget constraint from Problem 1 in Section 2.1 becomes:

$$\begin{aligned} p_t^* c_t + P_{st} s_t + P_{bt} b_t + p_t^* \sum_{i=1}^{k} m_{it} &\leq s_{t-1}(d_t p_t^* + P_{st}) + P_{b,t-1} b_{t-1}(1 + R_{t-1}) \\ &+ \sum_{i=1}^{k} m_{i,t-1}(1 + r_{i,t-1}) p_{t-1}^* + \sum_{i=1}^{k} [X_{it} - (1 + r_{i,t-1}) X_{i,t-1}], \end{aligned} \tag{8.2}$$

where s_t is the quantity of equity, and b_t is the quantity of bonds held during time period t. [33]

As in the earlier sections of this paper, we continue to introduce money through the money-in-utility-function approach, with the utility function viewed as the derived utility function that must exist, if money has positive value in equilibrium. Alternatively, in a cash-in-advance model, it is difficult to justify the existence of multiple monetary asset that pay different interest rates. In cash-in-advance models, there exists only one monetary asset in the equilibrium of an economy.

We first assume that R_t and r_{it} are known to consumers at the beginning of the time period t, and in Section 8.4 we will relax this assumption. Let π_t be the user cost price aggregate that is dual to the exact monetary quantity aggregate, M_t. According to Fisher's factor reversal test, expenditure on an exact aggregate must equal expenditure on the components, so that π_t must satisfy

$$\sum_{i=1}^{k} \pi_{it} m_{it} = \pi_t M_t, \tag{8.3}$$

where

$$\pi_{it} = \frac{R_t - r_{it}}{1+R_t}, \tag{8.4}$$

as in equations (3.1). We now define R_{mt} such that

$$\sum_{i=1}^{k} (1+r_{it}) m_{it} = (1+R_{mt}) M_t. \tag{8.5}$$

Note that R_{mt} can be interpreted as the aggregate rate of return dual to the exact monetary aggregate. To formalize this important concept, we state the following definition:

Definition 2: The aggregate rate of return, R_{mt}, dual to the exact monetary aggregate M_t is the solution for R_{mt} to equation (8.5).

Dividing equation (8.5) by $(1+R_t)$ and adding the resulting equation to equation (8.3), we have

$$\sum_{i=1}^{k} m_{it} = \left(\pi_t + \frac{1+R_{mt}}{1+R_t}\right) M_t \tag{8.6}$$

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[33] For convenience, we drop the subscript 1 from k_1, so that k now is the k_1 of section 3.

Let M_t^n be the nominal supply-side exact monetary aggregate. We assume that the monetary asset markets are in equilibrium when $M_t^n = M_t p_t^*$ where $M_t p_t^*$ is the nominal demand side exact monetary aggregate.[34] Under these assumptions the budget constraint can be written as

$$\begin{aligned} & p_t^* c_t + P_{st} s_t + P_{bt} b_t + p_t^* M_t (\pi_t + \tfrac{1+R_{mt}}{1+R_t}) \le s_{t-1}(d_t p_t^* + P_{st}) \\ & + p_{t-1}^* (1 + R_{m,t-1}) M_{t-1} + b_{t-1}(1 + R_{t-1}) P_{b,t-1} + M_t^n (\pi_t + \tfrac{1+R_{mt}}{1+R_t}) \\ & - M_{t-1}^n (1 + R_{mt-1}) \end{aligned} \qquad (8.7)$$

The representative agent chooses controls $\mathbf{u}_t = (c_t, M_t, s_t, b_t)$ for $t \ge 1$ to maximize expected lifetime utility (8.1) subject to constraint (8.7) with given M_0 and d_0. The exact monetary aggregate appears in both the utility function and budget constraint, and the macroeconomic "dimension reduction" is completely consistent with the microeconomic theory of monetary aggregation.

Let $\mathbf{z}_t = (M_{t-1}, s_{t-1}, b_{t-1}, c_{t-1}, p_t^*, P_{st}, P_{bt}, R_t, M_t^n)$ be the set of state variables, and define $T(\mathbf{z}_t) = \sum_{t=0}^{\infty} \rho^t F(c_t, M_t)$. , where recall $F(c_t, M_t) = V(c_t, \mathbf{m}_t)$. In equilibrium, $T(\mathbf{z}_t)$ must satisfy the following Bellman's equation

$$T(\mathbf{z}_t) = \max_{\mathbf{u}_t} \{F(c_t, M_t) + \beta E_t T(\mathbf{z}_{t+1})\}$$

when the following market conditions are satisfied:

$$c_t = d_t,$$

$$s_t = 1,$$

$$b_t = 0,$$

and

$$M_t^n = M_t p_t^* .$$

The equilibrium condition on equities is a normalization, while the equilibrium condition on bonds states that bonds are privately issued by some consumers and bought by others, and the net demand for bonds is zero in equilibrium. Recall that the representative agent is aggregated over consumers under Gorman's conditions for the

[34] Actually there is a possible regulatory wedge between the supply and demand side aggregator functions, when required reserves pay no interest and thereby produce an implicit tax on the supply side. For more discussion of this issue, see Barnett (1987), who provides the formulas for both the demand and supply side exact monetary aggregator function.

existence of a representative consumer. Hence b_t is net per capita borrowing among consumers, where lending is negative borrowing. If interest rates are out of equilibrium, net borrowing need not be zero.[35]

8.3. Money Velocity with No Nominal Interest Rate Risk

In this section we derive the necessary first order conditions and the equations for money velocitry, under the assumption that nominal interest rates are known, although other sources of risk still exist, including unknown future income and money supply. The first order conditions for the maximization problem are

$$F_{ct} = \lambda_t p_t^* \tag{8.9}$$

$$F_{Mt} = \lambda_t p_t^* (\pi_t + \frac{1 + R_{mt}}{1 + R_t}) - \rho E_t[\lambda_{t+1}](1 + R_{mt}) p_t^*, \tag{8.10}$$

$$\lambda_t P_{st} = \rho E_t[\lambda_{t+1}(d_{t+1} p_{t+1}^* + P_{s,t+1})], \tag{8.11}$$

$$\lambda_t = \rho E_t[\lambda_{t+1}](1 + R_t), \tag{8.12}$$

where F_{ct} and F_{Mt} are the marginal utilities of consumption goods and aggregate monetary services respectively, and λ_t is the Lagrange multiplier of the budget constraint (8.7). Equation (8.10) is the first order condition for monetary services. Equations (8.11) and (8.12) are standard Euler equations for stocks and bonds.

From equations (8.9) and (8.12), we have:

$$\pi_t = \frac{F_{Mt}}{F_{ct}}. \tag{8.13}$$

That is, the marginal rate of substitution between consumption goods and aggregate monetary services equals the aggregate user cost of the monetary services.[36] In the aggregate case, observe the analogy to Definition 1 in section 3. Assume that the utility function takes the constant relative risk aversion form

$$F(c_t, M_t) = \frac{1}{1-\phi}(c_t^s M_t^{1-s})^{1-\phi}, \quad \text{when } \phi \neq 1$$

$$= \ln(c_t^s M_t^{1-s}), \quad \text{when } \phi = 1,$$

where, as above, M_t is the real exact monetary aggregate, $s \in (0,1)$ is a constant, and $\phi \in (0, \infty)$ is the coefficient of relative risk aversion. We get the following relationship:

$$\pi_t = \frac{1-s}{s}\frac{c_t}{M_t}. \tag{8.14}$$

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[35]Similarly, see Marshall (1992, p. 1321) and Boyle (1990, p. 1042).

[36] We can get the same result if we start with the maximization problem in terms of the original disaggregated monetary assets. See Barnett and Xu (1995).

When solved for M_t, equation (8.14) is the equation of demand for the exact monetary aggregate. Given $c_t = d_t$ and the parameter s, the only determinant of the demand for monetary services in equilibrium is the user cost aggregate, π_t. Although other factors, such as the inflation rate, are not in this equation directly, they may affect the demand for money through the user cost π_t, which is a function of the nominal interest rates R_t and r_{it}. Given these equilibrium conditions, we can examine the behavior of money velocity.

Traditional money velocity is usually defined as the ratio of nominal income to the simple sum monetary aggregate

$$V_t = \frac{p_t^* d_t}{X_t}.$$

We define the aggregation theoretic exact money velocity by replacing the simple sum monetary aggregate with the exact monetary aggregate to get

$$v_t = \frac{d_t}{M_t}.$$

Using the definition $\pi_{it} = (R_t - r_{it})/(1+ R_t)$, the identity $\pi_t M_t = \sum_{i=1}^k \pi_{it} m_{it}$, and the equilibrium condition $X_{it} = p_t^*$ m_{it}, we have the following results from equation (8.14):

$$v_t = \frac{s}{1-s}\pi_t \qquad (8.15)$$

$$V_t = \frac{s}{1-s}\Pi_t \qquad (8.16)$$

where

$$\Pi_t = \sum_{i=1}^{k} \frac{\pi_{it} X_{it}}{\sum_{i=1}^k X_{it}} = (R_t - R_{smt})/(1+ R_t)$$

with

$$R_{smt} = \sum_{i=1}^k \theta_{it} r_{it},$$

and

$$\theta_{it} = X_{it} / \sum_{j=1}^k X_{jt}.$$

Observe that Π_t is the weighted average of the user costs π_{it}, which are the opportunity costs of holding monetary assets instead of the bond. Equations (8.15) and (8.16) both say that money velocity is a function of the user costs $\{\pi_{it}, i = 1,2, ..., k\}$. If we define velocity in the traditional way as V_t, then the corresponding determinant should be a weighted average of the user costs π_{it}, with the weights being ratios of the X_{it} to the simple sum aggregate X_t. If alternatively we define money velocity relative to the theoretic exact monetary aggregate, then the relevant determinant is the user cost aggregate π_t dual to the exact monetary aggregate. Both velocity functions have the same form with the key elements being the user costs of monetary assets. The equivalence of the forms of the two velocity function (8.15) and (8.16) depends upon our specification of the utility function.

Note that if all monetary assets yield no interest so that $r_{it} = 0$ for all i, then $\pi_t = \Pi_t = R_t/(1+R_t)$, as in Boyle (1990) and LeRoy (1984). In this special case, the inverse of money velocity is a linear function of the inverse of interest rates:

$$\frac{1}{v_t} = \frac{s}{1-s} + \frac{s}{1-s}\frac{1}{R_t}\,.$$

This equation was first estimated be Latane (1954) without rigious derivation and reestimated by Christ (1993) for M1 velocity. According to this equation, variations in velocity are caused solely by fluctuations in the benchmark interest rates R_t. But when monetary assets themselves yield interest rates r_{it}, velocity could fluctuate even when the benchmark rate does not vary.

In short, money velocity is a variable rather than a constant in the model developed in this section. The opportunity cost and the taste parameters determine the stochastic behavior of money velocity. Observe that equation (8.15) or (8.16) have no intercepts and have constant slope. These implications conflict with many published results. In the next section, when interest rate uncertainty is introduced, we show that the intercept becomes nonzero, and the slope may be time-varying, if time-varying risk is present.

8.4 Money Velocity With Nominal Interest Risk

In this section we relax the assumption that nominal interest rates are known, but we keep the assumption that economic agents are risk averse. We focus on the question of whether the model can use the interest rate uncertainty to explain the instability of money velocity reported in the literature.

We start with the monetary aggregation problem. In the previous section, the exact monetary quantity aggregator function $M_t = M(\mathbf{m_t})$ can be tracked accurately by the Divisia monetary aggregate, $M_t{}^d$, since that tracking ability is known under perfect certainty. However, when nominal interest rates are uncertain, we have seen above that the unadjusted Divisia monetary aggregate's tracking ability can be somewhat compromised. That compromise is eliminated by using the extended Divisia monetary aggregate defined by (4.12). Let M_t^G denote the extended Divisia monetary aggregate over the monetary assets. The only difference between M_t^G and M_t^d is the user cost formula to compute the prices in the Divisia index formula.

In accordance with Fisher's factor reversal test, we define the extended (or generalized) aggregate user cost π_t^G dual to M_t^G to be the solution for π_t^G to:

$$\sum_{i=1}^{k} \pi_{it}^G m_{it} = \pi_t^G m_t^G,$$

and let

$$\Pi_t^G = \sum_{i=1}^{k} \frac{\pi_{it}^G X_{it}}{\sum_{i=1}^{k} X_{it}}$$

be the weighted average of the indidual generalized user costs of monetary assets. We have the following proposition:

Proposition: When nominal interest rates (R_t, r_{it}) are not known with certainty at the beginning of period t, an equation analogous to equation (8.13) still holds, and money velocity is a function of the aggregate generalized user cost, which equals the marginal rate of substitution between consumption goods and monetary services, so that

$$\frac{F_{Mt}}{F_{ct}} = \pi_t^G \qquad (8.17)$$

$$v_t = \frac{s}{1-s}\pi_t^G \qquad (8.18)$$

$$V_t = \frac{s}{1-s}\Pi_t^G \qquad (8.19)$$

Proof: See Barnett and Xu (1995).

Equation (8.17) can also be proven from the maximization problem without prior aggregation over monetary assets. See Barnett and Xu (1995).

We can simplify the generalized user cost π_{it}^G to get a more intuitive equation for money velocity V_t. Dropping the remainder term of a second order Taylor series approximation to the objective function, Barnett and Xu (1995) prove the following. Let k_1 be defined such that

$$k_1 = \left(-c_t \frac{\partial^2 T}{\partial c_{t+1}^2}\Big|_t\right) / \left(\frac{\partial T}{\partial c_t}\right) = \beta(1-s+s\phi)$$

where T is as defined in (8.1) and where $|_t$ denotes evaluation of the derivative at time t. Then k_1 is the discounted relative risk aversion parameter, which in our specification is a constant. Similarly, define k_2 such that

$$k_2 = (M_t^G \frac{\partial^2 T}{\partial c_{t+1} \partial M_{t+1}^G} |_t) / (\frac{\partial T}{\partial c_t}) = \beta(1-s)(1-\phi)$$

Now assume that

$$\sigma_{rc} = \mathrm{Cov}(R_t, c_{t+1} / c_t)$$

and

$$\sigma_{rm} = \mathrm{Cov}(R_t, M_{t+1}^G / M_t^G)$$

are constant, so that the effect of R_t risk on the consumption of goods and monetary services is time-invariant. Let

$$\theta_1 = \frac{\mathrm{Cov}(R_t, \partial T / \partial c_{t+1})}{\partial T / \partial c_t}$$

and

$$\theta_{2i} = \frac{\mathrm{Cov}(r_{it}, \partial T / \partial c_{t+1})}{\partial T / \partial c_t} .$$

Then it follows that

$$\theta_1 = k_2 \sigma_{rm} - k_1 \sigma_{rc}$$

$$\theta_{2i} = k_2 \sigma_{im} - k_1 \sigma_{ic}$$

where

$$\sigma_{ic} = \mathrm{Cov}(r_{it}, c_{t+1} / c_t)$$

$$\sigma_{im} = \mathrm{Cov}(r_{it}, M_{t+1}^G / M_t^G)$$

are assumed to be time-invariant. It follows that

$$\pi_{it}^G = \pi_{it}^e + (1-\pi_{it}^e)\theta_1 - \theta_{2i}$$

$$= (1-\theta_1)\pi_{it}^e + (\theta_1 - \theta_{2i})$$

Let θ_2 be the weighted average of the θ_{2i} such that

$$\theta_2 = \sum_{i=1}^{k} \frac{\theta_{it} X_{it}}{\sum_{i=1}^{k} X_{it}}$$

Barnett and Xu (1995) find that

$$V_t = a_0 + a_1 \Pi_t^e \tag{8.20}$$

where

$$a_0 = \frac{s}{1-s} (\theta_1 - \theta_2),$$

$$a_1 = \frac{s}{1-s}(1-\theta_1),$$

and

$$\Pi_t^e = \sum_{i=1}^{k} \frac{\pi_{it}^e X_{it}}{\sum_{i=1}^{k} X_{it}}.$$

Note that if the covariances σ_{ic}, σ_{im}, σ_{rc} and σ_{rm} are time-varying, then a_0 and a_1 are also time-varying, and equation (8.20) is a time-varying or random coefficient model. Hence, a time-varying coefficient model of money velocity can be justified by the fact that interest rate uncertainty may change over time, as for example from an ARCH process.

It is worthwhile to look at the effect of risk aversion and interest rate uncertainty on money velocity in more detail. Note that under our assumptions about the parameters of the model, we have

$$k_1 > 0,$$

and

$$k_2 > 0 \text{ if } \phi < 0.$$

For expositional purposes, we call $\phi > 1$ high risk aversion and $\phi < 1$ low risk aversion.

In the low risk aversion case, if $\sigma_{rm} < 0$, then the larger the value of $|\sigma_{rm}|$, the larger the value of a_1, and the smaller the value of a_0. The net effect of an increase of $|\sigma_{rm}|$ in this case is to reduce the money velocity, since $\Pi_t^e < 1$ and the magnitude of the decrease of the intercept is larger than that of the increase of $a_1 \Pi_t^e$. Therefore, if increased money growth variability raises the value of $|\sigma_{rm}|$, and if $|\sigma_{im}|$ are not affected, then money velocity will decline. In the high risk aversion case, the results are just the opposite. An increase of $|\sigma_{rm}|$ will lead to higher money velocity. It follows that Friedman's (1983) hypothesis that the increased money growth variability causes money velocity to decline can be justified in our model by either: (1) $\phi < 1$ and $\sigma_{rm} < 0$, or (2) $\phi > 1$ and $\sigma_{rm} > 0$. Also note that the magnitude of the effect of money growth variability on money velocity depends upon other parameters of the model, such as the parameter s.

But there remains the effect of uncertainty of the individual monetary assets' own rates of return. Note that the covariances σ_{im} are not important in determining the magnitude of slope a_1, although those covariances are important in determining the value of the velocity function's intercept. Therefore, a shift in the value of the $|\sigma_{im}|$ will lead to a shift in the intercept of the money velocity function. If increased money growth variability raises both

$|\sigma_{rm}|$ and $|\sigma_{im}|$, and if σ_{rm} and σ_{im} have the same sign, then the effect of money growth variability on money velocity through σ_{rm} will be partially offset. The complicated nature of the effect of money growth variability on money velocity partially explains the comtroversies in the empirical literature.

In short, if $\phi \neq 1$, the money growth variability will affect money velocity, but the direction and magnitude of the effect depend upon the degree of risk aversion and the correlation between interest rates and real money growth. If all the covariances are zero, as would be the case under perfect certainty, then (8.20) reduces to (8.16), as we would expect.

To further explore the economic interpretation of the coefficients in the money velocity function, note that from the first order condition on the bond price

$$\lambda_t = \rho E_t[\lambda_{t+1}(1+R_t)],$$

the parameter θ_1 can be written as

$$\theta_1 = 1 - E_t(1+R_t)E_t[\frac{T_{c,t+1}}{T_{ct}}],$$

where $E_t[\frac{T_{c,t+1}}{T_{ct}}]$ is the expected growth rate of the marginal utility of consumption goods. Therefore, the slope coefficient of the money velcoity function is

$$a_1 = \frac{s}{1-s} E_t(1+R_t)E_t[\frac{T_{c,t+1}}{T_{ct}}].$$

If we use $E_t(R_t - R_{smt})$ as the independent variable in the money velocity function, rather than the user cost Π_t^e , we have

$$V_t = a_{0t} + b_t E_t(R_t - R_{smt}) \qquad (8.21)$$

where

$$b_t = \frac{s}{1-s} E_t[\frac{T_{c,t+1}}{T_{ct}}].$$

The subscript t in b_t and a_{0t} is used to indicate that the values of b_t and a_{0t} may not be time constant. In our model, given the specification of the utility function, we have

$$b_t = \frac{\beta s}{1-s} E_t[(\frac{c_{t+1}}{c_t})^{s(1-\phi)-1}(\frac{M_{t+1}^G}{M_t^G})^{(1-\phi)(1-s)}]. \qquad (8.22)$$

From equation (8.22), it can be seen that the slope coefficient b_t depends upon both the growth rate of consumption and the growth rate of the real money stock. If the conditional expectation operator depends upon the second or higher moments of the growth rate processes, the slope process b_t will depend on the variability of both the consumption growth rate and the real money growth rate. If $\phi = 1$, the expected growth rate of the real money stock will not affect the coefficient b_t, since in that case, the marginal utility of consumption goods does not depend on the real money stock.

This section provides a theoretical avenue to examine the effects of money growth rate and consumption growth rate and their variability on the stochastic behavior of money velocity. It is shown in this section that the expected real money growth rate or its variability will affect money velocity by shifting the coefficients of the traditional money velocity function. The magnitude of this effect is also determined by other parameters in the representative agent's preference, such as risk aversion. If the representative agent in this model is risk neutral, the changes in money growth rate and its variability will not affect the stability of the money velocity function in equilibrium. On the other side, the higher the degree of risk aversion, the larger the effect of real money growth rate and consumption growth rate and their variabilities on the stability of the money velocity function.

8.5 Some Empirical Results

In this section, we first simulate the model using quarterly data over the period of 1960.1 to 1992.4 for some specifications of parameters, to examine the stability of the coefficients in the traditional money velocity function implied by our theoretical model. We then estimate a random coefficient model of moeny velocity to examine the stability of the coefficients empirically. The random coefficient model approach we follow is Swamy and Tinsley's (1980). The results from both the empirical estimation and the theoretical simulation are compared to see whether the empirical behavior of the money velocity can be explained by the model developed in this paper.

The data on monetary assets and their corresponding yields were provided by the Federal Reserve Bank of St. Louis. Output data are GNP. The inflation rate is the growth rate of the price deflator for GNP. The benchmark asset return path is approximated by the upper envelope of the three month Treasury bill rate path and the time paths of each individual monetary asset's own rate of return.[37] The growth rate of consumption is replaced by the real GNP growth rate. With M1, which includes no assets having high risk rates of return, the regular Divisia

[37] Even the upper envelope is too low, since the theoretical benchmark asset is completely illiquid and therefore must have higher expected yield than the upper envelope over any yield-curve-adjusted rates of return on monetary assets providing any monetary services.

monetary aggregate closely tracks the generalized Divisia monetary aggregates. Hence we use ordinary Divisia M1 to measure the theoretical monetary quantity aggregate.

To simulate the process b_t in equation (8.21), we first set $\beta = 0.99$, $s = 0.972$, and $\phi \in \{0.5, 2, 5\}$. The three different values of ϕ are chosen to capture the influence of different degrees of risk aversion on the stability of the coefficient b_t. The parameter s mainly affects the sample mean of b_t. We estimate a VAR (vector autoregressive) model of real moeny and GNP growth rates using quarterly data from 1960.1 to 1992.4. The estimated VAR model is then used to estimate the conditional expectation

$$E_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{s(1-\phi)-1}\left(\frac{M_{t+1}}{M_t}\right)^{(1-\phi)(1-s)}\right].$$

The estimated b_t process is plotted in Figure 15. From Figure 15, it can be seen that when $\phi = 0.5$ (low risk aversion case), the simulated slope process b_t is almost constant. When the value of ϕ increases, the variability in b_t also increases. When $\phi = 5$, the process b_t shows a lot of variability. There are two periods during which b_t is extremely volatile. One is from 1972 to 1976 and the other is from 1979-1982. The latter period approximately corresponds with the episode of the "monetarist experiment" of the Federal Reserve System. The b_t process also shows some variability in the recent years of 1991-1992. These simulation results confirm the theoretical prediction that if the degree of risk aversion is higher, the traditional money velocity function will be less stable.

These theoretical results from model simulation can be compared with empirical estimation. We estimate equation (8.21) with stochastically varying coefficients. We use the Swamy and Tinsley (1980) asymptotically efficient estimation procedure. Letting $\alpha_t = (a_{ot}, b_t)$, we assume, as in Swamy and Tinsley (1980):

$$\alpha_t = \alpha^0 + e_t,$$

where α^0 is a vestor of constants, and

$$e_t = \Phi_1 e_{t-1} + \Phi_2 e_{t-2} + u_t\ ,$$

where Φ_1 and Φ_2 are matrices of parameters to be estimated, and u_t is a random vector with mean zero and convariance matrix Ω. The estimated b_t process for M1 money velocity is plotted in Figure 16. We estimate the b_t process with both prior information incorporated and with no prior information (i.e., with diffuse prior information) about b_t. The two processes show almost the same movements. From Figure 16, it can be seen that b_t is very volatile during the periods from 1972 to 1974 and from 1979 to 1982. This approximately coincides with the simulation result with moderate risk aversion.

Overall, the results from model simulation and from model estimation of a random coefficient model indicate that our model can capture the main features of the coefficient process b_t in the traditional money velocity function.

9. Conclusion

In earlier research on aggregation theoretic foundations for monetary modeling, the primary issue has been to explore the degree of the tracking error of the unadjusted Divisia monetary index, derived from the first order conditions under perfect certaintly, to the degree of risk aversion. In this paper, we develop a CCAPM adjustment to user costs that permits the Divisia index to be derived directly from the Euler equations under risk, so that the tracking error produced from risk aversion disappears, since the risk is internalized within the resulting extended Divisia index. Further details regarding that derivation can be found in the closely related working paper by Barnett and Liu (1995).

Using the components of the usual monetary aggregates, we find that the CCAPM adjustment to user costs is very small, and the gain from moving from the unadjusted Divisia index to the extended index seems slight, at least with those relatively low risk components. These results are very similar to related results on the equity premium puzzle, since the small adjustment in both cases result from the very low covariance between rates of return and the consumption stream. We then explore the possibility that aggregation over economic agents may have smoothed that covariance, or the possibility that earlier results on the tracking error of the unadjusted Divisia index may have been produced by specification error in the parametric model of the utility function. We do so by producing simulated data from a modeled rational consumer. The procedure used to solve the Euler equations is that in Jensen (1995). We find that the tracking error of the unadjusted index is nontrivial under risk aversion, and depends upon the degree of risk aversion. Hence the CCAPM adjustment to user costs and the risk-adjusted Divisia index can be expected to be needed, when modeling the behavior of one economic agent under risk aversion. How to progress further in the direction of aggregation over economic agents is not explored in this paper.

We then used the above theory to explore the determinants of money velocity. The effects of risk aversion and interest rate uncertainty on money velocity are examined within a monetary general equilibrium model. This paper indicates that if covariances between interest rates and consumption growth or between interest rates and money growth are generated by an ARCH type process, the traditional moeny velocity function will become unstable. Both model simulation and estimation produce significant varability in the slope of the traditional money velocity fucntion, especially during 1972-1974 and 1979-1982 periods. This study sheds some new light on the nature of the instability of traditional money velocity functions. For further details regarding this aspect of this ongoing project, see Barnett and Xu (1995).

APPENDIX: Aggregation Theorem

It is clear that equation (2.3) does define the exact monetary aggregate in the welfare sense, since M_s measures the consumer's subjective evaluation of the services that he receives from holding $\mathbf{m}_s$. However it also can be shown that equation 2.6 defines the exact monetary aggregate in the aggregation theoretic sense. In particular, the stochastic process M_s , $s \geq t$, contains all of the information about $\mathbf{m}_s$ that is needed by the consumer to solve the rest of his decision problem. This conclusion is based upon the following theorem, which we call the consumer's aggregation theorem.

Let $D_s = I_s + \sum_{i=1}^{k_1} [(1+r_{i,s-1}) p^*_{s-1} \; m_{i,s-1} - p^*_s \; m_{is}]$,

and let

$$D(s) = \{ (\mathbf{h}_s, \mathbf{x}_s, A_s) \in Y: \sum_{i=1}^{n} p_{is} x_{is} =$$

$$+ \sum_{i=1}^{k_2} [(1+\mathbf{i}_{i,s-1}) p^*_{s-1} \; h_{i,s-1} - p^*_s \; h_{is}] + (1+R_{s-1}) p^*_{s-1} \; A_{s-1} - p^*_s \; A_s + D_s \}. \qquad \text{(A.1)}$$

Let $(\mathbf{a}^*_s, \mathbf{x}^*_s, A^*_s)$, $s \geq t$, solve Problem 1 , and assume that the utility function $u(\mathbf{m}_s, \mathbf{h}_s, \mathbf{x}_s)$ is weakly separable in $\mathbf{m}_s$, so that there exists aggregator function M and utility function U such that $U(M(\mathbf{m}_s), \mathbf{h}_s, \mathbf{x}_s) = u(\mathbf{m}_s, \mathbf{h}_s, \mathbf{x}_s)$. Consider the following decision problem, which is conditional upon prior knowledge of the aggregate process $M^*_s = M(\mathbf{m}^*_s)$, although not upon the component processes $\mathbf{m}^*_s$.

<u>Problem 2</u>: Choose the deterministic point ($\mathbf{h}_t$, $\mathbf{x}_t$, A_t) and the stochastic process $(\mathbf{h}_s, \mathbf{x}_s, A_s)$, $s=t+1, ..., \infty$, to maximize

$$U(M^*_t, \mathbf{h}_t, \mathbf{x}_t) + E_t \left[\sum_{s=t+1}^{\infty} \left(\frac{1}{1+\xi}\right)^{s-t} U(M^*_s, \mathbf{h}_s, \mathbf{x}_s) \right] \qquad \text{(A.2)}$$

subject to $(\mathbf{h}_s, \mathbf{x}_s, A_s) \in D(s)$ for $s \geq t$, and also subject to

$$\lim_{s \to \infty} E_t \left(\frac{1}{1+\xi}\right)^{s-t} A_s = 0,$$

with the process M^*_s given for $s \geq t$.

<u>Theorem A1</u> (Consumer's Aggregation Theorem): Let the deterministic point $(\mathbf{m}_t,\mathbf{h}_t,\mathbf{x}_t,A_t)$ and the stochastic process $(\mathbf{m}_s, \mathbf{h}_s, \mathbf{x}_s, A_s)$, s=t+1, ..., t+T solve Problem 1. Then the deterministic point $(\mathbf{h}_t,\mathbf{x}_t,A_t)$ and the stochastic process $(\mathbf{h}_s,\mathbf{x}_s,A_s)$, s = t+1, ..., t+T, will solve Problem 2 conditionally upon $M_s^* = M(\mathbf{m}_s)$ for s = t, ... , t+T.

<u>Proof</u>: Let $(\mathbf{m}_s,\mathbf{h}_s,\mathbf{x}_s, A_s)$, s^3t solve problem 1, but let $(\mathbf{h}_s,\mathbf{x}_s,A_s)$, s^3t, not solve problem 2 conditionally upon the process $M_s^* = M(\mathbf{m}_s)$ given for s = t, ... , t+T. Then there exist $(\tilde{\mathbf{h}}_s, \tilde{\mathbf{x}}_s, \tilde{A}_s) \in D(s)$, s^3t, satisfying the transversality condition, such that (A.2) evaluated at $(\tilde{\mathbf{h}}_s, \tilde{\mathbf{x}}_s, \tilde{A}_s)$, s^3t, is strictly greater than (A.2) evaluated at $(\mathbf{h}_s,\mathbf{x}_s,A_s)$, s^3t, conditionally upon $M_s^* = M(\mathbf{m}_s)$.

Hence (2.1) evaluated at $(\mathbf{m}_s, \tilde{\mathbf{h}}_s, \tilde{\mathbf{x}}_s, \tilde{\mathbf{A}}_s)$, s^3t. is strictly greater than (2.1) evaluated at $(\mathbf{m}_s,\mathbf{h}_s, \mathbf{x}_s,A_s)$, s^3t. But since $(\tilde{\mathbf{h}}_s, \tilde{\mathbf{x}}_s, \tilde{A}_s)$, s^3t, is feasible for problem 2 conditionally upon $M_s=M(\mathbf{m}_s)$, it follows that $(\mathbf{m}_s, \tilde{\mathbf{h}}_s, \tilde{\mathbf{x}}_s, \tilde{\mathbf{A}}_s)$, s^3t, is feasible for problem 1. Our assumption that $(\mathbf{m}_s,\mathbf{h}_s,\mathbf{x}_s, A_s)$, s^3t, solves problem 1 is contradicted.

Q.E.D.

Clearly this proof by contradiction applies not only when M_s is produced by voluntary behavior, but also when the M_s process is exogenously imposed upon the consumer, as through a perfectly inelastic supply function for M_s imposed by central bank policy. In that case, Problem 2 describes optimal behavior by the consumer in the remaining variables. Clearly the information about M_s is needed in the solution of problem 2 for the processes $(\mathbf{h}_s,\mathbf{x}_s,A_s)$. Alternatively information about the usual simple sum monetary aggregate over the components of $\mathbf{m}_s$ is of no use in solving either problem 1 or 2. unless the monetary aggregator function M happens to be a simple sum. In other words, the simple sum aggregate contains useful information about behavior only if the components of $\mathbf{m}_s$ are perfect substitutes in identical ratios (linear aggregation with <u>equal</u> coefficients).

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