Some Applications of Malliavin Calculus to SPDE and Convergence of Densities

By

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# Abstract

Some applications of Malliavin calculus to stochastic partial differential equations (SPDEs) and to normal approximation theory are studied in this dissertation.

In Chapter 3, a Feynman-Kac formula is established for a stochastic heat equation driven by Gaussian noise which is, with respect to time, a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. To establish such a formula, we introduce and study a nonlinear stochastic integral of the Gaussian noise. The existence of the Feynman-Kac integral then follows from the exponential integrability of this nonlinear stochastic integral. Then, techniques from Malliavin calculus is used to show that the Feynman-Kac integral is the weak solution to the stochastic heat equation.

In Chapter 4, the density formula in Malliavin calculus is used to study the joint Hölder continuity of the solution to a nonlinear SPDE arising from the study of one dimensional super-processes. Dawson, Vaillancourt and Wang [Ann. Inst. Henri. Poincaré Probab. Stat., **36** (2000) 167-180] proved that the solution of this SPDE gives the density of the branching particles in a random environment. The time-space joint continuity of the density process was left as an open problem. Li, Wang, Xiong and Zhou [Probab. Theory Related Fields **153** (2012), no. 3-4, 441–469] proved that this solution is joint Hölder continuous with exponent up to $\frac{1}{10}$ in time and up to $\frac{1}{2}$ in space. Using our new method of Malliavin calculus, we improve their result and obtain the optimal exponent $\frac{1}{4}$ in time.

In Chapter 5, we study the convergence of densities of a sequence of random variables to a normal density. The random variables considered are nonlinear functionals of a Gaussian process, in particular, the multiple integrals. They are assumed to be non-degenerate so that their probability densities exist. The tool we use is the Malliavin calculus, in particular, the density formula, the integration by parts formula and the Stein's method. Applications to the convergence of densities of the least square estimator for the drift parameter in Ornstein-Uhlenbeck is also considered.

In Chapter 6, we apply an upper bound estimate from small deviation theory to prove the non-degeneracy of some functional of fractional Brownian motion.

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# Contents

# Chapter 1

# Introduction

In this dissertation, we study some applications of Malliavin calculus to stochastic partial differential equations (SPDEs) and to normal approximation. It is a collection of my joint works with my advisors, Yaozhong Hu and David Nualart.

The Malliavin calculus, also known as the stochastic calculus of variations, is an infinite-dimensional differential calculus on the Wiener space. One can distinguish two parts in the Malliavin calculus. First is the theory of differential operators defined on suitable Sobolev spaces of Wiener functionals. A crucial fact in this theory is *the integration by parts formula*, which relates the derivative operator on the Wiener space and the Skorohod extended stochastic integral. The second part of this theory establishes general criteria, such as *the density formulae*, in terms of existence of negative moments of "Malliavin covariance matrix" for a given random vector to possess a density or, a smooth density. This gives a probabilistic proof and extensions to the Hörmander's theorem about the existence and smoothness of the density for a solution of a stochastic differential equation.

In the applications of Malliavin calculus to SPDE, we shall use the Malliavin differential operators to study existence and representation of solutions in Chapter 3, and use the general criteria for density to study the regularity of probability laws of solutions in Chapter 4.

Through the Gaussian integration by parts formula, Malliavin calculus can be combined with Stein's method (a general method to obtain bounds on the distance between two probability distributions with respect to a probability metric) to study normal approximation theory. Their interaction has led to some remarkable new results involving central and non-central limit theorems for functionals of infinite dimensional Gaussian fields. While these central limit results study the convergence in distribution, we shall use the density formulae to study the convergences of densities of the random variables in Chapter 5. As mentioned above, a crucial (and sufficient) condition for a random vector to possess a density is the existence of negative moments of its "Malliavin covariance matrix". In Chapter 6 we shall address this problem for some functionals of fractional Brownian motion.

In the following we give a brief introduction to Chapter 3–6.

Chapter 3 is taken from [13], in which we derive a Feynman–Kac formula for a SPDE driven by Gaussian noise which is, with respect to time, a fractional Brownian motion (fBm) with Hurst parameter $H < 1/2$. More precisely, we consider the stochastic heat equation (SHE) on $\mathbb{R}^d$

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\frac{\partial W}{\partial t}(t,x) , \quad t \geq 0 \, , \, x \in \mathbb{R}^d \, , \\ u(0,x) = u_0(x) \, , \end{cases}$$

where $u_0$ is a bounded measurable function and $W = \{W(t,x), \, t \geq 0, x \in \mathbb{R}^d\}$ is a fBm of Hurst parameter $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$ in time and it has a spatial covariance $Q(x,y)$, which is locally $\gamma$-Hölder continuous, with $\gamma > 2-4H$. We shall show that the solution to this

SHE is given by

$$u(t,x) = E^B \left[ u_0(B_t^x) \exp \int_0^t W(ds, B_{t-s}^x) \right],$$

where $B = \{B_t^x = B_t + x, t \geq 0, x \in \mathbb{R}^d\}$ is a $d$ -dimensional Brownian motion starting at $x \in \mathbb{R}^d$, independent of $W$.

In 1949, Kac [18] proved the famous Feynman-Kac formula representing the solution of a heat equation by the expectation of a stochastic process, and hence established a link between PDE and Probability. Since then, stochastic heat equation (a heat equation with random potentials) has been extensively studied. However, the case of random potentials defined by means of fractional Brownian sheet (fBs) has been investigated only recently by Hu, Nualart and Song [17], where the time Hurst index $H > \frac{1}{2}$, the spatial Hurst indices are relatively large and space-time derivative are considered. Due to singularity of the fractional Brownian sheet noise, a Feynman-Kac formula can not be written if $H \leq \frac{1}{2}$. In order to study the case of fractional noise potential with time Hurst index $H < \frac{1}{2}$, we let the noise to be regular in space variables. Then we construct a solution to the Cauchy problem via a generalized Feynman-Kac formula. The difficulty lies in the lack of fractional smoothness of the fBm in time. To overcome the difficulty, we compensate it with regularity in space by introducing a nonlinear stochastic integral $\int_0^t W(ds, \phi(s))$ for all $\phi$ which is Hölder continuous of order $\alpha > \frac{1}{\gamma}(1-2H)$. The main tool in the proof is fractional calculus. We then show the exponential integrability of the nonlinear stochastic integral. The study of nonlinear stochastic integral is of independent interest and might be used in other related problems.

The second effort is to show that the Feynman–Kac formula provides a solution to the SPDE. The approach of approximation with techniques from Malliavin calculus is used. The solution is in the weak sense, and the integral with respect to the noise is in the Stratonovich sense. The Feynman-Kac expression allows us a rather complete analysis of the statistical properties of the solution. We prove that the solution is $L^p$ integrable for all $p \geq 1$. We also prove that the solution is Hölder continuous of order $(H - \frac{1}{2} + \frac{1}{4}\gamma)$ a.s. if $u_0$ is Lipschitz and bounded.

Chapter 4 is devoted to study the joint Hölder continuity of the solution to the following nonlinear SPDE arising from the study of one dimensional superprocesses:

$$\begin{aligned} X_t(x) &= \mu(x) + \int_0^t \Delta X_u(x) dr - \int_0^t \int_{\mathbb{R}} \nabla_x \left(h\left(y-x\right) X_u\left(x\right)\right) W\left(du, dy\right) \\ &\quad + \int_0^t \sqrt{X_u\left(x\right)} \frac{V\left(du, dx\right)}{dx}, \end{aligned}$$

where $V$ and $W$ are two independent Brownian sheets on $\mathbb{R}_+ \times \mathbb{R}$. Dawson, Vaillancourt and Wang [7]) proved that the solution $X_t(x)$ gives the density of the branching particles in the random environment $W$. The time-space joint continuity of $X_t(x)$ was left as an open problem. Recently, Li et al [25] proved that this solution is almost surely joint Hölder continuous with exponent up to $\frac{1}{10}$ in time and up to $\frac{1}{2}$ in space. The methods they used are fractional integration by parts technique and Krylov's $L_p$ theory (cf. Krylov [20]). Comparing to the Hölder continuity for the stochastic heat equation which has the Hölder continuity of $1/4$ in time, it is conjectured that the optimal exponent of Hölder continuity of $X_t(x)$ should also be $1/4$.

By introducing the techniques from Malliavin calculus, we shall give an affirmative answer to this conjecture. We also give a short proof of the Hölder continuity with exponent $\frac{1}{2}$ in space using our method. We anticipate that our method would be potentially useful for the study of other interactive branching diffusions. This chapter is taken from [12].

In Chapter 5, we study the convergence of densities of a sequence of random variables to a normal density. The motivation is the following. The central limit theorem (CLT) gives the convergence of the distribution functions of a sequence of random variables. A natural question to ask is, what can we say about the probability densities if they exist?

Inspired by the following recent developments of CLT for multiple integrals (see the definition of the multiple integral in Section 2.1), we shall give a partial answer to this question.

Consider a sequence of random variables $F_n = I_q(f_n)$ ( in the $q$–th Wiener chaos with $q \geq 2$ and $EF_n^2 = 1$. Nualart and Peccati [41] and Nualart and Ortiz-Latorre [40] have proved that $F_n$ converges in distribution to the normal law $N(0,1)$ as $n \to \infty$ if and only if one of the following three equivalent conditions holds:

(i) $\lim_{n\to\infty} \mathbb{E}[F_n^4] = 3$,

(ii) For all $1 \leq r \leq q-1$, $\lim_{n\to\infty} \|f_n \otimes_r f_n\|_{\mathfrak{H}^{\otimes 2(q-r)}} = 0$,

(iii) $\|DF_n\|_{\mathfrak{H}}^2 \to q$ in $L^2(\Omega)$ as $n \to \infty$.

where $DF_n$ is the Malliavin derivative of $F_n$ (see for definition in the following sections). The first two conditions were proved in [41] using the method of moments and comulants, which are called "*the fourth moment theorem*", and (iii) was proved in [40] using characteristic function. Later, two new proofs of the fourth moment theorem were given by Nourdin and Peccati [34] by bringing together Stein's method and Malliavin calculus and by Nourdin [32] using free Brownian motion. Multidimensional versions of this characterization were studied by Peccati and Tudor [47] and Nualart and Ortiz-Latorre [40]. Different extensions and applications of these results can be found in the extensive literature, among them we mention Hu and Nualart [14] and Peccati and Taqqu [45, 46].

Subsequently, quantitative bounds to the fourth moment theorem is provided by Nourdin and Peccati [34] (see also Nourdin and Peccati [35]). Bringing together Stein's method with Malliavin calculus, they proved a bound for the total variation distance,

$$d_{TV}(F_n,N) := \sup_{B\in\mathscr{B}(\mathbb{R})} |P(F_n \in B) - P(N \in B)| \leq 2\sqrt{|EF_n^4 - 3|}.$$

The aim of this chapter is to study the convergence of the densities of $F_n = I_q(f_n)$. It is well-known in Malliavin calculus that each $F_n$ has a density $f_{F_n}$ if its Malliavin derivative $\|DF_n\|_{\mathfrak{H}}$ has negative moments (we say a random vaiable is *non-degenerate* if its Malliavin derivative has negative moments). Assuming further that

$$M := \sup_n \mathbb{E}[\|DF_n\|_{\mathfrak{H}}^{-6}] < \infty,$$

we shall prove that $f_{F_n} \in C(\mathbb{R})$ for each $n$ and there exists a constant $C_{q,M}$ depending only $q$ and $M$ such that

$$\sup_{x\in\mathbb{R}} |f_{F_n}(x) - \phi(x)| \leq C_{q,M}\sqrt{|EF_n^4 - 3|},$$

where we denote $\phi(x)$ the density of $N(0,1)$.

Assuming higher order of negative moments, we also show the uniform convergence of the derivatives of the densities. The convergence of densities for random vectors has also been studied. Applications to the convergence of densities of the least square estimator for the drift parameter in Ornstein-Ulenbeck is also considered.

The main ingredients in proof are the density formulae in Malliavin calculus and the combination of Stein's method and Malliavin calculus. In the application of multivariate Stein's method, we gave estimates on the solution to the multivariate Stein's equation with non-smooth unbounded test functions. Our result extends the relatively few current results for non-smooth test functions (see e.g. [4, 50]) for multivariate Stein's method.

As we have seen in the above chapters, non-degeneracy of a random variable plays a fondamental role when we apply density formula. Chapter 6 is a short note on non-degeneracy of the following functional of fractional Brownian motion $B^H$:

$$F = \int_0^1 \int_0^1 \frac{\left|B_t^H - B_{t'}^H\right|^{2p}}{|t-t'|^q} dt dt',$$

where $q \geq 0$ and the integer $p$ satisfies $(2p-2)H > q-1$.

In the case of $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ is a Brownian motion, and the random variable $F$ is the Sobolev norm on the Wiener space considered by Airault and Malliavin in [1]. This norm plays a central role in the construction of surface measures on the Wiener space. Fang [9] showed that $F$ is non-degenerate. Then it follows from the well-known density formula in Malliavin calculus that the law of $F$ has a smooth density.

In Chapter 6, using an upper bound estimates in small deviation theory, we shall show that $F$ is non-degenerate for all $H \in (0,1)$.

# Chapter 2

# Preliminaries

We introduce some basic elements of Gaussian analysis and Malliavin calculus, for which we refer to [39, 35] for further details.

## 2.1 Isonormal Gaussian process and multiple integrals

Let $\mathfrak{H}$ be a real separable Hilbert space (with its inner product and norm denoted by $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$, respectively). For any integer $q \geq 1$, let $\mathfrak{H}^{\otimes q}(\mathfrak{H}^{\odot q})$ be the $q$th tensor product (symmetric tensor product) of $\mathfrak{H}$. Let $X = \{X(h), h \in \mathfrak{H}\}$ be an isonormal Gaussian process associated with the Hilbert space $\mathfrak{H}$, defined on a complete probability space $(\Omega, \mathscr{F}, P)$. That is, $X$ is a centered Gaussian family of random variables such that $E[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}$ for all $h, g \in \mathfrak{H}$.

For every integer $q \geq 0$, the *qth Wiener chaos* (denoted by $\mathscr{H}_q$) of $X$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\left\{H_q(X(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\right\}$, where $H_q$ is the $q$th *Hermite polynomial* recursively defined by $H_0(x) = 1, H_1(x) = x$ and

$$H_{q+1}(x) = xH_q(x) - qH_{q-1}(x), \quad q \geq 1. \tag{2.1}$$

For every integer $q \geq 1$, the mapping $I_q(h^{\otimes q}) = H_q(X(h))$ , where $\|h\|_{\mathfrak{H}} = 1$, can be extended to a linear isometry between $\mathfrak{H}^{\odot q}$ (equipped with norm $\sqrt{q!}\,\|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and $\mathscr{H}_q$ (equipped with $L^2(\Omega)$ norm). For $q = 0$, $\mathscr{H}_0 = \mathbb{R}$, and $I_0$ is the identity map.

It is well-known (Wiener chaos expansion) that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathscr{H}_q$. That is, any random variable $F \in L^2(\Omega)$ has the following chaotic expansion:

$$F = \sum_{q=0}^{\infty} I_q(f_q), \tag{2.2}$$

where $f_0 = E[F]$, and $f_q \in \mathfrak{H}^{\odot q}, q \geq 1$, are uniquely determined by $F$. For every $q \geq 0$ we denote by $J_q$ the orthogonal projection on the $q$th Wiener chaos $\mathscr{H}_q$, so $I_q(f_q) = J_q F$.

Let $\{e_n, n \geq 1\}$ be a complete orthonormal basis of $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot q}$ and $g \in \mathfrak{H}^{\odot p}$, for $r = 0, \ldots, p \wedge q$ the *r–th contraction* of $f$ and $g$ is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.3}$$

Notice that $f \otimes_r g$ is not necessarily symmetric. We denote by $f \widetilde{\otimes}_r g$ its symmetrization. Moreover, $f \otimes_0 g = f \otimes g$, and for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$. For the product of two multiple integrals we have the multiplication formula

$$I_p(f) I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g). \tag{2.4}$$

In the particular case $H = L^2(A, \mathscr{A}, \mu)$, where $(A, \mathscr{A})$ is a measurable space and $\mu$ is a $\sigma$–finite and nonatomic measure, one has that $H^{\otimes q} = L^2(A^q, \mathscr{A}^{\otimes q}, \mu^{\otimes q})$ and $H^{\odot q}$ is the space of symmetric and square-integrable functions on $A^q$. Moreover, for every $f \in H^{\odot q}$, $I_q(f)$ coincides with the $q$th *multiple Wiener–Itô integral* of $f$ with respect to

$X$, and (2.3) can be written as

$$f \otimes_r g\left(t_1,\ldots,t_{p+q-2r}\right) = \int_{A^r} f\left(t_1,\ldots,t_{q-r},s_1,\ldots,s_r\right) \times g\left(t_{1+q-r},\ldots,t_{p+q-r},s_1,\ldots,s_r\right) d\mu(s_1)\ldots d\mu(s_r). \tag{2.5}$$

## 2.2 Malliavin operators

We introduce some basic facts on Malliavin calculus with respect to the Gaussian process $X$. Let $\mathscr{S}$ denote the class of smooth random variables of the form $F = f(X(h_1),\ldots,X(h_n))$, where $h_1,\ldots,h_n$ are in $\mathfrak{H}$, $n \geq 1$, and $f \in C_p^\infty(\mathbb{R}^n)$, the set of smooth functions $f$ such that $f$ itself and all its partial derivatives have at most polynomial growth. Given $F = f(X(h_1),\ldots,X(h_n))$ in $\mathscr{S}$, its Malliavin derivative $DF$ is the $\mathfrak{H}$-valued random variable given by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(h_1),\ldots,X(h_n))h_i.$$

The derivative operator $D$ is a closable and unbounded operator on $L^2(\Omega)$ taking values in $L^2(\Omega;\mathfrak{H})$. By iteration one can define higher order derivatives $D^k F \in L^2(\Omega;\mathfrak{H}^{\odot k})$. For any integer $k \geq 0$ and any $p \geq 1$ and we denote by $\mathbb{D}^{k,p}$ the closure of $\mathscr{S}$ with respect to the norm $\|\cdot\|_{k,p}$ given by:

$$\|F\|_{k,p}^p = \sum_{i=0}^k E(\left|\left|D^i F\right|\right|_{\mathfrak{H}^{\otimes i}}^p).$$

For $k = 0$ we simply write $\|F\|_{0,p} = \|F\|_p$. For any $p \geq 1$ and $k \geq 0$, we set $\mathbb{D}^{\infty,p} = \cap_{k\geq 0}\mathbb{D}^{k,p}$ and $\mathbb{D}^{k,\infty} = \cap_{p\geq 1}\mathbb{D}^{k,p}$.

We denote by $\delta$ (the *divergence* operator) the adjoint operator of $D$, which is an unbounded operator from a domain in $L^2(\Omega;\mathfrak{H})$ to $L^2(\Omega)$. An element $u \in L^2(\Omega;\mathfrak{H})$

belongs to the domain of $\delta$ if and only if it verifies

$$\left|E[\langle DF,u\rangle_{\mathfrak{H}}]\right| \le c_u\sqrt{E[F^2]}$$

for any $F\in\mathbb{D}^{1,2}$, where $c_u$ is a constant depending only on $u$. In particular, if $u\in \text{Dom }\delta$, then $\delta(u)$ is characterized by the following duality relationship

$$E(\delta(u)F)=E(\langle DF,u\rangle_{\mathfrak{H}}) \tag{2.6}$$

for any $F\in\mathbb{D}^{1,2}$. This formula extends to the multiple integral $\delta^q$, that is, for $u\in\text{Dom }\delta^q$ and $F\in\mathbb{D}^{q,2}$ we have

$$E(\delta^q(u)F)=E(\langle D^qF,u\rangle_{\mathfrak{H}^{\otimes q}}).$$

We can factor out a scalar random variable in the divergence in the following sense. Let $F\in\mathbb{D}^{1,2}$ and $u\in\text{Dom }\delta$ such that $Fu\in L^2(\Omega;\mathfrak{H})$. Then $Fu\in\text{Dom }\delta$ and

$$\delta(Fu)=F\delta(u)-\langle DF,u\rangle_{\mathfrak{H}}, \tag{2.7}$$

provided the right hand side is square integrable. The operators $\delta$ and $D$ have the following commutation relationship

$$D\delta(u)=u+\delta(Du) \tag{2.8}$$

for any $u\in\mathbb{D}^{2,2}(\mathfrak{H})$ (see [39, page 37]).

The following version of Meyer's inequality (see [39, Proposition 1.5.7]) will be used frequently in this paper. Let $V$ be a real separable Hilbert space. We can also introduce Sobolev spaces $\mathbb{D}^{k,p}(V)$ of $V$-valued random variables for $p\ge1$ and integer $k\ge1$. Then, for any $p>1$ and $k\ge1$, the operator $\delta$ is continuous from $\mathbb{D}^{k,p}(V\otimes\mathfrak{H})$ into $\mathbb{D}^{k-1,p}(V)$.

That is,

$$\|\delta(u)\|_{k-1,p} \le C_{k,p} \|u\|_{k,p}. \tag{2.9}$$

The operator $L$ defined on the Wiener chaos expansion as $L = \sum_{q=0}^{\infty}(-q)J_q$ is the infinitesimal generator of the Ornstein–Uhlenbeck semigroup. Its domain in $L^2(\Omega)$ is

$$\text{Dom}\, L = \left\{ F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \left\|J_q F\right\|_2^2 < \infty \right\} = \mathbb{D}^{2,2}.$$

The relation between the operators $D$, $\delta$ and $L$ is explained in the following formula (see [39, Proposition 1.4.3]). For $F \in L^2(\Omega)$, $F \in \text{Dom}\, L$ if and only if $F \in \text{Dom}(\delta D)$ (i.e., $F \in \mathbb{D}^{1,2}$ and $DF \in \text{Dom}\, \delta$), and in this case

$$\delta D F = -LF. \tag{2.10}$$

For any $F \in L^2(\Omega)$, we define $L^{-1}F = -\sum_{q=1}^{\infty} q^{-1} J_q(F)$. The operator $L^{-1}$ is called the pseudo-inverse of $L$. Indeed, for any $F \in L^2(\Omega)$, we have that $L^{-1}F \in \text{Dom}\, L$, and

$$LL^{-1}F = F - E[F].$$

We list here some properties of multiple integrals which will be used in Section 4. Fix $q \ge 1$ and let $f \in \mathfrak{H}^{\odot q}$. We have $I_q(f) = \delta^q(f)$ and $DI_q(f) = qI_{q-1}(f)$, and hence $I_q(f) \in \mathbb{D}^{\infty,2}$. The multiple integral $I_q(f)$ satisfies *hypercontractivity* property:

$$\left\|I_q(f)\right\|_p \le C_{q,p} \left\|I_q(f)\right\|_2 \text{ for any } p \in [2,\infty). \tag{2.11}$$

This easily implies that $I_q(f) \in \mathbb{D}^{\infty}$ and for any $1 \le k \le q$ and $p \ge 2$,

$$\|I_q(f)\|_{k,p} \le C_{q,k,p} \left\|I_q(f)\right\|_2.$$

As a consequence, for any $F \in \oplus_{l=1}^{q} \mathscr{H}_l$, we have

$$\|F\|_{k,p} \leq C_{q,k,p} \|F\|_2 . \tag{2.12}$$

For any random variable $F$ in the chaos of order $q \geq 2$, we have (see [35], Equation (5.2.7))

$$\frac{1}{q^2}\mathrm{Var}\left(\|DF\|_{\mathfrak{H}}^2\right) \leq \frac{q-1}{3q}\left(E[F^4]-(E[F^2])^2\right) \leq (q-1)\mathrm{Var}\left(\|DF\|_{\mathfrak{H}}^2\right). \tag{2.13}$$

In the case where $\mathfrak{H}$ is $L^2(A, \mathscr{A}, \mu)$, for a integrable random variable $F = \sum_{q=0}^{\infty} I_q(f_q) \in \mathbb{D}^{1,2}$, its derivative can be represented as an element in of $L^2(A \times \Omega)$ given by

$$D_t F = \sum_{q=1}^{\infty} q I_q(f_q(\cdot, t)).$$

## 2.3 A Density Formula

The following lemma gives an explicit expression of the probability density of a random variable.

**Lemma 2.1.** *Let $F$ be a random variable in the space $\mathbb{D}^{1,2}$, and suppose that $\frac{DF}{\|DF\|_H^2}$ belongs to the domain of the operator $\delta$ in $L^2(\Omega)$. Then the law of $F$ has a continuous and bounded density given by*

$$p(x) = E\left[\mathbf{1}_{\{F>x\}} \delta\left(\frac{DF}{\|DF\|_H^2}\right)\right].$$

From $E\delta(u) = 0$ for any $u \in Dom(\delta)$ and the Hölder inequality it follows that

**Lemma 2.2.** *Let $F$ be a random variable and let $u \in \mathbb{D}^{1,q}(H)$ with $q > 1$. Then for the conjugate pair $p$ and $q$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$),*

$$\left| E\left[ \mathbf{1}_{\{F > x\}} \delta(u) \right] \right| \leq \left( P\left( |F| > |x| \right) \right)^{\frac{1}{p}} \|\delta(u)\|_{L^q(\Omega)}. \tag{2.14}$$

# Chapter 3

# Feynman-Kac Formula for the Heat Equation Driven by fractional Noise with Hurst parameter $H < 1/2$

In this chapter, a Feynman-Kac formula is established for a stochastic partial differential equation driven by Gaussian noise which is, with respect to time, a fractional Brownian motion with Hurst parameter $H < 1/2$. To establish such a formula, we introduce and study a nonlinear stochastic integral from the given Gaussian noise. The existence of the Feynman-Kac integral then follows from the exponential integrability of nonlinear stochastic integral. Then, the approach of approximation with techniques from Malliavin calculus is used to show that the Feynman-Kac integral is the weak solution to the stochastic partial differential equation.

## 3.1 Introduction

Consider the stochastic heat equation on $\mathbb{R}^d$

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\frac{\partial W}{\partial t}(t,x), \quad t \geq 0, \; x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{3.1}$$

where $u_0$ is a bounded measurable function and $W = \{W(t,x),\ t \geq 0, x \in \mathbb{R}^d\}$ is a fractional Brownian motion of Hurst parameter $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$ in time and it has a spatial covariance $Q(x,y)$, which is locally $\gamma$-Hölder continuous (see Section 3.2 for precise meaning of this condition), with $\gamma > 2-4H$. We shall show that the solution to (3.1) is given by

$$u(t,x) = E^B\left[u_0(B_t^x)\exp\int_0^t W(ds, B_{t-s}^x)\right], \tag{3.2}$$

where $B = \{B_t^x = B_t + x, t \geq 0, x \in \mathbb{R}^d\}$ is a $d$–dimensional Brownian motion starting at $x \in \mathbb{R}^d$, independent of $W$.

This is a generalization of the well-known Feyman–Kac formula to the case of a random potential of the form $\frac{\partial W}{\partial t}(t,x)$. Notice that the integral $\int_0^t W(ds, B_{t-s}^x)$ is a nonlinear stochastic integral with respect to the fractional noise $W$. This type of Feynman-Kac formula was mentioned as a conjecture by Mocioalca and Viens in [29].

There exists an extensive literature devoted to Feynman-Kac formulae for stochastic partial differential equations. Different versions of the Feynman-Kac formula have been established for a variety of random potentials. See, for instance, a Feynman-Kac formula for anticipating SPDE proved by Ocone and Pardoux [43]. Ouerdiane and Silva [44] give a generalized Feynman-Kac formula with a convolution potential by introducing a generalized function space. Feynman-Kac formulae for Lévy processes are presented by Nualart and Schoutens [42].

However, only recently a Feynman-Kac formula has been established by Hu *et al.* [17] for random potentials associated with the fractional Brownian motion. The authors consider the following stochastic heat equation driven by fractional noise

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u\frac{\partial^{d+1} W}{\partial t \partial x_1 \cdots \partial x_d}(t,x), \quad t \geq 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{3.3}$$

where $W = \{W(t,x),\ t \geq 0, x \in \mathbb{R}^d\}$ is fractional Brownian sheet with Hurst parameter $(H_0, H_1, \ldots, H_d)$. They show ([17], Theorem 4.3) that if $H_1, \ldots, H_d \in (\frac{1}{2}, 1)$, and $2H_0 + H_1 + \cdots + H_d > d + 1$, then the solution $u(t,x)$ to the above stochastic heat equation is given by

$$u(t,x) = E^B \left[ f(B_t^x) \exp\left( \int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy) \right) \right], \tag{3.4}$$

where $B = \{B_t^x = B_t + x, t \geq 0, x \in \mathbb{R}^d\}$ is a $d$–dimensional Brownian motion starting at $x \in \mathbb{R}^d$, independent of $W$. The condition $2H_0 + H_1 + \cdots + H_d > d + 1$ is shown to be sharp in that framework. Since the $H_i$, $i = 1, \ldots, d$ cannot take value greater or equal to 1, this condition implies that $H_0 > \frac{1}{2}$.

We remark that if $B^{H_0} = \{B_t^{H_0}, t \geq 0\}$ is a fractional Brownian motion with Hurst parameter $H_0 > \frac{1}{2}$, then the stochastic integral $\int_0^T f(t) dB_t^{H_0}$ is well defined for a suitable class of distributions $f$, and in this sense the above integral $\int_0^t \int_{\mathbb{R}^d} \delta(B_{t-r}^x - y) W(dr, dy)$ is well defined for any trajectory of the Brownian motion $B$. If $H_0 < \frac{1}{2}$, this is no longer true and we can integrate only functions satisfying some regularity conditions. For this reason, it is not possible to write a Feynman-Kac formula for the Equation (3.3) with $H_0 < \frac{1}{2}$.

Notice that for $d = 1$ and $H_0 = H_1 = \frac{1}{2}$ (space-time white noise) a Feynman-Kac formula can not be written for Equation (3.3), but this equation has a unique mild solution when the stochastic integral is interpreted in the Itô sense. A renormalized Feynman-Kac formula with Wick exponential has been obtained in this case by Bertinin and Cancrini [2]. More generally, if the product appearing in (3.3) is replaced by Wick product, Hu and Nualart [15] showed that a formal solution can be obtained using chaos expansions.

In this chapter, we are concerned with the case $H_0 < \frac{1}{2}$, but we use a random potential of the form $\frac{\partial W}{\partial t}(t,x)$. One of the main obstacles to overcome is to define the stochastic integral $\int_0^t W(ds, B_{t-s}^x)$. We start with the construction of a general nonlinear stochastic integral $\int_0^t W(ds,\phi_s)$ where $\phi$ is a Hölder continuous function of order $\alpha > \frac{1}{\gamma}(1-2H)$. It turns out that the irregularity in time of $W(t,x)$ is compensated by the above Hölder continuity of $\phi$ through the covariance in space, with an appropriate application of the fractional integration by parts technique. Let us point out that $\int_0^t W(ds,\phi_s)$ is well-defined for all Hölder continuous function $\phi$ with $\alpha > \frac{1}{\gamma}(\frac{1}{2}-H)$, and we consider here only the case $\alpha > \frac{1}{\gamma}(1-2H)$ because this condition is required when we show that $u(t,x)$ is a weak solution to (3.1). Furthermore, the condition $\alpha > \frac{1}{\gamma}(1-2H)$ also allows us to obtain an explicit formula for the variance of $\int_0^t W(ds,\phi_s)$. Contrary to [17], it is rather simpler to show that $\int_0^t W(ds,B_{t-s}^x)$ is exponentially integrable. A by-product is that $u(t,x)$ defined by (3.2) is almost surely Hölder continuous of order which can be arbitrarily close to $H-\frac{1}{2}+\frac{\gamma}{4}$ from below. Let us also mention recent work on stochastic integral [11] and [19] with general Gaussian processes which can be applied to the case $H<\frac{1}{2}$.

Another main effort is to show that $u(t,x)$ defined by (3.2) is a solution to (3.1) in a weak sense (see Definition 3.12). As in [17], this is done by using an approximation scheme together with techniques of Malliavin calculus. Let us point out that in the definition of $\int_0^t W(ds,\phi_s)$ one can use a one-side approximation, but it is necessary to use symmetric approximations (as well as the condition $H>\frac{1}{2}-\frac{\gamma}{4}$) to show the convergence of the trace term (3.59).

We also discuss the corresponding Skorohod-type equation, which corresponds to taking the Wick product in [15]. We show that a unique mild solution exists for $H \in (\frac{1}{2}-\frac{\gamma}{4}, \frac{1}{2})$.

This chapter is organized as follows. Section 3.2 contains some preliminaries on the fractional noise $W$ and some results on fractional calculus which is needed in the paper. We also list all the assumptions that we make for the noise $W$ in this section.

In Section 3.3, we study the nonlinear stochastic integral appeared in Equation (3.2) by using smooth approximation and we derive some basic properties of this integral. Section 3.4 verifies the integrability and Hölder continuity of $u(t,x)$. Section 3.5 is devoted to show that $u(t,x)$ is a solution to (3.1) in a weak sense. Section 3.6 gives a solution to the Skorohod type equation. The last section is the Appendix with some technical results.

## 3.2 Preliminaries

Fix $H \in (0, \frac{1}{2})$ and denote by $R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$ the covariance function of the fractional Brownian motion of Hurst parameter $H$. Suppose that $W = \{W(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a mean zero Gaussian random field, defined on a probability space $(\Omega, \mathscr{F}, P)$, whose covariance function is given by

$$E(W(t,x)W(s,y)) = R_H(t,s)Q(x,y),$$

where $Q(x,y)$ satisfies the following properties for some $M < 2$ and $\gamma \in (0,1]$:

**(Q1)** $Q$ is locally bounded: there exists a constant $C_0 > 0$ such that for any $K > 0$

$$Q(x,y) \leq C_0\,(1+K)^M$$

for any $x, y \in \mathbb{R}^d$ such that $|x|, |y| \leq K$.

**(Q2)** $Q$ is locally $\gamma$-Hölder continuous: there exists a constant $C_1 > 0$ such that for any $K > 0$

$$|Q(x,y) - Q(u,v)| \leq C_1\,(1+K)^M \left(|x-u|^{\gamma} + |y-v|^{\gamma}\right),$$

for any $x, y, u, v \in \mathbb{R}^d$ such that $|x|, |y|, |u|, |v| \leq K$.

Denote by $\mathscr{E}$ the vector space of all step functions on $[0,T]$. On this vector space $\mathscr{E}$ we introduce the following scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}_0} = R_H(t,s).$$

Let $\mathfrak{H}_0$ be the closure of $\mathscr{E}$ with respect to the above scalar product. Denote by $C^\alpha([a,b])$ the set of all functions which is Hölder continuous of order $\alpha$, and denote by $||\cdot||_\alpha$ the $\alpha$-Hölder norm. It is well known that $C^\alpha([0,T]) \subset \mathfrak{H}_0$ for $\alpha > \frac{1}{2} - H$.

Let $\mathfrak{H}$ be the Hilbert space defined by the completion of the linear span of indicator functions $\mathbf{1}_{[0,t]\times[0,x]}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ under the scalar product

$$\langle \mathbf{1}_{[0,t]\times[0,x]}, \mathbf{1}_{[0,s]\times[0,y]} \rangle_{\mathfrak{H}} = R_H(t,s)Q(x,y).$$

In the above formula, if $x_i < 0$ we assume by convention that $\mathbf{1}_{[0,x_i]} = -\mathbf{1}_{[-x_i,0]}$. The mapping $W : \mathbf{1}_{[0,t]\times[0,x]} \to W(t,x)$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian space spanned by $W$. Then, $\{W(h),\ h \in \mathfrak{H}\}$ is an isonormal Gaussian process.

We recall the following notations of Malliavin caluclus. Let $\mathscr{S}$ be the space of random variables $F$ of the form: $F = f(W(\varphi_1),...,W(\varphi_n))$, where $\varphi_i \in \mathfrak{H}$, $f \in C^\infty(\mathbb{R}^n)$, $f$ and all its partial derivatives have polynomial growth. The Malliavin derivative $DF$ of $F \in \mathscr{S}$ is an $\mathfrak{H}$–valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(\varphi_1),...,W(\varphi_n))\varphi_i.$$

The divergence operator $\delta$ is the adjoint of the derivative operator $D$, determined by the duality relationship

$$E(\delta(u)F) = E(\langle DF, u\rangle_{\mathfrak{H}}), \quad \text{for any } F \in \mathbb{D}^{1,2}.$$

$\delta(u)$ is also called the Skorohod integral of $u$. For any random variable $F \in \mathbb{D}^{1,2}$ and $\phi \in \mathfrak{H}$,

$$FW(\phi) = \delta(F\phi) + \langle DF, \phi\rangle_{\mathfrak{H}}. \tag{3.5}$$

Since we deal with the case of Hurst parameter $H \in (0, 1/2)$, we shall use intensively the fractional calculus. We recall some basic definitions and properties. For a detailed account, we refer to [51].

Let $a, b \in \mathbb{R}$, $a < b$. Let $f \in L^1(a,b)$ and $\alpha > 0$. The left and right-sided fractional integral of $f$ of order $\alpha$ are defined for $x \in (a,b)$, respectively, as

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\, dy$$

and

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)\, dy.$$

Let $I_{a+}^{\alpha}(L^p)$(resp. $I_{b-}^{\alpha}(L^p)$) the image of $L^p(a,b)$ by the operator $I_{a+}^{\alpha}$(resp. $I_{b-}^{\alpha}$).

If $f \in I_{a+}^{\alpha}(L^p)$(resp. $I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$ then the left and right-sided fractional derivatives are defined by

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right), \tag{3.6}$$

and

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} dy \right) \tag{3.7}$$

for all $x \in (a,b)$ (the convergence of the integrals at the singularity $y = x$ holds point-wise for almost all $x \in (a,b)$ if $p = 1$ and moreover in $L^p$-sense if $1 < p < \infty$).

It is easy to check that if $f \in I^1_{a+(b-)}\left(L^1\right)$,

$$D^{\alpha}_{a+}D^{1-\alpha}_{a+} f = Df\,, \quad D^{\alpha}_{b-}D^{1-\alpha}_{b-} f = Df \tag{3.8}$$

and

$$(-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f(x)\, g(x)\, dx = \int_a^b f(x)\, D^{\alpha}_{b-} g(x)\, dx \tag{3.9}$$

provided that $0 \le \alpha \le 1$, $f \in I^{\alpha}_{a+}(L^p)$ and $g \in I^{\alpha}_{b-}(L^q)$ with $p \ge 1, q \ge 1, \frac{1}{p} + \frac{1}{q} \le 1 + \alpha$.

It is clear that $D^{\alpha} f$ exists for all $f \in C^{\beta}([a,b])$ if $\alpha < \beta$. The following proposition was proved in [55].

**Proposition 3.1.** *Suppose that $f \in C^{\lambda}([a,b])$ and $g \in C^{\mu}([a,b])$ with $\lambda + \mu > 1$. Let $\lambda > \alpha$ and $\mu > 1 - \alpha$. Then the Riemann Stieltjes integral $\int_a^b f dg$ exists and it can be expressed as*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D^{\alpha}_{a+} f(t)\, D^{1-\alpha}_{b-} g_{b-}(t)\, dt, \tag{3.10}$$

*where $g_{b-}(t) = g(t) - g(b)$.*

## 3.3 Nonlinear stochastic integral

In this section, we introduce the nonlinear stochastic integral that appears in the Feynman-Kac formula (3.2) and obtain some properties of this integral which are useful in the following sections. The main idea to define this integral is to use an appropriate approximation scheme. In order to introduce our approximation, we need to extend the fractional Brownian field to $t < 0$. This can be done by defining $W = \{W(t,x), t \in \mathbb{R}, x \in \mathbb{R}^d\}$ as a mean zero Gaussian process with the following covariance

$$E\left[W\left(t,x\right)W\left(s,y\right)\right]=\frac{1}{2}\left(\left|t\right|^{2H}+\left|s\right|^{2H}-\left|t-s\right|^{2H}\right)Q\left(x,y\right).$$

For any $\varepsilon>0$, we introduce the following approximation of $W(t,x)$:

$$W^{\varepsilon}(t,x)=\int_0^t \dot{W}^{\varepsilon}(s,x)ds, \tag{3.11}$$

where $\dot{W}^{\varepsilon}(s,x)=\frac{1}{2\varepsilon}\left(W(s+\varepsilon,x)-W(s-\varepsilon,x)\right)$.

**Definition 3.2.** *Given a continuous function $\phi$ on $[0,T]$, define*

$$\int_0^t W(ds,\phi_s)=\lim_{\varepsilon\to 0}\int_0^t \dot{W}^{\varepsilon}(s,\phi_s)ds,$$

*if the limit exits in $L^2(\Omega)$.*

Now we want to find conditions on $\phi$ such that the above limit exists in $L^2(\Omega)$. To this end, we set $I_\varepsilon(\phi)=\int_0^t \dot{W}^{\varepsilon}(s,\phi_s)ds$ and compute $E\left(I_\varepsilon(\phi)I_\delta(\phi)\right)$ for $\varepsilon,\delta>0$. Denote

$$V_{\varepsilon,\delta}^{2H}\left(r\right)=\frac{1}{4\varepsilon\delta}\left(\left|r+\varepsilon-\delta\right|^{2H}-\left|r+\varepsilon+\delta\right|^{2H}-\left|r-\varepsilon-\delta\right|^{2H}+\left|r-\varepsilon+\delta\right|^{2H}\right).$$

Using the fact that $Q\left(x,y\right)=Q\left(y,x\right)$, we have

$$\begin{aligned} E\left(I_\varepsilon(\phi)I_\delta(\phi)\right) &= \frac{1}{4\varepsilon\delta}\int_0^t\int_0^\theta Q(\phi_\theta,\phi_\eta)[\left|\theta-\eta+\varepsilon-\delta\right|^{2H}-\left|\theta-\eta+\delta+\varepsilon\right|^{2H} \\ &\quad -\left|\theta-\eta-\varepsilon-\delta\right|^{2H}+\left|\theta-\eta-\varepsilon+\delta\right|^{2H}]d\eta d\theta. \end{aligned}$$

Making the substitution $r=\theta-\eta$ and using the notation $V_{\varepsilon,\delta}^{2H}$, we can write

$$E\left(I_\varepsilon(\phi)I_\delta(\phi)\right)=\int_0^t\int_0^\theta Q(\phi_\theta,\phi_{\theta-r})V_{\varepsilon,\delta}^{2H}\left(r\right)drd\theta. \tag{3.12}$$

We need the following two technical lemmas.

**Lemma 3.3.** *For any bounded function $\psi : [0,T] \to \mathbb{R}$, we have*

$$\left| \int_0^t \psi(s) \int_0^s V_{\varepsilon,\delta}^{2H}(r)\, drds - 2H \int_0^t \psi(s) s^{2H-1} ds \right| \leq 4 \, ||\psi||_\infty (\varepsilon + \delta)^{2H}. \qquad (3.13)$$

*Proof.* Let $g(s) := \int_0^s |r|^{2H} dr$ and $f_{\varepsilon,\delta}(t) := \int_0^t \psi(s) \int_0^s V_{\varepsilon,\delta}^{2H}(r)\, drds$. Note that $g''$ exists everywhere except at 0 and $g''(r) = 2H sign(r) |r|^{2H-1}$ for $r \neq 0$. Then,

$$\begin{aligned} f_{\varepsilon,\delta}(t) &= \frac{1}{4\varepsilon\delta} \int_0^t \psi(s) \left[ g(s+\varepsilon-\delta) - g(s+\varepsilon+\delta) - g(s-\varepsilon-\delta) + g(s-\varepsilon+\delta) \right] ds \\ &= \frac{1}{4} \int_{-1}^1 \int_{-1}^1 \int_0^t \psi(s) g''(s + \eta\varepsilon - \xi\delta)\, dsd\xi d\eta \\ &= \frac{1}{4} \int_{-1}^1 \int_{-1}^1 \int_0^t \psi(s) g''(s - \Delta)\, dsd\xi d\eta, \end{aligned}$$

where $\Delta = \xi\delta - \eta\varepsilon$.

*Case i)*: If $\Delta \leq 0$, we have

$$\begin{aligned} & \left| \int_0^t \psi(s) \left( g''(s-\Delta) - 2Hs^{2H-1} \right) ds \right| \\ \leq \; & 2H \, ||\psi||_\infty \int_0^t \left( s^{2H-1} - (s-\Delta)^{2H-1} \right) ds \qquad (3.14) \\ = \; & ||\psi||_\infty \left[ t^{2H} - (t-\Delta)^{2H} + (-\Delta)^{2H} \right] \leq 2 \, ||\psi||_\infty |\Delta|^{2H}. \end{aligned}$$

*Case ii)*: If $\Delta > 0$, we assume that $\Delta < t$ (the case $\Delta \geq t$ follows easily). Then

$$\int_0^t \psi(s) g''(s-\Delta)\, ds = -2H \int_0^\Delta \psi(s) (\Delta - s)^{2H-1} ds + 2H \int_\Delta^t \psi(s) (s-\Delta)^{2H-1} ds.$$

Therefore,

$$\left| \int_0^t \psi(s) \left( g''(s-\Delta) - 2Hs^{2H-1} \right) ds \right| \leq F_\Delta^1 + F_\Delta^2, \qquad (3.15)$$

where

$$F_{\Delta}^{1} := 2H \int_{0}^{\Delta} \psi(s) \left[ (\Delta - s)^{2H-1} + s^{2H-1} \right] ds \le 2\|\psi\|_{\infty} |\Delta|^{2H} \tag{3.16}$$

and

$$\begin{aligned} F_{\Delta}^{2} &:= 2H \int_{\Delta}^{t} \psi(s) \left[ (s-\Delta)^{2H-1} - s^{2H-1} \right] ds \\ &\le 2H \|\psi\|_{\infty} \int_{\Delta}^{t} \left[ (s-\Delta)^{2H-1} - s^{2H-1} \right] ds \le 2\|\psi\|_{\infty} |\Delta|^{2H}. \end{aligned} \tag{3.17}$$

Then (3.13) follows from (3.14)–(3.17). □

**Lemma 3.4.** *Let* $\psi \in C([0,T]^2)$ *with* $\psi(0,s) = 0$, *and* $\psi(\cdot,s) \in C^{\alpha}([0,T])$ *for any* $s \in [0,T]$. *Assume* $\alpha + 2H > 1$ *and* $\sup_{s\in[0,T]} ||\psi(\cdot,s)||_{\alpha} < \infty$. *Then for any* $1-2H < \gamma < \alpha$ *and* $t \le T$ *we have*

$$\begin{aligned} &\left| \int_{0}^{t} \int_{0}^{s} \psi(r,s) \left[ V_{\varepsilon,\delta}^{2H}(r) - 2H(2H-1) r^{2H-2} \right] dr ds \right| \\ \le\ & C \sup_{s\in[0,T]} ||\psi(\cdot,s)||_{\alpha} (\varepsilon+\delta)^{2H+\gamma-1}, \end{aligned} \tag{3.18}$$

*where the constant* $C$ *depends on* $H$, $\gamma$, $\alpha$ *and* $T$, *but it is independent of* $\delta, \varepsilon$ *and* $\psi$.

*Proof.* Along the proof, we denote by $C$ a generic constant which depends on $H$, $\gamma$, $\alpha$ and $T$. Set $h(r) := |r|^{2H}$. Then $h'(r)$ exists everywhere except at 0 and $h'(r) = 2H sign(r) |r|^{2H-1}$ if $r \neq 0$. Using (3.8) and (3.10) we have

$$\begin{aligned} & f_{\varepsilon,\delta}(t) := \int_{0}^{t} \int_{0}^{s} \psi(r,s) V_{\varepsilon\delta}^{2H}(r) dr ds \\ =\ & \frac{1}{4\varepsilon} \int_{-1}^{1} \int_{0}^{t} \int_{0}^{s} \psi(r,s) \frac{\partial}{\partial r} [h(r+\varepsilon-\xi\delta) - h(r-\varepsilon-\xi\delta)] dr ds d\xi \\ =\ & (-1)^{\alpha'} \frac{1}{4\varepsilon} \int_{-1}^{1} \int_{0}^{t} \int_{0}^{s} D_{0+}^{\alpha'} \psi(r,s) D_{s-}^{1-\alpha'} [h(r+\varepsilon-\xi\delta) - h(r-\varepsilon-\xi\delta)] dr ds d\xi \\ =\ & (-1)^{\alpha'} \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \int_{0}^{t} \int_{0}^{s} D_{0+}^{\alpha'} \psi(r,s) D_{s-}^{1-\alpha'} h'(r+\eta\varepsilon-\xi\delta) dr ds d\xi d\eta, \end{aligned}$$

where $\gamma < \alpha' < \alpha$. On the other hand, we also have

$$2H(2H-1)\int_0^t\int_0^s \psi(r,s)r^{2H-2}dr = (-1)^{\alpha'}\int_0^t\int_0^s D_{0+}^{\alpha'}\psi(r,s)D_{s-}^{1-\alpha'}h'(r)\,drds\,.$$

Thus,

$$\begin{aligned} I_{\varepsilon,\delta} &:= \left|\int_0^t\int_0^s \psi(r,s)\left[V_{\varepsilon,\delta}^{2H}(r) - 2H(2H-1)r^{2H-2}\right]drds\right| \\ &\le \frac{1}{4}\int_{-1}^1\int_{-1}^1\int_0^t\int_0^s \left|D_{0+}^{\alpha'}\psi(r,s)\right|\left|D_{s-}^{1-\alpha'}h'(r+\eta\varepsilon-\xi\delta) - D_{s-}^{1-\alpha'}h'(r)\right| drdsd\xi d\eta. \end{aligned}$$

Denote $\Delta = \xi\delta - \eta\varepsilon$ and

$$f_\Delta(t) := \int_0^t\int_0^s \left|D_{0+}^{\alpha'}\psi(r,s)\right|\left|\left[D_{s-}^{1-\alpha'}h'(r-\Delta) - D_{s-}^{1-\alpha'}h'(r)\right]\right| drds. \tag{3.19}$$

Then we may write

$$I_{\varepsilon,\delta} \le \frac{1}{4}\int_{-1}^1\int_{-1}^1 f_\Delta(t)d\xi d\eta. \tag{3.20}$$

Hence, in order to prove (3.18) it suffices to prove

$$f_\Delta(t) \le C\sup_{s\in[0,T]}||\psi(\cdot,s)||_\alpha |\Delta|^{2H+\gamma-1}\,. \tag{3.21}$$

By (3.6), we have

$$\begin{aligned} \left|D_{0+}^{\alpha'}\psi(r,s)\right| &= \frac{1}{\Gamma(1-\alpha')}\left|\frac{\psi(r,s)}{r^{\alpha'}} + \alpha'\int_0^r \frac{\psi(r,s)-\psi(u,s)}{(r-u)^{\alpha'+1}}du\right| \\ &\le C\sup_{s\in[0,T]}||\psi(\cdot,s)||_\alpha\,. \end{aligned} \tag{3.22}$$

Therefore,

$$f_\Delta(t) \le C\sup_{s\in[0,T]}||\psi(\cdot,s)||_\alpha\left(F_\Delta^1 + F_\Delta^2\right), \tag{3.23}$$

where

$$
\begin{aligned}
F_\Delta^1 &= \int_0^t \int_0^s \frac{|h'(r-\Delta)-h'(r)|}{(s-r)^{1-\alpha'}} drds, \\
F_\Delta^2 &= \int_0^t \int_0^s \int_r^s \frac{|h'(r-\Delta)-h'(u-\Delta)-h'(r)+h'(u)|}{(u-r)^{2-\alpha'}} dudrds.
\end{aligned}
$$

As in the proof of Lemma 3.3, we consider the two cases separately: $\Delta \le 0$ and $\Delta > 0$.

*Case i)*: If $\Delta \le 0$, we can write

$$
\begin{aligned}
\left| \frac{h'(r-\Delta)-h'(r)}{(s-r)^{1-\alpha'}} \right| &\le C(s-r)^{\alpha'-1}|\Delta| \int_0^1 (r-\xi\Delta)^{2H-2} d\xi \\
&\le C(s-r)^{\alpha'-1} r^{-\gamma} |\Delta|^{2H+\gamma-1},
\end{aligned}
$$

which implies

$$
F_\Delta^1 \le C|\Delta|^{2H+\gamma-1}. \tag{3.24}
$$

For $0 < r < u$, we have

$$
\begin{aligned}
& \left| h'(r-\Delta)-h'(u-\Delta)-h'(r)+h'(u) \right| \\
= \;& C|\Delta| \int_0^1 \int_0^1 (r-\xi\Delta+\theta(u-r))^{2H-3} d\theta d\xi \; (r-u) \\
\le \;& C r^{2H-1-\beta_1-\beta_2} (u-r)^{\beta_1} |\Delta|^{\beta_2}
\end{aligned}
$$

for any $\beta_1, \beta_2 > 0$ such that $\beta_1+\beta_2 < 2H$. If $\alpha'+\beta_1 > 1$, we obtain

$$
\begin{aligned}
& \int_r^s \frac{|h'(r-\Delta)-h'(u-\Delta)-h'(r)+h(u)|}{(u-r)^{2-\alpha'}} du \\
\le \;& C r^{2H-1-\beta_1-\beta_2} (s-r)^{\alpha'+\beta_1-1} |\Delta|^{\beta_2},
\end{aligned}
$$

which implies, taking $\beta_2 = 2H + \gamma - 1$,

$$F_\Delta^2 \leq C|\Delta|^{2H+\gamma-1}. \tag{3.25}$$

Substituting (3.24) and (3.25) into (3.23), we get (3.21).

*Case ii)*: Now let $\Delta > 0$. We assume that $\Delta < t$ (the case $t \leq \Delta$ is simpler and omitted). Let us first consider the term $F_\Delta^1$. Define the sets

$$D_{11} = \{0 < r < s < \Delta\}, \quad D_{12} = \{0 < r < \Delta < s < t\}, \quad D_{13} = \{\Delta < r < s < t\}.$$

Then

$$F_\Delta^1 = F_\Delta^{11} + F_\Delta^{12} + F_\Delta^{13},$$

where

$$F_\Delta^{1i} = \int_{D_{1i}} \frac{|h'(r-\Delta) - h'(r)|}{(s-r)^{1-\alpha'}} drds, \quad i = 1,2,3.$$

It is easy to see that

$$F_\Delta^{11} \leq C \int_0^\Delta \int_0^s \left[(\Delta - r)^{2H-1} + r^{2H-1}\right](s-r)^{\alpha'-1} drds \leq C\Delta^{2H+\alpha'} \tag{3.26}$$

and

$$F_\Delta^{12} \leq C \int_\Delta^t \int_0^\Delta \left[(\Delta - r)^{2H-1} + r^{2H-1}\right](s-r)^{\alpha'-1} drds \leq C\Delta^{2H}. \tag{3.27}$$

As for $F_\Delta^{13}$, we have

$$F_\Delta^{13} = \int_\Delta^t \int_\Delta^s \frac{|h'(r-\Delta) - h'(r)|}{(s-r)^{1-\alpha}} drds = \int_0^{t-\Delta} \int_0^u \frac{|h'(v) - h'(v+\Delta)|}{(u-v)^{1-\alpha'}} dvdu.$$

Using the estimate

$$|h'(v) - h'(v+\Delta)| \leq Cv^{2H-\beta-1}\Delta^\beta$$

for all $0 < \beta < 2H$, we obtain

$$F_\Delta^{13} \leq C\, \Delta^\beta . \tag{3.28}$$

Thus, (3.26)–(3.28) yield

$$F_\Delta^{1} \leq C \Delta^\beta , \quad \text{for all } \ 0 < \beta < 2H . \tag{3.29}$$

Now we study the second term $F_\Delta^2$. Denote

$$D_{21} = \{0 < r < u < s < \Delta < t\} , \quad D_{22} = \{0 < r < u < \Delta < s < t\} ,$$
$$D_{23} = \{0 < r < \Delta < u < s < t\} , \quad D_{24} = \{0 < \Delta < r < u < s < t\} .$$

Then

$$F_\Delta^2 = F_\Delta^{21} + F_\Delta^{22} + F_\Delta^{23} + F_\Delta^{24} ,$$

where for $i = 1,2,3,4$,

$$F_\Delta^{2i} = \int_{D_{2i}} \frac{|h'(r-\Delta) - h'(u-\Delta) - h'(r) + h'(u)|}{(u-r)^{2-\alpha'}} dudrds .$$

Consider first the term $F_\Delta^{21}$. We can write

$$\begin{aligned} & \frac{1}{2H} \left| h'(r-\Delta) - h'(u-\Delta) \right| = \left| (\Delta - u)^{2H-1} - (\Delta - r)^{2H-1} \right| \\ \leq \ & C (u-r) \int_0^1 (\Delta - u + \theta (u-r))^{2H-2} d\theta \\ \leq \ & C (u-r)^{1-\beta} (\Delta - u)^{2H+\beta-2} , \end{aligned}$$

where $1-2H<\beta<\alpha'$. Similarly, we have

$$\left|h'(r)-h'(u)\right| \le Cr^{2H+\beta-2}(u-r)^{1-\beta}.$$

As a consequence,

$$\begin{aligned} F_{\Delta}^{21} &\le C\int_0^{\Delta}\int_0^s\int_r^s (u-r)^{\alpha'-\beta-1}(\Delta-u)^{2H+\beta-2}\,dudrds \\ &\le C\int_0^{\Delta}\int_0^{\Delta}\int_r^{\Delta} (u-r)^{\alpha'-\beta-1}(\Delta-u)^{2H+\beta-2}\,dudrds \qquad (3.30)\\ &\le C\Delta^{2H+\alpha'}. \end{aligned}$$

In a similar way we can prove that

$$F_{\Delta}^{22} \le C\int_{\Delta}^{t}\int_0^{\Delta}\int_r^{\Delta}(u-r)^{\alpha'-\beta-1}(\Delta-u)^{2H+\beta-2}\,dudrds \le C\Delta^{2H+\alpha'-1}. \qquad (3.31)$$

For $F_{\Delta}^{23}$, notice that when $r<\Delta<u$,

$$\left|h'(r-\Delta)-h'(u-\Delta)-h'(r)+h'(u)\right| = (\Delta-r)^{2H-1}+(u-\Delta)^{2H-1}+r^{2H-1}+u^{2H-1}$$

and

$$\begin{aligned} (u-r)^{\alpha'-2} &= (u-\Delta+\Delta-r)^{\alpha'-2} \\ &\le (u-\Delta)^{-\beta}(\Delta-r)^{\alpha'+\beta-2}\wedge(u-\Delta)^{-\beta-2H+1}(\Delta-r)^{2H+\alpha'+\beta-3}, \end{aligned}$$

where we can take any $\beta\in(0,1)$ satisfying $2H+\beta+\alpha'>2$. Then,

$$\begin{aligned} F_{\Delta}^{23} &\le C\int_{D_{23}}\left[(\Delta-r)^{2H-1}+(u-\Delta)^{2H-1}+r^{2H-1}+u^{2H-1}\right](u-r)^{\alpha'-2}\,dudrds \\ &\le C\int_{D_{23}}\left[(\Delta-r)^{2H+\alpha'+\beta-3}(u-\Delta)^{-\beta}+r^{2H-1}(u-\Delta)^{-\beta}(\Delta-r)^{\alpha'+\beta-2}\right]dudrds \end{aligned}$$

$$\leq \quad C\Delta^{2H+\alpha'+\beta-2}.$$

Taking $\beta = 1+\gamma-\alpha'$, we obtain

$$\left|F_\Delta^{23}\right| \leq C\Delta^{2H+\alpha'-1}. \tag{3.32}$$

Finally we consider the last term $F_\Delta^{24}$. Making the substitutions $x = r-\Delta$, $y = u-\Delta$ we can write

$$\begin{aligned} F_\Delta^{24} &= \int_{D_{24}} \frac{|h'(r-\Delta) - h'(u-\Delta) - h'(r) + h'(u)|}{(u-r)^{2-\alpha'}} dudrds \\ &= \int_\Delta^t \int_0^{s-\Delta} \int_x^{s-\Delta} \frac{|h'(x) - h'(y) - h'(x+\Delta) + h'(y+\Delta)|}{(y-x)^{2-\alpha'}} dydxds. \end{aligned}$$

Note that for $0 < x < y$ and $\Delta > 0$,

$$\begin{aligned} & \left|h'(x) - h'(y) - h'(x+\Delta) + h'(y+\Delta)\right| \\ = \; & x^{2H-1} - y^{2H-1} - (x+\Delta)^{2H-1} + (y+\Delta)^{2H-1} \\ = \; & C \int_0^1 \int_0^1 (x+\theta(y-x)+\tilde{\theta}\Delta)^{2H-3} d\theta d\tilde{\theta} \\ \leq \; & C x^{2H+\beta_1+\beta_2-3} (y-x)^{1-\beta_1} \Delta^{1-\beta_2}, \end{aligned}$$

where

$$0 < \beta_1, \beta_2 < 1, \quad 2H + \beta_1 + \beta_2 > 2, \quad \text{and} \quad \beta_1 < \alpha'.$$

Taking $\beta_2 = 2-2H-\gamma$ we get

$$F_\Delta^{24} \leq C\, \Delta^{2H+\gamma-1}. \tag{3.33}$$

From (3.30)–(3.33), we see that

$$F_{\Delta}^{2} \leq C\Delta^{2H+\gamma-1}\,. \tag{3.34}$$

This completes the proof of the lemma. □

**Theorem 3.5.** *Suppose that* $\phi \in C^{\alpha}([0,T])$ *with* $\gamma\alpha > 1-2H$ *on* $[0,T]$. *Then, the nonlinear stochastic integral* $\int_0^t W(ds,\phi_s)$ *exists and*

$$\begin{aligned} E\left(\int_0^t W(ds,\phi_s)\right)^2 &= 2H\int_0^t \theta^{2H-1}Q(\phi_\theta,\phi_\theta)d\theta \\ +2H(2H-1)\int_0^t\int_0^\theta & r^{2H-2}\left(Q(\phi_\theta,\phi_{\theta-r})-Q(\phi_\theta,\phi_\theta)\right)drd\theta. \end{aligned} \tag{3.35}$$

*Furthermore, for any* $\frac{1-2H}{\gamma} < \alpha' < \alpha$, *we have*

$$\begin{aligned} &\sup_{0\leq t\leq T} E\left(\left|\int_0^t \dot{W}^{\varepsilon}(s,\phi_s)ds - \int_0^t W(ds,\phi_s)\right|^2\right) \\ &\leq\; C\left(1+\|\phi\|_\infty\right)^M\left(1+||\phi||_\alpha^\gamma\right)\varepsilon^{2H+\gamma\alpha'-1}, \end{aligned} \tag{3.36}$$

*where the constant* $C$ *depends on* $H$, $T$, $\gamma$, $\alpha$, $\alpha'$ *and the constants* $C_0$ *and* $C_1$ *appearing in (Q1) and (Q2).*

*Proof.* We can write (3.12) as

$$\begin{aligned} E\left(I_\varepsilon(\phi)I_\delta(\phi)\right) &= \int_0^t\int_0^\theta \left(Q(\phi_\theta,\phi_{\theta-r})-Q(\phi_\theta,\phi_\theta)\right)V_{\varepsilon,\delta}^{2H}(r)\,drd\theta \\ &\quad+\int_0^t\int_0^\theta Q(\phi_\theta,\phi_\theta)V_{\varepsilon,\delta}^{2H}(r)\,drd\theta\,. \end{aligned} \tag{3.37}$$

Due to the local boundedness of $Q$ (see (Q1)) and applying Lemma 3.3 to $\psi(\theta) = Q(\phi_\theta, \phi_\theta)$, we see that the second integral converges to

$$\lim_{\varepsilon,\delta\to 0}\int_0^t \int_0^\theta Q(\phi_\theta, \phi_\theta) V_{\varepsilon,\delta}^{2H}(r)\, dr d\theta = 2H \int_0^t Q(\phi_\theta, \phi_\theta)\theta^{2H-1} d\theta\,.$$

On the other hand, using the local Hölder continuity of $Q$ (see (Q2)) and applying Lemma 3.4, to $\psi(r,\theta) = Q(\phi_\theta, \phi_{\theta-r}) - Q(\phi_\theta, \phi_\theta)$, we see that the first integral converges to

$$\begin{aligned}
&\lim_{\varepsilon,\delta\to 0}\int_0^t \int_0^\theta \left(Q(\phi_\theta, \phi_{\theta-r}) - Q(\phi_\theta, \phi_\theta)\right) V_{\varepsilon,\delta}^{2H}(r)\, dr d\theta \\
= \;& 2H\,(2H-1)\int_0^t \int_0^\theta \left(Q(\phi_\theta, \phi_{\theta-r}) - Q(\phi_\theta, \phi_\theta)\right) r^{2H-2} dr d\theta\,.
\end{aligned}$$

This implies that $\{I_{\varepsilon_n}(\phi)\,, n \geq 1\}$ is a Cauchy sequence in $L^2(\Omega)$ for any sequence $\varepsilon_n \downarrow 0$. As a consequence, $\lim_{\varepsilon\to 0} I_\varepsilon(\phi)$ exists in $L^2(\Omega)$ and is denoted by $I(\phi) := \int_0^t W(ds, \phi_s)$. Letting $\varepsilon, \delta \to 0$ in (3.37), we obtain (3.35).

From (3.37), Lemma 3.3 and Lemma 3.4, we have for any $\alpha' < \alpha$,

$$\left|E\left(I_\varepsilon(\phi) I_\delta(\phi)\right) - E(I^2(\phi))\right| \leq C\,(1+\|\phi\|_\infty)^M\,(1+\|\phi\|_\alpha^\gamma)(\varepsilon+\delta)^{2H+\gamma\alpha'-1}\,. \tag{3.38}$$

In Equation (3.38), let $\delta \to 0$ and notice that $I_\delta(\phi) \to I(\phi)$ in $L^2(\Omega)$. Then

$$\left|E\left(I_\varepsilon(\phi) I(\phi)\right) - E(I^2(\phi))\right| \leq C\,(1+\|\phi\|_\infty)^M\,(1+\|\phi\|_\alpha^\gamma)\varepsilon^{2H+\gamma\alpha'-1}\,.$$

On the other hand, if we let $\varepsilon = \delta$ in (3.38), we obtain

$$\left|E I_\varepsilon^2(\phi) - E(I^2(\phi))\right| \leq C\,(1+\|\phi\|_\infty)^M\,(1+\|\phi\|_\alpha^\gamma)\varepsilon^{2H+\gamma\alpha'-1}\,.$$

Thus we have

$$E\left|I_\varepsilon(\phi)-I(\phi)\right|^2=\left[E\left(I_\varepsilon^2(\phi)\right)-E\left(I^2(\phi)\right)\right]-2\left[E\left(I_\varepsilon(\phi)I(\phi)\right)-E\left(I^2(\phi)\right)\right].$$

Applying the triangular inequality, we obtain (3.36). □

The following proposition can be proved in the same way as (3.35).

**Proposition 3.6.** *Suppose $\phi,\psi\in C^\alpha([0,T])$ with $\alpha\gamma>1-2H$. Then*

$$\begin{aligned}&E\left(\int_0^t W(dr,\phi_r)\int_0^t W(dr,\psi_r)\right)=2H\int_0^t\theta^{2H-1}Q(\phi_\theta,\psi_\theta)d\theta\\&+H(2H-1)\int_0^t\int_0^\theta r^{2H-2}\left(Q(\phi_\theta,\psi_{\theta-r})-Q(\phi_\theta,\psi_\theta)\right)drd\theta\\&+H(2H-1)\int_0^t\int_0^\theta r^{2H-2}\left(Q(\phi_{\theta-r},\psi_\theta)-Q(\phi_\theta,\psi_\theta)\right)drd\theta.\end{aligned}\tag{3.39}$$

The following proposition provides the Hölder continuity of the indefinite integral.

**Proposition 3.7.** *Suppose $\phi\in C^\alpha([0,T])$ with $\alpha\gamma>1-2H$. Then for all $0\le s<t\le T$,*

$$E\left(\int_0^t W(dr,\phi_r)-\int_0^s W(dr,\phi_r)\right)^2\le C\left(1+\|\phi\|_\infty\right)^M(t-s)^{2H},\tag{3.40}$$

*where the constant $C$ depends on $H$, $T$, $\gamma$, $\alpha$ and the constants $C_0$ and $C_1$ appearing in (Q1) and (Q2). As a consequence, the process $X_t=\int_0^t W(dr,\phi_r)$ is almost surely $(H-\delta)$-Hölder continuous for any $\delta>0$.*

*Proof.* We shall first show that

$$E\left(\int_0^t \dot W^\varepsilon(r,\phi_r)dr-\int_0^s \dot W^\varepsilon(r,\phi_r)dr\right)^2\le C\left(1+\|\phi\|_\infty\right)^M(t-s)^{2H}.\tag{3.41}$$

We can write

$$
\begin{aligned}
& E\left(\int_0^t W^{\varepsilon}(dr,\phi_r) - \int_0^s W^{\varepsilon}(dr,\phi_r)\right)^2 = E\left(\int_s^t W^{\varepsilon}(dr,\phi_r)\right)^2 \\
= \; & \frac{1}{4\varepsilon^2}\int_s^t\int_s^t E\left[\left(W(\theta+\varepsilon,\phi_\theta) - W(\theta-\varepsilon,\phi_\theta)\right)\left(W(\eta+\varepsilon,\phi_\eta) - W(\eta-\varepsilon,\phi_\eta)\right)\right] d\theta d\eta \\
= \; & \frac{1}{8\varepsilon^2}\int_s^t\int_s^t Q(\phi_\theta,\phi_\eta)\,[|\eta-\theta|^{2H} - |\eta-\theta-2\varepsilon|^{2H} - |\eta-\theta+2\varepsilon|^{2H}] d\theta d\eta \\
= \; & \frac{1}{8\varepsilon^2}\int_0^{t-s}\int_0^{t-s} Q(\phi_{s+\theta},\phi_{s+\eta})\,[|\eta-\theta|^{2H} - |\eta-\theta-2\varepsilon|^{2H} - |\eta-\theta+2\varepsilon|^{2H}] d\theta d\eta \\
= \; & \frac{1}{4\varepsilon^2}\int_0^{t-s}\int_0^{\theta} Q(\phi_{s+\theta},\phi_{s+\theta-r})\left[2r^{2H} - |r+2\varepsilon|^{2H} - |r-2\varepsilon|^{2H}\right] dr d\theta.
\end{aligned}
$$

The inequality (3.41) follows from the assumption (Q1) and the inequality (3.65) obtained in the Appendix. Finally, the inequality (3.40) follows from (3.41), Proposition 3.5 and the Fatou's lemma. □

## 3.4 Feynman-Kac integral

In this section, we show that the random field $u(t,x)$ given by (3.2) is well-defined and study its Hölder continuity. Since the Brownian motion $B_t$ has Hölder continuous trajectories of order $\delta$ for any $\delta \in \left(0,\frac{1}{2}\right)$, by Lemma 3.5 the nonlinear stochastic integral $\int_0^t W(ds, B^x_{t-s})$ can be defined for any $H > \frac{1}{2} - \frac{\gamma}{4}$. The following theorem shows that it is exponentially integrable and hence $u(t,x)$ is well-defined.

Set $||B||_{\infty,T} = \sup_{0\le s\le T}|B_s|$ and $||B||_{\delta,T} = \sup_{0\le s<t\le T}\frac{|B_t - B_s|}{|t-s|^{\delta}}$ for $\delta \in \left(0,\frac{1}{2}\right)$.

**Theorem 3.8.** *Let $H > \frac{1}{2} - \frac{\gamma}{4}$ and let $u_0$ be bounded. For any $t \in [0,T]$ and $x \in \mathbb{R}^d$, the random variable $\int_0^t W(ds, B^x_{t-s})$ is exponentially integrable and the random field $u(t,x)$ given by (3.2) is in $L^p(\Omega)$ for any $p \ge 1$.*

*Proof.* Suppose first that $p = 1$. By (3.40) with $s = 0$ and the Fernique's theorem we have

$$\begin{aligned} E^W |u(t,x)| &\leq ||u_0||_\infty E^B E^W [\exp \int_0^t W(ds, B^x_{t-s})] \\ &\leq ||u_0||_\infty E^B [e^{Ct^{2H}(1+||B||_{\infty,T})^M}] < \infty. \end{aligned}$$

The $L^p$ integrability of $u(t,x)$ follows from Jensen's inequality

$$\begin{aligned} E^W |u(t,x)|^p &\leq ||u_0||_\infty E^B E^W \exp\left( p \int_0^t W(dr, B^x_{t-r}) \right) \\ &\leq ||u_0||_\infty E^B [\exp\left( Cp(1+||B||_{\infty,T})^M T^{2H} \right)] < \infty. \end{aligned} \tag{3.42}$$

□

To show the Hölder continuity of $u(\cdot,x)$, we need the following lemma.

**Lemma 3.9.** *Assume that $u_0$ is Lipschitz continuous. Then for $0 \leq s < t \leq T$ and for any $\alpha < 2H - 1 + \frac{1}{2}\gamma$,*

$$E^W \left| \int_0^s W(dr, B^x_{t-r}) - \int_0^s W(dr, B^x_{s-r}) \right|^2 \leq C(1 + ||B||_{\infty,T})^M \|B\|^\gamma_{\delta,T} (t-s)^\alpha,$$

*where the constant $C$ depends on $H$, $T$, $\gamma$ and the constant $C_1$ appearing in $(Q2)$.*

*Proof.* Suppose $\delta \in \left(0, \frac{1}{2}\right)$. For $0 \leq u < v < s \leq T$, denote

$$\Delta Q(s,t,u,v) := Q(B^x_{t-u}, B^x_{t-v}) - Q(B^x_{t-u}, B^x_{t-u}) - Q(B^x_{t-u}, B^x_{s-v}) + Q(B^x_{t-u}, B^x_{s-u}).$$

Note that $(Q2)$ implies

$$|\Delta Q(s,t,u,v)| \leq 2C_1(1 + ||B||_{\infty,T})^M \|B\|^\gamma_{\delta,T} (t-s)^{\gamma\delta},$$

and

$$|\Delta Q(s,t,u,v)| \leq 2C_1(1+||B||_{\infty,T})^M \|B\|_{\delta,T}^{\gamma} |u-v|^{\gamma\delta},$$

which imply that for any $\beta \in (0,1)$,

$$|\Delta Q(s,t,u,v)| \leq 2C_1(1+||B||_{\infty,T})^M \|B\|_{\delta,T}^{\gamma} (t-s)^{\beta\gamma\delta} |u-v|^{(1-\beta)\gamma\delta}.$$

Applying (3.39) and using $Q(x,y) = Q(y,x)$, we get

$$\begin{aligned}
& E^W \left| \int_0^s W(dr, B_{t-r}^x) - \int_0^s W(dr, B_{s-r}^x) \right|^2 \\
= \; & 2H(2H-1) \int_0^s \int_0^\theta r^{2H-2} \left[\Delta Q(s,t,\theta,\theta-r) + \Delta Q(t,s,\theta,\theta-r)\right] dr d\theta \\
& +2H \int_0^s \theta^{2H-1} \left[Q(B_{t-\theta}^x, B_{t-\theta}^x) - 2Q(B_{t-\theta}^x, B_{s-\theta}^x) + Q(B_{s-\theta}^x, B_{s-\theta}^x)\right] d\theta \\
\leq \; & C(1+||B||_{\infty,T})^M \|B\|_{\delta,T}^{\gamma} (t-s)^{\beta\gamma\delta},
\end{aligned}$$

for any $\beta$ such that $(1-\beta)\gamma\delta > 1-2H$, i.e. $\beta\gamma\delta < 2H - 1 + \gamma\delta$. Taking $\beta$ and $\delta$ such that $\beta\gamma\delta = \alpha$, we get the lemma. □

**Theorem 3.10.** *Suppose $u_0$ is Lipschitz continuous and bounded. Then for each $x \in \mathbb{R}^d$, $u(\cdot,x) \in C^{H_1}([0,T])$ for any $H_1 \in \left(0, H - \frac{1}{2} + \frac{1}{4}\gamma\right)$.*

*Proof.* For $0 \leq s < t \leq T$, from the Minkowski's inequality it follows that

$$\begin{aligned}
& E^W \left[|u(t,x) - u(s,x)|^p\right] \\
\leq \; & \left[E^B \left(E^W \left| u_0(B_t^x) e^{\int_0^t W(dr, B_{t-r}^x)} - u_0(B_s^x) e^{\int_0^s W(dr, B_{s-r}^x)} \right|^p\right)^{\frac{1}{p}}\right]^p \\
\leq \; & C||u_0||_\infty \left[E^B \left(E^W \left| e^{\int_0^t W(dr, B_{t-r}^x)} - e^{\int_0^s W(dr, B_{s-r}^x)} \right|^p\right)^{\frac{1}{p}}\right]^p \\
& +C \left[E^B \left(E^W \left| (u_0(B_t^x) - u_0(B_s^x)) e^{\int_0^s W(dr, B_{s-r}^x)} \right|^p\right)^{\frac{1}{p}}\right]^p.
\end{aligned} \tag{3.43}$$

Since $u_0$ is Lipschitz continuous, using (3.42) and Hölder's inequality we have

$$\left[E^B\left(E^W\left(|u_0(B_t^x)-u_0(B_s^x)|\,e^{\int_0^s W(dr,B_{s-r}^x)}\right)^p\right)^{\frac{1}{p}}\right]^p \leq C(t-s)^{\frac{p}{2}}. \tag{3.44}$$

For the first term in (3.43), denoting $\left[E^W(\exp\int_0^t W(dr,B_{t-r}^x)+\exp\int_0^s W(dr,B_{s-r}^x))^{2p}\right]^{\frac{1}{2}}$ by $K$, using the formula that $|e^a-e^b|\leq(e^a+e^b)|a-b|$ for $a,b\in\mathbb{R}$ and Hölder's inequality we get

$$\begin{aligned} & E^W\left[\left|\exp\int_0^t W(dr,B_{t-r}^x)-\exp\int_0^s W(dr,B_{s-r}^x)\right|^p\right] \\ \leq\ & K\left[E^W\left|\int_0^t W(dr,B_{t-r}^x)-\int_0^s W(dr,B_{s-r}^x)\right|^{2p}\right]^{\frac{1}{2}}. \end{aligned} \tag{3.45}$$

Applying Lemma 3.7 and Lemma 3.9, we obtain

$$\begin{aligned} & E^W\left|\int_0^t W(dr,B_{t-r}^x)-\int_0^s W(dr,B_{s-r}^x)\right|^2 \leq 2E^W\left|\int_s^t W(dr,B_{t-r}^x)\right|^2 \\ & +2E^W\left|\int_0^s W(dr,B_{t-r}^x)-\int_0^s W(dr,B_{s-r}^x)\right|^2 \\ \leq\ & C(1+||B||_{\infty,T})^M\|B\|_{\delta,T}^{\gamma}(t-s)^{2H_1}. \end{aligned} \tag{3.46}$$

Noting that conditional to $B$, $\int_0^t W(dr,B_{t-r}^x)-\int_0^s W(dr,B_{s-r}^x)$ is Gaussian, and using (3.45), (3.46) and (3.42) we get

$$\begin{aligned} & \left[E^B\left(E^W\left|\exp\int_0^t W(dr,B_{t-r}^x)-\exp\int_0^s W(dr,B_{s-r}^x)\right|^p\right)^{\frac{1}{p}}\right]^p \\ \leq\ & C\left[E^B\left(E^W\left|\int_0^t W(dr,B_{t-r}^x)-\int_0^s W(dr,B_{s-r}^x)\right|^2\right)^{\frac{1}{2}}\right]^p \\ \leq\ & C(t-s)^{pH_1}. \end{aligned} \tag{3.47}$$

From (3.43), (3.44) and (3.47), we can see that for any $p \geq 1$,

$$E^W\left[|u(t,x) - u(s,x)|^p\right] \leq C(t-s)^{pH_1}. \tag{3.48}$$

Now Kolmogorov's continuity criterion implies the theorem. $\square$

## 3.5 Validation of the Feynman-Kac Formula

In the last section, we have proved that $u(t,x)$ given by (3.2) is well-defined. In this section, we shall show that $u(t,x)$ is a weak solution to Equation (3.1).

To give the exact meaning about what we mean by a weak solution, we follow the idea of [15] and [17]. First, we need a definition of the Stratonovich integral.

**Definition 3.11.** *Given a random field $v = \{v(t,x),\ t \geq 0, x \in \mathbb{R}^d\}$ such that for all $t > 0$ $\int_0^t \int_{\mathbb{R}^d} |v(s,x)|dxds < \infty$ a.s., the Stratonovich integral*

$$\int_0^t \int_{\mathbb{R}^d} v(s,x)W(ds,x)dx$$

*is defined as the following limit in probability if it exists*

$$\lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^d} v(s,x)\dot{W}^{\varepsilon}(s,x)dsdx$$

*where $W^{\varepsilon}(t,x)$ is introduced in (3.11).*

The precise meaning of the weak solution to equation (3.1) is given below.

**Definition 3.12.** *A random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is a weak solution to Equation (3.1) if for any $\varphi \in C_0^{\infty}\left(\mathbb{R}^d\right)$, we have*

$$\int_{\mathbb{R}^d} (u(t,x) - u_0(x))\varphi(x)dx \quad = \quad \int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\varphi(x)dxds$$

$$+\int_0^t \int_{\mathbb{R}^d} u(s,x)\,\varphi(x)W(ds,x)dx \qquad (3.49)$$

*almost surely, for all $t \geq 0$, where the last term is a Stratonovich stochastic integral in the sense of Definition 3.11.*

The following theorem justifies the Feynman-Kac formula (3.2).

**Theorem 3.13.** *Suppose $H > \frac{1}{2} - \frac{1}{4}\gamma$ and $u_0$ is a bounded measurable function. Let $u(t,x)$ be the random field defined in (3.2). Then for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, $u(t,x)\varphi(x)$ is Stratonovich integrable and $u(t,x)$ is a weak solution to Equation (3.1) in the sense of Definition 3.12.*

*Proof.* We prove this theorem by a limit argument. We divide the proof into three steps.
*Step 1.* Let $u^\varepsilon(t,x)$ be the unique solution to the following equation:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \frac{1}{2}\triangle u^\varepsilon + u^\varepsilon \frac{\partial W^\varepsilon}{\partial t}(t,x),\ t>0, x\in\mathbb{R}^d, \\ u^\varepsilon(0,x) = u_0(x). \end{cases} \qquad (3.50)$$

Since $W^\varepsilon(t,x)$ is differentiable, the classical Feynman-Kac formula holds for the solution to this equation, that is,

$$u^\varepsilon(t,x) := E^B[u_0(B_t^x)e^{\int_0^t \dot{W}^\varepsilon(s,B_{t-s}^x)ds}].$$

The fact that $u^\varepsilon(t,x)$ is well-defined follows from (3.41) and Fernique's theorem. In fact, we have (c.f. the argument in the proof of Lemma 3.8)

$$\begin{aligned} E^W|u^\varepsilon(t,x)|^p &\leq ||u_0||_\infty E^B E^W \exp\left(p\int_0^t \dot{W}^\varepsilon(r,B_{t-r}^x)dr\right) \\ &\leq ||u_0||_\infty E^B[\exp\left(Cp(1+||B||_{\infty,T})^M t^{2H}\right)] < \infty. \qquad (3.51) \end{aligned}$$

Introduce the following notations

$$
\begin{aligned}
g^{\varepsilon}_{s,x}(r,z) :=& \quad \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,\ s+\varepsilon]}(r)\mathbf{1}_{[0,x]}(z),\\
g^{B}_{s,x}(r,z) :=& \quad \mathbf{1}_{[0,s]}(r)\mathbf{1}_{[0,B^x_{s-r}]}(z),\\
g^{\varepsilon,B}_{s,x}(r,z) :=& \quad \int_0^s \frac{1}{\varepsilon}\mathbf{1}_{[\theta-\varepsilon,\ \theta+\varepsilon]}(r)\mathbf{1}_{[0,B^x_{s-\theta}]}(z)d\theta\,.
\end{aligned}
$$

From the results of Section 3, we see that $g^{\varepsilon}_{s,x}$, $g^{B}_{s,x}$, $g^{\varepsilon,B}_{s,x} \in \mathfrak{H}$ ($\mathfrak{H}$ is introduced in Section 2), and we can write

$$
\dot{W}^{\varepsilon}(s,x) = W\left(\frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,\ s+\varepsilon]}(r)\mathbf{1}_{[0,x]}(z)\right) = W\left(g^{\varepsilon}_{s,x}\right),
$$

$$
\int_0^s W\left(d\theta, B^x_{s-\theta}\right) = W\left(g^{B}_{s,x}\right), \quad \int_0^s \dot{W}^{\varepsilon}\left(\theta, B^x_{s-\theta}\right)d\theta = W\left(g^{\varepsilon,B}_{s,x}\right).
$$

Set

$$
\widetilde{u}^{\varepsilon}(s,x) := u^{\varepsilon}(s,x) - u(s,x)\,.
$$

*Step 2*. We prove the following claim:

$u^{\varepsilon}(s,x) \to u(s,x)$ in $\mathbb{D}^{1,2}$ as $\varepsilon \downarrow 0$, uniformly on any compact subset of $[0,T]\times\mathbb{R}^d$, that is, for any compact $K \subseteq \mathbb{R}^d$

$$
\sup_{s\in[0,T],x\in K} E^W\left[|\widetilde{u}^{\varepsilon}(s,x)|^2 + \|D\widetilde{u}^{\varepsilon}(s,x)\|^2_{\mathfrak{H}}\right] \to 0 \text{ as } \varepsilon \downarrow 0. \tag{3.52}
$$

Since $u_0$ is bounded, without loss of generality, we may assume $u_0 \equiv 1$. Let $B^1$ and $B^2$ be two independent Brownian motions, both independent of $W$. Using the inequality $|e^a - e^b| \le (e^a + e^b)|a-b|$, Hölder inequality and the fact that $W(g^{\varepsilon,B}_{t,x})$ and $W(g^{B}_{t,x})$ are Gaussian conditioning to $B$, we have

$$
E^W\left(u^{\varepsilon}(t,x) - u(t,x)\right)^2 = E^W[E^B(e^{W(g^{\varepsilon,B}_{t,x})} - e^{W(g^{B}_{t,x})})]^2
$$

$$
\begin{aligned}
&\le E^B E^W \left| e^{W(g_{t,x}^{\varepsilon,B})} - e^{W(g_{t,x}^{B})} \right|^2 \\
&\le E^B \left[ E^W \left( e^{W(g_{t,x}^{\varepsilon,B})} + e^{W(g_{t,x}^{B})} \right)^4 \right]^{\frac{1}{2}} \left[ E^W \left| W(g_{t,x}^{\varepsilon,B}) - W(g_{t,x}^{B}) \right|^4 \right]^{\frac{1}{2}} \\
&\le C \left[ E^B E^W \left( e^{4W(g_{t,x}^{\varepsilon,B})} + e^{4W(g_{t,x}^{B})} \right) \right]^{\frac{1}{2}} E^B E^W \left| W(g_{t,x}^{\varepsilon,B}) - W(g_{t,x}^{B}) \right|^2 .
\end{aligned}
$$

Note that (3.42) and (3.51) imply

$$
E^B E^W \left( e^{pW(g_{t,x}^{\varepsilon,B})} + e^{pW(g_{t,x}^{B})} \right) < \infty \tag{3.53}
$$

for any $p \geq 1$. On the other hand, applying Theorem 3.5, we have

$$
\sup_{0 \le t \le T, x \in K} E^B E^W \left| W(g_{t,x}^{\varepsilon,B}) - W(g_{t,x}^{B}) \right|^2 \to 0 \text{ as } \varepsilon \downarrow 0. \tag{3.54}
$$

Then it follows that as $\varepsilon \downarrow 0$

$$
\sup_{0 \le t \le T, x \in K} E^W \left| \widetilde{u}^{\varepsilon}(t,x) \right|^2 = \sup_{0 \le t \le T, x \in K} E^W \left( u^{\varepsilon}(t,x) - u(t,x) \right)^2 \to 0.
$$

For the Malliavin derivatives, we have

$$
\begin{aligned}
Du^{\varepsilon}(s,x) &= E^B \left[ \exp\left( W\left( g_{s,x}^{\varepsilon,B} \right) \right) g_{s,x}^{\varepsilon,B} \right], \\
Du(s,x) &= E^B \left[ \exp\left( W\left( g_{s,x}^{B} \right) \right) g_{s,x}^{B} \right].
\end{aligned}
$$

Then

$$
\begin{aligned}
& E^W \| Du^{\varepsilon}(s,x) - Du(s,x) \|_{\mathfrak{H}}^2 \\
= \; & E^W \left\| E^B \left[ \left( \exp\left( W\left( g_{s,x}^{\varepsilon,B} \right) \right) g_{s,x}^{\varepsilon,B} - \exp\left( W\left( g_{s,x}^{B} \right) \right) g_{s,x}^{B} \right) \right] \right\|_{\mathfrak{H}}^2 \\
\le \; & 2 E^W E^B \left[ \exp\left( 2W\left( g_{s,x}^{\varepsilon,B} \right) \right) \| g_{s,x}^{\varepsilon,B} - g_{s,x}^{B} \|_{\mathfrak{H}}^2 \right]
\end{aligned}
$$

$$+2E^W E^B\left[\left|\exp\left(W\left(g^{\varepsilon,B}_{s,x}\right)\right)-\exp\left(W\left(g^{B}_{s,x}\right)\right)\right|^2 ||g^B_{s,x}||^2_{\mathfrak{H}}\right].$$

Note that $||g^{\varepsilon,B}_{t,x}-g^B_{t,x}||^2_{\mathfrak{H}}=E^W\left|W(g^{\varepsilon,B}_{t,x})-W(g^B_{t,x})\right|^2$. Then it follows again from (3.53) and (3.54) that as $\varepsilon\downarrow 0$

$$\sup_{0\le t\le T,x\in K} E^W\,||Du^\varepsilon(s,x)-Du(s,x)||^2_{\mathfrak{H}}\to 0.$$

*Step 3*. From Equation (3.50) and (3.52), it follows that $\int_0^t\int_{\mathbb{R}^d} u^\varepsilon(s,x)\,\varphi(x)\dot{W}^\varepsilon(s,x)dsdx$ converges in $L^2$ to some random variable as $\varepsilon\downarrow 0$. Hence, if

$$V_\varepsilon := \int_0^t\int_{\mathbb{R}^d}(u^\varepsilon(s,x)-u(s,x))\varphi(x)\dot{W}^\varepsilon(s,x)dsdx. \tag{3.55}$$

converges to zero in $L^2$, then

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\mathbb{R}^d} u(s,x)\,\varphi(x)\dot{W}^\varepsilon(s,x)dsdx=\lim_{\varepsilon\to 0}\int_0^t\int_{\mathbb{R}^d} u^\varepsilon(s,x)\varphi(x)\dot{W}^\varepsilon(s,x)dsdx,$$

that is, $u(s,x)\,\varphi(x)$ is Stratonovich integrable and $u(s,x)$ is a weak solution to Equation (3.1). Thus, it remains to show that $V_\varepsilon$ converges to zero in $L^2$.

In order to show the convergence to zero of (3.55) in $L^2$, first we write $\widetilde{u}^\varepsilon(s,x)W(g^\varepsilon_{s,x})$ as the sum of a divergence integral and a trace term (see (3.5))

$$\widetilde{u}^\varepsilon(s,x)W(g^\varepsilon_{s,x})=\delta(\widetilde{u}^\varepsilon(s,x)g^\varepsilon_{s,x})-\left\langle D\widetilde{u}^\varepsilon(s,x),g^\varepsilon_{s,x}\right\rangle_{\mathfrak{H}}.$$

Then we have

$$\begin{aligned} V_\varepsilon &= \int_0^t\int_{\mathbb{R}^d}\widetilde{u}^\varepsilon(s,x)\varphi(x)W(g^\varepsilon_{s,x})dsdx \\ &= \int_0^t\int_{\mathbb{R}^d}\left(\delta(\widetilde{u}^\varepsilon(s,x)g^\varepsilon_{s,x})-\left\langle D\widetilde{u}^\varepsilon(s,x),g^\varepsilon_{s,x}\right\rangle_{\mathfrak{H}}\right)\varphi(x)dsdx \end{aligned}$$

$$= \quad \delta(\psi^{\varepsilon}) - \int_0^t \int_{\mathbb{R}^d} \left\langle D\widetilde{u}^{\varepsilon}(s,x), g^{\varepsilon}_{s,x} \right\rangle_{\mathfrak{H}} \varphi(x) ds dx =: V^1_{\varepsilon} - V^2_{\varepsilon},$$

where

$$\psi^{\varepsilon}(r,z) = \int_0^t \int_{\mathbb{R}^d} \widetilde{u}^{\varepsilon}(s,x) g^{\varepsilon}_{s,x}(r,z) \varphi(x) ds dx.$$

For the term $V^1_{\varepsilon}$, using the estimates on $L^2$ norm of the Skorohod integral (see (1.47) in [39]), we obtain

$$E\left[\left|V^1_{\varepsilon}\right|^2\right] \leq E\left[||\psi^{\varepsilon}||^2_{\mathfrak{H}}\right] + E\left[||D\psi^{\varepsilon}||^2_{\mathfrak{H}\otimes\mathfrak{H}}\right]. \tag{3.56}$$

Denoting $\operatorname{supp}(\varphi)$ the support of $\varphi$, we have

$$\begin{aligned}
& E\left[||\psi^{\varepsilon}||^2_{\mathfrak{H}}\right] \\
= \; & E\int_0^t \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{u}^{\varepsilon}(s_1,x_1)\widetilde{u}^{\varepsilon}(s_2,x_2) \left\langle g^{\varepsilon}_{s_1,x_1}, g^{\varepsilon}_{s_2,x_2} \right\rangle_{\mathfrak{H}} \varphi(x_1)\varphi(x_2) ds_1 ds_2 dx_1 dx_2 \\
\leq \; & M_1 \int_0^t \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left\langle g^{\varepsilon}_{s_1,x_1}, g^{\varepsilon}_{s_2,x_2} \right\rangle_{\mathfrak{H}} \varphi(x_1)\varphi(x_2) ds_1 ds_2 dx_1 dx_2 \\
= \; & M_1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E^W \left[W^{\varepsilon}(t,x_1) W^{\varepsilon}(t,x_2)\right] \varphi(x_1)\varphi(x_2) dx_1 dx_2,
\end{aligned}$$

where $M_1 := \sup_{s\in[0,T], x\in \operatorname{supp}(\varphi)} E\left[|\widetilde{u}^{\varepsilon}(s,x)|^2\right]$. Note that

$$\begin{aligned}
& \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E^W \left[W^{\varepsilon}(t,x_1) W^{\varepsilon}(t,x_2)\right] \varphi(x_1)\varphi(x_2) dx_1 dx_2 \\
& = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E^W \left[W(t,x_1) W(t,x_2)\right] \varphi(x_1)\varphi(x_2) dx_1 dx_2 \\
& = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} t^{2H} Q(x_1,x_2) \varphi(x_1)\varphi(x_2) dx_1 dx_2 < \infty.
\end{aligned} \tag{3.57}$$

Thus by (3.52), we get $E\left[||\psi^{\varepsilon}||^2_{\mathfrak{H}}\right] \to 0$ as $\varepsilon \downarrow 0$.

On the other hand, setting $M_2 := \sup_{s\in[0,T], x\in \operatorname{supp}(\varphi)} E\left[||D\widetilde{u}^{\varepsilon}(s,x)||^2_{\mathfrak{H}}\right]$, we have

$$E\left[||D\psi^{\varepsilon}||^2_{\mathfrak{H}\otimes\mathfrak{H}}\right]$$

$$
\begin{aligned}
&= E\int_0^t\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left\langle D\widetilde{u}^\varepsilon(s_1,x_1)\otimes g^\varepsilon_{s_1,x_1}, D\widetilde{u}^\varepsilon(s_2,x_2)\otimes g^\varepsilon_{s_2,x_2}\right\rangle_{\mathfrak{H}}\varphi(x_1)\varphi(x_2)d\vec{s}d\vec{x} \\
&= E\int_0^t\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\langle D\widetilde{u}^\varepsilon(s_1,x_1), D\widetilde{u}^\varepsilon(s_2,x_2)\rangle_{\mathfrak{H}}\left\langle g^\varepsilon_{s_1,x_1}, g^\varepsilon_{s_2,x_2}\right\rangle_{\mathfrak{H}}\varphi(x_1)\varphi(x_2)d\vec{s}d\vec{x} \\
&\leq M_2\int_0^t\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left\langle g^\varepsilon_{s_1,x_1}, g^\varepsilon_{s_2,x_2}\right\rangle_{\mathfrak{H}}\varphi(x_1)\varphi(x_2)ds_1ds_2dx_1dx_2.
\end{aligned}
$$

Then (3.52) and (3.57) imply that $E\left[||D\psi^\varepsilon||^2_{\mathfrak{H}\otimes\mathfrak{H}}\right]$ converges to zero as $\varepsilon\downarrow 0$.

Finally, we deal with the trace term

$$
\begin{aligned}
V^2_\varepsilon &= \int_0^t\int_{\mathbb{R}^d}\left(\left\langle Du^\varepsilon(s,x), g^\varepsilon_{s,x}\right\rangle_{\mathfrak{H}} - \left\langle Du(s,x), g^\varepsilon_{s,x}\right\rangle_{\mathfrak{H}}\right)\varphi(x)dsdx \qquad (3.58) \\
&=: T^\varepsilon_1 - T^\varepsilon_2,
\end{aligned}
$$

where

$$
\begin{aligned}
T^\varepsilon_1 &= \int_0^t\int_{\mathbb{R}^d}\left\langle Du^\varepsilon(s,x), g^\varepsilon_{s,x}\right\rangle_{\mathfrak{H}}\varphi(x)dsdx, \\
T^\varepsilon_2 &= \int_0^t\int_{\mathbb{R}^d}\left\langle Du(s,x), g^\varepsilon_{s,x}\right\rangle_{\mathfrak{H}}\varphi(x)dsdx.
\end{aligned}
$$

We will show that $T^\varepsilon_1$ and $T^\varepsilon_2$ converge to the same random variable as $\varepsilon\downarrow 0$.

We start with the term $T^\varepsilon_2$. Note that

$$
\begin{aligned}
\left\langle g^B_{s,x}, g^\varepsilon_{s,x}\right\rangle &= \left\langle \mathbf{1}_{[0,s]}(r)\mathbf{1}_{[0,B^x_{s-r}]}(z), \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,\ s+\varepsilon]}(r)\mathbf{1}_{[0,x]}(z)\right\rangle_{\mathfrak{H}} \\
&= \left\langle \mathbf{1}_{[0,s]}(r)Q\left(B^x_{s-r},x\right), \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,\ s+\varepsilon]}(r)\right\rangle_{\mathfrak{H}}.
\end{aligned}
$$

Since $Q\left(B^x_{s-\cdot},x\right)\in C^{\frac{1}{2}-\delta}\left([0,T]\right)$ for any $0<\delta<\frac{1}{2}$, noticing that $H>\frac{1}{2}-\frac{\gamma}{4}$ and applying Lemma 3.20 we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}T_2^{\varepsilon} &= E^B\int_0^t\int_{\mathbb{R}^d}u_0\left(B_s^x\right)\exp\left(W\left(g^B_{s,x}\right)\right)\left\langle g^B_{s,x},g^{\varepsilon}_{s,x}\right\rangle_{\mathfrak{H}}\varphi(x)dsdx\\ &= E^B\int_0^t\int_{\mathbb{R}^d}u_0\left(B_s^x\right)\exp\left(W\left(g^B_{s,x}\right)\right)\varphi(x)[Q\left(x,x\right)Hs^{2H-1}\\ &\quad +H\left(2H-1\right)\int_0^s\left(Q\left(B^x_{s-r},x\right)-Q\left(x,x\right)\right)r^{2H-2}dr]dsdx.\end{aligned}\tag{3.59}$$

On the other hand, for the term $T_1^{\varepsilon}$, note that

$$\begin{aligned}\left\langle g^{\varepsilon,B}_{s,x},g^{\varepsilon}_{s,x}\right\rangle &= \left\langle\int_0^{2\varepsilon}\frac{1}{2\varepsilon}\mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]}(r)\mathbf{1}_{[0,B^x_{s-\theta}]}(z)d\theta,\frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,\ s+\varepsilon]}(r)\mathbf{1}_{[0,x]}(z)\right\rangle_{\mathfrak{H}}\\ &= \left\langle\int_0^{2\varepsilon}\frac{1}{2\varepsilon}\mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]}(r)Q\left(B^x_{s-\theta},x\right)d\theta,\frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,\ s+\varepsilon]}(r)\right\rangle_{\mathfrak{H}}.\end{aligned}$$

Applying Lemma 3.21, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}T_1^{\varepsilon} &= E^B\int_0^t\int_{\mathbb{R}^d}u_0\left(B_s^x\right)\exp\left(W\left(g^{\varepsilon,B}_{s,x}\right)\right)\left\langle g^{\varepsilon,B}_{s,x},g^{\varepsilon}_{s,x}\right\rangle_{\mathfrak{H}}\varphi(x)dsdx\\ &= E^B\int_0^t\int_{\mathbb{R}^d}u_0\left(B_s^x\right)\exp\left(W\left(g^B_{s,x}\right)\right)\varphi(x)[Q\left(x,x\right)Hs^{2H-1}\\ &\quad +H\left(2H-1\right)\int_0^s\left(Q\left(B^x_{s-r},x\right)-Q\left(x,x\right)\right)r^{2H-2}dr]dsdx.\end{aligned}\tag{3.60}$$

The convergence in $L^2$ to zero of $V_{\varepsilon}^2$ follows from (3.60) and (3.59). □

## 3.6 Skorohod type equation and Chaos expansion

In this section, we consider the following heat equation on $\mathbb{R}^d$

$$\begin{cases}\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+u\diamond\frac{\partial W}{\partial t}(t,x), \quad t\geq 0,\ x\in\mathbb{R}^d,\\ u(0,x)=u_0(x).\end{cases}\tag{3.61}$$

The difference between the above equation and Equation (3.1) is that here we use the Wick product $\diamond$. This equation is studied in Hu and Nualart [15] for the case $H_1 = \cdots = H_d = \frac{1}{2}$, and in [17] for the case $H_1, \ldots, H_d \in (\frac{1}{2}, 1)$, $2H_0 + H_1 + \cdots + H_d > d+1$. As in that paper, we can define the following notion of solution.

**Definition 3.14.** *An adapted random field $u = \{u(t,x), t \geq 0,\ x \in \mathbb{R}^d\}$ with $Eu^2(t,x) < \infty$ for all $(t,x)$ is a (mild) solution to Equation (3.61) if for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$, the process $\left\{p_{t-s}(x-y)\, u(s,y)\, \mathbf{1}_{[0,t]}(s), s \geq 0,\ y \in \mathbb{R}^d\right\}$ is Skorohod integrable, and the following equation holds*

$$u(t,x) = p_t f(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\, u(s,y)\, \delta W_{s,y}, \tag{3.62}$$

*where $p_t(x)$ denotes the heat kernel and $p_t f(x) = \int_{\mathbb{R}^d} p_t(x-y) f(y)\, dy$.*

From [15], we know that the solution to Equation (3.61) exists with an explicit Wiener chaos expansion if and only if the Wiener chaos expansion converges. Note that $g^B_{t,x}(r,z) := \mathbf{1}_{[0,t]}(r)\mathbf{1}_{[0,B^x_{t-r}]}(z) \in \mathfrak{H}$. Formally, we can write $g^B_{t,x}(r,z) = \delta\left(B^x_{t-r} - z\right)$ and we have

$$\int_0^t W\left(dr, B^x_{s-r}\right) = W\left(g^B_{t,x}\right) = \int_0^t \int_{\mathbb{R}^d} \delta\left(B^x_{t-r} - z\right) W(dr,z)\, dz.$$

Then in the same way as in Section 8 in [17] we can check that $u(t,x)$ given by (3.63) below has the suitable Wiener chaos expansion, which has to be convergent because $u(t,x)$ is square integrable. We state it as the following theorem.

**Theorem 3.15.** *Suppose $H > \frac{1}{2} - \frac{1}{4}\gamma$ and $u_0$ is a bounded measurable function. Then the unique (mild) solution to Equation (3.61) is given by the process*

$$u(t,x) = E^B\left[u_0(B^x_t)\exp(W\left(g^B_{t,x}\right) - \frac{1}{2}\,||g^B_{t,x}||^2_{\mathfrak{H}})\right]. \tag{3.63}$$

**Remark 3.16.** *We can also obtain a Feynman-Kac formula for the coefficients of the chaos expansion of the solution to Equation (3.1)*

$$u(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(h_n(t,x)),$$

*with*

$$h_n(t,x) = E^B \left[ u_0(B_t^x) g_{t,x}^B(r_1,z_1) \cdots g_{t,x}^B(r_n,z_n) \exp\left( \frac{1}{2} ||g_{t,x}^B||_{\mathfrak{H}}^2 \right) \right].$$

## 3.7 Appendix

In this section, we denote by $B^H = \{B_t^H, t \in \mathbb{R}\}$ a mean zero Gaussian process with covariance $E(B_t^H B_s^H) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$. Denote by $\mathscr{E}$ the space of all step functions on $[-T,T]$. On $\mathscr{E}$, we introduce the following scalar product $\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}_0} = R_H(t,s)$, where if $t<0$ we assume that $\mathbf{1}_{[0,t]} = -\mathbf{1}_{[t,0]}$. Let $\mathfrak{H}_0$ be the closure of $\mathscr{E}$ with respect to the above scalar product.

For $r>0$, $\varepsilon>0$ and $\beta>0$, let

$$f^{\varepsilon}(r) := \frac{1}{4\varepsilon^2}\left[2r^{\beta} - |r-2\varepsilon|^{\beta} - (r+2\varepsilon)^{\beta}\right].$$

It is easy to see that

$$\lim_{\varepsilon \downarrow 0} f^{\varepsilon}(r) = \beta(\beta-1) r^{\beta-2}. \tag{3.64}$$

**Lemma 3.17.** *For any $r>0$, $\varepsilon>0$ and $0<\beta<2$,*

$$|f^{\varepsilon}(r)| \leq 64 r^{\beta-2}. \tag{3.65}$$

*Proof.* If $0<r<4\varepsilon$, then $|r-2\varepsilon|^{\beta}<(2\varepsilon)^{\beta}$, $(r+2\varepsilon)^{\beta}<(6\varepsilon)^{\beta}$, and hence (noting that $\beta<2$)

$$|f^{\varepsilon}(r)| \leq 4^{\beta+1}\varepsilon^{\beta-2} \leq 64r^{\beta-2}.$$

On the other hand, if $r\geq 4\varepsilon$, then

$$\begin{aligned} r^{\beta}-|r-2\varepsilon|^{\beta} &= 2\varepsilon\beta\int_0^1 (r-2\lambda\varepsilon)^{\beta-1}\,d\lambda, \\ r^{\beta}-(r+2\varepsilon)^{\beta} &= -2\varepsilon\beta\int_0^1 (r+2\lambda\varepsilon)^{\beta-1}\,d\lambda, \end{aligned}$$

and hence

$$\begin{aligned} f^{\varepsilon}(r) &= \frac{1}{2\varepsilon}\beta\int_0^1 \left[(r-2\lambda\varepsilon)^{\beta-1}-(r+2\lambda\varepsilon)^{\beta-1}\right]d\lambda \\ &= 2\beta(\beta-1)\int_0^1\int_0^1 \lambda\,(r-2\lambda\varepsilon+4\mu\lambda\varepsilon)^{\beta-2}\,d\mu\, d\lambda. \end{aligned}$$

Therefore, using $\beta<2$ and $r\geq 4\varepsilon$ we obtain

$$|f^{\varepsilon}(r)| \leq 2\beta\,(r-2\varepsilon)^{\beta-2} \leq 4r^{\beta-2}\left(\frac{r-2\varepsilon}{r}\right)^{\beta-2} \leq 16r^{\beta-2}.$$

□

**Lemma 3.18.** *For any $s>0$, $0<\beta<1$ and $\phi\in C^{\alpha}([0,T])$ with $\alpha>1-\beta$,*

$$\lim_{\varepsilon\to 0}\int_0^s \phi(r)\,f^{\varepsilon}(r)\,dr = \phi(0)\,\beta s^{\beta-1}+\beta(\beta-1)\int_0^s(\phi(r)-\phi(0))\,r^{\beta-2}dr. \tag{3.66}$$

*Moreover,*

$$\left|\int_0^s \phi(r)\,f^{\varepsilon}(r)\,dr\right| \leq C(\beta,\alpha)\left(\|\phi\|_{\infty}s^{\beta-1}+\|\phi\|_{\alpha}s^{\alpha+\beta-1}\right). \tag{3.67}$$

*Proof.* The lemma follows easily from (3.68) and (3.65) if we rewrite

$$\int_0^s \phi(r) f^{\varepsilon}(r)\, dr = \phi(0) \int_0^s f^{\varepsilon}(r)\, dr + \int_0^s [\phi(r) - \phi(0)] f^{\varepsilon}(r)\, dr.$$

□

**Lemma 3.19.** *For any bounded function $\phi \in \mathfrak{H}_0$ and any $s,t \geq 0$, we have*

$$\left\langle \mathbf{1}_{[0,s]}\phi, \mathbf{1}_{[0,t]} \right\rangle_{\mathfrak{H}_0} = H \int_0^s \phi(r) \left[ r^{2H-1} + \operatorname{sign}(t-r)\, |t-r|^{2H-1} \right] dr. \tag{3.68}$$

*If $u < s < t$, we have*

$$\left\langle \mathbf{1}_{[0,s]}\phi, \mathbf{1}_{[u,t]} \right\rangle_{\mathfrak{H}_0} = H \int_0^s \phi(r) \left[ (t-r)^{2H-1} - \operatorname{sign}(u-r)\, |u-r|^{2H-1} \right] dr. \tag{3.69}$$

*Proof.* We only have to prove (3.68) since (3.69) follows easily. Without loss of generality, assume that $\phi = \Sigma_{i=1}^n a_i \mathbf{1}_{[t_{i-1},t_i]}$, where $0 = t_0 \leq t_1 \leq ... \leq t_n = s$. (If $t < s$, we assume that $t = t_i$ for some $0 < i < n$.) Then

$$\begin{aligned} \left\langle \mathbf{1}_{[0,s]}\phi, \mathbf{1}_{[0,t]} \right\rangle_{\mathfrak{H}_0} &= E\Sigma_{i=1}^n a_i \left( B_{t_i}^H - B_{t_{i-1}}^H \right) B_t^H \\ = & \Sigma_{i=1}^n a_i \frac{1}{2} \left( t_i^{2H} - t_{i-1}^{2H} + |t-t_{i-1}|^{2H} - |t-t_i|^{2H} \right) \\ = & H \int_0^s \phi(r) \left[ r^{2H-1} + \operatorname{sign}(t-r)\, |t-r|^{2H-1} \right] dr. \end{aligned}$$

□

Using Lemma 3.19 and similar arguments to those in the proof of Lemma 3.18, we can prove the following lemma.

**Lemma 3.20.** *For any $s > 0$, for any $\phi \in C^{\alpha}([0,T])$ with $\alpha > 1-2H$,*

$$\lim_{\varepsilon\to 0}\left\langle \mathbf{1}_{[0,s]}\phi, \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\right\rangle_{\mathfrak{H}_0} = \phi(s)Hs^{2H-1} + c_0\int_0^s (\phi(s-r)-\phi(s))\,r^{2H-2}dr,$$

*where $c_0 = H(2H-1)$. Moreover,*

$$\left|\left\langle \mathbf{1}_{[0,s]}\phi, \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\right\rangle_{\mathfrak{H}_0}\right| \le C(H,\alpha)\left(||\phi||_{\infty}s^{2H-1} + ||\phi||_{\alpha}s^{\alpha+2H-1}\right). \tag{3.70}$$

*Proof.* Applying Lemma 3.19 and making a substitution, we get

$$\begin{aligned}
&\left\langle \mathbf{1}_{[0,s]}\phi, \frac{1}{2\varepsilon}\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\right\rangle_{\mathfrak{H}_0} \\
= \ & \frac{H}{2\varepsilon}\int_0^s \phi(s-u)\left[(u+\varepsilon)^{2H-1} - \operatorname{sign}(u-\varepsilon)|u-\varepsilon|^{2H-1}\right]du \\
= \ & : H\phi(s)\int_0^s g^{\varepsilon}(u)\,du + H\int_0^s [\phi(s-u)-\phi(s)]\,g^{\varepsilon}(u)\,du,
\end{aligned}$$

where we let

$$g^{\varepsilon}(u) = \frac{1}{2\varepsilon}\left[(u+\varepsilon)^{2H-1} - \operatorname{sign}(u-\varepsilon)|u-\varepsilon|^{2H-1}\right].$$

If $0 < u < 2\varepsilon$, we have $|g^{\varepsilon}(u)| \le 16r^{2H-2}$. On the other hand, if $u > 2\varepsilon$,

$$\begin{aligned}
|g^{\varepsilon}(u)| &= \left|\frac{1}{2\varepsilon}\left[(u-\varepsilon)^{2H-1} - (u+\varepsilon)^{2H-1}\right]\right| \\
&= \frac{1}{2}(1-2H)\int_{-1}^{1}(u-\lambda\varepsilon)^{2H-2}d\lambda \le (1-2H)\,u^{2H-2}.
\end{aligned}$$

Then the lemma follows by noticing that $\lim_{\varepsilon\to 0} g^{\varepsilon}(u) = (2H-1)\,u^{2H-2}$. □

**Lemma 3.21.** *For any* $\phi \in C^{\alpha}([0,T])$ *with* $\alpha > 1-2H$, *for any* $s > 0$,

$$\begin{aligned} & \lim_{\varepsilon \to 0} \left\langle \frac{1}{2\varepsilon} \int_0^s \mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]} \phi(\theta) \, d\theta, \frac{1}{2\varepsilon} \mathbf{1}_{[s-\varepsilon,s+\varepsilon]} \right\rangle_{\mathfrak{H}_0} \\ = \;& \phi(s) H s^{2H-1} + H(2H-1) \int_0^s (\phi(s-r) - \phi(s)) \, r^{2H-2} dr. \end{aligned} \tag{3.71}$$

*Moreover,*

$$\begin{aligned} & \left| \left\langle \frac{1}{2\varepsilon} \int_0^s \mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]} \phi(\theta) \, d\theta, \frac{1}{2\varepsilon} \mathbf{1}_{[s-\varepsilon,s+\varepsilon]} \right\rangle_{\mathfrak{H}_0} \right| \\ \leq \;& C(H,\alpha) \left( ||\phi||_{\infty} H s^{2H-1} + ||\phi||_{\alpha} s^{\alpha+2H-1} \right). \end{aligned} \tag{3.72}$$

*Proof.* By Fubini's theorem and making a substitution, we have

$$\begin{aligned} & \left\langle \frac{1}{2\varepsilon} \int_0^s \mathbf{1}_{[\theta-\varepsilon,\theta+\varepsilon]} \phi(\theta) \, d\theta, \frac{1}{2\varepsilon} \mathbf{1}_{[s-\varepsilon,s+\varepsilon]} \right\rangle_{\mathfrak{H}_0} \\ = \;& \frac{1}{4\varepsilon^2} E \left[ \int_0^s \phi(\theta) \left( B^H_{\theta+\varepsilon} - B^H_{\theta-\varepsilon} \right) \left( B^H_{s+\varepsilon} - B^H_{s-\varepsilon} \right) d\theta \right] \\ = \;& \frac{1}{8\varepsilon^2} \int_0^s \phi(s-\theta) \left[ 2r^{2H} - |r-2\varepsilon|^{2H} - (r+2\varepsilon)^{2H} \right] dr. \end{aligned}$$

Then (3.72) and (3.73) follow from Lemma 3.18. □

# Chapter 4

# Hölder continuity of the solution for a class of nonlinear SPDE arising from one dimensional superprocesses

The Hölder continuity of the solution $X_t(x)$ to a nonlinear stochastic partial differential equation (see (4.2) below) arising from one dimensional super process is studied in this chapter. It is proved that the Hölder exponent in time variable is as close as to $1/4$, improving the result of $1/10$ in [25]. The method is to use the Malliavin calculus. The Hölder continuity in spatial variable $x$ of exponent $1/2$ is also obtained by using this new approach. This Hölder continuity result is sharp since the corresponding linear heat equation has the same Hölder continuity.

## 4.1 Introduction

Consider a system of particles indexed by multi-indexes $\alpha$ in a random environment whose motions are described by

$$x_\alpha(t) = x_\alpha + B^\alpha(t) + \int_0^t \int_{\mathbb{R}} h(y - x_\alpha(u)) W(du, dy), \tag{4.1}$$

where $h \in L^2(\mathbb{R})$, $(B^\alpha(t); t \geq 0)_\alpha$ are independent Brownian motions and $W$ is a Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}$ independent of $B^\alpha$. For more detail about this model, we refer to Wang ([53], [54]) and Dawson, Li and Wang [6]. Under some specifications for the branching mechanism and in the limiting situation, Dawson, Vaillancourt and Wang [7] obtained that the density of the branching particles satisfies the following stochastic partial differential equation (SPDE):

$$\begin{aligned} X_t(x) &= \mu(x) + \int_0^t \Delta X_u(x) dr - \int_0^t \int_{\mathbb{R}} \nabla_x \left(h(y-x) X_u(x)\right) W(du, dy) \\ &\quad + \int_0^t \sqrt{X_u(x)} \frac{V(du, dx)}{dx}, \end{aligned} \tag{4.2}$$

where $V$ is a Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}$ independent of $W$. The joint Hölder continuity of $(t,x) \longmapsto X_t(x)$ is left as an open problem in [7].

Let $H_2^k(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}); u^{(i)} \in L^2(\mathbb{R}) \text{ for } i = 1, 2, \ldots, k \right\}$, the Sobolev space with norm $\|h\|_{k,2}^2 = \sum_{i=0}^k \left\| h^{(k)} \right\|_{L^2(\mathbb{R})}^2$. In a recent paper, Li, Wang, Xiong and Zhou [25] proved that $X_t(x)$ is almost surely jointly Hölder continuous, under the condition that $h \in H_2^2(\mathbb{R})$ with $\|h\|_{1,2}^2 < 2$ and $X_0 = \mu \in H_2^1(\mathbb{R})$ is bounded. More precisely, they showed that for fixed $t$ its Hölder exponent in $x$ is in $(0, 1/2)$ and for fixed $x$ its Hölder exponent in $t$ is in $(0, 1/10)$. Comparing to the Hölder continuity for the stochastic heat equation which has the Hölder continuity of $1/4$ in time, it is conjectured that the Hölder continuity of $X_t(x)$ should also be $1/4$.

The aim of this chapter is to provide an affirmative answer to the above conjecture. Here is the main result.

**Theorem 4.1.** *Suppose that $h \in H_2^2(\mathbb{R})$ and $X_0 = \mu \in L^2(\mathbb{R})$ is bounded. Then the solution to $X_t(x)$ is jointly Hölder continuous with the Hölder exponent in $x$ in $(0, 1/2)$ and with the Hölder exponent in $t$ in $(0, 1/4)$. That is, for any $0 \leq s < t \leq T$, $x, y \in \mathbb{R}$ and*

$p \geq 1$, *there exists a constant* $C$ *depending only on* $p, T$, $\|h\|_{2,2}$ *and* $\|\mu\|_{L^2(\mathbb{R})}$ *such that*

$$E\left|X_t(y) - X_s(x)\right|^{2p} \leq C(1+t^{-p})(|x-y|^{p-\frac{1}{2}} + (t-s)^{\frac{p}{2}-\frac{1}{4}}). \tag{4.3}$$

Note that the term $t^{-p}$ in the right hand side of (4.3) implies that the Hölder norm of $X_t(x)$ blows up as $t \to 0$. This problem arises naturally since we only assume $X_0 = \mu \in L^2(\mathbb{R})$.

When $h = 0$ the equation (4.2) is reduced to the famous Dawson-Watanabe equation (process). The study on the joint Hölder continuity for this equation has been studied by Konno and Shiga [21] and Reimers [49]. The starting point is to interpret the equation (when $h = 0$) in mild form with the heat kernel associated with the Laplacian $\Delta$ in (4.2). Then the properties of the heat kernel (Gaussian density) can be fully used to analyze the Hölder continuity.

The straightforward extension of the mild solution concept and technique to general nonzero $h$ case in (4.2) meets a substantial difficulty. To overcome this difficulty, Li et al [25] replace the heat kernel by a random heat kernel associated with

$$\int_0^t \Delta X_u(x)dr - \int_0^t \int_{\mathbb{R}} \nabla_x (h(y-x)X_u(x))W(du,dy).$$

The random heat kernel is given by the conditional transition function of a typical particle in the system with $W$ given. To be more precise, consider the spatial motion of a typical particle in the system:

$$\xi_t = \xi_0 + B_t + \int_0^t \int_{\mathbb{R}} h(y-\xi_u)W(du,dy), \tag{4.4}$$

where $(B_t; t \geq 0)$ is a Brownian motion.

For $r \leq t$ and $x \in \mathbb{R}$, define the conditional (conditioned by $W$) transition probability by

$$P_t^{r,x,W}(\cdot) \equiv P^W(\xi_t \in \cdot | \xi_r = x). \tag{4.5}$$

Denote by $p^W(r,x;t,y)$ the density of $P_t^{r,x,W}(\cdot)$. It is proved that $X_t(y)$ has the following convolution representation:

$$\begin{aligned} X_t(y) &= \int_{\mathbb{R}} \mu(z) p^W(0,z;t,y)\,dz + \int_0^t \int_{\mathbb{R}} p^W(r,z;t,y) Z(dr,dz) \\ &\equiv X_{t,1}(y) + X_{t,2}(y), \end{aligned} \tag{4.6}$$

where $Z(dr,dz) = \sqrt{X_r(z)}V(dr,dz)$. Then they introduce a fractional integration by parts technique to obtain the Hölder continuity estimates, using Krylov's $L_p$ theory (cf. Krylov [20]) for linear SPDE.

In this chapter, we shall use the techniques from Malliavin calculus to obtain more precise estimates for the conditional transition function $p^W(r,x;t,y)$. This allows us to improve the Hölder continuity in the time variable for the solution $X_t(x)$.

The rest of the chapter is organized as follows: First we derive moment estimates for the conditional transition function in Section 4.2. Then we study the Hölder continuity in spatial and time variables of $X_t(x)$ in Section 4.3 and Section 4.4 respectively. The proof of Theorem 4.1 is concluded in Section 4.4.

Along the paper, we shall use the following notations; $\|\cdot\|_H$ denotes the norm on Hilbert space $H = L^2([0,T])$, $\|\cdot\|$ (and $\|\cdot\|_p$) denotes the norm on $L^2(\mathbb{R})$ (and on $L^p(\Omega)$). The expectation on $(\Omega, \mathscr{F}, P)$ is denoted by $E$ and the conditional expectation with respect to the process $W$ is denoted by $E^B$.

We denote by $C$ a generic positive constant depending only on $p$, $T$, $\|h\|_{2,2}$ and $\|\mu\|_{L^2(\mathbb{R})}$.

## 4.2 Moment estimates

In this section, we derive moment estimates for the derivatives of $\xi_t$ and the conditional transition function $p^W(r,x;t,y)$.

Recall that $\xi_t = \xi_t^{r,x}$ with initial value $\xi_r = x$ is given by

$$\xi_t = x + B_r^t + I_r^t(h)\,,\ \ 0 \le r < t \le T\,, \tag{4.7}$$

where we introduced the notations

$$B_r^t \equiv B_t - B_r,\ \text{and}\ I_r^t(h) \equiv \int_r^t \int_{\mathbb{R}} h(y-\xi_u)\,W(du,dy)\,. \tag{4.8}$$

Since $h \in H_2^2(\mathbb{R})$, by using the standard Picard iteration scheme, we can prove that such a solution $\xi_t$ to the stochastic differential equation (4.7) exists, and by a regularization argument of $h$ we can prove that $\xi_t \in \mathbb{D}^{2,2}$ (here the Malliavin derivative is with respect to $B$). Taking the Malliavin derivative $D_\theta$ with respect to $B$, we have

$$D_\theta \xi_t = \mathbf{1}_{[r,t]}(\theta)\left[1 - \int_\theta^t \int_{\mathbb{R}} h'(y-\xi_u)\,D_\theta \xi_u W(du,dy)\right]. \tag{4.9}$$

Note that

$$M_{\theta,t} := \int_\theta^t \int_{\mathbb{R}} h'(y-\xi_u)\,W(du,dy)$$

is a martingale with quadratic variation $\langle M \rangle_{\theta,t} = \|h'\|^2 (t-\theta)$ for $t > \theta$. Thus

$$D_\theta \xi_t = \mathbf{1}_{[r,t]}(\theta)\exp\left(M_{\theta,t} - \frac{1}{2}\left\|h'\right\|^2 (t-\theta)\right). \tag{4.10}$$

As a result, we have

$$D_\eta D_\theta \xi_t = \mathbf{1}_{[r,t]}(\theta) \exp\left(M_{\theta,t} - \frac{1}{2}\left\|h'\right\|^2 (t-\theta)\right) D_\eta M_{\theta,t} = D_\theta \xi_t \cdot D_\eta M_{\theta,t}, \tag{4.11}$$

where $D_\eta M_{\theta,t} = \mathbf{1}_{[\theta,t]}(\eta) \int_\eta^t \int_{\mathbb{R}} h''(y-\xi_u) D_\eta \xi_u W(du,dy)$.

The next lemma gives estimates for the moments of $D\xi_t$ and $D^2\xi_t$.

**Lemma 4.2.** *For any $0 \le r < t \le T$ and $p \ge 1$, we have*

$$\| \|D\xi_t\|_H \|_{2p} \le \exp\left((2p-1)\left\|h'\right\|^2 (t-r)\right)(t-r)^{\frac{1}{2}}, \tag{4.12}$$

$$\left\| \left\|D^2\xi_t\right\|_{H\otimes H} \right\|_{2p} \le C_p \left\|h''\right\| \exp\left((4p-1)\left\|h'\right\|^2 (t-r)\right)(t-r)^{\frac{3}{2}}, \tag{4.13}$$

*and for any $\gamma > 0$,*

$$E(\|D\xi_t\|_H^{-2\gamma}) \le \exp\left(\left(2\gamma^2+\gamma\right)\left\|h'\right\|^2 (t-r)\right)(t-r)^{-\gamma}. \tag{4.14}$$

*Proof.* Note that for any $p \ge 1$ and $r \le \theta < t$,

$$\begin{aligned} \|D_\theta \xi_t\|_{2p}^2 &= \left(E\exp\left[2p\left(M_{\theta,t} - \frac{1}{2}\left\|h'\right\|^2 (t-\theta)\right)\right]\right)^{\frac{1}{p}} \\ &= \exp\left((2p-1)\left\|h'\right\|^2 (t-\theta)\right). \end{aligned} \tag{4.15}$$

Then (4.12) follows from Minkowski's inequality and (4.15) since

$$\| \|D\xi_t\|_H \|_{2p}^2 = \left[E\left(\int_r^t |D_\theta \xi_t|^2 d\theta\right)^p\right]^{\frac{1}{p}} \le \int_r^t \|D_\theta \xi_t\|_{2p}^2 d\theta.$$

Applying the Burkholder-Davis-Gundy inequality we have for any $r \leq \theta \leq \eta < t$

$$\begin{aligned} \left\|D_\eta M_{\theta,t}\right\|_{2p}^2 &\leq C_p \left(E\left|\int_\eta^t \int_{\mathbb{R}} \left|h''(y-\xi_u) D_\eta \xi_u\right|^2 dydu\right|^p\right)^{\frac{1}{p}} \\ &\leq C_p \left\|h''\right\|^2 \int_\eta^t \left\|D_\eta \xi_u\right\|_{2p}^2 du. \end{aligned} \tag{4.16}$$

Combining (4.11), (4.15) and (4.16) yields for any $r \leq \theta \leq \eta < t$

$$\begin{aligned} \left\|D_\eta D_\theta \xi_t\right\|_{2p}^2 &= \left\|D_\theta \xi_t D_\eta M_{\theta,t}\right\|_{2p}^2 \leq \left\|D_\theta \xi_t\right\|_{4p}^2 \left\|D_\eta M_{\theta,t}\right\|_{4p}^2 \\ &\leq C_p \left\|h''\right\|^2 \exp\left(2(4p-1)\left\|h'\right\|^2(t-\theta)\right)(t-\eta). \end{aligned} \tag{4.17}$$

An application of Minkowski's inequality implies that

$$\left\|\left\|D^2\xi_t\right\|_{H\otimes H}\right\|_{2p}^2 \leq \int_r^t \int_r^t \left\|D_\eta D_\theta \xi_t\right\|_{2p}^2 d\theta d\eta.$$

This yields (4.13).

For the negative moments of $\|D\xi_t\|_H$, by Jensen's inequality we have

$$E\left(\|D\xi_t\|_H^{-2\gamma}\right) = E\left(\int_r^t |D_\theta \xi_t|^2 d\theta\right)^{-\gamma} \leq (t-r)^{-\gamma-1} \int_r^t E\left|D_\theta \xi_t\right|^{-2\gamma} d\theta.$$

Then, (4.14) follows immediately. □

The moment estimates of the Malliavin derivatives of the difference $\xi_t - \xi_s$ can also be obtained in a similar way. The next lemma gives these estimates.

**Lemma 4.3.** *For $0 \leq s < t \leq T$ and $p \geq 1$, we have*

$$\|\|D(\xi_t - \xi_s)\|_H\|_{2p} < C(t-s)^{\frac{1}{2}}, \tag{4.18}$$

*and*

$$\left\| \left\| D^2 (\xi_t - \xi_s) \right\|_{H\otimes H} \right\|_{2p} < C (t-s)^{\frac{3}{2}}. \tag{4.19}$$

*Proof.* Similar to (4.9), we have

$$\begin{aligned} D_\theta \xi_t &= D_\theta \xi_s + \mathbf{1}_{[s,t]}(\theta) - \int_{\theta\vee s}^t \int_{\mathbb{R}} h'(y-\xi_u) D_\theta \xi_u W(du,dy) \\ &= D_\theta \xi_s + \mathbf{1}_{[s,t]}(\theta) - I_\theta^t \left( h' D_\theta \xi_\cdot \right), \end{aligned} \tag{4.20}$$

where henceforth for any process $Y = (Y_t, 0 \le t \le T)$ and $f \in L^2(\mathbb{R})$, we denote

$$I_\theta^t (fY_\cdot) = \mathbf{1}_{[s,t]}(\theta) \int_\theta^t \int_{\mathbb{R}} f(y-\xi_u) Y_u W(du,dy).$$

Applying the Burkholder-Davis-Gundy inequality with (4.15), we obtain for $s \le \theta \le t$

$$\begin{aligned} \left\| I_\theta^t \left( h' D_\theta \xi_\cdot \right) \right\|_{2p}^2 &\le \left( E \left| \int_\theta^t \int_{\mathbb{R}} \left| h'(y-\xi_u) D_\theta \xi_u \right|^2 du dy \right|^p \right)^{\frac{1}{p}} \\ \le \quad & \left\| h' \right\|^2 \exp\Big( (2p-1) \left\| h' \right\|^2 (t-\theta) \Big) (t-\theta). \end{aligned} \tag{4.21}$$

Then (4.18) follows from (4.20) and (4.21) since

$$\begin{aligned} \left( E \left\| D\xi_t - D\xi_s \right\|_H^{2p} \right)^{\frac{1}{p}} &= \left[ E \left( \int_0^T \left| \mathbf{1}_{[s,t]}(\theta) + I_\theta^t \left( h' D_\theta \xi_\cdot \right) \right|^2 d\theta \right)^p \right]^{\frac{1}{p}} \\ \le \quad & \int_0^T \left( E \left| \mathbf{1}_{[s,t]}(\theta) + I_\theta^t \left( h' D_\theta \xi_\cdot \right) \right|^{2p} \right)^{\frac{1}{p}} d\theta \\ \le \quad & 2(t-s) + 2 \int_s^t \left( E \left| I_\theta^t \left( h' D_\theta \xi_\cdot \right) \right|^{2p} \right)^{\frac{1}{p}} d\theta \\ \le \quad & 2 \Big( 1 + \left\| h' \right\|^2 \exp\Big( (2p-1) \left\| h' \right\|^2 (t-s) \Big) \Big) (t-s). \end{aligned}$$

For moments of $D^2(\xi_t - \xi_s)$, from (4.20) we have

$$D^2_{\eta,\theta} (\xi_t - \xi_s) = -D_\eta I_\theta^t \left( h' D_\theta \xi_\cdot \right) = I_\eta^t \left( h'' D_\theta \xi_\cdot D_\eta \xi_\cdot \right) - I_\eta^t \left( h' D^2_{\eta,\theta} \xi_\cdot \right). \tag{4.22}$$

In a similar way as above we can get (4.19). $\square$

Next we derive some estimates for the density $p^W(r,x;t,y)$ of the conditional transition probability defined in (4.5). Denote

$$u_t \equiv \frac{D\xi_t}{\|D\xi_t\|_H^2}. \tag{4.23}$$

The next two lemmas give estimates of the divergence of $u_t$ and $u_t - u_s$, which are important to derive the moment estimates of $p^W(r,x;t,y)$.

**Lemma 4.4.** *For any $p \geq 1$ and $0 \leq r < t \leq T$, we have*

$$\|\delta(u_t)\|_p \leq C(t-r)^{-\frac{1}{2}}. \tag{4.24}$$

*Proof.* Using the estimate (2.9) we obtain

$$\begin{aligned}
\|\delta(u_t)\|_p &= (E|\delta(u_t)|^p)^{\frac{1}{p}} \leq \left[E\left(E^B|\delta(u_t)|^p\right)\right]^{\frac{1}{p}} \\
&\leq C_p\left(E\left[\left\|E^B u_t\right\|_H^p + \left(E^B\|Du_t\|_{H\otimes H}^p\right)\right]\right)^{\frac{1}{p}} \\
&\leq C_p\left(\|\|u_t\|_H\|_p + \left\|\|Du_t\|_{H\otimes H}\right\|_p\right).
\end{aligned}$$

We have

$$Du_t = \frac{D^2\xi_t}{\|D\xi_t\|_H^2} - 2\frac{\left\langle D^2\xi_t, D\xi_t\otimes D\xi_t\right\rangle_{H\otimes H}}{\|D\xi_t\|_H^4},$$

and consequently $\|Du_t\|_{H\otimes H} \leq \frac{3\left\|D^2\xi_t\right\|_{H\otimes H}}{\|D\xi_t\|_H^2}$. Hence, for any positive number $\alpha, \beta > 1$ such that $\frac{1}{a} + \frac{1}{\beta} = \frac{1}{p}$, applying (4.13) and (4.14) we obtain (4.24):

$$\begin{aligned}
\|\delta(u_t)\|_p &\leq C_p\left(\left\|\|D\xi_t\|_H^{-1}\right\|_p + 3\left\|D^2\xi_t\right\|_{L^\alpha(\Omega,H\otimes H)}\left\|\|D\xi_t\|_H^{-2}\right\|_\beta\right) \\
&\leq C\left(p,\left\|h'\right\|,\left\|h''\right\|,T\right)\left((t-r)^{-\frac{1}{2}} + (t-r)^{\frac{3}{2}}(t-r)^{-1}\right).
\end{aligned}$$

This proves the lemma. $\square$

**Lemma 4.5.** *For $p \geq 1$, and $0 \leq r < s < t \leq T$,*

$$\left\|\delta\left(u_t - u_s\right)\right\|_{2p} \leq C\left(t-s\right)^{\frac{1}{2}}\left(s-r\right)^{-\frac{1}{2}}\left(t-r\right)^{-\frac{1}{2}}. \tag{4.25}$$

*Proof.* Using (4.20) we can write

$$u_t - u_s = \frac{D\xi_t}{\|D\xi_t\|_H^2} - \frac{D\xi_s}{\|D\xi_s\|_H^2} = A_1 + A_2 + A_3,$$

where

$$A_1 = D\xi_s\left(\frac{1}{\|D\xi_s\|_H^2} - \frac{1}{\|D\xi_t\|_H^2}\right), A_2 = \frac{\mathbf{1}_{[s,t]}(\theta)}{\|D\xi_t\|_H^2}, A_3 = \frac{I_\theta^t\left(h' D_\theta \xi_\cdot\right)}{\|D\xi_t\|_H^2}.$$

As a consequence, we have

$$\left\|\delta\left(u_t - u_s\right)\right\|_{2p} \leq \sum_{i=1}^{3}\left\|\delta A_i\right\|_{2p}. \tag{4.26}$$

For simplicity we introduce the following notation

$$V_t \equiv \|D\xi_t\|_H, N_t \equiv \left\|D^2\xi_t\right\|_{H\otimes H}, Y_i = \left\|D^i\left(\xi_t - \xi_s\right)\right\|_{H^{\otimes i}}, i = 1, 2.$$

Note that

$$\|A_1\|_H = \frac{\langle D\xi_t - D\xi_s, D\xi_t + D\xi_s\rangle}{\|D\xi_s\|_H \|D\xi_t\|_H^2} \leq Y_1\left(V_t^{-2} + V_s^{-1}V_t^{-1}\right),$$

and

$$\begin{aligned} \|DA_1\|_{H\otimes H} &= \left\|D\left(\frac{D\xi_s\langle D\xi_t - D\xi_s, D\xi_t + D\xi_s\rangle}{\|D\xi_s\|_H^2\|D\xi_t\|_H^2}\right)\right\|_{H\otimes H} \\ &\leq Y_1 N_s\left(V_s^{-2}V_t^{-1} + V_s^{-1}V_t^{-2}\right) + Y_2\left(V_s^{-1}V_t^{-1} + V_t^{-2}\right) \end{aligned}$$

$$+Y_1\left(N_t+N_s\right)V_s^{-1}V_t^{-2}$$
$$+2Y_1\left[N_s\left(V_s^{-2}V_t^{-1}+V_s^{-1}V_t^{-2}\right)+N_t\left(V_t^{-3}+V_s^{-1}V_t^{-2}\right)\right].$$

As a consequence, applying (2.9) and Hölder's inequality we get

$$\begin{aligned}
&\left\|\delta\left(A_1\right)\right\|_{2p}\leq C\left(\left\|\left\|A_1\right\|_H\right\|_{2p}+\left\|\left\|DA_1\right\|_{H\otimes H}\right\|_{2p}\right)\\
\leq\ & C\left\|Y_1\right\|_{4p}\left(\left\|V_t^{-2}\right\|_{4p}+\left\|V_t^{-1}\right\|_{8p}\left\|V_s^{-1}\right\|_{8p}\right)\\
&+C\left\|Y_1\right\|_{8p}\left\|N_s\right\|_{8p}\left(\left\|V_t^{-1}\right\|_{8p}\left\|V_s^{-2}\right\|_{8p}+\left\|V_s^{-1}\right\|_{8p}\left\|V_t^{-2}\right\|_{8p}\right)\\
&+C\left\|Y_2\right\|_{4p}\left(\left\|V_t^{-1}\right\|_{8p}\left\|V_s^{-1}\right\|_{8p}+\left\|V_t^{-2}\right\|_{4p}\right)\\
&+C\left\|Y_1\right\|_{8p}\left(\left\|N_s\right\|_{8p}+\left\|N_t\right\|_{8p}\right)\left\|V_t^{-2}\right\|_{8p}\left\|V_s^{-1}\right\|_{8p}\\
&+2C\left\|Y_1\right\|_{8p}\left\|N_s\right\|_{8p}\left(\left\|V_t^{-1}\right\|_{8p}\left\|V_s^{-2}\right\|_{8p}+\left\|V_t^{-2}\right\|_{8p}\left\|V_s^{-1}\right\|_{8p}\right)\\
&+2C\left\|Y_1\right\|_{8p}\left\|N_t\right\|_{8p}\left(\left\|V_t^{-3}\right\|_{4p}+\left\|V_t^{-2}\right\|_{8p}\left\|V_s^{-1}\right\|_{8p}\right).
\end{aligned}$$

From Lemma 4.2 and Lemma 4.3 it follows that

$$\left\|\delta\left(A_1\right)\right\|_{2p}\leq C\left(t-s\right)^{\frac{1}{2}}\left(s-r\right)^{-\frac{1}{2}}\left(t-r\right)^{-\frac{1}{2}}. \tag{4.27}$$

Note that $\left\|A_2\right\|_H=\left\|\frac{\mathbf{1}_{[s,t]}(\theta)}{\|D\xi_t\|_H^2}\right\|_H=\|D\xi_t\|_H^{-2}\left(t-s\right)^{\frac{1}{2}}$ and

$$\left\|DA_2\right\|_{H\otimes H}\leq 2\left\|D\xi_t\right\|_H^{-3}\left\|D^2\xi_t\right\|_{H\otimes H}\left(t-s\right)^{\frac{1}{2}}.$$

Then, by (2.9), Hölder's inequality and Lemma 4.2 we see that

$$\begin{aligned}
&\left\|\delta\left(A_2\right)\right\|_{2p}\leq C\left(\left\|\left\|A_2\right\|_H\right\|_{2p}+\left\|DA_2\right\|_{2p}\right)\\
\leq\ & C\left(t-s\right)^{\frac{1}{2}}\left(\left\|V_t^{-2}\right\|_{2p}+\left\|D^2\xi_t\right\|_{4p}\left\|V_t^{-1}\right\|_{4p}\right)\\
\leq\ & 2C\left(t-s\right)^{\frac{1}{2}}\left(\left(t-r\right)^{-1}+1\right).
\end{aligned} \tag{4.28}$$

For the term $A_3$, we apply Minkowski's inequality and the Burkholder-Davis-Gundy inequality and use (4.21). Thus for any $p \geq 1$,

$$
\begin{aligned}
& \left\| \left\| I_\theta^t \left( h' D_\theta \xi_\cdot \right) \right\|_H \right\|_{2p} = \left( E \left| \int_s^t I_\theta^t (h' D_\theta \xi_\cdot)^2 d\theta \right|^p \right)^{\frac{1}{2p}} \\
\leq \; & C_p \left( \int_s^t \left\| I_\theta^t \left( h' D \xi_\cdot \right) \right\|_{2p}^2 d\theta \right)^{\frac{1}{2}} \\
\leq \; & C_p \left\| h' \right\| \exp\left( (2p-1) \left\| h' \right\|^2 (t-r) \right) (t-s)^{\frac{1}{2}} . \qquad (4.29)
\end{aligned}
$$

From (4.22) it follows that

$$
\begin{aligned}
\left\| DA_3 \right\|_{H \otimes H} \leq \; & \left\| D^2 (\xi_t - \xi_s) \right\|_{H \otimes H} \left\| D\xi_t \right\|_H^{-2} \\
& + 2 \left| \left| I_\theta^t \left( h' D \xi_\cdot \right) \right| \right|_H \left\| D^2 \xi_t \right\|_{H \otimes H} \left\| D\xi_t \right\|_H^{-3} .
\end{aligned}
$$

Combining this with (2.9), Hölder's inequality, Lemma 4.3 and (4.29) we deduce

$$
\begin{aligned}
& \left\| \delta(A_3) \right\|_{2p} \leq C_p \left( \left\| \left\| A_3 \right\|_H \right\|_{2p} + \left\| DA_3 \right\|_{2p} \right) \\
\leq \; & C_p \left\| \left\| I_s^t \left( h' D_\theta \xi_\cdot \right) \right\|_H \right\|_{2p} \left\| V_t^{-2} \right\|_{2p} + \left\| Y_2 \right\|_{4p} \left\| V_t^{-2} \right\|_{4p} \\
& + 2C_p \left\| \left\| I_s^t \left( h' D_\theta \xi_\cdot \right) \right\|_H \right\|_{4p} \left\| V_t^{-3} \right\|_{8p} \left\| N_t \right\|_{8p} \\
\leq \; & C (t-s)^{\frac{1}{2}} (t-r)^{-1} . \qquad (4.30)
\end{aligned}
$$

Substituting (4.27), (4.28) and (4.30) into (4.26) yields (4.25). □

Now we provide the moment estimates for the conditional transition probability density $p^W(r,x;t,y)$.

**Lemma 4.6.** *Let $c = 1 \vee \|h\|^2$. For any $0 \leq r < t \leq T$, $y \in \mathbb{R}$ and $p \geq 1$,*

$$
\left( E \left| p^W(r,x;t,y) \right|^{2p} \right)^{\frac{1}{2p}} \leq 2 \exp\left( -\frac{(x-y)^2}{64pc(t-r)} \right) \left\| \delta(u_t) \right\|_{4p} . \qquad (4.31)
$$

*Proof.* By Lemma 2.1 we can write

$$p^W(r,x;t,y) = E^B\left(\mathbf{1}_{\{\xi_t>y\}}\delta(u_t)\right) = E^B[\mathbf{1}_{\{B_r^t+I_r^t(h)>y-x\}}\delta(u_t)], \tag{4.32}$$

where $B_r^t$ and $I_r^t(h)$ are defined in (4.8). Then, (2.14) implies

$$\begin{aligned}
& \left(E\left|p^W(r,x;t,y)\right|^{2p}\right)^{\frac{1}{2p}} \\
\leq\ & \left(E\left[\left(P^B\left(\left|B_r^t+I_r^t(h)\right|>|y-x|\right)\right)^p\left(E^B|\delta(u_t)|^2\right)^p\right]\right)^{\frac{1}{2p}} \\
\leq\ & \|\delta(u_t)\|_{4p}\left(E\left(P^B\left(\left|B_r^t+I_r^t(h)\right|>|y-x|\right)\right)^{2p}\right)^{\frac{1}{4p}}.
\end{aligned} \tag{4.33}$$

Applying Chebyshev and Jensen's inequalities, we have for $p \geq 1$,

$$\begin{aligned}
& E\left|P^B\left(\left|B_r^t+I_r^t(h)\right|>|y-x|\right)\right|^{2p} \\
\leq\ & \exp\left(\frac{-2p(x-y)^2}{32pc(t-r)}\right)E\left|E^B\exp\frac{(B_r^t+I_r^t(h))^2}{32pc(t-r)}\right|^{2p} \\
\leq\ & \exp\left(\frac{-(x-y)^2}{16c(t-r)}\right)E\exp\frac{(B_r^t+I_r^t(h))^2}{16c(t-r)}.
\end{aligned} \tag{4.34}$$

Using the fact that for $0 \leq \nu < 1/8$ and Gaussian random variables $X, Y$,

$$Ee^{\nu(X+Y)^2} \leq Ee^{2\nu\left(X^2+Y^2\right)} \leq \left(Ee^{4\nu X^2}\right)^{\frac{1}{2}}\left(Ee^{4\nu Y^2}\right)^{\frac{1}{2}} = (1-8\nu)^{-\frac{1}{2}},$$

and noticing that $B_r^t$ and $I_r^t(h)$ are Gaussian, we have

$$E\exp\frac{(B_r^t+I_r^t(h))^2}{16c(t-r)} \leq \left(1-\frac{1}{2c}\right)^{-\frac{1}{2}} \leq \sqrt{2}. \tag{4.35}$$

Combining (4.33)–(4.35), we get (4.31). □

## 4.3 Hölder continuity in spatial variable

In this section, we obtain the Hölder continuity of $X_t(y)$ with respect to $y$. More precisely, we show that for $t>0$ fixed, $X_t(y)$ is almost surely Hölder continuous in $y$ with any exponent in $(0,1/2)$. This result was proved in [25]. Here we provide a different proof based on Malliavin calculus. We continue to use the notations $B^t_r$, $I^t_r(h)$ (defined by (4.8)) and $u_t$ (defined by (4.23)).

**Proposition 4.7.** *Suppose that $h\in H^2_2(\mathbb{R})$ and $X_0=\mu\in L^2(\mathbb{R})$ is bounded. Then, for any $t\in(0,T]$, $\alpha\in(0,1)$ and $p>1$, there exists a constant $C$ depending only on $p,T$, $\|h\|_{2,2}$ and $\|\mu\|_{L^2(\mathbb{R})}$ such that*

$$E\left|X_t\left(y_2\right)-X_t\left(y_1\right)\right|^{2p}\le C(1+t^{-p})\left(y_2-y_1\right)^{\alpha p}. \tag{4.36}$$

*Proof.* We will use the convolution representation (4.6), where the two terms $X_{t,1}(y)$ and $X_{t,2}(y)$ will be estimated separately.

We start with $X_{t,2}(y)$. Suppose $y_1<y_2\in\mathbb{R}$. Note first that $\mathbf{1}_{\{\xi_t>y_1\}}-\mathbf{1}_{\{\xi_t>y_2\}}=\mathbf{1}_{\{y_1<\xi_t\le y_2\}}$ and

$$E^B\mathbf{1}_{\{y_1<\xi_t\le y_2\}}=P^B\left\{y_1<\xi_t\le y_2\right\}=\int_{y_1}^{y_2}p^W\left(r,x;t,z\right)dz.$$

Therefore by (4.32) we have

$$\begin{aligned}&\left|p^W\left(r,x;t,y_1\right)-p^W\left(r,x;t,y_2\right)\right|^2=\left|E^B\left[\mathbf{1}_{\{y_1<\xi_t<y_2\}}\delta\left(u_t\right)\right]\right|^2\\ \le\ & E^B\left|\delta\left(u_t\right)\right|^2\int_{y_1}^{y_2}p^W\left(r,x;t,z\right)dz.\end{aligned}$$

Hence,

$$
\begin{aligned}
& \left(E\left|p^W(r,x;t,y_1)-p^W(r,x;t,y_2)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}} \\
\leq\; & \|\delta(u_t)\|_{4(2p-1)}^2 \int_{y_1}^{y_2} \left\|p^W(r,x;t,z)\right\|_{2(2p-1)} dz. \qquad (4.37)
\end{aligned}
$$

Lemma 4.4 and Lemma 4.6 yield

$$
\begin{aligned}
& \int_{\mathbb{R}} \left(E\left|p^W(r,x;t,y_1)-p^W(r,x;t,y_2)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}} dx \\
\leq\; & C\int_{\mathbb{R}} \|\delta(u_t)\|_{4(2p-1)}^3 \int_{y_1}^{y_2} \exp\left(\frac{-(z-x)^2}{32(2p-1)c(t-r)}\right) dz dx \\
\leq\; & C(t-r)^{-1}(y_2-y_1). \qquad (4.38)
\end{aligned}
$$

On the other hand, the left hand side of (4.38) can be estimated differently again by using Lemma 4.6:

$$
\begin{aligned}
& \int_{\mathbb{R}} \left(E\left|p^W(r,x;t,y_1)-p^W(r,x;t,y_2)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}} dx \\
\leq\; & 2\int_{\mathbb{R}} \Sigma_{i=1,2} \left(E\left|p^W(r,x;t,y_i)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}} dx \\
\leq\; & C_p \int_{\mathbb{R}} \Sigma_{i=1,2} \|\delta(u_t)\|_{4(2p-1)}^2 \exp\left(\frac{-(y_i-x)^2}{64pc(t-r)}\right) dx \leq C(t-r)^{-\frac{1}{2}}. \qquad (4.39)
\end{aligned}
$$

Then (4.38) and (4.39) yield that for any $\alpha,\beta>0$ with $\alpha+\beta=1$

$$
\int_{\mathbb{R}} \left(E\left|p^W(r,x;t,y_1)-p^W(r,x;t,y_2)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}} dx \leq C(t-r)^{-\alpha-\frac{1}{2}\beta}(y_2-y_1)^{\alpha}. \qquad (4.40)
$$

Since $\mu$ is bounded, it follows from [25, Lemma 4.1] that

$$
E\left|\int_0^t \int_{\mathbb{R}} \left(p^W(r,x;t,y_2)-p^W(r,x;s,y_1)\right)^2 Z(drdx)\right|^{2p}
$$

$$\leq C\left(E\left|\int_0^t\int_{\mathbb{R}}\left(p^W(r,x;t,y_2)-p^W(r,x;s,y_1)\right)^2drdx\right|^{2p-1}\right)^{\frac{p}{2p-1}}, \quad (4.41)$$

for any $p\geq 1$, $0\leq s\leq t\leq T$ and $y_1,y_2\in\mathbb{R}$. Then, applying Minkowski's inequality we obtain for any $0<\alpha<1$,

$$\begin{aligned}&\left(E\left|X_{t,2}(y_2)-X_{t,2}(y_1)\right|^{2p}\right)^{\frac{1}{p}}\\ \leq\ &\int_0^t\int_{\mathbb{R}}\left(E\left|p^W(r,x;t,y_1)-p^W(r,x;t,y_2)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}}dxdr\\ \leq\ &C\int_0^t(t-r)^{-\alpha-\frac{1}{2}\beta}(y_2-y_1)^\alpha dr\leq C(y_2-y_1)^\alpha\end{aligned}$$

since $(t-r)^{-\alpha-\frac{1}{2}\beta}=(t-r)^{-(1+\alpha)/2}$ is integrable for all $0<\alpha<1$.

Now we consider $X_{t,1}(y)$ in (4.6). Applying Minkowski's inequality and using (4.37) with $2p-1$ replaced by $p$ we get

$$\begin{aligned}&E\left|X_{t,1}(y_2)-X_{t,1}(y_1)\right|^{2p}\\ \leq\ &\left(\int_{\mathbb{R}}\left(E\left|p(0,x;t,y_1)-p^W(0,x;t,y_2)\right|^{2p}\right)^{\frac{1}{2p}}\mu(x)dx\right)^{2p}\\ \leq\ &C\left\{\int_{\mathbb{R}}\left(\int_{y_1}^{y_2}\left\|p^W(0,x;t,z)\right\|_{2p}dz\right)^{1/2}\|\delta(u_t)\|_{4p}\mu(x)dx\right\}^{2p}\\ \leq\ &C\|\delta(u_t)\|_{4p}^{2p}\|\mu\|_{L^2(\mathbb{R})}^{2p}\left(\int_{\mathbb{R}}\int_{y_1}^{y_2}\exp\left(-\frac{(z-x)^2}{64pct}\right)dzdx\right)^p\\ \leq\ &C\|\mu\|_{L^2(\mathbb{R})}^{2p}t^{-p}(y_2-y_1)^p.\end{aligned}$$

This completes the proof. □

## 4.4 Hölder continuity in time variable

In this section we show that for any fixed $y \in \mathbb{R}$, $X_t(y)$ is Hölder continuous in $t$ with any exponent in $(0, 1/4)$.

**Proposition 4.8.** *Suppose that $h \in H_2^2(\mathbb{R})$ and $X_0$ has a bounded density $\mu \in L^2(\mathbb{R})$. Then, for any $p \geq 1$, $0 \leq s < t \leq T$ and $y \in \mathbb{R}$,*

$$E\left|X_t(y) - X_s(y)\right|^{2p} \leq C(1 + t^{-p})(t - s)^{\frac{p}{2} - \frac{1}{4}},$$

*where the constant $C$ depending only on $p, T$, $\|h\|_{2,2}$ and $\|\mu\|_{L^2(\mathbb{R})}$.*

We need some preparations to prove the above result.

Suppose $0 < s < t$. We start by estimating $X_{\cdot,2}(y)$ in (4.6) and we write

$$X_{t,2}(y) - X_{s,2}(y) = \int_0^s \int_{\mathbb{R}} \left(p^W(r,x;t,y) - p^W(r,x;s,y)\right) Z(drdx)$$
$$+ \int_s^t \int_{\mathbb{R}} p^W(r,x;t,y) Z(drdx). \tag{4.42}$$

We are going to estimate the two terms separately.

**Lemma 4.9.** *For any $0 \leq s < t \leq T$, $y \in \mathbb{R}$ and $p \geq 1$, we have*

$$E\left(\int_0^s \int_{\mathbb{R}} \left(p^W(r,x;t,y) - p^W(r,x;s,y)\right) Z(drdx)\right)^{2p} \leq C(t-s)^{\frac{p}{2} - \frac{1}{4}}. \tag{4.43}$$

*Proof.* From (4.32), we have for $0 < r < s < t \leq T$,

$$p^W(r,x;t,y) - p^W(r,x;s,y) = E^B\left[\mathbf{1}_{\{\xi_t > y\}} \delta(u_t) - \mathbf{1}_{\{\xi_s > y\}} \delta(u_s)\right]$$
$$= E^B\left[\left(1_{\{\xi_t > y\}} - \mathbf{1}_{\{\xi_s > y\}}\right) \delta(u_t) + \mathbf{1}_{\{\xi_s > y\}} \delta(u_t - u_s)\right],$$

Let $I_1 \equiv \left(1_{\{\xi_t>y\}} - \mathbf{1}_{\{\xi_s>y\}}\right)\delta(u_t)$ and $I_2 \equiv \mathbf{1}_{\{\xi_s>y\}}\delta(u_t-u_s)$. Then (4.41) implies

$$\begin{aligned}
& E\left(\int_0^s \int_{\mathbb{R}} \left(p^W(r,x;t,y) - p^W(r,x;s,y)\right) Z(drdx)\right)^{2p} \\
\leq\ & \left[E\left(\int_0^s \int_{\mathbb{R}} \left(E^B[I_1+I_2]\right)^2 drdx\right)^{2p-1}\right]^{\frac{p}{2p-1}} \\
\leq\ & C\sum_{i=1,2}\left[E\left(\int_0^s \int_{\mathbb{R}} \left(E^B I_i\right)^2 drdx\right)^{2p-1}\right]^{\frac{p}{2p-1}}. \qquad (4.44)
\end{aligned}$$

First, we study the term $I_1$. Note that

$$\left(1_{\{\xi_t>y\}} - \mathbf{1}_{\{\xi_s>y\}}\right)^2 = 1_{\{\xi_s\leq y<\xi_t\}} + \mathbf{1}_{\{\xi_t\leq y<\xi_s\}} =: A_1 + A_2.$$

Then we can write

$$\begin{aligned}
& \left[E\left(\int_0^s \int_{\mathbb{R}} E^B I_1^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}} \\
=\ & \left[E\left(\int_0^s \int_{\mathbb{R}} E^B\left[(A_1+A_2)\,\delta(u_t)\right]^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}} \\
\leq\ & 2\sum_{i=1,2}\left[E\left(\int_0^s \int_{\mathbb{R}} E^B\left[A_i\delta(u_t)\right]^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}}. \qquad (4.45)
\end{aligned}$$

Applying Minkowski, Jensen and Hölder's inequalities we deduce that for $i=1,2$ and for any conjugate pair $(p_1,q_1)$

$$\begin{aligned}
& \left[E\left(\int_0^s \int_{\mathbb{R}} E^B\left[A_i\delta(u_t)\right]^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}} \\
\leq\ & \int_0^s \left(E\left|\int_{\mathbb{R}} A_i\left|\delta(u_t)\right|^2 dx\right|^{2p-1}\right)^{\frac{1}{2p-1}} dr
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^s \left\| \left( \int_{\mathbb{R}} A_i dx \right)^{\frac{1}{p_1}} \left( \int_{\mathbb{R}} A_i \left| \delta\left(u_t\right) \right|^{2q_1} dx \right)^{\frac{1}{q_1}} \right\|_{2p-1} dr \\
&\le \int_0^s \left\| \left| \int_{\mathbb{R}} A_i dx \right|^{\frac{1}{p_1}} \right\|_{2(2p-1)} \left\| \left( \int_{\mathbb{R}} A_i \left| \delta\left(u_t\right) \right|^{2q_1} dx \right)^{\frac{1}{q_1}} \right\|_{2(2p-1)} dr. \qquad (4.46)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\{\xi_s \le y < \xi_t\} &= \left\{ y - B_r^t - I_r^t(h) < x \le y - B_r^s - I_r^s(h) \right\}, \\
\{\xi_t \le y < \xi_s\} &= \left\{ y - B_r^s - I_r^s(h) < x \le y - B_r^t - I_r^t(h) \right\}.
\end{aligned}
$$

Then, for $i = 1, 2$, we have

$$
\left| \int_{\mathbb{R}} A_i dx \right| = \left| B_s^t + I_s^t(h) \right|.
$$

Hence for $p_1 = 1 - \frac{1}{2p}$,

$$
\left\| \left| \int_{\mathbb{R}} A_i dx \right|^{\frac{1}{p_1}} \right\|_{2(2p-1)} \le C\left(t-s\right)^{\frac{1}{2}-\frac{1}{4p}}. \qquad (4.47)
$$

On the other hand, we have

$$
\begin{aligned}
\{\xi_s \le y < \xi_t\} &= \left\{ B_r^s + I_r^s(h) \le y - x < B_r^t + I_r^t(h) \right\} \\
&\subset \left\{ |x-y| \le \left| B_r^t + I_r^t(h) \right| + \left| B_r^s + I_r^s(h) \right| \right\}.
\end{aligned}
$$

Similarly

$$
\{\xi_t \le y < \xi_s\} \subset \left\{ |x-y| \le \left| B_r^t + I_r^t(h) \right| + \left| B_r^s + I_r^s(h) \right| \right\}.
$$

Applying Chebyshev's inequality and (4.35), we deduce that for $i = 1, 2$,

$$
E\left(A_i\right) \le E P^B \left\{ |x-y| \le \left| B_r^t + I_r^t(h) \right| + \left| B_r^s + I_r^s(h) \right| \right\}
$$

$$
\begin{aligned}
&\leq \exp\left(\frac{-(x-y)^2}{32c(t-r)}\right) E\exp\left(\frac{|B_r^t+I_r^t(h)|^2}{16c(t-r)}+\frac{|B_r^s+I_r^s(h)|^2}{16c(s-r)}\right)\\
&\leq 2\exp\left(-\frac{(x-y)^2}{32c(t-r)}\right). \qquad (4.48)
\end{aligned}
$$

Using Minkowski and Hölder's inequalities, from (4.48) and Lemma 4.4 we obtain that for $q_1 = 2p \leq 2(2p-1)$,

$$
\begin{aligned}
&\left\| \left(\int_{\mathbb{R}} A_i \left|\delta(u_t)\right|^{2q_1} dx\right)^{\frac{1}{q_1}} \right\|_{2(2p-1)} \leq \left(\int_{\mathbb{R}} \left\| A_i \left|\delta(u_t)\right|^{2q_1} \right\|_{\frac{2(2p-1)}{q_1}} dx\right)^{\frac{1}{q_1}}\\
&\leq \left(\int_{\mathbb{R}} (EA_i)^{\frac{q_1}{4(2p-1)}} \left\|\delta(u_t)\right\|_{8(2p-1)}^{2q_1} dx\right)^{\frac{1}{q_1}} \leq C(t-r)^{\frac{1}{4p}-1}. \qquad (4.49)
\end{aligned}
$$

Substituting (4.47) and (4.49) into (4.46) we obtain

$$
\begin{aligned}
&\left[E\left(\int_0^s \int_{\mathbb{R}} E^B \left[A_i \delta(u_t)\right]^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}}\\
&\leq C(t-s)^{\frac{1}{2}-\frac{1}{4p}} \int_0^s (t-r)^{\frac{1}{4p}-1} dr \leq C(t-s)^{\frac{1}{2}-\frac{1}{4p}}. \qquad (4.50)
\end{aligned}
$$

Combining (4.45) and (4.50), we have

$$
\left[E\left(\int_0^s \int_{\mathbb{R}} E^B I_1^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}} \leq C(t-s)^{\frac{1}{2}-\frac{1}{4p}}. \qquad (4.51)
$$

We turn into the term $I_2$. From Lemma 2.2 we can deduce as in Lemma 4.6 that

$$
\left(E\left(E^B I_2\right)^{2(2p-1)}\right)^{\frac{1}{2p-1}} \leq 2\exp\left(\frac{-(x-y)^2}{32(2p-1)c(s-r)}\right) \left\|\delta(u_t-u_s)\right\|_{4(2p-1)}^2.
$$

Then applying Minkowski's inequality and Lemma 4.5, we obtain

$$
\begin{aligned}
&\left[E\left(\int_0^s \int_{\mathbb{R}} \left(E^B I_2\right)^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}} \leq \int_0^s \int_{\mathbb{R}} \left(E\left(E^B I_2\right)^{2(2p-1)}\right)^{\frac{1}{2p-1}} drdx \\
\leq\ & 2\int_0^s \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{32\,(2p-1)\,(s-r)}\right) \|\delta\,(u_t-u_s)\|^2_{4(2p-1)}\, dxdr \\
\leq\ & C\,(t-s)\int_0^s (s-r)^{\frac{1}{2}-1}\,(t-r)^{-1}\, dr \leq C\,(t-s)^{\frac{1}{2}-\frac{1}{4p}}, \qquad (4.52)
\end{aligned}
$$

where in the last step we used that $(t-r)^{-1} \leq (t-s)^{-\frac{1}{2}-\varepsilon}\,(s-r)^{-\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$.

Substituting (4.51) and (4.52) in (4.44) we obtain (4.43). □

**Lemma 4.10.** *For any* $0 \leq s < t \leq T$ *and any* $y \in \mathbb{R}$ *and* $p \geq 1$*, we have*

$$
E\left(\int_s^t \int_{\mathbb{R}} p^W\,(r,x;t,y)\,Z\,(drdx)\right)^{2p} \leq C\,(t-s)^{\frac{p}{2}}. \qquad (4.53)
$$

*Proof.* Since $\mu$ is bounded, it follows from [25, Lemma 4.1] that

$$
E\left|\int_s^t \int_{\mathbb{R}} p^W\,(r,x;t,y)^2\, Z\,(drdx)\right|^{2p} \leq C\left(E\left|\int_s^t \int_{\mathbb{R}} p^W\,(r,x;t,y)^2\, drdx\right|^{2p-1}\right)^{\frac{p}{2p-1}}, \qquad (4.54)
$$

for any $p \geq 1$ and $y \in \mathbb{R}$. Applying Minkowski's inequality, Lemma 4.6 and Lemma 4.4 we obtain

$$
\begin{aligned}
&\left[E\left(\int_0^s \int_{\mathbb{R}} \left|p^W\,(r,x;t,y)\right|^2 drdx\right)^{2p-1}\right]^{\frac{1}{2p-1}} \\
\leq\ & C\int_s^t \int_{\mathbb{R}} \left(E\left|p^W\,(r,x;t,y)\right|^{2(2p-1)}\right)^{\frac{1}{2p-1}} drdx \\
\leq\ & C\int_s^t \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{32c\,(t-r)}\right) \|\delta\,(u_t)\|^2_{4(2p-1)}\, drdx \\
\leq\ & C\int_s^t (t-r)^{\frac{1}{2}-1}\, dr \leq C\,(t-s)^{\frac{1}{2}}.
\end{aligned}
$$

Then (4.53) follows immediately. □

In summary of the above two lemmas, we get

**Proposition 4.11.** *For any $p \geq 1$, $0 \leq s < t \leq T$ and $y \in \mathbb{R}$, we have*

$$E\left|X_{t,2}(y) - X_{s,2}(y)\right|^{2p} \leq C(t-s)^{\frac{p}{2}-\frac{1}{4}}.$$

Now we consider $X_{t,1}(y)$. Note that

$$E\left|X_{t,1}(y) - X_{s,1}(y)\right|^{2p} = E\left|\int_{\mathbb{R}} \left(p^W(0,z;t,y) - p^W(0,z;t,y)\right)\mu(z)dz\right|^{2p}$$
$$= E\left|\int_{\mathbb{R}} \left(E^B\left[\mathbf{1}_{\{\xi_t > y\}}\delta(u_t) - \mathbf{1}_{\{\xi_s > y\}}\delta(u_s)\right]\right)\mu(z)dz\right|^{2p}.$$

Then, similar to the proof for $X_{\cdot,2}(y)$ we get estimates for $X_{\cdot,1}(y)$.

**Proposition 4.12.** *For any $p \geq 1$, $0 \leq s < t \leq T$ and any $y \in \mathbb{R}$, we have*

$$E\left|X_{t,1}(y) - X_{s,1}(y)\right|^{2p} \leq C\left(1 + t^{-p}\right)(t-s)^{\frac{1}{2}p}. \tag{4.55}$$

*Proof.* Let $I_1 \equiv \left(1_{\{\xi_t > y\}} - \mathbf{1}_{\{\xi_s > y\}}\right)\delta(u_t)$ and $I_2 \equiv \mathbf{1}_{\{\xi_s > y\}}\delta(u_t - u_s)$. Then,

$$E\left|X_{t,1}(y) - X_{s,1}(y)\right|^{2p} = E\left|\int_{\mathbb{R}} \mu(x)E^B[I_1 + I_2]dx\right|^{2p}.$$

Noticing that $\left|1_{\{\xi_t > y\}} - \mathbf{1}_{\{\xi_s > y\}}\right| = 1_{\{\xi_s \leq y < \xi_t\}} + \mathbf{1}_{\{\xi_t \leq y < \xi_s\}} =: A_1 + A_2$, and applying Fubini's theorem, Jensen, Hölder and Minkowski's inequalities, we obtain

$$E\left|\int_{\mathbb{R}} \mu(x)E^B|I_1|dx\right|^{2p} \leq \sum_{i=1,2} E\left|\int_{\mathbb{R}} \mu(x)E^B[A_i\delta(u_t)]dx\right|^{2p}$$
$$\leq \sum_{i=1,2} E[\left(\int_{\mathbb{R}} |\mu(x)\delta(u_t)|^2 dx\right)^p \left|\int_{\mathbb{R}} A_i dx\right|^p]$$

$$
\begin{aligned}
&\leq \sum_{i=1,2} \left( \int_{\mathbb{R}} |\mu(x)|^2 \|\delta(u_t)\|_{4p}^2 \, dx \right)^p \left( E\left|B_s^t + I_s^t(h)\right|^{2p} \right)^{\frac{1}{2}} \\
&\leq C\left(1 + \|h\|^2\right) \|\mu\|_{L^2}^{2p} t^{-p} (t-s)^{\frac{1}{2}p}.
\end{aligned}
$$

For the term $I_2$, using Minkowski's inequality, (4.31) and (4.25) with $r = 0$ we have

$$
\begin{aligned}
&E\left|\int_{\mathbb{R}} \mu(x) E^B |I_2| \, dx\right|^{2p} = \left( \int_{\mathbb{R}} |\mu(x)| \left( E\left|E^B \mathbf{1}_{\{\xi_s > y\}} \delta(u_t - u_s)\right|^{2p} \right)^{\frac{1}{2p}} dx \right)^{2p} \\
&\leq C \|\mu\|_\infty^{2p} \left( \int_{\mathbb{R}} \exp\left( -\frac{(x-y)^2}{32cs} \right) \|\delta(u_t - u_s)\|_{4p} \, dx \right)^{2p} \\
&\leq C \|\mu\|_\infty^{2p} t^{-p} (t-s)^p.
\end{aligned}
$$

Then we can conclude (4.55). □

*Proof of Proposition 4.8.* It follows from Proposition 4.11 and Proposition 4.12. □

*Proof of Theorem 4.1.* It follows from Proposition 4.7 and Proposition 4.8. □

# Chapter 5

# Convergence of densities of some functionals of Gaussian processes

The aim of this chapter is to establish the uniform convergence of the densities of a sequence of random variables, which are functionals of an underlying Gaussian process, to a normal density. Precise estimates for the uniform distance are derived by using the techniques of Malliavin calculus, combined with Stein's method for normal approximation. We need to assume some non-degeneracy conditions. First, the study is focused on random variables in a fixed Wiener chaos, and later, the results are extended to the uniform convergence of the derivatives of the densities and to the case of random vectors in some fixed chaos, which are uniformly non-degenerate in the sense of Malliavin calculus. Explicit upper bounds for the uniform norm are obtained for random variables in the second Wiener chaos, and an application to the convergence of densities of the least square estimator for the drift parameter in Ornstein-Uhlenbeck processes is discussed.

## 5.1 Introduction

There has been a recent interest in studying normal approximations for sequences of multiple stochastic integrals. Consider a sequence of multiple stochastic integrals of order $q \geq 2$, $F_n = I_q(f_n)$, with variance $\sigma^2 > 0$, with respect to an isonormal Gaussian process $X = \{X(h), h \in \mathfrak{H}\}$ associated with a Hilbert space $\mathfrak{H}$. It was proved by Nualart and Peccati [41] and Nualart and Ortiz-Latorre [40] that $F_n$ converges in distribution to the normal law $N(0, \sigma^2)$ as $n \to \infty$ if and only if one of the following three equivalent conditions holds:

(i) $\lim_{n\to\infty} E[F_n^4] = 3\sigma^4$ (convergence of the fourth moments).

(ii) For all $1 \leq r \leq q-1$, $f_n \otimes_r f_n$ converges to zero, where $\otimes_r$denotes the contraction of order $r$ (see equation (2.5)).

(iii) $\|DF_n\|_{\mathfrak{H}}^2$ (see definition in Section 2) converges to $q\sigma^2$ in $L^2(\Omega)$ as $n$ tends to infinity.

A new methodology to study normal approximations and to derive quantitative results combining Stein's method with Malliavin calculus was introduced by Nourdin and Peccati [34] (see also Nourdin and Peccati [35]). As an illustration of the power of this method, let us mention the following estimate for the total variation distance between the law $\mathscr{L}(F)$ of $F = I_q(f)$ and distribution $\gamma = N(0, \sigma^2)$, where $\sigma^2 = E[F^2]$:

$$d_{TV}(\mathscr{L}(F), \gamma) \leq \frac{2}{q\sigma^2}\sqrt{\mathrm{Var}\left(\|DF\|_{\mathfrak{H}}^2\right)} \leq \frac{2\sqrt{q-1}}{\sigma^2\sqrt{3q}}\sqrt{E[F^4] - 3\sigma^4}.$$

This inequality can be used to show the above equivalence (i)-(iii). A recent result of Nourdin and Poly [38] says that the convergence in law for a sequence of multiple stochastic integrals of order $q \geq 2$ is equivalent to the convergence in total variation if the limit is not constant. As a consequence, for a sequence $F_n$ of nonzero multiple stochastic integrals of order $q \geq 2$, the limit in law to is equivalent to the limit of the densities in $L^1(\mathbb{R})$, provided the limit is not constant. A multivariate extension of this result has been derived in [33].

The aim of this paper is to study the uniform convergence of the densities of a sequence of random vectors $F_n$ to the normal density using the techniques of Malliavin calculus, combined with Stein's method for normal approximation. It is well-known that to guarantee that each $F_n$ has a density we need to assume that the norm of the Malliavin derivative of $F_n$ has negative moments. Thus, a natural assumption to obtain uniform convergence of densities is to assume uniform boundedness of the negative moments of the corresponding Malliavin derivatives. Our first result (Theorem 5.10) says that if $F$ is a multiple stochastic integral of order $q \geq 2$ such that $E[F^2] = \sigma^2$ and $M := E(\|DF\|_{\mathfrak{H}}^{-6}) < \infty$, we have

$$\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq C\sqrt{E[F^4] - 3\sigma^4}, \tag{5.1}$$

where $f_F$ is the density of $F$, $\phi$ is the density of the normal law $N(0, \sigma^2)$ and the constant $C$ depends on $q$, $\sigma$ and $M$. We can also replace the expression in the right-hand side of (5.1) by $\sqrt{\operatorname{Var}\left(\|DF\|_{\mathfrak{H}}^2\right)}$. The main idea to prove this result is to express the density of $F$ using Malliavin calculus:

$$f_F(x) = E[\mathbf{1}_{\{F>x\}} q\|DF\|_{\mathfrak{H}}^{-2} F] - E[\mathbf{1}_{\{F>x\}} \langle DF, D(\|DF\|_{\mathfrak{H}}^{-2}) \rangle_{\mathfrak{H}}].$$

Then, one can find an estimate of the form (5.1) for the terms $E[|\langle DF, D(\|DF\|_{\mathfrak{H}}^{-2}) \rangle_{\mathfrak{H}}|]$ and $E[|q\|DF\|_{\mathfrak{H}}^{-2} - \sigma^{-2}|]$. On the other hand, taking into account that

$$\phi(x) = \sigma^{-2} E[\mathbf{1}_{\{N>x\}} N],$$

it suffices to estimate the difference

$$E[\mathbf{1}_{\{F>x\}} F] - E[\mathbf{1}_{\{N>x\}} N],$$

which can be done by Stein's method. The estimate (5.1) leads to the uniform convergence of the densities in the above equivalence of conditions (i) to (iii) if we assume that $\sup_n E(\|DF_n\|_{\mathfrak{H}}^{-6}) < \infty$.

This methodology is extended in the paper in several directions. We consider the uniform approximation of the $m$th derivative of the density of $F$ by the corresponding densities $\phi^{(m)}$, in the case of random variables in a fixed chaos of order $q \geq 2$. In Theorem 5.13 we obtain an inequality similar to (5.1) assuming that $E(\|DF\|_{\mathfrak{H}}^{-\beta}) < \infty$ for some $\beta > 6m + 6\left(\lfloor \frac{m}{2} \rfloor \vee 1\right)$. Again the proof is obtained by a combination of Malliavin calculus and the Stein's method. Here we need to consider Stein's equation for functions of the form of $h(x) = \mathbf{1}_{\{x>a\}} p(x)$, where $p$ is a polynomial.

For a $d$ dimensional random vector $F = (F^1, \ldots, F^d)$ whose components are multiple stochastic integrals of orders $q_1, \ldots, q_d$, $q_i \geq 2$, we assume non degeneracy condition $E[\det \gamma_F^{-p}] < \infty$ for all $p \geq 1$, where $\gamma_F = (\langle DF, , DF \rangle)_{1 \leq i,j \leq d}$ denotes the Malliavin matrix of $F$. Then, for any multi-index $\beta = (\beta_1, \ldots, \beta_k)$, $1 \leq \beta_i \leq d$, we obtain the estimate (see Theorem 5.17)

$$\sup_{x \in \mathbb{R}^d} \left| \partial_\beta f_F(x) - \partial_\beta \phi(x) \right| \leq C \left( |V - I|^{\frac{1}{2}} + \sum_{j=1}^{d} \sqrt{E[F_j^4] - 3(E[F_j^2])^2} \right),$$

where $V$ is the covariance matrix of $F$, $\phi$ is the standard $d$ dimensional normal density, and $\partial_\beta = \frac{\partial^k}{\partial x_{\beta_1} \cdots \partial x_{\beta_k}}$. As a consequence, we derive the uniform convergence of the densities and their derivatives for a sequence of vectors of multiple stochastic integrals, under the the assumption $\sup_n E[\det \gamma_{F_n}^{-p}] < \infty$ for all $p \geq 1$. A multivariate extension of Stein's method is required for noncontinuous functions with polynomial growth (see Proposition 5.25). While univariate Stein's equations with nonsmooth test functions have been extensively studied, relatively few results are available for the multivariate case, see [5, 4, 28, 37, 48, 50], so this result has its own interest.

We also consider the case of random variables $F$ such that $E[F]=0$ and $E[F^2]=\sigma^2$, belonging to the Sobolev space $\mathbb{D}^{2,s}$ for some $s>4$. In this case, under a nondegeneracy assumption of the form $E[|\langle DF, -DFL^{-1}F\rangle_{\mathfrak{H}}|^{-r}|]<\infty$ for some $r>2$, we derive an estimate for the uniform distance between the density of $F$ and the density of the normal law $N(0,\sigma^2)$.

The chapter is organized as follows. Section 5.2 briefly introduces Stein's method for normal approximations. Section 5.3 is devoted to density formulae with elementary estimates using Malliavin calculus. The density formulae themselves are well-known results, but we present explicit formulae with useful estimates, such as the Hölder continuity and boundedness estimates in theorems 5.3 and 5.5. The boundedness estimates enable us to prove the $L^p$ convergence of the densities (see (5.43)). The Hölder continuity estimates can be used to provide a short proof for the convergence of densities based on a compactness argument, assuming convergence in law (see Theorem 5.31). Section 5.4 proves the convergence of densities of random variables in a fixed Wiener chaos, and Section 5.5 discusses convergence of densities for random vectors. In Section 5.6, the convergence of densities for sequences of general centered square integrable random variables are studied.

The main difficulty in the application of the above results is to verify the existence of negative moments for the determinant of the Malliavin matrix. We provide explicit sufficient conditions for this condition for random variables in the second Wiener chaos in Section 5.7. As an application we derive the uniform convergence of the densities and their derivatives to the normal distribution, as time goes to infinity, for the least squares estimator of the parameter $\theta$ in the Ornstein-Uhlenbeck process: $dX_t=-\theta X_t dt+\gamma dB_t$, where $B=\{B_t, t\geq 0\}$ is a standard Brownian motion. Some technical results and proofs are included in Section 5.8.

Along this paper, we denote by $C$ (maybe with subindices) a generic constant that might depend on quantities such as the order of multiple stochastic integrals $q$, the order of the derivatives $m$, the variance $\sigma^2$ or the negative moments of the Malliavin derivative. We denote by $\|\cdot\|_p$ the norm in the space $L^p(\Omega)$.

## 5.2 Stein's method of normal approximation

We shall now give a brief account of Stein's method of univariate normal approximation and its connection with Malliavin calculus. For a more detailed exposition we refer to [5, 35, 52].

Let $F$ be an arbitrary random variable and let $N$ be a $N(0,\sigma^2)$ distributed random variable, where $\sigma^2 > 0$. Consider the distance between the law of $F$ and the law of $N$ given by

$$d_{\mathscr{H}}(F,N) = \sup_{h\in\mathscr{H}} |E[h(F)-h(N)]| \tag{5.2}$$

for a class of functions $\mathscr{H}$ such that $E[h(F)]$ and $E[h(N)]$ are well-defined for $h \in \mathscr{H}$. Notice first the following fact (which is usually referred as *Stein's lemma*): a random variable $N$ is $N(0,\sigma^2)$ distributed if and only if $E[\sigma^2 f'(N) - Nf(N)] = 0$ for all absolutely continuous functions $f$ such that $E[|f'(N)|] < \infty$. This suggests that the distance of $E[\sigma^2 f'(F) - Ff(F)]$ from zero may quantify the distance between the law of $F$ and the law of $N$. To see this, for each function $h$ such that $E[|h(N)|] < \infty$, Stein [52] introduced the *Stein's equation*:

$$f'(x) - \frac{x}{\sigma^2} f(x) = h(x) - E[h(N)] \tag{5.3}$$

for all $x \in \mathbb{R}$. For a random variable $F$ such that $E[|h(F)|] < \infty$, any solution $f_h$ to Equation (5.3) verifies

$$\frac{1}{\sigma^2} E[\sigma^2 f_h'(F) - F f_h(F)] = E[h(F) - h(N)], \tag{5.4}$$

and the distance defined in (5.2) can be written as

$$d_{\mathscr{H}}(F,N) = \frac{1}{\sigma^2} \sup_{h \in \mathscr{H}} \left| E[\sigma^2 f_h'(F) - F f_h(F)] \right|. \tag{5.5}$$

The unique solution to (5.3) verifying $\lim_{x \to \pm\infty} e^{-x^2/(2\sigma^2)} f(x) = 0$ is

$$f_h(x) = e^{x^2/(2\sigma^2)} \int_{-\infty}^{x} \{h(y) - E[h(N)]\} e^{-y^2/(2\sigma^2)} dy. \tag{5.6}$$

From (5.5) and (5.6), one can get bounds for probability distances like the total variation distance, where we let $\mathscr{H}$ consist of all indicator functions of measurable sets, Kolmogorov distance, where we consider all the half-line indicator functions and Wasserstein distance, where we take $\mathscr{H}$ to be the set of all Lipschitz-continuous functions with Lipschitz-constant equal to 1.

In the present chapter, we shall consider the case when $h : \mathbb{R} \to \mathbb{R}$ is given by $h(x) = \mathbf{1}_{\{x>z\}} H_k(x)$ for any integer $k \geq 1$ and $z \in \mathbb{R}$, where $H_k(x)$ is the $k$th Hermite polynomial. More generally, we have the following lemma whose proof can be found in the Appendix.

**Lemma 5.1.** *Suppose $|h(x)| \leq a|x|^k + b$ for some integer $k \geq 0$ and some nonnegative numbers $a, b$. Then, the solution $f_h$ to the Stein's equation (5.3) given by (5.6) satisfies*

$$\left| f_h'(x) \right| \leq a C_k \sum_{i=0}^{k} \sigma^{k-i} |x|^i + 4b$$

*for all $x \in \mathbb{R}$, where $C_k$ is a constant depending only on $k$.*

Nourdin and Peccati [34, 35] combined Stein's method with Malliavin calculus to estimate the distance between the distributions of regular functionals of an isonormal Gaussian process and the normal distribution $N(0,\sigma^2)$. The basic ingredient is the following integration by parts formula. For $F \in \mathbb{D}^{1,2}$ with $E[F]=0$ and any function $f \in C^1$ such that $E[|f'(F)|] < \infty$, using (2.10) and (2.6) we have

$$\begin{aligned} E[Ff(F)] &= E[LL^{-1}Ff(F)] = E[-\delta DL^{-1}Ff(F)] \\ &= E[\langle -DL^{-1}F, Df(F)\rangle] = E[f'(F)\langle -DL^{-1}F, DF\rangle_{\mathfrak{H}}]. \end{aligned}$$

Then, it follows that

$$E[\sigma^2 f'(F) - Ff(F)] = E[f'(F)(\sigma^2 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}})]. \tag{5.7}$$

Combining Equation (5.7) with (5.4) and Lemma 5.1 we obtain the following result.

**Lemma 5.2.** *Suppose $h : \mathbb{R} \to \mathbb{R}$ verifies $|h(x)| \le a|x|^k + b$ for some $a, b \ge 0$ and some integer $k \ge 0$. Let $N \sim N(0,\sigma^2)$ and let $F \in \mathbb{D}^{1,2k}$ with $\|F\|_{2k} \le c\sigma$ for some $c > 0$. Then there exists a constant $C_{k,c}$ depending only on $k$ and $c$ such that*

$$|E[h(F) - h(N)]| \le \sigma^{-2}[aC_{k,c}\sigma^k + 4b] \left\| \sigma^2 - \langle DF, -DL^{-1}F\rangle_{\mathfrak{H}} \right\|_2.$$

*Proof.* From (5.4), (5.7) and Lemma (5.1), it suffices to notice that $\left\| \sum_{i=0}^k \sigma^{k-i} |F|^i \right\|_2 \le \sum_{i=0}^k \|F\|_{2k}^i \sigma^{k-i} \le C_{k,c}\sigma^k$. □

## 5.3 Density formulae

In this section, we present explicit formulae for the density of a random variable and its derivatives, using the techniques of Malliavin calculus.

### 5.3.1 Density formulae

We shall present two explicit formulae for the density of a random variable, with estimates of its uniform and Hölder norms.

**Theorem 5.3.** *Let $F \in \mathbb{D}^{2,s}$ such that $E[|F|^{2p}] < \infty$ and $E[\|DF\|_{\mathfrak{H}}^{-2r}] < \infty$ for $p,r,s > 1$ satisfying $\frac{1}{p}+\frac{1}{r}+\frac{1}{s}=1$. Denote*

$$w = \|DF\|_{\mathfrak{H}}^2,\ u = w^{-1}DF.$$

*Then $u \in \mathbb{D}^{1,p'}$ with $p' = \frac{p}{p-1}$ and $F$ has a density given by*

$$f_F(x) = E\left[\mathbf{1}_{\{F>x\}}\delta(u)\right]. \tag{5.8}$$

*Furthermore, $f_F(x)$ is bounded and Hölder continuous of order $\frac{1}{p}$, that is*

$$f_F(x) \le C_p \left\|w^{-1}\right\|_r \|F\|_{2,s}\left(1 \wedge (|x|^{-2}\|F\|_{2p}^2)\right), \tag{5.9}$$

$$|f_F(x) - f_F(y)| \le C_p \left\|w^{-1}\right\|_r^{1+\frac{1}{p}} \|F\|_{2,s}^{1+\frac{1}{p}} |x-y|^{\frac{1}{p}} \tag{5.10}$$

*for any $x,y \in \mathbb{R}$, where $C_p$ is a constant depending only on $p$.*

*Proof.* Note that

$$Du = w^{-1}D^2F - 2w^{-2}\left(D^2F \otimes_1 DF\right) \otimes DF.$$

Applying Meyer's inequality (2.9) and Hölder's inequality we have

$$\begin{aligned}
\|\delta(u)\|_{p'} &\le C_p\|u\|_{1,p'} \le C_p(\|u\|_{p'} + \|Du\|_{p'}) \\
&\le C_p(\left\|w^{-1}\|DF\|_{\mathfrak{H}}\right\|_{p'} + 3\left\|w^{-1}\left\|D^2F\right\|_{\mathfrak{H}\otimes\mathfrak{H}}\right\|_{p'}) \\
&\le 3C_p\left\|w^{-1}\right\|_r(\|DF\|_s + \left\|D^2F\right\|_s).
\end{aligned} \tag{5.11}$$

Then $u \in \mathbb{D}^{1,p'}$ and the density formula (5.8) holds (see, for instance, Nualart [39, Proposition 2.1.1]). From $E[\delta(u)] = 0$ and Hölder's inequality it follows that

$$\left|E\left[\mathbf{1}_{\{F>x\}}\delta(u)\right]\right| \le P(|F| > |x|)^{\frac{1}{p}} \|\delta(u)\|_{p'} \le \left(1 \wedge (|x|^{-2p} \|F\|_{2p}^{2p})\right)^{\frac{1}{p}} \|\delta(u)\|_{p'} . \tag{5.12}$$

Then (5.9) follows from (5.12) and (5.11).

Finally, for $x < y \in \mathbb{R}$, noticing that $\mathbf{1}_{\{F>x\}} - \mathbf{1}_{\{F>y\}} = \mathbf{1}_{\{x<F\le y\}}$, we have

$$|f_F(x) - f_F(y)| \le \left(E[\mathbf{1}_{\{x<F\le y\}}]\right)^{\frac{1}{p}} \|\delta(u)\|_{p'} .$$

Applying (5.9) and (5.11) with the fact that $E[\mathbf{1}_{\{x<F\le y\}}] = \int_x^y f_F(z)\,dz$ one gets (5.10).

□

With the exact proof of [39, Propositions 2.1.1], one can prove the following slightly more general result.

**Proposition 5.4.** *Let $F \in \mathbb{D}^{1,p}$ and $h : \Omega \to \mathfrak{H}$, and suppose that $\langle DF, h\rangle_{\mathfrak{H}} \neq 0$ a.s. and $\frac{h}{\langle DF,h\rangle_{\mathfrak{H}}} \in \mathbb{D}^{1,q}(\mathfrak{H})$ for some $p, q > 1$. Then the law of $F$ has a density given by*

$$f_F(x) = E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{h}{\langle DF, h\rangle_{\mathfrak{H}}}\right)\right]. \tag{5.13}$$

Our next goal is to take $h$ to be $-DL^{-1}F$ in formula (5.13) and get a result similar to Theorem 5.3. First, to get a sufficient condition for $\frac{-DL^{-1}F}{\langle DF,-DL^{-1}F\rangle_{\mathfrak{H}}} \in \mathbb{D}^{1,p'}$ for some $p' > 1$, we need some technical estimates on $DL^{-1}F$ and $D^2L^{-1}F$. Estimates of this type have been obtained by Nourdin, Peccati and Reinert [36] (see also Nourdin and Peccati's book [35, Lemma 5.3.8]), when proving an infinite-dimensional Poincaré inequality. More precisely, by using Mehler's formula, they proved that for any $p \geq 1$, if $F \in \mathbb{D}^{2,p}$, then

$$E[\left\|DL^{-1}F\right\|_{\mathfrak{H}}^{p}] \leq E[\|DF\|_{\mathfrak{H}}^{p}]. \tag{5.14}$$

$$E[\left\|D^2L^{-1}F\right\|_{op}^{p}] \leq 2^{-p}E[\left\|D^2F\right\|_{op}^{p}], \tag{5.15}$$

where $\left\|D^2F\right\|_{op}$ denotes the operator norm of the Hilbert-Schmidt operator from $\mathfrak{H}$ to $\mathfrak{H} : f \mapsto f \otimes_1 D^2F$. Furthermore, the operator norm $\left\|D^2F\right\|_{op}$ satisfies the following "*random contraction inequality*"

$$\left\|D^2F\right\|_{op}^{4} \leq \left\|D^2F \otimes_1 D^2F\right\|_{\mathfrak{H}^{\otimes 2}}^{2} \leq \left\|D^2F\right\|_{\mathfrak{H}^{\otimes 2}}^{4}. \tag{5.16}$$

The next proposition gives a density formula with estimates similar to Theorem 5.3. Let

$$\bar{w} = \left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{H}}, \ \bar{u} = -\bar{w}^{-1}DL^{-1}F.$$

**Proposition 5.5.** *Let $F \in \mathbb{D}^{2,s}$, $E[|F|^{2p}] < \infty$ and suppose that $E[|\bar{w}|^{-r}] < \infty$, where $p > 1$, $r > 2$, $s > 3$ satisfy $\frac{1}{p} + \frac{2}{r} + \frac{3}{s} = 1$. Then $\bar{u} \in \mathbb{D}^{1,p'}$ with $p' = \frac{p}{p-1}$ and the law of $F$ has a density given by*

$$f_F(x) = E\left[\mathbf{1}_{\{F>x\}}\delta(\bar{u})\right]. \tag{5.17}$$

*Furthermore, $f_F(x)$ is bounded and Hölder continuous of order $\frac{1}{p}$, that is*

$$f_F(x) \leq K_0\left(1 \wedge (|x|^{-2}\|F\|_{2p}^{2})\right), \tag{5.18}$$

$$|f_F(x) - f_F(y)| \leq K_0^{1+\frac{1}{p}}|x-y|^{\frac{1}{p}} \tag{5.19}$$

*for any $x, y \in \mathbb{R}$, where $K_0 = C_p\left\|\bar{w}^{-1}\right\|_r\|F\|_{2,s}(\left\|\bar{w}^{-1}\right\|_r\|DF\|_s^2 + 1)$, and $C_p$ depends only on $p$.*

*Proof.* Note that $D\bar{w} = -D^2F \otimes_1 DL^{-1}F - DF \otimes_1 D^2L^{-1}F$. Then, applying (5.14) and (5.15) we obtain

$$\|D\bar{w}\|_{\frac{s}{2}} \le (1+2^{-s}) \left\| \|D^2F\|_{op} \right\|_s \|DF\|_s . \tag{5.20}$$

From $\bar{u} = -\bar{w}^{-1}DL^{-1}F$ we get $D\bar{u} = -\bar{w}^{-1}D^2L^{-1}F + w^{-2}D\bar{w} \otimes DL^{-1}F$. Then, using (5.14)–(5.16) we have for $t > 0$ satisfying $\frac{1}{p'} = \frac{1}{r} + \frac{1}{t}$,

$$\|\bar{u}\|_{p'} \le \left\| \bar{w}^{-1} \|DL^{-1}F\|_{\mathfrak{H}} \right\|_{p'} \le \left\|\bar{w}^{-1}\right\|_r \|DF\|_t ,$$

and

$$\begin{aligned} \|D\bar{u}\|_{p'} &\le \left\| \bar{w}^{-1} \left\|D^2L^{-1}F\right\|_{\mathfrak{H}\otimes\mathfrak{H}} \right\|_{p'} + \left\| \bar{w}^{-2} \|D\bar{w}\|_{\mathfrak{H}} \left\|DL^{-1}F\right\|_{\mathfrak{H}} \right\|_{p'} \\ &\le \left\|\bar{w}^{-1}\right\|_r \left\|D^2F\right\|_t + \left\|\bar{w}^{-2}\right\|_r \|D\bar{w}\|_{\frac{s}{2}} \|DF\|_s , . \end{aligned}$$

Noticing that $\left\|D^2F\right\|_t \le \left\|D^2F\right\|_s$ because $t < s$, and applying Meyer's inequality (2.9) with (5.20) and (5.16) we obtain

$$\|\delta(\bar{u})\|_{p'} \le C_p \|\bar{u}\|_{1,p'} \le K_0. \tag{5.21}$$

Then $u \in \mathbb{D}^{1,p'}$ and the density formula (5.17) holds. As in the proof of Theorem 5.3, (5.18) and (5.19) follow from (5.21) and

$$\left|E\left[\mathbf{1}_{\{F>x\}}\delta(\bar{u})\right]\right| \le P(|F| > |x|)^{\frac{1}{p}} \|\delta(\bar{u})\|_{p'} \le \left(1 \wedge (|x|^{-2} \|F\|_{2p}^2)\right) \|\delta(\bar{u})\|_{p'} ,$$

$$|f_F(x) - f_F(y)| \le \left(E[\mathbf{1}_{\{x<F\le y\}}]\right)^{\frac{1}{p}} \|\delta(u)\|_{p'} .$$

□

### 5.3.2 Derivatives of the density

Next we present a formula for the derivatives of the density function, under additional conditions. A sequence of recursively defined random variables given by $G_0 = 1$ and $G_{k+1} = \delta(G_k u)$ where $u$ is an $\mathfrak{H}$-valued process, plays an essential role in the formula. The following technical lemma gives an explicit formula for the sequence $G_k$, relating it to Hermite polynomials. To simplify the notation, for an $\mathfrak{H}$–valued random variable $u$, we denote

$$\delta_u = \delta(u),\; D_u G = \langle DG, u\rangle_{\mathfrak{H}},\; D_u^k G = \left\langle D\left(D_u^{k-1}G\right), u\right\rangle_{\mathfrak{H}}. \tag{5.22}$$

Recall $H_k(x)$ denotes the $k$th Hermite polynomial. For $\lambda > 0$ and $x \in \mathbb{R}$, we define the generalized $k$th Hermite polynomial as

$$H_k(\lambda, x) = \lambda^{\frac{k}{2}} H_k\left(\frac{x}{\sqrt{\lambda}}\right). \tag{5.23}$$

From the property $H_k'(x) = kH_{k-1}(x)$ it follows by induction that the $k$th Hermite polynomials has the form $H_k(x) = \sum_{0\le i\le \lfloor k/2\rfloor} c_{k,i}x^{k-2i}$, where we denote by $\lfloor k/2\rfloor$ the largest integer less than or equal to $k/2$. Then (5.23) implies

$$H_k(\lambda, x) = \sum_{0\le i\le \lfloor k/2\rfloor} c_{k,i}x^{k-2i}\lambda^i. \tag{5.24}$$

**Lemma 5.6.** *Fix an integer $m \ge 1$ and a number $p > m$. Suppose $u \in \mathbb{D}^{m,p}(\mathfrak{H})$. We define recursively a sequence $\{G_k\}_{k=0}^m$ by $G_0 = 1$ and $G_{k+1} = \delta(G_k u)$. Then, these variables are well-defined and for $k = 1, 2, \dots, m$, $G_k \in \mathbb{D}^{m-k,\frac{p}{k}}$ and*

$$G_k = H_k(D_u\delta_u, \delta_u) + T_k, \tag{5.25}$$

*where we denote by $T_k$ the higher order derivative terms which can be defined recursively as follows: $T_1 = T_2 = 0$ and for $k \geq 2$,*

$$T_{k+1} = \delta_u T_k - D_u T_k - \partial_\lambda H_k(D_u\delta_u, \delta_u) D_u^2 \delta_u. \tag{5.26}$$

The following remark is proved in the Appendix.

**Remark 5.7.** *From (5.26) we can deduce that for $k \geq 3$*

$$T_k = \sum_{(i_0,\dots,i_{k-1})\in J_k} a_{i_0,i_1,\dots,i_{k-1}} \delta_u^{i_0} (D_u\delta_u)^{i_1} \left(D_u^2\delta_u\right)^{i_2} \cdots \left(D_u^{k-1}\delta_u\right)^{i_{k-1}}, \tag{5.27}$$

*where the coefficients $a_{i_0,i_1,\dots,i_{k-1}}$ are real numbers and $J_k$ is the set of multi-indices $(i_0, i_1, \dots, i_{k-1}) \in \mathbb{N}^k$ satisfying the following three conditions*

$$(a)\ i_0 + \sum_{j=1}^{k-1} j i_j \leq k-1;\ (b)\ i_2 + \cdots + i_{k-1} \geq 1;\ (c)\ \sum_{j=1}^{k-1} i_j \leq \lfloor \frac{k-1}{2} \rfloor.$$

*From $(b)$ we see that every term in $T_k$ contains at least one factor of the form $D_u^j\delta_u$ with some $j \geq 2$. We shall show this type of factors will converge to zero. For this reason we call these terms high order terms.*

*Proof of Lemma 5.6.* First, we prove by induction on $k$ that the above sequence $G_k$ is well-defined and $G_k \in \mathbb{D}^{m-k,\frac{p}{k}}$. Suppose first that $k = 1$. Then, Meyer's inequality implies that $G_1 = \delta_u \in \mathbb{D}^{m-1,p}$. Assume now that for $k \leq m-1$, $G_k \in \mathbb{D}^{m-k,\frac{p}{k}}$. Then it follows from Meyer's and Hölder's inequalities (see [39, Proposition 1.5.6]) that

$$\|G_{k+1}\|_{m-k-1,\frac{p}{k+1}} \leq C_{m,p} \|G_k u\|_{m-k,\frac{p}{k+1}} \leq C'_{m,p} \|G_k\|_{m-k,\frac{p}{k}} \|u\|_{m-k,p} < \infty.$$

Let us now show, by induction, the decomposition (5.25). When $k = 1$ (5.25) is true because $G_1 = \delta_u$ and $T_1 = 0$. Assume now (5.25) holds for $k \leq m-1$. Noticing that

$\partial_x H_k(\lambda,x) = kH_{k-1}(\lambda,x)$ (since $H_k'(x) = kH_{k-1}(x)$), we get

$$D_u H_k(D_u\delta_u,\delta_u) = kH_{k-1}(D_u\delta_u,\delta_u)D_u\delta_u + \partial_\lambda H_k(D_u\delta_u,\delta_u)D_u^2\delta_u.$$

Hence, applying the operator $D_u$ to both sides of (5.25),

$$D_u G_k = kH_{k-1}(D_u\delta_u,\delta_u)D_u\delta_u + \widetilde{T}_{k+1},$$

where

$$\widetilde{T}_{k+1} = D_u T_k + \partial_\lambda H_k(D_u\delta_u,\delta_u)D_u^2\delta_u. \tag{5.28}$$

From the definition of $G_{k+1}$ and using (2.7) we obtain

$$\begin{aligned} G_{k+1} &= \delta(uG_k) = G_k\delta_u - D_u G_k \\ &= \delta_u H_k(D_u\delta_u,\delta_u) + \delta_u T_k - kH_{k-1}(D_u\delta_u,\delta_u)D_u\delta_u - \widetilde{T}_{k+1}. \end{aligned}$$

Note that $H_{k+1}(x) = xH_k(x) - kH_{k-1}(x)$ implies $xH_k(\lambda,x) - k\lambda H_{k-1}(\lambda,x) = H_{k+1}(\lambda,x)$. Hence,

$$G_{k+1} = H_{k+1}(D_u\delta_u,\delta_u) + \delta_u T_k - \widetilde{T}_{k+1}.$$

The term $T_{k+1} = \delta_u T_k - \widetilde{T}_{k+1}$ has the form given in (5.26). This completes the proof. □

Now we are ready to present some formulae for the derivatives of the density function under certain sufficient conditions on the random variable $F$. For a random variable $F$ in $\mathbb{D}^{1,2}$ and for any $\beta \geq 1$ we are going to use the notation

$$M_\beta(F) = \left(E[\|DF\|_{\mathfrak{H}}^{-\beta}]\right)^{\frac{1}{\beta}}. \tag{5.29}$$

**Proposition 5.8.** *Fix an integer $m \geq 1$. Let $F$ be a random variable in $\mathbb{D}^{m+2,\infty}$ such that $M_\beta(F) < \infty$ for some $\beta > 3m + 3(\lfloor \frac{m}{2} \rfloor \vee 1)$. Denote $w = \|DF\|_{\mathfrak{H}}^2$ and $u = \frac{DF}{w}$. Then, $u \in \mathbb{D}^{m+1,p}(\mathfrak{H})$ for some $p > 1$, and the random variables $\{G_k\}_{k=0}^{m+1}$ introduced in Lemma 5.6 are well-defined. Under these assumptions, $F$ has a density $f$ of class $C^m$ with derivatives given by*

$$f_F^{(k)}(x) = (-1)^k E[\mathbf{1}_{\{F>x\}} G_{k+1}] \tag{5.30}$$

*for $k = 1, \ldots, m$.*

*Proof.* It is enough to show that $\{G_k\}_{k=0}^{m+1}$ are well-defined, since it follows from [39, Exercise 2.1.4] that the $k$th derivative of the density of $F$ is given by (5.30). To do this we will show that $G_k$ defined in (5.25) are in $L^1(\Omega)$ for all $k = 1, \ldots, m+1$. From (5.25) we can write

$$E[|G_k|] \leq E[|H_k(D_u\delta_u, \delta_u)|] + E[|T_k|].$$

Recall the explicit expression of $H_k(\lambda, x)$ in (5.24). Since $\beta > 3(m+1)$, we can choose $r_0 < \frac{\beta}{3}, r_1 < \frac{\beta}{6}$ such that

$$1 \geq \frac{k-2i}{r_0} + \frac{i}{r_1} > \frac{3\,(k-2i)}{\beta} + \frac{6i}{\beta} = \frac{3k}{\beta},$$

for any $0 \leq i \leq \lfloor k/2 \rfloor$ and $1 \leq k \leq m+1$. Then, applying Hölder's inequality with (5.24), (5.147) and (5.148) we have

$$E[|H_k(D_u\delta_u, \delta_u)|] \leq C_k \sum_{0 \leq i \leq \lfloor k/2 \rfloor} \|\delta_u\|_{r_0}^{k-2i} \|D_u\delta_u\|_{r_1}^{i} < \infty.$$

To prove that $E[|T_k|] < \infty$, applying Hölder's inequality to the expression (5.27) and choosing $r_j > 0$ for $0 \le j \le k-1$ such that

$$1 \ge \frac{i_0}{r_0} + \sum_{j=1}^{k-1} \frac{i_j}{r_j} > \frac{3i_0}{\beta} + \sum_{j=1}^{k-1} \frac{(3j+3)i_j}{\beta},$$

we obtain that, (assuming $k \ge 3$, otherwise $T_k = 0$)

$$E[|T_k|] \le C \sum_{(i_0,\dots,i_k)\in J_k} \|\delta_u\|_{r_0}^{i_0} \prod_{j=1}^{k-1} \left\|D_u^j \delta_u\right\|_{r_j}^{i_j}.$$

Due to (5.147) and (5.148), this expression is finite, provided $r_j < \frac{\beta}{3j+3}$ for $0 \le j \le k-1$. We can choose $(r_j, 0 \le j \le k-1)$ satisfying the above conditions because $\beta > 3(k-1) + 3\lfloor \frac{k-1}{2} \rfloor$ for all $1 \le k \le m+1$, and from properties (a) and (c) of $J_k$ in Remark 5.7 we have

$$\frac{3i_0}{\beta} + \sum_{j=1}^{k-1} \frac{(3j+3)i_j}{\beta} \le \frac{3(k-1) + 3\lfloor \frac{k-1}{2} \rfloor}{\beta}.$$

This completes the proof. □

**Example 5.9.** *Consider a random variable in the first Wiener chaos $N = I_1(h)$, where $\in \mathfrak{H}$ with $\|h\|_{\mathfrak{H}} = \sigma$. Then $N$ has the normal distribution $N \sim N(0, \sigma^2)$ with density denoted by $\phi(x)$. Clearly $\|DN\|_{\mathfrak{H}} = \sigma$, $u = \frac{h}{\sigma^2}$, $\delta_u = \frac{N}{\sigma^2}$ and $D_u \delta_u = \frac{h}{\sigma^2}$. Then $G_k = H_k(\frac{1}{\sigma^2}, \frac{N}{\sigma^2})$ and from (5.30) we obtain the formula*

$$\phi^{(k)}(x) = (-1)^k E\left[\mathbf{1}_{\{N>x\}} H_{k+1}\left(\frac{1}{\sigma^2}, \frac{N}{\sigma^2}\right)\right], \tag{5.31}$$

*which can also be obtained by analytic arguments.*

## 5.4 Random variables in the $q$th Wiener chaos

In this section we establish our main results on uniform estimates and uniform convergence of densities and their derivatives. We shall deal first with the convergence of densities and later we consider their derivatives.

### 5.4.1 Uniform estimates of densities

Let $F = I_q(f)$ for some $f \in \mathfrak{H}^{\odot q}$ and $q \geq 2$. To simplify the notation, along this section we denote

$$w = \|DF\|_{\mathfrak{H}}^2, \; u = w^{-1}DF.$$

Note that $LF = -qF$ and using (2.7) and (2.10) we can write

$$\delta_u = \delta(u) = qFw^{-1} - \left\langle Dw^{-1}, DF \right\rangle_{\mathfrak{H}}. \tag{5.32}$$

**Theorem 5.10.** *Let $F = I_q(f)$, $q \geq 2$, for some $f \in \mathfrak{H}^{\odot q}$ be a random variable in the $q$th Wiener chaos with $E[F^2] = \sigma^2$. Assume that $M_6(F) < \infty$, where $M_6(F)$ is defined in* (5.29)*. Let $\phi(x)$ be the density of $N \sim N(0, \sigma^2)$. Then $F$ has a density $f_F(x)$ given by* (5.8)*. Furthermore,*

$$\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq C\sqrt{E[F^4] - 3\sigma^4}, \tag{5.33}$$

*where the constant $C$ has the form $C = C_q\left(\sigma^{-1}M_6(F)^2 + M_6(F)^3 + \sigma^{-3}\right)$ and $C_q$ depends only on $q$.*

We begin with a lemma giving an estimate for the contraction $D^kF \otimes_1 D^lF$ with $k + l \geq 3$.

**Lemma 5.11.** *Let $F = I_q(f)$ be a random variable in the qth Wiener chaos with $E[F^2] = \sigma^2$. Then for any integers $k \geq l \geq 1$ satisfying $k + l \geq 3$, there exists a constant $C_{k,l,q}$ depending only on $k, l, q$ such that*

$$\left\| D^k F \otimes_1 D^l F \right\|_2 \leq C_{k,l,q} \left\| q\sigma^2 - \|DF\|_{\mathfrak{H}}^2 \right\|_2 . \tag{5.34}$$

*Proof.* Note that $D^k F = q(q-1)\cdots(q-k+1)I_{q-k}(f)$. Applying (2.4), we get

$$\begin{aligned} D^k F \otimes_1 D^l F &= q^2(q-1)^2 \cdot \cdots \cdot (q-l+1)^2 (q-l)\cdots(q-k+1) \\ &\quad \times \sum_{r=0}^{q-k} r! \binom{q-k}{r} \binom{q-l}{r} I_{2q-k-l-2r}(f \widetilde{\otimes}_{r+1} f). \end{aligned}$$

Taking into account the orthogonality of multiple integrals of different orders, we obtain

$$\begin{aligned} E[\left\| D^k F \otimes_1 D^l F \right\|_{\mathfrak{H}^{\otimes(k+l-2)}}^2] &= \frac{(q!)^4}{(q-l)!^2 (q-k)!^2} \\ &\quad \times \sum_{r=0}^{q-k} r!^2 \binom{q-k}{r}^2 \binom{q-l}{r}^2 (2q-k-l-2r)! \left\| f \widetilde{\otimes}_{r+1} f \right\|_{\mathfrak{H}^{\otimes 2q-2-2r}}^2 . \end{aligned} \tag{5.35}$$

Applying (5.35) with $k = l = 1$, we obtain

$$\begin{aligned} & E[\|DF\|_{\mathfrak{H}}^4] = E[|DF \otimes_1 DF|^2] \\ = \; & q^4 \sum_{r=0}^{q-1} r!^2 \binom{q-1}{r}^4 (2q-2-2r)! \left\| f \widetilde{\otimes}_{r+1} f \right\|_{\mathfrak{H}^{\otimes 2q-2-2r}}^2 \\ = \; & q^4 \sum_{r=0}^{q-2} r!^2 \binom{q-1}{r}^4 (2q-2-2r)! \left\| f \widetilde{\otimes}_{r+1} f \right\|_{\mathfrak{H}^{\otimes 2q-2-2r}}^2 + q^2 q!^2 \|f\|_{\mathfrak{H}^{\otimes q}}^4 . \end{aligned} \tag{5.36}$$

Taking into account that $\sigma^2 = E[F^2] = q! \|f\|_{\mathfrak{H}^{\otimes q}}^2$, we obtain that for any $k + l \geq 3$, there exists a constant $C_{k,l,q}$ such that

$$E[\left\| D^k F \otimes_1 D^l F \right\|_{\mathfrak{H}^{\otimes(k+l-2)}}^2] \leq C_{k,l,q}^2 E[\|DF\|_{\mathfrak{H}}^4 - q^2\sigma^4].$$

Meanwhile, it follows from $E[\|DF\|_{\mathfrak{H}}^2] = q\|f\|_{\mathfrak{H}^{\otimes q}}^2 = q\sigma^2$ that

$$E[\|DF\|_{\mathfrak{H}}^4 - q^2\sigma^4] = E[\|DF\|_{\mathfrak{H}}^4 - 2q\sigma^2\|DF\|_{\mathfrak{H}}^2 + q^2\sigma^4] = E[(\|DF\|_{\mathfrak{H}}^2 - q\sigma^2)^2]. \quad (5.37)$$

Combining (5.35), (5.36) and (5.37) we have

$$E[\left\|D^kF \otimes_1 D^lF\right\|_{\mathfrak{H}^{\otimes(k+l-2)}}^2] \leq C_{k,l,q}^2 E[(\|DF\|_{\mathfrak{H}}^2 - q\sigma^2)^2],$$

which completes the proof. □

***Proof of Theorem*** 5.10. It follows from Theorem 5.3 that $F$ admits a density $f_F(x) = E\left[\mathbf{1}_{\{F>x\}}\delta(u)\right]$. By (5.31) with $k=1$ we can write $\phi(x) = \frac{1}{\sigma^2}E[\mathbf{1}_{\{N>x\}}N]$. Then, using (5.32), for all $x \in \mathbb{R}$ we obtain

$$\begin{aligned} & f_F(x) - \phi(x) = E\left[\mathbf{1}_{\{F>x\}}\delta(u)\right] - \sigma^{-2}E[\mathbf{1}_{\{N>x\}}N] \\ = \; & E[\mathbf{1}_{\{F>x\}}(F(\frac{q}{w} - \sigma^{-2}) - \langle Dw^{-1}, DF\rangle_{\mathfrak{H}})] + \sigma^{-2}E\left[F\mathbf{1}_{\{F>x\}} - N\mathbf{1}_{\{N>x\}}\right] \\ = \; & A_1 + A_2. \end{aligned} \quad (5.38)$$

For the first term $A_1$, Hölder's inequality implies

$$\begin{aligned} |A_1| &= \left|E[\mathbf{1}_{\{F>x\}}(F(\frac{q}{w} - \sigma^{-2}) - \langle Dw^{-1}, DF\rangle_{\mathfrak{H}})]\right| \\ &\leq \sigma^{-2}E\left[\left|Fw^{-1}(w - q\sigma^2)\right|\right] + 2E[w^{-\frac{3}{2}}\left\|D^2F \otimes_1 DF\right\|_{\mathfrak{H}}] \\ &\leq \sigma^{-2}\left\|w^{-1}\right\|_3 \|F\|_3 \left\|w - q\sigma^2\right\|_3 + 2\left\|w^{-\frac{3}{2}}\right\|_2 \left\|\left\|D^2F \otimes_1 DF\right\|_{\mathfrak{H}}\right\|_2. \end{aligned}$$

Note that (2.12) implies

$$\left\|w - q\sigma^2\right\|_3 \leq C\left\|w - q\sigma^2\right\|_2$$

and $\|F\|_3 \le C\|F\|_2 = C\sigma$. Combining these estimates with (5.34) we obtain

$$|A_1| \le C(\sigma^{-1}\left\|w^{-1}\right\|_3 + \left\|w^{-1}\right\|_3^{\frac{3}{2}})\left\|w - q\sigma^2\right\|_2. \tag{5.39}$$

For the second term $A_2$, applying Lemma 5.2 to the function $h(z) = z\mathbf{1}_{\{z>x\}}$, which satisfies $|h(z)| \le |z|$, we have

$$\begin{aligned} |A_2| &= \sigma^{-2}\left|E\left[F\mathbf{1}_{\{F>x\}} - N\mathbf{1}_{\{N>x\}}\right]\right| \\ &\le C\sigma^{-3}\left\|\sigma^2 - \left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{H}}\right\|_2 \le C\sigma^{-3}\left\|q\sigma^2 - w\right\|_2. \end{aligned} \tag{5.40}$$

Combining (5.38) with (5.39)–(5.40) we obtain

$$\sup_{x\in\mathbb{R}}|f_F(x) - \phi(x)| \le C(\sigma^{-1}\left\|w^{-1}\right\|_3 + \left\|w^{-1}\right\|_3^{\frac{3}{2}} + \sigma^{-3})\left\|w - q\sigma^2\right\|_2.$$

Then (5.33) follows from (2.13). This completes the proof. □

Using the estimates shown in Theorem 5.10 we can deduce the following uniform convergence and convergence in $L^p$ of densities for a sequence of random variables in a fixed $q$th Wiener chaos.

**Corollary 5.12.** *Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of random variables in the $q$th Wiener chaos with $q \ge 2$. Set $\sigma_n^2 = E[F_n^2]$ and assume that $\lim_{n\to\infty}\sigma_n^2 = \sigma^2$, $0 < \delta \le \sigma_n^2 \le K$ for all $n$, $\lim_{n\to\infty} E[F_n^4] = 3\sigma^4$ and*

$$M := \sup_n \left(\mathbb{E}[\|DF_n\|_{\mathfrak{H}}^{-6}]\right)^{1/6} < \infty. \tag{5.41}$$

*Let $\phi(x)$ be the density of the law $N(0,\sigma^2)$. Then, each $F_n$ admits a density $f_{F_n} \in C(\mathbb{R})$ and there exists a constant $C$ depending only on $q, \sigma, \delta$ and $M$ such that*

$$\sup_{x\in\mathbb{R}} |f_{F_n}(x) - \phi(x)| \le C \left( \left|E[F_n^4] - 3\sigma_n^4\right|^{\frac{1}{2}} + |\sigma_n - \sigma| \right). \tag{5.42}$$

*Furthermore, for any $p \ge 1$ and $\alpha \in (\frac{1}{2}, p)$,*

$$\|f_{F_n} - \phi\|_{L^p(\mathbb{R})} \le C \left( \left|E[F_n^4] - 3\sigma_n^4\right|^{\frac{1}{2}} + |\sigma_n - \sigma| \right)^{\frac{p-\alpha}{p}}, \tag{5.43}$$

*where $C$ is a constant depending on $q, \sigma, M, p, \alpha$ and $K$.*

*Proof.* Let $\phi_n(x)$ be the density of $N(0,\sigma_n^2)$. Then Theorem 5.10 implies that

$$\sup_{x\in\mathbb{R}} |f_{F_n}(x) - \phi_n(x)| \le C \left|E[F_n^4] - 3\sigma_n^4\right|^{\frac{1}{2}}.$$

On the other hand, if $N_n \sim N(0,\sigma_n^2)$, it is easy to see that

$$\sup_{x\in\mathbb{R}} |\phi_n(x) - \phi(x)| \le C\, |\sigma_n - \sigma|.$$

Then (5.42) follows from triangle inequality. To show (5.43), first notice that (5.9) implies

$$f_{F_n}(x) \le C(1 \wedge |x|^{-2}).$$

Therefore, if $\alpha > \frac{1}{2}$ the function $(f_{F_n}(x) + \phi(x))^{\alpha}$ is integrable. Then, (5.43) follows from (5.42) and the inequality

$$|f_{F_n}(x) - \phi(x)|^p \le |f_{F_n}(x) - \phi(x)|^{p-\alpha} (f_{F_n}(x) + \phi(x))^{\alpha}.$$

□

### 5.4.2 Uniform estimation of derivatives of densities

In this subsection, we establish the uniform convergence for derivatives of densities of random variables to a normal distribution. We begin with the following theorem which estimates the uniform distance between the derivatives of the densities of a random variable $F$ in the $q$th Wiener chaos and the normal law $N(0,E[F^2])$.

**Theorem 5.13.** *Let $m \geq 1$ be an integer. Let $F$ be a random variable in the $q$th Wiener chaos, $q \geq 2$, with $E[F^2] = \sigma^2$ and $M_\beta := M_\beta(F) < \infty$ for some $\beta > 6m + 6(\lfloor \frac{m}{2} \rfloor \vee 1)$ (Recall the definition of $M_\beta(F)$ in (5.29)). Let $\phi(x)$ be the density of $N \sim N(0,\sigma^2)$. Then $F$ has a density $f_F(x) \in C^m(\mathbb{R})$ with derivatives given by (5.30). Moreover, for any $k = 1,\dots,m$*

$$\sup_{x\in\mathbb{R}} \left| f_F^{(k)}(x) - \phi^{(k)}(x) \right| \leq \sigma^{-k-3} C \sqrt{E[F^4] - 3\sigma^2},$$

*where the constant $C$ depends on $q$, $\beta$, $m$, $\sigma$ and $M_\beta$ with polynomial growth in $\sigma$ and $M_\beta$.*

To prove Theorem 5.13, we need some technical results. Recall the notation we introduced in (5.22), where we denote $\delta_u = \delta(u)$, $D_u\delta_u = \langle D\delta_u, u\rangle_{\mathfrak{H}}$.

**Lemma 5.14.** *Let $F$ be a random variable in the $q$th Wiener chaos with $E[F^2] = \sigma^2$. Let $w = \|DF\|_{\mathfrak{H}}^2$ and $u = w^{-1}DF$.*

*(i) If $M_\beta(F) < \infty$ for some $\beta > 6$, then for any $1 < r \leq \frac{2\beta}{\beta+6}$*

$$\left\| \delta_u - \sigma^{-2}F \right\|_r \leq C\sigma^{-1}(M_\beta^3 \vee 1) \left\| q\sigma^2 - w \right\|_2. \tag{5.44}$$

*(ii) If $M_\beta(F) < \infty$ for some $\beta > 12$, then for any $1 < r < \frac{2\beta}{\beta+12}$*

$$\left\| D_u\delta_u - \sigma^{-2} \right\|_r \leq C\sigma^{-2}(M_\beta^6 \vee 1) \left\| q\sigma^2 - w \right\|_2, \tag{5.45}$$

*where the constant C depends on $\sigma$.*

*Proof.* Recall that $\delta_u = qFw^{-1} - D_{DF}w^{-1}$. Using Hölder's inequality and (5.139) we can write

$$\begin{aligned} \left\|\delta_u - \sigma^{-2}F\right\|_r &\leq \left\|\sigma^{-2}Fw^{-1}(q\sigma^2 - w)\right\|_r + \left\|D_{DF}w^{-1}\right\|_r \\ &\leq C\left(\sigma^{-2}\|Fw^{-1}\|_s + (M_\beta^3 \vee 1)\right)\left\|q\sigma^2 - w\right\|_2, \end{aligned}$$

provided $\frac{1}{r} = \frac{1}{s} + \frac{1}{2}$. By the hypercontractivity property (2.11) $\|F\|_\gamma \leq C_{q,\gamma}\|F\|_2$ for any $\gamma \geq 2$. Thus, by Hölder's inequality, if $\frac{1}{s} = \frac{1}{\gamma} + \frac{1}{p}$

$$\|Fw^{-1}\|_s \leq \|F\|_\gamma \|w^{-1}\|_p \leq C_{q,\gamma}\sigma M_{2p}^2.$$

Choosing $p$ such that $2p < \beta$ we get (5.44).

We can compute $D_u\delta_u$ as

$$D_u\delta_u = qw^{-1} + qFw^{-1}D_{DF}w^{-1} - w^{-1}D_{DF}^2w^{-1} - w^{-1}\left\langle D^2F, DF \otimes Dw^{-1}\right\rangle_{\mathfrak{H}}.$$

Applying Hölder's inequality we obtain

$$\begin{aligned} &\left\|D_u\delta_u - \sigma^{-2}\right\|_r \leq \left\|w^{-1}\left[\sigma^{-2}\left(q\sigma^2 - w\right) + qFD_{DF}w^{-1} - D_{DF}^2w^{-1}\right]\right\|_r \\ \leq\ & \sigma^{-2}\|w^{-1}\|_{\frac{2r}{2-r}}\left\|q\sigma^2 - w\right\|_2 + C_\sigma\|w^{-1}\|_p\left(\left\|D_{DF}w^{-1}\right\|_s + \left\|D_{DF}^2w^{-1}\right\|_s\right), \end{aligned}$$

if $\frac{1}{r} > \frac{1}{p} + \frac{1}{s}$. Then, using (5.139) and (5.140) with $k = 2$ and assuming that $s < \frac{2\beta}{\beta+8}$ and that $2p < \beta$ we obtain (5.45). □

***Proof of Theorem*** 5.13. Proposition 5.8 implies that $f_F(x) \in C^{m-1}(\mathbb{R})$ and for $k = 0, 1, \ldots, m-1$,

$$f_F^{(k)}(x) = (-1)^k E[\mathbf{1}_{\{F>x\}}G_{k+1}],$$

where $G_0 = 1$ and $G_{k+1} = \delta(G_k u) = G_k \delta(u) - \langle DG_k, u \rangle_{\mathfrak{H}}$. From (5.31),

$$\phi^{(k)}(x) = (-1)^k E[\mathbf{1}_{\{N>x\}} H_{k+1}(\sigma^{-2}, \sigma^{-2} N)].$$

Then, the identity $G_{k+1} = H_{k+1}(D_u \delta_u, \delta_u) + T_{k+1}$ (see formula (5.25)), suggests the following triangle inequality

$$\begin{aligned}
\left| f_F^{(k)}(x) - \phi^{(k)}(x) \right| &= \left| E[\mathbf{1}_{\{F>x\}} G_{k+1} - \mathbf{1}_{\{N>x\}} H_{k+1}(\sigma^{-2}, \sigma^{-2} N)] \right| \\
&\leq \left| E[\mathbf{1}_{\{F>x\}} G_{k+1} - \mathbf{1}_{\{F>x\}} H_{k+1}(\sigma^{-2}, \sigma^{-2} F)] \right| \\
&\quad + \left| E[\mathbf{1}_{\{F>x\}} H_{k+1}(\sigma^{-2}, \sigma^{-2} F) - \mathbf{1}_{\{N>x\}} H_{k+1}(\sigma^{-2}, \sigma^{-2} N)] \right| \\
&= A_1 + A_2 .
\end{aligned}$$

We first estimate the term $A_2$. Note that $\|F\|_{2k+2} \leq C_{q,k} \|F\|_2 = C_{q,k}\sigma$ by the hypercontractivity property (2.11). Applying Lemma 5.2 with $h(z) = \mathbf{1}_{\{z>x\}} H_{k+1}(\sigma^{-2}, \sigma^{-2} z)$, which satisfies $|h(z)| \leq C_k(|z|^{k+1} + \sigma^{-k-1})$, we obtain

$$\begin{aligned}
A_2 &= |E[h(F) - h(N)]| \\
&\leq C_{q,k} \sigma^{-2} \left| \sigma^k + 4\sigma^{-k-1} \right| \left\| \sigma^2 - \left\langle DF, -DL^{-1}F \right\rangle_{\mathfrak{H}} \right\|_2 \\
&\leq C_{q,k,\sigma} \sigma^{-k-3} \left\| q\sigma^2 - w \right\|_2 , \qquad (5.46)
\end{aligned}$$

where in the second inequality we used the fact that $\left\langle DF, -DL^{-1}F \right\rangle_{\mathfrak{H}} = \frac{w}{q}$.

For the term $A_1$, Lemma 5.6 implies

$$A_1 \leq E[\left| H_{k+1}(D_u \delta_u, \delta_u) - H_{k+1}(\sigma^{-2}, \sigma^{-2} F) \right|] + E[|T_{k+1}|]. \qquad (5.47)$$

To proceed with the first term above, applying (5.24) we have

$$\begin{aligned}
&\left|H_{k+1}(D_u\delta_u,\delta_u)-H_{k+1}(\sigma^{-2},\sigma^{-2}F)\right| \qquad (5.48)\\
\le\ & \sum_{i=0}^{\lfloor\frac{k+1}{2}\rfloor}|c_{k,i}|\left|\delta_u^{k+1-2i}(D_u\delta_u)^i-\left(\sigma^{-2}F\right)^{k+1-2i}\sigma^{-2i}\right|\\
\le\ & C_k\sum_{i=0}^{\lfloor\frac{k+1}{2}\rfloor}\left[\left|\delta_u^{k+1-2i}-\left(\sigma^{-2}F\right)^{k+1-2i}\right||D_u\delta_u|^i+\left|\sigma^{-2}F\right|^{k+1-2i}\left|(D_u\delta_u)^i-\sigma^{-2i}\right|\right].
\end{aligned}$$

Using the fact that $\left|x^k-y^k\right|\le C_k|x-y|\sum_{0\le j\le k-1}|x|^{k-1-j}|y|^j$ and applying Hölder's inequality and the hypercontractivity property (2.11) we obtain

$$\begin{aligned}
&E\left[\left|\delta_u^{k+1-2i}-\left(\sigma^{-2}F\right)^{k+1-2i}\right||D_u\delta_u|^i\right]\\
\le\ & C_kE\left[\left|\delta_u-\sigma^{-2}F\right||D_u\delta_u|^i\sum_{0\le j\le k-2i}|\delta_u|^{k-2i-j}\left|\sigma^{-2}F\right|^j\right]\\
\le\ & C_{q,k,\sigma}\left\|\delta_u-\sigma^{-2}F\right\|_r\|D_u\delta_u\|_s^i\sum_{0\le j\le k-2i}\|\delta_u\|_p^{k-2i-j}\sigma^{-j}, \qquad (5.49)
\end{aligned}$$

provided $1\ge\frac{1}{r}+\frac{i}{s}+\frac{k-2i-j}{p}$, which is implied by $1\ge\frac{1}{r}+\frac{i}{s}+\frac{k-2i}{p}$. In order to apply the estimates (5.44), (5.148) (with $k=1$) and (5.147) we need $\frac{1}{r}>\frac{3}{\beta}+\frac{1}{2}$, $\frac{1}{s}>\frac{6}{\beta}$ and $\frac{1}{p}>\frac{3}{\beta}$, respectively. These are possible because $\beta>6k+6$. Then we obtain an estimate of the form

$$E\left[\left|\delta_u^{k+1-2i}-\left(\sigma^{-2}F\right)^{k+1-2i}\right||D_u\delta_u|^i\right]\le C_{q,k,\sigma}\sigma^{-k}(M_\beta^{3k+3}\vee 1)\|q\sigma^2-w\|_2. \qquad (5.50)$$

Similarly,

$$\begin{aligned}
&E\left[\left|\sigma^{-2}F\right|^{k+1-2i}\left|(D_u\delta_u)^i-\sigma^{-2i}\right|\right]\\
\le\ & C_{q,k,\sigma}E\left[\left|\sigma^{-2}F\right|^{k+1-2i}\left|D_u\delta_u-\sigma^{-2}\right|\sum_{0\le j\le i-1}|D_u\delta_u|^j\sigma^{-2(i-1-j)}\right]
\end{aligned}$$

$$\leq \quad C_{q,k,\sigma}\sigma^{-(k-1)} \left\|D_u\delta_u - \sigma^{-2}\right\|_r \sum_{0\leq j\leq i-1} \|D_u\delta_u\|_s^j, \tag{5.51}$$

provided $1 > \frac{1}{r} + \frac{j}{s}$. In order to apply the estimates (5.45) and (5.148) (with $k = 1$) we need $\frac{1}{r} > \frac{6}{\beta} + \frac{1}{2}$ and $\frac{1}{s} > \frac{6}{\beta}$, respectively. This implies

$$\frac{1}{r} + \frac{j}{s} > \frac{6+6j}{\beta} + \frac{1}{2}.$$

Notice that $6+6j \leq 6i \leq 3k+3$. So, we need $1 > \frac{1}{2} + \frac{3k+3}{\beta}$. The above $r, s$ and $p$ exist because $\beta > 6k+6$. Thus, we obtain an estimate of the form

$$E\left[\left|\sigma^{-2}F\right|^{k+1-2i}\left|(D_u\delta_u)^i - \sigma^{-2i}\right|\right] \leq C_{q,k,\sigma,\beta}\sigma^{-(k-1)}(M_\beta^{3k+3}\vee 1)\|q\sigma^2 - w\|_2. \tag{5.52}$$

Combining (5.50) and (5.52) we have

$$E\left[\left|H_{k+1}(D_u\delta_u, \delta_u) - H_{k+1}(\sigma^{-2}, \sigma^{-2}F)\right|\right] \leq C_{q,k,\sigma,\beta}\sigma^{-k}(M_\beta^{3k+3}\vee 1)\|q\sigma^2 - w\|_2. \tag{5.53}$$

Applying Hölder's inequality to the expression (5.27) we obtain, (assuming $k \geq 2$ , otherwise $T_{k+1} = 0$)

$$E\left[|T_{k+1}|\right] \leq C_{q,k,\sigma,\beta} \sum_{(i_0,\dots,i_k)\in J_{k+1}} \|\delta_u\|_{r_0}^{i_0} \prod_{j=1}^k \left\|D_u^j\delta_u\right\|_{r_j}^{i_j},$$

where $1 = \frac{i_0}{r_0} + \sum_{j=1}^k \frac{i_j}{r_j}$. From property (b) in Remark 5.7 there is at least one factor of the form $\left\|D_u^j\delta_u\right\|_{s_j}$ with $j \geq 2$. We apply the estimate (5.149) to one of these factors, and the estimate (5.148) to all the remaining factors. We also use the estimate (5.147) to control $\|\delta_u\|_{r_0}$. Notice that

$$1 = \frac{i_0}{r_0} + \sum_{j=1}^k \frac{i_j}{r_j} > \frac{3i_0}{\beta} + \sum_{j=1}^k \frac{i_j(3j+3)}{\beta} + \frac{1}{2},$$

and, on the other hand, using properties (a) and (c) in Remark 5.7

$$\frac{3i_0}{\beta}+\sum_{j=1}^{k}\frac{i_j(3j+3)}{\beta}+\frac{1}{2}\leq\frac{3k+3\lfloor\frac{k}{2}\rfloor}{\beta}+\frac{1}{2}.$$

We can choose the $r_j$'s satisfying the above properties because $\beta>6k+6\lfloor\frac{k}{2}\rfloor$, and we obtain

$$E\left|T_{k+1}\right|\leq C_{q,k,\sigma,\beta}(M_{\beta}^{3k+3\lfloor\frac{k}{2}\rfloor}\vee 1)\left\|q\sigma^2-w\right\|_2. \tag{5.54}$$

Combining (5.53) and (5.54) we complete the proof. □

**Corollary 5.15.** *Fix an integer $m\geq 1$. Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of random variables in the qth Wiener chaos with $q\geq 2$ and $E[F_n^2]=\sigma_n^2$. Assume $\lim_{n\to\infty}\sigma_n=\sigma$, $0<\delta\leq\sigma_n^2\leq K$ for all n, $\lim_{n\to\infty}E[F_n^4]=3\sigma^4$ and*

$$M:=\sup_n\left(\mathbb{E}[\|DF_n\|_{\mathfrak{H}}^{-\beta}]\right)^{\frac{1}{\beta}}<\infty \tag{5.55}$$

*for some $\beta>6(m)+6(\lfloor\frac{m}{2}\rfloor\vee 1)$. Let $\phi(x)$ be the density of $N(0,\sigma^2)$. Then, each $F_n$ admits a probability density function $f_{F_n}\in C^m(\mathbb{R})$ with derivatives given by (5.30) and for any $k=1,\dots,m$,*

$$\sup_{x\in\mathbb{R}}\left|f_{F_n}^{(k)}(x)-\phi^{(k)}(x)\right|\leq C\left(\sqrt{E[F_n^4]-3\sigma_n^4}+|\sigma_n-\sigma|\right),$$

*where the constant C depends only on $q,m,\beta,M,\sigma,\delta$ and K.*

*Proof.* Let $\phi_n(x)$ be the density of $N(0,\sigma_n^2)$. Then Theorem 5.13 implies that

$$\sup_{x\in\mathbb{R}}\left|f_{F_n}^{(k)}(x)-\phi_n^{(k)}(x)\right|\leq C_{q,m,\beta,M,\sigma}\sqrt{E[F_n^4]-3\sigma_n^4}.$$

On the other hand, by the mean value theorem we can write

$$\left|\phi_n^{(k)}(x)-\phi^{(k)}(x)\right| \le |\sigma_n-\sigma| \sup_{\gamma\in[\frac{\sigma}{2},2\sigma]} \left|\partial_\gamma \phi_\gamma^{(k)}(x)\right| = \frac{1}{2}|\sigma_n-\sigma| \sup_{\gamma\in[\frac{\sigma}{2},2\sigma]} \gamma \left|\phi_\gamma^{(k+2)}(x)\right|,$$

where $\phi_\gamma(x)$ is the density of the law $N(0,\gamma^2)$. Then, using the expression

$$\phi_\gamma^{(k+2)}(x) = E[\mathbf{1}_{N>x} H_{k+3}(\gamma^{-2},\gamma^{-2}Z)],$$

where $Z \sim N(0,\gamma^2)$ and the explicit form of $H_{k+3}(\lambda,x)$, we obtain

$$\sup_{\gamma\in[\frac{\sigma}{2},2\sigma]} \gamma \left|\phi_\gamma^{(k+2)}(x)\right| \le C_{k,\sigma}.$$

Therefore,

$$\sup_{x\in\mathbb{R}} \left|\phi_n^{(k)}(x)-\phi^{(k)}(x)\right| \le C_{k,\sigma}\,|\sigma_n-\sigma|\,.$$

This completes the proof. □

## 5.5 Random vectors in Wiener chaos

### 5.5.1 Main result

In this section, we study the multidimensional counterpart of Theorem 5.15. We begin with a density formula for a smooth random vector.

A random vector $F=(F_1,\dots,F_d)$ in $\mathbb{D}^\infty$ is called *nondegenerate* if its *Malliavin matrix* $\gamma_F = (\langle DF_i, DF_j\rangle_{\mathfrak{H}})_{1\le i,j\le d}$ is invertible a.s. and $(\det\gamma_F)^{-1} \in \cap_{p\ge1} L^p(\Omega)$. For any multi-index

$$\beta = (\beta_1,\beta_2,\dots,\beta_k) \in \{1,2,\dots,d\}^k$$

of length $k \geq 1$, the symbol $\partial_\beta$ stands for the partial derivative $\frac{\partial^k}{\partial x_{\beta_1} \ldots \partial x_{\beta_k}}$. For $\beta$ of length 0 we make the convention that $\partial_\beta f = f$. We denote by $\mathscr{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing smooth functions, that is, the space of all infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\sup_{x \in \mathbb{R}^d} |x|^m \left|\partial_\beta f(x)\right| < \infty$ for any nonnegative integer $m$ and for all multi-index $\beta$. The following lemma (see Nualart [39, Proposition 2.1.5]) gives an explicit formula for the density of $F$.

**Lemma 5.16.** *Let $F = (F_1, \ldots, F_d)$ be a nondegenerate random vector. Then, $F$ has a density $f_F \in \mathscr{S}(\mathbb{R}^d)$, and $f_F$ and its partial derivative $\partial_\beta f_F$, for any multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ of length $k \geq 0$, are given by*

$$f_F(x) = E[\mathbf{1}_{\{F>x\}} H_{(1,2,\ldots,d)}(F)], \tag{5.56}$$

$$\partial_\beta f_F(x) = (-1)^k E[\mathbf{1}_{\{F>x\}} H_{(1,2,\ldots,d,\beta_1,\beta_2,\ldots,\beta_k)}(F)], \tag{5.57}$$

*where $\mathbf{1}_{\{F>x\}} = \prod_{i=1}^d \mathbf{1}_{\{F_i > x_i\}}$ and the elements $H_\beta(F)$ are recursively defined by*

$$\begin{cases} H_\beta(F) = 1, & \text{if } k = 0; \\ H_{(\beta_1,\beta_2,\ldots,\beta_k)}(F) = \sum_{j=1}^d \delta\left(H_{(\beta_1,\beta_2,\ldots,\beta_{k-1})}(F) \left(\gamma_F^{-1}\right)^{\beta_1 j} DF_j\right), & \text{if } k \geq 1. \end{cases} \tag{5.58}$$

Fix $d$ natural numbers $1 \leq q_1 \leq \cdots \leq q_d$. We will consider a random vector of multiple stochastic integrals: $F = (F_1, \ldots, F_d) = (I_{q_1}(f_1), \ldots, I_{q_d}(f_d))$, where $f_i \in \mathfrak{H}^{\odot q_i}$. Denote

$$V = \left(E[F_i F_j]\right)_{1 \leq i,j \leq d}, \quad Q = \text{diag}(q_1, \ldots, q_d) \quad \text{(diagonal matrix of elements } q_1, \ldots, q_d). \tag{5.59}$$

Along this section, we denote by $N = (N_1, \ldots, N_d)$ a standard normal vector given by $N_i = I_1(h_i)$, where $h_i \in \mathfrak{H}$ are orthonormal. We denote by $I$ the $d$-dimensional identity matrix, and by $|\cdot|$ the Hilbert-Schmidt norm of a matrix. The following is the main theorem of this section.

**Theorem 5.17.** *Let $F = (F_1, \ldots, F_d) = (I_{q_1}(f_1), \ldots, I_{q_d}(f_d))$ be nondegenerate and let $\phi$ be the density of $N$. Then for any multi-index $\beta$ of length $k \geq 0$, the density $f_F$ of $F$ satisfies*

$$\sup_{x\in\mathbb{R}^d} \left|\partial_\beta f_F(x) - \partial_\beta \phi(x)\right| \leq C\left(|V - I| + \sum_{1\leq j\leq d} \sqrt{E[F_j^4] - 3(E[F_j^2])^2}\right), \qquad (5.60)$$

*where the constant $C$ depends on $d, V, Q, k$ and $\left\|(\det\gamma_F)^{-1}\right\|_{(k+4)2^{k+3}}$.*

*Proof.* Note that $\partial_\beta \phi(x) = (-1)^k E[\mathbf{1}_{\{N>x\}} H_{(1,2,\ldots,d,\beta_1,\beta_2,\ldots,\beta_k)}(N)]$. Then, in order to estimate the difference between $\partial_\beta f_{F_n}$ and $\partial_\beta \phi$, it suffices to estimate

$$E[\mathbf{1}_{\{F>x\}} H_\beta(F)] - E[\mathbf{1}_{\{N>x\}} H_\beta(N)]$$

for all multi-index $\beta$ of length $k$ for all $k \geq d$.

Fix a multi-index $\beta$ of length $k$ for some $k \geq d$. For the above standard normal random vector $N$, we have $\gamma_N = I$ and $\delta(DN_i) = N_i$. We can deduce from the expression (5.58) that $H_\beta(N) = g_\beta(N)$, where $g_\beta(x)$ is a polynomial on $\mathbb{R}^d$ (see Remark 5.19). Then,

$$\begin{aligned}
&\left|E[\mathbf{1}_{\{F>x\}} H_\beta(F)] - E[\mathbf{1}_{\{N>x\}} H_\beta(N)]\right| \\
\leq\ &\left|E[\mathbf{1}_{\{F>x\}} g_\beta(F)] - E[\mathbf{1}_{\{N>x\}} g_\beta(N)]\right| + E\left[\left|H_\beta(F) - g_\beta(F)\right|\right] \\
=\ &A_1 + A_2. \qquad (5.61)
\end{aligned}$$

The term $A_1 = \left|E[\mathbf{1}_{\{F>x\}} g_\beta(F) - \mathbf{1}_{\{N>x\}} g_\beta(N)]\right|$ will be studied in Subsection 5.5.3 by using the multivariate Stein's method. Proposition 5.25 will imply that $A_1$ is bounded by the right-hand side of (5.60).

Consider the term $A_2 = E\left[\left|H_\beta(F) - g_\beta(F)\right|\right]$. We introduce an auxiliary term $K_\beta(F)$, which is defined similar to $H_\beta(F)$ with $\gamma_F^{-1}$ replaced by $(VQ)^{-1}$. That is, for any multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ of length $k \geq 0$, we define

$$\begin{cases} K_\beta(F) = 1 & \text{if } k = 0; \\ \\ K_\beta(F) = \delta\left(K_{(\beta_1,\beta_2,\ldots,\beta_{k-1})}(F)\left(\left(VQ\right)^{-1} DF\right)_{\beta_k}\right) & \text{if } k \geq 1. \end{cases} \tag{5.62}$$

We have

$$A_2 \leq E\left[\left|H_\beta(F) - K_\beta(F)\right|\right] + E\left[\left|K_\beta(F) - g_\beta(F)\right|\right] =: A_3 + A_4\,. \tag{5.63}$$

Lemma 5.26 below shows that the term $A_3 = E\left[\left|H_\beta(F) - K_\beta(F)\right|\right]$ is bounded by the right-hand of (5.60).

It remains to estimate $A_4$. For this we need the following lemma which provides an explicit expression for the term $K_\beta(F)$. Before stating this lemma we need to introduce some notation. For any multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$, $k \geq 1$, denote by $\widehat{\beta}_{i_1\ldots i_m}$ the multi-index obtained from $\beta$ after taking away the elements $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_m}$. For example, $\widehat{\beta}_{14} = (\beta_2, \beta_3, \beta_5, \ldots, \beta_k)$. For any $d$-dimensional vector $G$ we denote by $G_\beta$ the product $G_{\beta_1} G_{\beta_2} \cdots G_{\beta_k}$ and set $G_\beta = 1$ if the length of $\beta$ is 0. Denote by $\left(S_k^m; 0 \leq m \leq \lfloor \frac{k}{2} \rfloor\right)$ the following sets

$$\begin{cases} S_k^{-1} = S_k^0 = \varnothing \\ S_k^m = \left\{ \begin{array}{l} \{(i_1,i_2),\dots,(i_{2m-1},i_{2m})\} \in \{1,2,\dots,k\}^{2m}: \\ i_{2l-1} < i_{2l} \text{ for } 1 \le l \le m \text{ and } i_l \neq i_j \text{ if } l \neq j \end{array} \right\} \end{cases} \tag{5.64}$$

For each element $\{(i_1,i_2),\dots,(i_{2m-1},i_{2m})\} \in S_k^m$, we emphasize that the $m$ pairs of indices are unordered. In other words, for $m \geq 1$, the set $S_k^m$ can be viewed as the set of all partitions of $\{1,2,\dots,k\}$ into $m$ pairs and $k-2m$ singletons.

Denote $M = V^{-1}\gamma_F V^{-1} Q^{-1}$ for $V$ and $Q$ given in (5.59) and denote $M_{ij}$ the $(i,j)$-th entry of $M$. Denote by $D_{\beta_i}$ the Malliavin derivative in the direction of $\left(V^{-1}Q^{-1}DF\right)_{\beta_i} = V^{-1}Q^{-1}DF_{\beta_i}$, that is,

$$D_{\beta_i} G = \left\langle DG, \left(V^{-1}Q^{-1}DF\right)_{\beta_i} \right\rangle_{\mathfrak{H}} \tag{5.65}$$

for any random variable $G \in \mathbb{D}^{1,2}$.

**Lemma 5.18.** *Let $F$ be a nondegenerate random vector. For a multi-index $\beta = (\beta_1,\dots,\beta_k)$ of length $k \geq 0$, $K_\beta(F)$ defined by (5.62) can be computed as follows:*

$$K_\beta(F) = G_\beta(F) + T_\beta(F), \tag{5.66}$$

*where*

$$G_\beta(F) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \sum_{\{(i_1,i_2),\dots,(i_{2m-1},i_{2m})\} \in S_k^m} \left(V^{-1}F\right)_{\widehat{\beta}_{i_1 \cdots i_{2m}}} M_{\beta_{i_1}\beta_{i_2}} \cdots M_{\beta_{i_{2m-1}}\beta_{2m}}, \tag{5.67}$$

*and $T_\beta(F)$ are defined recursively by*

$$T_\beta(F) \;\; = \;\; \left(V^{-1}F\right)_{\beta_k} T_{\widehat{\beta}_k}(F) - D_{\beta_k} T_{\widehat{\beta}_k}(F) \tag{5.68}$$

$$-\sum_{\{(i_1,i_2),\dots,(i_{2m-1},i_{2m})\}\in S_{k-1}^m}\left(V^{-1}F\right)_{\widehat{\beta}_{ki_1\cdots i_{2m}}} D_{\beta_k}\left(M_{\beta_{i_1}\beta_{i_2}}\cdots M_{\beta_{i_{2m-1}}\beta_{2m}}\right),$$

*for* $k\geq 2$ *and* $T_1(F)=T_2(F)=0$.

*Proof.* For simplicity, we write $K_\beta$, $G_\beta$ and $T_\beta$ for $K_\beta(F)$, $G_\beta(F)$ and $T_\beta(F)$, respectively. By using the fact that $\delta\left(\left((VQ)^{-1}DF\right)_{\beta_i}\right)=\left(V^{-1}F\right)_{\beta_i}$ we obtain from (5.62) that

$$K_\beta=\left(V^{-1}F\right)_{\beta_k}K_{\widehat{\beta}_k}-D_{\beta_k}K_{\widehat{\beta}_k}. \tag{5.69}$$

If if $k=1$, namely, $\beta=(\beta_1)$, then

$$K_\beta=\left(V^{-1}F\right)_{\beta_1}=G_\beta.$$

If $k=2$, namely, $\beta=(\beta_1,\beta_2)$, then

$$K_\beta=\left(V^{-1}F\right)_\beta-M_{\beta_1\beta_2}=G_\beta\,.$$

Hence, the identity (5.66) is true for $k=1,2$. Assume now (5.66) is true for all multi-index of length less than or equal to $k$. Let $\beta=(\beta_1,\dots,\beta_{k+1})$. Then, (5.69) implies

$$K_\beta=\left(V^{-1}F\right)_{\beta_{k+1}}\left(G_{\widehat{\beta}_{k+1}}+T_{\widehat{\beta}_{k+1}}\right)-D_{\beta_{k+1}}\left(G_{\widehat{\beta}_{k+1}}+T_{\widehat{\beta}_{k+1}}\right). \tag{5.70}$$

Noticing that

$$D_{\beta_{k+1}}\left(V^{-1}F\right)_{\widehat{\beta}_{(k+1)i_1\cdots i_{2m}}}=\sum_{j\in\{1,\dots,k\}\setminus\{i_1,\dots,i_{2m}\}}\left(V^{-1}F\right)_{\widehat{\beta}_{(k+1)ji_1\cdots i_{2m}}}M_{\beta_j\beta_{k+1}},$$

we have

$$D_{\beta_{k+1}}G_{\widehat{\beta}_{k+1}}=B_\beta \tag{5.71}$$

$$+\sum_{m=0}^{\lfloor k/2 \rfloor}(-1)^m \sum_{\{(i_1,i_2),\dots,(i_{2m-1},i_{2m})\}\in S_k^m} \left(V^{-1}F\right)_{\widehat{\beta}_{(k+1)i_1\cdots i_{2m}}} D_{\beta_{k+1}}\left(M_{\beta_{i_1}\beta_{i_2}}\cdots M_{\beta_{i_{2m-1}}\beta_{2m}}\right),$$

where we let

$$B_\beta = \sum_{m=0}^{\lfloor k/2 \rfloor}(-1)^m \sum_{\substack{\{(i_1,i_2),\dots,(i_{2m-1},i_{2m})\}\in S_k^m,\\ j\in\{1,\dots,k\}\setminus\{i_1,\dots,i_{2m}\}}} \left(V^{-1}F\right)_{\widehat{\beta}_{j(k+1)i_1\cdots i_{2m}}} M_{\beta_j\beta_{k+1}} M_{\beta_{i_1}\beta_{i_2}}\cdots M_{\beta_{i_{2m-1}}\beta_{i_{2m}}}.$$

Substituting the expression (5.71) for $D_{\beta_{k+1}}G_{\widehat{\beta}_{k+1}}$ into (5.70) and using (5.68) we obtain

$$K_\beta = \left(V^{-1}F\right)_{\beta_{k+1}} G_{\widehat{\beta}_{k+1}} - B_\beta + T_\beta.$$

To arrive at (5.66) it remains to verify

$$G_\beta = \left(V^{-1}F\right)_{\beta_{k+1}} G_{\widehat{\beta}_{k+1}} - B_\beta. \tag{5.72}$$

Introduce the following notation

$$C_{k+1}^m = \left\{\{(i_1,i_2),\dots,(i_{2m-3},i_{2m-2}),(j,k+1)\} : \{(i_1,i_2),\dots,(i_{2m-3},i_{2m-2})\}\in S_k^{m-1}\right\} \tag{5.73}$$

for $1\le m\le \lfloor \frac{k}{2}\rfloor$. Then, $S_{k+1}^m$ can be decomposed as follows

$$S_{k+1}^m = S_k^m \cup C_{k+1}^m. \tag{5.74}$$

We consider first the case when $k$ is even. In this case, noticing that for any element in $\{(i_1,i_2),\dots,(i_{2m-1},i_{2m})\}\in S_k^{\lfloor \frac{k}{2}\rfloor}$, $\{1,\dots,k\}\setminus\{i_1,\dots,i_{2m}\}=\varnothing$. For any collection of indices $i_1,\dots,i_{2m}\subset\{1,2,\dots,k\}$, we set

$$\Phi_{i_1\dots i_{2m}} = \left(V^{-1}F\right)_{\widehat{\beta}_{i_1\cdots i_{2m}}} M_{\beta_{i_1}\beta_{i_2}}\cdots M_{\beta_{i_{2m-1}}\beta_{i_{2m}}}.$$

Then, we have

$$
\begin{aligned}
-B_\beta = & \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor - 1} (-1)^{m+1} \sum_{\substack{\{(i_1,i_2),\ldots,(i_{2m-1},i_{2m})\}\in S_k^m, \\ j\in\{1,\ldots,k\}\setminus\{i_1,\ldots,i_{2m}\}}} \Phi_{j(k+1)i_1\ldots i_{2m}} \\
= & \sum_{m=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^{n} \sum_{\substack{\{(i_1,i_2),\ldots,(i_{2m-3},i_{2m-2})\}\in S_k^{m-1}, \\ j\in\{1,\ldots,k\}\setminus\{i_1,\ldots,i_{2n-2}\}}} \Phi_{j(k+1)i_1\ldots i_{2m-2}} \qquad (5.75) \\
= & \sum_{m=1}^{\lfloor \frac{k+1}{2} \rfloor} (-1)^{m} \sum_{\substack{\{(i_1,i_2),\ldots,(i_{2m-3},i_{2m-2}),(j,k+1)\} \\ \in C_{k+1}^m}} \Phi_{j(k+1)i_1\ldots i_{2m-2}},
\end{aligned}
$$

where in the last equality we used (5.73) and the fact that $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k+1}{2} \rfloor$ since $k$ is even. Taking into account that $\left(V^{-1}F\right)_{\beta_{k+1}} \left(V^{-1}F\right)_{\widehat{\beta}_{(k+1)i_1\cdots i_{2m}}} = \left(V^{-1}F\right)_{\widehat{\beta}_{i_1\cdots i_{2m}}}$, we obtain from (5.67) that

$$
\left(V^{-1}F\right)_{\beta_{k+1}} G_{\widehat{\beta}_{k+1}} = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \sum_{\{(i_1,i_2),\ldots,(i_{2m-1},i_{2m})\}\in S_k^m} \left(V^{-1}F\right)_{\widehat{\beta}_{i_1\cdots i_{2m}}} M_{\beta_{i_1}\beta_{i_2}} \cdots M_{\beta_{i_{2m-1}}\beta_{i_{2m}}}. \qquad (5.76)
$$

Now combining (5.75) and (5.76) with (5.74) and using again $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k+1}{2} \rfloor$ we obtain

$$
\begin{aligned}
\left(V^{-1}F\right)_{\beta_{k+1}} G_{\widehat{\beta}_{k+1}} - B_\beta = & \sum_{m=0}^{\lfloor (k+1)/2 \rfloor} (-1)^m \sum_{\{(i_1,i_2),\ldots,(i_{2m-1},i_{2m})\}\in S_k^m} \Phi_{i_1\ldots i_{2m}} \\
& + \sum_{m=1}^{\lfloor (k+1)/2 \rfloor} (-1)^m \sum_{\{(i_1,i_2),\ldots,(i_{2m-3},i_{2m-2}),(j,k+1)\}\in C_{k+1}^m} \Phi_{i_1\ldots i_{2m}} \\
= & \sum_{m=0}^{\lfloor (k+1)/2 \rfloor} (-1)^m \sum_{\{(i_1,i_2),\ldots,(i_{2m-1},i_{2m})\}\in S_{k+1}^m} \Phi_{i_1\ldots i_{2m}} \\
= & \; G_\beta
\end{aligned}
$$

as desired. This verifies (5.72) for the case $k$ is even. The case when $k$ is odd can be verified similarly. Thus, we have proved (5.66) by induction. □

**Remark 5.19.** *For the random vector $N \sim N(0,I)$, we have $\gamma_N = VQ = I$, so $H_\beta(N) = K_\beta(N)$. Then, it follows from Lemma* 5.18 *that $H_\beta(N) = K_\beta(N) = g_\beta(N)$ with the function $g_\beta(x) : \mathbb{R}^d \to \mathbb{R}$ given by*

$$g_\beta(x) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \sum_{\{(i_1,i_2),\dots,(i_{2m-1},i_{2m})\} \in S_k^m} x_{\widehat{\beta}_{i_1 \cdots i_{2m}}} \delta_{\beta_{i_1}\beta_{i_2}} \cdots \delta_{\beta_{i_{2m-1}}\beta_{2m}}, \tag{5.77}$$

*where we used $\delta_{ij}$ to denote the Kronecker symbol (without confusion with the divergence operator). Notice that*

$$g_\beta(x) = \prod_{i=1}^{d} H_{k_i}(x_i),$$

*where $H_{k_i}$ is the $k_i$th Hermite polynomial and for each $i = 1,\dots,d$, $k_i$ is the number of components of $\beta$ equal to $i$.*

Let us return to the proof of Theorem 5.17 of estimating the term $A_4$. From (5.66) we can write

$$A_4 = E\left[\left|K_\beta(F) - g_\beta(F)\right|\right] \le E\left[\left|G_\beta(F) - g_\beta(F)\right|\right] + E\left[\left|T_\beta(F)\right|\right]. \tag{5.78}$$

Observe from the expression (5.68) that $T_\beta(F)$ is the sum of terms of the following form

$$\left(V^{-1}F\right)_{\beta_{i_1}\beta_{i_2}\cdots\beta_{i_s}} D_{\beta_{k_1}} D_{\beta_{k_2}} \cdots D_{\beta_{k_t}} \left(\prod_{i}^{r} M_{\beta_{j_i}\beta_{l_i}}\right) \tag{5.79}$$

for some $\{i_1,\dots,i_s,k_1,\dots,k_t,j_1,l_1,\dots j_r,l_r\} \subset \{1,2,\dots,k\}$ and $t \ge 1$. Applying Lemma 5.20 with (2.11) and (2.12) we obtain

$$E\left[\left|T_\beta(F)\right|\right] \le C \sum_{1 \le l \le d} \left\| \|DF_l\|_{\mathfrak{H}}^2 - q_l E[F_l^2] \right\|_2^{\frac{1}{2}}. \tag{5.80}$$

In order to compare $g_\beta(F)$ with $G_\beta(F)$, from (5.77) we can write $g_\beta(F)$ as

$$g_\beta(F) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \sum_{\{(i_1,i_2),\dots,(i_{2m-1},i_{2m})\} \in S_k^m} F_{\widehat{\beta}_{i_1 \cdots i_{2m}}} \delta_{\beta_{i_1}\beta_{i_2}} \cdots \delta_{\beta_{i_{2m-1}}\beta_{2m}} .$$

Then, it follows from hypercontractivity property (2.11) that

$$E\left[\left|G_\beta(F) - g_\beta(F)\right|\right] \le C\left(\left|V^{-1} - I\right| + \|M - I\|_2\right),$$

where the constant $C$ depends on $k, V$ and $Q$. From $V^{-1} - I = V^{-1}(I - V)$ we have $\left|V^{-1} - I\right| \le C|V - I|$, where $C$ depends on $V$. We also have $M - I = V^{-1}(\gamma_F - VQ)V^{-1}Q^{-1} + V^{-1} - I$. Then, Lemma 5.20 implies that

$$\begin{aligned} \|M - I\|_2 &\le C\left(\|\gamma_F - VQ\|_2 + |V^{-1} - I|\right) \\ &\le C\left(\sum_{1 \le l \le d} \left\| \|DF_l\|_{\mathfrak{H}}^2 - q_l E F_l^2 \right\|_2 + |V - I|\right), \end{aligned}$$

where the constant $C$ depends on $k, V$ and $Q$. Therefore

$$E\left[\left|G_\beta(F) - g_\beta(F)\right|\right] \le C\left(|V - I| + \sum_{1 \le l \le d} \left\| \|DF_l\|_{\mathfrak{H}}^2 - q_l E F_l^2 \right\|_2^{\frac{1}{2}}\right). \qquad (5.81)$$

Combining it with (5.80) we obtain from (5.78) that

$$A_4 \le C\left(|V - I| + \sum_{1 \le l \le d} \left\| \|DF_l\|_{\mathfrak{H}}^2 - q_l E F_l^2 \right\|_2^{\frac{1}{2}}\right),$$

where the constant $C$ depends on $d, V, Q$. This completes the estimation of the term $A_4$. □

### 5.5.2 Sobolev norms of the inverse of the Malliavin matrix

In this subsection we estimate the Sobolev norms of $\gamma_F^{-1}$, the inverse of the Malliavin matrix $\gamma_F$ for a random variable $F$ of multiple stochastic integrals. We begin with the following estimate on the variance and Sobolev norms of $(\gamma_F)_{ij} = \langle DF_i, DF_j \rangle_{\mathfrak{H}}$, $1 \le i, j \le d$, following the approach of [31, 35, 37].

**Lemma 5.20.** *Let $F = I_p(f)$ and $G = I_q(g)$ with $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$ for $p, q \ge 1$. Then for all $k \ge 0$ there exists a constant $C_{p,q,k}$ such that*

$$\begin{aligned} &\left\| D^k \left( \langle DF, DG \rangle_{\mathfrak{H}} - \sqrt{pq} E[FG] \right) \right\|_2 \qquad (5.82) \\ \le\; & C_{p,q,k}(\|F\|_2^2 + \|G\|_2^2)\left( \left\| \|DF\|_{\mathfrak{H}}^2 - pE\left[F^2\right] \right\|_2^{\frac{1}{2}} + \left\| \|DG\|_{\mathfrak{H}}^2 - pE\left[G^2\right] \right\|_2^{\frac{1}{2}} \right). \end{aligned}$$

*Proof.* Without lost of generality, we assume $p \le q$. Applying (2.4) with the fact that $DI_p(f) = pI_{p-1}(f)$ we have

$$\begin{aligned} \langle DF, DG \rangle_{\mathfrak{H}} &= pq \left\langle I_{p-1}(f), I_{q-1}(g) \right\rangle_{\mathfrak{H}} \qquad (5.83) \\ &= pq \sum_{r=0}^{p-1} r! \binom{p-1}{r} \binom{q-1}{r} I_{p+q-2-2r}\left( f \widetilde{\otimes_{r+1}} g \right) \\ &= pq \sum_{r=1}^{p} (r-1)! \binom{p-1}{r-1} \binom{q-1}{r-1} I_{p+q-2r}\left( f \widetilde{\otimes}_r g \right). \end{aligned}$$

Note that $E[FG] = 0$ if $p < q$ and $E[FG] = \langle f, g \rangle_{\mathfrak{H}^{\otimes p}} = f \widetilde{\otimes}_p g$ if $p = q$. Then

$$\langle DF, DG \rangle_{\mathfrak{H}} - \sqrt{pq} E(FG) = pq \sum_{r=1}^{p} (1 - \delta_{qr})(r-1)! \binom{p-1}{r-1} \binom{q-1}{r-1} I_{p+q-2r}\left( f \widetilde{\otimes}_r g \right),$$

where $\delta_{qr}$ is again the Kronecker symbol. It follows that

$$E\left[ \langle DF, DG \rangle_{\mathfrak{H}} - \sqrt{pq} E[FG] \right]^2 \qquad (5.84)$$

$$= \quad p^2q^2\sum_{r=1}^{p}(1-\delta_{qr})(r-1)!^2\binom{p-1}{r-1}^2\binom{q-1}{r-1}^2(p+q-2r)!\left\|f\widetilde{\otimes}_r g\right\|^2_{\mathfrak{H}^{\otimes(p+q-2r)}}.$$

Note that if $r<p\leq q$, then (see also [35, (6.2.7)])

$$\begin{aligned}\left\|f\widetilde{\otimes}_r g\right\|^2_{\mathfrak{H}^{\otimes(p+q-2r)}} &\leq \|f\otimes_r g\|^2_{\mathfrak{H}^{\otimes(p+q-2r)}} = \left\langle f\otimes_{p-r} f, g\otimes_{q-r} g\right\rangle_{\mathfrak{H}^{\otimes 2r}} \\ &\leq \frac{1}{2}(\left\|f\otimes_{p-r} f\right\|^2_{\mathfrak{H}^{\otimes 2r}}+\left\|g\otimes_{q-r} g\right\|^2_{\mathfrak{H}^{\otimes 2r}}),\end{aligned} \tag{5.85}$$

and if $r=p<q$,

$$\left\|f\widetilde{\otimes}_p g\right\|^2_{\mathfrak{H}^{\otimes(q-p)}} \leq \left\|f\otimes_p g\right\|^2_{\mathfrak{H}^{\otimes(q-p)}} \leq \|f\|^2_{\mathfrak{H}^{\otimes p}}\left\|g\otimes_{q-p} g\right\|_{\mathfrak{H}^{\otimes 2p}}. \tag{5.86}$$

From (5.36) and (5.37) it follows that

$$\left\|\|DF\|^2_{\mathfrak{H}}-pE\left[F^2\right]\right\|_2^2 = p^4\sum_{r=1}^{p-1}(r-1)!^2\binom{p-1}{r-1}^2(2p-2r)!\,\|f\otimes_r f\|^2_{\mathfrak{H}^{\otimes(2p-2r)}}. \tag{5.87}$$

Combining (5.84)–(5.87) we obtain

$$\begin{aligned}&E\left[\langle DF,DG\rangle_{\mathfrak{H}}-\sqrt{pq}E\left[FG\right]\right]^2 \\ \leq\ & C_{p,q}(\left\|\|DF\|^2_{\mathfrak{H}}-pE\left[F^2\right]\right\|_2^2+\|F\|_2^2\left\|\|DG\|^2_{\mathfrak{H}}-pE\left[G^2\right]\right\|_2).\end{aligned}$$

Then (5.82) with $k=0$ follows from $\left\|\|DF\|^2_{\mathfrak{H}}-pE\left[F^2\right]\right\|_2 \leq C_p\|F\|_2^2$, which is implied by (2.12). From (5.83) we deduce

$$D^k\langle DF,DG\rangle_{\mathfrak{H}} = pq\sum_{r=1}^{p\wedge\frac{p+q-k}{2}}(r-1)!\binom{p-1}{r-1}\binom{q-1}{r-1}\frac{p+q-2r}{p+q-k-2r}I_{p+q-k-2r}\left(f\widetilde{\otimes}_r g\right).$$

Then it follows from (5.85)–(5.87) that

$$
\begin{aligned}
& E\left\|D^k\langle DF,DG\rangle_{\mathfrak{H}}\right\|^2_{\mathfrak{H}^{\otimes k}} \\
= \;& p^2q^2\sum_{r=1}^{p\wedge[(p+q-k)/2]}(r-1)!^2\binom{p-1}{r-1}^2\binom{q-1}{r-1}^2\frac{(p+q-2r)!^2}{(p+q-k-2r)!}\left\|f\widetilde{\otimes}_r g\right\|^2_{\mathfrak{H}^{\otimes(p+q-2r)}} \\
\le \;& C_{p,q}\Big(\left\|\|DF\|^2_{\mathfrak{H}}-pE\left[F^2\right]\right\|^2_2+\|F\|^2_2\left\|\|DG\|^2_{\mathfrak{H}}-pE\left[G^2\right]\right\|_2\Big).
\end{aligned}
$$

This completes the proof. □

The following lemma gives estimates on the Sobolev norms of the entries of $\gamma_F^{-1}$.

**Lemma 5.21.** *Let $F=(F_1,\dots,F_d)=(I_{q_1}(f_1),\dots,I_{q_d}(f_d))$ be nondegenerate and let $\gamma_F=\left(\langle DF_i,DF_j\rangle_{\mathfrak{H}}\right)_{1\le i,j\le d}$. Set $V=\left(E[F_iF_j]\right)_{1\le i,j\le d}$. Then for any real number $p>1$,*

$$\left\|\gamma_F^{-1}\right\|_p\le C\left\|(\det\gamma_F)^{-1}\right\|_{2p}, \tag{5.88}$$

*where the constant $C$ depends on $q_1,\dots,q_d,d,p$ and $V$. Moreover, for any integer $k\ge1$ and any real number $p>1$*

$$\left\|\gamma_F^{-1}\right\|_{k,p}\le C\left\|(\det\gamma_F)^{-1}\right\|^{k+1}_{(k+2)2p}\sum_{i=1}^{d}\left\|\|DF_i\|^2_{\mathfrak{H}}-q_iE\left[F_i^2\right]\right\|_2, \tag{5.89}$$

*where the constant $C$ depends on $q_1,\dots,q_d,d,p,k$ and $V$.*

*Proof.* Let $\gamma_F^*$ be the adjugate matrix of $\gamma_F$. Note that Hölder inequality and (2.12) imply

$$\left\|\langle DF_i,DF_j\rangle_{\mathfrak{H}}\right\|_p\le\|DF_i\|_{2p}\left\|DF_j\right\|_{2p}\le C_{V,p}$$

for all $1\le i,j\le d,\ p\ge1$. Applying Holder's inequality we obtain that the $p$ norm of $\gamma_F^*$ is also bounded by a constant. A further application of Holder's inequality to

$\gamma_F^{-1} = (\det \gamma_F)^{-1} \gamma_F^*$ yields

$$\left\|\gamma_F^{-1}\right\|_p \leq \left\|(\det \gamma_F)^{-1}\right\|_{2p} \| \gamma_F^* \|_{2p} \leq C_{V,p} \left\|(\det \gamma_F)^{-1}\right\|_{2p}, \tag{5.90}$$

which implies (5.88).

Since $F$ is nondegenerate, then (see [39, Lemma 2.1.6]) $\left(\gamma_F^{-1}\right)_{ij}$ belongs to $\mathbb{D}^\infty$ for all $i, j$ and

$$D\left(\gamma_F^{-1}\right)_{ij} = -\sum_{m,n=1}^{d} \left(\gamma_F^{-1}\right)_{im} \left(\gamma_F^{-1}\right)_{nj} D\left(\gamma_F\right)_{mn}. \tag{5.91}$$

Then, applying Hölder's inequality we obtain

$$\begin{aligned} \left\|D\left(\gamma_F^{-1}\right)\right\|_p &\leq \left\|\gamma_F^{-1}\right\|_{3p}^2 \|D\gamma_F\|_{3p}. \\ &\leq C_{V,p} \left\|(\det \gamma_F)^{-1}\right\|_{6p}^2 \sum_{i=1}^{d} \left\| \|DF_i\|_{\mathfrak{H}}^2 - q_i E\left[F_i^2\right] \right\|_2, \end{aligned}$$

where in the second inequality we used (5.88) and

$$\|D\gamma_F\|_{3p} \leq C_{V,p} \|D\gamma_F\|_2 \leq C_{V,p} \sum_{i=1}^{d} \left\| \|DF_i\|_{\mathfrak{H}}^2 - q_i E\left[F_i^2\right] \right\|_2$$

for all $p \geq 1$, which follows from (2.12) and (5.82). This implies (5.89) with $k = 1$. For higher order derivatives, (5.89) follows from repeating the use of (5.91), (2.12) and (5.82). □

The following lemma estimates the difference $\gamma_F^{-1} - V^{-1}Q^{-1}$.

**Lemma 5.22.** *Let $F = (F_1, \ldots, F_d) = (I_{q_1}(f_1), \ldots, I_{q_d}(f_d))$ be a nondegenerate random vector with $1 \leq q_1 \leq \cdots \leq q_d$ and $f_i \in \mathfrak{H}^{\odot q_i}$. Let $\gamma_F$ be the Malliavin matrix of $F$. Recall the notation of $V$ and $Q$ in (5.59). Then, for every integer $k \geq 1$ and any real number*

*$p > 1$ we have*

$$\left\|\gamma_F^{-1} - V^{-1}Q^{-1}\right\|_{k,p} \le C \left\|(\det \gamma_F)^{-1}\right\|_{(k+2)2p}^{k+1} \sum_{1\le l\le d} \left\| \|DF_l\|_{\mathfrak{H}}^2 - q_l E\left[F_l^2\right]\right\|_2^{\frac{1}{2}}, \quad (5.92)$$

*where the constant $C$ depends on $d, V, Q, p$ and $k$.*

*Proof.* In view of Lemma 5.21, we only need to consider the case when $k = 0$ because $V$ and $Q$ are deterministic matrices. Note that

$$\gamma_F^{-1} - V^{-1}Q^{-1} = \gamma_F^{-1}\,(VQ - \gamma_F)V^{-1}Q^{-1}.$$

Then, applying Holder's inequality we have

$$\left\|\gamma_F^{-1} - V^{-1}Q^{-1}\right\|_p \le C_{V,Q}\left\|\gamma_F^{-1}\right\|_{2p} \|VQ - \gamma_F\|_{2p}.$$

Note that (2.12) and (5.82) with $k = 0$ imply

$$\|VQ - \gamma_F\|_{2p} \le C_{V,Q,p}\|VQ - \gamma_F\|_2 \le C_{V,Q,p}\sum_{i=1}^{d}\left\|\|DF_i\|_{\mathfrak{H}}^2 - q_i E\left[F_i^2\right]\right\|_2^{\frac{1}{2}}.$$

Then, applying (5.90) we obtain

$$\left\|\gamma_F^{-1} - V^{-1}Q^{-1}\right\|_p \le C_{d,V,Q,p}\sum_{i=1}^{d}\left\|\|DF_i\|_{\mathfrak{H}}^2 - q_i E\left[F_i^2\right]\right\|_2^{\frac{1}{2}} \quad (5.93)$$

as desired. □

### 5.5.3 Technical estimates

In this subsection, we study the terms $A_1 = |E[h(F)] - E[h(N)]|$ in Equation (5.61) and $A_3 = E\left[\left|H_\beta(F) - K_\beta(F)\right|\right]$ in (5.63). For $A_1$, we shall use the multivariate Stein's method to give an estimate for a large class of non-smooth test functions $h$.

**Lemma 5.23.** *Let $h : \mathbb{R}^d \to \mathbb{R}$ be an almost everywhere continuous function such that $|h(x)| \le c(|x|^m + 1)$ for some $m, c > 0$. Let $F = (F_1, \ldots, F_d)$ be nondegenerate with $E[F_i] = 0, 1 \le i \le d$ and denote $N \sim N(0, I)$. Then there exists a constant $C_{m,c}$ depending on $m$ and $c$ such that*

$$|E[h(F)] - E[h(N)]| \le C_{m,c}(\|F\|_{2m}^m + 1) \sum_{i,j,k=1}^d \left\| \delta\left(A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right)\right\|_2, \tag{5.94}$$

*where $\gamma_F^{-1}$ is the inverse of the Malliavin matrix of $F$ and*

$$A_{ij} = \delta_{ij} - \left\langle DF_j, -DL^{-1}F_i\right\rangle_{\mathfrak{H}}. \tag{5.95}$$

*Proof.* For $\varepsilon > 0$, let

$$h_\varepsilon(x) = (\mathbf{1}_{\{|\cdot|<\frac{1}{\varepsilon}\}} h) * \rho_\varepsilon(x) = \int_{\mathbb{R}^d} \mathbf{1}_{|y|<\frac{1}{\varepsilon}} h(y)\rho_\varepsilon(x-y)dy.$$

where $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho(\frac{x}{\varepsilon})$, with $\rho(x) = C\mathbf{1}_{\{|x|<1\}}\exp(\frac{1}{|x|^2-1})$ and the constant $C$ is such that $\int_{\mathbb{R}^d}\rho(x)dx = 1$. Then $h_\varepsilon$ is Lipschitz continuous. Hence, the solution $f_\varepsilon$ to the following *Stein's equation*:

$$\Delta f_\varepsilon(x) - \langle x, \nabla f_\varepsilon(x)\rangle_{\mathbb{R}^d} = h_\varepsilon(x) - E[h_\varepsilon(N)] \tag{5.96}$$

exists and its derivative has the following expression [35, Page 82]

$$\partial_i f_\varepsilon(x) \quad = \quad \frac{\partial}{\partial x_i}\int_0^1 \frac{1}{2t} E[h_\varepsilon(\sqrt{t}x + \sqrt{1-t}N)]dt \tag{5.97}$$

$$= \int_0^1 E[h_\varepsilon(\sqrt{t}x+\sqrt{1-t}N)N_i]\frac{1}{2\sqrt{t}\sqrt{1-t}}dt.$$

It follows directly from the polynomial growth of $h$ that

$$|h_\varepsilon(x)| \le C_1 |x|^m + C_2 \tag{5.98}$$

for all $\varepsilon < 1$, where $C_1, C_2 > 0$ are two constants depending on $c$ and $m$. Then, from (5.96) we can write

$$|\partial_i f_\varepsilon(x)| \le C_1 |x|^m + C_2,$$

with two possibly different constants $C_1$, and $C_2$. Hence,

$$\|\partial_i f_\varepsilon(F)\|_2 \le C_1 \|F\|_{2m}^m + C_2. \tag{5.99}$$

Meanwhile, note that for $1 \le i \le d$,

$$\begin{aligned} E[F_i\partial_i f_\varepsilon(F)] &= E[LL^{-1}F_i\partial_i f_\varepsilon(F)] = E[\langle -DL^{-1}F_i, D\partial_i f_\varepsilon(F)\rangle] \\ &= \sum_{j=1}^d E[\langle -DL^{-1}F_i, \partial_{ij} f_\varepsilon(F)DF_j\rangle]. \end{aligned}$$

Then, replacing $x$ by $F$ and taking expectation in Equation (5.96) yields

$$|E[h_\varepsilon(F)] - E[h_\varepsilon(N)]| = \left|\sum_{i,j=1}^d E\left[\partial_{ij}^2 f_\varepsilon(F)A_{ij}\right]\right|. \tag{5.100}$$

Notice that

$$\langle DF_i, D\partial_i f_\varepsilon(F)\rangle_{\mathfrak{H}} = \left\langle DF_i, \sum_{j=1}^d \partial_{ij}^2 f_\varepsilon(F)DF_j\right\rangle_{\mathfrak{H}} = \sum_{j=1}^d \partial_{ij}^2 f_\varepsilon(F)\left\langle DF_i, DF_j\right\rangle_{\mathfrak{H}}$$

for all $1 \le i,k \le d$, which implies

$$\partial_{ij} f_\varepsilon(F) = \sum_{k=1}^{d} \left(\gamma_F^{-1}\right)_{jk} \langle DF_k, D\partial_i f_\varepsilon(F)\rangle_{\mathfrak{H}},$$

and hence

$$\begin{aligned}
\sum_{i,j=1}^{d} E\left[\partial_{ij}^2 f_\varepsilon(F) A_{ij}\right] &= \sum_{i,j=1}^{d} E\left[A_{ij}\left\langle \sum_{k=1}^{d} \left(\gamma_F^{-1}\right)_{jk} DF_k, D\partial_i f_\varepsilon(F)\right\rangle_{\mathfrak{H}}\right] \\
&= \sum_{i,j=1}^{d} E\left[\partial_i f_\varepsilon(F)\delta\left(A_{ij}\sum_{k=1}^{d}\left(\gamma_F^{-1}\right)_{jk} DF_k\right)\right].
\end{aligned}$$

Substituting this expression in (5.100) and using (5.99) we obtain

$$\begin{aligned}
|E[h_\varepsilon(F)] - E[h_\varepsilon(N)]| &= \sum_{i,j,k=1}^{d} E\left[\partial_i f_\varepsilon(F)\delta\left(A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right)\right] \\
&\le \sum_{i,j,k=1}^{d} \|\partial_i f_\varepsilon(F)\|_2 \left\|\delta\left(A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right)\right\|_2 \\
&\le \left(C_1 \|F\|_{2m}^m + C_2\right) \sum_{i,j,k=1}^{d} \left\|\delta\left(A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right)\right\|_2.
\end{aligned}$$

Then, we can conclude the proof by observing that

$$\lim_{\varepsilon\to 0} |E[h_\varepsilon(F)] - E[h_\varepsilon(N)]| = |E[h(F)] - E[h(N)]|,$$

which follows from (5.98) and the fact that $h_\varepsilon \to h$ almost everywhere. □

The next lemma gives an estimate for $\left\|\delta\left(A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right)\right\|_2$ when $F$ is a vector of multiple stochastic integrals.

**Lemma 5.24.** *Let $F = (F_1,\dots,F_d) = (I_{q_1}(f_1),\dots,I_{q_d}(f_d))$, where $f_i \in \mathfrak{H}^{\odot q_i}$, be nondegenerate and denote $N \sim N(0,I)$. Recall the notation of $V$ and $Q$ in* (5.59) *and $A_{ij}$ in*

(5.95). *Then, for all* $1 \leq i,j,k \leq d$ *we have*

$$\begin{aligned} \left\| \delta\left(A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right)\right\|_2 &\leq C \left\|(\det \gamma_F)^{-1}\right\|_{12}^3 \\ &\quad \times \left( |V-I| + \sum_{i=1}^d \left\| \|DF_i\|_{\mathfrak{H}}^2 - q_i E\left[F_i^2\right]\right\|_2^{\frac{1}{2}} \right), \end{aligned} \tag{5.101}$$

*where the constant* $C$ *depends on* $d, V, Q$.

*Proof.* Applying Meyer's inequality (2.9) we have

$$\left\| \delta\left(A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right)\right\|_2 \leq \left\| A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right\|_2 + \left\| D\left(A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right)\right\|_2.$$

Applying Holder's inequality and (2.12) we have

$$\left\| A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right\|_2 \leq \|A_{ij}\|_2 \left\|\left(\gamma_F^{-1}\right)_{jk}\right\|_4 \|DF_k\|_4 \leq C_{d,V,Q} \|A_{ij}\|_2 \left\|\left(\gamma_F^{-1}\right)_{jk}\right\|_4.$$

Similarly, Holder's inequality and (2.12) imply

$$\begin{aligned} \left\| D\left(A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right)\right\|_2 &\leq C_{d,V,Q}[\, \|DA_{ij}\|_2 \left\|\left(\gamma_F^{-1}\right)_{jk}\right\|_4 \\ &\quad + \|A_{ij}\|_2 \left\| D\left(\gamma_F^{-1}\right)_{jk}\right\|_4 + \|A_{ij}\|_2 \left\|\left(\gamma_F^{-1}\right)_{jk}\right\|_4 ]. \end{aligned}$$

Combining the above inequalities we obtain

$$\left\| \delta\left(A_{ij}\left(\gamma_F^{-1}\right)_{jk} DF_k\right)\right\|_2 \leq C_{d,V,Q} \|A_{ij}\|_{1,2} \left\|\left(\gamma_F^{-1}\right)_{jk}\right\|_{1,4}.$$

Note that

$$A_{ij} = \delta_{ij} - \left\langle DF_j, -DL^{-1}F_i \right\rangle_{\mathfrak{H}} = \delta_{ij} - V_{ij} + V_{ij} - \frac{1}{q_i}\left\langle DF_j, -DF_i \right\rangle_{\mathfrak{H}}.$$

Then, it follows from Lemma 5.20 that

$$\left\|A_{ij}\right\|_{1,2} \le C_{d,V,Q}\left(|V-I| + \sum_{i=1}^{d}\left\|\|DF_i\|_{\mathfrak{H}}^2 - q_iE\left[F_i^2\right]\right\|_2^{\frac{1}{2}}\right).$$

Then, the lemma follows by taking into account of (5.89) with $k=1$. □

As a consequence of the above two lemmas, we have the following result.

**Proposition 5.25.** *Let $h:\mathbb{R}^d \to \mathbb{R}$ be an almost everywhere continuous function such that $|h(x)| \le c\,(|x|^m + 1)$ for some $m, c > 0$. Let $F = (F_1, \dots, F_d) = (I_{q_1}(f_1), \dots, I_{q_d}(f_d))$, where $f_i \in \mathfrak{H}^{\odot q_i}$, be nondegenerate and denote $N \sim N(0, I)$. Recall the notation of $V$ and $Q$ in* (5.59). *Then*

$$\begin{aligned} |E[h(F)] - E[h(N)]| \quad &\le \quad C\left\|(\det\gamma_F)^{-1}\right\|_{12}^{3} \\ &\quad \times\left(|V-I| + \sum_{i=1}^{d}\left\|\|DF_i\|_{\mathfrak{H}}^2 - q_iE\left[F_i^2\right]\right\|_2^{\frac{1}{2}}\right), \end{aligned} \tag{5.102}$$

*where the constant $C$ depends on $d, V, Q, m, c$.*

In the following, we estimate the term $A_3 = E\left[\left|H_\beta(F) - K_\beta(F)\right|\right]$ in (5.63), where $H_\beta(F)$ and $K_\beta(F)$ are defined in (5.58) and (5.62), respectively.

**Lemma 5.26.** *Let $F = (F_1, \dots, F_d) = (I_{q_1}(f_1), \dots, I_{q_d}(f_d))$ be nondegenerate. Let $\beta = (\beta_1, \dots, \beta_k)$ be a multi-index of length $k \ge 1$. Let $H_\beta(F)$ and $K_\beta(F)$ be defined by* (5.58) *and* (5.62)*, respectively. Then there exists a constant $C$ depending on $d, V, Q, k$ such that*

$$\begin{aligned} E\left[\left|H_\beta(F) - K_\beta(F)\right|\right] \quad &\le \quad C\left\|(\det\gamma_F)^{-1}\right\|_{(k+4)2^{k+3}}^{k(k+2)} \\ &\quad \times\sum_{i=1}^{d}\left\|\|DF_i\|_{\mathfrak{H}}^2 - q_iE\left[F_i^2\right]\right\|_2^{\frac{1}{2}}. \end{aligned} \tag{5.103}$$

*Proof.* To simplify notation, we write $H_\beta$ and $K_\beta$ for $H_\beta(F)$ and $K_\beta(F)$, respectively. From (5.58) and (5.62) we see that

$$H_\beta - K_\beta = \delta\left(H_{\widehat{\beta}_k}\left(\gamma_F^{-1}DF\right)_{\beta_k} - K_{\widehat{\beta}_k}\left(\left(VQ\right)^{-1}DF\right)_{\beta_k}\right),$$

where $\widehat{\beta}_k = (\beta_1, \ldots, \beta_{k-1})$. For any $s \geq 0, p > 1$, using Meyer's inequality (2.9) we obtain

$$\begin{aligned} & \left\|H_\beta - K_\beta\right\|_{s,p} \\ \leq\ & C_{s,p}\left\|H_{\widehat{\beta}_k}\left(\gamma_F^{-1}DF\right)_{\beta_k} - K_{\widehat{\beta}_k}\left(\left(VQ\right)^{-1}DF\right)_{\beta_k}\right\|_{s+1,p} \\ \leq\ & C_{s,p}\left\|\left(H_{\widehat{\beta}_k} - K_{\widehat{\beta}_k}\right)\left(\left(VQ\right)^{-1}DF\right)_{\beta_k}\right\|_{s+1,p} \\ & + C_{s,p}\left\|H_{\widehat{\beta}_k}\left(\left(\gamma_F^{-1} - (VQ)^{-1}\right)DF\right)_{\beta_k}\right\|_{s+1,p}. \end{aligned}$$

Then, Hölder's inequality yields

$$\begin{aligned} & \left\|H_\beta - K_\beta\right\|_{s,p} \\ \leq\ & \left\|H_{\widehat{\beta}_k} - K_{\widehat{\beta}_k}\right\|_{s+1,2p}\left\|\left(\left(VQ\right)^{-1}DF\right)_{\beta_k}\right\|_{s+1,2p} \\ & + \left\|H_{\widehat{\beta}_k}\right\|_{s+1,2p}\left\|\left(\left(\gamma_F^{-1} - (VQ)^{-1}\right)DF\right)_{\beta_k}\right\|_{s+1,2p}. \end{aligned}$$

Note that (2.12) implies $\left\|\left(\left(VQ\right)^{-1}DF\right)_{\beta_k}\right\|_{s+1,2p} \leq C_{d,V,Q,s,p}$. Also note that (2.12), Hölder's inequality and (5.92) indicate

$$\left\|\left(\left(\gamma_F^{-1} - (VQ)^{-1}\right)DF\right)_{\beta_k}\right\|_{s+1,2p} \leq C_{d,V,Q,s,p}\Delta\left\|\left(\det\gamma_F\right)^{-1}\right\|_{(s+3)8p}^{s+2}.$$

where we denote

$$\Delta := \sum_{1\leq l\leq d}\left\|\|DF_l\|_{\mathfrak{H}}^2 - q_l E\left[F_l^2\right]\right\|_2^{\frac{1}{2}}$$

to simplify notation. Thus we obtain

$$\begin{aligned}\left\|H_\beta - K_\beta\right\|_{s,p} &\leq C_{d,V,Q,s,p}\left\|H_{\widehat{\beta}_k} - K_{\widehat{\beta}_k}\right\|_{s+1,2p} \\ &\quad + C_{d,V,Q,s,p}\Delta\left\|H_{\widehat{\beta}_k}\right\|_{s+1,2p}\left\|(\det\gamma_F)^{-1}\right\|_{(s+3)8p}^{s+2}.\end{aligned} \tag{5.104}$$

Similarly, from Meyer's inequality (2.9), Hölder's inequality and (2.12) we obtain by iteration

$$\begin{aligned}\left\|H_\beta\right\|_{s,p} &\leq C_{s,p}\left\|H_{\widehat{\beta}_k}\left(\gamma_F^{-1}DF\right)_{\beta_k}\right\|_{s+1,p} \\ &\leq C_{d,V,Q,s,p}\left\|H_{\widehat{\beta}_k}\right\|_{s+1,2p}\left\|(\det\gamma_F)^{-1}\right\|_{(s+3)8p}^{s+2} \\ &\quad \cdots \\ &\leq C_{d,V,Q,s,p,k}\left\|(\det\gamma_F)^{-1}\right\|_{(s+k+1)2^{k+2}p}^{k(s+k)}.\end{aligned} \tag{5.105}$$

Applying (5.105) into (5.104) and by iteration we can obtain

$$\left\|H_\beta - K_\beta\right\|_{s,p} \leq C_{d,V,Q,s,p,k}\left\|(\det\gamma_F)^{-1}\right\|_{(2s+k+4)2^{k+2}p}^{k(2s+k+2)}\Delta.$$

Now (5.103) follows by taking $s = 0$, $p = 2$ in the above inequality. □

## 5.6 Uniform estimates for densities of general random variables

In this section, we study the uniform convergence of densities for general random variables. We first characterize the convergence of densities with quantitative bounds for a sequence of centered random variables, using the density formula (5.17). In the second part of this section, a short proof of the uniform convergence of densities (without quantitative bounds) is given, using a compactness argument based on the assumption that the sequence converges in law.

### 5.6.1 Convergence of densities with quantitative bounds

In this subsection, we estimate the rate of uniform convergence for densities of general random variables. The idea is to use the density formula (5.17).

We use the following notations throughout this section.

$$\bar{w} = \left\langle DF, -DL^{-1}F \right\rangle_{\mathfrak{H}}, \; \bar{u} = -\bar{w}^{-1}DL^{-1}F.$$

The following technical lemma is useful.

**Lemma 5.27.** *Let $F \in \mathbb{D}^{2,s}$ with $s \geq 4$ such that $E[F] = 0$ and $E[F^2] = \sigma^2$. Let $m$ be the largest even integer less than or equal to $\frac{s}{2}$. Then there is a positive constant $C_m$ such that for any $t \leq m$,*

$$\left\|\bar{w} - \sigma^2\right\|_t \leq \left\|\bar{w} - \sigma^2\right\|_m \leq C_m \left\|D\bar{w}\right\|_m \leq C_m \left\|D\bar{w}\right\|_{s/2}. \tag{5.106}$$

*Proof.* It suffices to show the above second inequality. From the integration by parts formula in Malliavin calculus it follows

$$\sigma^2 = E[F^2] = E\left[\left\langle DF, -DL^{-1}F \right\rangle_{\mathfrak{H}}\right] = E\left[\bar{w}\right].$$

Note that from (5.16) and (5.20) we have $\bar{w} \in \mathbb{D}^{1,\frac{s}{2}}$. Then the lemma follows from the following infinite-dimensional Poincaré inequality [35, Lemma 5.3.8]:

$$E[(G - E[G])^m] \leq (m-1)^{m/2} E\left[\|DG\|_{\mathfrak{H}}^m\right],$$

for any even integer $m$ and $G \in \mathbb{D}^{1,m}$. □

The next theorem gives a bound for the uniform distance between the density of a random variable $F$ and the normal density.

**Theorem 5.28.** *Let $F \in \mathbb{D}^{2,s}$ with $s \geq 8$ such that $E[F] = 0$, $E[F^2] = \sigma^2$. Suppose $M^r := E\left[|\bar{w}|^{-r}\right] < \infty$, where $\bar{w} = \left\langle DF, -DL^{-1}F\right\rangle_{\mathfrak{H}}$ and $r > 2$. Assume $\frac{2}{r} + \frac{4}{s} = 1$. Then $F$ admits a density $f_F(x)$ and there is a constant $C_{r,s,\sigma,M}$ depending on $r, s, \sigma$ and $M$ such that*

$$\sup_{x\in\mathbb{R}} |f_F(x) - \phi(x)| \leq C_{r,s,\sigma,M} \|F\|_{1,s}^2 \left\| \left\|D^2F\right\|_{op}\right\|_{0,s}, \tag{5.107}$$

*where $\phi(x)$ is the density of $N \sim N(0,\sigma^2)$ and $\left\|D^2F\right\|_{op}$ indicates the operator norm of $D^2F$ introduced in* (5.16).

*Proof.* It follows from Proposition 5.5 that $F$ admits a density $f_F(x) = E\left[\mathbf{1}_{\{F>x\}}\delta(\bar{u})\right]$. Then

$$\sup_{x\in\mathbb{R}} |f_F(x) - \phi(x)| = \sup_{x\in\mathbb{R}} \left|E\left[\mathbf{1}_{\{F>x\}}\delta(\bar{u})\right] - \sigma^{-2}E[\mathbf{1}_{\{N>x\}}N]\right|. \tag{5.108}$$

Note that, from (2.7)

$$\delta(\bar{u}) = \delta(-DL^{-1}F\bar{w}^{-1}) = F\bar{w}^{-1} + \left\langle D\bar{w}^{-1}, DL^{-1}F\right\rangle_{\mathfrak{H}}.$$

Then

$$\begin{aligned} &\left|E\left[\sigma^2\mathbf{1}_{\{F>x\}}\delta(\bar{u})\right] - E[\mathbf{1}_{\{N>x\}}N]\right| \\ \leq\ & E\left[\left|F\bar{w}^{-1}(\sigma^2 - \bar{w})\right|\right] + \sigma^2 E\left[\left|\left\langle D\bar{w}^{-1}, DL^{-1}F\right\rangle_{\mathfrak{H}}\right|\right] \\ &+ \left|E\left[F\mathbf{1}_{\{F>x\}} - N\mathbf{1}_{\{N>x\}}\right]\right|. \end{aligned} \tag{5.109}$$

Note that for $t = \left(\frac{1}{r}+\frac{3}{s}\right)^{-1}$, we have $\frac{s}{2}-t \geq 2$, so there exists an even integer $m \in [t, \frac{s}{2}]$. Also, we have $\frac{1}{r}+\frac{1}{s}+\frac{1}{t}=1$. Then, we can apply Hölder's inequality and (5.106) to obtain

$$\begin{aligned} E\left[\left|F\bar{w}^{-1}(\bar{w}-\sigma^2)\right|\right] &\leq \|F\|_s \left\|\bar{w}^{-1}\right\|_r \left\|\bar{w}-\sigma^2\right\|_t \\ &\leq C_{r,s} \|F\|_s \left\|\bar{w}^{-1}\right\|_r \|D\bar{w}\|_{s/2}. \end{aligned} \tag{5.110}$$

Meanwhile, applying Hölder's inequality and (5.14) we have

$$\begin{aligned} E\left[\left|\bar{w}^{-2}\left\langle D\bar{w}, -DL^{-1}F\right\rangle_{\mathfrak{H}}\right|\right] &\leq \left\|\bar{w}^{-1}\right\|_r^2 \|D\bar{w}\|_{\frac{s}{2}} \left\|DL^{-1}F\right\|_{\frac{s}{2}} \\ &\leq \left\|\bar{w}^{-1}\right\|_r^2 \|D\bar{w}\|_{\frac{s}{2}} \|DF\|_s. \end{aligned} \tag{5.111}$$

Also, applying Lemma 5.2 for $h(y) = y\mathbf{1}_{\{y>x\}}$ and (5.106) we have

$$|E[F\mathbf{1}_{F>x} - N\mathbf{1}_{N>x}]| \leq C_\sigma \left\|\sigma^2 - \bar{w}\right\|_2 \leq C_\sigma \|D\bar{w}\|_{s/2}. \tag{5.112}$$

Applying the estimates (5.110)-(5.112) to (5.109) we have

$$\left|E\left[\sigma^2 \mathbf{1}_{F>x}\delta(\bar{u})\right] - E[\mathbf{1}_{N>x}N]\right| \leq C_{r,s,\sigma,M} \|F\|_{1,s} \|D\bar{w}\|_{s/2}. \tag{5.113}$$

Combining (5.108), (5.113) and (5.20) one gets

$$\sup_{x\in\mathbb{R}} |f_F(x) - \phi(x)| \leq C_{r,s,\sigma,M} \|F\|_{1,s}^2 \left\|\left\|D^2F\right\|_{op}\right\|_s.$$

This completes the proof. □

**Corollary 5.29.** *Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathbb{D}^{2,s}$ with $s \geq 8$ such that $E[F_n] = 0$ and $\lim_{n\to\infty} E[F_n^2] = \sigma^2$. Assume $E[F_n^2] \geq \delta > 0$ for all $n$. For $r > 2$ such that $\frac{2}{r}+\frac{4}{s} = 1$, assume* *(i)* $M_1 = \sup_n \|F_n\|_{1,s} < \infty$.

*(ii)* $M_2 = \sup_n E \left| \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}} \right|^{-r} < \infty$.

*(iii)* $E \left\| D^2 F_n \right\|_{op}^s \to 0$ *as* $n \to \infty$.

*Then each* $F_n$ *admits a density* $f_{F_n}(x)$ *and ,*

$$\sup_{x\in\mathbb{R}} |f_{F_n}(x) - \phi(x)| \le C \left( \left\| \left\| D^2 F_n \right\|_{op} \right\|_s + \left| E[F_n^2] - \sigma^2 \right| \right), \tag{5.114}$$

*where the constant* $C$ *depends on* $\sigma, M_1, M_2$ *and* $\delta$*. Moreover, if* $M_3 = \sup_n \|F_n\|_{2s} < \infty$*, then for any* $k \ge 1$ *and* $\alpha \in (\frac{1}{2}, k)$*,*

$$\|f_{F_n} - \phi\|_{L^k(\mathbb{R})} \le C \left( \left\| \left\| D^2 F_n \right\|_{op} \right\|_s + \left| E[F_n^2] - \sigma^2 \right| \right)^{\frac{k-\alpha}{k}},$$

*where the constant* $C$ *depends on* $\sigma, M_1, M_2, M_3, \alpha$ *and* $\delta$*.*

**Remark 5.30.** *By the "random contraction inequality"* (5.16)*, a sufficient condition for* (iii) *is* $E \left\| D^2 F_n \otimes_1 D^2 F_n \right\|_{\mathfrak{H}^{\otimes 2}}^{s/2} \to 0$ *or* $E \left\| D^2 F_n \right\|_{\mathfrak{H}^{\otimes 2}}^s \to 0$.

*Proof of Corollary* 5.29. It follows from Theorem 5.28 and Proposition 5.5 with an argument similar to Corollary 5.12. □

### 5.6.2 Compactness argument

In general, convergence in law does not imply convergence of the corresponding densities even if they exist. The following theorem specifies some additional conditions which ensure that convergence in law will imply convergence of densities.

**Theorem 5.31.** *Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of random variables in $\mathbb{D}^{2,s}$ satisfying any one of the following two conditions:*

$$\sup_n \|F_n\|_{2,s} + \sup_n \|F_n\|_{2p} + \sup_n \left\| \|DF_n\|_{\mathfrak{H}}^{-2} \right\|_r < \infty \tag{5.115}$$

*for some $p,r,s>1$ satisfying $\frac{1}{p}+\frac{1}{r}+\frac{1}{s}=1$, or*

$$\sup_n \|F_n\|_{2,s} + \sup_n \left\| \left| \langle DF_n, -DL^{-1}F_n \rangle_{\mathfrak{H}} \right|^{-1} \right\|_r < \infty \tag{5.116}$$

*for some $r,s>1$ satisfying $\frac{2}{r}+\frac{4}{s}=1$.*

*Suppose in addition that $F_n \to N \sim N(0,\sigma^2)$ in law. Then each $F_n$ admits a density $f_{F_n} \in C(\mathbb{R})$ given by either (5.8) or (5.17), and*

$$\sup_{x\in\mathbb{R}} |f_{F_n}(x) - \phi(x)| \to 0$$

*as $n\to\infty$, where $\phi$ is the density of $N$.*

*Proof.* We assume (5.115). The other condition can be treated identically. From Theorem 5.3 it follows that the density formula (5.8) holds for each $n$ and for all $x,y\in\mathbb{R}$

$$|f_{F_n}(x)| \le C(1\wedge x^{-2}),$$

$$|f_{F_n}(x) - f_{F_n}(y)| \le C\,|x-y|^{\frac{1}{p}}.$$

Hence the sequence $\{f_{F_n}\} \subset C(\mathbb{R})$ is uniformly bounded and equicontinuous. Then applying Azelà-Ascoli theorem, we obtain a subsequence $\{f_{F_{n_k}}\}$ which converges uniformly to a continuous function $f$ on $\mathbb{R}$ such that $0 \le f(x) \le C(1\wedge x^{-2})$. Then $f_{F_{n_k}} \to f$ in $L^1(\mathbb{R})$ as $k\to\infty$ with $\|f\|_{L^1(\mathbb{R})} = \lim_k \left\| f_{F_{n_k}} \right\|_{L^1(\mathbb{R})} = 1$. This implies that $f$ is a density function. Then $f$ must be $\phi$ because $F_n$ converges to $N$ in law. Since the limit is unique for any subsequence, we get the uniform convergence of $f_{F_n}$ to $\phi$. $\square$

**Corollary 5.32.** *Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of centered random variables in $\mathbb{D}^{2,4}$ with the following Wiener chaos expansions: $F_n = \sum_{q=1}^{\infty} J_q F_n$. Suppose that*

*(i)* $\lim_{Q\to\infty} \limsup_{n\to\infty} \sum_{q=Q+1}^{\infty} E[|J_q F_n|^2] = 0$.

*(ii) for every $q \geq 1$, $\lim_{n\to\infty} E[(J_q F_n)^2] = \sigma_q^2$.*

*(iii)* $\sum_{q=1}^{\infty} \sigma_q^2 = \sigma^2$.

*(iv) for all $q \geq 1$, $\langle D(J_q F_n), D(J_q F_n)\rangle_{\mathfrak{H}} \longrightarrow q\sigma_q^2$, in $L^2(\Omega)$ as $n \to \infty$.*

*(v)* $\sup_n \|F_n\|_{2,4} + \sup_n E[\|DF_n\|_{\mathfrak{H}}^{-8}] < \infty$.

*Then each $F_n$ admits a density $f_{F_n}(x)$ and*

$$\sup_{x\in\mathbb{R}} |f_{F_n}(x) - \phi(x)| \to 0$$

*as $n \to \infty$, where $\phi$ is the density of $N(0,\sigma^2)$.*

*Proof.* It has been proved by Nualart and Ortiz-Latorre in [40, Theorem 8] that under conditions (i)–(iv), $F_n$ converges to $N \sim N(0,\sigma^2)$ in law. The condition (v) implies (5.115) with $s = 4, p = 2, r = 4$. Then we can conclude from Theorem 5.31. $\square$

## 5.7 Applications

The main difficulty in applying Theorem 5.10 or Theorem 5.17 is the verification of the non-degeneracy condition of the Malliavin matrix: $\sup_n E[\|DF_n\|_{\mathfrak{H}}^{-p}] < \infty$ or $\sup_n E[|\det \gamma_{F_n}|^{-p}] < \infty$, respectively. In this section we consider the particular case of random variables in the second Wiener chaos and we find sufficient conditions for $\sup_n E[\|DF_n\|_{\mathfrak{H}}^{-p}] < \infty$. As an application we consider the problem of estimating the drift parameter in an Ornstein-Uhlenbeck process.

A general approach to verify $E[G^{-p}] < \infty$ for some positive random variable and for some $p \geq 1$ is to obtain a small ball probability estimate of the form

$$P(G \leq \varepsilon) \leq C\varepsilon^{\alpha} \quad \text{for some } \alpha > p \text{ and for all } \varepsilon \in (0, \varepsilon_0), \tag{5.117}$$

where $\varepsilon_0 > 0$ and $C > 0$ is a constant that may depend on $\varepsilon_0$ and $\alpha$. We refer to the paper by Li and Shao [23] for a survey on this topic. However, finding upper bounds of this type is a challenging topic, and the application of small ball probabilities to Malliavin calculus is still an unexplored domain.

### 5.7.1 Random variables in the second Wiener chaos

A random variable $F$ in the second Wiener chaos can always be written as $F = I_2(f)$ where $f \in \mathfrak{H}^{\odot 2}$. Without loss of generality we can assume that

$$f = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i, \tag{5.118}$$

where $\{\lambda_i, i \geq 1\}$ verifying $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \geq \ldots$ are the eigenvalues of the Hilbert-Schmidt operator corresponding to $f$ and $\{e_i, i \geq 1\}$ are the corresponding eigenvectors forming an orthonormal basis of $\mathfrak{H}$. Then, we have $F = I_2(f) = \sum_{i=1}^{\infty} \lambda_i (I_1(e_i)^2 - 1)$,

$$DF = 2\sum_{i=1}^{\infty} \lambda_i I_1(e_i) e_i \tag{5.119}$$

and

$$\|DF\|_{\mathfrak{H}}^2 = 4\sum_{i=1}^{\infty} \lambda_i^2 I_1(e_i)^2. \tag{5.120}$$

The following lemma provides necessary and sufficient conditions for a random variable of the form (5.120) to have negative moments.

**Lemma 5.33.** *Let* $G = \left(\sum_{i=1}^{\infty} \lambda_i^2 X_i^2\right)^{\frac{1}{2}}$, *where* $\{\lambda_i\}_{i\geq 1}$ *satisfies* $|\lambda_i| \geq |\lambda_{i+1}|$ *for all* $i \geq 1$ *and* $\{X_i\}_{i\geq 1}$ *are i.i.d standard normal. Fix an* $\alpha > 1$. *Then,* $E[G^{-2\alpha}] < \infty$ *if and only if there exists an integer* $N > 2\alpha$ *such that* $|\lambda_N| > 0$ *and in this case there exists a constant* $C_\alpha$ *depending only on* $\alpha$ *such that*

$$E[G^{-2\alpha}] \leq C_\alpha N^{-\alpha} |\lambda_N|^{-2\alpha}. \tag{5.121}$$

*Proof.* Notice $\lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda y} y^{\alpha-1} dy$ and $E[e^{-tX_i^2}] = \frac{1}{\sqrt{1+2t}}$ for all $t > 0$. If there exists $N > 2\alpha$ such that $|\lambda_N| > 0$, then

$$\begin{aligned} E[G^{-2\alpha}] &\leq E\left[\left(\sum_{i=1}^{N} \lambda_i^2 X_i^2\right)^{-\alpha}\right] = \frac{1}{\Gamma(\alpha)} E\left[\int_0^\infty e^{-y\sum_{i=1}^N \lambda_i^2 X_i^2} y^{\alpha-1} dy\right] \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} \prod_{i=1}^{N} (1+2\lambda_i^2 y)^{-\frac{1}{2}} dy. \end{aligned} \tag{5.122}$$

Since $\lambda_i^2$ is non increasing in $i$ and $N > 2\alpha$, using the change of variables $1 + 2\lambda_N^2 y = z$ we have

$$\begin{aligned} &\int_0^\infty y^{\alpha-1} \prod_{i=1}^{N} (1+2\lambda_i^2 y)^{-\frac{1}{2}} dy \leq \int_0^\infty y^{\alpha-1} (1+2\lambda_N^2 y)^{-\frac{N}{2}} dy \\ = &\ \left(2\lambda_N^2\right)^{-\alpha} \int_1^\infty (z-1)^{\alpha-1} z^{-\frac{N}{2}} dz = \left(2\lambda_N^2\right)^{-\alpha} \int_1^\infty \left(\frac{z-1}{z}\right)^{\alpha-1} z^{\alpha-1-\frac{N}{2}} dz \\ = &\ \left(2\lambda_N^2\right)^{-\alpha} \int_0^1 (1-x)^{\alpha-1} x^{\frac{N}{2}-\alpha-1} dx = \left(2\lambda_N^2\right)^{-\alpha} \frac{\Gamma(\alpha)\Gamma(\frac{N}{2}-\alpha)}{\Gamma(N/2)}, \end{aligned}$$

which implies (5.121).

On the other hand, if $|\lambda_i| = 0$ for all $i > 2\alpha$, let $N \leq 2\alpha$ be the largest nonnegative integer such that $|\lambda_N| > 0$. Then, the inequality in (5.122) becomes an equality. Using again that $\left\{\lambda_i^2\right\}_{i\geq 1}$ is a decreasing sequence we have

$$\int_0^\infty y^{\alpha-1}\prod_{i=1}^N (1+2\lambda_i^2 y)^{-\frac{1}{2}}dy \geq (1+2\lambda_1^2)^{-\frac{N}{2}}(\int_0^1 y^{\alpha-1}dy+\int_1^\infty y^{\alpha-1-\frac{N}{2}}dy)=\infty,$$

and we conclude that $E[G^{-2\alpha}]=\infty$. This completes the proof. □

The following theorem describes the distance between the densities of $F=I_2(f)$ and $N(0,E[F^2])$.

**Theorem 5.34.** *Let $F=I_2(f)$ with $f\in\mathfrak{H}^{\odot 2}$ given in (5.118). Assume that there exists $N>6m+6\left(\lfloor\frac{m}{2}\rfloor\vee 1\right)$, for some integer $m\geq 0$, such that $\lambda_N\neq 0$. Then $F$ admits an $m$ times continuously differentiable density $f_F$. Furthermore, if $\phi(x)$ denotes the density of $N(0,E[F^2])$, then for $k=0,1,\dots,m$,*

$$\sup_{x\in\mathbb{R}}\left|f_F^{(k)}(x)-\phi^{(k)}(x)\right|\leq C\left(\sum_{i=1}^\infty \lambda_i^4\right)^{\frac{1}{2}}\leq C\left(E[F^4]-3\left(E[F^2]\right)^2\right)^{\frac{1}{2}},$$

*where the constant $C$ depends on $N$ and $\lambda_N$.*

*Proof.* Taking into account of (5.120), we have

$$\operatorname{Var}\left(\|DF\|_{\mathfrak{H}}^2\right)=E\left|4\sum_{i=1}^\infty \lambda_i^2\left(I_1(e_i)^2-1\right)\right|^2=32\sum_{i=1}^\infty\lambda_i^4. \tag{5.123}$$

From (5.120) and Lemma 5.33 it follows that

$$E[\|DF\|_{\mathfrak{H}}^{-\beta}]\leq C_{\beta/2}N^{-\beta/2}|\lambda_N|^{-\beta}, \tag{5.124}$$

for all $\beta<N$. Then, the theorem follows from Theorem 5.13, taking into account (5.123).

□

Now we are ready to prove convergence of densities of random variables in the second Wiener chaos. Consider a sequence $F_n = I_2(f_n)$ with $f_n \in \mathfrak{H}^{\odot 2}$, which can be written as

$$f_n = \sum_{i=1}^{\infty} \lambda_{n,i} e_{n,i} \otimes e_{n,i}, \tag{5.125}$$

where $\{\lambda_{n,i}, i \geq 1\}$ verifies $|\lambda_{n,i}| \geq \left|\lambda_{n,i+1}\right|$ for all $i \geq 1$ and $\{e_{n,i}, i \geq 1\}$ are the corresponding eigenvectors.

**Theorem 5.35.** *Let $F_n = I_2(f_n)$ with $f_n \in \mathfrak{H}^{\odot 2}$ given by* (5.125)*. Assume that $\{\lambda_{n,i}\}_{n,i\in\mathbb{N}}$ satisfies*

*(i)* $\sigma^2 := 2\lim_{n\to\infty}\sum_{i=1}^{\infty}\lambda_{n,i}^2 > 0$*;*

*(ii)* $\lim_{n\to\infty}\sum_{i=1}^{\infty}\lambda_{n,i}^4 = 0$*;*

*(iii)* $\inf_n \left(\sup_{i>6m+6\left(\lfloor\frac{m}{2}\rfloor\vee 1\right)} |\lambda_{n,i}|\sqrt{i}\right) > 0$ *for some integer $m \geq 0$.*

*Then, each $F_n$ admits a density function $f_{F_n} \in C^m(\mathbb{R})$. Furthermore, for $k = 0, 1, \ldots, m$ and if $\phi$ denotes the density of the law $N(0,\sigma^2)$, the derivatives of $f_{F_n}^{(k)}$ converge uniformly to the the derivatives of $\phi$ with a rate given by*

$$\sup_{x\in\mathbb{R}}\left|f_{F_n}^{(k)}(x) - \phi^{(k)}(x)\right| \leq C\left[\left(\sum_{i=1}^{\infty}\lambda_{n,i}^4\right)^{\frac{1}{2}} + \left|2\sum_{i=1}^{\infty}\lambda_{n,i}^2 - \sigma^2\right|^{\frac{1}{2}}\right],$$

*where $C$ is a constant depending only on $m$ and the infimum appearing in condition (iii).*

***Proof of Theorem*** 5.35. Note that $E[(I_1(e_{n,i})^2-1)(I_1(e_{n,j})^2-1)] = 2\delta_{ij}$. Thus,

$$\sum_{i=1}^{\infty}\lambda_{n,i}^2 = \|f_n\|_{\mathfrak{H}^{\odot 2}}^2 = \frac{1}{2}E[F_n^2].$$

Then, the result follows from (5.123), (5.124) and Corollary 5.15. □

Condition (iii) in Theorem 5.35 means that there exist a positive constant $\delta > 0$ such that for each $n$ we can find an index $i(n) > 6m + 6\left(\lfloor \frac{m}{2} \rfloor \vee 1\right)$ with $|\lambda_{n,i(n)}|\sqrt{i(n)} \geq \delta$.

### 5.7.2 Parameter estimation in Ornstein-Uhlenbeck processes

Consider the following Ornstein-Uhlenbeck process

$$X_t = -\theta \int_0^t X_s ds + \gamma B_t,$$

where $\theta > 0$ is an unknown parameter, $\gamma > 0$ is known and $B = \{B_t, 0 \leq t < \infty\}$ is a standard Brownian motion. Assume that the process $X = \{X_t, 0 \leq t \leq T\}$ can be observed continuously in the time interval $[0,T]$. Then the least squares estimator (or the maximum likelihood estimator) of $\theta$ is given by $\widehat{\theta}_T = \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt}$. It is known (see for example, [26], [22]) that, as $T$ tends to infinity, $\widehat{\theta}_T$ converges to $\theta$ almost surely and

$$\sqrt{T}(\widehat{\theta}_T - \theta) = -\frac{TF_T}{\int_0^T X_t^2 dt} \xrightarrow{\mathscr{L}} N(0, 2\theta), \tag{5.126}$$

where

$$F_T = I_2(f_T) = \int_0^T \int_0^T f_T(t,s) dB_t dB_s, \tag{5.127}$$

with

$$f_T(t,s) = \frac{\gamma^2}{2\sqrt{T}} e^{-\theta|t-s|}. \tag{5.128}$$

Recently, Hu and Nualart [16] extended this result to the case where $B$ is a fractional Brownian motion with Hurst parameter $H \in [\frac{1}{2}, \frac{3}{4})$, which includes the standard Brownian motion case. Since $\frac{1}{T}\int_0^T X_t^2 dt \to \frac{1}{2}\gamma^2\theta^{-1}$ almost surely as $T$ tends to infinity, the main effort in proving (5.126) is to show the convergence in law of $F_T$ to the normal law

$N(0, \frac{\gamma^4}{2\theta})$. We shall prove that the density of $F_T$ converges as $T$ tends to infinity to the density of the normal distribution $N(0, \frac{\gamma^4}{2\theta})$.

**Theorem 5.36.** *Let $F_T$ be given by* (5.128) *and let $\phi$ be the density of the law $N(0, \sigma^2)$, where $\sigma^2 = \frac{\gamma^4}{2\theta}$. Then for each $T > 0$, $F_T$ has a smooth probability density $f_{F_T}$ and for any $k \geq 0$,*

$$\sup_{x\in\mathbb{R}} \left| f_{F_T}^{(k)}(x) - \phi^{(k)}(x) \right| \leq CT^{-\frac{1}{2}},$$

*where the constant $C$ depends on $k$, $\gamma$ and $\theta$.*

Before proving the theorem, let us first analyze the asypmototic behavior of the eigenvalues of $f_T$. The Hilbert space corresponding to Brownian motion $B$ is $\mathfrak{H} = L^2([0,T])$. Let $Q_T : L^2([0,T]) \to L^2([0,T])$ be the Hilbert-Schmidt operator associated to $f_T$, that is,

$$(Q_T \varphi)(t) = \int_0^T f_T(t,s)\varphi(s)ds \tag{5.129}$$

for $\varphi \in L^2[0,T]$. The operator $Q_T$ has eigenvalues $\lambda_{T,1} > \lambda_{T,2} > \cdots \geq 0$ and $\sum_{i=1}^{\infty} \lambda_{T,i}^2 < \infty$. The following lemma provides upper and lower bounds for these eigenvalues.

**Lemma 5.37.** *Fix $T > 0$. Let $f_T$ be given by* (5.128) *and $Q_T$ be given by* (5.129)*. The eigenvalues $\lambda_{T,i}$ of $Q_T$ (except maybe one) satisfy the following estimates*

$$\frac{\gamma^2\theta}{\sqrt{T}\left(\theta^2 + \left(\frac{i\pi + \frac{\pi}{2}}{T}\right)^2\right)} < \lambda_{T,i} < \frac{\gamma^2\theta}{\sqrt{T}\left(\theta^2 + \left(\frac{i\pi - \frac{\pi}{2}}{T}\right)^2\right)}. \tag{5.130}$$

*Proof.* Consider the eigenvalue problem $Q_T\varphi = \lambda\varphi$, that is,

$$\int_0^T f_T(t,s)\varphi(s)ds = \frac{\gamma^2}{2\sqrt{T}} \left( \int_0^t e^{-\theta(t-s)}\varphi(s)ds + \int_t^T e^{-\theta(s-t)}\varphi(s)ds \right) = \lambda\varphi(t). \tag{5.131}$$

Then, $\phi$ is differentiable and

$$\frac{\gamma^2\theta}{2\sqrt{T}}\left(-\int_0^t e^{-\theta(t-s)}\varphi(s)ds+\int_t^T e^{-\theta(s-t)}\varphi(s)ds\right)=\lambda\varphi'(t). \tag{5.132}$$

Differentiating again we have

$$\frac{\gamma^2\theta}{2\sqrt{T}}\left(-2\varphi(t)+\theta\int_0^t e^{-\theta(t-s)}\varphi(s)ds+\theta\int_t^T e^{-\theta(s-t)}\varphi(s)ds\right)=\lambda\varphi''(t).$$

Comparing this expression with (5.131), we obtain

$$(\theta^2-\frac{\gamma^2\theta}{\sqrt{T}\lambda})\varphi(t)=\varphi''(t). \tag{5.133}$$

Also, from (5.131) and (5.132) it follows that

$$\varphi(0)=\theta\varphi'(0),\ \varphi(T)=-\theta\varphi'(T). \tag{5.134}$$

Equations (5.133) and (5.134) form a Sturm-Liouville system. Its general solution is of the form

$$\varphi(t)=C_1\sin\mu t+C_2\cos\mu t,$$

where $C_1$ and $C_2$ are constants, and $\mu>0$ is an eigenvalue of the Sturm-Liouville system. By eliminating the constants $C_1$ and $C_2$ from (5.133) and (5.134) we obtain

$$-\mu^2=\theta^2-\frac{\gamma^2\theta}{\sqrt{T}\lambda}. \tag{5.135}$$

Then, the desired estimates on the eigenvalues of $Q_T\varphi = \lambda\varphi$will follow form estimates on $\mu$. Note that the Neumann condition (5.134) yields

$$(\mu^2\theta^2 - 1)\sin\mu T = 2\mu\theta\cos\mu T.$$

If we write $x = \mu\theta > 0$ (since $\mu, \theta > 0$), the above equation becomes

$$(x^2 - 1)\sin\frac{x}{\theta}T = 2x\cos\frac{x}{\theta}T.$$

The solution $x = 1$ corresponds to the eigenvalue $\mu = \frac{1}{\theta}$. If $x \neq 1$, then $\cos\frac{x}{\theta}T \neq 0$ and

$$\tan\frac{x}{\theta}T = \frac{2x}{x^2 - 1}. \tag{5.136}$$

For any $i \in \mathbb{Z}_+$, there is exactly one solution $x_i$ to (5.136) such that $\frac{x_i}{\theta}T \in (i\pi - \frac{\pi}{2}, i\pi + \frac{\pi}{2})$. Corresponding to each $x_i$ is an eigenvalue $\mu_i = \frac{x_i}{\theta}$ of the Sturm-Liouville system, satisfying $\dfrac{i\pi - \frac{\pi}{2}}{T} < \mu_i < \dfrac{i\pi + \frac{\pi}{2}}{T}$. The corresponding eigenvalue $\lambda_i$ of $Q_T$ obtained from Equation (5.135) satisfies the estimate (5.130). □

*Proof of Theorem* 5.36. For each $T$, let us compute the second moment of $F_T$.

$$\begin{aligned} E\left[F_T^2\right] &= \|f_T\|^2_{\mathfrak{H}^{\otimes 2}} = \int_0^T \int_0^T f_T(t,s)^2 dsdt \\ &= \frac{\gamma^4}{4T}\int_0^T \int_0^t e^{-2\theta(t-s)}dsdt = \frac{\gamma^4}{2\theta} - \frac{\gamma^4}{8\theta T}(1 - e^{-2\theta T}). \end{aligned}$$

Also, noticing that $F_T = I_2(f_T) = \delta^2(f_T)$ and

$$D_s D_t F_T^3 = 3F_T^2 f_T(t,s) + 6F_T I_1(f(\cdot,t)) \otimes I_1(f(\cdot,s)),$$

and using the duality between $\delta$ and $D$, we can write

$$\begin{aligned} E\left[F_T^4\right] &= E\left[\left\langle f_T, D^2 F_T^3 \right\rangle_{\mathfrak{H}^{\otimes 2}}\right] = 3E\left[F_T^2 \langle f_T, f_T\rangle_{\mathfrak{H}^{\otimes 2}}\right] \\ &\quad +6E\left[F_T \langle f_T(t,s), I_1(f_T(\cdot,t)) \otimes I_1(f_T(\cdot,s))\rangle_{\mathfrak{H}^{\otimes 2}}\right] \\ &= 3\left(E\left[F_T^2\right]\right)^2 + 6A, \end{aligned}$$

where

$$\begin{aligned} A &= E\left[F_T \langle f_T(t,s), I_1(f_T(\cdot,t)) \otimes I_1(f_T(\cdot,s))\rangle_{\mathfrak{H}^{\otimes 2}}\right] \\ &= \left\langle f_T(u,v), \langle f_T(t,s), f_T(u,t) \otimes f_T(v,s)\rangle_{\mathfrak{H}^{\otimes 2}} \right\rangle_{\mathfrak{H}^{\otimes 2}} \\ &= \frac{\gamma^8}{16T^2} \int_0^T \int_0^T \int_0^T \int_0^T e^{-\theta(|u-v|+|t-s|+|u-t|+|v-s|)} dudvdtds. \end{aligned}$$

Because the integrand is symmetric, we have

$$A = \frac{\gamma^8}{16T^2} 4! \int_0^T du \int_0^u dv \int_0^v ds \int_0^s dt \; e^{-2\theta(u-t)} \leq CT^{-1}.$$

Then, in order to complete the proof by applying Corollary 5.15, we only need to verify that condition (iii) of Theorem 5.35 holds for any integer $m \geq 1$, which implies the uniform boundedness of the negative moments

$$\sup_{T>0} E\left[\|DF_T\|_{\mathfrak{H}}^{-\beta}\right] < \infty$$

for any $\beta > 0$. Fix $\beta > 0$, and for each $T$, let $i(T) = \lfloor \beta + 1 \rfloor + \lfloor T \rfloor$. Then, the lower bound in (5.130) yields

$$\sqrt{i(T)}\lambda_{T,i(T)} \geq \frac{\sqrt{i(T)}\gamma^2/\theta}{\sqrt{T}\left(1 + \left(\frac{(i+1/2)\pi}{T\theta}\right)^2\right)}$$

$$\geq \quad \frac{\sqrt{i(T)}\gamma^2/\theta}{\sqrt{T}\left(1+\left(\frac{i(T)}{T}\right)^2 4\frac{\pi^2}{\theta^2}\right)} \geq \frac{\gamma^2/\theta}{\max_{(\beta+2)^{-1}\leq r\leq 1} g(r)} > 0,$$

where in the last inequality we made the substitution $r^{-1} = \frac{i(T)}{T}$ and set

$$g(r) := \sqrt{r}(1 + r^{-2}4\frac{\pi^2}{\theta^2}).$$

This implies condition (iii) and the proof of the theorem is complete. □

## 5.8 Appendix

In this section, we present the omitted proofs and some technical results.

*Proof of Lemma 5.1.* Since $\int_{-\infty}^{\infty}\{h(y) - E[h(N)]\}e^{-y^2/(2\sigma^2)}dy = 0$, we have

$$\int_{-\infty}^{x}\{h(y) - E[h(N)]\}e^{-y^2/(2\sigma^2)}dy = -\int_{x}^{\infty}\{h(y) - E[h(N)]\}e^{-y^2/(2\sigma^2)}dy.$$

Hence

$$\left|\int_{-\infty}^{x}\{h(y) - E[h(N)]\}e^{-y^2/(2\sigma^2)}dy\right| \leq \int_{|x|}^{\infty}[ay^k + b + E|h(N)|]e^{-y^2/(2\sigma^2)}dy.$$

By using the representation (5.6) of $f_h$ and Stein's equation (5.3) we have

$$\begin{aligned}
\left|f_h'(x)\right| &\leq |h(x) - E[h(N)]| + \frac{|x|}{\sigma^2}e^{x^2/(2\sigma^2)}\left|\int_{-\infty}^{x}\{h(y) - E[h(N)]\}e^{-y^2/(2\sigma^2)}dy\right| \\
&\leq a|x|^k + b + E|h(N)| + \frac{1}{\sigma^2}e^{x^2/(2\sigma^2)}\int_{|x|}^{\infty} y[ay^k + b + E|h(N)|]e^{-y^2/(2\sigma^2)}dy \\
&= a|x|^k + (b + E|h(N)|)\left(1 + \frac{1}{\sigma^2}s_1(x)\right) + \frac{a}{\sigma^2}s_{k+1}(x), \qquad (5.137)
\end{aligned}$$

where we let $s_k(x) = e^{x^2/(2\sigma^2)}\int_{|x|}^{\infty} y^k e^{-y^2/(2\sigma^2)}dy$ for any integer $k \geq 0$.

Note that $E|h(N)| \le aE|N|^k + b \le C_k a\sigma^k + b$ and

$$s_1(x) = e^{x^2/(2\sigma^2)} \int_x^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \sigma^2$$

for all $x \in \mathbb{R}$. Using integration by parts, we see by induction that for any integer $k \ge 1$,

$$\begin{aligned} s_{k+1}(x) &= e^{x^2/(2\sigma^2)} \int_{|x|}^{\infty} y^{k+1} e^{-y^2/(2\sigma^2)} dy \\ &= \sigma^2 e^{x^2/(2\sigma^2)} \int_{|x|}^{\infty} y^k d(-e^{-y^2/(2\sigma^2)}) = \sigma^2[|x|^k + k\, s_{k-1}(x)]. \end{aligned}$$

Then if $k \ge 1$ is even, we have

$$s_{k+1}(x) \le C_k \sigma^2 [|x|^k + \sigma^2 |x|^{k-2} + \cdots + \sigma^{k-2} s_1(x)] \le C_k \sigma^2 \sum_{i=0}^{k} \sigma^{k-i} |x|^i .$$

If $k \ge 1$ is odd, we have

$$s_{k+1}(x) \le C_k \sigma^2 [|x|^k + \sigma^2 |x|^{k-2} + \cdots + \sigma^{k-1}(|x| + s_0(x))] \le C_k \sigma^2 \sum_{i=0}^{k} \sigma^{k-i} |x|^i ,$$

where we used the fact that $s_0(x) \le s_0(0) = \sqrt{\frac{\pi}{2}}\sigma$ for all $x \in \mathbb{R}$ (indeed, when $x \ge 0$ we have $s_0'(x) = \frac{x}{\sigma^2} e^{x^2/(2\sigma^2)} \int_x^{\infty} e^{-y^2/(2\sigma^2)} dy - 1 \le e^{x^2/(2\sigma^2)} \int_x^{\infty} \frac{y}{\sigma^2} e^{-\frac{y^2}{2\sigma^2}} dy - 1 = 0$; similarly when $x < 0$, $s_0'(x) \ge 0$). Putting the above estimates into (5.137) we complete the proof.

□

*Proof of Remark* 5.7. We shall prove these properties by induction. From $T_1 = T_2 = 0$, (5.24) and (5.26) we know that $T_3 = D_u^2 \delta_u$, with $J_3 = \{(0,0,1)\}$; and $T_4 = \delta_u D_u^2 \delta_u + D_u^3 \delta_u$, with $J_4 = \{(1,0,1,0),(0,0,0,1)\}$. Now suppose the statement is true for all $T_l$ with $l \le k-1$ for $k \ge 5$. We want to prove the multi-indices of $T_k$ satisfy $(a)$–$(c)$. This will be done by studying the three operations, $\delta_u T_{k-1}$, $D_u T_{k-1}$ and $\partial_\lambda H_{k-1}(D_u\delta_u,\delta_u)D_u^2\delta_u$, in expression (5.26).

For the term $\partial_\lambda H_{k-1}(D_u\delta_u,\delta_u)D_u^2\delta_u$, we observe from (5.24) that

$$\partial_\lambda H_{k-1}(D_u\delta_u,\delta_u)D_u^2\delta_u = D_u^2\delta_u \sum_{1\le i\le \lfloor (k-1)/2\rfloor} i c_{k-1,i}\delta_u^{k-1-2i}(D_u\delta_u)^{i-1},$$

whose terms have multi-indices $(k-1-2i, i-1, 1, 0, \ldots, 0) \in \mathbb{N}^k$ for $1 \le i \le \lfloor \frac{k-1}{2} \rfloor$. Then, it is straightforward to check that these multi-indices satisfy $(a)$, $(b)$ and $(c)$.

The term $\delta_u T_{k-1}$ shifts the multi-index $(i_0, i_1, \ldots, i_{k-2}) \in J_{k-1}$ to multi-index $(i_0 + 1, i_1, \ldots, i_{k-2}, 0) \in \mathbb{N}^k$, which obviously satisfies $(a)$, $(b)$ and $(c)$, due to the induction hypothesis.

The third term $D_u T_{k-1}$ shifts the multi-index $(i_0, i_1, \ldots, i_{k-2}) \in J_{k-1}$ to either $\alpha = (i_0 - 1, i_1 + 1, \ldots, i_{k-2}, 0) \in \mathbb{N}^k$ if $i_0 \ge 1$, or to

$$\beta = \begin{cases} (i_0, i_1, \ldots, i_{j_0} - 1, i_{j_0+1} + 1, \ldots, i_{k-2}, 0), & \text{for } 1 \le j_0 \le k-3; \\ (i_0, i_1, \ldots, i_{j_0} - 1, 1), & \text{for } j_0 = k-2, \end{cases}$$

if $i_{j_0} \ge 1$. It is easy to check that $\beta$ satisfies properties $(a)$, $(b)$ and $(c)$ and $\alpha$ satisfies properties $(b)$ and $(c)$. We are left to verify that $\alpha$ satisfies property $(c)$. That is, we want to show that

$$1 + \sum_{j=1}^{k-2} i_j \le \lfloor \frac{k-1}{2} \rfloor. \tag{5.138}$$

If $k$ is odd, say $k = 2m+1$ for some $m \ge 2$, (5.138) is true because $(i_0, i_1, \ldots, i_{k-2}) \in J_{k-1}$, which implies by induction hypothesis that $\sum_{j=1}^{k-2} i_j \le \lfloor \frac{k-2}{2} \rfloor = m-1$. If $k$ is even, say $k = 2m+2$, (5.138) is true because the following claim asserts that if $i_0 \ge 1$, then $\sum_{j=1}^{k-2} i_j < \lfloor \frac{k-2}{2} \rfloor = m$.

**Claim:** For $(i_0, i_1, \ldots, i_{2m}) \in J_{2m+1}$ with $m \ge 1$, if $\sum_{j=1}^{2m} i_j = m$ then $i_0 = 0$.

Indeed, suppose $(i_0,i_1,\dots,i_{2m})\in J_{2m+1}$, $\sum_{j=1}^{2m} i_j=m$ and $i_0\ge 1$. We are going to show that leads to a contradiction. First notice that $i_1\ge 1$, otherwise $i_1=0$ and $\sum_{j=2}^{2m} i_j=m$, which is not possible because

$$i_0+2m\le i_0+\sum_{j=1}^{2m} ji_j\le 2m.$$

Also, we must have $i_{2m}=0$, because otherwise property $(a)$ implies $i_{2m}=1$ and $i_0=i_1=\cdots=i_{2m-1}=0$. Now we trace back to its parent multi-indices in $J_{2m}$ by reversing the three operations. Of the three operations, we can exclude $\partial_\lambda H_{2m}(D_u\delta_u,\delta_u)D_u^2\delta_u$ and $\delta_u T_{2m}$, because $\partial_\lambda H_{2m}(D_u\delta_u,\delta_u)D_u^2\delta_u$ generates $(2m-2j,j-1,1,0,\dots,0)$ with $1\le j\le m$, where $j$ must be $m$; and $\delta_u T_{2m}$ traces it back to $(i_0-1,i_1,\dots,i_{2m-1})\in J_{2m}$, where $i_1+\cdots+i_{2m-1}=m>\lfloor\frac{2m-1}{2}\rfloor$. Therefore, its parent multi-index in $J_{2m}$ must come from the operation $D_uT_{2m}$ and hence must be $(i_0+1,i_1-1,\dots,i_{2m-1})\in J_{2m}$. Note that for this multi-index, $i_1-1+\cdots+i_{2m-1}=m-1$. Repeating the above process we will end up at $(i_0+i_1,0,i_2\dots,i_{2m-i_1})\in J_{2m+1-i_1}$ with $i_2+\cdots+i_{2m-i_1}=m-i_1$, which contradicts the property $(b)$ of $J_{2m+1-i_1}$ because

$$i_0+2m-i_1\le i_0+i_1+\sum_{j=2}^{2m-i_1} ji_j\le 2m-i_1.$$

□

Recall that for any $k\ge 2$ we denote $D_{DF}w^{-1}=\left\langle Dw^{-1},DF\right\rangle_{\mathfrak{H}}$ and $D_{DF}^k w^{-1}=\left\langle D(D_{DF}^{k-1}w^{-1}),DF\right\rangle_{\mathfrak{H}}$. The following lemma estimates the $L^p(\Omega)$ norms of $D_{DF}^k w^{-1}$.

**Lemma 5.38.** *Let $F=I_q(f)$ with $q\ge 2$ satisfying $E[F^2]=\sigma^2$. For any $\beta\ge 1$ we define and $M_\beta=\left(E\,\|DF\|_{\mathfrak{H}}^{-\beta}\right)^{1/\beta}$. Set $w=\|DF\|_{\mathfrak{H}}^2$.*

*(i) If $M_\beta < \infty$ for some $\beta \geq 6$, then for any $1 \leq r \leq \frac{2\beta}{\beta+6}$*

$$\left\| D_{DF} w^{-1} \right\|_r \leq C M_\beta^3 \left\| q\sigma^2 - w \right\|_2 . \tag{5.139}$$

*(ii) If $k \geq 2$ and $M_\beta < \infty$ for some $\beta \geq 2k+4$, then for any $1 < r < \frac{2\beta}{\beta+2k+4}$*

$$\left\| D_{DF}^k w^{-1} \right\|_r \leq C \left( \sigma^{2k-2} \vee 1 \right) \left( M_\beta^{k+2} \vee 1 \right) \left\| q\sigma^2 - w \right\|_2 . \tag{5.140}$$

*(iii) If $k \geq 1$and $M_\beta < \infty$ for any $\beta > k+2$, then for any $1 < r < \frac{\beta}{k+2}$*

$$\left\| D_{DF}^k w^{-1} \right\|_r \leq C \left( \sigma^{2k} \vee 1 \right) \left( M_\beta^{k+2} \vee 1 \right) . \tag{5.141}$$

*Proof.* Note that $D_{DF} w^{-1} = \left\langle Dw^{-1}, DF \right\rangle_{\mathfrak{H}} = -2w^{-2} \left\langle D^2 F \otimes_1 DF, DF \right\rangle$. Then

$$\left| D_{DF} w^{-1} \right| \leq 2 w^{-\frac{3}{2}} \left\| D^2 F \otimes_1 DF \right\|_{\mathfrak{H}} .$$

Applying Hölder's inequality with $\frac{1}{r} = \frac{1}{p} + \frac{1}{2}$, yields

$$\left\| D_{DF} w^{-1} \right\|_r \leq 2 \left( E(w^{-\frac{3p}{2}}) \right)^{\frac{1}{p}} \left\| D^2 F \otimes_1 DF \right\|_2 ,$$

which implies (5.139) by choosing $p \leq \beta/3$ and taking into account (5.34). Notice that we need $1 \geq \frac{1}{r} \geq \frac{3}{\beta} + \frac{1}{2} = \frac{\beta+6}{2\beta}$.

Consider now the case $k \geq 2$. From the pattern indicated by the first three terms,

$$\begin{aligned}
D_{DF} w^{-1} &= \left\langle Dw^{-1}, DF \right\rangle_{\mathfrak{H}} , \\
D_{DF}^2 w^{-1} &= \left\langle D^2 w^{-1}, (DF)^{\otimes 2} \right\rangle_{\mathfrak{H}^{\otimes 2}} + \left\langle Dw^{-1} \otimes DF, D^2 F \right\rangle_{\mathfrak{H}^{\otimes 2}} , \\
D_{DF}^3 w^{-1} &= \left\langle D^3 w^{-1}, (DF)^{\otimes 3} \right\rangle_{\mathfrak{H}^{\otimes 3}} + 3 \left\langle D^2 w^{-1} \otimes DF, D^2 F \otimes DF \right\rangle_{\mathfrak{H}^{\otimes 3}}
\end{aligned}$$

$$+\left\langle Dw^{-1}\otimes D^2F, D^2F\otimes DF\right\rangle_{\mathfrak{H}^{\otimes 3}}+\left\langle Dw^{-1}\otimes (DF)^{\otimes 2}, D^3F\right\rangle_{\mathfrak{H}^{\otimes 3}},$$

we can prove by induction that

$$\left|D^k_{DF}w^{-1}\right|\le C\sum_{i=1}^k \left\|D^iw^{-1}\right\|_{\mathfrak{H}^{\otimes i}}\|DF\|^i_{\mathfrak{H}}\left(\sum_{\sum_{j=1}^k i_j=k-i}\prod_{j=1}^k \left\|D^jF\right\|^{i_j}_{\mathfrak{H}^{\otimes j}}\right).$$

By (2.12), for any $p>1$, $\left\|D^jF\right\|_p\le C\|F\|_2=C\sigma$. Applying Hölder's inequality and assuming that $s>r$, we have,

$$\left\|D^k_{DF}w^{-1}\right\|_r\le C\sum_{i=1}^k\left\|\left\|D^iw^{-1}\right\|_{\mathfrak{H}^{\otimes i}}\|DF\|^i_{\mathfrak{H}}\right\|_s\sigma^{k-i}. \tag{5.142}$$

We are going to see that $\|DF\|^i_{\mathfrak{H}}$ will contribute to compensate the singularity of $\left\|D^iw^{-1}\right\|_{\mathfrak{H}^{\otimes i}}$. First by induction one can prove that for $1\le i\le m$, $D^iw^{-1}$ has the following expression

$$D^iw^{-1}=\sum_{l=1}^i(-1)^l\sum_{(\alpha,\beta)\in I_{i,l}}w^{-(l+1)}\bigotimes_{j=1}^l\left(D^{\alpha_j}F\otimes_1 D^{\beta_j}F\right), \tag{5.143}$$

where $I_{i,l}=\{(\alpha,\beta)\in\mathbb{N}^{2l}:\alpha_j+\beta_j\ge 3,\sum_{j=1}^l(\alpha_j+\beta_j)=i+2l\}$. In fact, for $i=1$,

$$Dw^{-1}=-2w^{-2}D^2F\otimes_1 DF,$$

which is of the above form because $I_{1,1}=\{(1,2),(2,1)\}$. Suppose that (5.143) holds for some $i\le m-1$. Then,

$$\begin{aligned} D^{i+1}w^{-1} &= \sum_{l=1}^i(-1)^{l+1}2(l+1)\sum_{(\alpha,\beta)\in I_{i,l}}w^{-(l+2)}(D^2F\otimes_1 DF)\bigotimes_{j=1}^l\left(D^{\alpha_j}F\otimes_1 D^{\beta_j}F\right)\\ &\quad+\sum_{l=1}^i(-1)^l\sum_{(\alpha,\beta)\in I_{i,l}}w^{-(l+1)}\sum_{h=1}^l(D^{\alpha_j+1}F\otimes_1 D^{\beta_j}F+D^{\alpha_j}F\otimes_1 D^{\beta_j+1}F)\end{aligned}$$

$$\times \bigotimes_{j=1, j\neq h}^{l} \left( D^{\alpha_j} F \otimes_1 D^{\beta_j} F \right),$$

which is equal to

$$\sum_{l=1}^{i+1} (-1)^l \sum_{(\alpha,\beta)\in I_{i+1,l}} w^{-(l+1)} \bigotimes_{j=1}^{l} \left( D^{\alpha_j} F \otimes_1 D^{\beta_j} F \right).$$

From (5.143) for any $i = 1, \ldots, k$ we can write

$$\left\| D^i w^{-1} \right\|_{\mathfrak{H}^{\otimes i}} \|DF\|_{\mathfrak{H}}^i \leq \sum_{l=1}^{i} w^{-(l+1)+\frac{i}{2}} \sum_{(\alpha,\beta)\in I_{i,l}} \prod_{j=1}^{l} \left\| D^{\alpha_j} F \otimes_1 D^{\beta_j} F \right\|_{\mathfrak{H}^{\otimes \alpha_j+\beta_j-2}}, \quad (5.144)$$

where $I_{i,l} = \{(\alpha,\beta) \in \mathbb{N}^l \times \mathbb{N}^l : \alpha_j + \beta_j \geq 3, \sum_{j=1}^{l} (\alpha_j + \beta_j) = i + 2l\}$. Note that by (2.12),

$$\left\| D^{\alpha_j} F \otimes_1 D^{\beta_j} F \right\|_p \leq C \|F\|_2^2 = C\sigma^2$$

for all $p \geq 1$ and all $\alpha_j, \beta_j$. This inequality will be applied to all but one of the contraction terms in the product $\prod_{j=1}^{l} \left\| D^{\alpha_j} F \otimes_1 D^{\beta_j} F \right\|_{\mathfrak{H}^{\otimes \alpha_j+\beta_j-2}}$. We decompose the sum in (5.144) into two parts. If the index $l$ satisfies $l \leq \frac{i}{2} - 1$, then the exponent of $w$ is nonnegative, and the $p$-norm of $w$ can be estimated by a constant times $\sigma^2$, while for $\frac{i}{2} - 1 < l$ this exponent is negative. Then, using Hölder's inequality and assuming that $\frac{1}{s} = \frac{1}{p} + \frac{1}{2}$, we obtain

$$\begin{aligned} &\left\| \left\| D^i w^{-1} \right\|_{\mathfrak{H}^{\otimes i}} \|DF\|_{\mathfrak{H}}^i \right\|_s \\ \leq\ & C \left( \mathbf{1}_{\{i\geq 2\}} \sigma^{i-2} + \sum_{\frac{i}{2}-1<l\leq i} \left\| w^{-(l+1)+\frac{i}{2}} \right\|_p \sigma^{2(l-1)} \right) \left\| D^{\alpha_1} F \otimes_1 D^{\beta_1} F \right\|_2 \end{aligned} \quad (5.145)$$

Note that for $l \le i \le k$, $l+1-\frac{i}{2} \le \frac{k}{2}+1$. Therefore, for $\frac{i}{2}-1 < l \le i$

$$\left\| w^{-(l+1)+\frac{i}{2}} \right\|_p = M^{2l+2-i}_{2(l+1-\frac{i}{2})p} \le M^{2l+2-i}_{(k+2)p} \le M^{k+2}_{(k+2)p} \vee 1.$$

Therefore, using (5.34) we obtain

$$\left\| \|D^i w^{-1}\|_{\mathfrak{H}^{\otimes i}} \|DF\|^i_{\mathfrak{H}} \right\|_s \le C\left( (\sigma^{2i-2} \vee 1)(M^{k+2}_{(k+2)p} \vee 1) \right) \|q\sigma^2 - w\|_2. \quad (5.146)$$

Combining (5.146) and (5.142) and choosing $p$ such that $(k+2)p \le \beta$ we get (5.140). Note that we need

$$1 > \frac{1}{r} > \frac{k+2}{\beta} + \frac{1}{2} = \frac{\beta+2k+4}{2\beta},$$

which holds if $1 < r < \frac{2\beta}{\beta+2k+4}$. The proof of part (iii) is similar and omitted. □

The next lemma gives estimates on $D^k_u \delta_u$ for $k \ge 0$.

**Lemma 5.39.** *Let $F = I_q(f)$ with $q \ge 2$ satisfying $E[F^2] = \sigma^2$. For any $\beta \ge 1$ we define $M_\beta = \left(E\|DF\|^{-\beta}_{\mathfrak{H}}\right)^{1/\beta}$ and denote $w = \|DF\|^2_{\mathfrak{H}}$.*

*(i) If $M_\beta < \infty$ for some $\beta > 3$, then for any $1 < s < \frac{\beta}{3}$,*

$$\|\delta_u\|_s \le C(\sigma^2 \vee 1)(M^3_\beta \vee 1). \quad (5.147)$$

*(ii) If $k \ge 1$ and $M_\beta < \infty$ for some $\beta > 3k+3$, then for any $1 < s < \frac{\beta}{3k+3}$,*

$$\|D^k_u \delta_u\|_s \le C_\sigma(M^{3k+3}_\beta \vee 1). \quad (5.148)$$

*(iii) If $k \ge 2$ and $M_\beta < \infty$ for some $\beta > 6k+6$, then for any $1 < s < \frac{2\beta}{\beta+6k+6}$,*

$$\|D^k_u \delta_u\|_s \le C_\sigma(M^{3k+3}_\beta \vee 1)\|q\sigma^2 - w\|_2. \quad (5.149)$$

*Proof.* Recall that $\delta_u = qFw^{-1} - D_{DF}w^{-1}$. Then for any $r > s$,

$$\|\delta_u\|_s \le C\left(\sigma\|w^{-1}\|_r + \|D_{DF}w^{-1}\|_s\right).$$

Then, $\|w^{-1}\|_r = M_{2r}^2$ and the result follows by applying Lemma 5.38 (iii) with $k = 1$ and by choosing $r < \frac{\beta}{3}$.

To show (ii) and (iii) we need to find a useful expression for $D_u^k\delta_u$. Consider the operator $D_u = w^{-1}D_{DF}$. We claim that for any $k \ge 1$ the iterated operator $D_u^k$ can be expressed as

$$D_u^k = \sum_{l=1}^{k} w^{-l} \sum_{\mathbf{i}\in I_{l,k}} b_{\mathbf{i}} \left[\prod_{j=1}^{k-l} D_{DF}^{i_j} w^{-1}\right] D_{DF}^{i_0}, \tag{5.150}$$

where $b_{\mathbf{i}} > 0$ are real numbers and

$$I_{l,k} = \{\mathbf{i} = (i_0, i_1, \dots, i_l) : i_0 \ge 1, i_j \ge 0\ \forall\ j = 1, \dots, l, \sum_{j=0}^{k-l} i_j = k\}.$$

In fact, this is clearly true for $k = 1$. Assume (5.150) holds for a given $k$. Then

$$\begin{aligned}
D_u^{k+1} &= w^{-1}D_{DF}D^k u \\
&= \sum_{l=1}^{k} l w^{-l} D_{DF} w^{-1} \sum_{\mathbf{i}\in I_{l,k}} b_{\mathbf{i}} \left[\prod_{j=1}^{k-l} D_{DF}^{i_j} w^{-1}\right] D_{DF}^{i_0} \\
&\quad + \sum_{l=1}^{k} w^{-l-1} \sum_{\mathbf{i}\in I_{l,k}} b_{\mathbf{i}} \left[\sum_{h=1}^{k-l} D_{DF}^{i_h+1} w^{-1} \prod_{j=1, j\ne h}^{k-l} D_{DF}^{i_j} w^{-1}\right] D_{DF}^{i_0} \\
&\quad + \sum_{l=1}^{k} w^{-l-1} \sum_{\mathbf{i}\in I_{l,k}} b_{\mathbf{i}} \left[\prod_{j=1}^{k-l} D_{DF}^{i_j} w^{-1}\right] D_{DF}^{i_0+1}.
\end{aligned}$$

Shifting the indexes, this can be written as

$$D_u^{k+1} = \sum_{l=1}^{k} l w^{-l} D_{DF} w^{-1} \sum_{\mathbf{i}\in I_{l,k}} b_{\mathbf{i}} \left[\prod_{j=1}^{k-l} D_{DF}^{i_j} w^{-1}\right] D_{DF}^{i_0}$$

$$+\sum_{l=2}^{k+1} w^{-l} \sum_{\mathbf{i}\in I_{l-1,k}} b_{\mathbf{i}} \left[\sum_{h=1}^{k+1-l} D_{DF}^{i_h+1} w^{-1} \prod_{j=1,j\neq h}^{k+1-l} D_{DF}^{i_j} w^{-1}\right] D_{DF}^{i_0}$$
$$+\sum_{l=2}^{k+1} w^{-l} \sum_{\mathbf{i}\in I_{l-1,k}} b_{\mathbf{i}} \left[\prod_{j=1}^{k+1-l} D_{DF}^{i_j} w^{-1}\right] D_{DF}^{i_0+1}.$$

It easy to check that this coincides with

$$\sum_{l=1}^{k+1} w^{-l} \sum_{\mathbf{i}\in I_{l,k+1}} b_{\mathbf{i}} \left[\prod_{j=1}^{k+1-l} D_{DF}^{i_j} w^{-1}\right] D_{DF}^{i_0}.$$

Also, note that $\delta_u = qFw^{-1} + D_{DF}w^{-1}$ and

$$D_{DF}\delta_u = q + qFD_{DF}w^{-1} + D_{DF}^2 w^{-1}.$$

By induction we can show that for any $i_0 \geq 1$

$$D_{DF}^{i_0}\delta_u = q\delta_{1i_0} + q\sum_{j=1}^{i_0-1} c_{i,j} D_{DF}^{i_0-1-j} w D_{DF}^{j} w^{-1} + qFD_{DF}^{i_0} w^{-1} + D_{DF}^{i_0+1} w^{-1}, \qquad (5.151)$$

where $\delta_{1i_0}$ is the Kronecker symbol. Combining (5.150) and (5.151) we obtain

$$D_u^k\delta_u = \sum_{l=1}^{k} w^{-l} \sum_{\mathbf{i}\in I_{l,k}} b_{\mathbf{i}} \left[\prod_{j=1}^{k-l} D_{DF}^{i_j} w^{-1}\right] \times \left[q\delta_{1i_0} + q\sum_{j=1}^{i_0-1} c_{i,0j} D_{DF}^{i_0-1-j} w D_{DF}^{j} w^{-1} + qFD_{DF}^{i_0} w^{-1} + D_{DF}^{i_0+1} w^{-1}\right].$$

Next we shall apply Hölder's inequality to estimate $\left\|D_u^k\delta_u\right\|_s$. Notice that for $l = k$, $i_0 = k \geq 2$. Therefore,

$$\left\|D_u^k\delta_u\right\|_s \leq C_\sigma \sum_{l=1}^{k-1}\sum_{\mathbf{i}\in I_{l,k}} \|w^{-l}\|_p \prod_{j=1}^{k-l} \|D_{DF}^{i_j} w^{-1}\|_{r_j} \left(\delta_{1i_0} + \max_{1\leq h\leq i_0+1} \|D_{DF}^{h} w^{-1}\|_{r_0}\right)$$
$$+C_\sigma \|w^{-k}\|_p \max_{1\leq h\leq k+1} \|D_{DF}^{h} w^{-1}\|_{\rho_0} = B_1 + B_2,$$

assuming that for $l = 1, \ldots, k-1$, $\frac{1}{s} > \frac{1}{p} + \sum_{j=0}^{k-l} \frac{1}{r_j}$ and $\frac{1}{s} > \frac{1}{p} + \frac{1}{\rho_0}$, and where $C_\sigma$ denotes a function of $\sigma$ of the form $C(1+\sigma^M)$.

Let us consider first the term $B_1$. Note that if $i_0 = 1$ there is at least one factor of the form $\|D_{DF}^{r_j} w^{-1}\|_{r_j}$ in the above product, because $\sum_{j=1}^{k-l} i_j = k-1 \geq 1$. Then, we will apply the inequality (5.140) to one of these factors and the inequality (5.141) to the remaining ones. The estimate (5.141) requires $\frac{1}{r_j} > \frac{i_j+2}{\beta}$ for $j = 1, \ldots, k-l$ and $\frac{1}{r_0} > \frac{i_0+3}{\beta}$. On the other hand, the estimate (5.140) requires $\frac{1}{r_j} > \frac{i_j+2}{\beta} + \frac{1}{2}$ for $j = 1, \ldots, k-l$ and $\frac{1}{r_0} > \frac{i_0+3}{\beta} + \frac{1}{2}$. Then, choosing $p$ such that $2pl < \beta$, and taking into account that $\sum_{j=0}^{k-l} i_j = k$ we obtain the inequalities

$$\frac{1}{s} > \frac{1}{p} + \sum_{j=1}^{k-l} \frac{i_j+2}{\beta} + \frac{i_0+3}{\beta} + \frac{1}{2} > \frac{3k+3}{\beta} + \frac{1}{2}.$$

Hence, if $s < \frac{2\beta}{\beta+6k+6}$ we can write

$$\begin{aligned} B_1 &\leq C_\sigma \sum_{l=1}^{k-1} M_\beta^{2l} \prod_{j=1}^{k-l} (M_\beta^{i_j+2} \vee 1)(M_\beta^{i_0+3} \vee 1)\|q\sigma^2 - w^{-1}\|_2 \\ &\leq C_\sigma (M_\beta^{3k+3} \vee 1)\|q\sigma^2 - w^{-1}\|_2. \end{aligned}$$

For the term $B_2$ we use the estimate (5.140) assuming $2pk < \beta$ and

$$\frac{1}{s} > \frac{1}{p} + \frac{k+3}{\beta} + \frac{1}{2} > \frac{3k+3}{\beta} + \frac{1}{2}.$$

This leads to the same estimate and the proof of (5.149) is complete. To show the estimate (5.148) we proceed as before but using the inequality (5.141) for all the factors. In this case the summand $\frac{1}{2}$ does not appear and we obtain (5.148). □

# Chapter 6

# Non-degeneracy of some Sobolev pseudo-norms of fBms

As we have seen in the previous chapters, non-degeneracy of a random variable plays an fundamental role when applying the density formula in Malliavin calculus. In this chapter, we shall apply an upper bound estimate in small deviation theory to prove the non-degeneracy of some fucntionals of fractional Brownian motion (fBm).

## 6.1 Introduction

Let $B^H = \left\{B_t^H : t \geq 0\right\}$ be a fractional Brownian motion (fBm) on $(\Omega, \mathscr{F}, P)$. That is, $\left\{B_t^H : t \geq 0\right\}$ is a centered Gaussian process of Hurst parameter $H \in (0,1)$ with covariance

$$R_H(t,s) = E(B_t^H B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \tag{6.1}$$

Consider the random variable $F$ given by a functional of $B^H$:

$$F = \int_0^1 \int_0^1 \frac{\left|B_t^H - B_{t'}^H\right|^{2p}}{|t-t'|^q} dt dt', \tag{6.2}$$

where $p, q \geq 0$ satisfy $(2p-2)H > q-1$.

In the case of $H = \frac{1}{2}$, $B^H$ is a Brownian motion, and the random variable $F$ is the Sobolev norm on the Wiener space considered by Airault and Malliavin in [1]. This norm plays a central role in the construction of surface measures on the Wiener space. Fang [9] showed that $F$ is non-degenerate (see its definition below). Then it follows from the well-known density formula in Malliavin calculus that the law of $F^{\frac{1}{2}}$ has a smooth density.

In this note we shall show that for all $H \in (0,1)$, $F$ is non-degenerate.

In order to state our result precisely, we need some notations from Malliavin calculus (for which we refer to Nualart [39, Section 1.2]). Denote by $\mathscr{E}$ the set of all step functions on $[0,1]$. Let $\mathfrak{H}$ be the Hilbert space defined as closure of $\mathscr{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t,s), \text{ for } s,t \in [0,1].$$

Then the mapping $\mathbf{1}_{[0,t]} \mapsto B_t^H$ extends to a linear isometry between $\mathfrak{H}$ and the Gaussian space spanned by $B^H$. We denote this isometry by $B^H$. Then, for any $h, g \in \mathfrak{H}$, $B^H(f)$ and $B^H(g)$ are two centered Gaussian random variables with $E[B^H(h)B^H(g)] = \langle h, g \rangle_{\mathfrak{H}}$. We define the space $\mathbb{D}^{1,2}$ as the closure of the set of smooth and cylindrical random variable of the form

$$G = f(B^H(h_1), \ldots, B^H(h_n))$$

with $h_i \in \mathfrak{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ ($f$ and all its partial derivatives has polynomial growth) under the norm

$$\|G\|_{1,2} = \sqrt{E[G^2] + E[\|DG\|_{\mathfrak{H}}^2]},$$

where the $DF$ is the Malliavin derivative of $F$ defined as

$$DG = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B^H(h_1), \ldots, B^H(h_n))h_i.$$

We say that a random vector $\mathbf{V} = (V_1, \ldots, V_d)$ whose components are in $\mathbb{D}^{1,2}$ is *non-degenerate* if its Malliavin matrix $\gamma_{\mathbf{V}} = \left(\left\langle DV_i, DV_j \right\rangle_{\mathfrak{H}}\right)$ is invertible a.s. and $(\det \gamma_{\mathbf{V}})^{-1} \in L^k(X)$, for all $k \geq 1$ (see for instance [39, Definition 2.1.1]). Our main result is the following theorem.

**Theorem 6.1.** *For all $H \in (0,1)$, the functional $F$ of a fBm $B^H$ given in (6.2) is non-degenerate. That is,*

$$\|DF\|_{\mathfrak{H}}^{-1} \in L^k(X), \text{ for all } k \geq 1. \tag{6.3}$$

We shall follow the same scheme introduced in [9] to prove Theorem 6.1. That is, it suffices to prove that for any integer $n$, there exists a constant $C_n$ such that

$$P(\|DF\|_{\mathfrak{H}} \leq \varepsilon) \leq C_n \varepsilon^n \tag{6.4}$$

for all $\varepsilon$ small. This kind of inequality is called upper bound estimate in small deviation theory (also called small ball probability theory, for which we refer to [23] and the reference there in), which is still a challenging topic. To prove (6.4), we will need an upper bound estimate of the small deviation for the path variance of the fBm, which is introduced in the following section.

We comment that Li and Shao [24, Theorem 4] proved that

$$P\Big(\int_0^1 \int_0^1 \frac{\left|B_t^H - B_s^H\right|^{2p}}{|t-s|^q} dtds \leq \varepsilon\Big) \leq \exp\{-\frac{C}{\varepsilon^\beta}\} \tag{6.5}$$

for $p > 0$, $0 \leq q < 1 + 2pH$, $q \neq 1$ and $\beta = 1/(pH - \max\{0, q-1\}$. But (6.5) gives the small ball probability of $F$, not of $\|DF\|_{\mathfrak{H}}$.

## 6.2 An estimate on the path variance of fBm

**Lemma 6.2** (Estimate of the path variance of the fBm). *Let $B^H = \{B_t^H : t \geq 0\}$ be a fBm. For $a > b \geq 0$, consider the path variance $V_{[a,b]}(B^H)$ defined by*

$$V_{[a,b]}(B^H) = \int_a^b \left|B_t^H\right|^2 \frac{dt}{b-a} - (\int_a^b B_t^H \frac{dt}{b-a})^2.$$

*Then for $c_H = H\left((2H+1)\sin\frac{\pi}{2H+1}\right)^{-\frac{2H+1}{2H}}(\Gamma(2H+1)\sin(\pi H))^{\frac{1}{2H}}$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{H}} \log P(V_{[a,b]}(B^H) \leq \varepsilon^2) = -(b-a)c_H. \tag{6.6}$$

Actually, we will only need

$$\limsup_{\varepsilon\to 0} \varepsilon^{\frac{1}{H}} \log P(V_{[a,b]}(B^H) \leq \varepsilon^2) < \infty. \tag{6.7}$$

In the case of $H = \frac{1}{2}$, this estimate of the path variance for Brownian motion was introduced by Malliavin [27, Lemma 3.3.2], by using the following Payley–Wiener expansion of Brownian motion:

$$B_t = tG + \sqrt{2}\sum_{k=1}^{\infty} \frac{1}{2\pi k}(X_k \cos 2\pi kt + Y_k \sin 2\pi kt), \text{ a.s. for all } t \in [0,1], \tag{6.8}$$

where $G$, $X_k, Y_k$, $k \in \mathbb{N}$, are i.i.d. standard Gaussian random variables. Then the estimate (6.7) follows by observing that $V_{[0,1]}(B) = \frac{1}{2\pi^2}\sum_{k=1}^{\infty} \frac{1}{2\pi k}(X_k^2 + Y_k^2)$, a sum of $\chi^2(1)$ random variables. The above expansion of Brownian motion can be obtained by integrating an expansion of white noise on the orthonormal basis $\left\{1, \sqrt{2}\cos 2\pi kt, \sqrt{2}\sin 2\pi kt\right\}$ of $L^2[0,1]$. Payley–Wiener expansion of fBm has been proved recently by Dzhaparidze and van Zanten [8]:

$$B_t^H = tX + \sum_{k=1}^{\infty} \frac{1}{\omega_k} \left[ X_k(\cos 2\omega_k t - 1) + Y_k \sin 2\omega_k t \right], \tag{6.9}$$

where $0 < \omega_1 < \omega_2 < \dots$ are the real zeros of $J_{-H}$ (the Bessel function of the first kind of order $-H$), and $X$, $X_k, Y_k$, $k \in \mathbb{N}$, are independent centered Gaussian random variables with variance

$$EX^2 = \sigma_H^2, EX_k^2 = EY_k^2 = \sigma_k^2,$$

with $\sigma_H^2 = \frac{\Gamma(\frac{3}{2}-H)}{2H\Gamma(H+\frac{1}{2})\Gamma(3-2H)}$ and $\sigma_k^2 = \sigma_H^2(2-2H)\Gamma^2(1-H)\left(\frac{\omega_k}{2}\right)^{2H} J_{-H}(\omega_k)$. Because the path variance $V_{[0,1]}(B^H)$ becomes difficult to evaluate, it is clear that the techniques of [27, Lemma 3.3.2] does no longer work.

Fortunately, the recent developments of small deviation theory makes (6.6) ready to be reached.

*Proof.* In [30, Theorem 3.1 and Remark 3.1] Nazarov and Nikitin proved that for any square integrable random variable $G$ and any nonnegative function $\psi \in L^1[0,1]$,

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{H}} \log P\left(\int_0^1 (B_t^H - G)^2 \psi(t) dt \le \varepsilon^2\right) = -c_H \left( \int_0^1 \psi(t)^{\frac{1}{2H+1}} dt \right)^{\frac{2H+1}{2H}}. \tag{6.10}$$

Noticing that by the self-similarity property of fBm,

$$V_{[a,b]}(B^H) = \int_a^b \left(B_t^H - \overline{B^H}\right)^2 \frac{dt}{b-a} = b \int_{a/b}^1 \left(B_{bu}^H - \overline{B^H}\right)^2 \frac{du}{b-a}$$

has the same distribution as $b^{2H+1} \int_{a/b}^1 \left(B_u^H - b^{-H}\overline{B^H}\right)^2 \frac{du}{b-a}$. Then, Lemma 6.2 follows from (6.10) by taking $G = b^{-H}\overline{B^H}$ and $\psi(t) = \mathbf{1}_{[a/b,1]}(t)$. □

We comment that Bronski [3] proved (6.10) for $G = 0$ (which can be dropped, since a random variable $G$ doesn't contribute to the asymptotics of the Karhunen–Loeve eigenvalues) and $\psi \equiv 1$ by estimating the asymptotics of the Karhunen–Loeve eigenvalues of fBm.

## 6.3 Proof of the main theorem

In this section we prove (6.3) by estimating $P(\|DF\|_{\mathfrak{H}} \le \varepsilon)$ for $\varepsilon$ small.

For simplicity, we denote

$$\begin{aligned} I &= \left\{ \left(t,t'\right) \in [0,1]^2, t' \le t \right\}, \\ \vec{t} &= \left(t,t'\right), \ d\vec{t} = dtdt'. \end{aligned}$$

**Lemma 6.3.** *Let $Q(\vec{t},\vec{s}) = \langle \mathbf{1}_{[t',t]}, \mathbf{1}_{[s',s]} \rangle_{\mathfrak{H}}$. Then the operator $Q$ defined by*

$$Qf(\vec{t}) = \int_I Q(\vec{t},\vec{s}) f(\vec{s}) d\vec{s}, \ f \in L^2(I)$$

*is a symmetric positive compact operator on $L^2(I)$.*

*Proof.* Compactness follows from that $Q(\vec{t},\vec{s}) \in L^2(I\times I)$. The function $Q(\vec{t},\vec{s})$ is symmetric, so is the operator $Q$. Finally, $Q$ is positive because for any $f \in L^2(I)$,

$$\langle Qf, f \rangle_{L^2(I)} = \int_I \int_I Q(\vec{t},\vec{s}) f(\vec{s}) d\vec{s} f(\vec{t}) d\vec{t} = \left\| \int_I \mathbf{1}_{[t',t]} f(\vec{t}) d\vec{t}, \right\|_{\mathfrak{H}}^2.$$

□

Then, it follows that $Q$ has a sequence of decreasing eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$, i.e. $\lambda_1 \ge \cdots \ge \lambda_n > 0$, and $\lambda_n \to 0$. The corresponding normalized eigen–functions $\{\varphi_n\}_{n\in\mathbb{N}}$ form an orthonormal basis of $L^2(I)$. Each of them are continuous because $\phi_n(\vec{t}) =$

$\lambda_n^{-1} \int_I Q(\vec{t},\vec{s})\phi_n(\vec{s})d\vec{s}$ and $Q(\vec{t},\vec{s})$ is continuous. We can write

$$Q(\vec{t},\vec{s}) = \sum_{n\geq 1} \lambda_n \varphi_n(\vec{t})\varphi_n(\vec{s}). \tag{6.11}$$

From the definition of Malliavin derivative we have

$$\begin{aligned} D_r F &= 2p \int_{[0,1]^2} \frac{\left|B_t^H - B_{t'}^H\right|^{2p-1}}{|t-t'|^q} \text{sign}(B_t^H - B_{t'}^H)\mathbf{1}_{[t',t]}(r)d\vec{t}' \\ &= 4p \int_I \frac{\left|B_t^H - B_{t'}^H\right|^{2p-1}}{|t-t'|^q} \text{sign}(B_t^H - B_{t'}^H)\mathbf{1}_{[t',t]}(r)d\vec{t}'. \end{aligned}$$

Then denoting $\Delta(t,t') = \frac{\left|B_t^H - B_{t'}^H\right|^{2p-1}}{|t-t'|^q}\text{sign}(B_t^H - B_{t'}^H)$ we have

$$\begin{aligned} \|DF\|_{\mathfrak{H}}^2 &= 16p^2 \left\| \int_I \mathbf{1}_{[t',t]}(\cdot)\Delta(t,t')dtdt' \right\|_{\mathfrak{H}}^2 \\ &= 16p^2 \int_{I\times I} \langle \mathbf{1}_{[t',t]}, \mathbf{1}_{[s',s]}\rangle_{\mathfrak{H}} \Delta(t,t')\Delta(s,s')d\vec{t}d\vec{s}. \end{aligned} \tag{6.12}$$

Applying (6.11) with (6.12) we have

$$\|DF\|_{\mathfrak{H}}^2 = 16p^2 \sum_{i\geq 1} \lambda_i V_i^2, \tag{6.13}$$

where we denote

$$V_i = \int_I \varphi_i(t,t') \frac{\left|B_t^H - B_{t'}^H\right|^{2p-1}}{|t-t'|^q} \text{sign}(B_t^H - B_{t'}^H)dtdt'. \tag{6.14}$$

For each $\beta = (\beta_1, \ldots, \beta_n) \in S^{n-1}$ (the unit sphere in $\mathbb{R}^n$), let $\Psi_\beta = \beta_1\varphi_1 + \cdots + \beta_n\varphi_n$. We denote

$$G_\beta = \int_I \Psi_\beta^2(\vec{t}) \frac{\left|B_t^H - B_{t'}^H\right|^{2p-2}}{|t-t'|^q} d\vec{t}. \tag{6.15}$$

**Lemma 6.4.** *There exists a constant $C_{p,H} > 0$ such that for all $\beta \in S^{n-1}$ and $\varepsilon > 0$,*

$$P\left(G_\beta \leq \varepsilon\right) \leq \exp\left\{-C_{p,H}\varepsilon^{-\frac{1}{2H(p-1)}}\right\}. \tag{6.16}$$

*Proof.* For any $\beta \in S^{n-1}$, $\Psi_\beta \not\equiv 0$ since $\varphi_i, \dots, \varphi_n$ are linearly independent. Since $\Psi_\beta$ is continuous on $I$, there exists $\vec{t}_\beta = (t'_\beta, t_\beta) \in I$, $\delta_\beta$ and $\rho_\beta$ such that

$$\Psi_\beta(\vec{t}) \geq \rho_\beta > 0, \text{ for all } \vec{t} \in A_\beta := [t'_\beta - \delta_\beta, t'_\beta + \delta_\beta] \times [t_\beta - \delta_\beta, t_\beta + \delta_\beta] \subset I.$$

Meanwhile, $C = 2\max_{i\in\{1,\dots,n\}} \sup_{\vec{t}\in I} \left|\varphi(\vec{t})\right| < \infty$ and

$$\left|\Psi^2_\beta(\vec{t}) - \Psi^2_{\beta'}(\vec{t})\right| \leq C \left\|\beta - \beta'\right\|.$$

Then for any $\vec{t} \in A_\beta$ and any $\beta, \beta'$ satisfying $\|\beta - \beta'\| \leq \rho_\beta/2C$, one has

$$\Psi^2_\beta(\vec{t}) \geq \Psi^2_{\beta'}(\vec{t}) - \left|\Psi^2_\beta(\vec{t}) - \Psi^2_{\beta'}(\vec{t})\right| \geq \rho_{\beta'}/2. \tag{6.17}$$

Note that $S^{n-1}$ has a finite cover $S^{n-1} \subset \cup_{i=1}^m B(\beta^i, \frac{\rho_{\beta^i}}{2C})$. Denote $\rho_i = \rho_{\beta^i}$, $\delta_i = \delta_{\beta^i}, \vec{t}_i = \vec{t}_{\beta^i}$ and $A_i = A_{\beta^i}$. Then it follows from (6.17) that for any $\beta \in S^{n-1}$, there exists a $\beta^i$ such that

$$\Psi^2_\beta(\vec{t}) \geq \rho_i/2, \text{ for all } \vec{t} \in A_i.$$

Then noticing that $|t - t'| \leq 1$ and applying Jensen's inequality we obtain

$$\begin{aligned} G_\beta &\geq \frac{\rho_i}{2}\int_{A_i} \frac{\left|B^H_t - B^H_{t'}\right|^{2p-2}}{|t-t'|^q} d\vec{t} \geq \frac{\rho_i}{2}\int_{A_i} \left|B^H_t - B^H_{t'}\right|^{2p-2} d\vec{t} \\ &\geq \frac{\rho_i}{2(2\delta_i)^{p-2}} \left(\int_{A_i} \left(B^H_t - B^H_{t'}\right)^2 d\vec{t}\right)^{p-1}. \end{aligned} \tag{6.18}$$

Note that for $f \in C[a,b]$ with average $\overline{f} = \frac{1}{b-a}\int_a^b f(\xi)d\xi$, we have

$$\frac{1}{b-a}\int_a^b (f(\xi)-\overline{f})^2 d\xi \le \frac{1}{b-a}\int_a^b (f(\xi)-c)^2 d\xi$$

for any number $c$. Then

$$\int_{A_i} \left(B_t^H - B_{t'}^H\right)^2 d\vec{t} = \int_{t_i-\delta_i}^{t_i+\delta_i} \int_{t_i'-\delta_i}^{t_i'+\delta_i} \left(B_t^H - B_{t'}^H\right)^2 dt dt' \ge 2\delta_i \int_{t_i-\delta_i}^{t_i+\delta_i} \left(B_t^H - \overline{B^H}\right)^2 dt \tag{6.19}$$

where $\overline{B^H} = \int_{t_i-\delta_i}^{t_i+\delta_i} B_t^H dt$. Combining (6.18)–(6.19) and applying Lemma 6.2 we obtain

$$\begin{aligned} P\left(G_\beta \le \varepsilon\right) &\le P(\int_{t_i-\delta_i}^{t_i+\delta_i} \left(B_t^H - \overline{B^H}\right)^2 dt \le (\rho_i \delta_i)^{-\frac{1}{p-1}} \varepsilon^{\frac{1}{p-1}}) \\ &\le \exp\{-c_H \delta_i (\rho_i \delta_i)^{\frac{1}{2H(p-1)}} \varepsilon^{-\frac{1}{2H(p-1)}}\}. \end{aligned}$$

for any $i = 1, \ldots, m$. Let $C_{p,H} = c_H \inf_i \delta_i (\rho_i \delta_i)^{\frac{1}{2H(p-1)}}$, one gets (6.16). □

**Remark:** At the first glance, it seems that (6.16) can be obtained by applying (6.5) to the first inequality in (6.18). But (6.5) can only be applied to square interval on the diagonal like $[a,b]\times[a,b]$ (after applying the scaling and self-similarity property of fBm), and here the interval $A_i = [t_i' - \delta_i, t_i' + \delta_i] \times [t_i - \delta_i, t_i + \delta_i]$ is off diagonal.

**Lemma 6.5.** *For any integer n, the random vector* $\mathbf{V} = (V_1, \ldots, V_n)$ *defined in* (6.14) *is non-degenerate.*

*Proof.* Denote by $M = \left(\left\langle DV_i, DV_j \right\rangle_{\mathfrak{H}}\right)$ the Malliavin covariance of $\mathbf{V}$. We want to show that $(\det M)^{-1} \in L^k$, for any $k \ge 1$. Note that $\det M \ge \gamma_1^n$, where $\gamma_1 > 0$ is the smallest eigenvalue of the positive definite matrix $M$. Then it suffices to show that $\gamma_1^{-1} \in L^{nk}$, for any $k \geq 1$, for which is enough to estimate $P(\gamma_1 \leq \varepsilon)$ for $\varepsilon$ small. We have

$$\gamma_1 = \inf_{\|\beta\|=1} (M\beta, \beta) = \inf_{\|\beta\|=1} \left\| D\left(\sum_{i=1}^n \beta_i V_i\right)\right\|_{\mathfrak{H}}^2. \tag{6.20}$$

For any $\beta = (\beta_1, \dots, \beta_n) \in S^{n-1}$, let $\Psi_\beta(\vec{t}) = \sum_{i=1}^n \beta_i \varphi_i(\vec{t})$,

$$D_r\left(\sum_{i=1}^n \beta_i V_i\right) = (2p-1)\int_I \Psi_\beta(\vec{t}) \frac{\left|B_t^H - B_{t'}^H\right|^{2p-2}}{|t-t'|^q} \mathbf{1}_{[t',t]}(r) d\vec{t}.$$

Applying (6.11) as we computed (6.13),

$$\begin{aligned}
\left\| D\left(\sum_{i=1}^n \beta_i V_i\right)\right\|_{\mathfrak{H}}^2 &= (2p-1)^2 \int_0^1 dr \left(\int_I \Psi_\beta(\vec{t}) \frac{\left|B_t^H - B_{t'}^H\right|^{2p-2}}{|t-t'|^q} \mathbf{1}_{[t',t]}(r) d\vec{t}\right)^2 \\
&= (2p-1)^2 \sum_{i\geq 1} \lambda_i \left(\int_I \varphi_i(\vec{t}) \Psi_\beta(\vec{t}) \frac{\left|B_t^H - B_{t'}^H\right|^{2p-2}}{|t-t'|^q} d\vec{t}\right)^2 \\
&\geq (2p-1)^2 \sum_{i=1}^n \lambda_i q_i^2,
\end{aligned}$$

where $q_i = \int_I \varphi_i(\vec{t}) \Psi_\beta(\vec{t}) \frac{\left(B_t^H - B_{t'}^H\right)^{2p-2}}{|t-t'|^q} d\vec{t}$. Recall (6.15), we have $G_\beta = \sum_{i=1}^n \beta_i q_i$. Since $\lambda_1 \geq \cdots \geq \lambda_n > 0$, we have

$$\sum_{i=1}^n \lambda_i q_i^2 \geq \lambda_n \sum_{i=1}^n q_i^2 \geq \lambda_n \sum_{i=1}^n \beta_i^2 q_i^2 \geq \frac{\lambda_n}{n} G_\beta^2,$$

where in the third inequality we used the fact that $\sum_{i=1}^n a_i^2 \geq \frac{1}{n}(\sum_{i=1}^n a_i)^2$. Therefore

$$\left\| D\left(\sum_{i=1}^n \beta_i V_i\right)\right\|_{\mathfrak{H}}^2 \geq (2p-1)^2 \frac{\lambda_n}{n} G_\beta^2. \tag{6.21}$$

Combining (6.20) and (6.21) we have

$$\gamma_1 = \inf_{\|\beta\|=1} (M\beta,\beta) \geq (2p-1)^2 \frac{\lambda_n}{n} \inf_{\|\beta\|=1} G_\beta^2. \tag{6.22}$$

For any $\varepsilon > 0$ and $0 < \alpha < \frac{1}{2H(p-1)}$, let

$$W_\beta = \left\{ G_\beta \geq \varepsilon \right\},$$

$$W_n = \left\{ \|DV_i\|_{\mathfrak{H}}^2 \leq \exp \varepsilon^{-\alpha}, i = 1, \dots, n \right\}.$$

On $W_n$, for any $\beta, \beta' \in S^{n-1}$ we have

$$\left|(M\beta,\beta) - \left(M\beta',\beta'\right)\right| \leq C_n \left\|\beta - \beta'\right\| \exp \frac{1}{\varepsilon^\alpha},$$

where $C_n$ is a constant independent of $\beta, \beta'$ and $\varepsilon$.

Note that we can find a finite cover $\cup_{i=1}^m B(\beta^i, \exp(-\frac{2}{\varepsilon^\alpha}))$ of $S^{n-1}$ with $\beta^i \in S^{n-1}$ and

$$m \leq C \exp \frac{2n}{\varepsilon^\alpha}.$$

Then on $W_n$, for any $\beta \in S^{n-1}$, there exists a $\beta^i$ such that

$$(M\beta,\beta) \geq \left(M\beta^i,\beta^i\right) - C_n \exp \frac{1}{\varepsilon^\alpha} \exp(-\frac{2}{\varepsilon^\alpha}).$$

On $W_{\beta^i} \cap W_n$, applying (6.22) with $A_n = (2p-1)^2 \frac{\lambda_n}{n}$ and taking $\varepsilon$ small enough,

$$(M\beta,\beta) \geq A_n \varepsilon^2 - C_n \exp(-\frac{1}{\varepsilon^\alpha}) \geq \frac{A_n}{2} \varepsilon^2.$$

Hence, on $\cap_{i=1}^m W_{\beta^i} \cap W_n$,

$$\gamma_1 = \inf_{\|\beta\|=1} (M\beta, \beta) \geq \frac{A_n}{2}\varepsilon^2 > 0. \tag{6.23}$$

On the other hand, applying Lemma 6.4, we have

$$\begin{aligned} P(\cup_{i=1}^m W_{\beta^i}^c) &\leq \sum_{i=1}^m P(\cup_{i=1}^m W_{\beta^i}^c) \leq m\sqrt{2}\exp(-\frac{C_{p,H}}{\varepsilon^{1/2H(p-1)}}) \\ &\leq C\exp\frac{2n}{\varepsilon^\alpha}\exp(-\frac{C_{p,\alpha}}{\varepsilon^{1/2H(p-1)}}) \leq C\exp(-\frac{C}{\varepsilon^{1/2H(p-1)}}). \end{aligned} \tag{6.24}$$

Also, by Chebyshev's inequality,

$$P(W_n^c) \leq C\exp(-\frac{1}{\varepsilon^\alpha}). \tag{6.25}$$

Then it follows from (6.23)–(6.25) that for $\varepsilon$ small,

$$P(\gamma_1 < \frac{A_n}{2}\varepsilon^2) \leq C\exp(-\frac{1}{\varepsilon^\alpha}).$$

This completes the proof. □

***Proof of Theorem*** 6.1. Note that

$$\|DF\|_{\mathfrak{H}}^2 = 16p^2 \sum_{i\geq 1} \lambda_i V_i^2 \geq 16p^2\lambda_n \sum_{i=1}^n V_i^2, \tag{6.26}$$

for any integer $n$. Then, denoting $|\mathbf{V}|^2 = \sum_{i=1}^n V_i^2$ we have

$$P(\|DF\|_{\mathfrak{H}} < \varepsilon) \leq P\left(|\mathbf{V}| < \frac{\varepsilon}{4p\sqrt{\lambda_n}}\right).$$

Since $\mathbf{V} = (V_1, \ldots, V_n)$ is nondegenerate, then it has a smooth density $f_{V_n}(x)$. Then we have

$$P\left(|\mathbf{V}| < \frac{\varepsilon}{4p\sqrt{\lambda_n}}\right) \leq C_{n,p}\varepsilon^n,$$

where $C_{n,p} = \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})}\left(4p\sqrt{\lambda_n}\right)^{-n} \max_{|x|\leq 1} f_{V_n}(x)$. Now the theorem follows. $\square$

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