The Rebate Value Process with Some Applications

By

Nathan M. Welch

Submitted to the Department of Mathematics and the
Faculty of the Graduate School of the University of Kansas
in partial fulfillment of the requirements for the degree of
Master of Arts

Prof. Bożenna Pasik-Duncan,
Chairperson

Committee members

Prof. Tyrone Duncan

Prof. Daniel Katz

Date defended: May 13, 2013

The Thesis Committee for Nathan M. Welch certifies
that this is the approved version of the following thesis :

The Rebate Value Process with Some Applications

Prof. Bożenna Pasik-Duncan, Chairperson

Date approved: ___ May 17, 2013 ___

Abstract

In the pricing of credit derivatives default is modelled as a stopping time and prices are typically determined by separation of cash-flows before and at default. In a general risk-neutral valuation setting, this technique suggests the decomposition of an asset which holds even if the asset is not credit-sensitive. The rebate value process is introduced and related to the price of an asset before and after default. The financial interpretation of this process is different depending on the type of asset decomposed. An interpretation of recovery is illustrated by pricing several standard credit-sensitive assets including a risky coupon bond and a credit default swap (CDS). An interpretation of insurance is illustrated by pricing the complements of the credit "building blocks" with respect to the stopping time. Several applications of these complements are presented including a risky interest rate swap and a full-recovery CDS.

Acknowledgements

I thank my advisor, Prof. Bożenna Pasik-Duncan, for her patience, help, hours of editing, guidance, and generous support. I also thank Prof. Tyrone Duncan for his time, help, thoughtful questions, and generous support. Without them, this work would not have been possible. I would also like to thank Prof. Daniel Katz for always treating me as a friend and for teaching me some algebra. Thank you to Profs. Lukasz Stettner, Jacek Jakubowski, and Tomasz Bielecki for their helpful comments on an earlier draft.

Thanks also to the Department of Mathematics for their professionalism and assistance. In particular, I want to thank Prof. Milena Stanislavova, Prof. Margaret Bayer, Prof. Erik Van Vleck, and Lori Springs for their help with my degree requirements.

Thank you to Basil Williams, for asking me how to hedge a risky annuity, which led to the ideas herein. Thanks to Aric, Vegas, and Kim-Yen for being my support group. Last, but not least, thanks to Mom and Dad for everything.

This work is dedicated to Prof. Bożenna Pasik-Duncan, for her unwavering faith in me.

Contents

Chapter 1

Preliminaries

We study an arbitrage-free economy on a filtered probability space $(\Omega, \mathscr{G}, (\mathscr{G}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is a *risk-neutral measure*. Associated with the risk-neutral measure is a *short-rate process* (i.e., a stochastic interest rate) $\{r_t\}_{t\geq 0}$, which is assumed non-negative and adapted, and a discount process

$$D_t := \exp\left(-\int_0^t r_s ds\right), \qquad t \geq 0. \tag{1.1}$$

Because $\mathbb{P}$ is a risk-neutral measure, if $\{S_t\}_{t\geq 0}$ is the price process of an asset that makes no cash-flow payments (dividends), then the discounted price process $\{D_t S_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale, see [20][p. 216].

Related to the stochastic discount factor D is a *Zero Coupon Bond* (ZCB). The value of a ZCB with maturity s at time $t \leq s$ is $B_t^s = \mathbb{E}\left\{\frac{D_s}{D_t}\middle|\mathscr{G}_t\right\}$, see also Sec. 1.2.

Recall that the *forward price at t of an asset X for delivery at $s \geq t$* is defined to be the $\mathscr{G}_t$-measurable value $F_X(t,s)$ that makes the value of the forward pay-off at time s, $X_s - F_X(t,s)$, equal to zero at time t, see [15][p. 339]. That is, $F_X(t,s)$ is the $\mathscr{G}_t$-measurable value satisfying

$$\mathbb{E}\left\{\frac{D_s}{D_t}(X_s - F_X(t,s))\middle|\mathscr{G}_t\right\} = 0. \tag{1.2}$$

We assume that $(\Omega, \mathscr{G})$ supports a stopping time, τ, with respect to the filtration $\mathbf{G} = (\mathscr{G}_t)_{t\geq 0}$. For example, τ may be a default or an earnings forecast. In this paper, τ can be interpreted as the default of a specific reference entity, like a corporation, and we may refer to τ as the *default time*.

For the purposes of pricing, we restrict our attention to a fixed finite time interval $[0,T]$. The purpose of this restriction is to fix a *final maturity* T, rather than to assume that all processes live on a finite interval. This is because we wish to accomodate applications where $(\Omega,\mathscr{G},(\mathscr{G}_t)_{t\geq 0},\mathbb{P})$ is an enlargement of some existing model, see Sec. 1.1, below. In particular, we do not assume that $\mathscr{G}=\mathscr{G}_T$, nor do we assume that $\tau \leq T$. The evolution variable $t \geq 0$ can be thought of as the present, and we will generally be concerned with finding the price of an asset at t. One goal of our study is to price assets consistently after default. Accordingly, we do not necessarily assume that $t \leq \tau$. Where it is natural to assume that the default has not occured for reasons of financial interpretation, we will be explicit about this assumption. We will occasionally have need for a forward-time interval $[T_1,T_2] \subseteq [0,T]$. In such case, it is natural to assume that $0 \leq t \leq T_1 \leq T_2 \leq T$, though as we shall see, the case $t > T_1$ is easily accomodated. For generic or dummy time variables we use s or u.

Our approach to pricing is that the value of an asset is the value of its discounted cash-flows. That is, we assume that each asset X in the economy is associated, implicitly or explicitly, with a *(cumulative) cash-flow process* $\{C_u\}_{u\in[0,T]}$, with specific assumptions below. The cash-flow process C may be dependent on τ. Thus, assets can be classified as *default-sensitive* or *not default-sensitive*, according to whether their cash-flows depend on τ. When an asset is default-sensitive, we sometimes say the asset is *credit-risky* or just *risky*. Similarly, assets that are not default-sensitive and moreover depend only on the short-rate process r are sometimes called *credit-riskless* or just *riskless*. Since r is assumed stochastic, this terminology should not be confused with "deterministic".

The (cumulative) cash-flow process C is assumed to be a real-valued semi-martingale on $[0,T]$, and satisfies $C_u = C_T$ for $u \geq T$. The exact specification of the process C is asset-dependent, but it is typically an Itô process or a jump-diffusion process. The assumption that C is a semi-martingale ensures that it is a "good integrator", see Protter (2004) [16]. A typical cash-flow process in this paper is given by

$$dC_u = \mathbb{1}_{\{u \leq T\}} \left(c\mathbb{1}_{\{\tau > u\}} du + \mathbb{1}_{\{\tau > u\}} dI_T(u) + Z_\tau dN_u \right), \tag{1.3}$$

where $c \geq 0$ is a constant, $I_T(u) = \mathbb{1}_{\{T \leq u\}}$ is the maturity indicator, $N_u = \mathbb{1}_{\{\tau \leq u\}}$ is the default indicator, and $Z_\tau \in \mathscr{G}_\tau$ is a non-negative random variable.

Let X_t be the value at time t of the contract entitling the holder to $\{C_u\}_{u \geq t}$. We assume that the price X_t under the risk-neutral measure $\mathbb{P}$ is given by

$$X_t = \frac{1}{D_t} \mathbb{E} \left\{ \int_{[t,T]} D_u dC_u \middle| \mathscr{G}_t \right\}, \tag{1.4}$$

and that

$$\mathbb{E} |X_t| < \infty, \tag{1.5}$$

for all $t \geq 0$. By (1.4), we adopt the convention that the price of an asset at t includes the value of any cash-flow at t. To clarify the inclusion of cash-flows we will write integrals over sets, where appropriate. Elsewhere, we may write integrals with an upper and lower limit of integration for notational convenience. Eqn. (1.4) is a type of risk-neutral valuation formula (see, for instance, Harrison and Pliska (1981) [8]). It means that X_t is a no-arbitrage price, although not necessarily the unique no-arbitrage price. Any equivalent martingale measure will give a no-arbitrage price. In other words, we do not assume that $\mathbb{P}$ is a unique risk-neutral measure, and the market may be incomplete. The valuation formula (1.4) occurs in Bielecki and Rutkowki (2002) [1][Eqn. (2.2), p. 35]. The authors comment that "the validity of the valuation formula [1](2.2) is not obvious *a priori*, so that it needs to be examined on a case by case basis" [1][p. 35]. For sufficient conditions under which (1.4) holds, see [1][p. 37-9].

$D_t X_t$ is not generally a martingale, because it may make cash-flow payments at any time, but if C is non-decreasing, $D_t X_t$ is a super-martingale. Indeed, if $s \leq t$, then

$$\mathbb{E}\left(D_t X_t | \mathscr{G}_s\right) = \mathbb{E}\left(\mathbb{E}\left\{ \int_t^T D_u dC_u \middle| \mathscr{G}_t \right\} \middle| \mathscr{G}_s \right)$$

$$
\begin{aligned}
&= \mathbb{E}\left\{\int_t^T D_u dC_u \middle| \mathscr{G}_s\right\} \\
&= D_s X_s - \mathbb{E}\left\{\int_s^t D_u dC_u \middle| \mathscr{G}_s\right\} \qquad (1.6) \\
&\leq D_s X_s.
\end{aligned}
$$

Similarly, we can obtain $F_X(t,s)$, the forward price at t for delivery of X_s at $s \geq t$.

$$
\begin{aligned}
\mathbb{E}\left\{\frac{D_s}{D_t}\left(F_X(t,s) - X_s\right) \middle| \mathscr{G}_t\right\} &= 0 \\
\mathbb{E}\left\{\frac{D_s}{D_t} \middle| \mathscr{G}_t\right\} F_X(t,s) &= \frac{1}{D_t}\mathbb{E}\left\{D_s X_s | \mathscr{G}_t\right\} \\
F_X(t,s) &= \frac{1}{B_t^s}\left(X_t - \mathbb{E}\left\{\int_t^s \frac{D_u}{D_t} dC_u \middle| \mathscr{G}_t\right\}\right),
\end{aligned}
$$

where we have used (1.6) and the identity $B_t^s = \mathbb{E}\left\{\frac{D_s}{D_t} \middle| \mathscr{G}_t\right\}$. These calculations show that we must be careful to value the intermediate cash-flows, just as in valuation involving any dividend-paying asset.

Let us note, however, the fact that the *discounted gain process* M is a martingale, where

$$
M_t := \mathbb{E}\left\{\int_0^T D_u dC_u \middle| \mathscr{G}_t\right\}, \qquad 0 \leq t \leq T. \qquad (1.7)
$$

1.1 Model Choice

Let $N_t = \mathbb{1}_{\{\tau \leq t\}}$ be the default indicator, as above, and let $\mathbf{N} = (\mathscr{N}_t, t \geq 0)$ be its associated filtration, where $\mathscr{N}_t = \sigma(N_s, 0 \leq s \leq t)$. The results in this paper are at the level of the filtration $\mathbf{G} = (\mathscr{G}_t, t \geq 0)$, where $\mathscr{N}_t \subseteq \mathscr{G}_t$ for all t. Thus the nature of the calculations herein primarily involve conditional expectations with respect to some $\mathscr{G}_t$. To evaluate these expectations requires the choice of a model. For example, we may think of $\mathscr{G}_t$ as an *enlargement of filtration*, in the following sense.

Suppose that we begin with a default-free model. Let $\mathbf{F} = (\mathscr{F}_t, t \geq 0)$ be the information available to the default-free market participants. For example, $\mathscr{F}_t$ may be the filtration generated by an $\mathbb{R}^d$-valued *state process* or a vector of *primary securities*. This default-free model is usually endowed with a risk-neutral measure $\mathbb{Q}$, and may also be complete. Now suppose a default time $\tau \notin \mathscr{F}_t$ is added as a source of risk in the economy. A new filtration, sometimes called the *full filtration*, $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{N}_t$, can then be considered. This is the so-called "enlargement of filtration" by the stopping time τ [11][p. 2]. Intuitively, the idea is to model all default-free dynamics within the default-free model and then to extend the space with the full filtration, thereby adding an auxiliary default time which supervenes on the default-free model.

More generally, we can think of $\mathscr{G}_t \supseteq \mathscr{F}_t \vee \mathscr{N}_t$, with a particular specification of **G** and **F** as constituting part of the model choice. Then the above assumption that $\tau \notin \mathscr{F}_t$ is often, but not always the case. For example, classical structural models, as in Merton (1974) [14], determine default based on the level of a *firm-value process*, V. As noted by Blanchet-Scalliet and Jeanblanc (2004) [3][p. 145], these models assume that $\mathscr{G}_t = \mathscr{F}_t = \mathscr{F}_t^V$ (and thus $\tau \in \mathscr{F}_t$), where $\mathscr{F}_t^V$ is the filtration generated by observing $\{V_s, 0 \leq s \leq t\}$. *Reduced-form models*, in contrast, assume that $\tau \notin \mathscr{F}_t$.

Remark An important technical consideration in the enlargement of filtration literature is the *martingale invariance hypothesis*:

(**H**) Any **F** square-integrable martingale is a **G** square-integrable martingale.

This hypothesis implies that the dynamics of the asset price are the same in the defaultable and default-free spaces [3][p. 150]. It is important in specifying conditions for the absence of arbitrage in an enlarged filtration. In this paper, we work at the level of **G** under the assumption that the valuation equation (1.4) holds, i.e., that the defaultable market is arbitrage-free. Outside of the present discussion of model choice, we are not concerned herein with whether **G** arises as an enlarged filtration. For a discussion of hypothesis (**H**), see [11][Sec. 2.7], [3][Sec. 3], and the references therein.

For $X \in \mathscr{G}_T$, the no arbitrage price at time $t \leq T$ of an asset which pays X at T is

$$\mathbb{E}\left\{ \frac{D_T}{D_t} X \mathbb{1}_{\{\tau > T\}} \middle| \mathscr{G}_t \right\}, \tag{1.8}$$

where the expectation is with respect to a $\mathscr{G}$-risk-neutral measure. If $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{N}_t$, such expectations can be further decomposed, see, for example, Jeanblanc and Rutkowski (2000a, 2000c) [10, 12]. In this case,

$$\mathbb{E}\left\{\frac{D_T}{D_t}X\mathbb{1}_{\{\tau>T\}}\middle|\mathscr{G}_t\right\} = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}\left\{\frac{D_T}{D_t}X\mathbb{1}_{\{\tau>T\}}\middle|\mathscr{F}_t\right\}}{\mathbb{P}(\tau>t|\mathscr{F}_t)}. \tag{1.9}$$

Next consider $F_t := \mathbb{P}(\tau \leq t|\mathscr{F}_t)$. If $F_t < 1$ a.s. for all $t \geq 0$, we can define the **F**-*hazard process* of τ,

$$\Gamma_t := -\ln(1-F_t).$$

Then, the no-arbitrage price becomes

$$\mathbb{E}\left\{\frac{D_T}{D_t}X\mathbb{1}_{\{\tau>T\}}\middle|\mathscr{G}_t\right\} = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left\{e^{-(\Gamma_T-\Gamma_t)}\frac{D_T}{D_t}X\middle|\mathscr{F}_t\right\}, \tag{1.10}$$

see [1, 10, 12]. If further, τ admits an intensity λ so that $\Gamma_t = \int_0^t \lambda_s ds$, the no-arbitrage price can be evaluated as

$$\mathbb{E}\left\{\frac{D_T}{D_t}X\mathbb{1}_{\{\tau>T\}}\middle|\mathscr{G}_t\right\} = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left\{\exp\left(-\int_t^T(r_s+\lambda_s)ds\right)X\middle|\mathscr{F}_t\right\}. \tag{1.11}$$

This is typically the case in reduced-form models, as in Lando (1998) [13].

Various cases where $\mathscr{G}_t \supseteq \mathscr{F}_t \vee \mathscr{D}_t$, particularly in the presence of a hazard process, have been extensively studied in work by Bielecki, Jeanblanc, Rutkowski and others. See, for example, [1, 10, 12].

1.2 Credit Derivatives Building Blocks

In the pricing of credit derivatives, certain quantities frequently arise in pricing, which are sometimes referred to as "building blocks", see, for example, Schönbucher (2003) [18][Sec. 3.3]. In Lando (1998) [13][Sec. 3], the author prices these fundamental credit securities presence of a model where default is given by the first jump of a Cox process. Here, we record these building blocks in the general setting above for later reference.[1]

The first two building blocks do not depend on τ, and so are credit-riskless. As mentioned above, a *Zero Coupon Bond* (ZCB), is an asset which makes a riskless payment of one unit at maturity T. The value of the ZCB at time $t \leq T$, B_t^T, is

$$B_t^T := \mathbb{E}\left\{ \frac{D_T}{D_t} \middle| \mathscr{G}_t \right\} = \mathbb{E}\left\{ \exp\left(-\int_t^T r_s ds \right) \middle| \mathscr{G}_t \right\}. \tag{1.12}$$

Similarly, a *(continuous) riskless annuity on* $(T_1, T_2]$ pays one unit per annum continuously on the interval $(T_1, T_2]$. I.e., the riskless annuity makes an infinitesmal payment $dC_u = \mathbb{1}_{\{T_1 < u \leq T_2\}} du$. Its value at time $t \geq 0$, $R(t; T_1, T_2)$, is given by

$$\begin{aligned} R(t; T_1, T_2) &:= \mathbb{E}\left\{ \int_{T_1 \vee (t \wedge T_2)}^{T_2} \frac{D_u}{D_t} du \middle| \mathscr{G}_t \right\} \\ &= \int_{T_1 \vee (t \wedge T_2)}^{T_2} B_t^u du, \end{aligned} \tag{1.13}$$

using (1.12). If $T_1 > t$, $R(t; T_1, T_2)$ is called a *forward-starting riskless annuity*. If $T_1 \leq t$, payments have already started and the annuity is equivalent to an annuity on $(t, T_2]$ and we can consistently define:

$$R_t^{T_2} := R(t; t, T_2) = \mathbb{E}\left\{ \int_{t \wedge T_2}^{T_2} \frac{D_u}{D_t} du \middle| \mathscr{G}_t \right\}. \tag{1.14}$$

A *risky zero coupon bond* is a promised payment by the debtor to the bondholder of one unit at maturity T if $\tau > T$. If $\tau \leq T$, the bondholder receives nothing ("zero recovery" from default). Let $\tilde{B}_t^T$ be the value of the risky zero coupon bond at some $t \leq T$. Then,

[1] Although several authors use the notion of "building blocks", the constituents of the set of building blocks is not standard. For example, Schönbucher (2003) includes a defaultable zero coupon bond but not a risky annuity. In place of a defaultable ZCB, Lando (1998) uses a general risky claim at maturity, $X\mathbb{1}_{\{\tau > T\}}$, and also includes a risky annuity which may have variable payments.

$$\tilde{B}_t^T := \mathbb{E}\left\{ \frac{D_T}{D_t} \mathbb{1}_{\{\tau > T\}} \middle| \mathscr{G}_t \right\}. \tag{1.15}$$

A *(continuous) risky annuity* on $(T_1, T_2]$ pays one unit per annum continuously during the interval $(T_1, T_2]$, so long as default has not occurred. I.e., the risky annuity makes an infinitesmal payment $dC_u = \mathbb{1}_{\{T_1 < u \le T_2\}} \mathbb{1}_{\{\tau > u\}} du$. Let $\tilde{R}(t; T_1, T_2)$ be the value at time $t \ge 0$ of a risky annuity on $(T_1, T_2]$. Then,

$$\begin{aligned} \tilde{R}(t; T_1, T_2) &:= \mathbb{E}\left\{ \int_{T_1 \vee (t \wedge T_2)}^{T_2} \frac{D_u}{D_t} \mathbb{1}_{\{\tau > u\}} du \middle| \mathscr{G}_t \right\} \\ &= \int_{T_1 \vee (t \wedge T_2)}^{T_2} \tilde{B}_t^u du, \end{aligned} \tag{1.16}$$

using (1.15). For the value of a risky annuity on $(t, T_2]$, we write $\tilde{R}_t^{T_2} := \tilde{R}(t; t, T_2)$.

Last, the contract for *unit payment at default in* $(T_1, T_2]$ pays one unit at τ, if $\tau \in (T_1, T_2]$. Let $N_t = \mathbb{1}_{\{\tau \le t\}}$ be the default indicator function. Then $dC_u = \mathbb{1}_{\{T_1 < u \le T_2\}} dN_u$, a jump-process. Denote the value at $t \ge 0$ of this security by $\hat{U}(t; T_1, T_2)$. Then,

$$\hat{U}(t; T_1, T_2) = \mathbb{E}\left\{ \frac{D_\tau}{D_t} \mathbb{1}_{\{T_1 < \tau \le T_2\}} \middle| \mathscr{G}_t \right\}. \tag{1.17}$$

For the value of the contract for unit payment at default in $(t, T_2]$, we write $\hat{U}_t^{T_2} := \hat{U}(t; t, T_2)$.

Chapter 2

Rebate Value Process

Recall from (1.7) that discounted gain process, M, is a martingale. Obviously, the discounted gain process can be split at τ into two martingales,

$$\tilde{M}_t := \mathbb{E}\left\{ \int_{[0,\tau)} D_u dC_u \middle| \mathscr{G}_t \right\}, \qquad 0 \leq t \leq T, \tag{2.1}$$

and,

$$\hat{M}_t := \mathbb{E}\left\{ \int_{[\tau,T]} D_u dC_u \middle| \mathscr{G}_t \right\}, \qquad 0 \leq t \leq T, \tag{2.2}$$

such that $M = \tilde{M} + \hat{M}$. We are concerned with valuing the claim to *future* cash-flows at some t, that is, with the price X_t, in (1.4). In terms of the discounted gain process, M, we have, for $t \geq 0$,

$$D_t X_t = M_t - \int_{[0,t)} D_u dC_u, \tag{2.3}$$

because the integral is $\mathscr{G}_t$-measurable. In terms of the martingale $\hat{M}$, we can also write,

$$D_t X_t = \mathbb{E}\left\{ \int_t^\tau D_u dC_u \middle| \mathscr{G}_t \right\} + \hat{M}_t, \tag{2.4}$$

where the integral is understood not to include any cash-flow at $t \vee \tau$. This equation holds for $t \geq 0$, but the financial interpretation of the conditional expectation changes at τ. Prior to τ, it is the value of the promised cash-flows. On $\{t \geq \tau\}$, the integral is $\mathscr{G}_t$-measurable and is the value of cashflows since τ:

$$D_t X_t = \hat{M}_t - \int_{[\tau,t)} D_u dC_u. \tag{2.5}$$

Note also that, on $\{t < \tau\}$,

$$\begin{aligned}
\hat{M}_t &= \mathbb{E}\left\{ \int_\tau^T D_u dC_u \middle| \mathscr{G}_t \right\} \\
&= \mathbb{E}\left\{ \mathbb{E}\left(\int_\tau^T D_u dC_u \middle| \mathscr{G}_\tau \right) \middle| \mathscr{G}_t \right\} \\
&= \mathbb{E}\left\{ D_\tau X_\tau | \mathscr{G}_t \right\}, \qquad (2.6)
\end{aligned}$$

from (1.4). Eqn. (2.6) is compatible with a lump sum recovery payment at τ, for example if $X_\tau = Z_\tau$, where Z_t, $0 \le t \le T$, is a *recovery process*. This is common in credit pricing, where recovery is typically treated as a terminal condition at default. See Section 2.1 for several examples of this approach. But (2.6) also shows that cash-flows can continue after default, in which case X_τ is just the present value at τ of the remaining cash-flows.

These considerations suggest a decomposition of X into two tradeable assets. However, we prefer to do this in a way that does not require a change of financial interpretation at τ. We define two assets as follows, for $t \ge 0$:

$$\tilde{X}_t := \frac{1}{D_t} \mathbb{E}\left\{ \int_t^T D_u \mathbb{1}_{\{\tau > u\}} dC_u \middle| \mathscr{G}_t \right\}, \tag{2.7}$$

and,

$$\hat{X}_t := \frac{1}{D_t} \hat{M}_{t \wedge \tau} = \frac{1}{D_t} \mathbb{E}\left\{ D_\tau X_\tau | \mathscr{G}_t \right\}, \tag{2.8}$$

where $\hat{M}_{t\wedge\tau}$ is the martingale $\hat{M}$ stopped at τ. Clearly, the process $D\hat{X}$ is a martingale. The financial interpretation of $\tilde{X}$ is the *value of the promised cash-flows* or the *value of the pre-default cash-flows* of X. The interpretation of $\hat{X}$ depends on the cash-flow process. If X is default-sensitive, then $\hat{X}_t$ is the present-value at t of the recovery, X_τ. If X is not default-sensitive, financially $\hat{X}$ is insurance on $\tilde{X}$. To capture both interpretations, we call $\hat{X}$ the *rebate value process*.[1] Note that $\tilde{X}$ and $\hat{X}$ are defined for all $t \geq 0$. On $\{\tau \leq t\}$, $\tilde{X}_t = 0$ and $\hat{X}_t = \frac{D_\tau X_\tau}{D_t}$.

Comparing $\tilde{X}_t$ and $\hat{X}_t$ with the corresponding quantities in (2.4) and (2.6), we have immediately the following proposition.

Proposition 2.0.1 *On* $\{\tau > t\}$,

$$X_t = \tilde{X}_t + \hat{X}_t. \tag{2.9}$$

In particular, we have

$$X_0 = \tilde{X}_0 + \mathbb{E}\left(D_\tau X_\tau | \mathscr{G}_0\right). \tag{2.10}$$

Also, on $\{\tau \leq t\}$,

$$\hat{M}_t - D_t \hat{X}_t = D_t X_t + \int_{[\tau,t)} D_u dC_u - D_\tau X_\tau, \tag{2.11}$$

and for all $0 \leq t \leq T$,

$$\mathbb{E}\{\hat{M}_t - D_t \hat{X}_t\} = 0. \tag{2.12}$$

Proof The claims generally follow from the preceding discussion by substitution, upon restricting to the appropriate set. From (2.4) restricted to $\{\tau > t\}$, using (2.6), (2.7), and (2.8), we obtain

$$\begin{aligned} D_t X_t &= \mathbb{E}\left\{ \int_t^\tau D_u dC_u \middle| \mathscr{G}_t \right\} + \hat{M}_t \\ &= D_t \tilde{X}_t + D_t \hat{X}_t, \end{aligned} \tag{2.13}$$

which yields (2.9). Eqn. (2.10) follows from (2.9) on $\{\tau > 0\}$. On $\{\tau = 0\}$, it follows from (2.7) and (2.8) since $\tilde{X}_t = 0$ if $\{\tau \leq t\}$.

On $\{\tau \leq t\}$, from (2.5), we have

$$\hat{M}_t = D_t X_t + \int_{[\tau,t)} D_u dC_u. \tag{2.14}$$

[1] Some authors use *rebate* as a technical term synonymous with *recovery*. It is hoped that the use of the term herein to also connote the interpretation of insurance will not cause confusion.

Also, $D_t\hat{X}_t = D_\tau X_\tau$ on $\{\tau \le t\}$, from (2.8). Combining these observations implies (2.11). Finally, $\mathbb{E}\left\{\hat{M}_t - D_t\hat{X}_t\right\} = 0$ follows because $D_t\hat{X}_t = \hat{M}_{t\wedge\tau}$. ∎

Thus, when pricing prior to default, we can use the decomposition (2.9), which holds whether the cash-flows of the asset are nominally credit-sensitive or not. For any $t \ge 0$, valuation may be simplified by the fact that $D\hat{X}$ and $\hat{M}$ are martingales. Clearly these martingales agree on $\{\tau > t\}$. Their difference after default is given by (2.11). Cash-flows need not terminate at default. Indeed, after default the price continues to be given by (1.4). Comparing (2.5) with (2.3), we see that the pricing problem after default is analogous to a credit-riskless valuation problem where the valuation "starts over" at τ.

Definition Let $t < \tau$, and suppose X has maturity T. The *forward price at t of an asset X for delivery at τ* is defined to be the $\mathcal{G}_t$-measurable value $F_X(t,\tau)$ satisfying

$$\mathbb{E}\left\{\frac{D_\tau}{D_t}\left(X_\tau - F_X(t,\tau)\right)\mathbb{1}_{\{t<\tau\le T\}}\middle|\mathcal{G}_t\right\} = 0. \tag{2.15}$$

Proposition 2.0.2 *Suppose $\mathbb{P}(\tau > t|\mathcal{G}_t) = 1$, and let $F_X(t,\tau)$ be the forward price at time t for delivery of X at τ. Then,*

$$\hat{X}_t = \hat{U}_t^T F_X(t,\tau), \tag{2.16}$$

where $\hat{U}_t^T = \mathbb{E}\left\{\frac{D_\tau}{D_t}\mathbb{1}_{\{t<\tau\le T\}}\middle|\mathcal{G}_t\right\}$ is the value at time t of one unit paid at τ if $\tau \le T$.

Proof From (2.15),

$$\begin{aligned}
\mathbb{E}\left\{\frac{D_\tau}{D_t}\left(X_\tau - F_X(t,\tau)\right)\mathbb{1}_{\{t<\tau\le T\}}\middle|\mathcal{G}_t\right\} &= 0 \\
\frac{1}{D_t}\mathbb{E}\left\{D_\tau X_\tau \mathbb{1}_{\{t<\tau\le T\}}\middle|\mathcal{G}_t\right\} &= \mathbb{E}\left\{\frac{D_\tau}{D_t}\mathbb{1}_{\{t<\tau\le T\}}\middle|\mathcal{G}_t\right\} F_X(t,\tau) \\
\hat{X}_t &= \hat{U}_t^T F_X(t,\tau).
\end{aligned}$$

∎

2.1 Application to Credit-Risky Assets

This section illustrates the use of the rebate value process decomposition to price several standard default-sensitive assets. When the asset to be decomposed is credit sensitive, the interpretation of the rebate value process is the present value of the asset's recovery. This section also illustrates how the building blocks arise naturally in pricing default-sensitive assets.

2.1.1 Risky Bonds

A *continuous-coupon risky bond* is a debt in which the debtor promises to pay the bondholder a constant coupon, c, continuously over the interval $(t,T]$ as well as one unit at maturity, T. If a default, τ, occurs prior to maturity, the promised payments cease and the bondholder instead receives a non-negative recovery, Z_τ. The cash-flow process is given by

$$dC_u = \mathbb{1}_{\{u\leq T\}}\left(c\mathbb{1}_{\{\tau>u\}}du + \mathbb{1}_{\{\tau>u\}}dI_T(u) + Z_\tau dN_u\right), \tag{2.17}$$

where, as before, $N_u = \mathbb{1}_{\{\tau\leq u\}}$ and $I_T(u) = \mathbb{1}_{\{T\leq u\}}$. Denote by Q_t^T the value at time t of the continuous-coupon risky bond. We assume $t < T$ and $\mathbb{P}(\tau > t|\mathscr{G}_t) = 1$. The value at t of the pre-default payments is

$$\begin{aligned}
\tilde{Q}_t^T &= \mathbb{E}\left\{\int_t^T \frac{D_u}{D_t}\mathbb{1}_{\{\tau>u\}}dC_u\middle|\mathscr{G}_t\right\} \\
&= \mathbb{E}\left\{\int_t^T \frac{D_u}{D_t}c\mathbb{1}_{\{\tau>u\}}du\middle|\mathscr{G}_t\right\} + \mathbb{E}\left\{\frac{D_T}{D_t}\mathbb{1}_{\{\tau>T\}}\middle|\mathscr{G}_t\right\} \\
&= c\tilde{R}_t^T + \tilde{B}_t^T,
\end{aligned}$$

from (1.15) and (1.16). The value of the post-default payment is

$$\begin{aligned}
\hat{Q}_t^T &= \mathbb{E}\left\{\frac{D_\tau}{D_t}Q_\tau^T\middle|\mathscr{G}_t\right\} \\
&= \mathbb{E}\left\{\frac{D_\tau}{D_t}Z_\tau\mathbb{1}_{\{\tau\leq T\}}\middle|\mathscr{G}_t\right\}.
\end{aligned} \tag{2.18}$$

Thus, the continuous-coupon risky bond has a value at time t of

$$\begin{aligned} Q_t^T &= c\tilde{R}_t^T + \tilde{B}_t^T + \mathbb{E}\left\{ \frac{D_\tau}{D_t} Z_\tau \mathbb{1}_{\{\tau \le T\}} \middle| \mathscr{G}_t \right\} \\ &= c\tilde{R}_t^T + \tilde{B}_t^T + \hat{Q}_t^T . \end{aligned} \tag{2.19}$$

$\hat{Q}_t^T$ is the *present value at t of recovery*, and will be useful in later calculations.

It is also useful to define to define the par coupon associated with the risky bond.

Definition The *risky par coupon* at time t associated with the risky bond Q_t^T is the coupon, which if issued at t, would make the value of Q_t^T unity. It is given by

$$c_t^T := \frac{1 - \hat{Q}_t^T - \tilde{B}_t^T}{\tilde{R}_t^T}. \tag{2.20}$$

The risky par coupon can be used to give another representation for the risky bond.

Lemma 2.1.1 *(Risky Bond Representation.) On* $\{t < \tau\}$,

$$Q_t^T = 1 + \left(c - c_t^T\right) \tilde{R}_t^T . \tag{2.21}$$

Proof The claim follows upon adding and subtracting the present value of the risky annuity with coupon c_t^T to (2.19), using (2.20):

$$\begin{aligned} Q_t^T &= c\tilde{R}_t^T + \tilde{B}_t^T + \hat{Q}_t^T + c_t^T \tilde{R}_t^T - c_t^T \tilde{R}_t^T \\ &= \left(c - c_t^T\right) \tilde{R}_t^T + \tilde{B}_t^T + \hat{Q}_t^T + \left(1 - \hat{Q}_t^T - \tilde{B}_t^T\right) \\ &= 1 + \left(c - c_t^T\right) \tilde{R}_t^T . \end{aligned} \tag{2.22}$$

∎

Eqn. (2.21) allows one to think in terms of risky par coupons instead of yields when comparing credit-sensitive bonds. Clearly, a *premium bond* (resp., *discount bond*) is one for which $c > c_t^T$

(resp., $c < c_t^T$). Eqn. (2.21) can also be used to imply a *risky par coupon curve* for an issuer of several bonds by plotting c_t^T as a function of maturity, T. See also Lemma 3.1.1, below.

2.1.2 Credit Default Swap

A credit default swap (CDS) on the interval $(T_1, T_2]$, where $T_2 \leq T$, is a contract between two counter-parties and is a kind of insurance on a reference risky bond. Specifically, the first counter-party, the *protection-buyer* pays a predetermined *premium*, often called the *CDS spread*, s_0, continuously on the interval $(\tau \wedge T_1, \tau \wedge T_2]$. If $\tau \in (T_1, T_2]$, the second counter-party, the *protection-seller*, makes a default payment of $1 - Z_\tau$ at τ, where Z_τ is the recovery (cf. (2.18)) of a reference risky bond. We assume $t \leq T_1$. If $T_1 > t$, the asset is sometimes called a *forward-starting CDS*. For a regular CDS, set $T_1 = t$ below.

From the perspective of the protection-seller, the cash-flow process is

$$dC_u = \mathbb{1}_{\{T_1 < u \leq T_2\}} \left(s_0 \mathbb{1}_{\{\tau > u\}} du - (1 - Z_\tau) dN_u \right). \tag{2.23}$$

Denoting the value at t of the CDS as $X_t^{s_0}$, the value of the pre-default payments is

$$\begin{aligned}
\tilde{X}_t^{s_0} &= \mathbb{E}\left\{ \int_t^T \frac{D_u}{D_t} \mathbb{1}_{\{\tau > u\}} dC_u \middle| \mathscr{G}_t \right\} \\
&= \mathbb{E}\left\{ \int_{T_1}^{T_2} \frac{D_u}{D_t} s_0 \mathbb{1}_{\{\tau > u\}} du \middle| \mathscr{G}_t \right\} \\
&= s_0 \tilde{R}(t; T_1, T_2).
\end{aligned}$$

The value at time t of the post-default payments is

$$\begin{aligned}
\hat{X}_t^{s_0} &= \mathbb{E}\left\{ \frac{D_\tau}{D_t} X_\tau^{s_0} \middle| \mathscr{G}_t \right\} \\
&= \mathbb{E}\left\{ \frac{D_\tau}{D_t} \left(-(1 - Z_\tau) \right) \mathbb{1}_{\{T_1 < \tau \leq T_2\}} \middle| \mathscr{G}_t \right\} \\
&= -\hat{U}(t; T_1, T_2) + \mathbb{E}\left\{ \frac{D_\tau}{D_t} Z_\tau \mathbb{1}_{\{T_1 < \tau \leq T_2\}} \middle| \mathscr{G}_t \right\},
\end{aligned}$$

from (1.17). Compare also (2.18). Thus, the value at time t of the credit default swap on the interval $(T_1, T_2]$ from the perspective of the protection seller is

$$X_t^{s_0} = s_0 \tilde{R}(t;T_1,T_2) - \left(\hat{U}(t;T_1,T_2) - \mathbb{E}\left\{ \frac{D_\tau}{D_t} Z_\tau \mathbb{1}_{\{T_1 < \tau \le T_2\}} \middle| \mathscr{G}_t \right\} \right). \tag{2.24}$$

2.1.3 Par CDS Spread

For $t \le T_1$, the *(par) CDS spread* on $(T_1, T_2]$ is defined to be the contractual premium $s(t;T_1,T_2)$ paid continuously on the interval $(\tau \wedge T_1, \tau \wedge T_2]$ that makes the value of the CDS in (2.24) equal to zero at time t. If $t < T_1$, $s(t;T_1,T_2)$ is sometimes called the *forward (par) CDS spread*. Thus,

$$s(t;T_1,T_2) := \frac{\hat{U}(t;T_1,T_2) - \mathbb{E}\left\{ \frac{D_\tau}{D_t} Z_\tau \mathbb{1}_{\{T_1 < \tau \le T_2\}} \middle| \mathscr{G}_t \right\}}{\tilde{R}(t;T_1,T_2)}. \tag{2.25}$$

Combining (2.24) and (2.25), the value of a CDS at t with a contractual premium s_0, not necessarily equal to $s(t;T_1,T_2)$, is given by

$$X_t^{s_0} = (s_0 - s(t;T_1,T_2))\, \tilde{R}(t;T_1,T_2). \tag{2.26}$$

Clearly, this is equal to the value of a risky annuity which pays $s_0 - s(t;T_1,T_2)$ on $(\tau \wedge T_1, \tau \wedge T_2]$. Indeed, a CDS with contractual CDS spread s_0 can be transformed into this annuity by taking an off-setting position in a new CDS at t with premium $s(t;T_1,T_2)$, at zero cost. Alternately, the position can also be settled with the original counter-party (or an intermediary) for a cash payment at $t < \tau$ equal to $X_t^{s_0}$ in (2.26).

If $T_1 \le t$, the swap is equivalent to a CDS on $(t, T_2]$, and we may write $s_t^{T_2} := s(t;t,T_2)$.

2.2 Complements of the Building Blocks

In this section we illustrate the application of the decomposition $X_t = \tilde{X}_t + \hat{X}_t$ to several riskless assets. In this case the rebate value process, $\hat{X}_t$, has the interpretation of the price of insurance on

$\tilde{X}_t$, which is a credit-risky asset. In particular, we derive the "complements" of the building blocks under the decomposition.

2.2.1 Risky Zero-Coupon Bond Insurance

The simplest application of the decomposition (2.9) to a riskless asset is the case of a zero-coupon bond (ZCB), see (1.12). The cash-flow process can be written as a jump process where the jump occurs deterministically at T:

$$dC_u = dI_T(u), \tag{2.27}$$

where $I_T(u) := \mathbb{1}_{\{T \le u\}}$ is a unit jump at T. In this case, the pre-τ payments are recognized as a risky ZCB as in (1.15), $\tilde{B}_t^T = \mathbb{E}\left\{ \frac{D_T}{D_t} \mathbb{1}_{\{\tau > T\}} \middle| \mathscr{G}_t \right\}$. If $\mathbb{P}(\tau \ge t|\mathscr{G}_t) = 1$, then from Proposition 2.0.1,

$$\begin{aligned} \hat{B}_t^T &= B_t^T - \tilde{B}_t^T \\ &= \mathbb{E}\left\{ \frac{D_T}{D_t} \mathbb{1}_{\{\tau \le T\}} \middle| \mathscr{G}_t \right\}. \end{aligned} \tag{2.28}$$

$\hat{B}_t^T$ has the interpretation of the price of insurance on a risky ZCB.

To give an example of a typical calculation, we can also calculate the martingale $D_t \hat{B}_t^T$ directly:

$$\begin{aligned} D_t \hat{B}_t^T &= \mathbb{E}\left\{ D_\tau B_\tau^T \middle| \mathscr{G}_t \right\} \\ &= \mathbb{E}\left\{ D_\tau B_\tau^T \mathbb{1}_{\{\tau \le T\}} \middle| \mathscr{G}_t \right\} \\ &= \mathbb{E}\left\{ D_\tau \mathbb{E}\left(\frac{D_T}{D_\tau} \middle| \mathscr{G}_\tau \right) \mathbb{1}_{\{\tau \le T\}} \middle| \mathscr{G}_t \right\} \\ &= \mathbb{E}\left\{ D_T \mathbb{1}_{\{\tau \le T\}} \middle| \mathscr{G}_t \right\}. \end{aligned} \tag{2.29}$$

2.2.2 Risky Annuity Insurance

Eqn. (2.26) shows that a risky annuity may arise naturally in closing out a CDS position. An investor has a choice to close out the position for a cash payment equal to the present value of the risky annuity or to actually receive the risky annuity payments on a running basis. This is a decision as to whether to assume default-timing risk. If default occurs relatively soon, the investor would be better off taking the up-front cash payment. If default occurs relatively late, the investor would be better off taking the running payments. Indeed, if default occurs after the maturity of the risky annuity, the running payments are the same as one would receive from a riskless annuity. These considerations motivate the development in this section.

Suppose an investor with a risky annuity, $\tilde{R}$ on $(T_1, T_2]$, could buy insurance on the risky annuity which would compensate for the risky cash-flows if the reference entity defaulted before T_2. If $\mathbb{P}(\tau \geq t | \mathcal{G}_t) = 1$, the value of this *risky annuity insurance* at $t \leq T_1$ is

$$\hat{R}(t; T_1, T_2) := R(t; T_1, T_2) - \tilde{R}(t; T_1, T_2), \tag{2.30}$$

where R and $\tilde{R}$ are as in (1.13) and (1.16). This is because an investor would be indifferent between the riskless annuity and a portfolio consisting of the risky annuity and the insurance. For general t, a representation can be obtained using the rebate value process. With τ again denoting the default time underlying the risky annuity, define

$$\tau_{T_1}^{T_2} := T_1 \vee (\tau \wedge T_2) = \begin{cases} T_1 & \text{if } \tau \leq T_1, \\ \tau & \text{if } \tau \in (T_1, T_2], \\ T_2 & \text{if } \tau > T_2. \end{cases} \tag{2.31}$$

The risky annuity insurance makes payments on the random interval $\left(\tau_{T_1}^{T_2}, T_2\right]$.

Proposition 2.2.1 *Let $\hat{R}(t; T_1, T_2)$ be the value at time $t \geq 0$ of risky annuity insurance for a risky annuity on $(T_1, T_2]$. Let $\tau_{T_1}^{T_2}$ be as in (2.31) and suppose $\mathbb{P}(\tau \geq t | \mathcal{G}_t) = 1$. Then,*

$$\hat{R}(t; T_1, T_2) = R(t; \tau_{T_1}^{T_2}, T_2). \tag{2.32}$$

Proof We have

$$D_t R(t; T_1, T_2) = \mathbb{E}\left\{ \int_{T_1 \vee (t \wedge T_2)}^{T_2} D_u du \middle| \mathcal{G}_t \right\}, \tag{2.33}$$

from (1.13). Then, on $\{\tau \geq t\}$,

$$\begin{aligned}
D_t \hat{R}(t;T_1,T_2) &= \mathbb{E}\{D_\tau R(\tau;T_1,T_2)|\mathscr{G}_t\} \\
&= \mathbb{E}\left\{\mathbb{E}\left(\int_{\tau_{T_1}^{T_2}}^{T_2} D_u du \middle| \mathscr{G}_\tau\right)\middle|\mathscr{G}_t\right\} \\
&= \mathbb{E}\left\{\int_{\tau_{T_1}^{T_2}}^{T_2} D_u du \middle| \mathscr{G}_t\right\} \\
&= D_t \mathbb{E}\left\{\int_{\tau_{T_1}^{T_2}\vee(t\wedge T_2)}^{T_2} \frac{D_u}{D_t} du \middle| \mathscr{G}_t\right\} \\
&= D_t R(t;\tau_{T_1}^{T_2},T_2),
\end{aligned}$$

since $\tau \geq t$ implies $\tau_{T_1}^{T_2} = \tau_{T_1}^{T_2} \vee (t \wedge T_2)$ in all cases. ∎

Note that the risky annuity insurance makes payments after default. These post-default cash-flows are not $\mathscr{G}_\tau$-measureable. Eqn. (2.32) simplifies dealing with the post-default cash-flows, justifying their treatment as recovery of a residual riskless annuity.

Corollary 2.2.2 *Let $\tau_{T_1}^{T_2}$ be as in (2.31). Then, for $\mathbb{P}(\tau \geq t|\mathscr{G}_t) = 1$ and $t \leq T_1$,*

$$\tilde{R}(t;T_1,T_2) = R(t;T_1,\tau_{T_1}^{T_2}). \tag{2.34}$$

Proof By the additivity of the integrals involved, for $T_1 \leq s \leq T_2$, we have $R(t;T_1,T_2) = R(t;T_1,s) + R(t;s,T_2)$. Thus,

$$\begin{aligned}
R(t;T_1,T_2) &= R(t;T_1,\tau_{T_1}^{T_2}) + R(t;\tau_{T_1}^{T_2},T_2) \\
&= R(t;T_1,\tau_{T_1}^{T_2}) + \hat{R}(t;T_1,T_2),
\end{aligned}$$

from (2.32). Comparing this with (2.30), we obtain

$$\tilde{R}(t;T_1,T_2) = R(t;T_1,\tau_{T_1}^{T_2}). \tag{2.35}$$

∎

Corollary 2.2.3

$$\tilde{R}(t;T_1,T_2) \leq R(t;T_1,\mathbb{E}\left\{\tau_{T_1}^{T_2}\middle|\mathscr{G}_t\right\}). \tag{2.36}$$

Proof This follows from Jensen's Inequality applied to (2.34). ∎

For example, take $t = T_1 = 0$ and $T_2 = 100$. Suppose the expected default time of XYZ is one year. Which is preferable, a one year riskless annuity or a 100 year risky annuity which ceases on default of XYZ? The above corollary states that one should prefer the riskless annuity.

Break-even Default Time

Recall the investor with the choice between taking the proceeds from a risky annuity as an up-front cash payment or as running risky payments. We quantify the intuition that if she chooses the risky running payments, and if moreover the reference entity survives past a certain time, then she is better off.

Proposition 2.2.4 *Let $t \leq T_1 < T_2$ and assume $\mathbb{P}(\tau \geq t|\mathscr{G}_t) = 1$. There exists a unique, non-stochastic $T^* \in [T_1,T_2]$ such that*

$$R(t;T_1,T^*) = \tilde{R}(t;T_1,T_2). \tag{2.37}$$

For such T^*,

$$\hat{R}(t;T_1,T^*) = \tilde{R}(t;T^*,T_2) \tag{2.38}$$

Proof Consider $f(s) = R(t;T_1,s)$, a continuous function on $[T_1,T_2]$. Since $f(T_2) \geq \tilde{R}(t;T_1,T_2) \geq 0$, and $f(T_1) = 0$, the existence of T^* satisfying (2.37) follows from the Intermediate Value Theorem. $f(s)$ is strictly increasing, so T^* is unique. For (2.38), note that

$$\tilde{R}(t;T_1,T^*) + \tilde{R}(t;T^*,T_2) \quad = \quad \tilde{R}(t;T_1,T_2)$$

$$
\begin{aligned}
&= R(t;T_1,T^*) \\
&= \tilde{R}(t;T_1,T^*) + \hat{R}(t;T_1,T^*)
\end{aligned}
$$

∎

Definition T^* is called the *break-even default time over* $(T_1,T_2]$.

The proposition implies that an investor with a risky annuity is better off taking running payments provided $\tau \geq T^*$. It also implies that the default timing risk associated with a running risky annuity can be immunized in a market that offers risky annuity insurance. This can be done by selling off the risky annuity after T^* and using the proceeds to buy risky annuity insurance on $(T_1,T^*]$.

Corollary 2.2.5 *For* $t \leq T_1 \leq T_2$*, if* $\mathbb{P}(\tau \geq t|\mathscr{G}_t) = 1$*, then*

$$
T^* \leq \mathbb{E}\left\{ \tau_{T_1}^{T_2} \middle| \mathscr{G}_t \right\}. \tag{2.39}
$$

Proof By Corollary 2.2.3 and Proposition 2.2.4,

$$
\begin{aligned}
R(t;T_1,\mathbb{E}\left\{ \tau_{T_1}^{T_2} \middle| \mathscr{G}_t \right\}) &\geq \tilde{R}(t;T_1,T_2) \\
&= R(t;T_1,T^*),
\end{aligned}
$$

whence,

$$
\mathbb{E}\left\{ \tau_{T_1}^{T_2} \middle| \mathscr{G}_t \right\} \geq T^*, \tag{2.40}
$$

because $R(t;T_1,s)$ is increasing in s. ∎

2.2.3 The Case of Unit Payment at Default

To derive the complement of the last building block, a unit payment at default, see (1.17), we first decompose the riskless asset known as a par floater.

A *par floater* is a riskless asset that pays the *floating rate* r_u continuously for $u \in (t,T]$, as well as one unit at maturity T. It is always worth par (one unit):

$$1 = \mathbb{E}\left\{ \frac{1}{D_t} \int_t^T r_u D_u du \middle| \mathscr{G}_t \right\} + B_t^T. \tag{2.41}$$

This is because

$$D_u = \exp\left(- \int_0^u r_s ds \right), \tag{2.42}$$

so $dD_u = -r_u D_u du$ and

$$\begin{aligned} \int_t^T r_u D_u du &= D_t - D_T & (2.43) \\ \frac{1}{D_t}\mathbb{E}\left\{ \int_t^T r_u D_u du \middle| \mathscr{G}_t \right\} &= 1 - \mathbb{E}\left\{ \frac{D_T}{D_t} \middle| \mathscr{G}_t \right\} \\ &= 1 - B_t^T. & (2.44) \end{aligned}$$

The following lemma gives the decomposition of a par floater in terms of the building blocks.

Lemma 2.2.6 *Suppose that $\mathbb{P}(\tau \geq t | \mathscr{G}_t) = 1$. $\hat{U}_t^T$ is the value of Par Floater Insurance. That is,*

$$\hat{U}_t^T = \mathbb{E}\left\{ \frac{1}{D_t} \mathbb{1}_{\{\tau \leq T\}} \int_\tau^T r_u D_u du \middle| \mathscr{G}_t \right\} + \hat{B}_t^T. \tag{2.45}$$

The value of a Risky Par Floater is

$$1 - \hat{U}_t^T = \mathbb{E}\left\{ \frac{1}{D_t} \int_t^T r_u D_u \mathbb{1}_{\{\tau > u\}} du \middle| \mathscr{G}_t \right\} + \tilde{B}_t^T. \tag{2.46}$$

Proof

$$\begin{aligned} \mathbb{E}\left\{ \frac{1}{D_t} \mathbb{1}_{\{\tau \leq T\}} \int_\tau^T r_u D_u du \middle| \mathscr{G}_t \right\} &= \mathbb{E}\left\{ \frac{1}{D_t} \mathbb{1}_{\{\tau \leq T\}} (D_\tau - D_T) \middle| \mathscr{G}_t \right\} \\ &= \mathbb{E}\left\{ \frac{D_\tau}{D_t} \mathbb{1}_{\{\tau \leq T\}} \middle| \mathscr{G}_t \right\} - \mathbb{E}\left\{ \frac{D_T}{D_t} \mathbb{1}_{\{\tau \leq T\}} \middle| \mathscr{G}_t \right\} \\ &= \hat{U}_t^T - \hat{B}_t^T, & (2.47) \end{aligned}$$

which establishes (2.45). Next,

$$
\begin{aligned}
1-B_t^T &= \mathbb{E}\left\{\frac{1}{D_t}\int_t^T r_u D_u du \middle| \mathscr{G}_t\right\} \\
&= \mathbb{E}\left\{\frac{1}{D_t}\int_t^T r_u D_u \mathbb{1}_{\{\tau>u\}} du \middle| \mathscr{G}_t\right\} + \left(\hat{U}_t^T - \hat{B}_t^T\right).
\end{aligned}
$$

So,

$$
\begin{aligned}
\mathbb{E}\left\{\frac{1}{D_t}\int_t^T r_u D_u \mathbb{1}_{\{\tau>u\}} du \middle| \mathscr{G}_t\right\} &= 1-B_t^T - \left(\hat{U}_t^T - \hat{B}_t^T\right) \\
&= 1-\hat{U}_t^T - \left(B_t^T - \hat{B}_t^T\right) \\
&= 1-\hat{U}_t^T - \tilde{B}_t^T, \qquad (2.48)
\end{aligned}
$$

which proves (2.46). ∎

Corollary 2.2.7 *Suppose that $\mathbb{P}(\tau \geq t|\mathscr{G}_t) = 1$. $\hat{U}_t^\infty$ is the value of a perpetual floating annuity starting at default (or the value of risky perpetual floating annuity insurance):*

$$
\hat{U}_t^\infty = \mathbb{E}\left\{\frac{1}{D_t}\int_\tau^\infty r_u D_u du \middle| \mathscr{G}_t\right\}. \qquad (2.49)
$$

$1-\hat{U}_t^\infty$ is the value of a risky perpetual floating annuity:

$$
1-\hat{U}_t^\infty = \mathbb{E}\left\{\frac{1}{D_t}\int_t^\tau r_u D_u du \middle| \mathscr{G}_t\right\}. \qquad (2.50)
$$

Proof $\hat{U}_t^T - \hat{B}_t^T$ is increasing in T and bounded above by (2.45) and (2.44). From (1.1) and (2.28), clearly $\lim_{T\to\infty} \hat{B}_t^T = 0$. So, the claims follow from (2.45) and (2.46) by letting $T \to \infty$, by the bounded convergence theorem. ∎

In other words, $\hat{U}^\infty$ is the rebate value process of one unit, and

$$
\mathbb{E}\left\{\frac{1}{D_t}\int_t^\infty r_u D_u du \middle| \mathscr{G}_t\right\} = (1-\hat{U}_t^\infty) + \hat{U}_t^\infty. \qquad (2.51)
$$

Chapter 3

Applications

3.1 Interest Rate Swap at Default

In this section, we price an interest rate swap with a continuous coupon and obtain the price of an interest rate swap at default.

A *continuous interest rate swap* on $(t,T]$ is an agreement between two counter-parties to exchange a *fixed payment* of ρ for a *floating payment* of r_u, both paid continuously over the interval $(t,T]$. From the point of view of the *payer* of the fixed payment, the cash-flow process is given by

$$dC_u = \mathbb{1}_{\{t<u\leq T\}}(r_u - \rho)\,du. \tag{3.1}$$

Denoting by X_t^{ρ} the value of a *payer interest rate swap* on $(t,T]$, we have

$$\begin{aligned} D_t X_t^{\rho} &= \mathbb{E}\left\{\int_t^T D_u dC_u \middle| \mathcal{G}_t\right\} \\ &= \mathbb{E}\left\{\int_t^T D_u (r_u - \rho) du \middle| \mathcal{G}_t\right\} \\ &= \mathbb{E}\left\{\int_t^T r_u D_u du \middle| \mathcal{G}_t\right\} - \rho \mathbb{E}\left\{\int_t^T D_u du \middle| \mathcal{G}_t\right\} \\ &= \mathbb{E}\left\{\int_t^T r_u D_u du \middle| \mathcal{G}_t\right\} - \rho D_t R_t^T. \end{aligned}$$

$$X_t^{\rho} \quad = \quad \mathbb{E}\left\{ \frac{1}{D_t} \int_t^T r_u D_u du \middle| \mathscr{G}_t \right\} - \rho R_t^T. \tag{3.2}$$

Combining this with (2.41), we obtain

$$X_t^{\rho} = (1 - B_t^T) - \rho R_t^T \tag{3.3}$$

as the value of the payer interest rate swap. For later reference we remark that the *swap rate* at time t is defined to be the fixed rate which makes the interest rate swap worth zero at time t. It is given by

$$\rho_t^T = \frac{1 - B_t^T}{R_t^T}. \tag{3.4}$$

Combining this with (3.3) gives

$$X_t^{\rho} = (\rho_t^T - \rho) R_t^T. \tag{3.5}$$

We now consider the decomposition of the interest rate swap into a risky interest rate swap and risky interest rate swap insurance.

Remark Our purpose here is to illustrate the decomposition of a riskless asset into a risky piece and an insurance piece, where the riskless asset is a stylized (continuous payment) interest rate swap, rather than to price specific credit derivatives. That is, our purpose is not to develop a general treatment of defaultable interest rate swaps. The latter task involves a careful stipulation of counterparty risk and recovery conventions. For example, is the default risk of both the payer and receiver non-negligible? What happens if the payer defaults and the value of the IRS is positive (or negative)? The reader interested in a treatment of these and related issues may consult Bielecki and Rutkowski (2002) [1][Ch. 14], and the references therein. For the reader familiar with such a treatment, the definitions and results herein correspond to the case of bilateral default risk with a "limited two-way payment rule" where all recovery rates are zero, see [1][p. 446-8].

We assume $\tau \geq t$ and first consider $\hat{X}_t^\rho$. Using (3.2) evaluated at τ,

$$\begin{aligned}
\hat{X}_t^\rho &= \mathbb{E}\left\{\frac{D_\tau}{D_t} X_\tau^\rho \mathbb{1}_{\{\tau \leq T\}} \middle| \mathscr{G}_t\right\} \\
&= \mathbb{E}\left\{\frac{1}{D_t}\mathbb{1}_{\{\tau \leq T\}}\left[\mathbb{E}\left(\int_\tau^T r_u D_u du \middle| \mathscr{G}_\tau\right) - \rho D_\tau R_\tau^T\right] \middle| \mathscr{G}_t\right\} \\
&= \mathbb{E}\left\{\frac{1}{D_t}\mathbb{1}_{\{\tau \leq T\}} \int_\tau^T r_u D_u du \middle| \mathscr{G}_t\right\} - \rho \hat{R}_t^T \\
\hat{X}_t^\rho &= \hat{U}_t^T - \hat{B}_t^T - \rho \hat{R}_t^T, \qquad (3.6)
\end{aligned}$$

where we have used (2.45) and the fact that $D_t \hat{R}_t^T = \mathbb{E}\left\{\mathbb{1}_{\{\tau \leq T\}} D_\tau R_\tau^T \middle| \mathscr{G}_t\right\}$, which is (2.8) applied to $X = R^T$.

The financial interpretation of $\hat{X}_t^\rho$ is the value at t of a *payer interest rate swap starting at τ*. This motivates the following definition.

Definition The *forward swap rate at default* at time $t \leq \tau$ is the fixed rate $\rho_\tau^T(t)$ which makes the value of a interest rate swap on $(\tau \wedge t, \tau \wedge T]$ equal to zero at time t.

From (3.6), the forward swap rate at default is given by

$$\rho_\tau^T(t) := \frac{\hat{U}_t^T - \hat{B}_t^T}{\hat{R}_t^T}. \qquad (3.7)$$

Moreover, combining (3.6) and (3.7) gives

$$\hat{X}_t^\rho = \left(\rho_\tau^T(t) - \rho\right) \hat{R}_t^T. \qquad (3.8)$$

Next, the *risky interest rate swap* has value

$$\begin{aligned}
\tilde{X}_t^\rho &= X_t^\rho - \hat{X}_t^\rho \\
&= \left(1 - B_t^T\right) - \rho R_t^T - \left(\hat{U}_t^T - \hat{B}_t^T - \rho \hat{R}_t^T\right) \\
&= \left(1 - \hat{U}_t^T\right) - \tilde{B}_t^T - \rho \tilde{R}_t^T. \qquad (3.9)
\end{aligned}$$

From (2.46), we recognize $1-\hat{U}^T-\tilde{B}^T$ as the value of a risky floating annuity. This prompts the following definition.

Definition The *risky swap rate* at time $t<\tau$ is defined to be the fixed coupon which makes the risky interest rate swap in (3.9) worth zero at time t. It is given by

$$\tilde{\rho}_t^T := \frac{1-\hat{U}_t^T-\tilde{B}_t^T}{\tilde{R}_t^T}. \tag{3.10}$$

In terms of the risky swap rate, the risky interest rate swap has representation

$$\tilde{X}_t^{\rho} = (\tilde{\rho}_t^T-\rho)\tilde{R}_t^T. \tag{3.11}$$

We also have the following relation between the various risky rates.

Lemma 3.1.1 *Let $\tilde{\rho}_t^T$, $s_t^T=s(t;t,T)$, and c_t^T be given by (3.10), (2.25), and (2.20), respectively. On $\{t<\tau\}$,*

$$c_t^T = \tilde{\rho}_t^T + s_t^T. \tag{3.12}$$

Proof We add and subtract s_t^T from c_t^T and obtain

$$\begin{aligned} c_t^T &= \frac{1-\hat{Q}_t^T-\tilde{B}_t^T}{\tilde{R}_t^T} - s_t^T + s_t^T \\ &= \frac{1-\hat{Q}_t^T-\tilde{B}_t^T-(\hat{U}_t^T-\hat{Q}_t^T)}{\tilde{R}_t^T} + s_t^T \\ &= \frac{1-\hat{U}_t^T-\tilde{B}_t^T}{\tilde{R}_t^T} + s_t^T \\ &= \tilde{\rho}_t^T + s_t^T. \end{aligned} \tag{3.13}$$

∎

The lemma shows that the risky par coupon of a risky bond can be decomposed into the risky swap rate and the par CDS spread. In the financial industry, traders often compare the CDS spread to the bond's yield or par coupon less some risk free rate. The lemma shows this comparison is only an approximation, since the interest rate implicit in the par coupon is risky, namely the risky swap rate.

3.2 Risky Bond Insurance

In this section, we apply the decomposition (2.9) to a credit-risky bond and suggest an alternative to a traditional credit default swap (CDS). We begin by considering a riskless continuous coupon bond.

A riskless continuous coupon bond has a cash-flow process

$$dC_u = \mathbb{1}_{\{u \leq T\}} \left(c du + dI_T(u) \right), \tag{3.14}$$

where $I_T(u) = \mathbb{1}_{\{T \leq u\}}$.

We assume that $\mathbb{P}(\tau \geq t | \mathscr{G}_t) = 1$ for the remainder of this section. The value of this riskless bond at time t can be written

$$\begin{aligned} P_t^T &= cR_t^T + B_t^T \\ &= \tilde{P}_t^T + \hat{P}_t^T \end{aligned} \tag{3.15}$$

where

$$\tilde{P}_t^T = c\tilde{R}_t^T + \tilde{B}_t^T, \tag{3.16}$$

and

$$\hat{P}_t^T = c\hat{R}_t^T + \hat{B}_t^T. \tag{3.17}$$

On the other hand, we know from (2.19) that a credit-risky bond with recovery Z_τ can be written

$$Q_t^T = c\tilde{R}_t^T + \tilde{B}_t^T + \mathbb{E}\left\{ \frac{D_\tau}{D_t} Z_\tau \mathbb{1}_{\{\tau \leq T\}} \middle| \mathscr{G}_t \right\}, \tag{3.18}$$

so that,

$$\tilde{Q}_t^T = c\tilde{R}_t^T + \tilde{B}_t^T, \tag{3.19}$$

and

$$\hat{Q}_t^T = \mathbb{E}\left\{ \frac{D_\tau}{D_t} Z_\tau \mathbb{1}_{\{\tau \le T\}} \middle| \mathscr{G}_t \right\}. \tag{3.20}$$

$\hat{Q}_t^T$ is the present value of recovery as in (2.18). Since $\tilde{Q}_t^T = \tilde{P}_t^T$, we can write the riskless bond as

$$\begin{aligned} P_t^T &= \tilde{Q}_t^T + \hat{P}_t^T + \left(\hat{Q}_t^T - \hat{Q}_t^T\right) \\ &= Q_t^T + \left(\hat{P}_t^T - \hat{Q}_t^T\right). \end{aligned} \tag{3.21}$$

Thus,

$$P_t^T - Q_t^T = \hat{P}_t^T - \hat{Q}_t^T = c\hat{R}_t^T + \hat{B}_t^T - \hat{Q}_t^T \tag{3.22}$$

is the value at time t of a kind of risky bond insurance which gives the holder of Q_t^T "full-recovery". That is, (3.21) shows that a payment of $cR_\tau^T + B_\tau^T - Z_\tau$ at default would immunize the holder of a risky bond to all default risk, including missed coupon payments. Note that Q_t^T is usually observable in the market, and P_t^T is easy to compute if one has a calibrated ZCB curve, $\{B_t^s\}_{s \ge t}$.

Before concluding our discussion of risky and riskless bonds, let us make explicit the relationship of the decomposition $X = \tilde{X} + \hat{X}$ to a common recovery convention. One version of the *recovery of Treasury* convention postulates that on $\{\tau \le T\}$

$$\hat{Q}_\tau^T = Z_\tau = \delta P_\tau^T, \tag{3.23}$$

where $\delta \in [0,1]$ is constant, see, for example, Jarrow and Turnbull (1995) [9]. That is, the recovery of the risky bond Q at default is equal to a fixed fraction of the equivalent default-free security P, which can be thought of as a *Treasury bond*. Then, from (3.20) and (3.23), clearly $\hat{Q}_t^T = \delta \hat{P}_t^T$ for

$t \le \tau$, and, since $\tilde{Q}_t^T = \tilde{P}_t^T$,

$$\begin{aligned} Q_t^T &= \tilde{Q}_t^T + \delta \hat{P}_t^T \\ &= \tilde{Q}_t^T + \delta \hat{P}_t^T + \delta \tilde{P}_t^T - \delta \tilde{Q}_t^T \\ &= (1-\delta)\,\tilde{Q}_t^T + \delta P_t^T. \end{aligned} \tag{3.24}$$

Eqn. (3.24) is Proposition 6.1 in Schönbucher (2003) [18][p. 134]. Some authors use a different form of the recovery of Treasury convention wherein a fraction of the equivalent default-free security is received at maturity T rather than at default, provided $\tau \le T$. For a discussion of both versions of the convention, see Bielecki and Rutkowski (2002) [1][pp. 5, 233-4].

3.3 Full-Recovery CDS

We have seen that a portfolio consisting of both a risky bond and risky bond insurance is riskless and equivalent to a riskless coupon bond with the same coupons and maturity. This motivates the following definition.

Definition If $\mathbb{P}(\tau \ge t|\mathscr{G}_t) = 1$, a *full-recovery CDS*, denoted $X_t^{\sigma_0}$, is a swap of $\sigma_0 du$ on $(\tau \wedge t, \tau \wedge T]$ for a contingent payment at $\tau \le T$ of $cR_\tau^T + B_\tau^T - Z_\tau$. The *full-recovery CDS spread or premium* at t is given by

$$\sigma_t^T := \frac{P_t^T - Q_t^T}{\tilde{R}_t^T} = \frac{c\hat{R}_t^T + \hat{B}_t^T - \hat{Q}_t^T}{\tilde{R}_t^T} \tag{3.25}$$

where $\hat{Q}_t^T$ is again the present value of recovery of the bond.

The value at time t of a full recovery CDS paying a contractual spread of σ_0, from the point of view of the protection seller, is

$$\begin{aligned} X_t^{\sigma_0} &= \sigma_0 \tilde{R}_t^T - \left(P_t^T - Q_t^T\right) \\ &= \left(\sigma_0 - \sigma_t^T\right)\tilde{R}_t^T, \end{aligned} \tag{3.26}$$

from (3.25).

A par CDS is another kind of risky bond insurance. Recall from (2.24) that its value is given by

$$X_t^{s_0} = s_0 \tilde{R}_t^T - \left(\hat{U}_t^T - \hat{Q}_t^T\right). \tag{3.27}$$

Whereas the default payment of X^{s_0} is $1 - Z_\tau$, the default payment of X^{σ_0} is $\left(cR_\tau^T + B_\tau^T\right) - Z_\tau$.

Proposition 3.3.1 *Suppose that $\mathbb{P}(\tau \geq t | \mathscr{G}_t) = 1$. Let $X_t^{s_0}$ and $X_t^{\sigma_0}$ be as above, and let $\rho_\tau^T(t)$ be as in (3.7). Then the difference between a par CDS and a full-recovery CDS of matched maturity and both referencing the same credit-risky bond with coupon c is given by*

$$X_t^{s_0} - X_t^{\sigma_0} = (s_0 - \sigma_0)\tilde{R}_t^T - \left[\hat{U}_t^T - \hat{B}_t^T - c\hat{R}_t^T\right]. \tag{3.28}$$

In particular, the value at t of the difference of the default payments,

$$\begin{aligned} \hat{X}_t^{s_0} - \hat{X}_t^{\sigma_0} &= -\left[\hat{U}_t^T - \hat{B}_t^T - c\hat{R}_t^T\right] \\ &= -\hat{X}_t^{\rho} \,|_{\rho=c} \\ &= \left(c - \rho_\tau^T(t)\right)\hat{R}_t^T, \end{aligned} \tag{3.29}$$

is the value of a forward receiver IRS at default which receives a fixed payment of c and pays a floating payment of r_u continuously on $(\tau \wedge t, \tau \wedge T]$. Alternately, it is the value of risky annuity insurance paying the holder a fixed payment of $c - \rho_\tau^T(t)$ continuously on the same interval.

Proof This follows directly from (3.26) and (3.27) by (3.7) and (3.22). ∎

A long position in the bond and an offsetting position in a matched-maturity CDS is known as a *basis trade*. Since (3.21) shows that a portfolio of the risky bond and a full recovery CDS has no default risk, (3.29) shows that a basis trade retains some default risk. In particular, insuring the bond with a par CDS exposes one to coupon risk. (3.29) also shows that this risk can be hedged by entering into a forward interest rate swap at default. Explicitly, the value of a basis trade is given in the following proposition.

Proposition 3.3.2 *The value of a basis trade portfolio at time t, where* $\mathbb{P}(\tau \geq t|\mathscr{G}_t) = 1$*, is*

$$\begin{aligned} Q_t^T - X_t^{s_0} &= (c - s_0)\tilde{R}_t^T + \tilde{B}_t^T + \hat{U}_t^T \\ &= 1 - \tilde{X}_t^{\rho}, \end{aligned} \tag{3.30}$$

where $\rho = c - s_0$.

Proof From (3.18), (3.20), and (3.27),

$$\begin{aligned} Q_t^T - X_t^{s_0} &= c\tilde{R}_t^T + \tilde{B}_t^T + \hat{Q}_t^T - s_0\tilde{R}_t^T + (\hat{U}_t^T - \hat{Q}_t^T) \\ &= (c - s_0)\tilde{R}_t^T + \tilde{B}_t^T + \hat{U}_t^T, \end{aligned}$$

which proves the first identity. Letting $\rho = c - s_0$, (3.9) can be rearranged to read

$$\rho\tilde{R}_t^T + \tilde{B}_t^T + \hat{U}_t^T = 1 - \tilde{X}_t^{\rho}, \tag{3.31}$$

which gives the second identity. ∎

By (3.30), the risky receiver IRS, $-\tilde{X}_t^{\rho}$, quantifies the "slippage" of the value of the basis trade from par. From (3.11) with $\rho = c - s_0$, we have

$$-\tilde{X}_t^{\rho} = \left((c - s_0) - \tilde{\rho}_t^T\right)\tilde{R}_t^T. \tag{3.32}$$

By the risky par coupon decomposition (3.12), we have $\tilde{\rho}_t^T = c_t^T - s_t^T$. So the difference of the basis trade and unity is the value of a risky swap which receives $c - s_0$ and pays $c_t^T - s_t^T$.

Chapter 4

τ-Forward Measure

In this section, we consider the change of numéraire suggested by Proposition 2.0.2. We recall (2.16):

$$\hat{X}_t = \hat{U}_t^T F_X(t,\tau), \tag{4.1}$$

where $\hat{X}_t$ is the rebate value process, given by (2.8), $\hat{U}_t^T = \mathbb{E}\left\{ \frac{D_\tau}{D_t} \mathbb{1}_{\{t<\tau\leq T\}} \middle| \mathscr{G}_t \right\}$ is the value of one unit paid at default from (1.17), and $F_X(t,\tau)$ is the forward price at $t<\tau$ for delivery of X at default, τ, defined in (2.15). It is easy to see that the forward prices for delivery at default of the building blocks are consistent with the complements of the building blocks discussed earlier. For example, taking $X_t = B_t^T$, a zero coupon bond (ZCB) with maturity T, we have $\hat{U}_t^T F_B(t,\tau) = \hat{B}_t^T$, see (1.12) and (2.28). Similarly, $\hat{U}_t^T F_R(t,\tau) = \hat{R}_t^T$, see (1.13), (1.16) and (2.30), and $\hat{U}_t^T F_U(t,\tau) = \hat{U}_t^T$, where $U=1$ is the value of a par floater as in (2.41).

Let us introduce the *money market account* or *accumulation factor*, β_t, defined by

$$\beta_t := D_t^{-1} = \exp\left\{ \int_0^t r_s ds \right\}, \qquad 0 \leq t \leq T, \tag{4.2}$$

where D_t is the discount factor from (1.1).

It is well-known, see, for example, Shreve (2004) [20][Ch. 9], that any strictly positive, non-dividend paying asset, $A_t > 0$, can be used as a numéraire. That is, if V_t is the price of an asset in units of the market currency, then V_t/A_t is the price of the asset denominated in units of A_t.

A measure $\mathbb{P}^A$ is called *risk-neutral for the numéraire A* if for every traded, non-dividend paying asset V_t, V_t/A_t is a $\mathbb{P}^A$-martingale. The "usual" risk-neutral measure $\mathbb{P}$ is risk-neutral for the money market account numéraire, β. The Radon-Nikodym density process

$$\eta_t := \frac{A_t/A_0}{\beta_t/\beta_0}, \qquad 0 \leq t \leq T, \tag{4.3}$$

defines the new measure $\mathbb{P}^A$ with Radon-Nikodym derivative $d\mathbb{P}^A/d\mathbb{P} = \eta_T$. Conditional expectations under this new measure are evaluated according to the formula

$$\mathbb{E}^A\{X|\mathcal{G}_t\} = \mathbb{E}\left\{X\frac{\eta_T}{\eta_t}\middle|\mathcal{G}_t\right\}. \tag{4.4}$$

In particular,

$$\begin{aligned} \mathbb{E}^A\left\{\frac{V_T}{A_T}\middle|\mathcal{G}_t\right\} &= \mathbb{E}\left\{\frac{V_T}{A_T}\frac{A_T/A_t}{\beta_T/\beta_t}\middle|\mathcal{G}_t\right\} \\ &= \frac{\beta_t}{A_t}\mathbb{E}\left\{\frac{V_T}{\beta_T}\middle|\mathcal{G}_t\right\} \\ &= \frac{V_t}{A_t}, \end{aligned} \tag{4.5}$$

so that the measure $\mathbb{P}^A$ is risk-neutral for A.

Eqn. (4.1) suggests the use of $\hat{U}_t^T$ as a numéraire. However, we need a minor redefinition first. By the formula

$$\hat{U}_t^T = \mathbb{E}\left\{\frac{\beta_t}{\beta_\tau}\mathbb{1}_{\{t<\tau\leq T\}}\middle|\mathcal{G}_t\right\}, \tag{4.6}$$

we see that $\hat{U}_t^T > 0$ on $\{t < \tau \leq T\}$ and $\hat{U}_t^T = 0$ on $\{\tau \leq t\}$. However, we note that (4.6) arises from (1.17) applied to $(T_1, T_2] = (t, T]$. Moreover,

$$\hat{U}_t^T = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left\{\frac{\beta_t}{\beta_\tau}\mathbb{1}_{\{\tau\leq T\}}\middle|\mathcal{G}_t\right\}. \tag{4.7}$$

From this we see that $\hat{U}_t^T = 0$ on $\{\tau \leq t\}$ is more an artifact of the definition (1.17) for forward intervals rather than an intrinsic property of the contract for unit payment at default. When the contract takes effect immediately, we can just as well define

$$\hat{U}_t^T := \mathbb{E}\left\{ \frac{\beta_t}{\beta_\tau} \mathbb{1}_{\{\tau \leq T\}} \middle| \mathcal{G}_t \right\}, \qquad 0 \leq t \leq T. \tag{4.8}$$

This redefinition agrees with the old one on $\{\tau > t\}$, whereas now on $\{\tau \leq t\}$, $\hat{U}_t^T = \beta_t/\beta_\tau$. Note also that in (2.15) and (2.16), we needed $\{\tau > t\}$ for reasons of financial interpretation.

Even with the redefinition, we still have $\hat{U}_t^T = 0$ on $\{\tau > T\}$. That is, we do not have a strictly positive asset to use as numéraire. Fortunately, this situation has been investigated by Schönbucher (2000, 2003b) [17, 19]. We follow Schönbucher (2003b) for the remainder of this discussion, wherein the author notes that (4.3) still defines a measure even for ω such that $A_t(\omega) = 0$ so long as $A_0(\omega) > 0$. This measure will no longer be an *equivalent* martingale measure, but it will be absolutely continuous with respect to $\mathbb{P}$.

Thus, we can define an absolutely continuous measure $\mathbb{P}^{\hat{U}}$, whose Radon-Nikodym derivative is given by

$$\frac{d\mathbb{P}^{\hat{U}}}{d\mathbb{P}} = \frac{\hat{U}_T^T/\hat{U}_0^T}{\beta_T/\beta_0} = \frac{\mathbb{1}_{\{\tau \leq T\}}}{\beta_\tau \mathbb{E}\left\{\frac{\mathbb{1}_{\{\tau \leq T\}}}{\beta_\tau}\right\}}, \tag{4.9}$$

since $\beta_0 = 1$ and $\hat{U}_T^T = \frac{\beta_T}{\beta_\tau}\mathbb{1}_{\{\tau \leq T\}}$. Under $\mathbb{P}^{\hat{U}}$, forward prices for delivery at default are martingales. By analogy with the T*-forward measure* introduced in Black (1976) [2], which uses the ZCB B_t^T as numéraire, we call $\mathbb{P}^{\hat{U}}$ the τ*-forward measure*.

Lastly, let us make explicit the connection between $\mathbb{P}^{\hat{U}}$ and the so-called *survival-measure* construction in Schönbucher (2003b). Consider a ZCB with random maturity τ:

$$B_t^\tau := \mathbb{E}\left\{ \frac{\beta_t}{\beta_\tau} \middle| \mathcal{G}_t \right\}, \qquad t \geq 0. \tag{4.10}$$

Next consider a defaultable security, $\bar{A}$, which pays B_T^τ at T in survival (if $\tau > T$). The value of this security at $t \leq T$ is

$$\bar{A}_t = \mathbb{E}\left\{ \frac{\beta_t}{\beta_T} B_T^\tau \mathbb{1}_{\{\tau > T\}} \middle| \mathscr{G}_t \right\} = \mathbb{E}\left\{ \frac{\beta_t}{\beta_\tau} \mathbb{1}_{\{\tau > T\}} \middle| \mathscr{G}_t \right\}, \tag{4.11}$$

where the last identity follows by iterated expectations. Then in the setup of Schönbucher (2003b), we have a promised payoff of $A'_T = B_T^\tau$ where $\bar{A}_T = A'_T \mathbb{1}_{\{T<\tau\}}$. Thus the equivalent martingale measure $\mathbb{P}^A$ for the default-free, non-negative numéraire A has a survival measure $\mathbb{P}^{\bar{A}}$ with defaultable numéraire $\bar{A}$. These measures are related to the τ-forward measure by

$$\mathbb{E}\left\{ \frac{\mathbb{1}_{\{\tau \leq T\}}}{\beta_\tau} \right\} \frac{d\mathbb{P}^{\hat{U}}}{d\mathbb{P}} = \mathbb{E}\left\{ \frac{1}{\beta_\tau} \right\} \frac{d\mathbb{P}^A}{d\mathbb{P}} - \mathbb{E}\left\{ \frac{\mathbb{1}_{\{\tau > T\}}}{\beta_\tau} \right\} \frac{d\mathbb{P}^{\bar{A}}}{d\mathbb{P}}. \tag{4.12}$$

Appendix A

Appendix: Delayed Recovery

This appendix sketches a direction for future research.

A case where the inclusion $\mathscr{F}_t \vee \mathscr{D}_t \subset \mathscr{G}_t$ may be proper is that of "delayed recovery". A real world bankruptcy process is usually not resolved for some time, and new information *does* become available during this resolution period. One feature of (2.8) is that on $\{t \geq \tau\}$,

$$\hat{X}_t = \frac{D_\tau}{D_t} X_\tau, \tag{A.1}$$

which is the t-present value of a default payment X_τ at τ, where the default payment proceeds have been invested in the money-market account. Several recent papers considered the asset price after default, see Guo, Jarrow, Lin (2008) [5], Guo, Jarrow, Zeng (2009a) [7], Guo, Jarrow, Larrard (2011) [6], and El Karoui, Jeanblanc, Jiao (2010) [4].

Let $\tau' = \tau + \theta$ be the random time of "final resolution" after default τ, where θ is a $\mathbf{G}$-stopping time modelling the time between default and final resolution. The ultimate recovery $X_{\tau'}$, is assumed to be $\mathscr{G}_{\tau'}$ measurable, but is not known at τ. To extend the rebate value process after τ, we assume that there are no cash-flows in $[\tau, \tau')$. On $\{s \geq \tau\}$, we define

$$\hat{X}_s := \frac{1}{D_s} \mathbb{E}\left(D_{\tau'} X_{\tau'} | \mathscr{G}_s\right), \qquad s \geq \tau. \tag{A.2}$$

In particular, we this means that

$$X_\tau = \hat{X}_\tau = \frac{1}{D_\tau}\mathbb{E}\left(D_{\tau'}X_{\tau'}|\mathscr{G}_\tau\right). \tag{A.3}$$

The extended rebate value process (A.2) coincides with the rebate value process defined in (2.8) on $\{t \le \tau\}$, since

$$\begin{aligned} D_t\hat{X}_t &= \mathbb{E}\left\{D_{\tau'}X_{\tau'}|\mathscr{G}_t\right\} \\ &= \mathbb{E}\left\{\mathbb{E}\left(D_{\tau'}X_{\tau'}|\mathscr{G}_\tau\right)|\mathscr{G}_t\right\} \\ &= \mathbb{E}\left\{D_\tau X_\tau|\mathscr{G}_t\right\}. \end{aligned} \tag{A.4}$$

Thus the discounted rebate value process remains a martingale and the delay is naturally accomodated for $t \le \tau$. During bankruptcy resolution, for $\tau \le s \le \tau'$, the discounted rebate value process continues to evolve given new information according to (A.2). The analogue to (A.1) is

$$\hat{X}_t = \frac{D_{\tau'}}{D_t}X_{\tau'}, \qquad \text{on } \left\{t \ge \tau'\right\}. \tag{A.5}$$

By making specific assumptions about the random variables θ and $X_{\tau'}$ conditional on the information during bankruptcy resolution, $(\mathscr{G}_s, \tau \le s \le \tau')$, one can further model delayed recovery.

Bibliography

[1] BIELECKI, T. R., AND RUTKOWSKI, M. *Credit Risk: Modeling, Valuation and Hedging*. Springer-Verlag Berlin Heidelberg, 2002.

[2] BLACK, F. The pricing of commodity contracts. *Journal of Financial Economics 3*, 2 (Mar. 1976), 167–179.

[3] BLANCHET-SCALLIET, C., AND JEANBLANC, M. Hazard rate for credit risk and hedging defaultable contingent claims. *Finance and Stochastics 8* (2004), 145–159.

[4] ELKAROUI, N., JEANBLANC, M., AND JIAO, Y. What happens after a default: The conditional density approach. *Stochastic Processes and their Applications 120*, 7 (2010), 1011 – 1032.

[5] GUO, X., JARROW, R., AND LIN, H. Distressed debt prices and recovery rate estimation. *Review of Derivatives Research 11* (2008), 171–204. 10.1007/s11147-009-9029-2.

[6] GUO, X., JARROW, R. A., AND DE LARRARD, A. The economic default time and the arcsine law. Quantitative Finance Papers 1012.0843, arXiv.org, 2011.

[7] GUO, X., JARROW, R. A., AND ZENG, Y. Modeling the recovery rate in a reduced form model. *Mathematical Finance 19*, 1 (2009a), 73–97.

[8] HARRISON, J. M., AND PLISKA, S. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and Their Applications 11* (1981), 215–260.

[9] JARROW, R. A., AND TURNBULL, S. M. Pricing derivatives on financial securities subject to credit risk. *The Journal of Finance 50*, 1 (Mar. 1995), 53–85.

[10] JEANBLANC, M., AND RUTKOWSKI, M. Modelling of default risk: mathematical tools. *Preprint, Department of Mathematics, Université d'Evry* (2000a).

[11] JEANBLANC, M., AND RUTKOWSKI, M. Default risk and hazard process. In *Mathematical Finance - Bachelier Congress 2000*. Springer-Verlag Berlin Heidelberg New York, 2000b, pp. 281–312. ISBN: 3-540-67781-X.

[12] JEANBLANC, M., AND RUTKOWSKI, M. Modelling of default risk: an overview. *Mathematical Finance: Theory and Practice* (2000c), 171–269. Beijing: Higher Education Press.

[13] LANDO, D. On Cox processes and credit risky securities. *Review of Derivatives Research 2* (1998), 99–120.

[14] MERTON, R. C. On the pricing of corportate debt: the risk structure of interest rates. *Journal of Finance 29* (May 1974), 449–470.

[15] MUSIELA, M., AND RUTKOWSKI, M. *Martingale Methods in Financial Modelling*, 2nd ed. Springer-Verlag Berlin Heidelberg New York, 2005.

[16] PROTTER, P. E. *Stochastic Integration and Differential Equations*, 2nd ed. Springer-Verlag Berlin Heidelberg New York, 2004.

[17] SCHÖNBUCHER, P. J. A libor market model with default risk. Available at SSRN: http://ssrn.com/abstract=261051., December 2000.

[18] SCHÖNBUCHER, P. J. *Credit Derivatives Pricing Models: Models, Pricing and Implementation*. Wiley, Mar. 2003.

[19] SCHÖNBUCHER, P. J. A note on survival measures and the pricing of options on credit default swaps, May 2003b.

[20] SHREVE, S. E. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer Finance, 2004.