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Optimal Stopping Rule for a Project with Uncertain Completion Time and Partial Salvageability

By Tailan Chi, John Liu, and Hong Chen
The University of Kansas, The University of Wisconsin-Milwaukee, The University of British Columbia

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Tailan Chi and John Liu
School of Business Administration, University of Wisconsin-Milwaukee,
Milwaukee, WI 53201-0742, USA

Hong Chen
Faculty of Commerce and Business Administration, University of British Columbia,
Vancouver, British Columbia, Canada V6T 1Z2

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I. Introduction

Firms often undertake projects that require time to develop and entail a continuous flow of investment during the development process. This type of investment projects include not only

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those for the development of new products but also those for the reengineering of manufacturing systems. Because it is impossible to anticipate all the technical problems that may arise during the development process, the time required to complete such a project is inevitably subject to a significant level of uncertainty. The presence of this kind of uncertainty means that the project's completion time can only be imperfectly estimated *ex ante* and that the estimation is likely to be revised later as new information is gathered in its development process. In such an uncertain and randomly evolving situation, the manager often faces serious difficulty in assessing whether the project is still worth continuing because established methods for project appraisal do not lend themselves to a dynamic evaluation of the project. As Ingersoll and Ross [5] have demonstrated, investment decisions based on the net-present-value (NPV) or internal-rate-of-return (IRR) calculations can be subject to very large biases when the value of the investment project is under the influence of dynamically evolving factors. Although the manager could try to correct such biases by supplementing or even supplanting the NPV or IRR calculations with his/her subjective judgments, the use of subjective judgments can also introduce certain systematic psychological biases [12]. It is, therefore, of both theoretical and practical importance to develop more accurate methods for the evaluation of development projects whose time to completion is uncertain.

The purpose of this paper is to develop a model for assessing such a project based on the theory of optimal stopping times. Optimal stopping models have in the past been used to evaluate uncertain investment projects that take time to build and provide no payoff if it is stopped before reaching a well defined point of completion [9], [11]. As will be explained in the next section, many development projects, especially those for the reengineering of manufacturing systems, can still yield some payoffs even if they are terminated before the originally envisioned performance objectives are achieved. The model developed in our paper accommodates this condition by allowing the project's potential payoff to be partially realizable in the case of termination without the attainment of its original performance objectives.

The paper is organized as follows. The next section describes the features of an actual development project to motivate our model. Section III sets up the optimal stopping model and

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derives its optimality conditions. By giving a specific form to the function that characterizes the buildup of the project's value, section IV examines in more detail the analytical and numerical methods for solving the model and the characteristics of the optimal decision rule. Section V scrutinizes the solutions of the model under alternative forms of the function that characterizes the buildup of the project's value. Section VI discusses methods for estimating the forms of the component functions used in the model. The last section summarizes and concludes the paper.

II. Description of a Specific Project

In order to motivate the model of the paper and clarify its application background, we use this section to describe in some detail the main features of a specific development project. These features will be incorporated in the model to be set up in the next section.

A project that provided much of the impetus for our research involved the reengineering of an automotive engine manufacturing plant that also needed expansion due to increased market demand. Although the owner of the plant had been quite successful in developing competitive automotive products, it felt that its own engineers did not possess the technology for the design of a state-of-the-art manufacturing system. Based on this assessment, the company contracted an outside engineering firm to redesign the plant. After a feasibility study, the engineering firm made a detailed proposal outlining the improvements that could be achieved through a major reengineering effort. Further negotiations led to the final contract between the two parties.

The contractor undertook the responsibilities for designing a new manufacturing system and assuring a smooth transition from the current to the new system. The final contract spelled out several standards that the reengineered plant was required to meet in terms of production efficiency and product quality (such as capacity, throughput rate, speed of model change-over and defect rate). The remuneration for the contractor's work was contingent on the attainment of the various standards spelled out in the contract. Specifically, the contractor would receive the full amount of a contractually stipulated consulting fee upon satisfaction of the prespecified standards; otherwise, it would incur penalties that were linked to the actual performance of the reengineered plant. So under this arrangement, the contractor would still get a portion of its contractual payment even if it decided to terminate the project before completing the originally envisioned tasks in the case of encountering unforeseen technical difficulties. However, because in many cases the benefit from an individual technical solution would be severely restricted if certain related technical problems were not solved, termination due to failure to solve important technical problems was expected to result in severe performance penalties.

Another important requirement of the contract was that the plant operate at approximately its original manufacturing capacity to guarantee timely fulfillment of all new orders during the reengineering process. Because of this requirement, unanticipated variation in the progress of the project would have a limited impact on the bottom line of the owner. Although it was expected to take about three years to complete the reengineering project, the actual completion time could be affected by variations in the timing and size of the new orders.

The main cost of the project from the contractor's viewpoint was the opportunity cost of its highly sought-after consultants assigned to this project. Because of the expected benefit from maintaining continuity, most of its consultants were expected to stay in the project from the start till the end. The main factor determining the total cost was thus the time required to complete the project. There were several sources of uncertainty concerning the project's completion time. The predominant source was due to uncertainty about the extent of difficulty in solving the various technical problems in the reengineering process. Since progress (or lack of it) in the project could yield new information about the prospect of its future development, this type of uncertainty also meant opportunities for learning more about the extent of difficulty in carrying out the remaining tasks. Other uncertain factors that could also affect the project's progress included the delivery time of new equipment ordered, the speed of technology transfer, and the timing and size of new orders that might arrive during the reengineering process. Although all these uncertainties would be resolved when the project was brought to completion, the consultants from the engineering firm expected that the extent of uncertainty and learning potential about its development process would remain roughly constant in its early stages.

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III. The Optimal Stopping Model

Taking into account the main features of the project discussed in the last section, we now draw up the specification of the optimal stopping model and derive its optimality conditions.

A. Model Specification

As suggested in the example described above, the most important uncertainty about a development project often concerns the extent of difficulty in solving the various technical problems of the project. Sometimes the solution of a seemingly easy problem takes a much longer time than expected, and sometimes an apparently difficult problem gets solved very quickly. So, we decided to focus our model on this type of technical uncertainty and use a stochastic variable X_t to represent the estimated remaining time to completion at time $t \geq 0$. Specifically, we assume the evolution of X_t to be characterized by a drifted Brownian motion

$$dX_t = -dt + \sigma\rho(X_t)dW_t, \tag{1}$$

where W_t is a Wiener process, σ is a constant, and $\rho(X_t)$ is an increasing function of X_t. Because the uncertainty and learning potential about the project's completion time necessarily diminishes as the project approaches completion (i.e., as X_t goes to zero), we require $\rho(0) = 0$ and $\rho'(X_t) > 0$.

The exact functional form of $\rho(X_t)$ should depend on how the extent of uncertainty and learning potential about the project's time to completion evolves over time.[1] If the potential for learning diminishes at a roughly constant rate throughout the project's development process, it will be reasonable to use a function that varies quite uniformly with X_t, such as a power function like $\sqrt{X_t}$ [11]. If the learning potential is not expected to fall significantly until some later stage of development, it will be more appropriate to employ a function that approaches an asymptote as X_t rises, such as $\sqrt{1-e^{-\lambda X_t}}$ or $\sqrt{X_t/(1+\lambda X_t)}$ where λ is a constant. In section VI, we will give a more detailed discussion of how to estimate $\rho(X_t)$ and its parameters for a given project. Suffice it here to note that our model can be solved using the procedures to be outlined in the next three

[1] Note that $\sigma\rho(X_t)dW_t \neq 0$ signifies an unanticipated adjustment in the project's estimated time to completion due to the arrival of new information about the length of time required to solve all the remaining technical problems. Given that $\sigma\rho(X_t)$ represents the standard deviation of such unanticipated adjustments, it reflects the potential for acquiring new information (i.e., learning) about the project's time to completion.

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sections under many different assumptions about the functional form of $\rho(X_t)$, including $\sqrt{X_t}$, $\sqrt{1-e^{-\lambda X_t}}$ and $\sqrt{X_t/(1+\lambda X_t)}$. For the model being developed here, we assume

$$\rho(X_t) = \sqrt{1-e^{-\lambda X_t}}\ , \tag{2}$$

where λ is a scaling coefficient; because this functional form not only fits the specific project that motivated our model but also represents an important pattern of evolution in a project's learning potential that has not been scrutinized in existent models.[2] It can be easily seen that $\rho(0)=0$ and $\rho(\infty)=1$ under this functional form and that the larger the value of λ the faster $\rho(X_t)$ approaches one as X_t increases. It can be shown that the mean of X_t is indeed X_t and that the variance of X_t can be approximated by $\sigma^2\left(1-e^{-\lambda X_t}\right)X_t$ under the specifications of (1) and (2). The mathematical proofs of these results are provided in the Appendix.

As explained earlier, the firm that undertakes a development project often has the option to terminate the project without achieving the originally envisioned performance objectives in case of encountering unforeseen technical difficulties. Let $p \in (0,\infty)$ represent the project's maximum possible value. If the firm that performs the development work is an outside contractor, p just represents the firm's full compensation in the absence of penalties for failure to meet the contractually stipulated performance standards. Since such a project normally can still yield some payoff even if it is stopped before achieving its original performance objectives, we assume that the realizable value of the project is built up gradually over time. Let $z(X_t)$ denote the realizable value of the project at termination. Given that the realizable value of the project is p if all the performance objectives are achieved and that X_t measures the distance from the attainment of those objectives, the basic requirements for the function $z(X_t)$ are $z(0) = p$ and $z'(X_t) < 0$. A more detailed specification of $z(X_t)$ requires the knowledge of how the speed of buildup in the value of the project changes as it moves toward completion (i.e., as X_t approaches zero).

We can borrow the framework of Kamien and Schwartz [6] for analyzing salvage values to categorize the process in which the project's value is built up, since the value buildup process

[2] Our experiments with alternative functional forms suggest that the qualitative results of our model are not sensitive to changes in the form of $\rho(X_t)$, so long as $\rho(0) = 0$ and $\rho'(X_t) > 0$.

is very much the converse of the process in which the salvage value of an asset depreciates with time of usage. Note that the second derivative of the terminal payoff function reflects the change in the speed of buildup. Based on the sign of $z''(X_t)$, we can distinguish three types of processes that do not have any reversal in the speed of buildup: $z''(X_t) < 0$ represents accelerated buildup, $z''(X_t) = 0$ represents constant buildup, and $z''(X_t) > 0$ represents decelerated buildup. In this paper, we will not give equal treatment to these three function types because some of them either have a trivial solution or imply a decision rule that seems unrealistic under most circumstances. Since a meaningful assessment of their significance requires the knowledge about the solution of the problem under each of the function types, we will defer our assessment till section V.

In this paper, we will first solve the optimal stopping problem for the case of accelerated buildup, i.e., $z''(X_t)<0$; because its solution not only has clearly realistic applications but is also fairly straightforward to derive. This kind of buildup process is most likely associated with the development of new technology and can be attributed to the condition that the potential benefits from most of the solutions found in the early stages may not be fully realizable until other related problems are also solved, possibly, in later stages. Due to the slow initial buildup, the investment in the project has a low salvage value if it is terminated in an early stage of development. So, a project that has this type of buildup process is subject to a high downside risk. In the next section, we will use a specific functional form, $z(X_t) = p\gamma^{X_t}$ with $\gamma \in (0,1)$, to approximate the realizable value of the project that is built up at an accelerating rate. Section V will explore the implications of constant and decelerated buildup as well as the case where the buildup accelerates in the early part and decelerates in the later part of the project.

In our model, two types of stopping time need to be distinguished: one is natural stopping due to completion and the other is forced stopping due to the project's lack of chance to reach a profitable completion. Let $\tau \equiv \inf\{t \geq 0 \mid X_t \leq 0\}$ denote the time at which the project is actually completed and let θ denote the time at which the project is terminated prior to completion. Note that θ is the decision variable in our problem. These definitions imply $\tau \leq \theta$ in the case of completion and $\theta < \tau$ in the case of termination before completion. Obviously, we have $X_\tau = 0$.

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As explained earlier, the realizable value of the project is $z(X_\theta)$ in the case of termination prior to completion and reaches its maximum value p upon completion. To reduce clutter in notation, we will use $x = X_0$ to denote the estimated remaining time to completion at $t = 0$.

Based on the above definitions, the expected profit from the project can be expressed as

$$E_x\left\{pe^{-\mu\tau}1[\tau \le \theta] + z(X_\theta)e^{-\mu\theta}1[\theta < \tau] - \int_0^{\theta\wedge\tau} ke^{-\mu t}dt - I\right\} \quad (3)$$

where k denotes a continuous flow of investment in the project, I represents an initial investment required to start the project, and μ is the applicable discount rate. The investment cost k can be considered as the opportunity cost of the resources (particularly engineering personnel) devoted to the project. Note that the payoff from the project equals $z(x) - I$ if it is stopped immediately after startup. It is reasonable to expect $I \ge z(x)$, so that the payoff from terminating the project immediately after startup is not positive. It should be noted that I becomes a sunk investment after the initiation of the project and thus has no bearing on the optimal stopping rule for an ongoing project, although this initial investment does affect the decision on whether to undertake the project in the beginning. The issue of determining the applicable discount rate for the project will be discussed later in the paper.

Given the specification outlined above, our objective function can be written as follows.

$$\pi(x) \equiv \max_\theta E_x\left\{pe^{-\mu\tau}1[\tau \le \theta] + z(X_\theta)e^{-\mu\theta}1[\theta < \tau] - \int_0^{\theta\wedge\tau} ke^{-\mu t}dt\right\} \quad (4)$$

Note that the initial investment I at the time of startup is dropped from (4) to reduce clutter in notation, since it has no effect on the optimal stopping rule. As defined in (4), our objective is to find a stopping time $\hat{\theta}$ such that the expectation given in (3) is maximized. Given our definitions in (1) and (2), this problem is stationary in the sense that the maximum profit function $\pi(x)$ is independent of time t. In such a case, the value and shape of $\pi(x)$ remains the same whether x is observed at $t = 0$ or any other time. So, after the project is started, we can treat any currently observed time to completion as x in our analysis.

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B. Optimality Conditions

As shown by Bensoussan and Lion [1], the maximum profit function given in (4) must satisfy

$$\frac{1}{2}\sigma^2\rho(x)^2\frac{d^2\pi(x)}{dx^2}-\frac{d\pi(x)}{dx}-\mu\pi(x)-k\le 0 \tag{5a}$$

$$\pi(x)-z(x)\ge 0 \tag{5b}$$

$$\left[\frac{1}{2}\sigma^2\rho(x)^2\frac{d^2\pi(x)}{dx^2}-\frac{d\pi(x)}{dx}-\mu\pi(x)-k\right]\left[\pi(x)-z(x)\right]=0 \tag{5c}$$

Condition (5a) can be called the marginal payoff condition and condition (5b) can be called the terminal payoff condition.[3] Note that, given the current state of the project *x*, we need to decide whether we should continue it for the next moment and further build up the value of the project or stop it immediately and get the terminal payoff $z(x)$. Condition (5a) measures the marginal change in the value of the maximum profit function $\pi(x)$ as the project is continued for the next moment. A strictly negative sign in (5a) means that the marginal buildup of the project's value due to continuation of its development falls short of the marginal cost of carrying it for the next moment, implying that the project should be stopped immediately. Since by definition $\pi(x)$ is the maximum profit, an equality sign in (5a) means that continuation is the optimal policy to follow. Condition (5b), in the meantime, evaluates the same decision by comparing the total maximum profit $\pi(x)$ with the total terminal payoff $z(x)$. Obviously, we should continue the project to build more value into it if $\pi(x) > z(x)$, and stop it only if $\pi(x) = z(x)$. Based on the above analysis, condition (5a) must be an equality when it is optimal to continue the project

[3] Bensoussan and Lion [1] derived this set of optimality conditions using the method of dynamic programming. As shown by Pindyck [9], an essentially identical set of conditions can also be derived using the method of contingent claims analysis. The main difference between the results obtained under these two approaches concerns the proper choice of the applicable discount rate μ. Under the dynamic programming approach, the proper discount rate should reflect the firm's marginal cost of capital, which may at times be difficult to determine. Under the contingent claims approach, one can use the risk-free rate of return as the applicable discount rate, so long as the risk involved in the project can be perfectly replicated in one or more forms of assets that are publicly traded on the open market (such as stocks or commodities). As pointed out by Kamrad [7], the contingent claims approach is advantageous if the risk primarily stems from fluctuations in the price of some tradable commodity (such as copper or wheat). In our case, however, the main risk of the project is due to the uncertain evolution of its time to completion, whose relationship with the price of any tradable asset is hard to tell. Since it seems rather difficult to select and maintain a portfolio of such assets that can perfectly mimic the risk of the project, we elect to use the dynamic programming approach in our derivations. An extensive comparison between these two approaches can be found in Dixit and Pindyck [3].

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while condition (5b) must be an equality when it is optimal to terminate the project. Hence, the two conditions given in (5a) and (5b) can not be both strictly inequalities if $\pi(x)$ represents the maximum profit function. This last condition is spelled out by the equality given in (5c).

IV. Solution of the Model under an Accelerating Buildup Process

As alluded to in the previous section, the buildup in the realizable value of the project can in theory accelerate, remain constant or decelerate as it moves toward completion. Because the procedure for solving the model varies with the nature of the buildup process, it is convenient to first examine in detail the solution under one of the alternative buildup processes. This section focuses on the solution under the accelerated buildup process that is characterized by $z''(x) < 0$, and the next section will examine the solutions under alternative buildup processes. The reason for looking at the case of accelerated buildup first is that this type of buildup process has clearly realistic applications and its solution is relatively straightforward to derive. The specific form of the terminal payoff function that we are going to use to derive the solution under the accelerated buildup process is $z(x) = p\gamma^x$ with $\gamma \in (0,1)$. Under this buildup process, the problem has an analytical solution for the special case of $\mu = 0$ and need to be solved numerically for the general case of $\mu \geq 0$. After deriving the conditions for obtaining the optimal control rule in the first part of this section, we proceed to present the analytical and numerical solutions of the model under the assumption of $z(x) = p\gamma^x$.

A. Operationalization of the Optimal Control Policy

As noted in our exposition of the optimality conditions, the control problem in managing the project is to determine, given its current state *x*, whether to continue or stop the project with a view to maximizing its profit. A larger *x* means that more time is needed to complete the project. As a rise in *x* increases the project's expected cost to completion while leaving its potential value at completion *p* unchanged, there must exist a value of *x*, denoted by $\hat{x}$, such that continuation of the project only brings a negative return for all $x \in [\hat{x}, \infty)$. In addition, since the rate of investment *k* is independent of *x* while the buildup of the project's value $z(x) = p\gamma^x$ accelerates as it moves

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closer to completion (i.e., as x becomes smaller), continuation necessarily yields a positive return for all $x \in (0, \hat{x})$ if it is so at $\hat{x} - \Delta x$ for an arbitrarily small Δx. This line of reasoning suggests that the optimal control policy in our problem is threshold control, i.e., there exists a threshold value $\hat{x}$ such that the project should be continued for all $x \in (0, \hat{x})$ and stopped for all $x \in [\hat{x}, \infty)$. We can refer to $\hat{x}$ as the threshold remaining time.

Hence, our approach to finding a solution to the optimal stopping problem is to search for an optimal profit function that satisfies the optimality conditions (5a) to (5c) under a threshold control policy. Since $z(x) = p\gamma^x$ becomes the optimal profit function $\pi(x)$ for $x \ge \hat{x}$ under a threshold control policy, our task is reduced to the construction of a function that satisfies the optimality conditions (5a) to (5c) for $x \in [0, \hat{x})$. As explained in the previous section, when continuation is optimal, (5a) must be an equality, i.e.,

$$\frac{1}{2}\sigma^2 \rho(x)^2 \frac{d^2\pi(x)}{dx^2} - \frac{d\pi(x)}{dx} - \mu\pi(x) - k = 0 \tag{6a}$$

for $x \in (0, \hat{x})$. It can be seen that equation (6a) is a second-order ordinary differential equation (ODE). Since the threshold remaining time $\hat{x}$ represents a free boundary that is to be determined together with the solution to the ODE, a unique solution of this free-boundary problem requires the knowledge of exactly three boundary conditions [13]. An obvious boundary condition is that the optimal profit function should equal the terminal payoff function $z(x) = p\gamma^x$ when the estimated remaining time equals the threshold remaining time, i.e.,

$$\pi(\hat{x}) = p\gamma^{\hat{x}}. \tag{6b}$$

A second boundary condition can be constructed using the Lagrange multiplier method. Rearrange (6b) as $\pi(\hat{x}) - p\gamma^x = 0$ and introduce a Lagrange multiplier $l \ne 0$ to write the terminal condition (6b) as a lagrangian function

$$L(\hat{x}, l) = l\left[\pi(\hat{x}) - p\gamma^{\hat{x}}\right].$$

Then, by the Lagrange multiplier theory, the optimal terminal profit $\pi(\hat{x})$ must satisfy

$$\frac{\partial L(\hat{x}, l)}{\partial \eta} = 0 \text{ and } \frac{\partial L(\hat{x}, l)}{\partial \hat{x}} = 0.$$

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The latter of the above yields a second terminal condition

$$\pi'(\hat{x}) = p\gamma^{\hat{x}} \ln\gamma \,. \tag{6c}$$

Since we know that the profit from the project must be equal to p if it is complete (i.e., $x = 0$), we obtain a third boundary condition as follows.

$$\pi(0) = p \tag{6d}$$

The conditions given in (6a) to (6d) define a complete free-boundary problem with the threshold remaining time $\hat{x}$ as the free boundary. The solution to this problem consists of the following steps.

1) Finding a solution to the ODE defined in (6a). Such a solution contains two constants from integration, whose values need to be determined by the known boundary conditions. Let $\hat{\pi}(x \mid \hat{x}, c_1, c_2)$ denote the solution to the ODE, with c_1 and c_2 representing the integral constants and with $\hat{x}$ representing the free boundary point.

2) Determine the values of c_1, c_2 and $\hat{x}$ using the boundary conditions (6b), (6c) and (6d), which we restate as

$$\hat{\pi}(0 \mid \hat{x}, c_1, c_2) = p \,, \tag{7a}$$

$$\hat{\pi}(\hat{x} \mid \hat{x}, c_1, c_2) = p\gamma^{\hat{x}} \,, \tag{7b}$$

$$\hat{\pi}'(\hat{x} \mid \hat{x}, c_1, c_2) = p\gamma^{\hat{x}} \ln\gamma \,. \tag{7c}$$

3) Represent the optimal profit function as

$$\pi(x) = \begin{cases} \hat{\pi}(x) & \text{for } x < \hat{x} \\ p\gamma^{x} & \text{for } x \geq \hat{x} \end{cases} . \tag{8}$$

This solution procedure is employed in the rest of this section to solve the optimal control problem for the development project. Both the analytical and numerical results confirm that the solution obtained using this procedure satisfies all the optimality conditions given in (5a) to (5c).

B. Optimal Stopping for a Special Case

In this subsection, we obtain analytical expressions of the function $\hat{\pi}(x)$ and its first and second derivatives $\hat{\pi}'(x)$ and $\hat{\pi}''(x)$ for the special case of $\mu = 0$. These analytical expressions not only are useful in characterizing the optimal stopping rule for the special case but also shed light on the solutions to the general case where $\mu \geq 0$. Specifically, we will use these analytical expressions to show two important results. First, there exists a simple rejection rule for the project when the discount rate is zero or negligible. Second, a threshold policy is optimal for the control of the development project when the discount rate is zero or negligible.

Denote $\omega(x) = \hat{\pi}'(x)$. Then, with $\mu = 0$, the ODE given in (6a) can be expressed as

$$\frac{1}{2}\sigma^2 \rho(x)^2 \omega'(x) = \omega(x) + k,$$

$$\frac{d\omega(x)}{\omega(x) + k} = \frac{2}{\sigma^2 \rho(x)^2} dx.$$

Taking integral on both sides, it becomes

$$\ln[\omega(x) + k] = \int \frac{2}{\sigma^2 \rho^2(x)} dx + c_1.$$

As $\omega(x) = \hat{\pi}'(x)$ by definition, the above equation can be rewritten as

$$\hat{\pi}'(x) = -k + c_1 e^{\frac{2}{\sigma^2}\int \frac{1}{\rho^2(x)}dx}. \tag{9}$$

Denoting $G(x) = \int \frac{1}{\rho^2(x)} dx$ and integrating (9) on both sides, we obtain

$$\hat{\pi}(x) = c_2 - kx + c_1 \int e^{\frac{2}{\sigma^2}G(x)} dx.$$

Denoting $F(x) = \int e^{\frac{2}{\sigma^2}G(x)} dx$, we can write

$$\hat{\pi}(x) = c_2 - kx + c_1 F(x).$$

Since the boundary condition of (6d) implies $\hat{\pi}(0) = p$, we have $c_2 = p - c_1 F(0)$. Substituting $p - c_1 F(0)$ for c_2 in the above expression, we get

$$\hat{\pi}(x) = p - kx + c_1[F(x) - F(0)],$$

which is equivalent to

$$\hat{\pi}(x) = p - kx + c_1 \int_0^x e^{\frac{2}{\sigma^2}G(y)} dy.$$

Given $\rho(X_t) = \sqrt{1 - e^{-\lambda X_t}}$, we can obtain the specific forms of $G(x)$ and $e^{\frac{2}{\sigma^2}G(x)}$.

$$G(x) = \int \frac{1}{1 - e^{-\lambda x}} dx = x + \frac{1}{\lambda}\ln\left(1 - e^{-\lambda x}\right),$$

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$$e^{\frac{2}{\sigma^2}G(x)} = e^{\frac{2}{\sigma^2}\left[x+\frac{1}{\lambda}\ln\left(1-e^{-\lambda x}\right)\right]} = e^{\frac{2}{\sigma^2}x}\left(1-e^{-\lambda x}\right)^{\frac{2}{\sigma^2\lambda}}.$$

Then, by the boundary condition defined in (6c) and the result given in (9) above, we can solve for c_1 as a function of $\hat{x}$:

$$c_1(\hat{x}) = \frac{k + p\gamma^{\hat{x}}\ln\gamma}{e^{\frac{2}{\sigma^2}\hat{x}}\left(1-e^{-\lambda\hat{x}}\right)^{\frac{2}{\sigma^2\lambda}}}. \tag{10}$$

Hence, the solution to the ODE (6a) that also satisfies the set of boundary conditions (7a) to (7c) is just

$$\hat{\pi}(x) = p - kx + c_1(\hat{x})\int_0^x e^{\frac{2}{\sigma^2}y}\left(1-e^{-\lambda y}\right)^{\frac{2}{\sigma^2\lambda}}dy. \tag{11}$$

Its first and second derivatives are, respectively

$$\hat{\pi}'(x) = -k + c_1(\hat{x})e^{\frac{2}{\sigma^2}x}\left(1-e^{-\lambda x}\right)^{\frac{2}{\sigma^2\lambda}}, \tag{12}$$

$$\hat{\pi}''(x) = \frac{2}{\sigma^2}c_1(\hat{x})e^{\frac{2}{\sigma^2}x}\left(1-e^{-\lambda x}\right)^{\frac{2}{\sigma^2\lambda}-1}. \tag{13}$$

Using the result in (10), the numerical value of $\hat{x}$ for a given set of parameter values can be obtained by solving the equation specified in (6a).

Lemma 1. For $x \in (0,\infty)$, $\hat{\pi}(x)$ is strictly convex if $c_1 > 0$ and strictly concave if $c_1 < 0$.

Proof. The result is immediate by (13); since, for $x \in (0,\infty)$, $\hat{\pi}''(x)$ is positive if $c_1 > 0$ and negative if $c_1 < 0$. □

By checking its first and second order derivatives, one can verify that the terminal payoff function $z(x) = p\gamma^x$ is strictly convex and decreasing over $x \in (0,\infty)$. Figure 1 depicts some numerical samples of $\hat{\pi}(x)$.

******** Figure 1 inserted around here ********

Theorem 1 (Rejection Rule). With $\mu = 0$, there exists a rejection rule such that if

$$-k \le p\ln\gamma \tag{14}$$

then reject the project immediately.

Proof. We show by contradiction that the only threshold value that solves equations (6a) to (6d) is $\hat{x} = 0$ if condition (14) holds. First, suppose $\hat{x} > 0$. Then, we know from (12) that $\hat{\pi}'(0) = -k$ if $\hat{x} > 0$. So, when condition (14) is true, we would have $\hat{\pi}'(0) \leq p \ln \gamma$ if $\hat{x} > 0$. Note that $\hat{\pi}'(0) \leq p \ln \gamma$ means $\hat{\pi}(0 + \Delta x) \leq p\gamma^{0+\Delta x}$ for an arbitrarily small Δx, which implies that the project should not be continued if $x > 0$, thus contradicting $\hat{x} > 0$. In addition, by (10), we have $c_1 > 0$ under condition (14). So by Lemma 1, $-k \geq p \ln \gamma$ implies that $\hat{\pi}(x)$ is strictly convex for $x \in (0, \infty)$. Hence, under condition (14), $\hat{\pi}(x)$ can never have a point of tangency to $p\gamma^x$ over $(0, \infty)$ if $\hat{x} > 0$, since $p\gamma^x$ is also strictly convex and tends toward zero as x approaches infinity. Therefore, the condition $-k \geq p \ln \gamma$ gives rise to an empty continuation set, i.e., $\hat{x} = 0$. □

The intuition behind Theorem 1 is straightforward: the project should not be accepted or continued if the instantaneous cost of construction (k) outweighs the instantaneous buildup of its value ($p \ln \gamma$). With this theorem, one can easily determine whether to continue or reject the project by just evaluating the key system parameters (i.e., the required rate of investment k, the project's potential value p, and the rate of development γ) when the discount rate is zero or negligible. The function $\hat{\pi}_0(x)$ in Figure 1 is a solution to the differential equation (6a) but has an empty continuation set. Using the data provided in the upper part of the graph, it can be checked that the parameters of $\hat{\pi}_0(x)$ conform to the Rejection Rule, i.e., $-k_0 < p \ln \gamma$.

Theorem 2 (Threshold Control). With $\mu = 0$, there exists a threshold remaining time $\hat{x}$ such that the optimal control policy is to continue the project whenever $x < \hat{x}$, and stop the project when $x \geq \hat{x}$.

Proof. First, by the result of Theorem 1, Theorem 2 immediately holds if condition (14) is true, in which case we have $\hat{x} = 0$. Next, we show by contradiction that there is a unique $\hat{x} \in (0, \infty)$ that solves (6a) to (6d) if condition (14) is false. Given $-k > p \ln \gamma$, we have $\hat{\pi}'(0) = -k > p \ln \gamma$ for $\hat{x} > 0$, which implies that $\hat{\pi}(0 + \Delta x) > p\gamma^{0+\Delta x}$ for an arbitrarily small Δx, so that the project should be continued for at least an infinitesimal time at

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$x = 0$. Also, we know from (10) that the sign of c_1 depends on the magnitude of $\hat{x}$ when condition (14) is false. Specifically, given $-k > p\ln\gamma$, c_1 is negative when $\hat{x}$ is small, and becomes positive as $\hat{x}$ grows sufficiently large. But by Lemma 1, $c_1 < 0$ implies that $\hat{\pi}(x)$ is strictly concave and thus can not have a point of tangency to $p\gamma^x$ at any $x > 0$. This clearly contradicts any purported solution of the threshold under which $c_1 < 0$. So, the value of $\hat{x}$ must be sufficiently large to make c_1 positive and $\hat{\pi}(x)$ strictly convex in order for $\hat{\pi}(x)$ to be a solution to (6a) to (6d). Given that $p\gamma^x$ is strictly convex and tends toward zero as x grows large, a strictly convex $\hat{\pi}(x)$ can have only one point of tangency to $p\gamma^x$ over $x \in (0,\infty)$. Therefore, when $-k > p\ln\gamma$, there exists a unique $\hat{x} \in (0,\infty)$ that solves (6a) to (6d). □

The function $\hat{\pi}_1(x)$ shown in Figure 1 is a solution to the differential equation (6a) that also satisfies the set of boundary conditions (7a) to (7c) at a $\hat{x} \in (0,\infty)$. As can be seen in the graph, $\hat{\pi}_1(x)$ is strictly convex over $x \in (0,\infty)$ and monotonically decreasing over $x \in [0,\hat{x}]$, so that its continuation set is convex, i.e., a continuous interval of $x \in (0,\hat{x})$. Using the data provided in the upper part of the graph, it can be checked that the parameters of $\hat{\pi}_1(x)$ do not conform to the Rejection Rule $-k_1 > p\ln\gamma$.

C. Parametric Variation of the Threshold Remaining Time

In this subsection, we present numerical results showing the effects of changes in the various parameters on the value of the threshold remaining time $\hat{x}$ in the general case of $\mu \geq 0$ given $z(x) = p\gamma^x$. Because the system of differential equations (6a) to (6d) does not have an analytical solution when $\mu > 0$, we need to use numerical methods to examine how the value of $\hat{x}$ varies with each of the parameters in our model. Since the second order ODE given in (6a) has a free boundary whose value is to be determined together with the solution to the ODE, regular numerical methods for solving ODEs would not work without appropriate modifications. Based on a method suggested by Press *et al*. [10], we converted the free-boundary problem to a regular boundary problem by introducing a new variable $q \in [0,1]$ and an additional equation $y = \hat{x}$ and substituting $q \cdot y$ for x in equations (6a) to (6d).

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Our numerical results indicate that $\hat{x}$ is an increasing function of p and σ and a decreasing function of γ, μ and k. The relationships of $\hat{x}$ with p and k suggest that the optimal stopping rule becomes less conservative when the potential payoff is greater, and becomes more conservative when the investment cost is higher. Since these results are quite obvious, we will focus on the effects on $\hat{x}$ of changes in γ, μ and σ in the rest of this section.[4]

******** Table I inserted around here ********

Table I reports 162 data points that demonstrate how $\hat{x}$ varies with γ, μ and σ.[5] It can be seen in the table that an increase in the value of γ reduces the value of $\hat{x}$. Note that the effect of a rise in γ can be viewed from two slightly different angles. First, if γ is small, the buildup of the project's value will become fast only when $\hat{x}$ moves close to zero, which means that the reward for waiting is greater when γ is smaller. Hence, a smaller γ justifies more patience in managing the project. Second, the larger γ is, the greater will be the realizable payoff from terminating the project prior to completion. Hence, a larger γ increases the reward for early termination and thus discourages patience in managing the project. Figure 2 graphically illustrates the relationship of $\hat{x}$ with γ under differing values of μ and σ.

******** Figure 2 inserted around here ********

The graph in the upper panel of Figure 2 is based on the data given in the middle section of Table I. As can be seen in the graph as well as in the table, an increase in the discount rate μ lowers the threshold remaining time $\hat{x}$. The intuition behind this result is that a higher discount

[4] A possible extension of the model is to endogenize the rate of investment k as a policy variable, since a project's time to completion could be influenced by the amount of resources devoted to its development. Given that the speed of development would not matter if the discount rate is zero, such an extension is meaningless unless the discount rate is positive. Because the problem does not have an analytical solution when the discount rate is positive, one needs to use numerical methods to endogenize k in the model.

[5] Based on our discussions with the consultants of the engineering firm alluded to in section II, the values of σ shown in the table reflect moderate to high levels of uncertainty for projects that take two to seven years to complete. We expect few firms to have a cost of capital that is beyond the range of μ used in our analyses. The values of γ cover essentially the parameter's feasible range of variation given the values of the other parameters. Although the table only reports the effects of changes in these parameters for $p = 5$, $k = 1$ and $\lambda = 0.5$, the qualitative relationships of these parameters with $\hat{x}$ are consistently observed under other values of p, k and λ. The 5:1 ratio between p and k was chosen on the basis of the actual project discussed in section II.

rate makes the payoff from the project less valuable in present value terms. Although the higher discount rate also lowers the present value of the total investment cost, its dampening effect on the project's payoff is greater because the payoff can only be realized in the end.

The graph in the lower panel of Figure 2 is based on data selected from all three sections of Table I to illustrate the relationship between σ and $\hat{x}$. Note that σ measures the variability of the estimation for the project's time to completion and that a larger σ implies a greater chance for the continuation of the project to yield new information and bring about significant corrections in the estimation later on. The positive relationship between σ and $\hat{x}$ shown in the graph suggests that the manager should be more explorative (or less conservative) when the continuation of the project is more likely to yield significant new information about how much time is required to complete it. The rationale behind this result can be explained as follows. Since a rise in σ means a symmetric increase in the project's risk on both the upside and the downside, such a change in the value of σ should have no effect on the value of the project if the decision rule excluded forced stopping, i.e., if the project were allowed to continue indefinitely till finish. The optimal decision rule derived from our model, however, does require forced stopping when the project's time to completion is too long to justify its eventual payoff. Under this decision rule, then, the adverse impact of a rise in the project's downside risk is curtailed because forced stopping in the face of unanticipated difficulties serves to save the portion of the development cost that has not yet been spent. In the meantime, this decision rule still allows the benefit from a strengthening of the project's upside potential to materialize fully because no intervention is exercised when there is an unanticipated fall in the project's estimated time to completion. Hence, the optimal control policy creates an asymmetry in the realization of the project's upside potential and downside risk, making it advantageous to exploit a rise in the variability of its estimated time to completion though the adoption of a less conservative control threshold. This result is also consistent with the results that Roberts and Weitzman [11] and Pindyck [9] obtained earlier.

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V. SOLUTIONS UNDER ALTERNATIVE BUILDUP PROCESSES

In this section, we first compare the solutions under the three types of the terminal payoff function that do not allow any reversal in the speed of buildup: accelerated buildup $z''(X_t) < 0$, constant buildup $z''(X_t) = 0$, and decelerated buildup $z''(X_t) > 0$. After giving an assessment of their relative significance for our problem, we then proceed to examine a more interesting case where the buildup accelerates in the early part and decelerates in the later part of the project.

The case of accelerated buildup has been analyzed in the preceding section. As shown in the results already obtained, this type of buildup process implies the existence of a threshold time to completion $\hat{x}$ such that the optimal decision rule is to continue the project unless $x \geq \hat{x}$. Such a decision rule also entails continuation of the project till completion so long as $x < \hat{x}$, because the ever accelerating buildup of the project's value necessarily justifies the constant cost of investment k once x passes to the left of the threshold $\hat{x}$.

The case of constant buildup implies an optimal decision rule that is trivial. To see this, let b denote the rate of buildup in the realizable value of the project. Then, depending on whether the rate of buildup is faster or slower than the required rate of investment k, the optimal decision rule is either to adopt and continue the project till completion (for $b > k$) or never to adopt the project (for $b < k$). This case, therefore, is analytically uninteresting.

Decelerated buildup implies a diminishing marginal return from developing the project. If there is not a predetermined completion time (e.g., satisfaction of a set of contractually specified performance standards), a project with diminishing return needs to be evaluated by defining the terminal payoff function in such a way that $z'(0) = 0$. With this definition of the terminal payoff function, $X_t = 0$ should no longer represent the point of completion in case of $k > 0$ because it is not economically rational to continue the project till its marginal return falls to zero in the face of a non-zero carrying cost. So, under diminishing return, there must exist a threshold value of x, denoted by $\bar{x}$, such that the net gain from continuing the project is negative for all $x \in [0, \bar{x}]$. The threshold $\bar{x}$ in the case of decelerating buildup is the counterpart of the threshold $\hat{x}$ in the case of accelerating buildup. In addition, since the rate of buildup is necessarily higher than the rate of

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investment for $x > \bar{x}$ when the project's realizable value is a concave function of *x*, the optimal decision rule is to adopt or continue the project so long as the currently observed *x* is larger than the threshold $\bar{x}$. This kind of decision rule is implied under a concave terminal payoff function because such a function rules out the possibility for the eventual payoff from the project to be too distant to justify further investment. This implication would generally appear too good to be true for projects aiming at technological innovation.[6]

Although a continuously decelerating buildup process is unlikely to characterize projects that involve the development of new technology, diminishing return toward the end of the project does appear reasonable for some innovation projects. This type of buildup process means that the effort in "fine-tuning" the project will exhibit diminishing return after most of the technical problems are solved. In such a case, the terminal payoff of the project can be approximated by a function that is concave for small *x* and convex for sufficiently large *x*. A functional form that possesses these features is

$$z(x) = p\cos\left\{0.5\pi\left[1 - \exp\left(-\kappa x^{\eta}\right)\right]\right\} \tag{15}$$

where η and κ are parameters. It is easy to see that under this function $z(0) = p$, $z(\infty) = 0$ and $z'(0) = z'(\infty) = 0$. The shape of the function can be easily adjusted by changes in the values of η and κ.[7] Because the combination of $z'(0) = 0$ and $k > 0$ makes it uneconomical to continue the project till $x = 0$, a solution of this problem must identify a threshold $\bar{x} > 0$ such that the project should be terminated if $x \le \bar{x}$. In addition, because $z'(\infty) = 0$ implies that the potential payoff from the project will be too distant to justify any investment for sufficiently large *x*, a solution of the problem must also identify a second threshold $\hat{x} < \infty$ such that the project should be stopped if $x \ge \bar{x}$. Thus, the problem entails solving differential equation (6a) between the two boundaries $\bar{x}$ and $\hat{x}$, which must be determined as part of the solution. So together with the two constants

[6] The terminal payoff function of a project could exhibit diminishing return (i.e., decelerating buildup) from the start if the project only involves the application of some simple solutions that have proven to bring about quick payoffs. An engineering consultant told us that he once encountered such a "lucky" project when he was hired by a firm that was owned by an incompetent heir and had neglected its manufacturing system for years.

[7] When η is sufficiently small, $z(x)$ becomes an essentially convex function.

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from integration c_1 and c_2, we have four unknowns and thus need four boundary conditions to get an exact solution of the differential equation. These four boundary conditions are $\pi(\bar{x}) = z(\bar{x})$, $\pi'(\bar{x}) = z'(\bar{x})$, $\pi(\hat{x}) = z(\hat{x})$ and $\pi'(\hat{x}) = z'(\hat{x})$. The last two conditions correspond to those given in (6b) and (6c) for the case of accelerated buildup analyzed earlier, and the first two conditions replace the one given in (6d), $\pi(0)=p$, as this condition is no longer relevant given that the new left boundary $\bar{x}$ is necessarily greater than zero.

Figure 3 provides a graphical representation of the solution under a given set of parameter values. Even though the optimal policy under the terminal payoff function defined in (15) takes the form of "bang-bang" control, the two control thresholds $\bar{x}$ and $\hat{x}$ could coincide at some x under certain parameter values, causing the solution $\hat{\pi}(x)$ to collapse into a point, just like the case of $\hat{x} = 0$ under $z(x) = p\gamma^x$. This will occur when the speed of buildup in the project's value is below the required speed of investment k throughout the relevant domain of x. Geometrically, this condition is represented by the absence of a sufficiently steep portion in $z(x)$ for the given value of k, and it is analogous to the violation of the rejection rule (14) derived for $z(x) = p\gamma^x$. Another interesting feature of the solution (not shown in the graph) is that a larger σ causes $\bar{x}$ to decrease and $\hat{x}$ to increase in value, suggesting that greater variability in the estimated time to completion justifies more experimentation both in the beginning and in the end of the project.

******** Figure 3 inserted around here ********

It should be noted that the two control thresholds $\bar{x}$ and $\hat{x}$ are in general not of equal significance. The reason is that the extent to which $\bar{x}$ and $\hat{x}$ deviate from the thresholds derived under the static NPV or payback method depends on the size of $\sigma\rho(x)$, which is an increasing function of x. Since $\bar{x}$ is generally a small value, it should be fairly close to the threshold derived under the static NPV or payback method and thus can be calculated fairly accurately without the aid of a stochastic optimization model. The value of $\hat{x}$, however, is generally much larger and thus likely to deviate considerably from the threshold derived under the static NPV or payback method, making it more difficult to guess without the use of a stochastic optimization model. In

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addition, there may be very little uncertainty about the threshold for completion if the standards for completion are clearly defined in a contract and the penalty for failure to meet those standards is specified as a step function of those standards (which is often the case).

VI. Application Issues

The purpose of this section is to suggest a number of ways to determine the forms of the component functions and the values of the parameters that appear in the model presented above. Under certain conditions, some of the parameters are exactly known or can be calculated quite accurately. For instance, the payoff of the project after completion, p, may be exactly specified in a contract when the firm that does the development work is hired to do so as a consultant. Also, the manager of the project may be able to compute the flow cost k fairly accurately based on the types and amounts of resources (e.g., personnel and equipment) required by the project.[8] But in general, the manager is not likely to have such information on all the elements of the model and may need to estimate some of them using data on past projects. Our discussion in this section will focus on two components of the model whose forms are unlikely to be known *a priori* and whose estimation may require specialized techniques. These two components are the function $\rho(X_t)$, together with the parameter σ, and the terminal payoff function $z(X_t)$.

A. Estimation of $\rho(X_t)$ and σ

As explained in section III, the form of $\rho(X_t)$ should depend on how the uncertainty and learning potential about the project's time to completion evolves in its development process. Because this uncertainty and learning potential necessarily diminishes as the project approaches completion, we need to require $\rho(0)=0$ and $\rho'(X_t)>0$. Within this broad specification, we have argued that $\rho(X_t)$ can vary with X_t in two different ways that are both reasonable. If the learning potential is expected to decrease at a roughly constant rate in the development process, $\rho(X_t)$ should be a function that varies quite uniformly with X_t, such as a power function like $\sqrt{X_t}$. In

[8] We can, of course, think of cases where the value of p or k is subject to a high degree of uncertainty and potential for learning. In such cases, the applicability of our model, which assumes both p and k to be known, will be severely constrained.

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the meantime, if the learning potential is not expected to fall significantly until some later stage of development, $\rho(X_t)$ can be better approximated by a function that approaches an asymptote as X_t increases, such as $\sqrt{1-e^{-\lambda X_t}}$ or $\sqrt{X_t/(1+\lambda X_t)}$.

When the form of $\rho(X_t)$ is not known *a priori* (which is perhaps true in most cases), one can estimate it using data on similar projects performed in the past. Specifically, the type of data that are needed to estimate the form of $\rho(X_t)$ are periodic reports on the progress of past projects, i.e., data on the evolution of the estimated time to completion X_t over time. Let s denote the lag between adjacent estimates of X_t, with $t = 0,1,\cdots,n$. Then, given that X_t theoretically follows a drifted Brownian motion defined in (1), its distribution conditioned on information at $t-1$ is just

$$X_t|\psi_{t-1} \sim N(X_{t-1}-s, h_t), \tag{16}$$

$$h_t(\alpha, X_{t-1}) = \sigma^2[\rho(X_{t-1})]^2 s, \tag{17}$$

where ψ_{t-1} denotes the information set available at $t-1$ and α denotes a set of parameters that determine the value of the variance h_t conditioned on ψ_{t-1}. Because the only unknown parameters of the conditional distribution specified above fall in the variance rather than the mean, standard estimation procedures such as the ordinary or general least squares methods are not applicable. A procedure that allows an unbiased estimation of the parameters α in the conditional variance h_t is the autoregressive conditional heteroscedasticity (ARCH) model suggested by Engle [4]. Using the results obtained by Engle, it is straightforward to show that the log likelihood function for the ARCH model corresponding to the specifications of (16) and (17) has the following form

$$L(\alpha) = -\sum_{t=1}^{n} \ln h_t(\alpha, X_{t-1}) - \sum_{t=1}^{n} \frac{(X_t - X_{t-1} + s)^2}{h_t(\alpha, X_{t-1})}. \tag{18}$$

Since each alternative form of $\rho(X_t)$ gives a unique specification of (18), the selection among the alternative forms of $\rho(X_t)$ can be made based on which of them fits the data best. The estimation of the function's form and any parameters in it can be carried out in the following manner.

1) Estimate each specification of the ARCH model corresponding to an alternative form of $\rho(X_t)$ by maximizing the log likelihood function (18) with respect to the parameters α.
2) Evaluate the fit of each specification based on their Akaike information criteria (AIC).

The AIC statistic is just the value of the log likelihood function (18) adjusted for the number of parameters in the model and is reported by most statistical packages as part of the regression results.[9] Although this regression procedure can be performed with data on a single project, the pooling of data on all the projects whose attributes are judged to be similar (perhaps based on the regression estimates for the individual projects) is likely to produce more reliable results.

The data that is needed to estimate $\rho(X_t)$ and σ statistically should be available in general if the firm collected periodic progress reports on similar projects in which it was involved in the past. If the firm did not collect such data but still wishes to use the model to assess an uncertain project, it can still estimate $\rho(X_t)$ and σ by utilizing the subjective judgments of those who have been involved in similar projects. The procedures for performing such subjective estimation can be based on established methods for eliciting expert opinions on the distribution of a stochastic phenomenon [2]. Briefly, the estimation of $\rho(X_t)$ can proceed in the following steps. First, the expert(s) can be asked to judge whether the learning potential is likely to diminish at a roughly constant rate in the project's development process or unlikely to fall significantly till some later stage of development. This judgment can be used to select between the two alternative functional forms of $\rho(X_t)$ suggested earlier. If a function such as $\sqrt{1-e^{-\lambda X_t}}$ or $\sqrt{X_t/(1+\lambda X_t)}$ is rated as the likely form of $\rho(X_t)$, the expert(s) can then be asked to identify the stage of development at which the learning potential is likely to start decreasing significantly with further development of the project. This information can help choose the value of λ in the function selected. Once the form of $\rho(X_t)$ and any parameter(s) in it are determined, the final step is to estimate the value of σ.

The estimation of σ can make use of the knowledge on the expression for the variance of $x = X_0$ (i.e., the unconditional variance of the project's completion time). The expression for this variance under $\rho(x) = \sqrt{1-e^{-\lambda x}}$ has been derived in the Appendix. Using the result of Pindyck

[9] In order to test whether the method suggested here can correctly select the right form of $\rho(X_t)$, we estimated two alternative specifications of the ARCH model corresponding to $\rho(X_t) = \sqrt{1-e^{-\lambda X_t}}$ and $\rho(X_t) = \sqrt{X_t}$, respectively, using data generated under a random process corresponding to one of these two forms of $\rho(X_t)$. Our results indicate that the ARCH model provides consistent estimates for the parameters and that the AIC statistic can distinguish the correct specification from the incorrect one in every instance.

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[9], it can be shown that the expression under $\rho(x) = \sqrt{x}$ is $[\sigma^2/(2-\sigma^2)]x^2$. The value of σ can be estimated objectively or subjectively depending on whether there is data on the actual lengths of time that similar projects in the past took to finish. Using this type of data, one can compute the variance of the completion time of similar projects, which should equal the variance of the current project's expected completion time *x*. Let *v* denote the variance computed from the data and let *V*(*x*) denote the expression for the variance of *x*. Then, using the calculated value of *v*, one can determine the value of σ based on the relationship *V*(*x*) = *v*. If recorded data is not available, one can still estimate a reasonable range for the variance of *x* subjectively (and thus a reasonable range for σ) to perform a sensitivity analysis based on the model. The subjective estimation of a reasonable value for *V*(*x*) can make use of the relationship between the variance of a normally distributed random variable and the chance for its realizations to fall within a given distance from its mean, e.g., there is about a 2/3 chance for the actual completion time to be within $x \pm \sqrt{V(x)}$.

B. Estimation of Terminal Payoff Function $z(X_t)$

The estimation of the terminal payoff function can also be made using either objective or subjective methods. Although it is common practice to require periodic reports on the progress of a development project, a particular firm's data on past projects may not contain estimates of their built-up values over time or the definition of built-up value in the data may not match that of *z*(*x*) in our model. With the right data, a nonlinear regression procedure can be used to test which of the alternative functional forms discussed earlier fits the data best. Here, again, the AIC statistics can serve as the benchmark for selecting the appropriate functional form, and the pooling of data on projects that are similar in nature is likely to produce more reliable results. In the absence of objective data, the firm can combine two subjective estimation methods to control estimation errors. One method is to ask experienced managers and engineers to judge the values of *z*(*x*) in various development stages of the project and use the results from this exercise to estimate the form of *z*(*x*). The second method, which should be applied after the first exercise is complete, is to show the plots of alternative functions to the judges and ask them to rate which one is most likely to characterize the project. The final estimation can be made using systematic procedures

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for combining expert opinions elicited under multiple methods [2]. If the judges identify a range of values to be likely for a given parameter, one may solve the problem using different values for the parameter to see how sensitive the optimal policy is to alternative assumptions.

VII. Concluding Remarks

The model developed in this paper offers a method for assessing development projects that require an uncertain time to build and can still provide some payoff if terminated without achieving the originally envisioned performance objectives. By allowing the possibility of getting a partial payoff in the case of termination without completion, we examined the impact that such a terminal payoff has on the optimal policy for controlling uncertain development projects. The analytical and numerical results we obtained indicate that the optimal policy is rather sensitive to how the terminal payoff evolves in a project's development process, pointing to the importance of carefully accounting for its impact in determining the control policy for the project.

In order to facilitate the application of the model to the management of specific projects, we also suggested a framework for categorizing the ways that the extent of uncertainty about a project evolves and the ways that its realizable value is built up. Based on this framework, we identified a number of functional forms that can reasonably characterize a project's uncertainty and value buildup process, and proposed methods for estimating the forms and parameters of those functions for a given project. As the model can be solved under the alternative functional forms identified, it seems to possesses considerable flexibility for accommodating variations in the features of the project.

As discussed in the previous section, the model will not be able to give a single correct decision rule if there exists significant uncertainty about the form of any component function or the value of any parameter used in the model. But sensitivity analysis based on the model should still provide more reliable reference points for decision making than sensitivity analysis based on such traditional models as the static NPV or IRR. The limitation of our knowledge about what functional forms are appropriate for what types of projects, however, does point to the need for further empirical investigation of these questions. The statistical methods suggested in our paper may also be employed in such empirical studies.

APPENDIX

Here we derive the expressions for the mean and the variance of the estimated time to completion when X_t is assumed to follow a diffusion process specified in (1) and (2), which we restate as

$$dX_t = -dt + \sigma\rho(X_t)dW_t, \tag{A.1}$$

$$\rho(X_t) = \sqrt{1-e^{-\lambda X_t}}\ . \tag{A.2}$$

The mean of $x = X_0$ by definition can be expressed as

$$M(x) = E_x\left[\int_0^\tau dt\right], \tag{A.3}$$

where τ, as defined earlier in section III, is the first passage time for $X_t = 0$. Based on Karlin and Taylor [8: 191-219], this functional must satisfy the following Kolmogorov backward equation.

$$\frac{1}{2}\sigma^2\left(1-e^{-\lambda x}\right)M_{xx} - M_x + 1 = 0 \tag{A.4}$$

Two boundary conditions are needed to obtain an exact solution of this differential equation. We require $M(0) = 0$ and $M'(\infty) < \infty$, i.e., the first derivative of the expectation with respect to x is finite. A general solution of (A.4) has the following form.[10]

$$M(x) = c_2 + x + c_1\int e^{\frac{2}{\sigma^2}x}\left(1-e^{-\lambda x}\right)^{\frac{2}{\sigma^2\lambda}}dx \tag{A.5}$$

The boundary condition $M(0) = 0$ requires $c_2 = 0$. Let $H(x) = c_1\int e^{\frac{2}{\sigma^2}x}\left(1-e^{-\lambda x}\right)^{\frac{2}{\sigma^2\lambda}}dx$. Because $H'(\infty) = \infty$, we must have $c_1 = 0$ in order to satisfy the second boundary condition $M'(\infty) < \infty$. Hence, the mean of x is just $M(x) = x$.

The variance of x by definition can be expressed as

$$V(x) = E_x\left[\left(\int_0^\tau dt\right)^2\right] - x^2 \tag{A.6}$$

[10] Note that (A.4) is very similar to the differential equation given in (6a) for $\mu = 0$. So we can make use of the results obtained in section IV to solve (A.4). As can be easily seen, the solution given in (A.5) is similar to (11), which is the solution of (6a) for $\mu = 0$.

Let $U(x) = E_x\left[\left(\int_0^\tau dt\right)^2\right]$. Again based on Karlin and Taylor [8: 191-219], this functional must satisfy the following Kolmogorov backward equation.

$$\frac{1}{2}\sigma^2\left(1-e^{-\lambda x}\right)U_{xx} - U_x + 2x = 0 \tag{A.7}$$

Since the infinitesimal standard deviation $\sigma\rho(x)$ remains finite as x approaches infinity, we can derive an exact solution of (A.7) again by making use of the boundary conditions $U(0) = 0$ and $U'(\infty) < \infty$. Following standard procedure, we obtain

$$U'(x) = c_1(x)e^{G(x)},$$

where $G(x) = x + \frac{1}{\lambda}\ln\left(1-e^{-\lambda x}\right)$ and $c_1(x) = c_0 - \int \frac{2x}{\sigma^2(1-e^{-\lambda x})e^{G(x)}}dx$ with c_0 being a constant.

Note that $G(0) = -\infty$ and $G(\infty) = \infty$. Let $f(x) = 2x$ and $g(x) = e^{-G(x)}$, so that $f'(x) = 2$ and $g'(x) = \frac{-e^{-G(x)}}{\sigma^2(1-e^{-\lambda x})}$ since $G'(x) = \frac{2}{\sigma^2(1-e^{-\lambda x})}$. Then, we can integrate $c_1(x)$ by parts and get

$$c_1(x) = 2xe^{-G(x)} + c_0 - \int 2e^{-G(x)}dx,$$

$$U'(x) = 2x + \left[c_0 - \int 2e^{-G(x)}dx\right]e^{G(x)},$$

$$U(x) = x^2 + \int\left[c_0 - \int^x 2e^{-G(y)}dy\right]e^{G(x)}dx + c_2.$$

The boundary condition $U(0) = 0$ requires $c_2 = 0$. Hence, by the definition of variance given in (A.6), we have

$$V(x) = \int\left[c_0 - \int^x 2e^{-G(y)}dy\right]e^{G(x)}dx \tag{A.8}$$

Differentiating (A.8) with respect to x, we get

$$V'(x) = c_0 e^{G(x)} + K(x)e^{G(x)}$$

where $K(x) = -\int 2e^{-G(x)}dx$. It can be verified that

$$\lim_{x\to\infty} K(x)e^{G(x)} = \lim_{x\to\infty}\frac{dK(x)/dx}{de^{-G(x)}/dx} = \lim_{x\to\infty}\sigma^2(1-e^{-\lambda x}) < \infty. \tag{A.9}$$

Because $e^{G(\infty)} = \infty$, we must have $c_0 = 0$ in order to satisfy the boundary condition $U'(\infty) < \infty$. Thus, we can write

$$V'(x) = K(x)e^{G(x)}.$$

It can also be verified that

$$\lim_{x\to 0} K(x)e^{G(x)} = \lim_{x\to 0}\frac{dK(x)/dx}{de^{-G(x)}/dx} = \lim_{x\to 0}\sigma^2(1-e^{-\lambda x}). \quad \text{(A.10)}$$

So, for x close to zero, we have

$$V'(x) = \sigma^2(1-e^{-\lambda x}) + o(x) \quad \text{(A.11)}$$

where $o(x)$ represents an infinitesimal error term. Then, for x close to zero, the Taylor expansion of $V(x)$ to the first order can be written as follows

$$V(x) = V(0) + \sigma^2(1-e^{-\lambda x})x + o(x). \quad \text{(A.12)}$$

We know $V(0) = 0$; so dropping the higher order terms in (A.12), the variance of x can be approximated by its first-order Taylor expansion

$$V(x) = \sigma^2(1-e^{-\lambda x})x. \quad \text{(A.13)}$$

It is interesting to note from (A.9) and (A.10) that the approximation (A.13) is more accurate as x becomes either very small or very large. Numerical tests confirm that the approximation given in (A.13) satisfies the differential equation (A.7) when x is small or large.

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Biographic Information about the Authors

Tailan Chi received his B.E. degree from the University of International Business and Economics, Beijing, China, his M.B.A. degree from University of San Francisco, and his M.S. degree in economics and Ph.D. degree in business administration from the University of Washington. Prior to coming to the United States for graduate study, he worked for four years in a trading company in China.

He is currently an assistant professor at the School of Business Administration, University of Wisconsin-Milwaukee. His research interests are in the area of organization and decision economics. His publications have appeared in *Management Science*, *Strategic Management Journal*, *Decision Sciences*, and *Omega—The International Journal of Management Science*. He is a member of the Academy of Management, Academy of International Business, American Economic Association, and Strategic Management Society.

John Liu received his M.S. degree in Engineering-Economic Systems from Stanford University and Ph.D. degree in Industrial Engineering from the Pennsylvania State University. He is currently an associate professor at the School of Business Administration, University of Wisconsin-Milwaukee.

His primary research interests are stochastic optimization models, manufacturing systems, and operations management. His current research focuses on manufacturing technology cycles, technology-based concurrent engineering, and R&D management. His publications have appeared in *Operations Research*, *Management Science*, *the European Journal of Operational Research*, and *the Journal of Production and Operations Management*. He is a member of INFORMS and IIE.

Hong Chen has been an associate professor at the Faculty of Commerce and Business Administration of the University of British Columbia since 1993. He received his Ph.D. degree in Engineering-Economic Systems from Stanford University (1987), and was Postdoctoral Fellow at Harvard University (1987/88).

He is a UBC Faculty Killam Research Fellow (1993/94), and a recipient of the Commerce Research Excellence Award from the Faculty of Commerce and Business Administration (1993). He received the Engineering Initiation Award from the National Science Foundation (USA) (1989-1991).

His research interests are primarily in the analysis, design and control of stochastic systems, such as manufacturing systems, communication networks, service processing networks, and economic systems. He is an author/co-author of over 25 refereed publications, including *IEEE Transactions on Automatic Control*, *Annals of Probability*, *Mathematics of Operations Research*, *Operations Research*, *Annals of Applied Probability*, and *Queuing Systems*. He is a principal investigator of many grants from federal sources. He was the chair for the INFORMS (Institute for Operations Research and Management Sciences) Section on Applied Probability (1995/96). He is an associate editor for Operations Research.