Normal form approach for dispersive equations with low-regularity data

By

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Doctor of Philosophy

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Abstract

In this dissertation, we examine applications of the normal form technique to non-linear dispersive equations with *rough* initial data. Working within the framework of Bourgain spaces, the normal form method often produces ample smoothing effects on the non-linearity. The extra gain in regularity is ideal for analysing solutions with low-regularity initial data, thus this approach can be used to overcome difficulties due to lack of smoothness in polynomial-type non-linearities. In particular, we will consider three canonical models in dispersive equations with quadratic and derivative quadratic non-linearities.

Informally, the normal form method starts with a preconditioning of the equation first (via a change of variables - normal form). The particular type of the normal form is a bilinear pseudo-differential operator, which solves explicitly the corresponding linear equation with the right-hand side consisting of the most troublesome terms in the original non-linearity. Then one needs to argue that the remaining component of the solution are better behaved, i.e. it gains *regularity* over the initial data. As a by-product of this approach, we can often obtain smoothing estimates on the non-linearity without much extra effort.

This technique was first introduced by Shatah [43] in the study of Klein-Gordon equation with a quadratic non-linearity. Recently this concept has been reformulated by the authors Germain, Masmoudi, Shatah, [21, 22] as an algorithm, referred as the *space-*

time resonance method, to obtain (mostly bilinear) dispersive estimates in a *small-data* regime. Furthermore, an expository article on this method was published by Shatah, [44]. A notable number of preprints appeared in the past year by Bernicot, Germain, [7]; Germain, [19]; Germain, Masmoudi, [20]; and Pusateri, Shatah, [42].

It should be noted, however, that space-time resonance often requires the initial data to be small and/or smooth, while the models in this dissertation concern non-smooth initial data. Thus, majority of these articles are concerned with the analysis with smooth initial data. In the *low-regularity* regime, the problem of *resonance* becomes a more delicate issue. This is perhaps the most interesting part of this investigation.

Apart from the space-time resonance method, the normal form transformation was also adapted by Takaoka, Tsutsumi, [47] and Nakanishi, Takaoka, Tsutsumi, [38], where the authors extend the well-posedness of modified Korteweg-de Vries equation to $H^{3/8+}$ and to $H^{1/3+}$ respectively. Furthermore, Babin, Ilyin, Titi, [1] also used this method to show the *unconditional well-posedness* of periodic Korteweg-de Vries equation in L^2. Here, the normal form transformation is referred as *differentiation by parts* in order to distinguish from the *space-time resonance* method. Erdogan and Tzirakis, [15] adapted the differentiation by parts computations to show a non-linear smoothing effect for the same equation.

In Chapter 2, we consider 1-D quadratic Schrödinger equation $u_t + iu_{xx} = \langle \nabla \rangle^{\beta} [u^2]$ with $\beta \in (0, 1/2)$, based on the result obtained in [41]. We establish local well-posedness in $H^{\beta-1+}$ (if $\beta = 0$, this matches, up to an endpoint, the sharp result of Bejenaru-Tao, [5]). Our approach differs significantly from the previous one - we use normal form transformation to analyze the worst interacting terms in the non-linearity and then show that the remaining terms are (much) smoother. In particular, this allows us to conclude that $u - e^{-it\partial_x^2} u(0) \in H^{-\frac{1}{2}}(\mathbf{R}^1)$, even though $u(0) \in H^{\beta-1+}$.

In addition, as a by-product of our normal form analysis, we obtain a Lipschitz continuity property in $H^{-\frac{1}{2}}$ of the solution operator (which originally acts on $H^{\beta-1+}$), which is new even in the case $\beta = 0$. As an easy corollary, we obtain local well-posedness results for $u_t + iu_{xx} = \langle\nabla\rangle^{\beta} z \langle\nabla\rangle^{\beta} z$. Also, we sketch an approach to obtain similar results for the equations $u_t + iu_{xx} = \langle\nabla\rangle^{\beta}[u\bar{u}]$ and $u_t + iu_{xx} = \langle\nabla\rangle^{\beta}[\bar{u}^2]$.

In Chapter 3, we consider the Korteweg-de Vries (KdV) equation $u_t + u_{xxx} = 6uu_x$ with periodic boundary condition, based on the result obtained in [39]. We prove a smoothing phenomena for low-regularity solutions by means of normal form transformation. As a corollary, the solution map from a ball on $H^{-\frac{1}{2}+}$ to $C_t^0([0,T], H^{-\frac{1}{2}+})$ can be shown to be Lipschitz in a H_x^{0+} topology, where the Lipschitz constant only depends on the rough norm $\|u_0\|_{H^{-\frac{1}{2}+}}$.

In Chapter 4, we consider the periodic "good" Boussinesq equation $u_{tt} + u_{xxxx} - u_{xx} + (u^2)_{xx} = 0$, based on the result obtained in [40]. We prove that the "good" Boussinesq model is locally well-posed in the space $H^{-\alpha} \times H^{-\alpha-2}$, $\alpha < \frac{3}{8}$ by means of normal forms, which allows us to explicitly extract the rougher part of the solution, while we show that the remainder is in the smoother space $C([0,T], H^{\beta}(\mathbf{T}))$, $\beta < \min(1-3\alpha, \frac{1}{2}-\alpha)$. This produces as a corollary the smoothing effect on the nonlinearity of order $\min(1-2\alpha, \frac{1}{2})$ in the case of mean-zero initial data. This is new even in the previously considered cases $\alpha \in (0, \frac{1}{4})$. After this work has been submitted for publication, N. Kishimoto [34] improved this result to a sharp index $\alpha \leq \frac{1}{2}$ by adapting the approach used in [25].

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To Saul Stahl, who showed me the beauty of mathematics

Contents

Chapter 1

Preliminaries

1.1 Notations

We adopt standard notations in approximate inequalities as follows: By $A \lesssim B$, we mean that there exists an absolute constant $C > 0$ with $|A| \le C|B|$. $A \ll B$ means that the implicit constant C is taken to be a *sufficiently* small positive number. For any number of quantities $\alpha_1, \ldots, \alpha_k$, $A \lesssim_{\alpha_1,\ldots,\alpha_k} B$ means that the implicit constant depends on $\alpha_1, \ldots, \alpha_k$. Finally, $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

For any $c \in \mathbf{R}$, $c+$ refers to the quantity $c + \delta$ for some $0 < \delta \ll 1$. We will use this notation only when δ does not affect the outcome.

We also introduce the Japanese bracket $\langle \cdot \rangle := (1 + |\cdot|^2)^{1/2}$. In particular, this is used to define weights on Sobolev spaces. For any $\xi \in \mathbf{R}$, note that $\langle \xi \rangle - |\xi| = O(1/|\xi|)$ as $|\xi| \to \infty$. Thus for large $|\xi| > 0$, we will consider $\langle \xi \rangle \sim \xi^2$. For small $|\xi|$, it is often easier to consider $\langle \xi \rangle \sim 1 + |\xi|$.

1.2 Fourier transform and Littlewood-Paley decomposition

Let $\mathbf{X}$ be an Euclidean space, $x \in \mathbf{X}$ such that $x = (x_1, \cdots, x_n)$. For any $k \in \mathbf{N}_0$, we define $C^k(\mathbf{X})$ to be the family of functions whose partial derivatives upto k^{th} order are continuous. We say a function is *smooth* if $f \in C^\infty(\mathbf{X}) := \cap_{k\in\mathbf{N}} C^k(\mathbf{X})$. If $\mathbf{X} = \mathbf{R}^n$, then we define Schwartz class of functions (denoted $\mathscr{S}(\mathbf{R}^n)$) to be

$$\mathscr{S}(\mathbf{R}^n) := \{f \in C^\infty(\mathbf{R}^n) : \sup_{x\in\mathbf{R}^n} |x|^{k_0} \partial_1^{k_1} \cdots \partial_n^{k_n} |f|(x) < \infty \quad \forall (k_0, \cdots, k_n) \in (\mathbf{N}_0)^{n+1}\}.$$

We fix φ to be a smooth time cut-off function which is supported on $[-2,2]$ and equals 1 on $[-1,1]$. In most cases, this can be replaced by an arbitrary Schwartz function $\varphi \in \mathscr{S}_t(\mathbf{R})$. Thus this notation will be relaxed, when the exact expression of φ does not influence the outcome. For any function space $\mathscr{Y}$, we denote the norm $\mathscr{Y}_T$ by the expression $\|u\|_{\mathscr{Y}_T} = \|\varphi(t/T)u\|_{\mathscr{Y}}$.

Let $\mathbf{Y} \in \{\mathbf{T}, \mathbf{R}\}$, where $\mathbf{T} = \mathbf{R}/2\pi$. Then for $f \in \mathscr{S}(\mathbf{Y})$ (where we denote $\mathscr{S}(\mathbf{T}) := C^\infty(\mathbf{T})$) and $u \in \mathscr{S}(\mathbf{R} \times \mathbf{Y})$, spatial and space-time Fourier transforms are defined as follows:

$$\widehat{f}(\xi) = \int_{\mathbf{Y}} f(x) e^{-ix\xi}\, dx, \qquad \xi \in \mathbf{Y}'$$
$$\widetilde{u}(\tau, \xi) = \int_{\mathbf{Y}\times\mathbf{R}} u(t,x) e^{-i(x\xi + t\tau)}\, dx\, dt \qquad (\tau, \xi) \in (\mathbf{R}, \mathbf{Y}')$$

where $\mathbf{Y}' = \mathbf{R}$ if $Y = \mathbf{R}$ and $\mathbf{Y}' = \mathbf{Z}$ if $Y = \mathbf{T}$. Also $g := g(\xi)$ or $v := v(\tau, \xi)$, we define the Fourier inverse operator for fast decaying functions as follows:

$$\mathscr{F}_\xi^{-1}[g](x) = \frac{1}{2\pi} \int_{\mathbf{Y}'} g(\xi) e^{ix\xi}\, d\xi, \qquad x \in \mathbf{Y}$$

$$\mathscr{F}^{-1}_{\tau,\xi}[v](t,x) = \frac{1}{4\pi^2}\int_{\mathbf{Y}'\times\mathbf{R}} v(\tau,\xi)e^{i(x\xi+t\tau)}\,d\xi\,d\tau \qquad (t,x)\in(\mathbf{R},\mathbf{Y}).$$

Given a *reasonable* function p of one variable, we define a differential operator $p(\partial_x)$ (sometimes written as $p(\nabla)$ instead) as an operator on $\mathscr{S}(\mathbf{Y})$, defined by $p(\partial_x)[u] = p(\nabla)[u] = \mathscr{F}^{-1}_{\xi}[p(i\xi)\widehat{u}]$.

For the case $\mathbf{Y}=\mathbf{R}$, we define the Littlewood-Paley decomposition. Let $\Phi:\mathbf{R}\to\mathbf{R}$ be a positive, smooth even function supported in $\{\xi : |\xi|\le 2\}$, and $\Phi(\xi)=1$ for all $|\xi|\le 1$. Define $\varphi(\xi)=\Phi(\xi)-\Phi(2\xi)$, which is supported in the annulus $1/2\le|\xi|\le 2$. This forms a partition of unity in the following two manners

$$\sum_{k\in\mathbf{Z}}\varphi(2^{-k}\xi)=1 \qquad \text{for } \xi\in\mathbf{R}\setminus\{0\}$$

$$\Phi(\xi)+\sum_{k\in\mathbf{Z}^+}\varphi(2^{-k}\xi)=1 \qquad \text{for } \xi\in\mathbf{R}.$$

The k^{th} Littlewood-Paley operator is defined via $\widehat{P_kf}(\xi)=\varphi(2^{-k}\xi)\widehat{f}(\xi)$ for $k\in\mathbf{Z}^+$. Similarly, $\widehat{P_{\le 0}f}(\xi)=\Phi(\xi)\widehat{f}(\xi)$. For a relation such as $\sim$, $\lesssim$, $\ll$, we define $P_{\sim k}f=\sum_{j:2^j\sim 2^k}P_jf$, etc. The notation f_k often will be used in place of P_kf, and $f_{\sim k}$ for $P_{\sim k}f$.

The kernels of P_k and of $P_{\le 0}$ are uniformly integrable in the following sense:

$$\sup_{k\in\mathbf{Z}}\int_{\mathbf{R}}\left|\mathscr{F}^{-1}_{\xi}[\varphi(2^{-k}\xi)](x)\right|dx+\int_{\mathbf{R}}\left|\mathscr{F}^{-1}_{\xi}[\Phi(\xi)](x)\right|dx\lesssim\int_{\mathbf{R}}|\widehat{\Phi}(\xi)|\,dx\le C_\Phi.$$

Thus, Young's inequality gives $P_k, P_{\le 0}: L^p\to L^p$ for $1\le p\le\infty$ with the bound equivalent to C_Φ (independent of k).

The same properties hold when φ or Φ above are replaced by general smooth functions $\psi\in\mathscr{S}(\mathbf{R})$. Therefore, when there is no possible confusion, the notation P_kf will be loosely translated to be $\widehat{P_kf}(\xi)=\psi_k(\xi)\widehat{f}(\xi)$, where $\psi_k\in\mathscr{S}(\mathbf{R})$ and $\psi_k(\xi)$ is supported on the annulus $2^{k-3}\le|\xi|\le 2^{k+3}$.

Next, we introduce a basic decomposition from the theory of the paraproducts. Given two functions $f,g \in \mathscr{S}(\mathbf{R})$ and $k \in \mathbf{Z}$,

$$P_k(fg) = P_k\left(\sum_{l,m} f_l g_m\right) = P_k\left(\sum_{|l-m|\leq 3} f_l g_m\right) + P_k\left(\sum_{|l-m|>3} f_l g_m\right).$$

Furthermore, in the first sum, we have the restriction $\min(l,m) > k-5$; and in the second sum, we have $|\max(l,m)-k| \leq 3$. Otherwise the supp $\widehat{f_l g_m}$ will be away from $\{\xi : |\xi| \sim 2^k\}$, and thus $P_k(f_l g_m) = 0$. We refer to the first summand as "high-high interaction" terms, and the second summand as "high-low interaction" terms.

1.3 Function spaces

Let $\mathbf{X}$ be an Euclidean space. For $1 \leq p < \infty$, the classical Lebesgue spaces are defined to be the spaces of functions f for which

$$\|f\|_{L^p(\mathbf{X})} = \left(\int_{\mathbf{X}} |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

If $p = \infty$, we define $\|f\|_{L^\infty(\mathbf{X})} = \operatorname{ess\,sup}_{x\in\mathbf{X}} |f(x)|$.

Let $\mathscr{Y}$ be a functional space and $\mathscr{Z}$ be a normed space. Then the mixed space $\mathscr{Y}_y\mathscr{Z}_z$ represents the space of functions $u := u(y,z)$ satisfying

1. For almost every y, $u(y,\cdot) \in \mathscr{Z}$.

2. $\|u(y,\cdot)\|_{\mathscr{Z}} \in \mathscr{Y}$.

3. Furthermore, if $\mathscr{Y}$ is a normed space, then the space is characterized by the norm $\|u\|_{\mathscr{Y}_y\mathscr{Z}_z} := \|\|u\|_{\mathscr{Z}_z}\|_{\mathscr{Y}_y}$.

In particular, $\mathscr{Z}_{t,x} := \mathscr{Z}_t \mathscr{Z}_x$ will be used when the ordering of the respective norms do not affect the outcome.

We introduce Sobolev-type norms as weighted L^2 norms on the Fourier transform of the given function. For any $s \in \mathbf{R}$, we define $H^s(\mathbf{Y})$ spaces to be the completion of $\mathscr{S}(\mathbf{Y})$ with respect to the norm

$$\|u\|_{H^s(\mathbf{Y})} := \left(\int_{\mathbf{Y}'} \langle \xi \rangle^{2s} |\widehat{f}|^2(\xi)\, d\xi \right)^{\frac{1}{2}}.$$

For $s \geq 0$, this forms a classical space of functions with s derivatives in L^2. However, for $s < 0$, this is a space of distributions. More specifically, this is the space of bounded linear functionals (or distributions) on H^{-s}. Note that for any $s < 0$, the operator $\langle \nabla \rangle^s : H^s \to L^2$ defined so that $\forall u \in H^s \cap \mathscr{S}(\mathbf{Y})$, $\langle \nabla \rangle^s(u) := \mathscr{F}_\xi^{-1}[\langle \xi \rangle^s \widehat{u}(\xi)]$ is an isometry. Therefore when working in the Sobolev spaces with negative indices, we will change variable to $v = \langle \nabla \rangle^s u \in L^2$ in place of $u \in H^s$ to avoid confusion.

Next, we define Bourgain spaces, which are also characterized by Sobolev-type norms. For $s, b \in \mathbf{R}$ and a real-valued function h of one variable, we define the (inhomogeneous) $X^{s,b}_{\tau=h(\xi)}$ spaces to be the completion of $\mathscr{S}(\mathbf{R} \times \mathbf{Y})$ with respect to the norm

$$\|u\|_{X^{s,b}_{\tau=h(\xi)}} = \left(\int_{\mathbf{R}\times\mathbf{Y}'} \langle \xi \rangle^{2s} \langle \tau - h(\xi) \rangle^{2b} |\widetilde{u}|^2(\tau, \xi)\, d\tau\, d\xi \right)^{\frac{1}{2}}. \tag{1.1}$$

From the definition above, we note the conjugation relation $\|u\|_{X^{s,b}_{\tau=h(\xi)}} = \|\overline{u}\|_{X^{s,b}_{\tau=-h(-\xi)}}$ and the duality relation $\left(X^{s,b}_{\tau=h(\xi)}\right)^* = X^{-s,-b}_{\tau=-h(-\xi)}$ given by

$$\|u\|_{X^{s,b}_{\tau=h(\xi)}} = \sup_{\|v\|_{X^{-s,-b}_{\tau=-h(-\xi)}}=1} \left| \int_{\mathbf{R}\times\mathbf{Y}} u(t,x) v(t,x)\, dt\, dx \right|. \tag{1.2}$$

In the definition (1.1), the *dispersion relation* $h(\xi)$ depends explicitly on the linear part of the given equation, thus the implied $X^{s,b}$ spaces will vary depending on the problem. But often we will denote $X^{s,b} = X^{s,b}_{\tau=h(\xi)}$ when the choice of h is clear from the context.

Consider a linear evolution equation $[\partial_t + p(\partial_x)]u(t,x) = 0$, where $p = p(x)$ is a function. Formally taking the Fourier transform of this equation gives $(\tau - ip(i\xi))\widetilde{u}(\tau,\xi) = 0$, which implies that $\widetilde{u}$ is supported on the curve $\tau = ip(i\xi)$. In this case, we set $h(\xi) = ip(i\xi)$, which is real-valued in all our examples. For instance, the Schrödinger equation $u_t + iu_{xx} = 0$ will give $h(\xi) = \xi^2$, whereas the Airy equation $u_t + u_{xxx} = 0$ will give $h(\xi) = \xi^3$.

From the previous heuristics, it is easy to see that the weight in $\tau - h(\xi)$ is *invisible* to solutions of the respective linear equation. This is the primary reason that this space has proven to be a powerful tool for the perturbation approach in nonlinear PDE's, where one treats a nonlinear equation as a perturbed linear one to establish local existence and uniqueness of solutions. Propositions 1 through 5 serve as a rigorous groundwork for these heuristics.

The next Proposition is from [49, Lemma 2.8].

Proposition 1. *Let* $\varphi \in \mathscr{S}_t(\mathbf{Y})$. *Then for* $f \in \mathscr{S}_x(\mathbf{Y})$,

$$\|\varphi(t)e^{ith(\nabla/i)}f\|_{X^{s,b}_{\tau=h(\xi)}} = \|\varphi\|_{H^b(\mathbf{R})}\|f\|_{H^s(\mathbf{Y})}.$$

Proof. Taking the space-time Fourier transform,

$$[\varphi(t)\widetilde{e^{ith(\nabla/i)}}f](\tau,\xi) = \widehat{f}(\xi)\int_{\mathbf{R}} \varphi(t)e^{-it(\tau-h(\xi))}\,dt = \widehat{f}(\xi)\widehat{\varphi}(\tau-h(\xi)).$$

$$
\begin{aligned}
\|\varphi(t)e^{ith(\nabla/i)}f\|^2_{X^{s,b}_{\tau=h(\xi)}} &= \int_{\mathbf{R}}\int_{\mathbf{Y}} \langle\xi\rangle^{2s}\langle\tau-h(\xi)\rangle^{2b}|\widehat{f}|^2(\xi)|\widehat{\varphi}|^2(\tau-h(\xi))\,d\xi\,d\tau \\
&= \int_{\mathbf{R}}\langle\tau\rangle^{2b}|\widehat{\varphi}|^2(\tau)\,d\tau\int_{\mathbf{Y}}\langle\xi\rangle^{2s}|\widehat{f}|^2(\xi)\,d\xi \\
&= \|\varphi\|^2_{H^b(\mathbf{R})}\|f\|^2_{H^s(\mathbf{Y})}
\end{aligned}
$$

□

The next two Propositions are from [49, Lemma 2.11]. I thank Terrence Tao for the correction of his original proof.

Proposition 2. *Let $\varphi \in \mathscr{S}_t(\mathbf{Y})$ and $u = u(t,x) \in \mathscr{S}_{t,x}(\mathbf{R}\times\mathbf{Y})$. Then for $s,b \in br$,*

$$
\|\varphi(t)u\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_b \left\|\langle\tau\rangle^{|b|}\widehat{\varphi}(\tau)\right\|_{L^1(\mathbf{R})}\|u\|_{X^{s,b}}.
$$

Proof. Taking the Fourier transform, $[\widetilde{\varphi(t)u}](\tau,\xi) = [\widehat{\varphi}*_\tau\widetilde{u}(\cdot,\xi)](\tau)$.

We use $\langle\tau-h(\xi)\rangle^b \lesssim_b \langle\tau-\sigma\rangle^{|b|}\langle\sigma-h(\xi)\rangle^b$ for all $b,\tau,\xi,\sigma\in\mathbf{R}$. Then by Young's inequality,

$$
\begin{aligned}
\|\varphi(t)u\|_{X^{s,b}_{\tau=h(\xi)}} &= \|\langle\xi\rangle^s\langle\tau-h(\xi)\rangle^b\,[\widehat{\varphi}*_\tau\widetilde{u}(\cdot,\xi)]\,(\tau)\|_{L^2_{\tau,\xi}} \\
&\lesssim_b \left\|\int_{\mathbf{R}}\langle\tau-\sigma\rangle^{|b|}|\widehat{\varphi}|(\tau-\sigma)\langle\xi\rangle^s\langle\sigma-h(\xi)\rangle^b|\widetilde{u}|(\sigma,\xi)\right\|_{L^2_{\tau,\xi}} \\
&\lesssim \left\|\langle\tau\rangle^{|b|}\widehat{\varphi}\right\|_{L^1_\tau}\|u\|_{X^{s,b}_{\tau=h(\xi)}}.
\end{aligned}
$$

□

Proposition 3. *Let φ be a smooth* time-cutoff *function such that $\varphi = 1$ on $[-1,1]$ and supported on $[-2,2]$. Then for $u = u(t,x)\in\mathscr{S}_{t,x}(\mathbf{R}\times\mathbf{Y})$ and $-1/2 < b' \le b < 1/2$,*

$$
\|\varphi(t/T)u\|_{X^{s,b'}_{\tau=h(\xi)}} \lesssim_{\varphi,b,b'} T^{b-b'}\|u\|_{X^{s,b}_{\tau=h(\xi)}}.
$$

Proof. By replacing $\langle\nabla\rangle^s u$ with u, we can assume without loss of generality that $s=0$. Also we claim that it suffices to show the claim for $0\le b'\le b<1/2$. Say that we have the claim for this range of b',b. If $-1/2<b'\le b\le 0$, then by duality,

$$\begin{aligned}\|\varphi(t/T)u\|_{X^{0,b'}_{\tau=h(\xi)}} &= \sup_{\|v\|_{X^{0,-b'}_{\tau=-h(-\xi)}}=1}\left|\int_{\mathbf{R}}\varphi(t/T)u\,v\,dt\,dx\right| \\ &\lesssim \sup_{\|v\|_{X^{0,-b'}_{\tau=-h(-\xi)}}=1}\|u\|_{X^{0,b}_{\tau=h(\xi)}}\|\varphi(t/T)v\|_{X^{0,-b}_{-\tau=h(-\xi)}}\lesssim_{\varphi,b,b'}T^{b-b'}\|u\|_{X^{0,b}}.\end{aligned}$$

If $-1/2<b'<0<b<1/2$, we can obtain the claim by setting $\varphi(t/T)=\varphi(t/T)\varphi(t/2T)$.

We write $u=u_1+u_2$, where the frequency of the respective component is supported on $\langle\tau-h(\xi)\rangle\ge 1/T$ for u_1, and $\langle\tau-h(\xi)\rangle\le 1/T$ for u_2. By triangular inequality it suffices to show $\left\|\varphi(t/T)u_j\right\|_{X^{0,b'}}\lesssim_\varphi T^{b-b'}\|u\|_{X^{s,b}}$ for $j=1,2$.

For u_1, we first observe the following algebraic inequality. If $|\sigma-h(\xi)|\ge 1/T$,

$$\langle\tau-h(\xi)\rangle^{b'}\lesssim_{b'}\langle T(\tau-\sigma)\rangle^{b'}\langle\sigma-h(\xi)\rangle^{b'}. \tag{1.3}$$

Note that $\tau-h(\xi)=(\tau-\sigma)+(\sigma-h(\xi))$. This means that the left side of (1.3) cannot be too large. So the only non-trivial case is $\tau-h(\xi)\sim\tau-\sigma$. Note that in this case, $|\tau-h(\xi)|\lesssim\frac{1}{T}\langle T(\tau-\sigma)\rangle\le\langle T(\tau-\sigma)\rangle\langle\sigma-h(\xi)\rangle$. So we have (1.3).

Using $\widehat{\varphi(\cdot/T)}(\tau)=T\widehat{\varphi}(T\tau)$ and Young's inequality,

$$\begin{aligned}\|\varphi(t/T)u_1\|_{X^{0,b'}_{\tau=h(\xi)}} &\lesssim_{b'} T^{1+b-b'}\left\|\int_{\mathbf{R}}\langle T(\tau-\sigma)\rangle^{b'}|\widehat{\varphi}|(T(\tau-\sigma))\langle\sigma-h(\xi)\rangle^{b}|\widetilde{u_1}|(\sigma,\xi)\,d\sigma\right\|_{L^2_{\tau,\xi}} \\ &\lesssim T^{1+b-b'}\int_{\mathbf{R}}\langle T\tau\rangle^{b'}|\widehat{\varphi}|(T\tau)\,d\tau\quad\|u_1\|_{X^{0,b}_{\tau=h(\xi)}} \\ &\le T^{b-b'}\int_{\mathbf{R}}\langle\tau\rangle^{b'}|\widehat{\varphi}|(\tau)\,d\tau\quad\|u\|_{X^{0,b}_{\tau=h(\xi)}}.\end{aligned}$$

For u_2, we introduce the operator $\mathbf{P}$ defined so that for any $v \in \mathscr{S}_{t,x}$, $\widetilde{\mathbf{P}v} = \chi_{[0,1/T]}(|\tau - h(\xi)|)\widetilde{v}(\tau,\xi)$. Then note $\|v\|_{X^{0,b'}_{\tau=h(\xi)}} \lesssim T^{-b'}\|v\|_{L^2_{t,x}}$ and hence

$$\begin{aligned}\|\mathbf{P}[\varphi(t/T)u_2]\|_{X^{0,b'}_{\tau=h(\xi)}} &\lesssim T^{-b'}\|u_2\|_{L^\infty_t L^2_x}\left(\int_{\mathbf{R}}|\varphi|^2(t/T)\,dt\right)^{\frac{1}{2}} \\ &= T^{\frac{1}{2}-b'}\|u\|_{L^\infty_t L^2_x}\|\varphi\|_{L^2_t}.\end{aligned}$$

By Minkowski inequality and Hölder's inequality,

$$\begin{aligned}\|u_2(t)\|_{L^2_x} &\lesssim \int_{|\tau-h(\xi)|\le\frac{1}{T}}\|\widetilde{u_2}(\tau)\|_{L^2_\xi}\,d\tau \\ &\lesssim \left(\int_{|\tau-h(\xi)|\le\frac{1}{T}}\langle\tau-h(\xi)\rangle^{-2b}\,d\tau\right)^{\frac{1}{2}}\|u_2\|_{X^{0,b}_{\tau=h(\xi)}} \lesssim_b T^{b-1/2}\|u\|_{X^{0,b}_{\tau=h(\xi)}}\end{aligned}$$

where the last inequality requires that $b < 1/2$. Combining these two estimates, we manage the term $\mathbf{P}[\varphi(t/T)u_2]$.

To deal with the other term, note that if $|\tau - h(\xi)| \ge 1/T$ and $|\sigma - h(\xi)| \le 1/T$,

$$\langle\tau-h(\xi)\rangle^{b'} \lesssim T^{1-b'}|\tau-h(\xi)| \le T^{1-b'}|\tau-\sigma| + T^{b-b'}\langle\sigma-h(\xi)\rangle^{b}.$$

Following similar computations as above, it is easy to see that the term involving $T^{b-b'}\langle\sigma-h(\xi)\rangle^b$ gives precisely the claim. So it now suffices to show

$$T^{1-b'}\left\|\widehat{\eta_T} *_\tau \widetilde{u_2}\right\|_{L^2_{\tau,\xi}} \lesssim_{\varphi,b,b'} T^{b-b'}\|u\|_{X^{0,b}_{\tau=h(\xi)}}$$

where $\widehat{\eta_T}(\tau) := |T\tau|\widehat{\varphi}(T\tau)$. Using Plancherel's identity, we estimate the norm on the left side of above.

$$\begin{aligned}\|\eta_T u_2\|_{L^2_{t,x}} &\lesssim \|\widehat{\eta_T}\|_{L^2_t}\|u_2\|_{L^\infty_t L^2_x} \\ &\lesssim T^{-\frac{1}{2}}\|\tau\widehat{\varphi}\|_{L^2_t} T^{b-\frac{1}{2}}\|u\|_{X^{0,b}_{\tau=h(\xi)}}\end{aligned}$$

where we have used the previously obtained bound on $\|u_2\|_{L^\infty_t L^2_x}$. Combining this with $T^{1-b'}$, we obtain the desired constant. □

The next Proposition is equivalent to [49, Proposition 2.12], but the proof presented here below takes a different approach.

Proposition 4. *Let φ be as in Proposition 3. Then for any $b > 1/2$,*

$$\left\| \varphi(t)\int_0^t e^{i(t-s)h(\nabla/i)}F(s)\,ds \right\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_{\varphi,b} \|F\|_{X^{s,b-1}_{\tau=h(\xi)}}.$$

Proof. It suffices show the claim for $s = 0$. First we write $F(s) = \frac{1}{2\pi}\int_{\mathbf{R}} \widehat{F}(\mu)e^{is\mu}\,d\mu$. Then taking the spatial Fourier transform of LHS of the given inequality, we obtain

$$\frac{e^{ith(\xi)}\varphi(t)}{2\pi}\int_{\mathbf{R}}\int_0^t e^{is(\mu-h(\xi))}\widetilde{F}(\mu,\xi)\,ds\,d\mu = \frac{e^{ith(\xi)}\varphi(t)}{2\pi}\int_{\mathbf{R}} \frac{e^{it(\mu-h(\xi))}-1}{i(\mu-h(\xi))}\widetilde{F}(\mu,\xi)\,d\mu.$$

Noting $\varphi(t) = \eta(t)\varphi(t)$ where $\eta(t) := \varphi(t/2)$, we take the Fourier transform in time and space variables

$$\begin{aligned}&\mathscr{F}_{t,x}\left[\varphi(t)\eta(t)\int_0^t e^{i(t-s)h(\nabla/i)}F(s)\,ds\right](\tau,\xi) \\ &\quad= \frac{1}{2\pi}\int_{\mathbf{R}}\widehat{\varphi}(\tau-\sigma-h(\xi))\int_{\mathbf{R}}\frac{\widehat{\eta}(\sigma-\mu+h(\xi))-\widehat{\eta}(\sigma)}{i(\mu-h(\xi))}\widetilde{F}(\mu,\xi)\,d\mu\,d\sigma.\end{aligned}$$

We use an algebraic inequality $\langle\tau-h(\xi)\rangle^b \lesssim_b \langle\tau-\sigma-h(\xi)\rangle^b\langle\sigma\rangle^b$ for $b\geq 0$. Then by Young's inequality,

$$\text{LHS} \lesssim \left\|\langle\cdot\rangle^b\widehat{\varphi}\right\|_{L^1_\sigma}\left\|\langle\sigma\rangle^b\int_{\mathbf{R}}\frac{\widehat{\eta}(\sigma-\mu+h(\xi))-\widehat{\eta}(\sigma)}{i(\mu-h(\xi))}\widetilde{F}(\mu,\xi)\,d\mu\right\|_{L^2_{\sigma,\xi}}$$

We decompose $F=F_1+F_2$ where $\widetilde{F_1}(\mu,\xi)$ is supported on $|\mu-\xi|\leq 1$, and $\widetilde{F_2}(\mu,\xi)$ is supported on $|\mu-\xi|\geq 1$. By triangular inequality, it suffices to show $\left\|T_b[F_j]\right\|_{L^2_{\sigma,\xi}}\lesssim_{b,\eta}\|F\|_{X^{0,b-1}_{\tau=h(\xi)}}$ for $j=1,2$ where

$$T_b[F_j](\sigma,\xi):=\langle\sigma\rangle^b\int_{\mathbf{R}}\frac{\widehat{\eta}(\sigma-\mu+h(\xi))-\widehat{\eta}(\sigma)}{i(\mu-h(\xi))}\widetilde{F}_j(\mu,\xi)\,d\mu.$$

For F_1, note $\|F_1\|_{L^2_{t,x}}\sim_b\|F_1\|_{X^{0,b}_{\tau=h(\xi)}}$ for any $b\in\mathbf{R}$. So it suffices to show $nT_b[F_1]L^2_{\sigma,\xi}\lesssim_{b,\eta}\|F_1\|_{L^2_{t,x}}$.

By the Mean-Value Theorem,

$$\frac{\widehat{\eta}(\sigma-\mu+h(\xi))-\widehat{\eta}(\sigma)}{\mu-h(\xi)}=\widehat{\eta}'(\sigma_0)\qquad\text{for some }\sigma_0\in[\sigma-1,\sigma+1].$$

Since $\eta\in\mathscr{S}(\mathbf{R})$ (thus $\widehat{\eta}\in\mathscr{S}(\mathbf{R})$, for any $N\in\mathbf{N}$, $\langle\sigma_0\rangle^N|\widehat{\eta}'|(\sigma_0)<C$ for some constant $C=C(N,\eta)$. Finally, noting $\langle\sigma_0\rangle^N\sim_N\langle\sigma\rangle^N$ (select $N\gg b$), we have

$$\begin{aligned}\|T_b[F_1]\|_{L^2_{\sigma,\xi}} &\lesssim \left\|\langle\sigma\rangle^b|\widehat{\eta}'|(\sigma_0)\int_{|\mu-h(\xi)|\leq 1}\widetilde{F_1}(\mu,\xi)\,d\mu\right\|_{L^2_{\sigma,\xi}}\\ &\lesssim_{N,\eta}\left\|\langle\sigma\rangle^{b-N}\right\|_{L^2_\sigma}\left\|\left(\int_{\mathbf{R}}|\widetilde{F_1}|^2(\mu,\xi)\,d\mu\right)^{\frac{1}{2}}\right\|_{L^2_\xi}\lesssim_{b,N}\|F_1\|_{L^2_{t,x}}\end{aligned}$$

For F_2, we use $\langle\sigma\rangle^b \lesssim_b \langle\sigma-\mu+h(\xi)\rangle^b\langle\mu-h(\xi)\rangle^b$. So if $|\mu-h(\xi)|\geq 1$,

$$\begin{aligned}\langle\sigma\rangle^b\left|\frac{\widehat{\eta}(\sigma-\mu+h(\xi))-\widehat{\eta}(\sigma)}{\mu-h(\xi)}\right| &\lesssim \frac{\langle\sigma\rangle^b|\widehat{\eta}|(\sigma-\mu+h(\xi))}{\langle\mu-h(\xi)\rangle^1}+\frac{\langle\sigma\rangle^b|\widehat{\eta}|(\sigma)}{\langle\mu-h(\xi)\rangle} \\ &\lesssim \frac{\langle\sigma-\mu+h(\xi)\rangle^b|\widehat{\eta}|(\sigma-\mu+h(\xi))}{\langle\mu-h(\xi)\rangle^{1-b}}+\frac{\langle\sigma\rangle^b|\widehat{\eta}|(\sigma)}{\langle\mu-h(\xi)\rangle}. \end{aligned} \tag{1.4}$$

For the component of $T[F_2]$ involving the first piece of the right side of (1.4), by Young's inequality

$$\begin{aligned}&\left\|\int_{\mathbf{R}}\frac{\langle\sigma-\mu+h(\xi)\rangle^b|\widehat{\eta}|(\sigma-\mu+h(\xi))}{\langle\mu-h(\xi)\rangle^{1-b}}\widetilde{F_2}(\mu,\xi)\,d\mu\right\|_{L^2_{\sigma,\xi}} \\ &\lesssim \left\|\langle\cdot\rangle^b\widehat{\eta}\right\|_{L^1_\mu}\left\|\langle\mu\rangle^{b-1}\widetilde{F_2}(\mu+h(\xi),\xi)\right\|_{L^2_{\mu,\xi}} \lesssim_{b,\eta} \|F\|_{X^{0,b-1}_{\tau=h(\xi)}}.\end{aligned}$$

Finally, the component of $T[F_2]$ with the second piece of (1.4) is

$$\begin{aligned}&\left\|\int_{\mathbf{R}}\frac{\langle\sigma\rangle^b|\widehat{\eta}|(\sigma)}{\langle\mu-h(\xi)\rangle}\widetilde{F_2}(\mu,\xi)\,d\mu\right\|_{L^2_{\sigma,\xi}} = \left\|\langle\cdot\rangle^b\widehat{\eta}\right\|_{L^2_\sigma}\left\|\langle\mu-h(\cdot)\rangle^{-1}\widehat{F_2}(\mu,\cdot)\,d\mu\right\|_{L^2_\xi} \\ &\lesssim_{\eta,b}\left(\int_{\mathbf{R}}\frac{1}{\langle\mu\rangle^{2b}}\,d\mu\right)^{\frac{1}{2}}\left\|\left(\int_{\mathbf{R}}\langle\mu-h(\xi)\rangle^{2b-2}|\widetilde{F_2}|^2(\mu,\xi)\,d\mu\right)^{\frac{1}{2}}\right\|_{L^2_\xi} \lesssim_b \|F\|_{X^{0,b-1}_{\tau=h(\xi)}}.\end{aligned}$$

□

The next Proposition is from [49, Lemma 2.9].

Proposition 5. *Let $\mathscr{X}$ be a normed space satisfying*

$$\sup_{\tau\in\mathbf{R}}\left\|e^{it\tau}e^{ith(\nabla/i)}f\right\|_{\mathscr{X}} \lesssim \|f\|_{H^s_x(\mathbf{Y})}$$

for all $f \in H^s$. Then for $b > 1/2$, we have the embedding

$$\|u\|_{\mathscr{X}} \lesssim_b \|u\|_{X^{s,b}_{\tau=h(\xi)}}.$$

Proof. By the inverse Fourier transform and the change of variable $\tau' = \tau - h(\xi)$,

$$u(t,x) = \frac{1}{4\pi^2} \int_{\mathbf{R}\times\mathbf{Y}} \widetilde{u}(\tau' + h(\xi), \xi) e^{it(\tau'+h(\xi))} e^{ix\xi}\, d\xi d\tau'.$$

Applying Minkowski's inequalty, the hypothesis on $\mathscr{X}$, Plancherel's theorem and Cauchy-Swartz inequality,

$$\begin{aligned}
\|u\|_{\mathscr{X}} &\lesssim \int_{\mathbf{R}} \left\| e^{it\tau'} e^{ith(\nabla/i)} \int_{\mathbf{Y}} \widetilde{u}(\tau' + h(\xi), \xi) e^{ix\xi}\, d\xi \right\|_{\mathscr{X}} d\tau' \\
&\lesssim \int_{\mathbf{R}} \left\| \langle \xi \rangle^s \widetilde{u}(\tau' + h(\xi), \xi) \right\|_{L^2_\xi} d\tau' \\
&\lesssim \left(\int_{br} \langle \tau' \rangle^{-2b}\, d\tau' \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}} \langle \tau' \rangle^{2b} \left\| \langle \cdot \rangle^s \widetilde{u}(\tau' + h(\cdot), \cdot) \right\|_{L^2_\xi} d\tau' \right)^{\frac{1}{2}} \lesssim_b \|u\|_{X^{s,b}_{\tau=h(\xi)}}.
\end{aligned}$$

□

Remark 1. *Note that the operator $e^{it\tau} e^{ith(\nabla/i)}$ is unitary as it maps $L^p_t H^s_x$ to H^s_x for any $1 \le p \le \infty$, as long as $h(\xi)$ is real-valued. In particular, we have the following useful embedding for $b > 1/2$*

$$\|u\|_{C^0_t H^s_x} \lesssim_b \|u\|_{X^{s,b}_{\tau=h(\xi)}}.$$

Furthermore, if we have a Strichartz inequality $\left\| e^{ith(\nabla/i)} f \right\|_{L^p_t L^q_x} \lesssim \|f\|_{L^2_x}$, then we also have

$$\|u\|_{L^p_t L^q_x} \lesssim_b \|u\|_{X^{0,b}_{\tau=h(\xi)}}$$

where $b > 1/2$.

1.4 On well-posedness

Roughly speaking, well-posedness of the initial-value problem for a partial differential equation depends on three factors:

- Existence of the solution;
- Uniqueness of the solution;
- Continuous dependence on the initial data.

Let $P(D)$ be a differential operator in x of order $k \in \mathbf{N}$. Given the initial value problem (IVP)

$$\left| \begin{array}{ll} \partial_t u + P(D)u = F; & (t,x) \in \mathbf{R} \times \mathbf{Y} \\ u(0,x) = u_0(x) & \end{array} \right. , \tag{1.5}$$

well-posedness of this problem can be understood in several different ways. In the classical setting, one requires a sufficient *smoothness* on the solution, such as $u \in C_t^1 C_x^k$ so that there is no problem interpreting the problem (1.5). These solutions are called *classical solutions* of (1.5). At times, one may need to require a further regularity and/or decay in order to justify the arguments for well-posedness. Under sufficient regularity and decay conditions, it is often easier to establish the well-posedness of (1.5).

But if the initial data is not *smooth*, we cannot expect the solution to satisfy the required regularity. Say that the initial data $u_0 \in \mathscr{X}$ for some normed space $\mathscr{X}$. In order to understand the IVP (1.5), we can reformulate the problem through Duhamel's Principle,

$$u(t) =_{\mathscr{X}} U(t)u_0 + \int_0^t U(t-s)F(s)\,ds \qquad \text{for } \forall t \in [0,T] \tag{1.6}$$

where $U(t)$ is the linear propagation map, i.e. $U(t)f$ solves the linear equation $\partial_t u + P(D)u = 0$. Note that the solution u of (1.6) may not be a differentiable function in the classical sense. (1.6) is called the *strong formulation* of (1.5), and a solution u of (1.6) is called a *strong solution* of (1.5). If there exists a unique $u \in C_t^0([0,T];\mathscr{X}_x)$ for every $f \in \mathscr{X}$, we can define the local-in-time solution map. If the local-in-time solution map is continuous with respect to $\mathscr{X}$ norm, we say that the original IVP (1.5) is locally *strongly* well-posed in $\mathscr{X}$. If T can be taken arbitrarily large, then we say that (1.5) is globally *strongly* well-posed in $\mathscr{X}$.

In a low-regularity setting, this task is particularly difficult. Take $\mathscr{X} = H^s$ for example. One often uses an auxiliary space $\mathscr{Y}$ and restrict the solution space to $C_t^0([0,T];H^s) \cap \mathscr{Y}$ to claim the uniqueness in the given space. Most commonly, the auxiliary spaces are Bourgain-type spaces $X^{s,b}$ which embed into $C_t^0([0,T];H^s)$. Using the Propositions from Section 1.3, one can often show that all classical solutions of (1.5) live in the $X^{s,b}$ space. Therefore the arguments involving Bourgain spaces produce the *minimal* notion of well-posedness as defined in [12]:

Definition 1. *If for every radius $R > 0$, there exists $T = T(R) > 0$ such that the solution map $\mathbf{S} : \mathscr{S}(\mathbf{Y}) \to C_t^\infty([0,T], \mathscr{S}_x)$ can be* uniformly continuously *and* uniquely *extended to a map $\mathbf{S}' : B_{H_x^s}(0,R) \to C_t^0([0,T];H_x^s)$ where $B_{H_x^s}(0,R) := \{u_0 \in H_x^s : \|u_0\|_{H^s} < R\}$, then we say that* (1.5) *is* locally well-posed *in H^s. If T can be made arbitrarily large, then we say* (1.5) *is* globally well-posed *in H^s.*

This is the notion of well-posedness adopted in this text. Note that it is not guaranteed that the solution will be unique in $C_t^0 H_x^s$, but the uniqueness as a limit of smooth solutions is obtained. If the uniqueness can be guaranteed in $C_t^0 H_x^s$, then we say that (1.5) is *unconditionally* well-posed.

We present here the general scheme of contraction arguments in $X^{s,b}_{\tau=h(\xi)}$ spaces which is used to prove local well-posedness statements. Let $P(D) = ih(\nabla/i)$. By taking spatial Fourier transform and solving an ODE problem, we can obtain that the linear solution operator is $U(t) = e^{ith(\nabla/i)}$. If φ is the smooth time-cutoff function as defined in Proposition 3, then the strong formulation (1.6) is equivalent to

$$u(t) = \varphi(t)e^{ith(\nabla/i)}u_0 + \varphi(t)\int_0^t e^{i(t-s)h(\nabla/i)}\varphi\left(\frac{s}{T}\right)F(s)\,ds \qquad \text{for } t \in [0,T] \quad (1.7)$$

when $0 < T \leq 1$. Define an operator Λ_T such that $[\Lambda_T u](t)$ is defined by the right side of (1.7) for $u \in \mathscr{S}_{t,x}$ and $t \in \mathbf{R}$. Once we show the continuity of Λ with respect to a certain $X^{s,b}$ norm, the definition can be extended to all of $X^{s,b}$ by density. Then finding the *strong* solution of (1.5) is equivalent to finding $u \in X^{s,b}_{\tau=h(\xi)}$ such that $\Lambda_T u = u$.

We recall that given a complete metric space $(\mathbf{X}, d)$ and an operator $T : \mathbf{X} \to \mathbf{X}$, T has a unique fixed point in $\mathbf{X}$ if T is a contraction map in $(\mathbf{X}, d)$. Than is, there exists an absolute constant $0 < C < 1$ so that $d(Tu, Tv) \leq Cd(u,v)$ for all $u, v \in \mathbf{X}$.

For our purpose, we select $\mathbf{X} = \{u \in X^{s,1/2+}_{\tau=h(\xi)} : \left\| u - \varphi(t)e^{ith(\nabla/i)}u_0 \right\|_{X^{s,1/2+}} \leq R/2\}$ equipped with the corresponding $X^{s,1/2+}$ norm, where $1/2+ = 1/2 + \delta$. We need to show two things:

1. Λ_T maps $\mathbf{X}$ to $\mathbf{X}$. That is, $\left\| \Lambda_T u(t) - \varphi(t)e^{ith(\nabla/i)}u_0 \right\|_{X^{s,1/2+}} \leq R/2$ for $\forall u \in \mathbf{X}$.

2. $\|\Lambda_T u - \Lambda_T v\|_{X^{s,1/2+}} \leq C\|u - v\|_{X^{s,1/2+}}$ for some $0 < C < 1$ and $\forall u, v \in \mathbf{X}$.

Let $R := \left\| \varphi(t)e^{ith(\nabla/i)}u_0 \right\|_{X^{s,1/2+\delta}_{\tau=h(\xi)}}$. From Proposition 1, we remark that $R \sim_\varphi \|u_0\|_{H^s}$. Then by the triangle inequality, $u \in \mathbf{X}$ implies $\|u\|_{X^{s,b}} \sim R$. To show the first condition, we apply Propositions 4 and 3. Given $u \in \mathbf{X}$,

$$\begin{aligned}\left\| \Lambda_T u(t) - \varphi_T(t)e^{ith(\nabla/i)}u_0 \right\|_{X^{s,\frac{1}{2}+\delta}} &= \left\| \varphi(t)\int_0^t e^{i(t-s)h(\nabla/i)}\varphi\left(\frac{s}{T}\right)F(s)\,ds \right\|_{X^{s,\frac{1}{2}+\delta}} \\ &\leq C_{\delta,\varphi}\|\varphi(t/T)F(t)\|_{X^{s,-\frac{1}{2}+\delta}}\end{aligned}$$

Assume for a moment that we have the following inequality for some $\alpha > 0$ and for any $u \in \mathbf{X}$,

$$\|F_u\|_{X_T^{s,-\frac{1}{2}+\delta}} \lesssim_{\delta,\varphi,\alpha} T^\alpha R^n \tag{1.8}$$

for some $n \in \mathbf{N}$ which depends on the nonlinearity F. For instance, one generally hopes to obtain (1.8) if the nonlinearity is in the form u^n. Selecting $T_R > 0$ small so that $C_{\delta,\varphi,\alpha} R^n T_R^\alpha \le R/2$, we can show $\Lambda u \in \mathbf{X}$. Note that $T \sim_{\delta,\varphi,\alpha} R^{-(n-1)/\alpha} \sim \|u_0\|_{H^s}^{-(n-1)/\alpha}$.

For the second condition (i.e. contraction property),

$$\begin{aligned}\|\Lambda_T u - \Lambda_T v\|_{X^{s,\frac{1}{2}+\delta}} &= \left\|\varphi(t)\int_0^t e^{i(t-s)h(\nabla/i)}\varphi(s/T)(F_u - F_v)(s)\,ds\right\|_{X^{s,\frac{1}{2}+\delta}} \\ &\le C_{\delta,\varphi}\|\varphi(\cdot/T)[F_u - F_v]\|_{X^{s,-\frac{1}{2}+\delta}}.\end{aligned}$$

Therefore if we assume for all $u, v \in \mathbf{X}$ (often a Corollary of the proof of (1.8))

$$\|F_u - F_v\|_{X_T^{s,-\frac{1}{2}+\delta}} \lesssim_{\delta,\varphi,\alpha} T^\alpha R^{n-1}\|u - v\|_{X^{s,b}} \tag{1.9}$$

where n is often the same as appears in (1.8), we can select a sufficiently small $T_R' > 0$ so that Λ is a contraction map on $\mathbf{X}$. Note once again $T_R' \sim_{\delta,\varphi,\alpha} R^{-(n-1)/\alpha}$. Taking $T = \min(T_R, T_R')$, we have shown that Λ is a contraction in $\mathbf{X}$. The fixed point theorem gives the uniqueness (in $\mathbf{X}$) and existence of the local-in-time strong solution of (1.5).

Finally, we assume (1.9) to show the Lipschitz continuity of the solution-map with respect to the $[H^s, C_t^0 H_x^s]$ norm. Given u_0, v_0 *close*, let u, v be the strong solutions given by (1.7). Then by Remark 1, Propositions 1 and 4,

$$\|u - v\|_{C_t^0 H^s} \lesssim_\delta \|u - v\|_{X^{s,\frac{1}{2}+\delta}}$$

$$\lesssim \left\| \varphi(t) e^{ith(\nabla/i)}[u_0 - v_0] \right\|_{X^{s,\frac{1}{2}+\delta}} + \left\| \varphi(t) \int_0^t e^{i(t-s)h(\nabla/i)} \varphi(s/T)[F_u - F_v](s)\, ds \right\|_{X^{s,\frac{1}{2}+\delta}}$$
$$\lesssim_{\varphi,\delta} \|u_0 - v_0\|_{H^s} + \|\varphi(\cdot/T)[F_u - F_v]\|_{X^{s,-\frac{1}{2}+\delta}}.$$

Finally, using (1.9) and the choice of $T \ll \max\left(\|u_0\|_{H^s}, \|v_0\|_{H^s}\right)^{-(n-1)/\alpha}$, we obtain that for some C with $0 < C < 1$,

$$\|u - v\|_{C_t^0 H^s} \lesssim \frac{1}{1-C} \|u_0 - v_0\|_{H^s}.$$

Therefore, the estimates (1.8) and (1.9) are very important in establishing these types of arguments. In the next section, we introduce a very useful tool for this types of estimates in the particular case where $x \in \mathbf{Y} = \mathbf{R}$ and nonlinearity is quadratic.

1.5 L^2 convolution-type operators

In this section, we introduce L^2 *convolution operator* as presented in [48]. This tool is very useful in obtaining estimates of the type (1.8) and (1.9) when F is a quadratic form in u. For example, we consider a simple estimate of the type

$$\|uv\|_{X^{0,-\frac{1}{2}+}_{\tau=h(\xi)}} \lesssim \|u\|_{X^{0,\frac{1}{2}+}_{\tau=h_1(\xi)}} \|v\|_{X^{0,\frac{1}{2}+}_{\tau=h_2(\xi)}} \tag{1.10}$$

where $h_j(\xi) = \varepsilon_j h(\varepsilon_j \xi)$ for $\varepsilon_j \in \{-1, 1\}$. In practice, ε_j is determined by complex conjugation operators. The product term uv in (1.10) is often replaced by some bilinear pseudo-differential operator $B(u,v)$ and $X^{0,1/2+}$ with $X^{s,b}$ for some $s, b \in \mathbf{R}$. Nonetheless, the computations presented here clearly demonstrate the essential ideas involved in these estimates.

Observe that for $u \in X^{s,b}_{\tau=h_1(\xi)}$ and $v \in X^{s,b}_{\tau=h_2(\xi)}$, by duality

$$\begin{aligned}
\|uv\|_{X^{0,-\frac{1}{2}+}_{\tau=h(\xi)}} &= \sup_{\|w\|_{X^{0,\frac{1}{2}-}_{\tau=-h(-\xi)}}=1} \left| \int_{\mathbf{R}\times\mathbf{R}} u(t,x)v(t,x)w(t,x)\,dt\,dx \right| \\
&= \sup_{\|w\|_{X^{0,\frac{1}{2}-}_{\tau=-h(-\xi)}}=1} \left| \int_{\mathbf{R}\times\mathbf{R}} \widetilde{uv}(\tau,\xi)\widetilde{w}(-\tau,-\xi)\,dt\,dx \right| \\
&= \sup_{\|w\|_{X^{0,\frac{1}{2}-}_{\tau=-h(-\xi)}}=1} \left| \int_{\substack{\tau_1+\tau_2+\tau_3=0;\\ \xi_1+\xi_2+\xi_3=0}} \widetilde{u}(\tau_1,\xi_1)\widetilde{v}(\tau_2,\xi_2)\widetilde{w}(\tau_3,\xi_3)\,d\sigma \right|. \qquad (1.11)
\end{aligned}$$

where $d\sigma$ is the measure on the given hyperplane inherited from $\mathbf{R}^3 \times \mathbf{R}^3$.

Motivated from the expression (1.11), we introduce the bilinear L^2 convolution operator norm $\|\cdot\|_{\mathscr{M}}$. Let $\tau = (\tau_1,\tau_2,\tau_3) \in \mathbf{R}^3$, $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbf{R}^3$. Let Γ be defined

$$\Gamma = \{(\tau,\xi) \in \mathbf{R}^3 \times \mathbf{R}^3 : \tau_1+\tau_2+\tau_3 = 0 \text{ and } \xi_1+\xi_2+\xi_3 = 0\}$$

and let $d\sigma$ be the measure on Γ inherited from $\mathbf{R}^3 \times \mathbf{R}^3$.

Given a function $m(\tau,\xi)$ defined on Γ, we define $||m||_{\mathscr{M}}$ to be the smallest constant C such that the following inequality holds:

$$\left| \int_\Gamma m(\tau,\xi)u(\tau_1,\xi_1)v(\tau_2,\xi_2)w(\tau_3,\xi_3)\,d\sigma \right| \le C\|u\|_{L^2_{\tau_1,\xi_1}}\|v\|_{L^2_{\tau_2,\xi_2}}\|w\|_{L^2_{\tau_3,\xi_3}}. \qquad (1.12)$$

From (1.12), we remark that the Comparison Principle holds in the following sense. If $|m(\tau,\xi)| \le M(\tau,\xi)$ for all $(\tau,\xi) \in \Gamma$, then $\|m\|_{\mathscr{M}} \le \|M\|_{\mathscr{M}}$.

To work with these multipliers, we introduce the following frequency and modulation localizing operators. Given $h_j(\xi) = \varepsilon_j h(\varepsilon_j \xi)$ for $j = 1,2,3$,

$$\widetilde{u_{N,L_j}}(\tau,\xi) = \chi_{[N,2N)}(|\xi|)\chi_{[L_j,2L_j)}(\langle \tau - h_j(\xi)\rangle)\widetilde{u}(\tau,\xi),$$

By using capitalized letters N, L, we will always assume that N, L are dyadic numbers (i.e. $N = 2^j$, $L = 2^l$ for some $j, l \in \mathbf{Z}$).

The following partition of unity will be used often to decompose expressions such as (1.11):

$$\chi^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} := \chi_{[H,2H)}(|h_1(\xi_1)+h_2(\xi_2)+h_3(\xi_3)|)\prod_{j=1}^{3}\chi_{[N_j,2N_j)}(|\xi_j|)\chi_{[L_j,2L_j)}(\langle\tau_j - h(\xi_j)\rangle). \tag{1.13}$$

Using the notations introduced in this section, we can write

$$\chi^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}\prod_{j=1}^{3}\widetilde{u^j}(\tau_j,\xi_j) = \chi_{[H,2H]}(|h_1(\xi_1)+h_2(\xi_2)+h_3(\xi_3)|)\prod_{j=1}^{3}\widetilde{u^j_{N_j,L_j}}(\tau_j,\xi_j).$$

Note that $\chi^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}$ depends on dyadic indices $H, N_1, N_2, N_3, L_1, L_2, L_3$ and that it forms a partition of unity via the summation

$$\sum_{N_1,N_2,N_3>0}\sum_{H>0}\sum_{L_1,L_2,L_3\geq 1}^{\infty}\chi^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} = 1 \qquad \text{for } (\tau,\xi)\in\Gamma\setminus\mathscr{R}$$

where $\mathscr{R} = \{(\tau,\xi)\in\Gamma : h_1(\xi_1)+h_2(\xi_2)+h_3(\xi_3) = 0, \text{ or } \xi_1\xi_2\xi_3 = 0\}$ is called the *resonant set*. Most often, $\mathscr{R}$ is a set of measure zero, so writing this as a partition of unity does not affect the outcome. However, our estimates will suffer greatly when $H \sim |h_1(\xi_1)+h_2(\xi_2)+h_3(\xi_3)| \ll 1$. Thus we will justify the use of this partition of unity by localizing spatial frequencies away from the resonant set.

Note that unless $N_{\max} \sim N_{\text{med}}$, the right side of (1.13) vanishes on Γ. Also, noting

$$\sum_{j=1}^{3}[\tau_j - h_j(\xi_j)] + [h_1(\xi_1)+h_2(\xi_2)+h_3(\xi_3)] = 0$$

for $(\tau,\xi)\in\Gamma$, we must have $L_{\max} \sim \max(H, L_{\text{med}})$. As we will see in later sections, this relation is essentially how the gain in the Bourgain weight $\langle\tau - h(\xi)\rangle^{-1}$ obtained in Proposition 4 translates to a gain of spatial derivatives. If we take advantage of the symmetry in the definition (1.12) and $\Gamma = -\Gamma$, then

$$\int_\Gamma \chi^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \prod_{j=1}^{3} u_j(\tau_j,\xi_j)\, d\sigma = \int_\Gamma \chi^{(-\varepsilon_1,-\varepsilon_2,-\varepsilon_3)} \prod_{j=1}^{3} u_j(-\tau_j,-\xi_j)\, d\sigma.$$

So $\left\|\chi^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}\right\|_{\mathscr{M}} = \left\|\chi^{(-\varepsilon_1,-\varepsilon_2,-\varepsilon_3)}\right\|_{\mathscr{M}}$. This can also be understood as taking complex conjugations of uv in (1.11) and $\|uv\|_{X^{s,b}_{\tau=h(\xi)}} = \|\overline{uv}\|_{X^{s,b}_{\tau=-h(-\xi)}}$.

Furthermore, by changing the order of u, v, w in (1.12) if necessary, it suffices to obtain bounds on $\left\|\chi^{(+,+,+)}\right\|_{\mathscr{M}}$ and $\left\|\chi^{(+,+,-)}\right\|_{\mathscr{M}}$.

Now we continue with the estimate (1.10). First, note that $\varepsilon_3 = -1$ is determined by $w \in X^{0,1/2-}_{\tau=-h(-\xi)}$. In order to use the partition of unity (1.13), we assume the functions involved are suitably localized away from $\mathscr{R}$. Then by the triangular inequality, the resulting integral of (1.11) is estimated

$$\begin{aligned}
&\lesssim \sum_{H,N_j,L_j} \int_\Gamma \chi^{\varepsilon_1,\varepsilon_2,-} \frac{[L_1^{\frac{1}{2}+}|\widetilde{u}|(\tau_1,\xi_1)][L_2^{\frac{1}{2}+}|\widetilde{v}|(\tau_2,\xi_2)][L_3^{\frac{1}{2}-}|\widetilde{w}|(\tau_3,\xi_3)]}{L_1^{\frac{1}{2}+}L_2^{\frac{1}{2}+}L_3^{\frac{1}{2}-}}\, d\sigma \\
&\lesssim \|u\|_{X^{0,\frac{1}{2}+}_{\tau=h_1(\xi)}} \|v\|_{X^{0,\frac{1}{2}+}_{\tau=h_2(\xi)}} \|w\|_{X^{0,\frac{1}{2}-}_{\tau=-h(-\xi)}} \sum_{H,N_j,L_j} \frac{\|\chi^{\varepsilon_1,\varepsilon_2,-}\|_{\mathscr{M}}}{L_1^{\frac{1}{2}+}L_2^{\frac{1}{2}+}L_3^{\frac{1}{2}-}}.
\end{aligned}$$

Therefore, in order to show (1.10), it suffices to show that

$$\sum_{H,N_j,L_j} \frac{\|\chi^{\varepsilon_1,\varepsilon_2,-}\|_{\mathscr{M}}}{L_1^{\frac{1}{2}+}L_2^{\frac{1}{2}+}L_3^{\frac{1}{2}-}} < \infty.$$

This type of estimates relies heavily on the measure of some three-dimensional object resulting from geometric arguments based on the exact expression of h as well as the signs of ε_j. This idea is represented in the next lemma, which is the main tool of obtaining these bounds.

Lemma 1. *Let* $A, B \subset \mathbf{R}^2$ *and*

$$\|\chi_A(x_1,y_1)\chi_B(x_2,y_2)\|_{\mathscr{M}} \lesssim \sup_{(x,y)\in\mathbf{R}^2} |\{(x_1,y_1)\in A : (x-x_1, y-y_1)\in B\}|^{\frac{1}{2}}.$$

Proof. Denote $M := \sup_{(x,y)\in\mathbf{R}^2} |\{(x_1,y_1)\in A : (x-x_1,y-y_1)\in B\}|^{1/2}$. We need to prove

$$\left|\int_\Gamma \chi_A(x_1,y_1)\chi_B(x_2,y_2)u(x_1,y_1)\,v(x_2,y_2)\,w(x_3,y_3)\,d\sigma\right| \lesssim M\|u\|_{L^2_{x,y}}\|v\|_{L^2_{x,y}}\|w\|_{L^2_{x,y}}. \tag{1.14}$$

Let $z_j = (x_j, y_j) \in \mathbf{R}^2$ for $j = 1,2,3$. Since $z_1+z_2+z_3 = 0$, the integral in (1.14) is

$$\begin{aligned}
&\lesssim \iint_{\mathbf{R}^2\times\mathbf{R}^2} \chi_A(z_1)\chi_B(-z_3-z_1)|u|(z_1)|v|(-z_3-z_1)|h|(z_3)\,dz_1\,dz_3\\
&\lesssim \int_{\mathbf{R}^2} \|\chi_A(\cdot)\chi_B(-z_3-\cdot)\|_{L^2_{z_1}}\|u(\cdot)v(-z_3-\cdot)\|_{L^2_{z_1}}|w|(z_3)\,dz_3\\
&\lesssim \|\chi_A(\cdot)\chi_B(-z_3-\cdot)\|_{L^\infty_{z_3}L^2_{z_1}}\left(\iint_{\mathbf{R}^2\times\mathbf{R}^2}|u|^2(z_1)|v|^2(-z_3-z_1)\,dz_1\,dz_3\right)^{\frac{1}{2}}\|w\|_{L^2_{z_3}}\\
&= M\|w\|_{L^2}\left(\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}|v|^2(-z_3-z_1)\,dz_3\,|u|^2(z_1)\,dz_1\right)^{\frac{1}{2}} = M\|u\|_{L^2}\|v\|_{L^2}\|w\|_{L^2}.
\end{aligned}$$

□

The following is a useful estimate, but it is a very coarse one since it does not take the geometry of these sets into consideration.

Corollary 1. *Given any* $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}$,

$$\left\|\chi^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}\right\|_{\mathscr{M}} \lesssim L_{\min}^{\frac{1}{2}} N_{\min}^{\frac{1}{2}}$$

Proof. Let $j,k \in \{1,2,3\}$ so that $N_j = N_{\min}$ and $L_k = L_{\min}$ if $L_j \neq L_{\min}$. If $L_j = L_{\min}$, then let $k \in \{1,2,3\} \setminus \{j\}$. We define

$$A = \{(\tau,\xi) \in \mathbf{R}^2 : N_j \leq |\xi| < 2N_j \text{ and } L_j \leq \langle \tau - h_j(\xi) \rangle < 2L_j\}$$
$$B = \{(\tau,\xi) \in \mathbf{R}^2 : N_k \leq |\xi| < 2N_k \text{ and } L_k \leq \langle \tau - h_k(\xi) \rangle < 2L_k\}.$$

Then note $\chi^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \leq \chi_A(\tau_j,\xi_j)\chi_B(\tau_k,\xi_k)$. By the comparison principle and Lemma 1,

$$\left\|\chi^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}\right\|_{\mathscr{M}} \lesssim \left\|\chi_A(\tau_j,\xi_j)\chi_B(\tau_k,\xi_k)\right\|_{\mathscr{M}} \lesssim \sup_{(\tau,\xi)\in\mathbf{R}^2} |\{(\tau_j,\xi_j) \in A : (\tau-\tau_j,\xi-\xi_j) \in B\}|^{\frac{1}{2}}.$$

For a fixed τ,ξ,ξ_j, we note that τ_j is restricted in a set of measure $O(\min(L_j,L_k)) = O(L_{\min})$ and ξ_j is contained in a set of measure $O(N_j) = O(N_{\min})$. □

Note that we did not use any information about ε_j or even h in the proof of Corollary 1. Surprisingly often, this suffices for the proof of (1.11) when, for example, $H \gtrsim N_{\max}^{\alpha} N_{\min}$ for some $\alpha > 0$ small. Recall that $H \lesssim L_{\max}$. Then

$$\frac{\|\chi^{\varepsilon_1,\varepsilon_2,\varepsilon_3}\|_{\mathscr{M}}}{L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta} L_3^{\frac{1}{2}-\delta}} \lesssim \frac{N_{\min}^{\frac{1}{2}} L_{\min}^{\frac{1}{2}}}{L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta} L_3^{\frac{1}{2}-\delta}} \lesssim \frac{N_{\min}^{\frac{1}{2}}}{L_{\max}^{\frac{1}{2}-\delta}} \lesssim \frac{N_{\min}^{\delta}}{L_{\max}^{\delta} N_{\max}^{(\frac{1}{2}-\delta)\alpha}}.$$

Therefore if $(\frac{1}{2}-\delta)\alpha > \delta$, this is summable in all indices when we are away from the resonant set $\mathscr{R}$ in the sense $N_j, H \gtrsim 1$.

In general, Corollary 1 is too rough, and one can obtain much better estimates by taking into consideration specific forms of h_j's. We will derive these estimates at the beginning of each section as needed.

1.6 Main theorems and background

1.6.1 Quadratic Schrödinger equation in one space dimension

Consider the 1-D quadratic Schrödinger equation

$$\left| \begin{array}{ll} u_t + iu_{xx} = Q(u,u): & (t,x) \in \mathbf{R}^1_+ \times \mathbf{R}^1 \\ u(0,x) = u_0 & \end{array} \right. \tag{1.15}$$

The problem has received a lot of attention in the last twenty years and a full account of the appropriate results and open questions is beyond of the scope of this text. We will however outline a selected list of recent works, which has some bearing on the problem that we are studying.

The classical results of the subject go back to Tsutsumi, [51], which establishes local well-posedness for data in $H^s, s \geq 0$ for all quadratic nonlinearities (i.e. $|Q(u,u)| \leq C|u|^2$). This is in a way optimal, since for Hamiltonian models (i.e. with $Q(e^{i\theta}u, e^{i\theta}u) = e^{i\theta}Q(u,u)$), it is well-known that there is ill-posedness in $H^s, s < 0$ - this is in the work of Kenig-Ponce-Vega, [29], see also Christ, [11] and Christ-Colliander-Tao, [12] for further results in this direction.

For the non-Hamiltonian model, several different models have been considered in the literature, the most popular being $Q(u,u) = u^2, u\bar{u}, \bar{u}^2$. Each of these comes with its own specifics and the corresponding local well-posedness results. Regarding the cases $u^2, \bar{u}^2$, it has been shown by Kenig-Ponce-Vega [29], that these are well posed in $H^{-\frac{3}{4}+}$ by means of bilinear estimates in $X^{s,b}$ spaces. Moreover, such bilinear estimates necessarily fail at the critical index $-3/4$, [29, 37]. Regarding the nonlinearity $u\bar{u}$, the problem is well-posed in $H^{-\frac{1}{4}+}$, [29] and this turns out to be sharp,[1] [35]. On the other

[1] At least as far as the uniform continuity of the solution map goes

hand, the results for $u\bar{u}$ may be improved to the really sharp index $s = -1/2+$, if one is willing to put some homogeneous Sobolev space requirements on the low-frequency portion of the data, [35].

Regarding the nonlinearity u^2, the results of [29] were extended to the sharp index $s \geq -1$,by Bejenaru-Tao [5], see also the work Bejenaru-Da Silva, [4] for the same result in two spatial dimensions. As we have mentioned, the spaces $X^{s,b}$ by themselves, could not accommodate such low regularity of solutions, so the authors had to come up with further refinements of these spaces, in which they were able to preform their fixed point arguments, see also Kishimoto, [30] for interesting commentary on these developments. For the nonlinearity $\bar{u}^2$, it has been shown that similar techniques may be used to obtain l.w.p. in $H^s, s \geq -1$, Kishimoto, [30]. It also should be noted that in all of these papers (with the exception of [32]), it is hard to show optimal l.w.p. for Schrödinger equations with nonlinearity of the form $Q(u,u) = c_1 u^2 + c_2 \bar{u}^2$, due to the specifics of the approaches. The result in [32] achieves this goal, at the expense of further layer of complexity, involved in the definition of the spaces and the corresponding bilinear estimates that need to be shown.

Our main result concerns the following specific generalization of the quadratic Schrödinger equation (1.15)

$$\left| \begin{array}{ll} u_t + iu_{xx} = \langle \nabla \rangle^{\beta} [u^2] : & (t,x) \in \mathbf{R}^1_+ \times \mathbf{R}^1 \\ u(0,x) = u_0 \in H^{-\alpha}, & \end{array} \right. \tag{1.16}$$

where $\beta \geq 0$. This model has been well-studied over the years, mainly the case $\beta = 0$, $\beta = 1$. We should first mention, that the corresponding equation is ill-posed for $\beta = 1$, in the sense that the flow map $u_0 \to u$ experiences norm inflation, Christ [11], see also [12] for related results. In the work of Stefanov, [46] existence of weak solutions in H^1

was shown, under the additional smallness requirement $\sup_x |\int_{-\infty}^x u_0(y)dy| \ll 1$. Similar results[2] (in $\mathbf{R}^n, n \geq 1$), were obtained for the more general Ginzburg-Landau equation in the work of Han-Wang-Guo, [24]. Finally, we mention some recent local well-posedness results, which were obtained for (not necessarily small) data in weighted Sobolev spaces by Bejenaru, [2, 3] and Bejenaru-Tataru, [6].

While some of the positive results mentioned above surely will extend to the case $\beta \in (0,1)$, it seems that this model has not been considered in the literature. One of the purposes of the current paper is to address the question for local well-posedness of this problem. An alternative goal is to develop an alternative proof of the known results in the case $\beta = 0$, which is within the framework of the standard $X^{s,b}$ spaces. We achieve that by the technique of normal forms.

Our main results recover (up to an endpoint) the sharp results of [5], [32] in the case $\beta = 0$, but also covers the new cases $\beta \in (0, 1/2)$, where the results are also arguably sharp. We also obtain the Lipschitz property (1.18) of the solution map, which is a new feature, even in the case $\beta = 0$. More specifically,

Theorem 1. *Let* $\alpha \in [\frac{1}{2}, 1)$ *and* $\beta, \gamma \geq 0$ *such that* $\beta + \gamma < \frac{1}{2}$, $\alpha - \gamma < \frac{1}{2}$ *and* $2\alpha + \beta < 2$. *Then the equation* (1.16) *is locally well-posed in* $H^{-\alpha}(\mathbf{R}^1)$. *More specifically, for every* $u_0 \in H^{-\alpha}(\mathbf{R}^1)$, *there exists a non-trivial* $T > 0$, *so that the equation* (1.16) *has an unique solution* $u \in C([0,T), H^{-\alpha})$.

Moreover, for fixed $\delta : 0 < \delta \ll 1$, *there is the following decomposition*

$$u = e^{-it\partial_x^2} u_0 + h + w, \tag{1.17}$$

where $h \in L_t^\infty H_x^{\frac{1}{2}-\alpha} \cap L_t^2 H_x^{1-\alpha}$, $w \in X^{\gamma-\alpha, \frac{1}{2}+\delta}$. *In particular* $u - e^{-it\partial_x^2} u_0 \in L_t^\infty H_x^{\gamma-\alpha}$.

[2] again for data small in L^1 sense and so that it belongs to some smooth modulation spaces

We also have the following Lipschitz property of the solution map of (1.16). *Let $N > 0$ and $u_0, v_0 \in H^{-\alpha}(\mathbf{R}^1) : \|u_0\|_{H^{-\alpha}} < N, \|v_0\|_{H^{-\alpha}} < N$, so that $u_0 - v_0 \in H^{-\frac{1}{2}}$. Then the corresponding solutions (defined on a common non-trivial time interval $(0, T_N)$) satisfy*

$$\|u - v\|_{L^\infty_T H^{\gamma-\alpha}_x} \leq C_N \|u_0 - v_0\|_{H^{\gamma-\alpha}_x}. \tag{1.18}$$

where C_N depends only on N.

Remarks:

- There are of course well-posedness results in the cases $\alpha \in (0, 1/2)$ and they are easier to obtain. We chose to include only those with $\alpha > 1/2$ in order to simplify our exposition.

- The Lipschitz property (1.18) is new even in the case $\beta = 0$.

- We do not obtain l.w.p. for the case $\alpha = 1 - \beta$, which in the case $\beta = 0$, will correspond to the endpoint case of $s = 1$, considered in [5]. Our arguments would imply such a statement, at least in the case of a Besov-1 space version of the main result.

- While our arguments fail at $\beta \geq 1/2$, we cannot claim sharpness in this regard. However, we very strongly suspect that this is the case. That is, we conjecture that some form of ill-posedness must occur, when one considers solutions to (1.16) with $\beta = 1/2$.

We also have the following corollary. Consider

$$\left| \begin{array}{ll} z_t + iz_{xx} = \langle \nabla \rangle^\beta z \langle \nabla \rangle^\beta z & (t,x) \in \mathbf{R}^1_+ \times \mathbf{R}^1 \\ z(0,x) = z_0 & \end{array} \right. \tag{1.19}$$

Setting $u = \langle \nabla \rangle^{\beta} z$ yields the equation

$$u_t + iu_{xx} = \langle \nabla \rangle^{\beta} [u^2], \tag{1.20}$$

for u. By Theorem 1, we conclude that (1.20) is well-posed in $H^{-\alpha}$, for all $\frac{1}{2} < \alpha < 1 - \beta$. Therefore, in terms of z, we have well-posedness in $H^{-\alpha+\beta}$

Corollary 2. *Let* $\beta \in [0, 1/2)$ *and* $0 < \alpha < 1 - 2\beta$. *The equation* (1.19) *is well posed in* $H^{-\alpha}$.

The proof of Theorem 1 is presented in Chapter 2, which is orgarnized as follows. In Section 2.1, we derive a more efficient bound for $\left\| \chi^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)} \right\|_{\mathscr{M}}$ in the Schrödinger setting (i.e. $h(\xi) = \xi^2$) and also apply these improved bounds to prove a few bilinear estimates. In Section 2.2, we construct the normal form transformation and provide basic estimates on the normal form. In Section 2.3, we state and prove the bilinear and trilinear estimates needed for the contraction argument. This is where the main technical difficulties lie. In Section 2.4, we conclude the proof, by reducing it to the estimates proved in Section 2.3. In Section 2.5, we give some ideas on how to approach the problem of local well-posedness for the problem with nonlinearities of the form $\langle \nabla \rangle^{\beta} [u\bar{u}]$ and $\langle \nabla \rangle^{\beta} [\bar{u}^2]$.

1.6.2 Korteweg-de Vries equation on the torus

Consider the periodic boundary initial value problem for the Korteweg-de Vries equation given by

$$\left| \begin{array}{ll} u_t + u_{xxx} = 6uu_x : & (t,x) \in [0,T] \times \mathbf{T} \\ u(0,x) = u_0 \in H^{-s}.(\mathbf{T}) & \end{array} \right. \tag{1.21}$$

This equation was derived by Korteweg and de Vries in 1895 as a model for the propagation of water waves along a long and narrow canal. Since then, (1.21) has been studies extensively along with a more generalized model where one replaces uu_x on the right hand side with $u^p u_x$ for some $p \geq 1$. The complete survey of this topic is outside the scope of this dissertation, but we offer a summary of the development for local and global well-posedness theories with low-regularity initial data.

Bourgain showed the local well-posedness of (1.21) L^2 by introducing $X^{s,b}$ spaces in [9], which implied the global well-posedness because L_x^2 norm is conserved for all L^2 solutions of (1.21). Kenig, Ponce, Vega extended this local well-posedness result to $H^{-\frac{1}{2}+}$ in [28]. In [13], Colliandar, Colliander, Keel, Staffilani, Takaoka and Tao extended the local well-posedness to the endpoint $H^{-\frac{1}{2}}$ and proved the corresponding global well-posedness by introducing the *I-method* of constructing almost-conserved quantities. The same authors in [14] discussed sharp a priori estimates for generalized KdV equations (i.e. nonlinearity uu_x replaced by $u^p u_x$ for some $p \in \mathbf{N}$), which is necessary to establish local well-posedness theories for these more general models.

In [12], Christ, Colliander, Tao proved that the solution map is no longer uniformly continuous in $H^{-1/2-}$, establishing the sharpness of the result in [13] with respect to the notion of well-posedness introduced in Section 1.4. However this notion of well-posedness of stronger than necessary, and improvements is possible by bypassing uniform continuity. Indeed, Kappeler and Topalov [26] proved the global well-posedness of (1.21) in H^{-1} by using inverse scattering method. This proof takes advantage the completely integrable structure of KdV, and it produces a very different non-Lipschitz, non-analytic continuity statement for the given problem.

In this dissertation, we only consider the smoothing estimates on the solution for (1.21) rather than well-posedness theories. We will take for granted the local/global well-posedness of (1.21) for $s > -\frac{1}{2}$ and derive *a posteriori* estimates on the solution.

The normal form method was first applied for this model in [1], where it is referred to as *differentiation by parts*. In [15], the differentiation-by-parts technique was used to show a global smoothing for the periodic KdV. Here, Erdogan and Tzirakis proved that the global solution u of (1.21) with initial data $u_0 \in H^{-s}$ for $s < \frac{1}{2}$ satisfies $u - e^{-t\partial_x^3} u_0 \in H^{-s+\gamma}$ where $\gamma < \min(-2s+1, 1)$. Close to L^2, this gives a gain of a full derivative, but as we approach $H^{-\frac{1}{2}}$, this gain gradually disappears.

We approach the problem from a different direction. To motivate our setting, we turn our attention toward the analysis of modified KdV. Bourgain, [9] noted that a trilinear resonant term causes trouble in the low regularity analysis of modified KdV. In fact, the *a priori* estimates for the non-resonant mKdV holds true for $H^{1/4+}$, whereas the estimate for the resonant portion fails below $H^{1/2}$. Also in [49, Exercise 4.21], Tao notes that the *non-resonant* solution causes the solution map to be not uniformly continuous. The non-uniform continuity in particular implies that the standard contraction arguments will fail in this setting. However, Takaoka, Tsutsumi, [47] and also Nakanishi, Takaoka, Tsutsumi, [38], the authors absorbs a part of resonance into the linear operator in order to extend the well-posedness theory of this equation beyond $H^{1/2}$ barrier.

While the idea of resonance has been central in the study of the periodic mKdV equation, it was not widely known or studied for the periodic KdV equation. This is due to the fact that the resonance in this setting is not even visible until the normal form transformation is applied. Thus, the resonance was first observed for (1.21) in [1] and also in [15], since the authors perform the normal form transform on the KdV equation. But the resonant solution as mentioned in [49, Exercise 4.21] for the mKdV setting was never discussed in these papers. Thus, it is worthwhile to investigate the properties of the solution of (1.21) in relation to the resonant solution.

In Section 3.1, we will construct the resonant solution $R^*[u_0]$ and derive a few interesting properties. Then we observe that there can be a smoothing of $1-s$ derivatives for $s \geq 0$, which is better than the $1-2s$ derivative gain achieved in [15] for the fully non-linear component. In Lemma 8, we show that the difference between the linear solution and the resonant solution is precisely $1-2s$ derivative smoother: that is, $R^*[u_0](t) - e^{-t\partial_x^3} u_0 \in H^{1-3s}$. Thus, as will be observed in Corollary 3, this type of smoothing can be regarded as a generalization of the non-linear smoothing.

The following is the main result of this work:

Theorem 2. *Let $0 \leq s < 1/2$, $0 \leq \gamma \leq 1-s$ and $0 < 10\delta < 1-s-\gamma$. For any real-valued $u_0 \in H^{-s}(\mathbf{T})$ with $\widehat{u_0}(0) = 0$, there exists a time interval $[0,T]$ with $T \sim \|u(0)\|_{H^{-s}(\mathbf{T})}^{-\alpha}$ for some $\alpha > 0$ so that the real-valued solution $u \in C_t^0([0,T]; H^{-s}(\mathbf{T}))$ to the periodic boundary value problem (**??**) can be decomposed in the following manner: $u = R^*[u_0] + h + w$ where*

$$R^*[u_0] \in L_t^\infty H_x^{-s}; \quad h \in L_t^\infty H_x^{-s+1}; \quad w \in X_T^{-s+\gamma,\frac{1}{2}+\delta}.$$

Furthermore, we can write the Lipschitz property of the solution map in a smoother space:

$$\|u-v\|_{C_t([0,T];H_x^{-s+\gamma})} \leq C_{N,T,\delta} \|u_0 - v_0\|_{H_x^{-s+\gamma}},$$

where $\|u_0\|_{H^{-s}} + \|v_0\|_{H^{-s}} < N$, and $C_{N,T,\delta}$ depends only on N, T and δ.

We remark that Theorem 2 implies $u - R^*[f] \in C_t^0([0,T]; H_x^{-s+\gamma})$ for $0 \leq s < 1/2$ and $\gamma \leq 1-s$, when we consider the embedding $X^{s,\frac{1}{2}+\delta} \subset C_0^t([0,T]; H_x^s)$ for any $\delta > 0$ and $s \in \mathbf{R}$. Further arguments in [15], this smoothing effect presumably extends to the global solution.. But we do not pursue this here.

Furthermore, in view of Lemma 8, this non-resonant smoothing can be interpreted as a generalization of non-linear smoothing (compare [15, Theorem 1.2]).

Corollary 3. *Let* $0 \leq s < 1/2$. *Then the solution* u *of* (**??**) *with the initial data in* H^{-s} *satisfies the following non-linear smoothing property:*

$$u(t) - e^{-t\partial_x^3} u_0 \in C_t^0([0,T]; H^{s_0})$$

where $s_0 \leq 1 - 3s$.

Proof.

$$\left\| u(t) - e^{-t\partial_x^3} u_0 \right\|_{C_t^0 H^{s_0}} \leq \| u(t) - R^*[u_0] \|_{C_t^0 H^{s_0}} + \left\| R^*[u_0] - e^{-t\partial_x^3} u_0 \right\|_{C_t^0 H^{s_0}}$$

where the first term on the right-hand side belongs in $C_t^0 H_x^{s_0}$ for $s_0 \leq 1 - 2s$ by Theorem 2, and the second term belongs to $C_t^0 H_x^{s_0}$ for $s_0 \leq 1 - 3s$ by Lemma 8. □

The proof of Theorem 2 is presented in Chapter 3, which is orgarnized as follows. In Section 3.1, we construct the normal form transformation and provide basic estimates on the normal form. Also we will define the resonant component $R^*[u_0]$ here. In Section 3.2, we state and prove the bilinear and trilinear estimates needed for the contraction argument, including an estimate on the Lipschitz continuity of R^*. In Section 3.3, we conclude the proof by reducing it to the estimates proved in Section 3.2.

1.6.3 Periodic "good" Boussinesq equation

We consider the Cauchy problem for the periodic "good" Boussinesq problem

$$\left| \begin{array}{l} u_{tt} + u_{xxxx} - u_{xx} + (u^p)_{xx} = 0, \quad (t,x) \in \mathbf{R}_+^1 \times \mathbf{T} \\ u(0,x) = u_0(x); u_t(0,x) = u_1(x) \end{array} \right. \tag{1.22}$$

This is a model that was derived by Boussinesq, [10], in the case $p = 2$ and belongs to a family of Boussinesq models, which all have the same level of formal validity. We will consider mostly the original model (i.e. with $p = 2$), but we state some previous results in this generality for completeness.

It was observed that (1.22) exhibits some desirable features, like local well-posedness in various function spaces. Let us take the opportunity to explain the known results. Most of these results concern the same equation on the real line. It seems that the earliest work on the subject goes back to Bona and Sachs, who have considered (1.22) and showed well posedness in $H^{\frac{5}{2}+}(\mathbf{R}^1) \times H^{\frac{3}{2}+}(\mathbf{R}^1)$, [8]. Interestingly, global well-posedness for (1.22) does not hold[3], even if one requires smooth initial data with compact support. In fact, there are "instability by blow-up" results for such unstable traveling waves for this equation.

Tsutsumi and Mathashi, [50], established local well-posedness of (1.22) in $H^1(\mathbf{R}^1) \times H^{-1}(\mathbf{R}^1)$. Linares lowered these smoothness requirement to $L^2(\mathbf{R}^1) \times H^{-2}(\mathbf{R}^1)$, $1 < p < 5$. In the same paper, Linares has showed the global existence of small solutions. Farah, [16] has shown well-posedness in $H^s(\mathbf{R}^1) \times \tilde{H}^{s-2}(\mathbf{R}^1)$, when $s > -1/4$ and the space $\tilde{H}^\alpha$ is defined via $\tilde{H}^\alpha = \{u : u_x \in H^{\alpha-1}(\mathbf{R}^1)\}$. Farah has also established ill-posedness (in the sense of lack of continuous dependence on initial data) for all $s < -2$. Kishimoto and Tsugava, [25] have further improved this result to $s > -1/2$, which seems to be the most general result currently available for this problem. Although, the authors prove the ill-posedness of the non-linear Schrödinger equation for $s < -1/2$, it is not clear how this translates to the ill-posedness of "good" Boussinesq equation. More recently, Geba, Himonas and Karapetyan, [18] showed that the solution map is discontinuous for $s < -7/4$.

[3]except for small data, see below

Regarding the case of periodic boundary conditions, Fang and Grillakis, [53] who have established local well-posedness in $H^s(\mathbf{T}) \times H^{s-2}(\mathbf{T})$, $s > 0$ (when $1 < p < 3$ in (1.22)). This result was later improved to $s > -1/4$ for the quadratic equation by Farah and Scialom, [17], by utilizing the optimal quadratic estimates (proved in the paper) in the Schrödinger $X^{s,b}$ spaces. In addition, he showed that these estimates fail below $s < -1/4$. Thus, local well-posedness for (1.22) in $H^{-1/4+}$ is the best possible result, *obtainable by this method*. Using the method from [25], Kishimoto, [34] showed the sharp well-posedness for $s \geq -1/2$ and ill-posedness for $s < -1/2$.

Next, we point out that the initial value problem for the Boussinesq problem (1.22) is very closely related to the corresponding problem for quadratic Schrödinger equation

$$iu_t + u_{xx} + F(u, \bar{u}) = 0, \tag{1.23}$$

where F is a bilinear form, which contains expressions in the form $u^2, u\bar{u}, \bar{u}^2$. Recall that Kenig, Ponce and Vega, [29] have established the local well-posedness in $H^{-1/4+}(\mathbf{R}^1)$ for (1.23), while later Kishimoto and Tsugava [25] (see also [31], [33]) have established the sharpness of this result on the line (when the nonlinearity is $u\bar{u}$).

Our main concern in this paper is to extend the results of Farah and Scialom [17] to even rougher initial data, namely in the class $H^s(\mathbf{T}) \times H^{s-2}(\mathbf{T})$, $s > -3/8$. As we have mentioned above, the method of Farah and Scialom is optimal as far as the estimates are concerned. Our approach is similar to the one given in Chapter 2 where we apply the method of normal forms. The idea is that the roughest part of the solution to the nonlinear equation is the free solution, while the rest is actually much smoother. This allows us to obtain a smoothing estimate for the solution, in the sense described in our main result, Theorem 3 below.

Before we state our results, we introduce the Schrödinger $X^{\varepsilon}_{s,b}, \varepsilon = \pm 1$ spaces, which will be relevant for our considerations, are defined via the completion of all sequences of Schwartz functions[4] $F = \{F_n\}_{n\in\mathscr{Z}\setminus\{0\}}$, $F_n : \mathbf{R}^1 \to \mathscr{C}$, in the norm

$$\|F\|_{X^{\varepsilon}_{s,b}} = \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{\mathbf{R}^1} (1+|\tau-\varepsilon n^2|)^{2b} < n >^{2s} |\widehat{F_n}(\tau)|^2 d\tau \right)^{1/2}. \tag{1.24}$$

In our definition (2.20), we have restricted to the space of functions with spatial-mean zero (i.e. $\int_{\mathbf{T}} f(t,x)\,dx = 0$ for all $t \in \mathbb{R}$). We will justify this reduction in Section 4.1.

Observe that we have the duality relation $(X^+_{s,b})^* = X^-_{-s,-b}$.

Theorem 3. *Let* $0 < \alpha < 3/8$ *and* $p = 2$*. The Cauchy problem* (1.22) *is locally well-posed in* $H^{-\alpha}(\mathbf{T}) \times H^{-\alpha-2}(\mathbf{T})$*. That is, given* $u_0 \in H^{-\alpha}$ *and* $u_1 \in H^{-\alpha-2}$*, there exists* $T := T(\|u_0\|_{H^{-\alpha}} + \|u_1\|_{-\alpha-2}) > 0$ *and a unique solution* $u \in C^0_t([0,T]; H^{-\alpha})$ *of* (1.22).

Furthermore, for data (u_0, u_1)*, so that* $\int_0^{2\pi} u_0(x)dx = 0 = \int_0^{2\pi} u_1(x)dx$*, we have*

$$u - \left[\cos(t\sqrt{\partial_x^4 - \partial_x^2})u_0 + \frac{\sin(t\sqrt{\partial_x^4 - \partial_x^2})}{\sqrt{\partial_x^4 - \partial_x^2}} u_1 \right] \in C^0_t([0,T]; H^{\beta}_x(\mathbb{T})) \tag{1.25}$$

for any $\beta : \beta < \min(1 - 3\alpha, \frac{1}{2} - \alpha)$.

Although the mean-zero condition gives a simplification to this problem, it is not essential. In our proof, we change variable to eliminate a technical issue caused by the zero-mode Fourier coefficient. Following is a more general version of (1.25).

Remark 2. *Given the data* $(u_0, u_1) \in H^{-\alpha} \times H^{-\alpha-2}$ *with* $0 < \alpha < 3/8$ *and* $p = 2$*, let* $A_0 = f12\pi \int_0^{2\pi} u_0\,dx$ *and* $A_1 = \frac{1}{2\pi}\int_0^{2\pi} u_1\,dx$*. Define an order-zero differential operator*

[4] Here, we take only those functions $F = (F_n)$, so that there exists N, so that $F_n(t) \equiv 0$, for all $|n| > N$.

$\mathscr{P} := \partial_{xx}\left(\sqrt{\partial_x^4 - \partial_x^3}\right)^{-1}$. *Then the solution u of* (1.22) *satisfies*

$$u(t) - e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}\left[\cos(t\sqrt{\partial_x^4 - \partial_x^2})\,(u_0 - A_0) + \frac{\sin(t\sqrt{\partial_x^4 - \partial_x^2})}{\sqrt{\partial_x^4 - \partial_x^2}}\,(u_1 - A_1)\right]$$

$\in C_t^0([0,T]; H_x^{\beta}(\mathbb{T}))$ *for any* $\beta : \beta < \min(1 - 3\alpha, \frac{1}{2} - \alpha)$.

Chapter 2

Schrödinger equation in 1D with nonlinearity u^2

2.1 Linear and bilinear estimates for Schrödinger equations on $\mathbf{R}^{1+1}$

We begin by collecting and sharpening necessary tools for the subsequent arguments. In view of the IVP (1.16), we note that the dispersion relation $h(\xi) = \xi^2$ corresponds to this model.

Recall the Strichartz estimates for Schrödinger equations from [23, 52, 27, 36]. Namely, $\|e^{-it\partial_x^2} f\|_{L_t^q L_x^r} \leq C_{Str.}\|f\|_{L^2}$, for all pairs $(q,r) : 2 \leq q,r \leq \infty$, $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ when $x \in \mathbf{R}^1$. Then by Remark 1,

$$\|u\|_{L_t^q L_x^r} \leq C_{Str.,\delta}\|u\|_{X^{0,\frac{1}{2}+\delta}} \tag{2.1}$$

for all Strichartz pairs $(q,r) : \frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ and for all $\delta > 0$. In particular, the admissible pair $(q,r) = (8,4)$ will be the essential tool for estimating the *resonance* involved in this problem.

Next we derive bounds on the L^2 convolution operator as introduced in Section 1.5. First consider $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +1$. Then it is easy to see that $H \sim \xi_1^2 + \xi_2^2 + \xi_3^2 \sim N_{\max}^2$, otherwise $\chi^{(+,+,+)}$ vanishes. On the other hand if $\varepsilon_1 = \varepsilon_2 = 1$ but $\varepsilon_3 = -1$, then $H \sim \xi_1^2 + \xi_2^2 - \xi_3^2 = \xi_1^2 + \xi_2^2 - (-\xi_1 - \xi_2)^2 = -\xi_1\xi_2 \sim N_1N_2$, otherwise $\chi^{(+,+,-)}$ vanishes.

The following statement is Proposition 11.1 in [48].

Proposition 6 ($(+,+,+)$ case)**.** *Let* $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$ *satisfy* $N_{\max} \sim N_{\text{med}}$, $H \sim N_{\max}^2$ *and* $L_{\max} \sim \max(H, L_{\text{med}})$. *Then we have the following estimates.*

1. *In the exceptional case where* $N_{\max} \sim N_{\min}$ *and* $L_{\max} \sim H$,

$$\|\chi^{(+,+,+)}\|_{\mathscr{M}} \leq C L_{\min}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{4}} \tag{2.2}$$

2. *Otherwise (i.e.* $N_{\min} \ll N_{\max}$ *or* $L_{\max} \sim L_{\text{med}} \gg H$*), there is an absolute constant* C*, so that*

$$\|\chi^{(+,+,+)}\|_{\mathscr{M}} \leq C \frac{L_{\min}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}}}{N_{\max}^{\frac{1}{2}}} \tag{2.3}$$

Next Proposition implicitly appears in [48], but we reformulate a portion of it to fit our needs.

Proposition 7 ($(+,+,-)$ case)**.** *Let* $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$ *satisfy* $N_{\max} \sim N_{med}$, $H \sim N_1N_2$ *and* $L_{\max} \sim \max(H, L_{\text{med}})$. *Then we have the following estimates.*

1. *There is an absolute constant* C *so that*

$$\|\chi^{(+,+,-)}\|_{\mathscr{M}} \leq C \min\left(\frac{L_1L_3}{N_2}, \frac{L_2L_3}{N_1}\right)^{\frac{1}{2}} \tag{2.4}$$

2. *In the special case* $N_{\max} \sim N_{\min}$*, then there is an absolute constant* C *so that*

$$\|\chi^{(+,+,-)}\|_{\mathscr{M}} \leq C L_{\min}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{4}}. \tag{2.5}$$

3. If $N_1 \sim N_2 \sim N_3$ does not hold and $L_3 = L_{\max}$, then there is an absolute constant C so that

$$\|\chi^{(+,+,-)}\|_{\mathscr{M}} \le C\frac{L_1^{\frac{1}{2}}L_2^{\frac{1}{2}}}{N_{\max}^{\frac{1}{2}}} \tag{2.6}$$

Proof. We will only prove (2.4). For the others, we refer to [48].

Define the following sets.

$$A_1 = \{(\tau,\xi) \in \mathbf{R}^2 : \xi \sim N_1, \tau - \xi^2 \sim L_1\}$$

$$A_2 = \{(\tau,\xi) \in \mathbf{R}^2 : \xi \sim N_2, \tau - \xi^2 \sim L_2\}$$

$$A_3 = \{(\tau,\xi) \in \mathbf{R}^2 : \xi \sim N_3, \tau + \xi^2 \sim L_3\}$$

To prove (2.4), let $N_j = N_{\min}$. Define $R = \{(\tau,\xi) \in \mathbf{R}^2 : |\xi| \le \varepsilon N_{\min}\}$ with $\varepsilon \ll 1$. Then we can find $m = O(1/\varepsilon)$ numbers $\xi_j^0 \sim N_j$, so that the sets of type $(0,\xi_j^0)+R$ covers the set A. Then we can apply the Box Localization [48, Corollary 3.13] so that

$$\|\chi^{(+,+,-)}\|_{\mathscr{M}} \le C\|\prod_{k=1}^{3}\chi_{A_k\cap[(0,\xi_k^0)+R]}(\tau_k,\xi_k)\|_{\mathscr{M}}$$

for some $\xi_k^0 \in A_k$ so that $\xi_1^0+\xi_2^0+\xi_3^0 \le \varepsilon N_{\min}$. Denote $A_k^0 = A_k \cap [(0,\xi_k^0)+R]$. Now by the Comparison Principle and Lemma 1,

$$\begin{aligned}\|\chi^{(+,+,-)}\|_{\mathscr{M}} &\le C\|\prod_{k=2}^{3}\chi_{A_k^0}(\tau_k,\xi_k)\|_{\mathscr{M}}\\ &\le C|\{(\tau_2,\xi_2)\in A_2^0 : (\tau,\xi)-(\tau_2,\xi_2)\in A_3^0\}|^{\frac{1}{2}}\end{aligned}$$

for some $(\tau,\xi) \in A_1 + 2R$. Note that $\xi \sim N_1$ for $\varepsilon > 0$ small.

We have $\tau_2 = \xi_2^2 + O(L_2)$ and $\tau - \tau_2 = -(\xi - \xi_2)^2 + O(L_3)$. First we can remove τ_2 by restricting it to an interval of length at most $O(\min(L_2, L_3))$ for a fixed ξ_2. Furthermore, these restrictions give $\xi_2^2 - (\xi - \xi_2)^2 = \tau + O(\max(L_2, L_3))$; that is $2\xi\xi_2 = \tau + \xi^2 + O(\max(L_2, L_3))$. So

$$\|\chi^{(+,+,-)}\|_{\mathscr{M}} \leq C[\min(L_2,L_3)|\{\xi_2 \sim N_2 : \xi_2 = \frac{\tau+\xi^2}{2\xi} + O(\max(L_2,L_3)/\xi)\}|]^{\frac{1}{2}}.$$

Clearly, ξ_2 above is contained in an interval of length at most $O(\max(L_2, L_3)/N_1)$. So we get the estimate

$$\|\chi^{(+,+,-)}\|_{\mathscr{M}} \leq C\left(\frac{L_2 L_3}{N_1}\right)^{\frac{1}{2}}.$$

By reversing the role of A_1, A_2 and following the same arguments, we also obtain

$$\|\chi^{(+,+,-)}\|_{\mathscr{M}} \leq C\left(\frac{L_1 L_3}{N_2}\right)^{\frac{1}{2}}.$$

This proves (2.4). □

We now apply Proposition 6 and Proposition 7 to deduce some bilinear estimates which we will need later.

Lemma 2. *Let $u, v \in \mathscr{S}(\mathbf{R}^{1+1})$. Then for $\delta > 0$, $k > 0$,*

$$\|(u_k v_{\ll k})_{\sim k}\|_{L^2_{t,x}} \lesssim_\delta 2^{(-\frac{1}{2}+\delta)k}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}} \tag{2.7}$$

$$\|(u_k v_{\sim k})_{\ll k}\|_{L^2_{t,x}} \lesssim_\delta 2^{(-\frac{1}{2}+\delta)k}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}} \tag{2.8}$$

$$\|u_k v\|_{L^2_{t,x}} \lesssim_\delta \|u\|_{X_T^{0,\frac{1}{2}+\delta}}\|v\|_{X_T^{0,\frac{1}{2}+\delta}} \tag{2.9}$$

In addition, there are the following estimates concerning the bilinear form $(u,v) \to u\bar{v}$

$$\|(u_k\overline{v_k})_k\|_{L^2_{tx}} \lesssim_\delta 2^{(-\frac{1}{2}+\delta)k}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}} \tag{2.10}$$

$$\|(u_k\overline{v_{\ll k}})_k\|_{L^2_{tx}} \lesssim_\delta 2^{(-\frac{1}{2}+\delta)k}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}} \tag{2.11}$$

$$\|u_k\bar{v}\|_{L^2_{tx}} \lesssim_\delta \|u\|_{X_T^{0,\frac{1}{2}+\delta}}\|v\|_{X_T^{0,\frac{1}{2}+\delta}} \tag{2.12}$$

Remark: It is easy to see from the arguments below that constants on the right side of (2.7), (2.8), (2.10) and (2.11) can be replaced by $2^{(-1/2+\varepsilon)k}C_{T,\delta,\varepsilon}$ for any $\varepsilon > 0$. But we will not take advantage of this in the sequel, thus we have allowed the constants to depend on $\delta > 0$ only to keep the involved parameters to the minimum.

Proof. We first dispense with the easy estimates (2.9) and (2.12). Indeed, taking into account the boundedness of $P_{\sim k}$ and $P_{\ll k}$ on all L^p spaces, we estimate both expressions by Hölder's and (2.1)

$$CT^{1/4}\|u\|_{L^8_TL^4_x}\|v\|_{L^8_TL^4_x} \lesssim_\delta T^{1/4}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}},$$

since $q = 8, r = 4$ is a Strichartz pair.

For the estimates (2.7) and (2.8), we use Proposition 6. We use the partition of unity $\chi^{(+,+,+)}$, where N_1,N_2,N_3 indicates the respective frequencies of u,v,uv. Denote by $\sum$, summation over $N_1,N_2,N_3,L_1,L_2,L_3 \geq 1$. Note that $N_{\max} \sim 2^k$ and the relation $L_{\max} \sim \max(L_{\text{med}},N^2_{\max})$, which holds by the constraints given in (1.13).

For (2.7), we apply (2.3) to obtain

$$\begin{aligned}\|(u_kv_{\ll k})_{\sim k}\|_{L^2_{t,x}} &= \sup_{\|w\|_{L^2_{t,x}}=1}\left|\int_{\mathbf{R}^1\times\mathbf{R}^1} u_k(t,x)v_{\ll k}(t,x)w_{\sim k}(t,x)\,dt\,dx\right| \\ &= \sup_{\|w\|_{L^2_{t,x}}=1}\left|\int_\Gamma \sum \chi^{(+,+,+)}\widetilde{u_k}(\tau_1,\xi_1)\widetilde{v_{\ll k}}(\tau_2,\xi_2)\widetilde{w_{\sim k}}(\tau_3,\xi_3)\,d\sigma\right|\end{aligned}$$

$$\leq C\sum \frac{1}{L_1^{\frac{1}{2}+\delta}L_2^{\frac{1}{2}+\delta}} \frac{L_{\min}^{\frac{1}{2}}L_{\text{med}}^{\frac{1}{2}}}{N_{\max}^{\frac{1}{2}}} \|u\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}}$$
$$\leq C\sum \frac{1}{L_{\text{med}}^{\delta}N_{\max}^{\frac{1}{2}}} \|u\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}}$$
$$\leq C_\delta 2^{(-\frac{1}{2}+\delta)k} \|u\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}}.$$

(2.8) is estimated exactly the same way as (2.7).

To prove (2.10)and (2.11), we use Proposition 7. We use the partition of unity $\chi^{(+,+,-)}$, where N_1, N_2, N_3 indicates the respective frequencies of $u, u\bar{v}, \bar{v}$. Denote by $\sum$, summation over $N_1, N_2, N_3, L_1, L_2, L_3 \geq 1$. Note $L_{\max} \sim \max(L_{\text{med}}, N_1 N_2)$ and $N_1 \sim 2^k$.

For both (2.10) and (2.11), $N_2 \sim N_{\max} \sim 2^k$. Since the calculations will be almost identical, we will only prove (2.10) here. We apply (2.4) to obtain

$$\|(u_k\overline{v_{\sim k}})_{\sim k}\|_{L^2} = \sup_{\|w\|_{L^2}=1} \left| \int_{\mathbf{R}^1\times\mathbf{R}^1} u_k(t,x)\overline{v_{\sim k}}(t,x) w_{\sim k}(t,x)\, dt\, dx \right|$$
$$= \sup_{\|w\|_{L^2}=1} \left| \int_\Gamma \sum \chi^{(+,+,-)} \widetilde{u_k}(\tau_1,\xi_1)\widetilde{\overline{v_{\sim k}}}(\tau_3,\xi_3)\widetilde{w_{\sim k}}(\tau_2,\xi_2)\, d\sigma \right|$$
$$\leq C\sum \frac{1}{L_1^{\frac{1}{2}+\delta}L_3^{\frac{1}{2}+\delta}} \frac{L_1^{\frac{1}{2}}L_3^{\frac{1}{2}}}{N_2^{\frac{1}{2}}} \|u_k\|_{L^2_{t,x}} \|\overline{v_{\sim k}}\|_{L^2_{t,x}}$$
$$\leq C_\delta 2^{(-\frac{1}{2}+\delta)k} \|u\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}}$$

□

We now provide a technical corollary, which allows us to put $\|v\|_{X^{0,\frac{1}{2}-\delta}}$ norms on the right hand sides of (2.10), (2.11) and (2.12) at the expense of slightly less gain in 2^k.

Corollary 4. *With the assumptions in Lemma 2, we have*

$$\|(u_k\overline{v_k})_k\|_{L^2_{tx}} \lesssim_\delta 2^{(-\frac{1}{2}+5\delta)k}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}-\delta}} \tag{2.13}$$

$$\|(u_k\overline{v_{\ll k}})_k\|_{L^2_{tx}} \lesssim_\delta 2^{(-\frac{1}{2}+5\delta)k}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}-\delta}} \tag{2.14}$$

$$\|u_k\bar{v}\|_{L^2_{tx}} \lesssim_\delta 2^{2\delta k}\|u\|_{X_T^{0,\frac{1}{2}+\delta}}\|v\|_{X_T^{0,\frac{1}{2}-\delta}} \tag{2.15}$$

Proof. We will show only (2.13), the others follow similar route. Indeed, we use a combination of Hölders with the Sobolev embedding $\|u_k\|_{L^\infty_x} \le C2^{k/2}\|u_k\|_{L^2_x}$ to obtain the following estimate

$$\|(u_k\overline{v_k})_k\|_{L^2_{tx}} \le \|u_k\|_{L^\infty_{tx}}\|v_k\|_{L^2_{tx}} \le C2^{k/2}\|u_k\|_{L^\infty_x L^2_x}\|v\|_{X^{0,0}} \le C_\delta 2^{k/2}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,0}}.$$

For a fixed function u, we are set to use complex interpolation between this and (2.10). Noting that $[X^{0,\frac{1}{2}+\delta}, X^{0,0}]_{4\delta} = X^{0,\frac{1}{2}-\delta-4\delta^2}$, we conclude

$$\|(u_k\overline{v_k})_k\|_{L^2_{tx}} \lesssim 2^{-(\frac{1}{2}-4\delta-\varepsilon+4\delta\varepsilon)k}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}-\delta-4\delta^2}} \lesssim 2^{-k(\frac{1}{2}-5\delta)}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}-\delta}}$$

where we have let $0 < \varepsilon \le \delta$ in the last inequality.

For the proof of (2.15), we interpolate between (2.12) and the estimate

$$\|u_k\bar{v}\|_{L^2_{tx}} \lesssim \|u_k\|_{L^\infty_{tx}}\|v\|_{L^2_{tx}} \lesssim_\delta 2^{k/2}\|u_k\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,0}}$$

□

The next lemma is new and addresses one situation in the (generally unfavorable) case (2.9), where one still can get a gain of almost half derivative. For a smooth function

$u \in \mathscr{S}(\mathbb{R}^{1+1})$, we define

$$\widehat{u^+}(t,\xi) := \widehat{u}(t,\xi)\chi_{(0,\infty)}(\xi), \qquad \widehat{u^-}(t,\xi) := \widehat{u}(t,\xi)\chi_{(-\infty,0)}(\xi).$$

Proposition 8. *For all $\delta > 0$ small and $k > 0$,*

$$||(u^+_{\sim k} v^-_{\sim k})_{\sim k}||_{L^2} \lesssim_\delta 2^{(-\frac{1}{2}+\delta)k} ||u||_{X^{0,\frac{1}{2}+\delta}} ||v||_{X^{0,\frac{1}{2}+\delta}}. \tag{2.16}$$

Proof. We present the argument for $||(u^+_k v^-_k)^-_k||_{L^2}$. The proof for the other case $||(u^+_k v^-_k)^+_k||_{L^2}$ is analogous.

First we define the following sets.

$$A := \{(\tau,\xi) | \xi > 0, \xi \sim 2^k, |\tau - \xi^2| \sim L_1\}$$
$$B := \{(\tau,\xi) | \xi < 0, \xi \sim 2^k, |\tau - \xi^2| \sim L_2\}$$
$$C := \{(\tau,\xi) | \xi < 0, \xi \sim 2^k\}$$

Then we need to show

$$||\chi_A(\tau_1,\xi_1)\chi_B(\tau_2,\xi_2)\chi_C(\tau_1+\tau_2,\xi_1+\xi_2)||_{\mathscr{M}} \lesssim \frac{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}}}{2^{\frac{k}{2}}}. \tag{2.17}$$

Note that if $\max(L_1,L_2) \gtrsim 2^{2k}$, then (2.2) gives us the desired statement. Otherwise, we have $\max(L_1,L_2) \ll 2^{2k}$.

For some $\varepsilon > 0$ small, we partition A (similarly B) into $m = O(1/\varepsilon)$ subsets $A_1, \cdots, A_m$ so that the diameter of A_j (similarly B_j) is less than $\varepsilon 2^k$ for all $1 \le j \le m$. Then, removing the terms when $\chi_{A_i}\chi_{B_j}\chi_C = 0$, we can omit χ_C from the expression (2.17).

By Lemma 1, the left side of (2.17) is bounded by

$$\sum_{i,j=1}^{m} \left|\{(\xi_1,\tau_1)\in A_i : (\tau,\xi)-(\tau_1,\xi_1)\in B_j\}\right|^{\frac{1}{2}} \tag{2.18}$$

where τ,ξ are fixed. Since $\chi_{A_i}\chi_{B_j}\chi_C \neq 0$, $(\tau,\xi)\in C+B_\varepsilon^k$ where $B_\varepsilon^k := \{(\tau,\xi)\in \mathbf{R}^2 : |\xi| \ll \varepsilon 2^k\}$. In particular, this implies that $\xi<0$ and $\xi\sim 2^k$. Writing out the condition of the set gives $\tau_1 = \xi_1^2 + O(L_1)$ and $\tau-\tau_1 = (\xi-\xi_1)^2 + O(L_2)$. So, for a fixed ξ_1, the τ_1 must be in an interval of length $O(\min(L_1,L_2))$. Then (2.18) is bounded by

$$\sum_{i,j=1}^{m} \min(L_1,L_2)^{\frac{1}{2}} \left|\{\xi_1>0, \xi_1\sim 2^k : \xi_1^2+(\xi-\xi_1)^2 = \tau + O(\max(L_1,L_2))\}\right|^{\frac{1}{2}}$$

We can write $\xi_1^2+(\xi-\xi_1)^2 = \dfrac{\xi^2+(2\xi_1-\xi)^2}{2}$. So the condition given above can be written as

$$(\xi_1-\frac{\xi}{2})^2 = C_{\tau,\xi} + O(\max(L_1,L_2))$$

where $C_{\tau,\xi} := \frac{2\tau-\xi^2}{4}$. Since ξ_1 and ξ have the opposite sign, the left hand side of the above is $\sim 2^{2k}$. On the other hand, $\max(L_1,L_2)\ll 2^{2k}$, so $C_{\tau,\xi}\sim 2^{2k}$. Then we have

$$\begin{aligned}\left|\xi_1-\left(\frac{\xi}{2}+\sqrt{C_{\tau,\xi}}\right)\right| &= \sqrt{C_{\tau,\xi}+O(\max(L_1,L_2))}-\sqrt{C_{\tau,\xi}}\\ &= \frac{O(\max(L_1,L_2))}{\sqrt{C_{\tau,\xi}+O(\max(L_1,L_2))}+\sqrt{C_{\tau,\xi}}}\\ &\lesssim \frac{O(\max(L_1,L_2))}{2^k}.\end{aligned}$$

So ξ_1 must be contained in an interval of length $\dfrac{O(\max(L_1,L_2))}{2^k}$. Using this bound in (2.18) gives the desired estimate (2.17). □

We remark that interpolation can be applied to Proposition 8 as in the proof of Corollary 4 to replace $X^{0,\frac{1}{2}+\delta}$ norm on the right side of (2.16) with $X^{0,\frac{1}{2}-\delta}$.

2.2 Normal form transformation

We begin by changing variables, which brings the function space to L^2. Namely, let $v : u = \langle\nabla\rangle^{\alpha} v$. A quick calculation then shows that (1.16) becomes

$$\left| \begin{array}{ll} v_t + i\partial_x^2 v = \langle\nabla\rangle^{\beta-\alpha}[\langle\nabla\rangle^{\alpha} v \langle\nabla\rangle^{\alpha} v] : & (t,x) \in \mathbf{R}^1 \times \mathbf{R}^1 \\ v(0,x) = \langle\nabla\rangle^{-\alpha} g =: f \in L^2(\mathbf{R}^1). & \end{array} \right. \tag{2.19}$$

Thus, we need to study the well-posedness of (2.19) in the L^2 setting.

Introduce $G(u,v) := \langle\nabla\rangle^{\beta-\alpha}[\langle\nabla\rangle^{\alpha} u \langle\nabla\rangle^{\alpha} v]$, so that the nonlinearity in (2.19) is of the form $G(v,v)$, note $G(u,v) = G(v,u)$. Observe that the bilinear form G may be written as follows

$$G(u,v)(x) = \frac{1}{4\pi^2} \int \frac{\langle\xi\rangle^{\alpha}\langle\eta\rangle^{\alpha}}{\langle\xi+\eta\rangle^{\alpha-\beta}} \widehat{u}(\xi)\widehat{v}(\eta) e^{i(\xi+\eta)x} d\xi d\eta.$$

We now decompose the form $G(v,v)$ as follows

$$\begin{aligned} G(v,v) &= G(v_{\leq 0}, v) + G(v_{>0}, v) = G(v_{\leq 0}, v) + G(v_{>0}, v_{\leq 0}) + G(v_{>0}, v_{>0}) = \\ &= G(v_{\leq 0}, (Id + P_{>0})v) + G(v_{>0}, v_{>0}). \end{aligned}$$

Next, we perform a change of variables $v \to z$, $v = e^{-it\partial_x^2} f + z$. Clearly, $z(0) = 0$ and

$$\begin{aligned} z_t + i\partial_x^2 z &= G([e^{-it\partial_x^2} f + z]_{\leq 0}, (Id + P_{>0})[e^{-it\partial_x^2} f + z]) + \\ &\quad + G([e^{-it\partial_x^2} f + z]_{>0}, [e^{-it\partial_x^2} f + z]_{>0}) \end{aligned}$$

Clearly, a lot of terms are generated by this transformation. We comment now on the form of various terms (especially the least favorable ones!), since this will influence our normal form analysis.

Heuristically, if we expect the z term to be smoother, then the least smooth term is expected to be $G(e^{-it\partial_x^2}f_{>0}, e^{-it\partial_x^2}f_{>0})$. Indeed, there are α derivatives acting on each of the two entries (which are free solutions and hence, in general, no better than L_x^2 smooth) and $\beta-\alpha$ derivatives acting on the product itself[1]. Thus, if we manage to build a smoother function h, which solves

$$(\partial_t + i\partial_x^2)h = G(e^{-it\partial_x^2}f_{>0}, e^{-it\partial_x^2}f_{>0}), \tag{2.20}$$

one would be compelled to change variables again, $z \to w$, where $z = h + w$. Define a bilinear operator T

$$T(u,v)(x) = \frac{1}{8\pi^2 i}\int \frac{\langle\xi\rangle^{\alpha}\langle\eta\rangle^{\alpha}}{\langle\xi+\eta\rangle^{\alpha-\beta}}\frac{1}{\xi\eta}\widehat{u_{>0}}(\xi)\widehat{v_{>0}}(\eta)e^{i(\xi+\eta)x}d\xi d\eta.$$

It is easy to check that for a pair of functions $u(t,x), v(t,x) \in \mathscr{S}(\mathbf{R}^1 \times \mathbf{R}^1)$,

$$(\partial_t + i\partial_x^2)T(u,v) = T((\partial_t + i\partial_x^2)u, v) + T(u, (\partial_t + i\partial_x^2)v) + G(u_{>0}, v_{>0}). \tag{2.21}$$

This last identity tells us that

$$(\partial_t + i\partial_x^2)T(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f) = G(e^{-it\partial_x^2}f_{>0}, e^{-it\partial_x^2}f_{>0})$$

[1] In fact this $\beta-\alpha$ derivatives on the product may not be of much help in "high-high to low" interaction scenario

which provides an explicit solution[2] of (2.20). Hence, set

$$h := T(e^{-it\partial_x^2} f, e^{-it\partial_x^2} f).$$

We now change variables $z = h + w = T(e^{-it\partial_x^2} f, e^{-it\partial_x^2} f) + w$, whence we get the following equation for w

$$\begin{aligned} w_t + i\partial_x^2 w \quad &= G([e^{-it\partial_x^2} f + h + w]_{\leq 0}, (Id + P_{>0})[e^{-it\partial_x^2} f + h + w]) + \\ &+ 2G((h+w)_{>0}, e^{-it\partial_x^2} f_{>0}) + G((h+w)_{>0}, (h+w)_{>0}) \end{aligned} \tag{2.22}$$

Note also that since $z(0) = 0$, it follows that the Schrödinger equation for w is supplemented by the following initial condition: $w(0) = -h(0) = -T(f, f)$. We have now prepared ourselves to close the argument in the w variable. More precisely, the proof of Theorem 1 reduces to establishing the local well-posedness of the Schrödinger equation (2.22) in an appropriate function space.

Fix $0 < \delta \ll 1$. For some $\gamma > 0$, consider the spaces

$$\begin{aligned} \mathscr{X} &= X^{\gamma, \frac{1}{2}+\delta}, \\ \mathscr{H} &= L_t^\infty H_x^{\frac{1}{2}} \cap X^{1-\delta, \delta}, \end{aligned}$$

Our strategy will be to show that the fixed point argument for w closes in the space $\mathscr{X}$, given that $f \in L^2$, $h \in \mathscr{H}$ and where we will occasionally need to use the particular form $h = T(e^{-it\partial_x^2} f, e^{-it\partial_x^2} f)$.

Next, we show the required smoothness of the normal form h, namely $h \in \mathscr{H}$. This will be done in two steps - in Lemma 3 and Lemma 4.

[2] Note that while the solution $h(t)$ of the Schrödinger equation (2.20) is not unique, it is completely determined by its value $h(0)$

Lemma 3. *If $\alpha+\beta<1$, then $T : L^2(\mathbb{R})\times L^2(\mathbb{R})\to H^{\frac{1}{2}}(\mathbb{R})$ continuously.*

Proof. Let $u,v\in\mathscr{S}(\mathbb{R})$. Then

$$\|T(u,v)\|_{H^{\frac{1}{2}}} \le \sum_{k\ge l+3}\|T(u_k,v_l)\|_{H^{\frac{1}{2}}} + \sum_{|k-l|<3}\|T(u_k,v_l)\|_{H^{\frac{1}{2}}} = I_1+I_2 \tag{2.23}$$

where $k,l>0$. Regarding the first sum in (2.23), we apply Hölder's and then the Sobolev embedding $\|u_l\|_{L^\infty}\le C2^{l/2}\|u_l\|_{L^2}$. We get

$$\begin{aligned} I_1 &\le C\sum_{l>0}\sum_{k\ge l}2^{\frac{k}{2}}\frac{2^{\alpha k+\alpha l}}{2^{(\alpha-\beta)k+k+l}}\|u_kv_l\|_{L^2} \le C\sum_{l>0}\sum_{k\ge l}2^{(\beta-\frac{1}{2})k+(\alpha-1/2)l}\|u_k\|_{L^2}\|v_l\|_{L^2} \\ &\le C\|u\|_{L^2}\|v\|_{L^2}\sum_{l>0}2^{(\alpha+\beta-1)l}\le C\|u\|_{L^2}\|v\|_{L^2} \end{aligned}$$

where we have need $\alpha+\beta<1$ for the sum. Similarly, we estimate the second sum in (2.23),

$$\begin{aligned} I_2 &\le C\sum_{k>0}\sum_{m\le k+2}2^{\frac{1}{2}m}\frac{2^{2\alpha k}}{2^{(\alpha-\beta)m+2k}}||P_m(u_kv_k)||_{L^2} \le C\sum_{k>0}\sum_{m\le k+2}2^{(\frac{1}{2}-\alpha+\beta)m+(2\alpha-2)k}2^{\frac{m}{2}}||u_kv_k||_{L^1} \\ &\le C\|u\|_{L^2}\|v\|_{L^2}\sum_{k>0}2^{(\alpha+\beta-1)k}\le C\|u\|_{L^2}\|v\|_{L^2} \end{aligned}$$

where we also need $\alpha+\beta<1$. □

The next lemma provides a different type of estimate, namely that if we measure $T(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f)$ in averages L^2_{tx} sense, we actually get a full spatial derivative gain. More precisely,

Lemma 4. *Let $u,v\in X^{0,\frac{1}{2}+\delta}_{\tau=\xi^2}$, then for $0\le\delta<1-\alpha-\beta$,*

$$\|T(u,v)\|_{X^{1-\delta,\delta}} \le C_\delta\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}.$$

Proof. By using the partition of unity $\chi^{(+,+,-)}$, we can localize spacial and time frequencies to their respective indices. (Here we localize u, v, uv respectively to N_1, N_2, N_3.) Also denote by the symbol $\sum$ to be the summation over $N_1, N_2, N_3, L_1, L_2, L_3 \geq 1$. We have

$$\begin{aligned}
\|T(u,v)\|_{X^{1-\delta,\delta}} &\leq C \sum \frac{N_3^{1-\alpha+\beta-\delta}}{N_1^{1-\alpha} N_2^{1-\alpha}} \|uv\|_{X^{0,\delta}} \\
&\leq C \sum \frac{N_3^{1-\alpha+\beta-\delta}}{N_1^{1-\alpha} N_2^{1-\alpha}} \sup_{\|w\|_{X^{0,-\delta}_{\tau=-\xi^2}}=1} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} u(t,x) v(t,x) w(t,x) \, dt \, dx \right| \\
&\leq C \sup_{\|w\|_{X^{0,-\delta}_{\tau=-\xi^2}}=1} \sum \frac{N_3^{1-\alpha+\beta-\delta}}{N_1^{1-\alpha} N_2^{1-\alpha}} \|\chi^{(+,+,-)}\|_{\mathscr{M}} \|u\|_{L^2_{t,x}} \|v\|_{L^2_{t,x}} \|w\|_{L^2_{t,x}} \\
&\leq C \|u\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \sum \frac{N_3^{1-\alpha+\beta-\delta} L_3^{\delta}}{N_1^{1-\alpha} N_2^{1-\alpha} L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta}} \|\chi^{(+,+,-)}\|_{\mathscr{M}}
\end{aligned} \tag{2.24}$$

We refer to Proposition 7. If $\max(L_1, L_2) = L_{\max}$, we use that $\max(N_1, N_2) \gtrsim N_3$ to simplify (2.24) and then apply the multiplier bound from Corollary 1. We estimate the sum in (2.24) by

$$\sum \frac{L_3^{\delta} N_3^{\beta-\delta}}{L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta}} L_{\min}^{\frac{1}{2}} N_{\min}^{\frac{1}{2}} \leq C \sum \frac{N_{\min}^{\frac{1}{2}-\delta} N_{\max}^{\beta}}{L_{\max}^{\frac{1}{2}}} \leq C_{\delta}$$

If $L_3 = L_{\max}$ and $N_1 \sim N_2 \sim N_3 \sim N$, then we can assume $L_3 \sim N^2$, so applying (2.5) yields the estimate

$$\sum \frac{N^{1-\alpha+\beta-\delta} L_3^{\delta}}{N^{1-\alpha} N^{1-\alpha} L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta}} L_{\min}^{\frac{1}{2}} L_{\mathrm{med}}^{\frac{1}{4}} \leq C \sum \frac{1}{N^{1-\alpha-\beta-\delta}} \leq C_{\delta},$$

provided $0<\delta<<1-\alpha-\beta$.

The remaining case is when $L_3 = L_{\max} \sim N_1 N_2$ with the bound (2.6). We have

$$\sum \frac{N_3^{1-\alpha+\beta-\delta} L_3^{\delta}}{N_1^{1-\alpha} N_2^{1-\alpha} L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta}} \frac{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}}}{N_{\max}^{\frac{1}{2}}} \leq C \sum N_{\max}^{\beta-\frac{1}{2}+\delta} \leq C.$$

□

2.3 Bilinear and trilinear estimates

Let us start with few words regarding strategy. All the terms (with an exception of one single term) in the right hand-side of (2.22), which contain at least one smooth term (i.e. in the form $u_{\leq 0}$) will be dealt with by relatively simple arguments, mainly based on Lemma 2. For all other terms, we shall need specific (bilinear and trilinear) estimates, which handle different type of configurations (i.e. h and w, h and $e^{-i\partial_x^2} f$) on the right-hand side of (2.22). In view of (1.8) and (1.9), we will show

$$\|G(u_{>0}, v_{>0})\|_{X_T^{\gamma,-\frac{1}{2}+\delta}} \lesssim_{T,\delta} \|u\|_{X^{\gamma,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \tag{2.25}$$

$$\|G(u_{>0}, v_{>0})\|_{X^{\gamma,-\frac{1}{2}+\delta}} \lesssim_{T,\delta} \|u\|_{X^{1-\delta,\delta}} \|v\|_{X^{1-\delta,\delta}}. \tag{2.26}$$

In addition, we would like to have $\|G(u_{>0}, v_{>0})\|_{X^{\gamma,\frac{1}{2}+\delta}} \lesssim_{T,\delta} \|u\|_{X^{1-\delta,\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}}$, but this estimate turns out to be false. On the other hand, note that in our case, this estimate can be replaced with the *tri-linear* estimate

$$\|G(T(u_{>0}, v_{>0}), w_{>0})\|_{X^{\gamma,-\frac{1}{2}+\delta}} \lesssim_{T,\delta} \|u\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}+\delta}}. \tag{2.27}$$

In order to gain the positive power of T, we begin the estimates (2.28), (2.26), (2.27) by

$$\|G(u,v)\|_{X_T^{\gamma,-\frac{1}{2}+\delta}} \lesssim T^{\delta}\|G(u,v)\|_{X^{\gamma,-\frac{1}{2}+2\delta}}$$

by applying Proposition 3.

The next lemma is useful when one deals with terms in the form $G(w_{>0}, e^{-i\partial_x^2} f_{>0})$ and $G(w_{>0}, w_{>0})$ on the right hand side of (2.22).

Lemma 5. *Let* $\alpha \in [1/2, 1)$, $2\alpha - \gamma < \frac{3}{2}$, $\alpha + \beta < 1$ *and* $\gamma + \beta < \frac{1}{2}$. *Given* $u, v \in \mathscr{S}(\mathbf{R}^{1+1})$,

$$\|G(u_{>0}, v_{>0})\|_{X^{\gamma,-\frac{1}{2}+2\delta}} \lesssim_{\delta} \|u\|_{X^{\gamma,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \tag{2.28}$$

for some $0 < \delta \ll 1$.

Remark: The case $\gamma = \beta = 0$ reproduces the result by Kenig, Ponce, Vega in [29], where the authors also show that (2.28) with $\gamma = 0$ is false if $\alpha > 3/4$. However, by letting $\gamma > 0$, the given estimate holds for $\alpha < \frac{3}{4} + \frac{1}{2}\gamma$. Thus, our goal is to achieve $\gamma < \frac{1}{2}$ in order to prove well-posedness upto the sharp index $\alpha < 1$.

Proof. By using the partition of unity $\chi^{(+,+,-)}$, we can localize spatial and time frequencies to their respective indices. (Here we localize u, v, uv respectively to N_1, N_2, N_3.) Also denote by the symbol $\sum$ to be the summation over $N_1, N_2, N_3, L_1, L_2, L_3 \geq 1$.

We obtain

$$\begin{aligned}
\|G(u_{>0}, v_{>0})\|_{X^{\gamma,-\frac{1}{2}+2\delta}_{\tau=\xi^2}} &= \left\| \langle\nabla\rangle^{\gamma+\beta-\alpha} [\langle\nabla\rangle^{\alpha} u \, \langle\nabla\rangle^{\alpha} v] \right\|_{X^{0,-\frac{1}{2}+2\delta}_{\tau=\xi^2}} \\
&= \sup_{\|w\|_{X^{0,\frac{1}{2}-2\delta}_{\tau=-\xi^2}}=1} \left| \int_{\mathbf{R}^{1+1}} \langle\nabla\rangle^{\alpha} u \, \langle\nabla\rangle^{\alpha} v \langle\nabla\rangle^{\gamma+\beta-\alpha} w \, dt \, dx \right|
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\|w\|_{X^{0,\frac{1}{2}-2\delta}_{\tau=-\xi^2}}=1} \left| \int_\Gamma [\langle \xi_1\rangle^\alpha \widetilde{u}]\,[\langle \xi_2\rangle^\alpha \widetilde{v}][\langle \xi_3\rangle^{\gamma+\beta-\alpha}\widetilde{w}]\,d\sigma \right| \\
&\lesssim \sum N_1^\alpha N_2^\alpha N_3^{\gamma+\beta-\alpha} \sup_{\|w\|_{X^{0,\frac{1}{2}-2\delta}_{\tau=-\xi^2}}=1} \left| \int_\Gamma \chi^{(+,+,-)} \widetilde{u_{N_1,L_1}}\,\widetilde{v_{N_2,L_2}}\,\widetilde{w_{N_3,L_3}}\,d\sigma \right| \\
&\lesssim \sum \frac{N_1^{\alpha-\gamma}N_2^\alpha N_3^{\gamma+\beta-\alpha}}{L_1^{\frac{1}{2}+\delta}L_2^{\frac{1}{2}+\delta}L_3^{\frac{1}{2}-2\delta}} \left\|\chi^{(+,+,-)}\right\|_{\mathscr{M}} \left\| L_1^{\frac{1}{2}+\delta} N_1^\gamma \widetilde{u_{N_1,L_1}} \right\|_{L^2_{\tau,\xi}} \left\| L_2^{\frac{1}{2}+\delta} \widetilde{v_{N_2,L_2}} \right\|_{L^2_{\tau,\xi}} \\
&\lesssim \|u\|_{X^{\gamma,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \sum \frac{N_1^{\alpha-\gamma}N_2^\alpha N_3^{\gamma+\beta-\alpha}}{L_1^{\frac{1}{2}+\delta}L_2^{\frac{1}{2}+\delta}L_3^{\frac{1}{2}-2\delta}} \left\|\chi^{(+,+,-)}\right\|_{\mathscr{M}}.
\end{aligned}
$$

Thus, the desired estimate is reduced by showing

$$
\sum \frac{N_1^{\alpha-\gamma}N_2^\alpha N_3^{\gamma+\beta-\alpha}}{L_1^{\frac{1}{2}+\delta}L_2^{\frac{1}{2}+\delta}L_3^{\frac{1}{2}-2\delta}} \left\|\chi^{(+,+,-)}\right\|_{\mathscr{M}} < \infty. \tag{2.29}
$$

Clearly, it suffices to show that the summand above is bounded by a constant muliple of $L_{\max}^{-\delta}$. We will show this below by splitting into cases.

Case 1. If $L_{\max} \sim L_{\text{med}} \gg H \sim N_1N_2$, we apply Corollary 1. Then the summand in (2.29) is estimated by

$$
\frac{N_1^{\alpha-\gamma}N_2^\alpha N_3^{\gamma+\beta-\alpha}}{L_1^{\frac{1}{2}+\delta}L_2^{\frac{1}{2}+\delta}L_3^{\frac{1}{2}-2\delta}} N_{\min}^{\frac{1}{2}} L_{\min}^{\frac{1}{2}} \le \frac{N_1^{\alpha-\gamma+\frac{1}{2}}N_2^\alpha N_3^{\gamma+\beta-\alpha}}{L_{\text{med}}^{\frac{1}{2}+\delta}L_{\max}^{\frac{1}{2}-2\delta}} \lesssim \frac{N_1^{\alpha-\gamma-\frac{1}{2}+2\delta}N_2^{\alpha-1+2\delta}N_3^{\gamma+\beta-\alpha}}{L_{\max}^\delta}.
$$

Note that only N_1 has a potentially positive exponent, so the worst case is when $N_1 = N_{\max}$. Recall $N_{\max} \sim N_{\text{med}}$. If $N_2 = N_{\text{med}}$, then RHS of the given quantity is bounded by $N_{\max}^{2\alpha-\gamma-3/2+4\delta}L_{\max}^{-\delta}$. If $N_3 = N_{\text{med}}$, then it is bounded by $N_{\max}^{\beta-1/2+2\delta}L_{\max}^{-\delta}$. Under the condition on α,β,γ given above, these are both bounded by $L_{\max}^{-\delta}$

Otherwise, we may assume that $L_{\text{med}} \ll L_{\max} \sim N_1N_2$.

Case 2. Assume $\max(L_1,L_2) = L_{\max}$. We recall (2.4) of Proposition 7.

If $L_1 = L_{\max}$, then $\|\chi^{(+,+,-)}\|_{\mathscr{M}} \lesssim \frac{L_2^{\frac{1}{2}} L_3^{\frac{1}{2}}}{N_1^{\frac{1}{2}}}$; if $L_2 = L_{\max}$, then $\|\chi^{(+,+,-)}\|_{\mathscr{M}} \lesssim \frac{L_1^{\frac{1}{2}} L_3^{\frac{1}{2}}}{N_2^{\frac{1}{2}}}$. Both cases will work out similarly, so we only show the former. When $L_1 = L_{\max}$, then the summand in (2.29) is estimated by

$$\frac{N_1^{\alpha-\gamma} N_2^{\alpha} N_3^{\gamma+\beta-\alpha}}{L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta} L_3^{\frac{1}{2}-2\delta}} \frac{L_2^{\frac{1}{2}} L_3^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \leq \frac{N_1^{\alpha-\gamma-\frac{1}{2}} N_2^{\alpha} N_3^{\gamma+\beta-\alpha}}{L_{\max}^{\frac{1}{2}-2\delta}} \lesssim \frac{N_1^{\alpha-\gamma-1+3\delta} N_2^{\alpha-\frac{1}{2}+3\delta} N_3^{\gamma+\beta-\alpha}}{L_{\max}^{\delta}}.$$

Note that the worst case here is if $N_2 = N_{\max}$. We use $N_{\max} \sim N_{\text{med}}$. If $N_1 = N_{\text{med}}$, then above is bounded by $N_{\max}^{2\alpha-\gamma-3/2+6\delta} L_{\max}^{-\delta}$. If $N_3 = N_{\text{med}}$, then above is bounded by $N_{\max}^{\gamma+\beta-1/2+2\delta} L_{\max}^{-\delta}$. With the given conditions on α, β, γ, these are both bounded by $L_{\max}^{-\delta}$.

For the remaining cases, we can assume $L_1, L_2 \ll L_3 \sim N_1 N_2$.

Case 3. If $N_{\max} \sim N_{\min} \sim N$, then we can estimate the summand in (2.29) via (2.5).

$$\frac{N_1^{\alpha-\gamma} N_2^{\alpha} N_3^{\gamma+\beta-\alpha}}{L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta} L_3^{\frac{1}{2}-2\delta}} L_{\min}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{4}} \lesssim \frac{N^{\alpha+\beta}}{L_{\max}^{\frac{1}{2}-2\delta}} \lesssim \frac{N^{\alpha+\beta-1+6\delta}}{L_{\max}^{\delta}}.$$

Since the conditions on α, β, γ imply $\alpha + \beta < 1$, above is bounded by $CL_{\max}^{-\delta}$.

Case 4. The remaining case satisfies condition for the estimate (2.6) of Proposition 7. Thus, we bound the summand in (2.29) by

$$\frac{N_1^{\alpha-\gamma} N_2^{\alpha} N_3^{\gamma+\beta-\alpha}}{L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta} L_3^{\frac{1}{2}-2\delta}} \frac{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}}}{N_{\max}^{\frac{1}{2}}} \leq \frac{N_1^{\alpha-\gamma} N_2^{\alpha-\frac{1}{2}} N_3^{\gamma+\beta-\alpha}}{L_{\max}^{\frac{1}{2}-2\delta}} \lesssim \frac{N_1^{\alpha-\gamma-\frac{1}{2}+3\delta} N_2^{\alpha-1+3\delta} N_3^{\gamma+\beta-\alpha}}{L_{\max}^{\delta}}.$$

By the same arguments as before, the quantity above is bounded by $L_{\max}^{-\delta}$ with given conditions on α, β, γ, we are done. □

The next lemma is useful when dealing with terms of the form $G(h,h)$ in the right side of (2.22).

Lemma 6. *Let $\alpha < \frac{5}{4}$ and $\gamma+\beta \leq \alpha$. Given $u, v \in \mathscr{S}(\mathbf{R}^{1+1})$,*

$$\|G(u_{>0}, v_{>0})\|_{X^{\gamma,-\frac{1}{2}+2\delta}} \lesssim_\delta \|u\|_{X^{1-\delta,\delta}} \|v\|_{X^{1-\delta,\delta}}$$

for some $0 < \delta \ll 1$.

Remark: Due to the smoothness of the normal form component, this estimate is far less restrictive on the indices α, β, γ. Note that the conditions given here are strictly weaker than the conditions given in Lemma 5.

Proof. By using the partition of unity $\chi^{(+,+,-)}$, we can localize spacial and time frequencies to their respective indices. (Here we localize u, v, uv respectively to N_1, N_2, N_3.) Also denote by the symbol $\sum$ to be the summation over $N_1, N_2, N_3, L_1, L_2, L_3 \geq 1$.

Following the computations in the proof of Lemma 5,

$$\begin{aligned}
\|G(u_{>0}, v_{>0})\|_{X^{\gamma,-\frac{1}{2}+2\delta}} &= \sup_{\|w\|_{X^{0,\frac{1}{2}-2\delta}_{\tau=-\xi^2}}=1} \left| \int_{\mathbf{R}^{1+1}} \langle\nabla\rangle^\alpha u \langle\nabla\rangle^\alpha v \langle\nabla\rangle^{\gamma+\beta-\alpha} w \, dt\, dx \right| \\
&\lesssim \sum N_1^\alpha N_2^\alpha N_3^{\gamma+\beta-\alpha} \sup_{\|w\|_{X^{0,\frac{1}{2}-2\delta}_{\tau=-\xi^2}}=1} \left| \int_\Gamma \chi^{(+,+,-)} \widetilde{u_{N_1,L_1}} \widetilde{v_{N_1,L_1}} \widetilde{w_{N_1,L_1}} \, d\sigma \right| \\
&\lesssim \sum \frac{N_1^{\alpha-1+\delta} N_2^{\alpha-1+\delta} N_3^{\gamma+\beta-\alpha}}{L_1^\delta L_2^\delta L_3^{\frac{1}{2}-2\delta}} \left\|\chi^{(+,+,-)}\right\|_{\mathscr{M}} \left\| N_1^{1-\delta} L_1^\delta \widetilde{u_{N_1,L_1}} \right\|_{L^2_{\tau,\xi}} \left\| N_2^{1-\delta} L_2^\delta \widetilde{v_{N_2,L_2}} \right\|_{L^2_{\tau,\xi}} \\
&\leq \|u\|_{X^{1-\delta,\delta}} \|v\|_{X^{1-\delta,\delta}} \sum \frac{N_1^{\alpha-1+\delta} N_2^{\alpha-1+\delta} N_3^{\gamma+\beta-\alpha}}{L_1^\delta L_2^\delta L_3^{\frac{1}{2}-2\delta}} \left\|\chi^{(+,+,-)}\right\|_{\mathscr{M}}. \qquad (2.30)
\end{aligned}$$

We apply Corollary 1 to estimate the summand in (2.30) by

$$\frac{N_1^{\alpha-1+\delta} N_2^{\alpha-1+\delta} N_3^{\gamma+\beta-\alpha}}{L_1^\delta L_2^\delta L_3^{\frac{1}{2}-2\delta}} N_{\min}^{\frac{1}{2}} L_{\min}^{\frac{1}{2}} \leq \frac{N_1^{\alpha-\frac{1}{4}+\delta} N_2^{\alpha-\frac{1}{4}+\delta}}{L_{\max}^{\frac{1}{2}-2\delta}} \lesssim \frac{N_1^{\alpha-\frac{5}{4}+3\delta} N_2^{\alpha-\frac{5}{4}+3\delta}}{L_{\max}^\delta}.$$

Since $\alpha < 5/4$, we are done. $\square$

Our next lemma treats all the terms on the right-hand side of (2.22) in the form $G(h, e^{-it\partial_x^2} f) = G(T(e^{-it\partial_x^2} f, e^{-it\partial_x^2} f), e^{-it\partial_x^2} f)$. Unfortunately, we cannot control such terms only with the *a posteriori* information $h \in \mathscr{H}, e^{-it\partial_x^2} f \in X^{0,\frac{1}{2}+\delta}$. Instead, we must treat the whole expression as a trilinear one, which then yields the desired control.

Lemma 7. *Let $\alpha \in (1/2, 1)$, $\gamma + \beta < \frac{1}{2}$ and $2\alpha + \beta < 2$. Given $u, v \in \mathscr{S}(\mathbf{R}^{1+1})$,*

$$\|G(T(u,v), w_{>0})\|_{X^{\gamma,-\frac{1}{2}+2\delta}} \lesssim_\delta \|u\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}+\delta}}.$$

Proof. First we decompose $T = T_1 + T_2$, where $T_1(u,v) = P_{>0} T(u,v)$ and $T_2(u,v) = P_{\leq 0} T(u,v)$. We first treat the component containing T_1.

Noting $T(u,v) = C\langle\nabla\rangle^{\beta-\alpha} \left[\frac{\langle\nabla\rangle^\alpha}{\nabla} u_{>0} \cdot \frac{\langle\nabla\rangle^\alpha}{\nabla} v_{>0}\right]$, we can write

$$G(T(u,v), w) = C\langle\nabla\rangle^{\beta-\alpha} \left[\langle\nabla\rangle^\beta \left[\frac{\langle\nabla\rangle^\alpha}{\nabla} u_{>0} \frac{\langle\nabla\rangle^\alpha}{\nabla} v_{>0}\right] \cdot \langle\nabla\rangle^\alpha w\right]$$

Applying duality,

$$\begin{aligned}
\|G(T_1(u,v), w_{>0})\|_{X^{\gamma,-\frac{1}{2}+2\delta}} &= C\left\|\langle\nabla\rangle^{\beta-\alpha} \left[\langle\nabla\rangle^\beta \left[\frac{\langle\nabla\rangle^\alpha}{\nabla} u_{>0} \frac{\langle\nabla\rangle^\alpha}{\nabla} v_{>0}\right] \cdot \langle\nabla\rangle^\alpha w_{>0}\right]\right\|_{X^{\gamma,-\frac{1}{2}+2\delta}} \\
&\lesssim \sum_{k,l>0} 2^{\beta k + \alpha l} \sum_{k_1,k_2>0} 2^{(\alpha-1)(k_1+k_2)} \|(u_{k_1} v_{k_2})_k w_l\|_{X^{\gamma+\beta-\alpha,-\frac{1}{2}+2\delta}} \\
&= \sum_{k,l>0} \sum_{k_1,k_2>0} 2^{(\alpha-1)(k_1+k_2)+\beta k+\alpha l} \sup_{\|z\|_{X_{\tau=\xi^2}^{\alpha-\gamma-\beta,\frac{1}{2}-2\delta}}=1} \left|\int_{\mathbf{R}^1\times\mathbf{R}^1} (u_{k_1} v_{k_2})_k w_l \bar{z}\, dt\, dx\right|.
\end{aligned}$$

Note that unless $k - 3 \leq \max(k_1, k_2)$, the integral above vanishes. We split the last sum $\sum_{k,l>0}$ into three parts

$$\sum_{l \leq k-6} \cdot + \sum_{k \leq l-6} \cdot + \sum_{|k-l|<6} \cdot$$

where $k,l>0$. We denote the corresponding terms by $I_1+I_2+I_3$. On each summand, we will apply Lemma 2 to obtain the desired estimate. We need to estimate the integral

$$2^{(\alpha-1)(k_1+k_2)+\beta k+\alpha l}\left|\int_{\mathbf{R}^1\times\mathbf{R}^1}(u_{k_1}v_{k_1})_k w_l\bar{z}\,dt\,dx\right| \tag{2.31}$$

where $z\in X^{\alpha-\gamma-\beta,\frac{1}{2}-2\delta}_{\tau=\xi^2}$.

For I_1, we have high-low interaction between $(u_{k_1}v_{k_2})_k$ and w_l, so $(u_{k_1}v_{k_2})_k w_l = P_{\sim k}[(u_{k_1}v_{k_2})_k w_l]$. Hence it suffices to control

$$2^{(\alpha+\beta)k+(\alpha-1)(k_1+k_2)}\left|\int_{\mathbf{R}^1\times\mathbf{R}^1}(u_{k_1}v_{k_2})_k(w_l\bar{z}_{\sim k})_{\sim k}\,dt\,dx\right|.$$

Thus by the Cauchy-Swartz inequality,

$$(2.31)\lesssim 2^{(\alpha+\beta)k+(\alpha-1)(k_1+k_2)}\|(u_{k_1}v_{k_2})_k\|_{L^2_{t,x}}\|(w_l\bar{z}_{\sim k})_{\sim k}\|_{L^2_{t,x}}.$$

Here we need to consider two cases: first when $2^{k_1}\sim 2^{k_2}\sim 2^k$, and second when this does not take place. In the first case, we apply (2.9) and (2.14)

$$\begin{aligned}
(2.31)&\lesssim 2^{(3\alpha+\beta-2)k}\|(u_{\sim k}v_{\sim k})_k\|_{L^2_{t,x}}\|(w_l\bar{z}_{\sim k})_{\sim k}\|_{L^2_{t,x}}\\
&\lesssim_\delta 2^{(3\alpha+\beta-\frac{5}{2}+10\delta)k}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}\|w\|_{X^{0,\frac{1}{2}+\delta}}\|z_{\sim k}\|_{X^{0,\frac{1}{2}-2\delta}}\\
&\lesssim 2^{(2\alpha+2\beta+\gamma-\frac{5}{2}+10\delta)k}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}\|w\|_{X^{0,\frac{1}{2}+\delta}}\|z\|_{X^{\alpha-\gamma-\beta,\frac{1}{2}-2\delta}}.
\end{aligned}$$

On the other hand, if $2^{k_1}\sim 2^{k_2}\sim 2^k$ does not take place, then we can gain almost $\frac{1}{2}$ derivative from $\|(u_{k_1}v_{k_2})_k\|_{L^2}$ via (2.7) or (2.8). Thus the estimate follows

$$(2.31)\lesssim 2^{(\alpha+\beta)k+(\alpha-1)(k_1+k_2)}\|(u_{k_1}v_{k_2})_k\|_{L^2_{t,x}}\|(w_l\bar{z}_{\sim k})_{\sim k}\|_{L^2_{t,x}}$$

$$\lesssim_\delta 2^{(\alpha+\beta-1+10\delta)k+(\alpha-1)(k_1+k_2)}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}\|w\|_{X^{0,\frac{1}{2}+\delta}}\|z_{\sim k}\|_{X^{0,\frac{1}{2}-2\delta}}$$
$$\lesssim 2^{(\gamma+2\beta-1+10\delta)k+(\alpha-1)(k_1+k_2)}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}\|w\|_{X^{0,\frac{1}{2}+\delta}}\|z\|_{X^{\alpha-\gamma-\beta,\frac{1}{2}-2\delta}}$$

Clearly in both cases, all dyadic sums are finite with the given conditions on α,β,γ.

For I_2, note that now $(u_{k_1}v_{k_2})_k w_l = P_{\sim l}[(u_{k_1}v_{k_2})_k w_l]$ and hence

$$(2.31) \lesssim 2^{(\alpha+\beta)l+(\alpha-1)(k_1+k_2)}\left|\int_{\mathbf{R}^1\times\mathbf{R}^1}(u_{k_1}v_{k_2})_k(w_l\bar{z}_{\sim l})_{\sim k}\,dt\,dx\right|$$

If $2^{k_1}\sim 2^{k_2}\sim 2^l$, then applying Cauchy-Schwartz inequality, (2.8) and (2.12)

$$\begin{aligned}(2.31) &\lesssim 2^{(3\alpha+\beta-2)l}\|(u_{\sim l}v_{\sim l})_k\|_{L^2_{t,x}}\|w_l\bar{z}_{\sim l}\|_{L^2_{t,x}}\\ &\lesssim_\delta 2^{(3\alpha+\beta-\frac{5}{2}+10\delta)l}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}\|w\|_{X^{0,\frac{1}{2}+\delta}}\|z_{\sim l}\|_{X^{0,\frac{1}{2}-2\delta}}\\ &\lesssim 2^{(2\alpha+2\beta+\gamma-\frac{5}{2}+10\delta)l}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}\|w\|_{X^{0,\frac{1}{2}+\delta}}\|z\|_{X^{\alpha-\gamma-\beta,\frac{1}{2}-2\delta}}.\end{aligned}$$

If $2^{k_1}\sim 2^{k_2}\sim 2^l$ does not hold, say $|k_1-l|>6$. Note that if $|k_2-l|\le 3$, then the term $(u_{k_1}v_{k_2})_k$ vanishes. Therefore, we must have $|k_2-l|>3$.

Now, we would like to estimate the last integral by $\|u_{k_1}w_l\|_{L^2_{t,x}}\|v_{k_2}\bar{z}_{\sim l}\|_{L^2_{t,x}}$ which would give us almost a full derivative gain. However, we cannot quite do that, since the integral in (2.31) is not a pointwise product, but rather the operator P_k acting on $u_{k_1}v_{k_2}$, which then is multiplied by $w_l\bar{z}_{\sim l}$.

The following calculation however provides a substitute for this, by Plancherel and triangle inequality

$$\left|\int_{\mathbf{R}^1\times\mathbf{R}^1}(u_{k_1}v_{k_2})_k w_l\bar{z}_{\sim l}\,dt\,dx\right| \le \int_{\mathbf{R}^1\times\mathbf{R}^1}|\widehat{u_{k_1}}*\widehat{v_{k_2}}(\xi)|\varphi(2^{-k}\xi)|[\widehat{w_l}*\widehat{\bar{z}_{\sim l}}](-\xi)|d\xi\,dt$$

$$\leq \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} Q[u_{k_1}]Q[v_{k_2}]Q[w_l]Q[\bar{z}_{\sim l}]dx\,dt \right|,$$

where $Q[h] := \mathscr{F}^{-1}[|\hat{h}|]$. Note that $Q[h]_k = Q[h_k]$ and $\|Q[h]\|_{X^{s,b}} = \|h\|_{X^{s,b}}$, by the definition of $\|\cdot\|_{X^{s,b}}$. In other words, we have managed to remove the Littlewood-Paley operator P_k (and to reduce to an expression as an integral of pointwise product of four functions), at the expense of introducing the operator Q, which does not really affect the $X^{s,b}$ norms of the entries. With that last reduction in mind, we continue our estimation of (2.31).

By (2.7) and either (2.13) or (2.14),

$$\begin{aligned}
(2.31) &\lesssim 2^{(\alpha+\beta)k+(\alpha-1)(k_1+k_2)} \|Q[u_{k_1}]Q[w_l]\|_{L^2_{t,x}} \|Q[v_{k_2}]Q[\bar{z}_{\sim l}]\|_{L^2_{t,x}} \\
&\lesssim_\delta 2^{(\alpha+\beta-1+10\delta)l+(\alpha-1)(k_1+k_2)} \|u\|_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \|z_{\sim l}\|_{X^{0,\frac{1}{2}-2\delta}} \\
&\lesssim 2^{(2\beta+\gamma-1+10\delta)l+(\alpha-1)(k_1+k_2)} \|u\|_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \|z\|_{X^{\alpha-\gamma-\beta,\frac{1}{2}-2\delta}}
\end{aligned}$$

This clearly is summable under the given conditions on α, β, γ.

For I_3, we have

$$(2.31) \lesssim 2^{(\alpha+\beta)k+(\alpha-1)(k_1+k_2)} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} (u_{k_1} v_{k_2})_k (w_{\sim k}\bar{z})_{\sim k}\,dt\,dx \right|.$$

At this point, let us discuss the frequency localization for $\bar{z}$. Clearly,

$$(w_{\sim k}\bar{z})_{\sim k} = (w_{\sim k}\bar{z}_{<k+3})_{\sim k} = (w_{\sim k}\bar{z}_{\sim k})_{\sim k} + (w_{\sim k}\bar{z}_{\ll k})_{\sim k}. \tag{2.32}$$

If $|k_1-k|\geq 3$ or $|k_2-k|\geq 3$, then by (2.7) or (2.8) (applied to $\|(u_{k_1}v_{k_2})_k\|_{L^2_{t,x}}$) and either (2.13) or (2.14) (applied to $\|(w_{\sim k}\bar{z})_{\sim k}\|_{L^2}$)

$$\begin{aligned}(2.31) &\lesssim 2^{(2\alpha+\beta-1)k}\|(u_{k_1}v_{k_2})_k\|_{L^2_{t,x}}(\|(w_{\sim k}\bar{z}_{\sim k})_{\sim k}\|_{L^2}+\|(w_{\sim k}\bar{z}_{\ll k})_{\sim k})\|_{L^2})\\ &\lesssim_\delta 2^{(2\alpha+\beta-2+10\delta)k}\|u\|_{X^{0,\frac12+\delta}}\|v\|_{X^{0,\frac12+\delta}}\|w\|_{X^{0,\frac12+\delta}}\|z\|_{X^{0,\frac12-\delta}}.\end{aligned}$$

Otherwise, $2^{k_1}\sim 2^{k_2}\sim 2^k$. In this case, we first handle the $z_{\sim k}$ term, which is easier due to the gain of $\alpha-\gamma-\beta$ derivatives. Applying Cauchy-Schwartz inequality, (2.9) and (2.13),

$$\begin{aligned}(2.31) &\lesssim 2^{(3\alpha+\beta-2)k}\|(u_{\sim k}v_{\sim k})_k\|_{L^2_{t,x}}\|(w_{\sim k}\bar{z}_{\sim k})_{\sim k}\|_{L^2_{t,x}}\\ &\lesssim_\delta 2^{(3\alpha+\beta-\frac52+10\delta)k}\|u\|_{X^{0,\frac12+\delta}}\|v\|_{X^{0,\frac12+\delta}}\|w\|_{X^{0,\frac12+\delta}}\|z_{\sim k}\|_{X^{0,\frac12-2\delta}}\\ &\lesssim 2^{(2\alpha+2\beta+\gamma-\frac52+10\delta)k}\|u\|_{X^{0,\frac12+\delta}}\|v\|_{X^{0,\frac12+\delta}}\|w\|_{X^{0,\frac12+\delta}}\|z\|_{X^{\alpha-\gamma-\beta,\frac12-2\delta}}.\end{aligned}$$

For the term with $z_{\ll k}$, we need a more refined analysis, which is possible thanks to the estimate (2.16). We can write

$$(u_{\sim k}v_{\sim k})_k=(u^+_{\sim k}v^-_{\sim k})_k+(u^-_{\sim k}v^+_{\sim k})_k+(u^+_{\sim k}v^+_{\sim k})^+_k+(u^-_{\sim k}v^-_{\sim k})^-_k. \tag{2.33}$$

For the first two terms, due to (2.16) and (2.14), we obtain

$$\begin{aligned}(2.31) &\lesssim 2^{(3\alpha+\beta-2)k}\|(u^+_{\sim k}v^-_{\sim k})_k\|_{L^2_{t,x}}\|(w_{\sim k}\bar{z}_{\ll k})_{\sim k}\|_{L^2_{t,x}}\\ &\lesssim_\delta 2^{(3\alpha+\beta-3+10\delta)k}\|f\|_{X^{0,\frac12+\delta}}\|g\|_{X^{0,\frac12+\delta}}\|v\|_{X^{0,\frac12+\delta}}\|w\|_{X^{0,\frac12-2\delta}}.\end{aligned}$$

To deal with the next two terms in (2.33), note that the integral in (2.31) is in the form $\int(u^+_kv^+_k)^+_k(w^-_k\bar{z}_{\ll k})^-_k$ or $\int(u^-_kv^-_k)^-(w^+_k\bar{z}_{\ll k})^+_k$. We estimate the first one, the second one being symmetrically equivalent to the first. We need to once again apply the bounds involving the operator Q.

$$\begin{aligned}
(2.31) &\lesssim 2^{(3\alpha+\beta-2)k}\left|\int_{\mathbf{R}^1\times\mathbf{R}^1}(u^+_{\sim k}v^+_{\sim k})_k(w^-_{\sim k}\bar{z}_{\ll k})^-_{\sim k}\,dt\,dx\right| \\
&\lesssim 2^{(3\alpha+\beta-2)k}\left|\int_{\mathbf{R}^1\times\mathbf{R}^1}Q[u^+_{\sim k}]Q[v^+_{\sim k}]Q[w^-_{\sim k}]Q[\bar{z}_{\ll k}]dt\,dx\right| \\
&\lesssim 2^{(3\alpha+\beta-2)k}\|Q[u^+_{\sim k}]Q[w^-_{\sim k}]\|_{L^2_{t,x}}\|Q[v^+_{\sim k}]Q[\bar{z}_{\ll k}]\|_{L^2_{t,x}} \\
&\lesssim_\delta 2^{(3\alpha+\beta-3+10\delta)k}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}\|w\|_{X^{0,\frac{1}{2}+\delta}}\|z\|_{X^{0,\frac{1}{2}-2\delta}}
\end{aligned}$$

where we have applied (2.16) for $\|Q[f^+_{\sim k}]Q[v^-_{\sim k}]\|_{L^2_{t,x}}$ and (2.14) for $\|Q[g^+_{\sim k}]Q[\overline{w}_{\ll k}]\|_{L^2_{t,x}}$.

Now to prove the same estimate for T_2, it is clear that the sums I_1 and I_3 do not appear (or is finite), since $k<0$ and $l>0$. So we need to regard the sum of type I_2. Note that we can consider $l \gg 1$, otherwise there is nothing to prove. With the restriction $P_{\le 0}(u_{k_1}v_{k_2})$, we must have $|k_1-k_2|\le 3$ since $k_1,k_2>0$.

$$(2.31) \lesssim 2^{\alpha l+(2\alpha-2)k_1}\left|\int_{\mathbf{R}^1\times\mathbf{R}^1}P_{\le 0}[u_{k_1}v_{\sim k_1}]v_l\overline{w}_{\sim l}\,dt\,dx\right|$$

If $|k_1-l|\le 6$, then by (2.8) and (2.15)

$$\begin{aligned}
(2.31) &\lesssim 2^{(3\alpha-2)l}\|(u_{\sim l}v_{\sim l})_{\le 0}\|_{L^2_{t,x}}\|w_l\bar{z}_{\sim l}\|_{L^2_{t,x}} \\
&\lesssim_\delta 2^{(3\alpha+\beta-\frac{5}{2}+10\delta)l}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}\|w\|_{X^{0,\frac{1}{2}+\delta}}\|z_{\sim l}\|_{X^{0,\frac{1}{2}-2\delta}} \\
&\lesssim 2^{(2\alpha+2\beta+\gamma-\frac{5}{2}+10\delta)l}\|u\|_{X^{0,\frac{1}{2}+\delta}}\|v\|_{X^{0,\frac{1}{2}+\delta}}\|w\|_{X^{0,\frac{1}{2}+\delta}}\|z\|_{X^{\alpha-\gamma-\beta,\frac{1}{2}-2\delta}}
\end{aligned}$$

Otherwise, we can assume that $|k_1-l|>6$, thus $|k_2-l|>3$. Then by (2.7) and either (2.13) or (2.14),

$$
\begin{aligned}
(2.31) &\lesssim 2^{\alpha l+(2\alpha-2)k_1} \|Q[u_{k_1}]Q[w_l]\|_{L^2_{t,x}} \|Q[v_{k_2}]Q[\bar{z}_l]\|_{L^2_{t,x}} \\
&\lesssim_\delta 2^{(\alpha-1+10\delta)l-(2\alpha-2)k_1} \|u\|_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \|z_{\sim l}\|_{X^{0,\frac{1}{2}-2\delta}} \\
&\lesssim 2^{(\gamma+\beta-1+10\delta)l-(2\alpha-2)k_1} \|u\|_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \|z\|_{X^{\alpha-\gamma-\beta,\frac{1}{2}-2\delta}}.
\end{aligned}
$$

Both of these cases are summable in $l > 0$, so we are done. □

2.4 Conclusion of the proof of Theorem 1

Now that we have the needed multilinear estimates, we perform a fixed point argument for the solution w of (2.22) in the space $\mathscr{X}$. For simplicity, we group the terms on the right-hand side of (2.22) as follows[3]

$$
\begin{aligned}
\mathscr{N}_1 &= G([e^{-it\partial_x^2} f + w]_{\le 0}, (Id+P_{>0})[e^{-it\partial_x^2} f + h + w]); \\
\mathscr{N}_2 &= G(h_{\le 0}, (Id+P_{>0})[h+w]) = \\
&= G(h_{\le 0}, (Id+P_{>0})[h]) + G(T(e^{-it\partial_x^2} f, e^{-it\partial_x^2} f)_{\le 0}, (Id+P_{>0})[w]); \\
\mathscr{N}_3 &= G(h_{\le 0}, e^{-it\partial_x^2}(Id+P_{>0})f) = G(T(e^{-it\partial_x^2} f, e^{-it\partial_x^2} f)_{\le 0}, e^{-it\partial_x^2}(Id+P_{>0})f); \\
\mathscr{N}_4 &= 2G(e^{-it\partial_x^2} f_{>0}, h_{>0}) = 2G(T(e^{-it\partial_x^2} f, e^{-it\partial_x^2} f)_{>0}, e^{-it\partial_x^2} f_{>0}); \\
\mathscr{N}_5 &= 2G(e^{-it\partial_x^2} f_{>0}, w_{>0}); \\
\mathscr{N}_6 &= G(h_{>0}, h_{>0}); \; \mathscr{N}_7 = G(h_{>0}, w_{>0}) = G(T(e^{-it\partial_x^2} f, e^{-it\partial_x^2} f)_{>0}, w_{>0}); \\
\mathscr{N}_8 &= G(w_{>0}, w_{>0}).
\end{aligned}
$$

[3] Recall that G is a bilinear form

In order to show that (2.22) is locally well-posed in H^γ, recall the heuristics in Section 1.4 which leads to the estimates of form (1.8) and (1.9). If we want to make a contraction argument within a ball of radius $R \sim \|f\|_{L^2}$ centered at $-\varphi(t)e^{-it\partial_x^2}T(f,f)$, we need to have

$$\sum_{j=1}^{8} \|\mathscr{N}_j\|_{X_T^{\gamma,-\frac{1}{2}+\delta}} \lesssim_\delta T^\delta C_R; \tag{2.34}$$

$$\sum_{j=1}^{8} \left\|\mathscr{N}_j^{w_1} - \mathscr{N}_j^{w_2}\right\|_{X_T^{\gamma,-\frac{1}{2}+\delta}} \lesssim_\delta T^\delta C_R' \|w_1 - w_2\|_{X^{\gamma,\frac{1}{2}+\delta}}. \tag{2.35}$$

By Lemma 3, we have $w_0 \in H^{\frac{1}{2}} \hookrightarrow H^\gamma$ when $\gamma \leq 1/2$.

Estimates for $\mathscr{N}_1$

Let us note first that for any two functions u, v,

$G(u_{\leq 0}, v) = \langle\nabla\rangle^{\beta-\alpha}(\langle\nabla\rangle^\alpha u_{\leq 0}\langle\nabla\rangle^\alpha v)$ behaves for all practical purposes like $u_{\leq 0}\langle\nabla\rangle^\beta v$. Thus,

$$\begin{aligned}\|\mathscr{N}_1\|_{X^{\gamma,-\frac{1}{2}+2\delta}} &\lesssim \left\|G([e^{-it\partial_x^2}f + w]_{\leq 0}, (Id + P_{>0})[e^{-it\partial_x^2}f + h + w])\right\|_{L_t^2 H_x^\gamma} \\ &\lesssim \left\|[e^{-it\partial_x^2}f + w]_{\leq 0} \cdot \langle\nabla\rangle^\beta (Id + P_{>0})[e^{-it\partial_x^2}f + h + w]\right\|_{L_t^2 H_x^\gamma}.\end{aligned}$$

By Hölder's inequality, we have

$$\|[e^{-it\partial_x^2}f + w]_{\leq 0} \cdot \langle\nabla\rangle^\beta (Id + P_{>0})[h]\|_{L_t^2 H_x^\gamma} \leq \|[e^{-it\partial_x^2}f + w]_{\leq 0}\|_{L_{t,x}^\infty} \|h\|_{L_t^2 H_x^{\gamma+\beta}}.$$

By the definition of $\mathscr{H}$ however, $\mathscr{H} \hookrightarrow X^{1-\delta,\delta} \hookrightarrow L_t^2 H_x^{\gamma+\beta}$ when $\gamma + \beta < 1$. Thus $\|h\|_{L_t^2 H_x^{\gamma+\beta}} \leq C\|h\|_{\mathscr{H}}$ and by Sobolev embedding and Remark 1

$$\left\|[e^{-it\partial_x^2}f + w]_{\leq 0}\right\|_{L_{t,x}^\infty} \lesssim \left\|e^{-it\partial_x^2}f + w\right\|_{L_t^\infty L_x^2} \lesssim_\delta \left\|e^{-it\partial_x^2}f + w\right\|_{X_{\tau=\xi^2}^{0,\frac{1}{2}+\delta}}$$

Regarding the remaining term in $\mathscr{N}_1$, we can again split in two terms $(Id+P_{>0})[e^{-it\partial_x^2}f+w]=[e^{-it\partial_x^2}f+w]_{\leq 0}+2[e^{-it\partial_x^2}f+w]_{>0}$. The low frequency term is easy to deal with by Sobolev embedding, whereas for the high-frequency term, we have by Lemma 2 (more specifically (2.7)) and given $\gamma+\beta<1/2$,

$$\begin{aligned}
&\left\|[e^{-it\partial_x^2}f+w]_{\leq 0}\cdot\langle\nabla\rangle^{\beta}[e^{-it\partial_x^2}f+w]_{>0}\right\|_{L_t^2H_x^{\gamma}}\\
&\lesssim_{\delta}\|[e^{-it\partial_x^2}f+w]_{\leq 0}\|_{X^{0,\frac{1}{2}+\delta}}\sum_{k>0}2^{-(\frac{1}{2}-\delta)k}2^{(\gamma+\beta)k}\left\|[e^{-it\partial_x^2}f+w]_k\right\|_{X^{0,\frac{1}{2}+\delta}}\\
&\lesssim_{\delta}(\|f\|_{L^2}+\|w\|_{X^{0,\frac{1}{2}+\delta}})^2.
\end{aligned}$$

Estimates for $\mathscr{N}_2$

Write $\mathscr{N}_2=\mathscr{N}_2^1+\mathscr{N}_2^2$, where $\mathscr{N}_2^1$ is the solution corresponding from the first term in $\mathscr{N}_2$. Then,

$$\begin{aligned}
\|\mathscr{N}_2^1\|_{X^{\gamma,-\frac{1}{2}+2\delta}}&\lesssim\|G(h_{\leq 0},(Id+P_{>0})[h])\|_{L_t^2H_x^{\gamma}}\leq C\|h_{\leq 0}\|_{L_{t,x}^{\infty}}\|h\|_{L_t^2H_x^{\gamma}}\\
&\leq C\|h_{\leq 0}\|_{L_t^{\infty}H_x^{\frac{1}{2}}}\|h\|_{\mathscr{H}}\leq C\|h\|_{\mathscr{H}}^2.
\end{aligned}$$

since $\mathscr{H}\hookrightarrow L_t^2H_x^{\gamma}\cap L_t^{\infty}H_x^{\frac{1}{2}}$.

As far as $\mathscr{N}_2^2$ is concerned, we apply Lemma 7, which yields

$$\|\mathscr{N}_2^2\|_{X^{\gamma,-\frac{1}{2}+2\delta}}\lesssim_{\delta}\|e^{it\partial_x^2}f\|_{X^{0,\frac{1}{2}+\delta}}^2\|(Id+P_{>0})w\|_{X^{0,\frac{1}{2}+\delta}}\leq C\|f\|_{L^2}^2\|w\|_{\mathscr{X}}.$$

Estimates for $\mathscr{N}_3$

The estimate for $\mathscr{N}_3$ is pretty similar to the one for $\mathscr{N}_2^2$. Also, the low frequency term $G(h_{\leq 0},e^{it\partial_x^2}f_{\leq 0})$ is easily estimated through Sobolev embedding, so here we estimate only the non-linearity $G(h_{\leq 0}, e^{it\partial_x^2} f_{>0})$. By Lemma 7,

$$\|\mathscr{N}_3\|_{X^{\gamma,-\frac{1}{2}+2\delta}} \lesssim_\delta \|e^{-it\partial_x^2} f\|^2_{X^{0,\frac{1}{2}+\delta}} \|e^{-it\partial_x^2} f\|_{X^{0,\frac{1}{2}+\delta}} \lesssim \|f\|^3_{L^2}.$$

Estimates for $\mathscr{N}_4$

We have by Lemma 7 that

$$\|\mathscr{N}_4\|_{X^{\gamma,-\frac{1}{2}+2\delta}} \lesssim_\delta \|e^{-it\partial_x^2} f_{>0}\|^3_{X^{0,\frac{1}{2}+\delta}} \lesssim \|f\|^3_{L^2}.$$

Estimates for $\mathscr{N}_5$

For $\mathscr{N}_5$, we apply Lemma 5, whence

$$\|\mathscr{N}_5\|_{X^{\gamma,-\frac{1}{2}+2\delta}} \lesssim_\delta \|w\|_{X^{\gamma,\frac{1}{2}+\delta}} \|e^{-it\partial_x^2} f\|_{X^{0,\frac{1}{2}+\delta}} \lesssim \|w\|_{\mathscr{X}} \|f\|_{L^2}.$$

Estimates for $\mathscr{N}_6$

The estimate for $\mathscr{N}_6$ follows form Lemma 6,

$$\|\mathscr{N}_6\|_{X^{\gamma,-\frac{1}{2}+2\delta}} \lesssim_\delta \|h\|^2_{X^{1-\delta,\delta}} \lesssim \|h\|^2_{\mathscr{H}}.$$

Estimates for $\mathscr{N}_7$

This term is in fact simpler than $\mathscr{N}_4$ (since w in the second component is smoother than the free solution in $\mathscr{N}_4$). We deal with it in the same way. Namely, by Lemma 7, we have

$$\|\mathscr{N}_7\|_{X^{\gamma,-\frac{1}{2}+2\delta}} \lesssim_\delta \|e^{-it\partial_x^2} f\|^2_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}+\delta}} \lesssim \|f\|^2_{L^2} \|w\|_{\mathscr{X}}.$$

Estimates for $\mathscr{N}_8$

Finally, the estimate for $\mathscr{N}_8$ follows from Lemma 5. We have

$$\|\mathscr{N}_8\|_{X^{\gamma,-\frac{1}{2}+2\delta}} \lesssim_\delta \|w\|_{X^{\gamma,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}+\delta}} \lesssim \|w\|_{\mathscr{X}}^2$$

Using the estimates for $\mathscr{N}_j$, $j = 1, \cdots, 8$, we conclude

$$\sum_{j=1}^{8} \|\mathscr{N}_j\|_{X_T^{\gamma,-\frac{1}{2}+2\delta}} \lesssim_\delta (\|f\|_{L^2} + \|h\|_{\mathscr{H}} + \|w\|_{\mathscr{X}})^2 (1 + \|f\|_{L^2} + \|w\|_{\mathscr{X}}).$$

By Lemma 3 and Lemma 4, note $\|h\|_{\mathscr{H}} \lesssim \|f\|_{L^2}^2 \sim R^2$ and $\|w\|_{\mathscr{X}} \lesssim \|f\|^2 + R \sim R + R^2$. Thus applying Proposition 3, we obtain (2.34) with $C_R \sim (R+R^2)^2(1+R+R^2)$. (2.35) follows via similar estimates.

2.5 Regarding non-linearities of the form $\langle\nabla\rangle^\beta[u\bar{u}]$ and $\langle\nabla\rangle^\beta[\bar{u}^2]$

We will just briefly sketch the analysis that one needs to undertake, in order to pursue well-posedness of the problem

$$u_t + iu_{xx} = \langle\nabla\rangle^\beta[u\bar{u}].$$

As a byproduct of this discussion, we will hopefully be able to shed some light on the issue with low regularity, which is present in this particular case, [35].

Following the ideas of Section 2.2, we need to construct T, so that (2.21) is satisfied, where of course $G(u,v) = \langle\nabla\rangle^{\beta-\alpha}[\langle\nabla\rangle^\alpha v \langle\nabla\rangle^\alpha \bar{v}]$. It is easy to see that the needed T is in the form

$$T(u,v)(x) = \frac{1}{8\pi^2 i}\int \frac{\langle\xi\rangle^\alpha\langle\eta\rangle^\alpha}{\langle\xi+\eta\rangle^{\alpha-\beta}}\frac{1}{\xi(\xi+\eta)}\widehat{u_{>0}}(\xi)\widehat{v}(\eta)e^{i(\xi+\eta)x}d\xi d\eta. \tag{2.36}$$

Note that this transformation may be performed only when the output function $T(u,v)$ is Fourier localized, so that its frequency satisfies $\gtrsim 1$, so that we do not run into trouble with the term $(\xi+\eta)^{-1}$ inside the symbol of T. This is the reason why, in general (and unless we impose some homogeneous Sobolev norms in small frequencies, as is done in [35]), we cannot do better than $H^{-\frac{1}{4}+}$ for the local well-posedness result.

It is also clear from the form (2.36), that in the case of "high-high" interactions, the (generally smoothing) term $(\xi+\eta)^{-1}$ is not of much help to achieve better smoothness of $T(u,v)$. Therefore, performing this transformation would be advantageous, only if $2\alpha < 1$. This is a simple (if a little naive) way to see the optimality of restriction $\alpha < 1/2$ in the results of [35].

For the nonlinearities of the form $\langle\nabla\rangle^\beta[\bar{u}^2]$, following the same ideas, we come up with the following normal form

$$T(u,v)(x) = \frac{1}{8\pi^2 i}\int \frac{\langle\xi\rangle^\alpha\langle\eta\rangle^\alpha}{\langle\xi+\eta\rangle^{\alpha-\beta}}\frac{1}{2(\xi^2+\eta^2+\xi\eta)}\widehat{u}(\xi)\widehat{v}(\eta)e^{i(\xi+\eta)x}d\xi d\eta. \tag{2.37}$$

Clearly, this normal form gains a derivative in each variable (very similar to the case $\langle\nabla\rangle^\beta[u^2]$ and hence, one may expect to get an identical result to Theorem 1 for this nonlinearity as well.

Chapter 3

Smoothing effects of the Korteweg-de Vries equation on T

Although the multilinear L^2 convolution estimates are applicable in the periodic case, it is not as efficient as in the real-line scenario. This is easily seen by noting that Fourier-side of the periodic space follows the counting measure. Therefore we cannot expect much gain by estimating the size of such measures, where we have gained in most cases $O\left(1/\sqrt{N_{\max}}\right)$ by these types of estimates in Chapter 2.

As an alternative, we take advantage of Strichartz estimates in the periodic setting, which are often obtained through number theoretic approach. These arguments are not within the scope of this dissertation, and we only refer to appropriate articles for the statements. In [9], Bourgain proved the following Strichartz estimates for the periodic Airy equation

$$\|u\|_{L^4_{t,x}(\mathbf{R}\times\mathbf{T})} \lesssim \|u\|_{X^{0,\frac{3}{8}}_{\tau=\xi^3}} \tag{3.1}$$

$$\|u\|_{L^6_{t,x}(\mathbf{R}\times\mathbf{T})} \lesssim_{\varepsilon,\delta} \|u\|_{X^{\varepsilon,\frac{1}{2}+\delta}_{\tau=\xi^3}} \tag{3.2}$$

for $\varepsilon, \delta > 0$ small. To minimize the number of parameters, we will let $\varepsilon = \delta > 0$ in the sequel.

In order to well-define the *resonance* arising from the normal form method, we need to assume that the solutions of (1.21) have the spatial mean-zero property. To this end, we define a closed subspace $Y^{s,b}$ of $X^{s,b}$ (with the same norm) as the image of orthogonal projection $\mathbf{P} : X^{s,b} \to Y^{s,b}$ defined by $\mathbf{P}(u)(x) := u(x) - \frac{1}{2\pi}\int_{\mathbf{T}} u\,dx$.

Fortunately, all smooth solutions to the periodic KdV equation (1.21) satisfy mean conservation (also known as "momentum conservation"), i.e.

$$\int_{\mathbf{T}} u(t,x)\,dx = \int_{\mathbf{T}} u(0,x)\,dx \qquad \text{for } \forall t \in \mathbf{R}.$$

If one assumes $u_0 \in Y^{s,b}$, the solution will be *a priori* remain in $Y^{s,b}$ due to the mean conservation. Thus we can justify the contraction argument based on the new mean-zero space, instead of $X^{s,b}$.

Furthermore, if $\frac{1}{2\pi}\int_{\mathbf{T}} u_0\,dx = M \neq 0$, a change of variable $v(t) := u(t) - M$ gives

$$\left| \begin{array}{l} v_t + v_{xxx} + 6Mv_x = 6vv_x \\ v(0) = v_0 \end{array} \right. \tag{3.3}$$

where $v_0 = u_0 - M$ clearly has mean-zero. It is possible to generalize the result obtained with mean-zero initial data via (3.3). See [9, 28, 45, 13, 1, 15] for this type of generalizations.

We briefly review the main result of [29] for the periodic KdV. Kenig, Ponce, Vega proved the bilinear estimate

$$\|\partial_x(uv)\|_{Y^{-s,-\frac{1}{2}}} \lesssim_{s,\delta} \|u\|_{Y^{-s,\frac{1}{2}}} \|v\|_{Y^{-s,\frac{1}{2}}} \tag{3.4}$$

for $0 \leq s < 1/2$ and $\delta > 0$ small. Since we are concerned here with local-in-time solution, we characterize the solution u of (1.21) over the time interval $[0,T]$ by the identity,

$$u = \eta(t)e^{-t\partial_x^3}u_0 + \eta(\frac{t}{T})\int_0^t e^{-(t-s)\partial_x^3}\partial_x(u^2)(s)\,ds. \tag{3.5}$$

Then the standard contraction argument gives that there exists $\alpha > 0$ large so that, for $T \sim \|u_0\|_{H^{-s}}^{-\alpha}$, the contraction argument will hold on a *small* ball in $Y^{-s,\frac{1}{2}}$ centered at $\eta(t)e^{-t\partial_x^3}u_0$. In particular, we have $\mathbf{P}u = u$ and $\|u\|_{Y^{-s,\frac{1}{2}}} \sim \|u_0\|_{H^{-s}}$.

This bilinear estimate is not ideal for local well-posedness theories, since it fails to guarantee that the solution v lives in $C_t^0 H_x^{-s}$. In this regard, the authors show in [29, Lemma 6.1] that (3.4) is false when $Y^{-s,\frac{1}{2}}$ is replaced by $Y^{-s,\frac{1}{2}+\delta}$ for any $\delta \neq 0$. Nonetheless, (3.4) does show that when the solution is properly controlled within $Y^{-s,\frac{1}{2}}$.

In [13], an improvement was made by introducing $Y^{-s,\frac{1}{2}} \cap L_x^2 L_\tau^1$. This norm controls $C_t^0 H_x^{-s}$ norm so that the contraction argument holds within this auxiliary space. Consequently, the local solution u of (3.5) is controlled in terms of $\|u_0\|_{H_x^{-s}}$ in the sense $\|u\|_{C_t^0([0,T];H_x^{-s}} \sim \|u_0\|_{H_x^{-s}}$.

Finally, if we assume u to be real-valued, then we have $\widehat{u}(-\xi) = \overline{\widehat{u}}(\xi)$. We will assume this relation in our proof.

3.1 Normal form transformation and resonance

Let u be the local-in-time solution of (1.21). Setting $v := \langle\nabla\rangle^s u$, we have

$$v_t + v_{xxx} = \mathcal{N}(v,v), \qquad v(0) = f \in L^2(\mathbf{T}) \tag{3.6}$$

where we assume $\int_{\mathbf{T}} f\,dx = 0$ and $\mathscr{N}(u,v) := \partial_x \langle\nabla\rangle^{-s}[\langle\nabla\rangle^s u \langle\nabla\rangle^s v]$. In particular, the bilinear operator $\mathscr{N}$ contains a spatial derivative, thus $\mathscr{N} \equiv \mathbf{P}\circ\mathscr{N}$ itself has mean-zero.

We construct the bilinear pseudo-differential operator T by the formula

$$T(u,v) := \sum_{\xi_1\xi_2(\xi_1+\xi_2)\neq 0} \frac{\langle\xi_1\rangle^s\langle\xi_2\rangle^s}{\langle\xi_1+\xi_2\rangle^s}\frac{1}{\xi_1\xi_2}\widehat{u}(\xi_1)\widehat{v}(\xi_2)e^{i(\xi_1+\xi_2)x}.$$

Then the Airy operator acts on T in the following manner:

$$(\partial_t+\partial_{xxx})T(u,v) = T((\partial_t+\partial_{xxx})u,v)+T(u,(\partial_t+\partial_{xxx})v)+\mathscr{N}(\mathbf{P}u,\mathbf{P}v).$$

If we write $h = T(v,v)$ where v solves (3.6) (recall $v = \mathbf{P}v$) and change variable by $v = h+z$, then z satisfies

$$\left|\begin{array}{l}(\partial_t+\partial_{xxx})z = -2T(\mathscr{N}(v,v),v);\\ z(0) = f - T(f,f).\end{array}\right. \tag{3.7}$$

For the right side of (3.7), we note that

$$T(\mathscr{N}(v,v),v) = \mathbf{P}\langle\nabla\rangle^{-s}(\mathbf{P}[\langle\nabla\rangle^s v\langle\nabla\rangle^s v]\frac{\langle\nabla\rangle^s}{\nabla}v).$$

We adapt the computations in [38] to simplify Fourier coefficients of the above expression as follows. For $\xi\neq 0$ (recall $\widehat{v}(0)=0$ and $\widehat{v}(-\xi)=\overline{\widehat{v}}(\xi)$),

$$\mathscr{F}[\mathbf{P}\langle\nabla\rangle^{-s}(\mathbf{P}[\langle\nabla\rangle^s v\langle\nabla\rangle^s v]\frac{\langle\nabla\rangle^s}{\nabla}v)](\xi) = \sum_{\substack{\xi_1+\xi_2\neq 0,\quad \xi_3\neq 0\\ \xi_1+\xi_2+\xi_3=\xi}} \frac{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}{i\xi_3\langle\xi\rangle^s}\widehat{v}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3)$$

$$= \sum_{\substack{(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_3+\xi_1)\neq 0\\ \xi_1+\xi_2+\xi_3=\xi,\quad \xi_3\neq 0}} \frac{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}{i\xi_3\langle\xi\rangle^s}\widehat{v}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3) + \frac{\langle\xi\rangle^{2s}}{-i\xi}\widehat{v}(\xi)\widehat{v}(\xi)\widehat{v}(-\xi)$$

$$+ \sum_{\xi_3\neq 0}\frac{\langle\xi\rangle^s\langle\xi_3\rangle^{2s}\langle\xi\rangle^s}{i\xi_3\langle\xi\rangle^s}\widehat{v}(-\xi_3)\widehat{v}(\xi)\widehat{v}(\xi_3) + \sum_{\xi_3\neq 0}\frac{\langle\xi\rangle^s\langle\xi_3\rangle^{2s}}{i\xi_3\langle\xi\rangle^s}\widehat{v}(\xi)\widehat{v}(-\xi_3)\widehat{v}(\xi_3)$$

$$= \sum_{\substack{(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_3+\xi_1)\neq 0\\ \xi_1+\xi_2+\xi_3=\xi,\quad \xi_3\neq 0}} \frac{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}{i\xi_3\langle\xi\rangle^s}\widehat{v}(\xi_1)\widehat{v}(\xi_2)\widehat{v}(\xi_3) - \frac{\langle\xi\rangle^{2s}}{i\xi}|\widehat{v}|^2(\xi)\widehat{v}(\xi).$$

We say that the first term on the right side of above is *non-resonant* and denoted $\mathscr{N}\mathscr{R}(\xi)$, and the second one is *resonant* and denoted $\mathscr{R}(\xi)$. Then we can rewrite (3.7) as

$$(\partial_t + \partial_{xxx})z = -2[\mathscr{F}_\xi^{-1}(\mathscr{N}\mathscr{R}) + \mathscr{F}_\xi^{-1}\mathscr{R}].$$

To deal with the *resonant* term, we construct a solution for

$$\begin{cases} (\partial_t+\partial_{xxx})v_* = -2\sum_{\xi\neq 0}\frac{\langle\xi\rangle^{2s}}{i\xi}|\widehat{v_*}(\xi)|^2\widehat{v_*}(\xi)e^{i\xi x} \\ v_*(0) = f\in L^2(\mathbf{T}). \end{cases}$$

We accomplish this by constructing a solution through the map $R : L^2 \to C_t^0L_x^2$ defined by

$$R[f](t,x) := \sum_{\xi\neq 0}\widehat{f}(\xi)e^{2i\frac{\langle\xi\rangle^{2s}}{\xi}|\widehat{f}(\xi)|^2 t}e^{i(\xi x+\xi^3 t)}. \tag{3.8}$$

We remark that $R^*[u_0]$ in the statement of Theorem 2 corresponds to $\langle\nabla\rangle^s R[\langle\nabla\rangle^{-s}u_0]$. For constructions and properties of such solution maps, refer to [49, Exercise 4.20-21].

In the next lemma, we compare $R^*[u_0]$ defined above against the linear solution $e^{-t\partial_x^3}u_0$. This is used to draw the corollary on the implications for non-linear smoothing.

Lemma 8. *Let $u_0 \in H^{-s}(\mathbf{T})$ for $0 \le s < 1/2$ with $\widehat{u_0} = 0$. Then for $s_0 \le 1-3s$,*

$$\left\| R^*[u_0] - e^{-t\partial_x^3} u_0 \right\|_{C_t^0 H^{s_0}} \lesssim_T \|u_0\|_{H^s}^3$$

Proof. Let $f := \langle \nabla \rangle^{-s} u_0 \in L^2(\mathbf{T})$. Since $\langle \nabla \rangle^s$ and $e^{-t\partial_x^3}$ commute, it suffices to show

$$\sup_{t\in[0,T]} \left\| R[f](t) - e^{-t\partial_x^3} f \right\|_{H_x^{s_0+s}} \lesssim_T \|f\|_{L^2}^3. \tag{3.9}$$

From (3.8), we have by mean-value theorem

$$\begin{aligned} \left\| R[f](t) - e^{-t\partial_x^3} f \right\|_{H_x^{s_0+s}} &\sim \left\| \langle \xi \rangle^{s_0+s} \widehat{f}(\xi) \left(e^{2i\frac{\langle \xi \rangle^{2s}}{\xi} |\widehat{f}(\xi)|^2 t} - 1 \right) \right\|_{l_\xi^2(\mathbb{Z}\setminus\{0\})} \\ &\lesssim \left\| \widehat{f}(\xi) \frac{\langle \xi \rangle^{3s+s_0}}{\xi} |\widehat{f}(\xi)|^2 t_\xi \right\|_{l_\xi^2(\mathbb{Z}\setminus\{0\})} \lesssim_T \|\widehat{f}\|_{l^2} \|\widehat{f}\|_{l_\xi^\infty}^2 \end{aligned}$$

since $3s + s_0 < 1$. Noting $\|\widehat{f}\|_{l^2} + \|\widehat{f}\|_{l_\xi^\infty} \lesssim \|f\|_{L^2}$, we are done. □

The next lemma guarantees that $R[f] \in X_T^{0,\frac{1}{2}+\delta} \subset C_t^0 L_x^2$, thus very close to the free solution $e^{-t\partial_x^3}$.

Lemma 9. *Given $f \in L^2$, $s \le \frac{1}{2}$, $b \ge 0$ and $\eta \in \mathscr{S}_t(\mathbf{R})$, we have*

$$\|\eta R[f]\|_{X^{0,b}} \lesssim \|\eta\|_{H^b} \max\left(\|f\|_{L^2}, \|f\|_{L^2}^{2b+1} \right)$$

Proof. From (3.8), it is easy to deduce

$$\widetilde{\eta \cdot R[f]}(\tau,\xi) = \widehat{f}(\xi)\widehat{\eta}\left(\tau - 2\frac{\langle\xi\rangle^{2s}}{\xi}|\widehat{f}(\xi)|^2 - \xi^3\right).$$

Let $a_\xi := 2\frac{\langle\xi\rangle^{2s}}{\xi}|\widehat{f}(\xi)|^2$, then

$$\begin{aligned}\|\eta R[f]\|_{X^{0,b}} &= \left\|\langle\tau-\xi^3\rangle^b\widehat{f}(\xi)\widehat{\eta}(\tau-a_\xi-\xi^3)\right\|_{L^2_\tau l^2_\xi} \\ &\lesssim \left\|\langle\tau-a_\xi-\xi^3\rangle^b\langle a_\xi\rangle^b\widehat{f}(\xi)\widehat{\eta}(\tau-a_\xi-\xi^3)\right\|_{l^2_\xi L^2_\tau} \\ &\lesssim \left\|\langle\tau\rangle^b\widehat{\eta}(\tau)\right\|_{L^2_\tau}\left\|\langle a_\xi\rangle^b\widehat{f}(\xi)\right\|_{l^2_\xi}.\end{aligned}$$

Noting that $\sup_\xi |a_\xi| \lesssim \|f\|_{L^2}$, we have the desired estimate. □

We perform another change of variable $z = R[f] + w$ to obtain the equation for w (recall that now $v = R[f] + h + w$),

$$\left|\begin{array}{l}(\partial_t + \partial_{xxx})w = -2\mathscr{F}_\xi^{-1}(\mathscr{N}\mathscr{R}) + 2\sum_{\xi\neq 0}\frac{\langle\xi\rangle^{2s}}{i\xi}B(\widehat{R[f]}(\xi),\widehat{h}(\xi),\widehat{w}(\xi))e^{i\xi x} \\ w(0) = -T(f,f) \in H^1(\mathbf{T})\end{array}\right. \tag{3.10}$$

where $B(x,y,z) := |x+y+z|^2(y+z) + x|y+z|^2 + x^2(\overline{y+z}) + |x|^2(y+z)$ for $x,y,z \in \mathbf{C}$. Heuristically $R[f]$ is the least smooth term among the three, so it is to our benefit that the particular tri-linear form in (3.10) excludes the Fourier coefficients $\left|\widehat{R[f]}\right|^2\widehat{R[f]}$.

3.2 Estimates on resonant and non-resonant forms

In this section, we establish necessary estimates for the contraction argument of (3.10) in $Y^{\frac{1}{2},\frac{1}{2}+\delta}$. First, we examine mapping properties of T. The following lemma gives that $T(f,f) \in H^1(\mathbf{T})$ and also $h \in L^\infty_t H^1_x$ since we know $v \in C^0_t([0,T];L^2(\mathbf{T}))$ from [13].

Lemma 10. *$T : L^2(\mathbf{T}) \times L^2(\mathbf{T}) \to H^1(\mathbf{T})$ is a bounded bilinear operator.*

Proof. Let $u, v \in C^\infty(\mathbf{T})$. Then

$$\|T(u,v)\|_{H^1} \sim \left\| \sum_{\xi_1(\xi-\xi_1)\neq 0} \frac{\langle \xi_1 \rangle^s \langle \xi - \xi_1 \rangle^s \langle \xi \rangle^{1-s}}{\xi_1(\xi-\xi_1)} \widehat{u}(\xi_1)\widehat{v}(\xi-\xi_1) \right\|_{l^2_\xi(\mathbf{Z}\setminus\{0\})}. \quad (3.11)$$

By symmetry, we can assume $|\xi_1| \geq |\xi - \xi_1|$. Then by Hölder and Sobolev embedding,

$$\begin{aligned}(3.11) &\lesssim M \| \sum_{\xi_1 \neq \xi} |\widehat{u}|(\xi_1) \frac{|\widehat{v}|(\xi-\xi_1)}{|\xi-\xi_1|^{\frac{1}{2}+\varepsilon}} \|_{l^2_\xi(\mathbf{Z})} \\ &\lesssim M \| \mathscr{F}^{-1}[|\widehat{u}|] |\partial_x|^{-\frac{1}{2}-\varepsilon} \mathscr{F}^{-1}[|\widehat{v}|] \|_{L^2_x(\mathbf{T})} \lesssim_\varepsilon M \|u\|_{L^2(\mathbf{T})} \|v\|_{L^2_x(\mathbf{T})}\end{aligned}$$

where

$$M := \sup_{\xi\xi_1(\xi-\xi_1)\neq 0} \frac{\langle \xi_1 \rangle^s \langle \xi - \xi_1 \rangle^s \langle \xi \rangle^{1-s}}{|\xi_1| |\xi - \xi_1|^{\frac{1}{2}-\varepsilon}}.$$

It is easy to see that M is a bounded quantity if $s < 1/2$, thus the claim follows. □

We derive the necessary estimates for the non-resonant term in the next lemma.

Lemma 11. *For $v \in Y^{0,\frac{1}{2}}$, $0 \leq s < 1/2$ and $\gamma \leq 1-s$, we have*

$$\|\mathscr{F}_\xi^{-1}(\mathscr{N}\mathscr{R})\|_{Y_T^{\gamma,-\frac{1}{2}+\delta}} \lesssim_{\delta,T} \|v\|^3_{Y^{0,\frac{1}{2}}}.$$

Proof. Note that for all the terms above, $\|\cdot\|_{X^{s,b}} = \|\cdot\|_{Y^{s,b}}$. Thus it will suffice to show the desired estimate with respect to the $X^{s,b}$ norm.

For this trilinear estimate, we use the embedding (3.2). First we localize each variable in terms of its dispersive frequencies, i.e. $\langle \tau_j - \xi_j^3 \rangle \sim L_j$ for $j = 1,2,3$ and $\langle \tau - \xi^3 \rangle \sim L$, where $L, L_j \gtrsim 1$ are dyadic indices. We only need to insure that the final estimate includes $L_{\max}^{-\varepsilon}$ for some $\varepsilon > 0$ so that sum in these indices (and also gain a small positive power of T).

First consider the case when $L \gg \max(L_1, L_2, L_3)$. From the identity $\sum_{j=1}^{3}(\tau_j - \xi_j^3) = (\tau - \xi^3) + 3(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_3+\xi_1)$ for every $\sum_{j=1}^{3}\xi_j = \xi$ and $\sum_{j=1}^{3}\tau_j = \tau$, we can deduce that $L \sim |\xi_1+\xi_2||\xi_2+\xi_3||\xi_3+\xi_1| \geq 1$.

For fixed L, L_1, L_2, L_3, apply Plancherel and Hölder, followed by (3.2) to obtain

$$\left\| \sum_{\substack{(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_3+\xi_1)\neq 0 \\ \xi_1+\xi_2+\xi_3=\xi, \quad \xi_3\neq 0}} \frac{\langle \xi_1\rangle^s \langle \xi_2\rangle^s \langle \xi_3\rangle^s \langle \xi\rangle^{\gamma-s}}{i\xi_3 \langle \tau - \xi^3\rangle^{\frac{1}{2}-\delta}} [\widetilde{v}(\xi_1) *_\tau \widetilde{v}(\xi_2) *_\tau \widetilde{v}(\xi_3)](\tau) \right\|_{L_\tau^2 l_\xi^2(\mathbf{Z}\setminus\{0\})}$$

$$\lesssim_\delta M' \|\widetilde{v_{-\delta}} * (\widetilde{v_{-\delta}} * \widetilde{v_{-\delta}})\|_{L_\tau^2 l_\xi^2} \sim M' \|(v_{-\delta})^3\|_{L_{t,x}^2} \lesssim M' \|v_{-\delta}\|_{L_{t,x}^6}^3 \lesssim_\delta M' \|v\|_{X^{0,\frac{1}{2}}}^3$$

where $\widetilde{v_{-\delta}}(\tau,\xi) := \langle \xi\rangle^{-\delta} \langle \tau - \xi^3\rangle^{-\delta/3} |\widetilde{v}|(\tau,\xi)$ and

$$M' := \sup_{\substack{(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_3+\xi_1)\neq 0 \\ \xi_1+\xi_2+\xi_3=\xi, \quad \xi_3\neq 0}} \frac{\langle \xi_1\rangle^{s+\delta} \langle \xi_2\rangle^{s+\delta} \langle \xi_3\rangle^{s+\delta} \langle \xi\rangle^{\gamma-s}}{|\xi_3| L^{\frac{1}{2}-2\delta}} \tag{3.12}$$

$$\lesssim \sup_{\substack{(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_3+\xi_1)\neq 0 \\ \xi_1+\xi_2+\xi_3=\xi}} \frac{\langle \xi_1\rangle^{s+\delta} \langle \xi_2\rangle^{s+\delta} \langle \xi\rangle^{\gamma-s}}{|\xi_3|^{1-s-\delta} (|\xi_1+\xi_2||\xi_2+\xi_3||\xi_3+\xi_1|)^{\frac{1}{2}-3\delta} L_{\max}^{\delta}}.$$

We split into two generic cases: 1) $|\xi| \sim |\xi_1| \sim |\xi_2| \sim |\xi_3|$, 2) $|\xi| \sim |\xi_1| \sim |\xi_2| \gg |\xi_3|$. Note that the other cases are easier and naturally follow from the given cases.

Case 1. If $|\xi| \sim |\xi_1| \sim |\xi_2| \sim |\xi_3|$, note $|\xi_1+\xi_2||\xi_2+\xi_3||\xi_3+\xi_1| \gtrsim \xi$.

$$M' \lesssim \sup_\xi \frac{\langle \xi\rangle^{2s+\gamma-1+3\delta}}{(|\xi_1+\xi_2||\xi_2+\xi_3||\xi_3+\xi_1|)^{\frac{1}{2}-3\delta} L_{\max}^{\delta}} \lesssim L_{\max}^{-\delta} \sup_\xi \langle \xi\rangle^{2s+\gamma-\frac{3}{2}+6\delta} \leq L_{\max}^{-\delta}.$$

Case 2. If $|\xi_1| \sim |\xi_2| \sim |\xi| \gg |\xi_3|$, note $|\xi_1+\xi_2||\xi_2+\xi_3||\xi_3+\xi_1| \gtrsim \langle\xi\rangle^2$.

$$M' \lesssim \sup_{\xi} \frac{\langle\xi\rangle^{s+\gamma+2\delta}}{(|\xi_1+\xi_2||\xi_2+\xi_3||\xi_3+\xi_1|)^{\frac{1}{2}-3\delta} L_{\max}^{\delta}} \lesssim L_{\max}^{-\delta} \sup_{\xi} \langle\xi\rangle^{s+\gamma-1+8\delta} \le L_{\max}^{-\delta}.$$

This concludes our estimate for the case $L \gtrsim \max(L_1, L_2, L_3)$. On the other hand, if $L_1 \gtrsim \max(L, L_2, L_3)$, we can use the same method as above after a brief justification. Note that in this case, $L_1 \gtrsim |\xi_1+\xi_2||\xi_2+\xi_3||\xi_3+\xi_1| \ge 1$. Thus the same estimates will follow once we can substitute L_1 in place of L in (3.12). The following computations can be used to justify such substitution: Let $u, v, w \in X^{0,\frac{1}{2}}$ be localized in frequency space with $L_1 \gtrsim L$.

$$\begin{aligned}
\|uvw\|_{X^{0,-\frac{1}{2}+\delta}} &\sim \left\| \frac{[\widetilde{u} * (\widetilde{v} * \widetilde{w})](\tau,\xi)}{\langle\tau-\xi^3\rangle^{\frac{1}{2}-\delta}} \right\|_{L^2_\tau l^2_\xi} \\
&\sim \sup_{\|z\|_{L^2_\tau l^2_\xi}=1} \left| \int_{\tau_1+\tau_2+\tau_3=\tau} \sum_{\xi_1+\xi_2+\xi_3=\xi} \frac{\widetilde{u}(\tau_1,\xi_1)\widetilde{v}(\tau_2,\xi_2)\widetilde{w}(\tau_3,\xi_3)}{\langle\tau-\xi^3\rangle^{\frac{1}{2}-\delta}} z(\tau,\xi)\, d\sigma \right| \\
&\lesssim \sup_{\|z\|_{L^2_\tau l^2_\xi}=1} \int_{\tau_1+\tau_2+\tau_3=\tau} \sum_{\xi_1+\xi_2+\xi_3=\xi} \frac{(L_1^{\frac{1}{2}}|\widetilde{u}|)|\widetilde{v}||\widetilde{w}||z|}{L^{\frac{1}{2}+\delta} L_1^{\frac{1}{2}-2\delta}} d\sigma \\
&\lesssim M^* \|u\|_{X^{0,\frac{1}{2}+\delta}} \sup_{\|z\|_{L^2_\tau l^2_\xi}=1} \left\| \left(\frac{|z|}{\langle\xi\rangle^{\delta} L^{\frac{1}{2}+\delta}}\right) * (\widetilde{v_{-\delta}} * \widetilde{w_{-\delta}}) \right\|_{L^2_{\tau_1} l^2_{\xi_1}} \\
&\sim M^* \|u\|_{X^{0,\frac{1}{2}+\delta}} \sup_{\|z\|_{L^2_\tau l^2_\xi}=1} \left\| \mathscr{F}^{-1}_{\tau_1,\xi_1} \left[\frac{|z|}{\langle\xi\rangle^{\delta} L^{\frac{1}{2}+\delta}} \right] v_{-\delta} w_{-\delta} \right\|_{L^2_{t,x}} \\
&\lesssim_\delta M^* \|u\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}+\delta}}
\end{aligned}$$

where we have used Hölder and (3.2) for the penultimate inequality, and

$$M^* := \sup_{\xi_1+\xi_2+\xi_3=\xi} \frac{\langle\xi\rangle^{\delta}\langle\xi_2\rangle^{\delta}\langle\xi_3\rangle^{\delta}}{L_1^{\frac{1}{2}-2\delta}}.$$

Although above computations were done without the pseudo-differential operator for simplicity, it is easy to see that similar arguments can be used to reduce the case $L_j \sim \max(L, L_1, L_2, L_3)$ for some $j = 1, 2, 3$ to the case $L \gg \max(L_1, L_2, L_3)$. This concludes the proof. □

The next lemma deals with the *resonant* terms in (3.10). To reduce the number of cases, we ignore the complex conjugation. This does not cause any problem in the proof, since we do not take advantage of cancellations from here on.

Lemma 12. *Let* $R[f]$ *be as defined in* (3.8) *and let* $h \in L_t^\infty H_x^1([0,T] \times \mathbf{T})$, $w \in X^{\gamma, \frac{1}{2}+\delta}$ *be arbitrary. Then for* $0 \le s < 1/2$ *and* $\gamma \ge 0$,

$$\left\| \frac{\langle \xi \rangle^{2s+\gamma}}{\xi} \widehat{R[f]}^2 (\widehat{h} + \widehat{w}) \right\|_{L_t^2 l_\xi^2([0,T] \times \mathbf{Z} \setminus \{0\})} \lesssim_{T,\delta} \|f\|_{L_x^2}^2 \left(\|h\|_{L_t^\infty H_x^1} + \|w\|_{X^{\gamma, \frac{1}{2}+\delta}} \right)$$

$$\left\| \frac{\langle \xi \rangle^{2s+\gamma}}{\xi} \widehat{R[f]} \widehat{h}^2 \right\|_{L_t^2 l_\xi^2([0,T] \times \mathbf{Z} \setminus \{0\})} \lesssim_{T,\delta} \|f\|_{L_x^2} \|h\|_{L_t^\infty H_x^1}^2$$

$$\left\| \frac{\langle \xi \rangle^{2s+\gamma}}{\xi} \widehat{R[f]} \widehat{w} (\widehat{h} + \widehat{w}) \right\|_{L_t^2 l_\xi^2([0,T] \times \mathbf{Z} \setminus \{0\})} \lesssim_{T,\delta} \|f\|_{L_x^2} \|h\|_{L_t^\infty H_x^1} \left(\|h\|_{L_t^\infty H_x^1} + \|w\|_{X^{\gamma, \frac{1}{2}+\delta}} \right)$$

$$\left\| \frac{\langle \xi \rangle^{2s+\gamma}}{\xi} \widehat{h} \widehat{w} (\widehat{h} + \widehat{w}) \right\|_{L_t^2 l_\xi^2([0,T] \times \mathbf{Z} \setminus \{0\})} \lesssim_{T,\delta} \|h\|_{L_t^\infty H_x^1} \|w\|_{X^{\gamma, \frac{1}{2}+\delta}} \left(\|h\|_{L_t^\infty H_x^1} + \|w\|_{X^{\gamma, \frac{1}{2}+\delta}} \right)$$

$$\left\| \frac{\langle \xi \rangle^{2s+\gamma}}{\xi} (\widehat{h}^3 + \widehat{w}^3) \right\|_{L_t^2 l_\xi^2([0,T] \times \mathbf{Z} \setminus \{0\})} \lesssim_{T,\delta} \|h\|_{L_t^\infty H_x^1}^3 + \|w\|_{X^{\gamma, \frac{1}{2}+\delta}}^3.$$

Proof. Note that $w, h \in L_t^2 H_x^\gamma([0,T] \times \mathbf{T})$, thus left sides of inequalities above contain at least one term which belongs to $L_t^2 H_x^\gamma$. This takes care of $\langle \xi \rangle^\gamma$ weight. Since $2s < 1$, all weight is property controlled.

Furthermore, for any smooth function u, note $\|\widehat{u}\|_{L_t^\infty l_\xi^\infty} \lesssim \|u\|_{L_t^\infty L_x^2}$. Thus the remaining terms (except for one which lives in $L_t^2 H_x^\gamma$) is controlled under $L_t^\infty L_x^2$ norm. Since $L_t^\infty L_x^2$ is controlled by the norms for $L_t^\infty H_x^1$ and $X^{\gamma, \frac{1}{2}+\delta}$, and since we have $\|R[f]\|_{L_t^\infty L_x^2} = \|f\|_{L_x^2}$, the estimates above follow. □

Finally, we establish the Lipschitz continuity of the map $R[f]$ on $L^2(\mathbf{T})$.

Lemma 13. *Let R be defined as in (3.8) with $s < 1/2$ and $\gamma \in \mathbf{R}$. Then for any $f, g \in L^2(\mathbf{T})$ with $f - g \in H^\gamma$,*

$$\|R[f] - R[g]\|_{C_t^0 H_x^\gamma([0,T]\times\mathbf{T})} \leq C_{N,T} \|f-g\|_{H^\gamma(\mathbf{T})}$$

where $\|f\|_{L^2} + \|g\|_{L^2} < N$.

Proof. First we write $\widehat{f}(\xi) = |\widehat{f}(\xi)|e^{i\alpha_\xi}$ and $\widehat{g}(\xi) = |\widehat{g}(\xi)|e^{i\beta_\xi}$. Denote $\theta_\xi := \alpha_\xi - \beta_\xi$ Then, the Law of cosines, triangle and Hölder's inequality gives

$$\begin{aligned}
\|R[f] - R[g]\|_{C_t^0 H_x^\gamma([0,T]\times\mathbf{T})} &= \sup_{t\in[0,T]} \left\| \langle\xi\rangle^\gamma \big(|\widehat{f}| e^{2it\frac{\langle\xi\rangle^{2s}}{\xi}(|\widehat{f}|^2 - |\widehat{g}|^2) + i\theta_\xi} - |\widehat{g}|\big) \right\|_{l^2_\xi(\mathbf{Z}\setminus\{0\})} \\
&= \sup_{t\in[0,T]} \left\| \langle\xi\rangle^\gamma \left(|\widehat{f}|^2 + |\widehat{g}|^2 - 2|\widehat{f}||\widehat{g}|\cos(2t\frac{\langle\xi\rangle^{2s}}{\xi}(|\widehat{f}|^2 - |\widehat{g}|^2) + \theta_\xi) \right)^{\frac{1}{2}} \right\|_{l^2_\xi(\mathbf{Z}\setminus\{0\})} \\
&\lesssim \|\langle\xi\rangle^\gamma(|\widehat{f}| - |\widehat{g}|)\|_{l^2_\xi} + 2\sup_{t\in[0,T]} \left\| \langle\xi\rangle^{2\gamma}|\widehat{f}||\widehat{g}|(1 - \cos(2t\frac{\langle\xi\rangle^{2s}}{\xi}(|\widehat{f}|^2 - |\widehat{g}|^2) + \theta_\xi) \right\|^{\frac{1}{2}}_{l^1_\xi(\mathbf{Z}\setminus\{0\})} \\
&\lesssim \|f-g\|_{H^\gamma} + 4\sup_{t\in[0,T]} \left\| \langle\xi\rangle^{2\gamma}|\widehat{f}||\widehat{g}|\sin^2\left(t\frac{\langle\xi\rangle^{2s}}{\xi}(|\widehat{f}|^2 - |\widehat{g}|^2) + \theta_\xi\right) \right\|^{\frac{1}{2}}_{l^1_\xi(\mathbf{Z}\setminus\{0\})}.
\end{aligned}$$

Using $\sin^2(A+B) \lesssim A^2 + \sin^2 B$ and the assumption $s < 1/2$, we need to estimate

$$\left\| \langle\xi\rangle^{2\gamma}|\widehat{f}||\widehat{g}|(|\widehat{f}|^2 - |\widehat{g}|^2)^2 \right\|^{\frac{1}{2}}_{l^1_\xi(\mathbf{Z}\setminus\{0\})}, \tag{3.13}$$

$$\left\| \langle\xi\rangle^{2\gamma}|\widehat{f}||\widehat{g}|\sin^2\theta_\xi \right\|^{\frac{1}{2}}_{l^1_\xi(\mathbf{Z}\setminus\{0\})}. \tag{3.14}$$

The bound for (3.13) is straight-forward. By Hölder's and triangle inequalty,

$$(3.13) \lesssim \|f\|^{\frac{1}{2}}_{L^2}\|g\|^{\frac{1}{2}}_{L^2} \left\| \langle\xi\rangle^\gamma(\widehat{f} - \widehat{g})(|\widehat{f}| + |\widehat{g}|) \right\|_{l^\infty_\xi} \lesssim \|f\|^{\frac{3}{2}}_{L^2}\|g\|^{\frac{3}{2}}_{L^2}\|f-g\|_{H^\gamma}.$$

For (3.14), we apply the Law of sines. Without loss of generality, we can assume $\theta_\xi \in (0,\pi)$. Noting that the triangle with side-lengths equal to $|\widehat{f}|, |\widehat{g}|, |\widehat{f}-\widehat{g}|$ has the angle θ_ξ which is opposite to the side with length $|\widehat{f}-\widehat{g}|$, we can deduce that $|\widehat{f}|\sin\theta_\xi \leq |\widehat{f}-\widehat{g}|$ and likewise for $|\widehat{g}|$. Thus,

$$(3.14) \leq \left\| \langle\xi\rangle^{2\gamma} |\widehat{f}-\widehat{g}|^2 \right\|_{l^1_\xi}^{\frac{1}{2}} \sim \left\| \langle\xi\rangle^{\gamma} |\widehat{f}-\widehat{g}| \right\|_{l^2_\xi} \sim \|f-g\|_{H^\gamma}.$$

□

3.3 Conclusion of the proof of Theorem 2

Now we turn to the proof of Theorem 2. Note that the existence and uniqueness of $w \in L^2_{t,x}([0,T]\times\mathbf{T})$ as the solution of (3.10) is given by the decomposition $w = v - R[f] - h$. However, to show that the solution w lives in a *smoother* space $Y^{\gamma,\frac{1}{2}+\delta}$ for $\gamma < 1-s$, we perform a fixed point argument for w of (3.10). We group the terms on the right-hand side of (3.10) as follows

$$\mathscr{N}_1 := -2\mathscr{F}_\xi^{-1}(\mathscr{N}\mathscr{R}); \qquad \mathscr{N}_2 := 2\sum_{\xi\neq 0} \frac{\langle\xi\rangle^{2s}}{\xi} B(\widehat{R[f]}(\xi), \widehat{h}(\xi), \widehat{w}(\xi)) e^{i\xi x}.$$

Let $\mathscr{B}$ be a ball in $Y^{\gamma,\frac{1}{2}+\delta}$ centered at $-\eta(t)e^{-t\partial_x^3}T^p(f,f)$ with *small* radius. Then our aim is to show that, for T small, Λ_T is a contraction map on $\mathscr{B}$.

From Proposition 4, we have

$$\|\Lambda_T w\|_{Y^{\gamma,\frac{1}{2}+\delta}} \lesssim_\eta \|T(f,f)\|_{H^\gamma} + \|\mathscr{N}_1\|_{Y_T^{\gamma,-\frac{1}{2}+\delta}} + \|\mathscr{N}_2\|_{Y_T^{\gamma,-\frac{1}{2}+\delta}}.$$

For, the first term, we apply Lemma 10,

$$\|T(f,f)\|_{H^\gamma_x} \lesssim \|T(f,f)\|_{H^1_x} \lesssim \|f\|^2_{L^2}. \tag{3.15}$$

For the non-linear term $\mathscr{N}_1$, we use Lemma 11 and $\|v\|_{Y^{0,\frac{1}{2}}} \sim \|f\|_{L^2}$,

$$\|\mathscr{N}_1\|_{Y_T^{\gamma,-\frac{1}{2}+\delta}} \lesssim_{\delta,T} \|v\|^3_{Y^{0,\frac{1}{2}}} \sim \|f\|^3_{L^2}. \tag{3.16}$$

For the last term $\mathscr{N}_2$, we apply Lemma 12 and $\|h\|_{L^\infty_t H^1_x} \lesssim \|v\|^2_{C^0_t([0,T];H^{-s})} \sim \|f\|^2_{L^2}$,

$$\begin{aligned} \|\mathscr{N}_2\|_{Y_T^{\gamma,-\frac{1}{2}+\delta}} &\lesssim_{\delta,T} \left\| \frac{\langle\xi\rangle^{2s+\gamma}}{\xi} B(\widehat{R[f]}, \widehat{h}, \widehat{w}) \right\|_{L^2_t l^2_\xi([0,T]\times\mathbf{Z}\backslash\{0\})} \\ &\lesssim_{\delta,T,\|f\|_{L^2}} \|w\|^3 + \|w\|^2 + \|w\| + 1 \end{aligned} \tag{3.17}$$

where the implicit constant in the last inequality involves a positive power of $\|f\|_{L^2}$. Thus making T suitably small with respect to $\|f\|_{L^2}$, we note that Λ_T is a contraction on $\mathscr{B}$.

To show Lipschitz property, let $R[f^k]$, h^k, w^k for $k = 1,2$ be the corresponding solutions with f replaced by f^k. Recall $v^k = R[f^k] + h^k + w^k$. Then we need to show

$$\|v^1 - v^2\|_{C_t([0,T];H^\gamma_x)} \lesssim_{N,\delta} \|f^1 - f^2\|_{H^\gamma_x}$$

where $\|f^1\|_{L^2} + \|f^2\|_{L^2} < N$. For the first term, we apply Lemma 13,

$$\left\|R[f^1] - R[f^2]\right\|_{C_t([0,T];H^\gamma_x)} \lesssim_{N,\delta} \|f^1 - f^2\|_{H^\gamma_x}.$$

For the second term, we use Lemma 10 and Lipschitz map of (3.5) from [13] to obtain

$$\|h^1 - h^2\|_{C([0,T];H_x^\gamma)} \lesssim \|T(v^1 + v^2, v^1 - v^2)\|_{L_t^\infty H_x^1} \lesssim \|v^1 + v^2\|_{L_t^\infty L_x^2} \|v^1 - v^2\|_{L_t^\infty L_x^2} \lesssim_N \|f^1 - f^2\|_{L^2}.$$

Thus, we have

$$\begin{aligned} \|w^1 - w^2\|_{C_t([0,T];H_x^\gamma)} &\lesssim \|w^1 - w^2\|_{Y^{\gamma,\frac{1}{2}+\delta}} \\ &\lesssim \|T(f^1, f^1) - T(f^2, f^2)\|_{H_x^\gamma} + \sum_{j=1}^{2} \|\mathcal{N}_j^1 - \mathcal{N}_j^2\|_{Y^{\gamma-\frac{1}{2}+\delta}} \end{aligned}$$

where for $\mathcal{N}_j^k$ is defined with respect to the initial data f^k for $k = 1, 2$. Then the desired estimate follows from estimates (3.15) through (3.17).

Chapter 4

Local Well-posedness for the periodic "good" Boussinesq equation

4.1 Reductions to mean-zero initial data

We first make a reduction of the Cauchy problem (1.22) to reduce to the case of mean value zero solutions, since this will be important for our argument.

Observe that if

$$u(t,x) = \sum_{n=-\infty}^{\infty} \hat{u}(t,n)e^{inx},$$

and if we consider the evolution of the zero mode, $\hat{u}(t,0)$, we find easily that

$$\frac{d^2\hat{u}(t,0)}{dt^2} = 0.$$

Equivalently, integrating the equation in x yields $\int_0^{2\pi} u_{tt}(t,x)dx = 0$, whence

$$\int_0^{2\pi} u(t,x)dx = \int_0^{2\pi} u(0,x)dx + t\int_0^{2\pi} u_t(0,x)dx$$

Thus, setting $w : u(t,x) = \frac{1}{2\pi}\int_0^{2\pi} u(t,x)dx + v(t,x)$, so that

$$u(t,x) = \frac{1}{2\pi}\left(\int_0^{2\pi} u(0,x)dx + t\int_0^{2\pi} u_t(0,x)dx\right) + v(t,x)$$

we conclude that $\int_0^{2\pi} v(t,x)dx = 0$. Denoting

$$A(t) = \frac{1}{2\pi}\left(\int_0^{2\pi} u(0,x)dx + t\int_0^{2\pi} u_t(0,x)dx\right),$$

we see that (1.22) is equivalent to the nonlinear problem

$$\left| \begin{array}{l} v_{tt} + v_{xxxx} - v_{xx} + (A(t)+v)^2_{xx} = 0 \\ v(0,x) = u_0(x) - \frac{1}{2\pi}\int_0^{2\pi} u_0(x)dx; v_t(0,x) = u_1(x) - \frac{1}{2\pi}\int_0^{2\pi} u_1(x)dx, \end{array} \right. \tag{4.1}$$

We would like to consider the problem with data in the Sobolev spaces $H^{-\alpha}$, but to make our notations simpler, we prefer to work in $L^2(\mathbf{T})$, so we transform the equation (4.1) in $L^2(\mathbf{T})$ context. Namely, we introduce $w = \langle\nabla\rangle^{-\alpha} v$, that is

$$w(t,x) = \sum_{n\neq 0} \frac{\hat{v}(t,n)}{\langle n\rangle^{\alpha}} e^{inx}.$$

Note that by construction $\int_0^{2\pi} w(t,x)dx = 0$. We can rewrite now (4.1) as follows

$$\left| \begin{array}{l} w_{tt} - w_{xx} + w_{xxxx} + 2A(t)w_{xx} + \langle\nabla\rangle^{-\alpha}\partial_x^2(\langle\nabla\rangle^{\alpha} w)^2 = 0, \qquad x \in \mathbf{T}, t > 0 \\ w(0,x) = f(x) \in L^2(\mathbf{T}); \quad w_t(0,x) = g(x) \in H^{-2}(\mathbf{T}) \end{array} \right. \tag{4.2}$$

where

$$f = \sum_{n\neq 0} \frac{\widehat{u_0}(n)}{\langle n\rangle^{\alpha}} e^{i\pi nx}, g = \sum_{n\neq 0} \frac{\widehat{u_1}(n)}{\langle n\rangle^{\alpha}} e^{i\pi nx}.$$

Note that $\int_{\mathbf{T}} f(x)dx = 0, \int_{\mathbf{T}} g(x)dx = 0$.

Set $L := \sqrt{\partial_x^4 - \partial_x^2}$. Note that $\widehat{Lh}(k) = |k|\sqrt{1+k^2}\hat{h}(k)$. Furthermore, in the space of functions with mean value zero, L is invertible, with inverse given by

$$L^{-1}h(x) = \sum_{k\neq 0} \frac{1}{|k|\sqrt{1+k^2}}\hat{h}(k)e^{ikx}.$$

By the Duhamel's principle, (4.2) is equivalent to

$$\begin{aligned} w(t,x) &= \cos(tL)f(x) + \sin(tL)[L^{-1}g] \\ &\quad + \int_0^t \sin((t-s)L)L^{-1}[2A(s)w_{xx} + \langle\nabla\rangle^{-\alpha}\partial_x^2(\langle\nabla\rangle^{\alpha}w(s,\cdot))^2]ds. \end{aligned} \tag{4.3}$$

Using Euler's formula, we can write $w = w^+ + w^-$, where

$$w^+(t,x) = \frac{e^{itL}f}{2} + \frac{e^{itL}L^{-1}g}{2i} + \frac{1}{2i}\int_0^t e^{i(t-s)L}[F(w^+) + \mathcal{N}(w^+ + w^-, w^+ + w^-)]\,ds$$

$$w^-(t,x) = \frac{e^{-itL}f}{2} - \frac{e^{-itL}L^{-1}g}{2i} - \frac{1}{2i}\int_0^t e^{-i(t-s)L}[F(w^-) + \mathcal{N}(w^+ + w^-, w^+ + w^-)]\,ds$$

where $F(w) = 2A(s)L^{-1}\partial_{xx}w$ and $\mathcal{N}(u,v) := L^{-1}\langle\nabla\rangle^{-\alpha}\partial_x^2(\langle\nabla\rangle^{\alpha}u\langle\nabla\rangle^{\alpha}v)$. Thus, we have replaced the single wave equation for w into a system of equations, involving w^+, w^-. Namely, denoting $\mathcal{L}(f,g) := \frac{1}{2}e^{itL}f + \frac{1}{2i}e^{itL}L^{-1}g$ (or $\mathcal{L}$ for short), we have

$$\begin{cases} (\partial_t - iL)w^+ & = F(w^+) + \mathcal{N}(w^+ + w^-, w^+ + w^-), \\ (\partial_t + iL)w^- & = F(w^-) + \mathcal{N}(w^+ + w^-, w^+ + w^-), \\ w^+(0,x) & = \mathcal{L}(0) = \frac{1}{2}f + \frac{1}{2i}L^{-1}g \in L^2 \\ w^-(0,x) & = \bar{\mathcal{L}}(0) = \frac{1}{2}\bar{f} - \frac{1}{2i}L^{-1}\bar{g} \in L^2 \end{cases} \tag{4.4}$$

The term $F(w^{\pm})$ creates certain complications, mostly of technical nature, which we now address. Write

$$F(w)(s) = 2A(s)L^{-1}\partial_{xx} = (A_0 + sA_1)\mathscr{P}w,$$

where $A_0 = \frac{1}{\pi}\int_0^{2\pi} u_0(x)dx, A_1 = \frac{1}{\pi}\int_0^{2\pi} u_1(x)dx$ are scalars and $\mathscr{P} := L^{-1}\partial_{xx}$ is an order zero differential operator, given by the symbol $-\frac{|k|}{<k>}$ and hence bounded on all Sobolev spaces. We now resolve the inhomogeneous equation $(\partial_t - iL - F)w^+ = G$ (for any right hand side G) in the following way. Introduce

$$w^{\pm}(s) = e^{(A_0 s + A_1 \frac{s^2}{2})\mathscr{P}}\tilde{w}^{\pm}(s),$$

where $e^{(A_0 s + A_1 \frac{s^2}{2})\mathscr{P}}$ is a bounded operator on any L^2 based Sobolev space, which can be represented for example by its power series. We have

$$\begin{aligned}(\partial_t - iL)w^+ &= e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}(\partial_t - iL)\tilde{w}^+ + (A_0 + tA_1)\mathscr{P}e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}\tilde{w}^+ = \\ &= e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}(\partial_t - iL)\tilde{w}^+ + F[w^+].\end{aligned}$$

Thus,

$$G = (\partial_t - iL - F)w^+ = e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}(\partial_t - iL)\tilde{w}^+,$$

whence[1]

$$(\partial_t - iL)\tilde{w}^+ = e^{-(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}G.$$

[1]Note that $e^{-(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}$ is the (bounded) inverse of $e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}$

Similar computations work for w^-. Thus, we have reduced (4.4) to the following equation for $\tilde{w}^+$

$$(\partial_t - iL)\tilde{w}^+ = e^{-(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} \mathscr{N}(e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}(\tilde{w}^+ + \tilde{w}^-), e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}(\tilde{w}^+ + \tilde{w}^-)), \tag{4.5}$$

and similar for w^-. Observe that $\tilde{w}^+(0) = \mathscr{L}(0)$ and $\tilde{w}^-(0) = \bar{\mathscr{L}}(0)$. For convenience, introduce the notation

$$\tilde{\mathscr{N}}(u,v) := e^{-(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} \mathscr{N}(e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} u, e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} v), \tag{4.6}$$

so that our main governing equation (4.5), now takes the form

$$(\partial_t - iL)\tilde{w}^+ = \tilde{\mathscr{N}}(\tilde{w}^+ + \tilde{w}^-, \tilde{w}^+ + \tilde{w}^-)$$

We note that the operators $e^{\pm(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}$ are mostly harmless, in the sense that they are bounded on all function spaces considered in the paper. At first reading, the reader may as well assume that $A_0 = A_1 = 0$ (which corresponds to the important case of mean value zero data) to avoid the cumbersome technical complications.

4.2 Construction of the normal forms

4.2.1 The case with mean value zero

We start with the case $A_0 = A_1 = 0$ in order to simplify matters. In the next section, we indicate how to handle the general case.

Clearly, we have

$$\|\mathscr{L}(f,g)\|_{L^2(\mathbf{T})} \leq \frac{1}{2}\|f\|_{L^2} + \frac{1}{2}\||\nabla|^{-1}\langle\nabla\rangle^{-1}g\|_{L^2} \sim \|f\|_{L^2} + \|g\|_{H^{-2}}. \tag{4.7}$$

We introduce further variables $z^{\pm}$, so that $w^+ = \mathscr{L} + z^+, w^- = \bar{\mathscr{L}} + z^-$. This yields a new set of two equations for the unknowns $z^{\pm}$. Furthermore, the nonlinearities take one of the following forms:

$$\mathscr{N}(\mathscr{L},\mathscr{L}), \quad \mathscr{N}(\mathscr{L},\overline{\mathscr{L}}), \quad \mathscr{N}(\overline{\mathscr{L}},\overline{\mathscr{L}}), \quad \mathscr{N}(\mathscr{L},z^{\pm}), \quad \mathscr{N}(\overline{\mathscr{L}},z^{\pm}), \quad \mathscr{N}(z^{\pm},z^{\pm}).$$

We construct an explicit solution, in the form of a bilinear pseudo-differential operator (i.e. a "normal form"), which will take care of the first three non-linearities, that is those in the form $\mathscr{N}(\mathscr{L},\mathscr{L})$, $\mathscr{N}(\mathscr{L},\overline{\mathscr{L}})$, $\mathscr{N}(\overline{\mathscr{L}},\overline{\mathscr{L}})$. That is, we are looking to solve for $\varepsilon = \pm 1$,

$$(\partial_t - i\varepsilon\, L)h^{\varepsilon} = \frac{1}{2i}\left[\mathscr{N}(\mathscr{L},\mathscr{L}) + 2\mathscr{N}(\mathscr{L},\overline{\mathscr{L}}) + \mathscr{N}(\overline{\mathscr{L}},\overline{\mathscr{L}})\right]. \tag{4.8}$$

In order to prepare us for our choice of h^{ε}, we need to display some algebraic relations for the symbols. More precisely, for $\varepsilon, \varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, we have

$$\begin{aligned}(\tau+\omega) - \varepsilon\sqrt{(\xi+\eta)^4 + (\xi+\eta)^2} &= (\tau - \varepsilon_1\sqrt{\xi^4+\xi^2}) + (\omega - \varepsilon_2\sqrt{\eta^4+\eta^2}) \\ &\quad + \varepsilon_1|\xi|\langle\xi\rangle + \varepsilon_2|\eta|\langle\eta\rangle - \varepsilon|\xi+\eta|\langle\xi+\eta\rangle.\end{aligned}$$

which implies that for every bilinear pseudo-differential operator Λ_σ with symbol $\sigma(\xi,\eta)$, that is $\Lambda_\sigma(u,v) = \sum_{\xi,\eta\in\mathscr{Z}} \sigma(\xi,\eta)\hat{u}(\xi)\hat{v}(\eta)e^{i(\xi+\eta)x}$, we have

$$(\partial_t - iL)\Lambda_\sigma(u,v) = -i(\Lambda_\sigma((\partial_t - iL)u,v) + \Lambda_\sigma(u,(\partial_t - iL)v) + \Lambda_\mu(u,v)),$$

$$\mu(\xi,\eta) = \sigma(\xi,\eta)(\varepsilon_1|\xi|\langle\xi\rangle + \varepsilon_2|\eta|\langle\eta\rangle - \varepsilon|\xi+\eta|\langle\xi+\eta\rangle).$$

In particular, if u, v are free solutions, i.e. $(\partial_t - iL)u = (\partial_t - iL)v = 0$, we get

$$(\partial_t - iL)\Lambda_\sigma(u,v) = -i\Lambda_\mu(u,v).$$

Thus, we define a bilinear pseudo-differential operator T by the formula

$$T^{\varepsilon;\varepsilon_1,\varepsilon_2}(u,v)(x) := -\frac{1}{2}\sum_{\xi\eta(\xi+\eta)\neq 0} \frac{|\xi+\eta|\langle\xi\rangle^\alpha\langle\eta\rangle^\alpha\widehat{u}(\xi)\widehat{v}(\eta)\,e^{i(\xi+\eta)x}}{\langle\xi+\eta\rangle^{1+\alpha}[\varepsilon_1|\xi|\langle\xi\rangle + \varepsilon_2|\eta|\langle\eta\rangle - \varepsilon|\xi+\eta|\langle\xi+\eta\rangle]} \tag{4.9}$$

we get that

$$\begin{aligned}(\partial_t - i\varepsilon L)T^{\varepsilon;+,+}(\mathscr{L},\mathscr{L}) &= \frac{1}{2i}\mathscr{N}(\mathscr{L},\mathscr{L}),\\ (\partial_t - i\varepsilon L)T^{\varepsilon;+,-}(\mathscr{L},\bar{\mathscr{L}}) &= \frac{1}{2i}\mathscr{N}(\mathscr{L},\bar{\mathscr{L}})\\ (\partial_t - i\varepsilon L)T^{\varepsilon;-,-}(\bar{\mathscr{L}},\bar{\mathscr{L}}) &= \frac{1}{2i}\mathscr{N}(\bar{\mathscr{L}},\bar{\mathscr{L}}),\end{aligned}$$

which allows us to get a solution of (4.8) in the form

$$h^\varepsilon = T^{\varepsilon;+,+}(\mathscr{L},\mathscr{L}) + 2T^{\varepsilon;+,-}(\mathscr{L},\overline{\mathscr{L}}) + T^{\varepsilon;-,-}(\overline{\mathscr{L}},\overline{\mathscr{L}}). \tag{4.10}$$

We perform another change of variables, $\Psi^\pm : z^\pm = h^\pm + \Psi^\pm$, so that

$$\left|\begin{aligned}&(\partial_t - iL)\Psi^+ = \mathscr{N}(\mathscr{L}+\overline{\mathscr{L}}, h^\pm + \Psi^\pm) + \mathscr{N}(h^\pm + \Psi^\pm, h^\pm + \Psi^\pm)\\ &\Psi^+(0,x) = -[T^{+,+}(\mathscr{L},\mathscr{L}) + 2T^{+,-}(\mathscr{L},\overline{\mathscr{L}}) + T^{-,-}(\overline{\mathscr{L}},\overline{\mathscr{L}})]|_{t=0},\end{aligned}\right. \tag{4.11}$$

similar formula holds for Ψ^-. In fact, from now on, we will set $\varepsilon = +1$, since the case $\varepsilon = -1$ can always be reduced to the case $\varepsilon = +1$. Thus, we drop ε from our notations, for example $T^{\varepsilon_1,\varepsilon_2}$ is used to denote $T^{+1;\varepsilon_1,\varepsilon_2}$ etc.

With that, we have largely prepared the nonlinear problem to its current form (4.11). Note that by our construction, $\Psi^\pm$ is a mean value zero function.

4.2.2 The general case

In the general case, and having in mind the particular form of the right-hand side of (4.5), we set $\tilde{w}^+ = \mathscr{L} + z^+, \tilde{w}^- = \bar{\mathscr{L}} + z^-$. Note $z^\pm(0) = 0$. Similar to (4.10), set

$$\begin{aligned} h^\varepsilon &= e^{-(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} T^{\varepsilon;+,+}(e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}\mathscr{L}, e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}\mathscr{L}) + \\ &+ 2e^{-(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} T^{\varepsilon;+,-}(e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}\mathscr{L}, e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}\overline{\mathscr{L}}) + \\ &+ e^{-(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} T^{\varepsilon;-,-}(e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}\overline{\mathscr{L}}, e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}\overline{\mathscr{L}}). \end{aligned}$$

With this assignment for h^ε, we will certainly not get the nice exact identity (4.8). However, we get something similar (up to an error term), which is good enough for our purposes. Namely,

$$(\partial_t - i\varepsilon L)h^\varepsilon = \tilde{\mathscr{N}}(\mathscr{L},\mathscr{L}) + 2\tilde{\mathscr{N}}(\mathscr{L},\bar{\mathscr{L}}) + \tilde{\mathscr{N}}(\bar{\mathscr{L}},\bar{\mathscr{L}}) + Err,$$

where the error term contains all the terms obtained when the time derivative hits the terms $e^{\pm(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}$ in the formula for h^ε. Thus, a typical error term will be

$$Err \sim e^{-(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}(-A_0 - A_1 t)\mathscr{P}[T^{\varepsilon;+,+}(e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}\mathscr{L}, e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}\mathscr{L})]. \quad (4.12)$$

Similar to Section 4.2.1 above, introduce the new variables $\Psi^\pm$, so that $z^\pm = h^\pm + \Psi^\pm$. That is, $\tilde{w}^+ = \mathscr{L} + h^+ + \Psi^+$, $\tilde{w}^- = \bar{\mathscr{L}} + h^- + \Psi^-$. We obtain the following equations for $\Psi^\pm$ (note the similarity to (4.11))

$$\left| \begin{array}{l} (\partial_t - iL)\Psi^+ = \tilde{\mathscr{N}}(\mathscr{L} + \overline{\mathscr{L}}, h^\pm + \Psi^\pm) + \tilde{\mathscr{N}}(h^\pm + \Psi^\pm, h^\pm + \Psi^\pm) - Err \\ \Psi^+(0,x) = -[T^{+,+}(\mathscr{L},\mathscr{L}) + 2T^{+,-}(\mathscr{L},\overline{\mathscr{L}}) + T^{-,-}(\overline{\mathscr{L}},\overline{\mathscr{L}})]|_{t=0}, \end{array} \right. \tag{4.13}$$

Note that for the initial data, that is at $t = 0$,

$$e^{-(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} T^{+,+}(e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} \mathscr{L}, e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} \mathscr{L})|_{t=0} = T^{+,+}(\mathscr{L},\mathscr{L})|_{t=0}.$$

etc. whence we get the same initial conditions in (4.13) and (4.11). Thus, our equation (4.13) will be the main object of interest for the remainder of the paper.

4.3 $X^{s,b}$ estimates and embeddings

We now need to state the relevant *a priori* estimates for the linear problem

Lemma 14. *Let m solve the linear inhomogeneous problem*

$$(\partial_t - i\varepsilon L)m = F, m(0) = m_0.$$

Then, for all $T > 0$, $s \in \mathbf{R}^1$ and $b > 1/2$, we have for all cut-off functions $\eta \in C_0^\infty$

$$\|\eta(t)m\|_{X^\varepsilon_{s,b}} \leq C_\eta(\|m_0\|_{H^s} + \|F\|_{X^\varepsilon_{s,b-1}}). \tag{4.14}$$

Proof. The proof is essentially contained in [49, Proposition3.12], which gives the estimate (4.14) for fairly general dispersive equations. As a result, we have

$$\|\eta(t)m\|_{Y^{\varepsilon}_{s,b}} \leq C_{\varphi}(\|m_0\|_{H^s} + \|F\|_{Y^{\varepsilon}_{s,b-1}}), \tag{4.15}$$

where

$$\|F\|_{Y^{\varepsilon}_{s,b}} = \left(\int_{\mathbf{R}^1} \sum_{n\in\mathbb{Z}\setminus\{0\}} (1+|\tau-\varepsilon|n|\langle n\rangle|)^{2b} \langle n\rangle^{2s} |\widehat{F}_n(\tau)|^2 d\tau\right)^{1/2}.$$

The difference between (4.15) and the estimate (4.14) is that we insist on using the standard Schrödinger $X_{s,b}$ spaces, instead of the less standard $Y_{s,b}$ spaces. But in fact, the two spaces are equivalent. That is, we claim that the symbols are equivalent in the following sense. More precisely, since $0 < |n|\langle n\rangle - n^2 < 1$, we have that the two norms $\|\cdot\|_{Y^{\varepsilon}_{s,b}}$ and $\|\cdot\|_{X^{\varepsilon}_{s,b}}$ are equivalent (for all values of the parameters ε, s, b) and hence (4.15) is equivalent to (4.14), and hence (4.14) is established. □

Next, there is the following important embedding result, due to Bourgain, [9].

Lemma 15. *The following embeddings hold:* $X^{\pm}_{0,\frac{3}{8}} \subset L^4_{t,x}$ *and* $X^{\pm}_{0+,\frac{1}{2}+} \subset L^6_{t,x}$.

The stability of the $X^{\varepsilon}_{s,b}$ norms with respect to products with smooth functions is the following standard

Lemma 16. *For a cut-off functions* $\eta \in C^{\infty}_0$*, there is* $C = C_{\eta}$*, so that*

$$\|\eta(t)m\|_{X^{\varepsilon}_{s,b}} \leq C\|m\|_{X^{\varepsilon}_{s,b}}.$$

Lemma 16 appears as [49, Lemma 2.11]. From the proof of Lemma 16, it can be inferred that for $b \in (1/2, 1)$, one can select $C_{\eta} = C(\|\eta\|_{L^1(\mathbf{R}^1)} + \|\eta''\|_{L^1(\mathbf{R}^1)})$ for some absolute constant C.

As a corollary, we derive the following estimate, which will be useful for us in the sequel

$$\|\eta(t)e^{(At+Bt^2)\mathscr{P}}m\|_{X^{\varepsilon}_{s,b}} \leq C_{\eta,A,B}\|m\|_{X^{\varepsilon}_{s,b}}. \tag{4.16}$$

For the proof of (4.16), take more generally a C^2 function $g(t)$ instead of $At+Bt^2$. One may expand the operator $e^{g(t)\mathscr{P}}$ in power series

$$e^{g(t)\mathscr{P}} = \sum_{k=0}^{\infty} \frac{g(t)^k \mathscr{P}^k}{k!}.$$

Thus, given that $\|\mathscr{P}\| \leq 1$, it is enough to show that $\|\eta(t)g(t)^k m\|_{X^{\varepsilon}_{s,b}} \leq C_k\|m\|_{X^{\varepsilon}_{s,b}}$, so that $\sum_k \frac{C_k}{k!} < \infty$. By the remark above, one could take

$$C_k = C(\|\eta(t)g(t)^k\|_{L^1(\mathbf{R}^1)}\| + \|(\eta(t)g(t)^k)''\|_{L^1(\mathbf{R}^1)} \leq Ck^2(1+\|g\|_{C^2(-M,M)})^k$$

where $supp\eta \subset (-M,M)$. Since $\sum_{k=1}^{\infty} \frac{k^2(1+\|g\|_{C^2(-M,M)})^k}{k!} < \infty$, (4.16) is established.

4.4 Multi-linear estimates and Proof of Theorem 3

We are now ready to take on the proof of Theorem 3. Let us recapitulate what we have done so far.

First, we have represented the original problem in the form of (4.2), which concern mean value zero L^2 solutions, that is we need to show well-posedness for $L^2 \times H^{-2}$ data for the problem (4.2). Next, instead of considering the second order in time equation, we have reduced to the first order in time system of equations for $w^{\pm}$, (4.4). By an additional change of variables, this was replaced by the system (4.5) for the slightly modified $\tilde{w}^{\pm}$. Next, we have constructed in Section 4.2.1 and 4.2.2 explicitly a solution $h^{\pm}$ to the linear inhomogeneous system with right hand sides involving the free solutions. That is,

$$\tilde{w}^+ = \mathscr{L} + z^+ = \mathscr{L} + h^+ + \Psi^+; \;\; \tilde{w}^- = \bar{\mathscr{L}} + z^- = \bar{\mathscr{L}} + h^- + \Psi^-.$$

In terms of $w^\pm$

$$w^+ = e^{-(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}[\mathscr{L} + h^+ + \Psi^+]; \;\; w^- = e^{-(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}[\bar{\mathscr{L}} + h^- + \Psi^-]. \quad (4.17)$$

Given that, as we pointed out earlier, the operators $e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}$ are harmless (i.e. they preserve the relevant function spaces) and the explicit structure of $\mathscr{L}, h^\pm$, it now remains to resolve the nonlinear equation for $\Psi^\pm$, (4.13). We will do that, as we have indicated earlier, in the spaces $X^\pm_{\gamma,\frac{1}{2}+}$, where $\gamma < \min(\frac{1}{2}, 1-2\alpha)$.

Our next lemma shows that the initial data $\Psi^+(0,x)$ is H^1 smooth.

Lemma 17. *For $0 < \alpha < 1/2$ and $\varepsilon_1, \varepsilon_2 \in \{+1, -1\}$, we have $T^{\varepsilon_1,\varepsilon_2} : L^2 \times L^2 \to H^1$*

Proof. We define the symbols $\sigma^{\varepsilon_1,\varepsilon_2}$ based on the expression (4.9) so that

$$\begin{aligned} T^{\varepsilon_1,\varepsilon_2}(u,v)(x) &= \sum_{\xi,\eta\in\mathbb{Z}} \sigma^{\varepsilon_1,\varepsilon_2}(\xi,\eta)\widehat{u}(\xi)\widehat{v}(\eta)e^{i(\xi+\eta)x} \\ &= \sum_{\xi\in\mathbb{Z}} \left[\sum_{\eta\in\mathbb{Z}} \sigma^{\varepsilon_1,\varepsilon_2}(\xi-\eta,\eta)\widehat{u}(\xi-\eta)\widehat{v}(\eta)\right] e^{i\xi x}. \end{aligned}$$

Note from the sum in (4.9) that $\sigma^{\varepsilon_1,\varepsilon_2} \equiv 0$ if $\xi\eta(\xi+\eta) = 0$. Otherwise, we have

$$\begin{aligned} \sigma^{-,-}(\xi,\eta) &\sim \frac{\langle\xi\rangle^\alpha\langle\eta\rangle^\alpha}{\langle\xi+\eta\rangle^\alpha \max(\xi^2,\eta^2)}; \\ \sigma^{+,+}(\xi,\eta) &\sim \frac{1}{\langle\xi+\eta\rangle^\alpha\langle\xi\rangle^{1-\alpha}\langle\eta\rangle^{1-\alpha}}; \\ \sigma^{+,-}(\xi,\eta) &\sim \frac{\langle\xi\rangle^\alpha}{\langle\xi+\eta\rangle^{\alpha+1}\langle\eta\rangle^{1-\alpha}}. \end{aligned}$$

The following estimates are based on the size of symbols $\sigma^{\pm,\pm}$. This is justified by taking absolute values on the Fourier side.

Let $u, v \in L^2(\mathbf{T})$. Then

$$\begin{aligned}
\|T^{+,+}(u,v)\|_{H^1} &\sim \|\sum_{\eta\in\mathbf{Z}} \frac{\langle\xi\rangle^{1-\alpha}}{\langle\xi-\eta\rangle^{1-\alpha}\langle\eta\rangle^{1-\alpha}}\widehat{u}(\xi-\eta)\widehat{v}(\eta)\|_{L^2_\xi(\mathbf{Z})} \\
&\lesssim \|\sum_{|\eta|\ll|\xi|} \frac{\widehat{u}(\xi-\eta)\widehat{v}(\eta)}{\langle\eta\rangle^{1-\alpha}}\|_{L^2_\xi} + \|\sum_{|\eta|\gtrsim|\xi|} \frac{\widehat{u}(\xi-\eta)\widehat{v}(\eta)}{\langle\xi-\eta\rangle^{1-\alpha}}\|_{L^2_\xi} \\
&\lesssim \|\widehat{u}\|_{L^2_\xi}\sum_{\eta\in\mathbf{Z}}\frac{|\widehat{v}|(\eta)}{\langle\eta\rangle^{1-\alpha}} + \|\frac{\widehat{u}(\cdot)}{\langle\cdot\rangle^{1-\alpha}}\|_{L^1_\xi}\|\widehat{v}\|_{L^2_\xi} \\
&\lesssim \|u\|_{L^2(\mathbf{T})}\|v\|_{L^2(\mathbf{T})}.
\end{aligned}$$

$$\begin{aligned}
\|T^{+,-}(u,v)\|_{H^1} &\sim \|\sum_{\eta\in\mathbf{Z}} \frac{\langle\xi-\eta\rangle^{\alpha}}{\langle\xi\rangle^{\alpha}\langle\eta\rangle^{1-\alpha}}\widehat{u}(\xi-\eta)\widehat{v}(\eta)\|_{L^2_\xi(\mathbf{Z})} \\
&\lesssim \|\sum_{|\eta|\ll|\xi|} \frac{1}{\langle\eta\rangle^{1-\alpha}}\widehat{u}(\xi-\eta)\widehat{v}(\eta)\|_{L^2_\xi} + \|\frac{1}{\langle\xi\rangle^{\alpha}}\sum_{|\eta|\gtrsim|\xi|} \frac{\widehat{u}(\xi-\eta)\widehat{v}(\eta)}{\langle\xi-\eta\rangle^{1-2\alpha}}\|_{L^2_\xi} \\
&\lesssim \|\widehat{u}\|_{L^2_\xi}\sum_{\eta\in\mathbf{Z}}\frac{|\widehat{v}|(\eta)}{\langle\eta\rangle^{1-\alpha}} + \|\langle\nabla\rangle^{-\alpha}[v\cdot\langle\nabla\rangle^{2\alpha-1}u]\|_{L^2_x(\mathbf{T})} \\
&\lesssim \|u\|_{L^2(\mathbf{T})}\|v\|_{L^2(\mathbf{T})}
\end{aligned}$$

where we have used Sobolev embedding and Hölder's inequality to obtain

$$\begin{aligned}
\|\langle\nabla\rangle^{-\alpha}[v\cdot\langle\nabla\rangle^{2\alpha-1}u]\|_{L^2_x(\mathbf{T})} &\lesssim \|v\cdot\langle\nabla\rangle^{2\alpha-1}u\|_{L_x^{\frac{2}{2\alpha+1}}(\mathbf{T})} \\
&\lesssim \|v\|_{L^2_x(\mathbf{T})}\|\langle\nabla\rangle^{2\alpha-1}u\|_{L_x^{\frac{1}{\alpha}}(\mathbf{T})} \\
&\lesssim \|u\|_{L^2_x(\mathbf{T})}\|v\|_{L^2_x(\mathbf{T})}.
\end{aligned}$$

The estimate for $T^{-,-}$ follows from the fact $\sigma^{-,-} \le \sigma^{+,+}$ and we are done. $\square$

4.4.1 Reducing the proof to bilinear and trilinear estimates

Assume for a moment that for some $\gamma > 0$, $\Psi^+ \in X^+_{\gamma,1/2+}$. Then in the equation (4.11) for Ψ^+, the right-hand side consists of nonlinearities in the form $\tilde{\mathscr{N}}(u,v)$ where

$$(u,v) \in [X^{\pm}_{\gamma,\frac{1}{2}+} \times X^{\pm}_{0,\frac{1}{2}+}] \cup \left[L^\infty_t H^1_x \times L^\infty_t H^1_x\right] \cup [L^\infty_t H^1_x \times X^{\pm}_{0,\frac{1}{2}+}].$$

Therefore, in order to prove the theorem (as a result of a contraction argument in $X^+_{\gamma,\frac{1}{2}+}$), we need to control the nonlinear terms in appropriate norms. More precisely, we shall need following estimates for $\varepsilon_1, \varepsilon_2 \in \{-1,1\}$ in order to proceed with the standard contraction argument:

$$\|\tilde{\mathscr{N}}(u,v)\|_{X^+_{\gamma,-\frac{1}{2}+}} \lesssim \|u\|_{X^{\varepsilon_1}_{\gamma,\frac{1}{2}+}} \|v\|_{X^{\varepsilon_2}_{0,\frac{1}{2}+}} \tag{4.18}$$

$$\|\tilde{\mathscr{N}}(u,v)\|_{X^+_{\gamma,-\frac{1}{2}+}} \lesssim \|u\|_{L^\infty_t H^1_x} \|v\|_{L^\infty_t H^1_x}. \tag{4.19}$$

In addition, we would have liked to have

$$\|\tilde{\mathscr{N}}(u,v)\|_{X^+_{\gamma,-\frac{1}{2}+}} \lesssim \|u\|_{L^\infty_t H^1_x} \|v\|_{X^{\varepsilon_1}_{0,\frac{1}{2}+}} \tag{4.20}$$

but *this estimate turns out to be false*. On the other hand, the entry u is not just an arbitrary $L^\infty_t H^1_x$ function, but rather a bilinear expression in the form $T^{\varepsilon_1,\varepsilon_2}(e^{\pm itL} f, e^{\pm itL} g)$. Due to this fact, we replace (4.20) with a *tri-linear* estimate, see Lemma 20 below.

We also make the observation that in what follows, we can replace $\tilde{\mathscr{N}}$ by $\mathscr{N}$. Indeed, referring to (4.6) and taking into account that $e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}}$ preserves $X^{\pm}_{s,b}$, we have

$$\|\eta(t)\tilde{\mathscr{N}}(u,v)\|_{X^{\pm}_{s,b}} \leq C_\eta \|\mathscr{N}(e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} u, e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} v)\|_{X^{\pm}_{s,b}}.$$

Note that for $\tilde{u} = e^{(A_0 t + A_1 \frac{t^2}{2})\mathscr{P}} u$, we have from (4.16) that $\|\tilde{u}\|_X \le C\|u\|_X$ for all function spaces that appear in (4.18) and (4.19) and hence, it suffices to establish (4.18) and (4.19) with $\tilde{\mathscr{N}}$ replaced by $\mathscr{N}$.

We state the following results, which will be our main technical tools in order to finish the proof of Theorem 3. In them, we assume $\gamma \ge 0$.

Our next lemma is a proof of (4.18).

Lemma 18. *For u, v smooth and $0 \le \alpha < 1/2$, let γ be such that $2\alpha - 1/2 < \gamma < 1/2$. Then*

$$\|\mathscr{N}(u,v)\|_{X^{+}_{\gamma,-\frac{1}{2}+}} \lesssim \|u\|_{X^{\varepsilon_1}_{\gamma,\frac{1}{2}+}} \|v\|_{X^{\varepsilon_2}_{0,\frac{1}{2}+}}.$$

The next lemma concerns (4.19). More precisely, we have

Lemma 19. *For u, v smooth and $0 \le \alpha < 1/2$, let $\gamma: \gamma < 1/2$.*

$$\|\mathscr{N}(u,v)\|_{X^{+}_{\gamma,-\frac{1}{2}+}} \lesssim \|u\|_{L^\infty_t H^1_x} \|v\|_{L^\infty_t H^1_x}.$$

Finally, we deal with the tri-linear case, which is necessitated due to the failure of the appropriate bilinear estimate.

Lemma 20. *For $0 \le \alpha < 1/2$ and $\gamma < \min(1 - 2\alpha, 1/2)$, and u, v, w smooth,*

$$\|\mathscr{N}(T^{\varepsilon_1,\varepsilon_2}(u,v),w)\|_{X^{+}_{\gamma,-\frac{1}{2}+}} \lesssim \|u\|_{X^{\varepsilon_1}_{0,\frac{1}{2}+}} \|v\|_{X^{\varepsilon_2}_{0,\frac{1}{2}+}} \|w\|_{X^{\varepsilon_3}_{0,\frac{1}{2}+}}.$$

Remarks:

- From Figure 4.1, we note that $\gamma = 0$ is permissible up to $\alpha < 1/4$. This leads to the case described in [17]. The restriction $\gamma > 2\alpha - 1/2$ comes from Lemma 18. It is easy to see from this graph where improvements can be made via the normal from method.

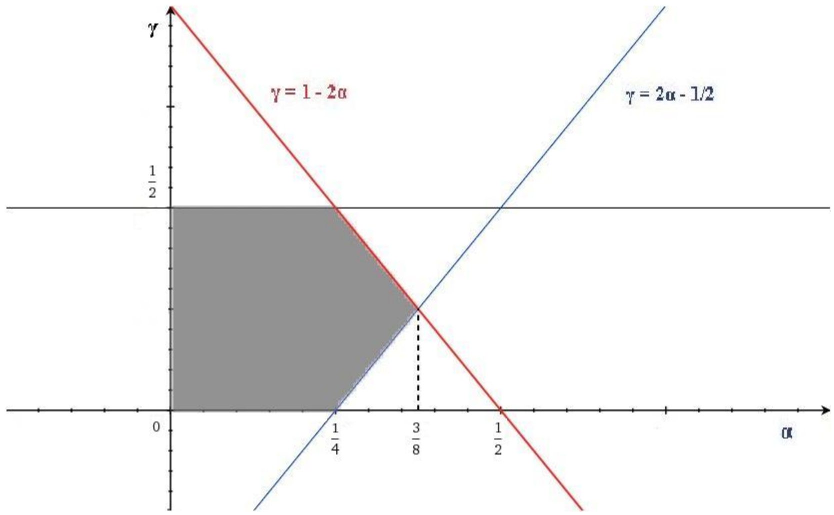

Figure 4.1: Permissible region for (α, γ)

- The restriction $\gamma < 1-2\alpha$ results from Lemma 20, and this is shown to be sharp in Section 4.5. This leads to the restriction $\alpha < 3/8$ instead of the sharp result $\alpha \leq 1/2$ obtained in [34].

For the purposes of these estimates, we treat $\mathcal{N}(u,v) \sim \langle\nabla\rangle^{-\alpha}(\langle\nabla\rangle^{\alpha}u\langle\nabla\rangle^{\alpha}v)$.

4.4.2 Proof of Lemma 18

Let $\lambda_j = \tau_j - \varepsilon_j\xi_j^2$ for $j = 1,2$ where $\tau = \tau_1 + \tau_2$ and $\xi = \xi_1 + \xi_2$. First we localize modulation $\tau - \varepsilon\xi^2$ of functions u, v by writing

$$\widetilde{u}(\tau,\xi) = \sum_{k=0}^{\infty} \chi_{[2^k,2^{k+1})}(\langle\tau - \varepsilon_1\xi^2\rangle)\widetilde{u}(\tau,\xi).$$

So in the following, we will assume that $\lambda_1 \sim L_1$, $\lambda_2 \sim L_2$ and $\tau - \xi^2 \sim L$ for some dyadic indices L_1, L_2, L. At the end of the estimate, we will have the bound in terms of summable constants in all dyadic indices (e.g. $L_{\max}^{-\delta/10}$ where $L_{\max} = \max(L, L_1, L_2)$).

We will show computations for the case $L_1 = L_{\max}$. It will be clear that the other cases follow in a similar manner. Applying the duality $(X^+_{s,b})^* = X^-_{s,b}$, we compute

$$\begin{aligned}
\|N(u,v)\|_{X^+_{\gamma,-\frac{1}{2}+\delta}} &\sim \sup_{\|w\|_{X^-_{0,\frac{1}{2}-\delta}}=1} \left| \int_{\mathbf{R}^1\times\mathbf{R}^1} \mathscr{N}(u,v)\,\langle\nabla\rangle^{\gamma} w\,dx\,dt \right| \\
&\sim \sup_{\|w\|_{X^-_{0,\frac{1}{2}-\delta}}=1} \left| \int_{\substack{\tau_1+\tau_2=\tau \\ \xi_1+\xi_2=\xi}} \frac{|\xi|\langle\xi_1\rangle^{\alpha}\langle\xi_2\rangle^{\alpha}}{\langle\xi\rangle^{1+\alpha-\gamma}} \widetilde{u}(\tau_1,\xi_1)\widetilde{v}(\tau_2,\xi_2)\,\widetilde{w}(\tau,\xi)\,d\sigma \right| \\
&\lesssim M_1 \sup_{\|w\|_{X^-_{0,\frac{1}{2}-\delta}}=1} \left| \int_{\substack{\tau_1+\tau_2=\tau \\ \xi_1+\xi_2=\xi}} \left[L_1^{\frac{1}{2}-\delta}\langle\xi_1\rangle^{\gamma}|\widetilde{u}| \right] |\widetilde{v}|\ |\widetilde{w}|\,d\sigma \right| \\
&\lesssim M_1 \sup_{\|w\|_{X^-_{0,\frac{1}{2}-\delta}}=1} \|\langle\lambda_1\rangle^{\frac{1}{2}-\delta}\langle\xi\rangle^{\gamma}\widetilde{u}\|_{L^2_\tau l^2_\xi} \left\| \mathscr{F}^{-1}_{\tau,\xi}|\widetilde{v}| \right\|_{L^4_{t,x}} \left\| \mathscr{F}^{-1}_{\tau,\xi}|\widetilde{w}| \right\|_{L^4_{t,x}} \\
&\lesssim M_1 \|u\|_{X^{\gamma,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}}
\end{aligned}$$

where

$$M_1 \sim \sup_{\substack{\xi_1+\xi_2=\xi \\ \xi\,\xi_1\,\xi_2\neq 0}} \frac{\langle\xi_1\rangle^{\alpha-\gamma}\langle\xi_2\rangle^{\alpha}}{\langle\xi\rangle^{\alpha-\gamma}L_{\max}^{\frac{1}{2}-\delta}}. \tag{4.21}$$

Note that we have used the embeddings $X^{\varepsilon}_{0,1/2+} \subset X^{\varepsilon}_{0,1/2-} \subset X^{\varepsilon}_{0,3/8} \subset L^4_{t,x}$ to obtain the last inequality above.

It suffices to show that M_1 is bounded by summable constants in $L_{\max}$. Let $N := \max(\xi_1,\xi_2)$ and note that $\xi \le 2N$ when $\xi_1+\xi_2=\xi$. Also we note the following

$$\lambda_1+\lambda_2 = \tau-\xi^2+\left[(\xi_1+\xi_2)^2-\varepsilon_1\xi_1^2-\varepsilon_2\xi_2^2\right].$$

Therefore, we must have $L_{\max} \gtrsim \left|(\xi_1+\xi_2)^2-\varepsilon_1\xi_1^2-\varepsilon_2\xi_2^2\right|$.

Case 1. When $\varepsilon_1 = \varepsilon_2 = -1$, we have $L_{max} \gtrsim N^2$. First if $\alpha \geq \gamma$, then $M_1 \lesssim N^{2\alpha-\gamma} L_{\max}^{-\frac{1}{2}+\delta} \lesssim N^{2\alpha-\gamma-1+4\delta} L_{\max}^{-\delta}$. Therefore, we need to have $\gamma > 2\alpha - 1$ and appropriately small $\gamma > 0$.

Otherwise, if $\alpha < \gamma$, then $M_1 \lesssim N^{\gamma} L_{\max}^{\frac{1}{2}-\delta} \lesssim N^{\gamma-1+4\delta} L_{\max}^{-\delta}$, so $\gamma < 1$ would suffice.

Case 2. If $\varepsilon_1 = \varepsilon_2 = +1$, then we have $L_{\max} \gtrsim \xi_1 \xi_2$. Then

$$M_1 \lesssim \frac{\langle \xi_1 \rangle^{\alpha-\gamma-\frac{1}{2}+2\delta} \langle \xi_2 \rangle^{\alpha-\frac{1}{2}+2\delta}}{\langle \xi \rangle^{\alpha-\gamma} L_{\max}^{\delta}}.$$

If $\alpha \geq \gamma$, then it suffices to require $\gamma \geq 0$ and $\alpha < 1/2$.

If $\alpha < \gamma$, then it suffices to require $\gamma < 1/2$ and $2\alpha - 1 < \gamma$.

Case 3. The remaining cases are either $\varepsilon_1 = +1, \varepsilon_2 = -1$ or $\varepsilon_1 = -1, \varepsilon_2 = +1$. The first case gives $L_{\max} \gtrsim \xi \xi_2$ and the second gives $L_{\max} \gtrsim \xi \xi_1$. So we have respectively

$$M_1 \lesssim \frac{\langle \xi_1 \rangle^{\alpha-\gamma} \langle \xi_2 \rangle^{\alpha-\frac{1}{2}+2\delta}}{\langle \xi \rangle^{\alpha-\gamma+\frac{1}{2}-2\delta} L_{\max}^{\delta}} \qquad \text{or} \qquad M_1 \lesssim \frac{\langle \xi_1 \rangle^{\alpha-\gamma-\frac{1}{2}+2\delta} \langle \xi_2 \rangle^{\alpha}}{\langle \xi \rangle^{\alpha-\gamma+\frac{1}{2}-2\delta} L_{\max}^{\delta}}.$$

In both cases, if $\xi \sim N$, then it suffices to require $\gamma < 1/2$.

If $\xi \ll N$, then both estimates give $M_1 \lesssim N^{2\alpha-\gamma-\frac{1}{2}+2\delta} L_{\max}^{-\delta}$. Therefore we need to require $2\alpha - 1/2 < \gamma$. We remark that this is the strongest bound which as appeared for this lemma.

Next, we prove Lemma 19.

4.4.3 Proof of Lemma 19

We will ignore the gain due to $\lambda^{1/2-}$ for this proof.

$$\|\mathcal{N}(u,v)\|_{L_T^2 H_x^{\gamma}} \sim \| \sum_{\xi_1+\xi_2=\xi} \frac{|\xi| \langle \xi_1 \rangle^{\alpha} \langle \xi_2 \rangle^{\alpha}}{\langle \xi \rangle^{1+\alpha-\gamma}} \widehat{u}(\xi_1) \widehat{v}(\xi_2) \|_{L_T^2 l_{\xi}^2}$$

$$\sim \left\| \sum_{\xi_1+\xi_2=\xi} \frac{|\xi|\langle\xi_1\rangle^{\alpha-1}\langle\xi_2\rangle^{\alpha-\frac{1}{2}+\delta}}{\langle\xi\rangle^{1+\alpha-\gamma}} [\langle\xi_1\rangle\widehat{u}(\xi_1)][\langle\xi_2\rangle^{\frac{1}{2}-\delta}\widehat{v}(\xi_2)] \right\|_{L^2_T l^2_\xi}$$

$$\lesssim M_2 \left\| \left|\widehat{\langle\nabla\rangle u}\right| *_\xi \left|\widehat{\langle\nabla\rangle^{\frac{1}{2}-\delta} v}\right| \right\|_{L^2_T l^2_\xi} \lesssim M_2 \left\| \mathscr{F}_\xi^{-1} \left|\widehat{\langle\nabla\rangle u}\right| \right\|_{L^\infty_T L^2_x} \left\| \mathscr{F}_\xi^{-1} \left|\widehat{\langle\nabla\rangle^{\frac{1}{2}-\delta} v}\right| \right\|_{L^2_T L^\infty_x}$$

$$\lesssim_\delta M_2 \|u\|_{L^\infty_t H^1_x} \|v\|_{L^2_T H^1_x} \lesssim_T M_2 \|u\|_{L^\infty_t H^1_x} \|v\|_{L^\infty_T H^1_x(\mathbf{T})}$$

where

$$M_2 \sim \sup_{\xi_1+\xi_2=\xi} \frac{\langle\xi_1\rangle^{\alpha-1}\langle\xi_2\rangle^{\alpha-\frac{1}{2}+\delta}}{\langle\xi\rangle^{\alpha-\gamma}}.$$

Note that we have used Sobolev embedding $H_x^{1/2+}(\mathbf{T}) \subset L_x^\infty(\mathbf{T})$ above. To prove the desired estimate, we need to bound M_2 by an absolute constant.

If $\alpha \geq \gamma$, then it suffices to have $\alpha < 1/2$.

If $\alpha < \gamma$, then it suffices to have $\gamma < 1/2$.

Lastly, we prove Lemma 20.

4.4.4 Proof of Lemma 20

In this proof, we will cover the cases when $\varepsilon_1 = +1$ and $\varepsilon_2 = -1$; or $\varepsilon_1 = \varepsilon_2 = +1$. The remaining case $\varepsilon_1 = \varepsilon_2 = -1$ is easier due to a faster decay in ξ_1, ξ_2, so it will not be argued here.

Case 1. First we consider the case where $\varepsilon_1 = +1$, $\varepsilon_2 = -1$. Let $\lambda_j = \tau_j - \varepsilon_j \xi_j^2$ for $j = 1,2,3,4$ where $\varepsilon_4 = -\varepsilon = -1$. As in the proof of Lemma 18, we localize modulations of each functions with respect to dyadic indices L_1, L_2, L_3, L_4 so that $\langle\tau_j - \varepsilon_j \xi_j\rangle \sim L_j$ for $j = 1,2,3,4$. In the end, we will have an estimate in terms of a summable bound for $L_{\max} := \max(L_1, L_2, L_3, L_4)$.

Let $\Gamma := \{(\tau,\xi) \in \mathbf{R}^4 \times \mathbf{Z}^4 : \tau_1+\tau_2+\tau_3+\tau_4 = 0, \xi_1+\xi_2+\xi_3+\xi_4 = 0\}$ and let $d\sigma$ be the inherited measure on Γ. Then

$$\|\mathscr{N}(T^{+,-}(u,v),w)\|_{X^+_{\gamma,-\frac{1}{2}+\delta}} \sim \sup_{\|z\|_{X^-_{0,\frac{1}{2}-\delta}}=1} \left| \int_\Gamma a(\xi)\,\widetilde{u}(\tau_1,\xi_1)\,\widetilde{v}(\tau_2,\xi_2)\,\widetilde{w}(\tau_3,\xi_3)\,\widetilde{z}(\tau_4,\xi_4)\,d\sigma \right|$$

where

$$a(\xi) \sim \frac{\langle\xi_1\rangle^\alpha \langle\xi_3\rangle^\alpha \langle\xi_4\rangle^{\gamma-\alpha}}{\langle\xi_1+\xi_2\rangle\langle\xi_2\rangle^{1-\alpha}} \qquad \text{if } \xi_1\xi_2\xi_4(\xi_1+\xi_2) \neq 0$$

and $a(\xi) = 0$ otherwise. If $L_{\max} \sim L_1$ for instance, the integral above can be estimated as follows.

$$\begin{aligned}
\int_\Gamma |a\widetilde{u}\widetilde{v}\widetilde{w}\widetilde{z}|\,d\sigma &\lesssim \int_\Gamma \left| \frac{a}{L_1^{\frac{1}{2}+}} \lambda_1^{\frac{1}{2}+} \widetilde{u}\widetilde{v}\widetilde{w}\widetilde{z} \right| d\sigma \\
&\lesssim \sup_\xi \left| \frac{a(\xi)\langle\xi_2\rangle^\delta\langle\xi_3\rangle^\delta\langle\xi_4\rangle^\delta}{L_{\max}^{\frac{1}{2}-\delta}} \right| \|L_1^{\frac{1}{2}+}u\|_{L^2_{t,x}} \|v_\delta w_\delta [L_4^{-2\delta} z_\delta]\|_{L^2_{t,x}} \\
&\lesssim \sup_\xi \left| \frac{a(\xi)N^{3\delta}}{L_{\max}^{\frac{1}{2}-\delta}} \right| \|u\|_{X^{0,\frac{1}{2}+\delta}} \|v_\delta\|_{L^6_{t,x}} \|w_\delta\|_{L^6_{t,x}} \|L_4^{-2\delta} z_\delta\|_{L^6_{t,x}} \\
&\lesssim M_3 \|u\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}+\delta}} \|z\|_{X^{0,\frac{1}{2}-\delta}}
\end{aligned}$$

where Let $N := \max(|\xi_1|,|\xi_2|,|\xi_3|,|\xi_4|)$, $u_\delta := \mathscr{F}^{-1}_{\tau,\xi}\left[\langle\xi\rangle^{-\delta}|\widetilde{u}|(\tau,\xi)\right]$ and

$$M_3 := \sup_{(\tau,\xi)\in\Gamma} \frac{\langle\xi_1\rangle^\alpha\langle\xi_3\rangle^\alpha\langle\xi_4\rangle^{\gamma-\alpha}N^{3\delta}}{\langle\xi_1+\xi_2\rangle\langle\xi_2\rangle^{1-\alpha}L_{\max}^{\frac{1}{2}-\delta}}$$

Note that we have used $X^{0+,\frac{1}{2}+} \subset L^6_{t,x}$ for the last inequality. Now it suffices to bound M_3 by a constant summable in $L_{\max}$. First we observe the following scenarios:

$$\varepsilon_3 = +1: \qquad \sum_{j=1}^4 \lambda_j = -\xi_1^2 + \xi_2^2 - \xi_3^2 + \xi_4^2 = 2(\xi_1+\xi_2)(\xi_2+\xi_3); \tag{4.22}$$

$$\varepsilon_3 = -1: \qquad \sum_{j=1}^4 \lambda_j = -\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = -2(\xi_2\xi_3 + \xi_3\xi_4 + \xi_4\xi_2). \tag{4.23}$$

We split into the following cases for this estimate:

Case 1A. If $|\xi_1+\xi_2| \gtrsim \max(|\xi_3|,|\xi_4|)$, then for $\alpha < 1/2$ and $\gamma < 1/2$,

$$M_3 \lesssim \frac{\langle \xi_1 \rangle^{\alpha} N^{3\delta}}{\langle \xi_1+\xi_2 \rangle^{1-\gamma} \langle \xi_2 \rangle^{1/2} L_{\max}^{\frac{1}{2}-\delta}} \lesssim \frac{1}{L_{\max}^{\frac{1}{2}-\delta}}.$$

So we are done. Negation of Case 1A gives $|\xi_1+\xi_2| \ll \max(|\xi_3|,|\xi_4|)$, which implies $\xi_3 \sim \xi_4$ because of the relation $\xi_1+\xi_2+\xi_3+\xi_4 = 0$. The next case covers the possibility that ξ_1, ξ_2 may be large with opposite signs.

Case 1B. Negation of Case 1A and also $\max(|\xi_1|,|\xi_2|) \sim N$. Note that since $|\xi_1 + \xi_2| \ll N$, we must have $\xi_1 \sim \xi_2$. Then

$$M_3 \lesssim \frac{\langle \xi_3 \rangle^{\gamma}}{N^{1-2\alpha-3\delta} L_{\max}^{\frac{1}{2}-\delta}}.$$

So we must have $\gamma < 1-2\alpha$. This is where the upper bound in Lemma 20 for γ originates from. We remark that this is completely necessary due to the cases such as

$$\xi_1 = N+1, \qquad \xi_2 = -N, \qquad \xi_3 = N, \qquad \xi_4 = -N-1. \tag{4.24}$$

Note that if above holds, $L_{\max}$ does not have to be comparable N in the case (4.22), thus the bound for M_3 cannot be improved. We have used (4.24) to construct a counter-example for the cases $\gamma > 1-2\alpha$.

By similar computations as above, the special case $\gamma = 1-2\alpha$ can be shown to be true if $X^{0,1/2+} \subset L^6_{t,x}$ were true. However, this is an open conjecture of Bourgain (see [9]) and it does not have a significant bearing on our conclusion, so we overlook this case.

Case 1C. Now the remaining case is when $\max(|\xi_1|,|\xi_2|) \ll N$. Recall that $N \sim \xi_3 \sim \xi_4$. This implies that $\xi_2 + \xi_3 \sim N$, so the case (4.22) gives that $L_{\max} \gtrsim N$. The case (4.23) is even better since this gives $L_{\max} \sim N^2$. So we take the lesser of these two bounds to estimate M_3 below. Since $|\xi_1| \leq 2\max(|\xi_1+\xi_2|,|\xi_2|)$,

$$M_3 \lesssim \frac{\langle \xi_1 \rangle^{\alpha}}{\langle \xi_1 + \xi_2 \rangle \langle \xi_2 \rangle^{1-\alpha}} \frac{N^{\gamma+3\delta}}{L_{\max}^{\frac{1}{2}-\delta}} \lesssim N^{\gamma - \frac{1}{2}+5\delta} L_{\max}^{-\delta}.$$

So it suffices to require $\gamma < 1/2$. This exhausts all cases for Case 1.

Case 2. Now we consider the case where $\varepsilon_1 = \varepsilon_2 = +1$. Following the same arguments as in the previous case, we have

$$a(\xi) \sim \frac{\langle \xi_3 \rangle^{\alpha} \langle \xi_4 \rangle^{\gamma-\alpha}}{\langle \xi_1 \rangle^{1-\alpha} \langle \xi_2 \rangle^{1-\alpha}} \qquad \text{if } \xi_1 \xi_2 \xi_4 (\xi_1 + \xi_2) \neq 0$$

and $a(\xi) = 0$ otherwise. By the same series of estimates, it suffices to estimate M_4 by a constant summable in $L_{\max}$ where

$$M_4 := \sup_{(\tau,\xi)\in\Gamma} \frac{\langle \xi_3 \rangle^{\alpha} \langle \xi_4 \rangle^{\gamma-\alpha} N^{3\delta}}{\langle \xi_1 \rangle^{1-\alpha} \langle \xi_2 \rangle^{1-\alpha} L_{\max}^{\frac{1}{2}-\delta}}.$$

In this case, we have the following scenarios:

$$\varepsilon_3 = +1: \qquad \sum_{j=1}^{4} \lambda_j = -\xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2 = 2(\xi_1\xi_2 + \xi_3[\xi_1 + \xi_2]); \tag{4.25}$$

$$\varepsilon_3 = -1: \qquad \sum_{j=1}^{4} \lambda_j = -\xi_1^2 - \xi_2^2 + \xi_3^2 + \xi_4^2 = 2(\xi_1 + \xi_3)(\xi_2 + \xi_3). \tag{4.26}$$

Case 2A. If $|\xi_1\xi_2| \gtrsim N$, then $M_4 \lesssim N^{\gamma+\alpha-1+3\delta} L_{\max}^{-1/2+\delta}$. So we are done since $\gamma < 1/2$ and $\alpha < 1/2$.

Case 2B. The remaining cases must have $|\xi_1\xi_2| \ll N$, which implies $\xi_3 \sim \xi_4 \sim N$. Then the case (4.25) gives $L_{\max} \gtrsim N$. On the other hand, the case (4.26) gives $L_{\max} \gtrsim N^2$. We use the lesser of these two to estimate

$$M_4 \lesssim \frac{N^{\gamma+3\delta}}{L_{\max}^{\frac{1}{2}-\delta}} \lesssim N^{\gamma-\frac{1}{2}+5\delta} L_{\max}^{-\delta}.$$

Since this is summable for $\gamma < 1/2$, we are done.

4.5 Proof of the sharpness of Lemma 20

In this section, we construct an explicit counter-example to show that the following estimate fails if $\gamma > 1 - 2\alpha$

$$\|\mathscr{N}(T^{+,-}(u,v),w)\|_{X^{+}_{\gamma,-\frac{1}{2}+\delta}} \leq C_\delta \|u\|_{X^{+}_{0,\frac{1}{2}+\delta}} \|v\|_{X^{-}_{0,\frac{1}{2}+\delta}} \|w\|_{X^{+}_{0,\frac{1}{2}+\delta}}. \tag{4.27}$$

Given $\eta \in \mathscr{S}_t(\mathbb{R})$ and $N \gg 1$, let u, v, w be defined as follows:

$$u(t,x) := \eta(t) e^{i(N+1)^2 t + i(N+1)x}; \quad v(t,x) = \eta(t) e^{-iN^2 t - iNx}; \quad w(t,x) = \eta(t) e^{iN^2 t + iNx}.$$

First, we remark that the right side of (4.27) is equal to $C\|\eta\|^3_{H_t^{1/2+\delta}}$, where C is independent of N. Substituting these functions to (4.9), we obtain

$$T^{+,-}(u,v)(t,x) = C_\alpha \eta^2(t) \frac{\langle N+1\rangle^\alpha \langle N\rangle^\alpha e^{i(2N+1)t+ix}}{N\left[\langle N+1\rangle - \langle N\rangle\right] + \langle N\rangle - \sqrt{2}}.$$

Recall $\mathscr{N}(u,v) = |\nabla|\langle\nabla\rangle^{-1-\alpha}\left[\langle\nabla\rangle^\alpha u \langle\nabla\rangle^\alpha v\right]$. Then writing $\phi = \eta^3$, we have

$$\mathscr{N}(T^{+,-}(u,v),w) = C_\alpha \phi(t) \frac{|N+1|\langle N\rangle^{2\alpha} e^{i(N+1)^2 t + i(N+1)x}}{\langle N+1\rangle(N\left[\langle N+1\rangle - \langle N\rangle\right] + \langle N\rangle - \sqrt{2})}.$$

Then

$$\|\mathscr{N}(T^{+,-}(u,v),w)\|_{X^+_{\gamma,-\frac{1}{2}+\delta}} = C(N,\alpha,\gamma)\left(\int_{\mathbb{R}} \frac{|\widehat{\phi}|^2(\tau-(N+1)^2)}{\langle \tau-(N+1)^2\rangle^{1-2\delta}}\,d\tau\right)^{\frac{1}{2}} \qquad (4.28)$$

where

$$C(N,\alpha,\gamma) := C\frac{|N+1|\langle N\rangle^{2\alpha}}{\langle N+1\rangle^{1-\gamma}(N\left[\langle N+1\rangle - \langle N\rangle\right] + \langle N\rangle - \sqrt{2})}.$$

Note that the integral in (4.28) becomes independent of N after a change of variable. Also, for large N, $C(N,\alpha,\gamma) \sim N^{2\alpha+\gamma-1}$. Since the right side of (4.27) is independent of N, the trilinear estimate cannot hold if $2\alpha+\gamma > 1$.

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