

SOLUTION PROPERTIES OF DETERMINISTIC AUCTIONS

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I. Introduction

A market can be imperfectly competitive for a variety of reasons; in the context of an auction or a contract awarding, imperfections may stem from the limited number of bidders involved. Bidders, recognizing that their behavior (or that of others) can affect the market outcome, may adopt strategies that are unlikely to lead to a Pareto efficient allocation. Such inefficiencies can occur in the absence of any collusive behavior on the part of bidders. If barriers to bid entry are removed, and bidders are sufficiently homogeneous, the likelihood increases that bids will reflect full (private) valuations of the auctioned goods. Under these conditions Pareto efficient allocations would be guided by a set of minimum prices: a "sale to the highest bidder" would be transacted at a price approximate to the valuation of the second highest bidder, and contracts would be awarded at the competitive supply price. Even when the number of bidders is restricted, auction procedures can be adopted which will insure efficiency to a degree. This efficiency is achieved by changing the motivations of the available bidders, and by providing incentives for bidders to reveal their full valuations of the objects being auctioned. This paper describes a set of auction procedures which achieve these ends.

Since auctions can be viewed as  $n$  person noncooperative games, some kinds of auctions call for strategic bidding based on the expected actions of others as well as one's own valuations. In these game situations, rational bidding entails the assessment of probable bids by others, and the balancing of the potential gain of a lower successful bid against the reduced (subjective) probability of success. These elements characterize what we call *probabilistic* auctions.

Much of the research relating to auctions has focused on these strategic aspects of particular bidding situations.<sup>1</sup> Some sharp results have emerged

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<sup>1</sup>The recent comprehensive bibliography compiled by Stark [11] bears out this claim.

from this line of inquiry (e.g., Vickrey [12], and Griesmer, Shubik and Levitan [5], but it is fair to say that very modest assumptions about players' (bidders') subjective beliefs give way to game characterizations that are quite complicated<sup>2</sup> or analytically intractable.

This paper is concerned with the solution properties of *deterministic* auctions. Here, unlike in *probabilistic* auctions, a bidder only needs to know his own valuation of an object to submit a rational bid. Indeed, such bids will equal one's full valuations of objects, unless the bidder is constrained by his resource limitations. As might be expected, the auction rules that establish transaction prices at the same time identify an auction as either *probabilistic* or *deterministic*. If prices are set by what we call the *second bid price mechanism*, the auction is *deterministic*; the resulting transaction prices are referred to as demand prices. The primary purposes of this paper are to establish the importance of demand prices in auction theory and to provide solution methods for a variety of *deterministic* auctions. The formal similarities between auctions and the linear assignment problem enable us to develop algorithms that solve for the desired commodity allocations and demand prices.

We first present some basic properties of auctions and define different types of auctions according to (1) the number of kinds of commodities offered, and (2) the rules which determine the price and allocation of the auctioned commodities. Then, in Section III, we discuss efficiency criteria for auction outcomes, and reconsider *deterministic* auctions as linear assignment problems. In Section IV we derive solutions for *deterministic* auctions.

## II. Auction Types

An auction can be defined as a specialized, dated market that is governed by an announced set of rules and procedures. The basic asymmetry in an auction is that sellers determine the rules and procedures of the market and minimum transaction prices,<sup>3</sup> and thereafter act passively; buyers accept the rules and procedures and then bid actively to determine transaction prices. There is a surprisingly large variety of auction procedures in use. To establish the setting for our subsequent analysis, we first offer some properties (P) of auction markets:

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<sup>2</sup>The case in which only two bidders draw a single object valuation from different rectangular distributions, and the highest bidder receives the award at a price equal to his submitted bid is analytically complex; see [5].

<sup>3</sup>To avoid confusion, we restrict our discussion here to the situation in which a higher bid dominates a lower one. Auctions where minimum bids dominate, such as in contract awardings, can be treated symmetrically.

- P1 The auctioned commodity (s) has individualistic characteristics and is in fixed supply. Usually the commodity is nonduplicable and "dated," e.g., antiques or securities of a particular issue.<sup>4</sup>
- P2 Market transactions occur at a specific point in time (the auction date); there is no continuous, prevailing price for auctioned commodities.
- P3 The organization of the market is dictated by the commodity sellers or their agents, the "auctioneers."
- P4 Bidders are able to make personal money valuations of commodities (sometimes only in a probabilistic sense), and their bidding actions are based on these (expected) valuations.
- P5 Bids carry contractual responsibility, if accepted. This property is included to assure the integrity of submitted bids (and to prevent bidders from bidding beyond their available resources). A sufficiently large penalty for renegeing can be assumed to exist.
- P6 Bidders act independently and in their own self-interest. A rational bid in this context maximizes one's expected utility of the auction outcome.

It is expositionally convenient to distinguish among auction types according to several characteristics (C).

C1. The Offering. Commodities are (usually) auctioned off in discrete units. An auction consists of either a single or multiple commodity offering, with single or multiple units of each commodity available. An *object* will refer to a single unit of some commodity.

C2. Bidding Mode. Two basic mechanisms are used to auction commodities. The *sealed bid* method requires the submission of bids before a set deadline, after which the bids are revealed and ranked. In some multiple commodity or multiple unit auctions, bidders are free to submit several bids. The *real time (progressive)* auction method requires participants to bid until a transaction is signaled, according to the rules defining the award mode.

C3. The Award Mode. The most distinctive characteristic of an auction is the way in which objects are awarded to bidders. The award mode refers to both the way a successful bidder is determined, and the price at which the transaction takes place. Two basic award modes characterize multiple unit, sealed bid auctions. Such an auction is *price discriminant* if each successful bidder must pay the price he bids. That is, if  $n$  units are available, the submitted bids are awarded a unit, with each unit sold at the individually bid price. In the event of a tie on the margin a fair coin is used to decide the awardee.

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<sup>4</sup> Supply may be fixed only in the sense of the Marshallian short run. Commodities may be dated only by the date of the auction itself, e.g., 91-day Treasury Bills.

Alternatively, the auction is *price competitive* if each successful bidder pays a uniform price equal to the  $n^{\text{th}}$  highest bid. In a single unit, sealed bid auction, both award modes imply a sale to the highest bidder.

The distinction between these two award modes is more subtle in real time auctions. The familiar "sale to the highest bidder" auction conforms to the price competitive award mode, since a successful bidder need only bid slightly more than any other bidder to obtain the object. The price discriminant award mode is conducted by first setting an arbitrarily high asking price for the object and allowing the price to fall steadily until a bid is made.<sup>5</sup> In this way a bidder can only capture some "consumer surplus" at the risk of losing the award.

The award modes just described are used in most auctions, although some variations are used to further segment the bidder market. For example, the weekly U.S. Treasury Bill auction is basically a price-discriminant sealed bid auction, with small investors permitted to buy at the average realized price (so-called "noncompetitive bids").

The second-bid price method is a variant of the price competitive award mode. In single unit, sealed bid auctions, the object would be awarded to the highest bidder at a price equal to the second highest bid submitted. For the multiple unit case, the highest unsuccessful bid would determine the uniform transaction price. In real time bidding, the award can be made by one of two methods: the familiar "sale to the highest bidder" could be transacted at a price equal to the penultimate bid. Alternatively a modified Dutch auction procedure could be used: As the ask price falls, bidders could register bids in secret with the auction terminating when the first unsuccessful bid is registered. The appeal of the second bid price mode is that it eliminates any incentive (in the absence of collusion) on the part of bidders to bid anything less than their full valuation of an object. In second bid price auctions, a bidder need not concern himself with the preferences and strategies of others; the transaction price will be independent of his bid if he is the successful bidder. This award mode, then, defines the class of deterministic auctions which we analyze in subsequent sections.

C4. Bidder Constraints. An obvious constraint facing a bidder is the limit of his resources.<sup>6</sup> Individual rationality (P6) precludes the submission of bids

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<sup>5</sup>This award mode has come to be known as a Dutch auction, for it is used in the Netherlands flower market. (See Vickrey [12].)

<sup>6</sup>Curiously the effect of resource constraints has been ignored in previous treatments of auction problems. These constraints are consequential in the solution of deterministic auctions.

in excess of one's resources. In a price-discriminant auction, since a successful bid becomes the transaction price, one's resources effectively limit his bids. Similarly, in a price-competitive auction the price setting bidder can ill afford to exceed his resource limit in view of P5. In second-bid price auctions, a bid in excess of one's resources introduces the probability of renegeing.<sup>7</sup>

A second kind of constraint is a limit on the number of objects a bidder is permitted to obtain in an auction. This constraint must be built into the auction rules, just as "limits per customer" are stipulated in promotional sales. The purchase limit can stimulate competition in multiple commodity auctions, wherein a resource constrained bidder can submit offers for a number of commodities with the understanding that he is contractually responsible for, say, only one commodity. This kind of auction situation can arise naturally from discreteness in consumer demand, say for housing or dowried brides, or in assignment problem situations.<sup>8</sup>

### III. The Allocative Efficiency of Auction Procedures

An important efficiency criterion that is applicable to any auction outcome is that of Pareto optimality: Considering the entire set of bidders and sellers, no reallocation (and prices) of the auctioned commodities is possible that would benefit someone without imposing a loss on another. This measure of efficiency is conditioned by the auction ground rules (e.g., a "one to a customer limit"), and generally by the resource limits of bidders and the codetermination of transaction prices for all objects up for auction. For the seller, Pareto optimality implies that the transaction price is no less than his valuation of the object (his reserve price). For the bidders the difference between one's valuation of an object and its transaction price is called a surplus. For a successful bidder that difference is his earned or realized surplus. Multiple

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<sup>7</sup>In both price-competitive and second-bid price auctions, an exceptional case calls for rational bidding beyond resource limits. If one knows with certainty that the first unsuccessful bid will exactly equal his resource limit in the absence of his bid, he can avoid ties by bidding slightly higher. P5 and P6 are intended to rule out this case.

<sup>8</sup>This kind of auction game has been used on an experimental basis to assign teaching responsibilities to faculty in university departments. Scrip are presented to bidders on some basis, and courses are awarded to the highest bidders subject to teaching load limits. Unspent scrip can be accumulated without interest. In the experiment that we are aware of the price-discriminant award mode was used. The second-bid price mode would have eliminated the gaming aspects of the auction, and some of its consumption value. A considerable amount of bidder collusion was in evidence.

object auctions then can be viewed as a discrete set of choices, with the opportunity cost of a successful bid being the maximum surplus that could have been earned on an alternative choice. Pareto optimality requires that each bidder's opportunity cost be no greater than his realized surplus.<sup>9</sup> In probabilistic auctions it is always rational to bid somewhat less than one's full valuation of an object, by an amount that is dependent on one's expectations of his competitors' bids. Here, miscalculations can lead to non-Pareto optimal outcomes.<sup>10</sup> In contrast, Pareto optimality is assured in deterministic auctions since submitted bids equal full valuations.

A second efficiency issue centers around the expected transaction prices that would be realized under alternative auction procedures. We have seen that rational bidding leads to a lower registered demand schedule (the array of bids for any commodity) in probabilistic auctions than in deterministic auctions. The extent of this downward shift in demand depends on (1) the characteristics of the (probabilistic) auction, (2) the number and nature of bidders, and (3) the availability of information on (expected) bidder behavior. The simplest characterization of this efficiency issue asks whether seller revenue is maximized in a single commodity, multiple unit auction under the price-discriminant or the second-bid price award mode. The answer depends of course on the circumstances mentioned; either mode can yield a higher average price.

This issue has produced a lively debate over the efficiency of the U.S. Treasury Bill auction. Advocates of the Treasury's price discriminant procedure (Brimmer [2], Goldstein [4] have argued that it maximizes revenue, while persuasive arguments for price-competitive or deterministic-auction alternatives have been made on both theoretic (Friedman [3] and Smith [10]), and empirical (Smith [9] and Bolten [1]) grounds. Essentially, the opponents argue that greater revenue would be realized in a deterministic auction through increased bidder participation and higher individual bids. The more basic efficiency issue surrounding this expected revenue debate, however, involves the real resource cost of gathering the information required to make rational bids in probabilistic auctions. For even if the deterministic auction would return a higher average price to the Treasury, the expected cost to the bidders need not

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<sup>9</sup> For a single object auction, this implies that the person placing the highest value on the object is the successful bidder.

<sup>10</sup> Some auction procedures insure nonoptimal allocations according to this criterion. For example, in sealed bid auctions of new securities in France, a percentage of the highest bids is rejected, and the remaining bids are fractionally accepted to achieve a target (uniform) transaction price. For a fascinating analysis of this auction, see McDonald and Jacquillat [7].

be higher. In probabilistic auctions, bidding costs drive a wedge between object valuations and the realized transaction prices. In the Treasury auction these costs are embodied in the trading specialists who bid for the bills, and then sell to less informed demanders in a secondary market. The elimination of such costs can represent a substantial efficiency gain.

In [12, p. 28], William Vickrey has established the efficiency of deterministic auctions involving identical objects and a "one to a customer" limit:

It is possible, by establishing in advance that the price is to be determined by the first rejected bid, to achieve the Pareto-optimal result. Moreover, in spite of this method's appearing to accord a lower price than necessary after the bids are in, the higher level of bids induced by this method results, on balance, in a price averaging-out at the same level as would be obtained under Dutch auction, individual bid pricing, or last-accepted-bid pricing methods, at least for cases where the bidders are symmetrical with respect to the *a priori* information which each one has about the probability distribution of the values or bids of the others. In such cases there is a rather strong presumption that a switch from other methods of negotiation to a first-rejected-bid pricing method would be to the long-run advantage of all concerned, the gain being derived from the greater certainty of obtaining a Pareto-optimal result and from the reduction in non-productive expenditure devoted to the sizing-up of the market by the bidders. To be sure, these conclusions are based on a model in which a high degree of rationality and sophistication is imputed to the bidders; nevertheless, in many markets the frequency of the dealings and the professional characteristics of the dealers are such as to make such an assumption not too far from reality; moreover, the change to the first-rejected-price method would substantially diminish the amount of sophistication required to achieve the optimal result.

Vickrey [12, pp. 24,26] erroneously rejected the applicability of demand pricing to auctions involving dissimilar objects or to auctions where bidders may purchase more than one object:

(Demand pricing) is applicable, however, only if the items are actually identical so that there is no problem of deciding who gets first choice, and no variation in the value imputed to the various items by a given bidder... (Further it) applies only to cases where each bidder is interested in a single unit, and there is no collusion among the bidders.

We now present a formal analysis of deterministic auction procedures and provide solutions for these rejected cases. Demand prices are determined via a variation of the optimal assignment problem in linear programming. For the case in which bidders are restricted to the purchase of at most one object, demand prices will be determined when potential buyers have resource constraints. The solutions developed here are offered on the one hand as normative prescriptions for auction procedures, and on the other hand as a description of price determination in approximately competitive markets such as residential housing.

#### IV. Deterministic Auction Solutions

Deterministic auctions are formally similar to the optimal assignment problem in linear programming. Our analytic approach is first to consider an auction in which a set of bidders  $I$  place true valuations  $c_{ij}$  on a set of distinct objects  $J$ , with each bidder constrained to obtain at most one object. This setting corresponds closely to the optimal assignment problem; we then consider variations of this problem that delimit the class of deterministic auctions. The optimal assignment problem can be written,

$$\begin{aligned}
 (1a) \quad & \text{Max } \sum_i \sum_j c_{ij} X_{ij} \\
 (1b) \quad & \sum_i X_{ij} \leq 1 \quad j \in J \\
 (1c) \quad & \sum_j X_{ij} \leq 1 \quad i \in I \\
 & c_{ij}, X_{ij} \geq 0
 \end{aligned}$$

For this problem, it is well known [6] that a set of prices exists that will support the optimal assignment. Further, neither the optimal assignment nor the set of supporting prices will be necessarily unique. Finally, given a set of supporting prices, the optimal assignment can be sustained by a competitive market. At the optimal assignment, the total (private) valuation of transaction is maximal, and no bidder regrets his choice at the supporting prices.<sup>11</sup>

It is important to distinguish between a set of prices that sustains an optimal assignment to (1) and the set of prices determined via a competitive mechanism. Demand prices associated with deterministic auctions are minimal supporting prices. In general commodity prices obtained in the solutions to (1) will not be minimal, due to the symmetric role of the prices associated with (1b) and (1c).

A. Single Purchase Auctions. We now develop the demand price solution. Call the commodity prices  $p_j$  and the earned surpluses  $w_i$ . In what follows we develop a transportation problem that solves the auction allocation and unambiguously

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<sup>11</sup>Pareto efficiency of the optimal assignment to sellers can be assured by making them at the same time bidders. Their *reserve prices* can be introduced as  $c_{ij}$  in the appropriate columns of their expanded valuation matrix  $C$ . If a seller is awarded an object, no transaction takes place; otherwise the minimal sale price is his reserve price.

determines the associated demand prices. The solution properties (2) are presented for the case in which bidder valuations are interconnected; that is, no degeneracies<sup>12</sup> exist in the BUY - BID sets defined below. To characterize the problem formally, let:

$$\begin{aligned} \text{BUY} &= \{(i,j)/i \text{ buys object } j\} \\ \text{BID} &= \{(i,j)/i \text{ is the highest unsuccessful bidder on object } j\} \\ I &= \{\text{the set of bidders in the auction}\} = \{1, \dots, m\} \\ J &= \{\text{the set of objects being auctioned}\} = \{1, \dots, n\} \end{aligned}$$

so that

$$(2a) \quad p_j = \max_{\substack{i \in I \\ \text{s.t. } (i,j) \notin \text{BUY}}} (c_{ij} - w_i) \quad \text{For all } j \in J$$

$$(2b) \quad w_i = 0 \quad \text{For all } i | (i,j) \notin \text{BUY for some } j$$

$$(2c) \quad c_{ij} = w_i + p_j \quad \text{For all } (i,j) \in \text{BUY or BID}$$

$$(2d) \quad w_i, p_j \geq 0 \quad \text{For all } i \in I, j \in J$$

$$(2e) \text{ If } (i,j) \in \text{BUY then } (i,t) \notin \text{BUY and } (s,j) \notin \text{BUY; for all } s \in I, t \in J.$$

Each bidder acts to maximize his owned earned surplus which is equal to zero if he obtains no object in the auction (2b). The selling price is set equal to the highest unsuccessful bid on any object (2a,c) subject to the single purchase limit (2e). Demand prices can be seen to be minimal: Suppose  $(\bar{p}_j, \bar{w}_i, \bar{x})$  is a demand price solution. For expositional purposes renumber the objects and bidders so that the BUY set is  $(i,i)$ . (That is, the awardees are associated with the main diagonal of the valuation matrix C.) From (2c)

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<sup>12</sup>In linear programming degeneracy can be resolved via standard solution techniques. In this context a degeneracy is said to exist for a proper subset  $K \subset J$ ,  $k \in K$  if

$$\begin{aligned} I_1 &= \{i/(i,j) \in \text{BUY } i \in I, j \in K\} \\ I_2 &= \{i/(i,j) \in \text{BID } i \in I, j \in K\} \end{aligned}$$

$$\text{and } I_1 = I_2.$$

$c_{ii} = \bar{p}_i + \bar{w}_i$  and  $c_{ki} = \bar{p}_i + \bar{w}_k$  for some  $k \in I, k \neq i$ . Arbitrarily, assume that a lower supporting price,  $\hat{p}_1$ , exists for the first object  $\hat{p}_1 = \bar{p}_1 - \delta$ . Since One's earned surplus is increased by  $\delta$ , his bids on any other object would be lower by  $\delta$ . Let  $(k_1, 1)$  be the associated member of BID. To accept  $\hat{p}_1$  his earned surplus must be correspondingly higher by the amount  $\delta$ , which implies  $\hat{p}_{k_1} = c_{k_1, k_1} - \hat{w}_{k_1} = \bar{p}_{k_1} - \delta$ . Similarly, for  $(k_2, k_1) \in \text{BID}$  this implies  $\hat{p}_{k_2} = \bar{p}_{k_2} - \delta$ , etc. Due to nondegeneracy, this eventually implies that  $\hat{w}_k = \delta$ ,  $(k, j) \notin \text{BUY}$  for any  $j$ , which contradicts (2b), or that  $\hat{p}_k < 0$  which contradicts (2d). Demand prices are thus the minimal support prices consistent with (2).

It will be useful to modify problem (2) to have  $z$  objects and  $z$  bidders, where  $z = \max [m, n+1]$ . This is done by adding bidders or objects as needed with  $c_{ij} = 0$  for  $i > m, j > n$ . In an obtained solution, if  $s$  is allocated an object  $t$  and  $s > m$ , then this object is not actually purchased. Similarly, if  $t > n$  then bidder  $s$  does not actually purchase any object. We show later that when  $c_{st} = 0$  for  $(s, t) \in \text{BUY}$  then  $w_s = 0$  and  $p_t = 0$ . Problem (2) now has  $m = n = z$ ; constraints (2a) - (2e) still apply and will hold for the original problem if they hold for the modified problem.

Now consider the following minimization problem:

$$\begin{aligned}
 (3a) \quad & \min z \sum_{j=1}^{z-1} p_j + (z-1) \sum_{i=1}^z w_i \\
 (3b) \quad & \text{s.t. } w_i + p_j \geq c_{ij} \text{ for } i \in Z \text{ and } j \in Z
 \end{aligned}$$

where  $Z = \{1, \dots, z\}$ .

We will show that the solution to problem (3) will yield the solution to problem (2). Before doing this we present the dual problem of (3). If we associate with each constraint (3b) a dual variable  $X_{ij}$ , then the dual problem is:

$$\begin{aligned}
 (4a) \quad & \text{Max } \sum_i^z \sum_j^z c_{ij} X_{ij} \\
 (4b) \quad & \text{s.t. } \sum_{j=1}^z X_{ij} = z-1 \quad \text{For all } i \in Z
 \end{aligned}$$

$$(4c) \quad \sum_{i=1}^z X_{ij} = z \quad \text{For all } j \in Z - \{z\}$$

$$(4d) \quad \sum_{i=1}^z X_{iz} = 0$$

$$(4e) \quad X_{ij} \geq 0 \quad \text{For all } i \in Z \text{ and } j \in Z.$$

Note that (3), (4) represent a transportation problem and its dual.

We now show the equivalence of problems (2) and (3):

Lemma 1: Any basic solution to the dual problem (4) will have two tight cells in each of the first  $z-1$  columns of the coefficient matrix,  $C$ , and one tight cell in column  $z$ .

Proof: By tight we mean that  $X_{ij}$  is in the basis and  $w_i + p_j = c_{ij}$ . Since (4) is a transportation problem with integer right-hand sides,  $X_{ij}$  will be integer values and  $X_{ij} \geq 0$  from (4e). By (4b) the largest value of  $X_{ij}$  in any column  $j$  must be  $\leq z-1$  but  $\sum_{i=1}^z X_{ij} = z$  from (4c) for all but column  $z$ . This implies at least two tight cells in the first  $z-1$  columns. Since this is a transportation problem, there will be at least one tight cell in column  $z$ . This accounts for at least  $2(z-1) + 1 = 2z-1$  tight cells but this is also the maximum number of tight cells in a transportation problem. Hence the Lemma.

Lemma 2: At the optimal solution to the primal-dual problem (3)-(4) if we set  $w_s = 0$ , for  $s \mid (s, z) \in \text{BUY}$ , then  $w_i \geq 0$  and  $p_j \geq 0$ , for all  $i, j$ .

Proof: By construction the matrix  $C$  contains at least one column,  $z$ , of zeros. Since the purchase of (contrived) object  $z$  by person  $s$  implies  $(s, z) \in \text{BUY}$ , then  $w_s = 0$ . (We can set this value because of the one degree of freedom in the transportation problem.) Since  $w_s + p_z = 0$ , it follows that  $p_z = 0$ .

But since  $w_i + p_z \geq 0$ , we have that  $w_i \geq 0$  for all  $i \in Z$ . Also  $w_s + p_j \geq c_{sj} \geq 0$  so that  $p_j \geq 0$  for all  $j \in Z$ .

Lemma 3: Solving (3)-(4) and setting  $w_s = 0$  for  $s \mid (s, z) \in \text{BUY}$  implies (2b).

Proof: In the modified  $z$  by  $z$  problem all bidders real and contrived will purchase an object (real or contrived). If  $s$  is the participant who acquires item  $z$ , then  $(s, z)$  is tight and  $w_s = 0$ . Since  $p_u \geq 0$  and  $w_v \geq 0$  we know that

$w_v = 0$ . Therefore, for all persons who do not purchase the original  $n$  items,  $w_i = 0$  which satisfies (2b).

Before proceeding it is necessary to show a mechanism for determining the sets BUY and BID from the solution to (3)-(4). Because of the tree structure of the optimal solution, we can find a row or column of  $C$  such that  $(i,j)$  is a singleton tight cell. Let  $(i,j) \in \text{BUY}$  and eliminate row  $i$  and column  $j$  from consideration. Because the integrity of the tree structure is maintained by this operation, we can find another singleton cell in some row or column and continue the process, until each row and column has a cell assigned to BUY. Once BUY is determined, BID is the set of tight cells in  $C$  which are not contained in BUY. The set BUY at the same time identified the optimal solution to the assignment problem (1), where  $X_{ij} = 1$  for  $(i,j) \in \text{BUY}$  ( $X_{ij} = 0$  otherwise). While the optimum allocations are identical for the problems (1) and (4), the supporting prices are in general different, with the demand prices  $p_j$  associated with (3)-(4) minimal.

Lemma 4: The solution to (3) with  $w_s = 0$  for  $(s,z)$  tight satisfies (2a), (2c) and (2e).

Proof: (2c) and (2e) follow from the construction of the sets BUY and BID from tight cells of the  $C$  matrix. (2a) follows since  $c_{uj} = w_u + p_j$  for  $(u,j) \in \text{BID}$  and since  $w_i + p_j \geq c_{ij}$  for  $i,j \notin \text{BUY}$  we have  $p_j = c_{uj} - w_u \geq c_{ij} - w_i$  for all  $i | (i,j) \notin \text{BUY}$ . Therefore,  $p_j = \max_{\substack{i | (i,j) \\ \notin \text{BUY}}} (c_{ij} - w_i)$

which completes the Lemma.

Theorem 1: The optimum solution to (3)-(4) yields the minimal demand price solution of problem (2).

Proof: This theorem follows immediately from Lemmas 1 through 4. The existence of a solution to transportation problems of the type (3)-(4) assures a solution to (2). It should be noted that alternate optimum allocations could exist for (3)-(4) but that the dual variables (demand prices and earned surplus) will remain unchanged as long as  $w_i = 0$  for  $(i,z) \in \text{BUY}$ .

B. Multiple Purchase Auctions. Consider now an auction in which each bidder  $i$  has the option of purchasing  $g_i$  objects,  $g_i \leq z$ . This entails a change in condition (2e) of the original auction problem:

(2e') If  $(i,j) \in \text{BUY}$ , then  $(s,j) \notin \text{BUY}$  and there are at most  $g_i - 1$  additional allocations  $(i,t) \in \text{BUY}$ ,  $t \in J$ .

It is also necessary to note that each object purchased by individual  $i$  has its own earned surplus. We call  $w_{i_v}$  the earned surplus of the  $v^{\text{th}}$  object awarded to  $i$ , as defined in (6) below. The multiple purchase auction is characterized by 2(a)-2(d) as amended by (6), and (2e'). Denote this problem as (2'). Its solution is obtained in two stages. First, the optimal allocation is determined via an analogous transportation problem to (4), where again  $z = [\max M, n + 1]$ :

$$(5a) \quad \text{Max.} \quad \sum_i \sum_j^z c_{ij} X_{ij}$$

$$(5b) \quad \text{s.t.} \quad \sum_{i=1}^z X_{ij} = 1 \quad \text{For all } j \in Z - \{z\}$$

$$(5c) \quad \sum_{j=1}^z X_{ij} = g_i \quad \text{For all } i \in Z$$

$$(5d) \quad \sum_{i=1}^z X_{iz} = \sum g_i - z + 1$$

$$(5e) \quad c_{ij}, X_{ij} \geq 0 \quad \text{For all } i \text{ and } j \in Z$$

The solution to this transportation problem gives the optimal assignment of objects to bidders, denoted  $\bar{X}_{ij}$ . However, the solution does not yield demand prices, since an individual purchasing several objects is "forced" in (5) to have the same earned surplus on each object. A new extended coefficient matrix, constructed from the optimal assignment  $\bar{X}_{ij}$ , is needed to allow for inframarginal earnings by any bidder. Then associated (minimal) commodity prices are demand prices.

For each  $i$ , such that  $\sum_{j=1}^{z-1} \bar{X}_{ij} = n_i \leq g_i$  create  $n_i$  new rows  $i_v$

such that  $v \in V(i)$ ,  $V(i) = \{1, 2, \dots, n_i\}$  in the following way: If

$(i,j)$  is the  $u^{\text{th}}$  cell ( $u \in V(i)$ ) of row  $i$  such that  $\bar{X}_{ij} = 1$  then

(6) let  $c_{i_v j} = c_{ij}$ . Also let  $c_{i_v j} = 0$  for  $v \in V(i) - \{u\}$ . Finally

let  $c_{i_v j} = c_{ij}$  for all  $v \in V(i)$  for all  $(i,j)$  such that  $\bar{X}_{ij} = 0$

or  $j = z$ .

This extended coefficient matrix (6) can be solved for demand prices according to (2)-(4). We now show the equivalence of this solution (6) to the desired solution (2').

Lemma 5: For each  $\bar{X}_{ij} = 1$  in the optimum solution to problem (5), the corresponding cell  $(i_{v}, j)$  will be a cell in BUY in the optimum solution of (6), where  $(i, j)$  is the  $v^{\text{th}}$  cell in row  $i$  such that  $\bar{X}_{ij} = 1$ .

Proof: Let  $S$  be the set  $(i_{v}, j)$  where  $(i, j)$  is the  $v^{\text{th}}$  cell in row  $i$  such that  $\bar{X}_{ij} = 1$  at the optimum solution to (5). It is easy to show that the set  $S$  satisfies the allocation properties of the set BUY (i.e., constraint 2e) for the extended matrix (6). This fact follows directly from the construction of matrix (6) in such a fashion that only one cell of  $S$  is located in any row or column of (6). It remains to be shown that the set  $S$  offers the optimal allocation of all objects to be auctioned. Assume that this is not true and that some other set,  $T$ , offers an allocation of resources with a strictly greater value for  $\sum c_{ij} X_{ij}$ . Because the set  $T$  has at most one cell in each row and column of (6),  $T$  also provides a solution to the original matrix (5) where  $X_{ij} = 1$  for each  $(i, j) \in T$ . But if  $T$  yields a larger objective function for (6), it must also represent a solution for (5) whose objective function is greater than the original optimum solution used to form  $S$ . This fact is clearly a contradiction, hence the Lemma.

Theorem 2: The solution to problem (5) yields a solution to problem (2').

From Lemma 5, the solution to problem (6) yields a solution to constraints (2a) through (2e). Note that for (2a):

$$P_j = \max_{i_v \in I} (c_{i_v j} - w_{i_v}) \text{ for all } j \in J$$

$$i_v \in I$$

$$(i_v, j) \notin \text{BUY}$$

with  $I$  being the indices of the expanded set of rows, and  $w_{i_v}$  the earned surplus of bidder  $i$  derived from the  $v^{\text{th}}$  object which he has been awarded. Since  $c_{i_v j} = 0$  for all but one of the set  $V(i)$ , ( $i$  awarded to  $j$ ) this constraint yields the multiple purchase analog to (2a). Similar interpretations hold for each of the other constraints. In particular, constraint (2e) in the expanded

Framework of problem (6) implies (2e') and the theorem is proved.

The foregoing analysis provides solutions to deterministic auctions when bidders face no resource constraints. Before turning to that case we note that the solutions obtained apply to both progressive and sealed bid auctions (see Appendix). In either case, rational bidder behavior entails bidding one's full valuations ( $c_{ij}$ ), knowing that demand prices determined according to (2)-(6) for commodities will be minimal, subject to any purchase limits in the auction.

C. Single Purchase Auctions with Resource Constraints. Since an auction takes place at a given point in time (P2) bidder resource limits can affect the auction outcome, in view of (P5). With the introduction of resource constraints, bidders again act to maximize their earned surpluses, but can bid no more than their resources,  $M_i$ , permit. The additional complexity of this auction situation cannot be resolved by the linear programming methods of section 4A, even though nonlinear problems analogous to (2)-(3) describe the solution properties of such auctions. Demand prices are again minimal, given the resource constraints, but neither the allocation nor the earned surpluses are now unique. In general more than one Pareto optimal allocation exists for this auction problem. These Pareto optimal allocations are associated with Kuhn-Tucker points<sup>13</sup> of the nonlinear model. Solutions are obtained via an algorithm based on the Dutch auction procedure, and are verified by the solution properties of the nonlinear model.

Formally the auction problem is similar to (2), with the additional stipulation that no bidder  $i$  can bid more than his resource limit  $M_i$ . Conditions (2b)-(2e) still apply. An additional condition,

$$(2f^*) \quad P_j \leq M_i \quad \text{for all } (i,j) \in \text{BUY}$$

limits transaction prices. Finally, condition (2a) which defines demand prices must be modified to reflect the fact that bidder resources limit the price determining bids:

$$(2a^*) \quad P_j = \max_{\substack{i \in I \\ (i,j) \notin \text{BUY}}} [\min(c_{ij} - w_i, M_i)]$$

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<sup>13</sup> See Mangasarian [8].

This auction problem is referred to as (2\*). Following the development of Section A, the modified  $z \times z$  problem consists of (3a) and (3b\*):

$$(3b^*) \quad w_i + p_j \geq \min (c_{ij}, M_i + w_i), \text{ for all } (i,j) \in Z.$$

This problem is referred to as (3\*). Now consider an auction allocation and associated prices  $(\bar{X}, \bar{p}, \bar{w})$  that satisfy (2\*) or (3\*). If we let  $d_{ij} = \min (c_{ij}, M_i + \bar{w}_i)$ , it is easy to show that the lemmata and Theorem 1 of section 4A apply at that solution point. The new constraint (2f\*) is satisfied since:

$$\begin{aligned} \bar{w}_i + \bar{p}_j &= d_{ij} = \min (c_{ij}, M_i + \bar{w}_i) \text{ for } (i,j) \in \text{BUY} \\ \text{if } c_{ij} &\geq M_i + \bar{w}_i \quad \text{then } \bar{w}_i + \bar{p}_j = M_i + \bar{w}_i \text{ and } \bar{p}_j = M_i \\ \text{if } c_{ij} &\leq M_i + \bar{w}_i \quad \text{then } \bar{w}_i + \bar{p}_j = c_{ij} \leq M_i + \bar{w}_i \text{ and } \bar{p}_j \leq M_i. \end{aligned}$$

Thus the solution properties of (3\*) conform to those of the desired solution (2\*). However (3\*) is difficult to solve, since the constraints (3b\*) do not define a convex region. To obtain solutions, we resort to an algorithm that is based on the Dutch auction procedure:<sup>14</sup>

Begin with a vector of arbitrarily high commodity prices  $p_j$  and let these prices fall, simultaneously or one at a time. Each price drops until two bids are registered, at which time the price is set equal to the second highest bid and the object is temporarily awarded to the highest bidder. At any time in the bidding process each participant  $i$  is willing to bid  $c_{ij} - w_i$  on object  $j$ , where  $w_i$  is his present (assured) earned surplus, provided this amount does not exceed his resources  $M_i$ . Once the price on object  $j$  is set, it will only begin to drop again if one of the two high bids is retracted. A bid may be retracted by a participant only after the temporary award of an object to that participant. A rational bidder will retract all bids on objects which if awarded at the present price will not provide as much earned surplus as some alternative. If an award is the second object awarded, one of the bids on the two awarded objects must be retracted. When no one is willing to bid further, the bidding is stopped and all awards are made final. Note that earned surplus for each bidder in the action is a strictly nondecreasing function throughout the auction

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<sup>14</sup> See Appendix for a more detailed description and example of this solution procedure.

process. It is obvious that when the algorithm stops the present solution is Pareto optimal and that this solution satisfies both (2\*) and (3\*). It is easy to show that this point is also associated with a Kuhn-Tucker solution. To show this fact, resolve the problem with constant right-hand sides  $d_{ij} = \min(c_{ij}, M_i + \bar{w}_i)$  for all  $(i, j)$ . The resulting Kuhn-Tucker solution found using the transportation problem (3)-(4) of section 4A is equivalent to that using the auction algorithm just described.

While the solution procedure just described yields a Pareto optimal allocation, all bidders may not be satisfied with the auction outcome. Bidder dissatisfaction cannot occur in the unconstrained case. The underlying reason for this phenomenon is that alternate (Kuhn-Tucker) allocations can be supported by the (minimal) demand prices, and at the same time, the earned surplus attached to these allocations can differ. Specifically, a bidder may be able to afford an object at its final transaction price, and were he to obtain the object, increase his earned surplus. When the situation occurs, the price of the desired object just equals the bidder's resource limit. We call this occurrence bidder regret. To illustrate the occurrence of bidder regret in resource constrained auctions, we offer the following single example: Assume there are three bidders, each of which can bid for one of two objects. The evaluation of each object by the bidders and their resource constraints are given below.

	object		$M_i =$ resource constraints
	1	2	
bidder 1	4	1	2
bidder 2	4	2	2
bidder 3	1	1	none

The following two auction outcomes are possible. (The awards are circled, prices are listed across the top and earned surplus along the side.) The unconstrained outcome is also given.

	2		1			2		1			3		1	
2	4	1	0	4	1	1	4	1	0	4	1	1	4	1
1	4	2	2	4	2	1	4	2	0	1	1	0	1	1
0	1	1	0	1	1	0	1	1	0	1	1	0	1	1
	Solution 1					Solution 2					Unconstrained Solution			

The first solution is obtained in the following way: When the price of the first object falls to 2, both bidders One and Two submit a bid, with the temporary award made to One. It is then never in One's interest to relinquish his purchase rights, and although Two would like to bid up to 3 for object one he cannot because of his resource constraint. This situation leads bidder Two to bid only 2 units for this object and therefore lose the award at a price he is willing to pay. One measure of his regret is the difference between his earned surplus and his surplus had he been awarded object two, i.e.,  $c_{21} - p_1 - w_2 = 1$ . If  $p_1$  had been strictly greater than 2 bidder Two's lost earned surplus would arise from his lack of resources--not the allocation mechanism; this of course cannot be construed as regret. In the second solution, bidder Two is awarded the first object. Bidder One is willing to bid 2 units on the first object and 1 on the second object. Yet bidder One obtains no object in the auction and thus incurs regret. In the unconstrained case if a participant was willing to bid for an object, but did not receive the object, then his bid (evaluation) set the price. This bidder's loss of earned surplus which resulted from not being awarded the object would have been zero. In the constrained case the same situation can occur with a nonzero loss of earned surplus, hence regret.<sup>15</sup>

The allocation rule determining the temporary awards in resource constraint auctions can be crucial to the outcome. In a progressive auction, the first bidder recognized could be designated the awardee, or alternatively, a fair coin could be used to break ties.<sup>16</sup> Still, this class of auctions is deterministic in the sense that rational bidders will submit their full valuations of objects, knowing that the allocation procedure will guarantee minimal transaction prices and a Pareto optimal outcome.

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<sup>15</sup>In this example, if One's resources were slightly larger then only the first solution would result. Similarly, if Two was slightly wealthier, the second solution would be assured. It might appear, then, that this situation would induce bidders to gather information and generate game strategies that increase the probability of their receiving an object. For assume that a bidder knows with certainty that he will tie for the high bid on an object at a price equal to his resource constraint, but less than his evaluation. The individual strategy would be to bid more than his resources allow in order to assure the first bid. The price would be set by the other bidder at the price he could just afford. However, in the absence of collusion, the possibility of reciprocal behavior by other bidders, and the cost of renegeing on an award (P5) would rule out such behavior.

<sup>16</sup>Priority numbers could be fairly assigned to bidders to settle all tie situations.

## V. Conclusion

Auction procedures play an important role in financial markets, particularly in the issuance of new securities. At present, participation in such auctions requires considerable expertise, including a knowledge of the likely behavior of other bidders in the market. The information requirements and the inherent risks involved in those probabilistic auctions limit the extent of direct bidder participation, calling forth the need for trading specialists and secondary markets for final demand distribution. The efficiency of these financial institutions has been the subject of a prolonged debate. As the technology for storing and transmitting information improves, deterministic auctions involving remote bidding by direct investors could become a viable alternative. This paper has been concerned with deriving the solution properties of deterministic auctions.

We have construed deterministic auctions as a set of assignment problems that can be solved via linear programming methods. The crucial property of these auction solutions is that the resulting commodity prices--demand prices--are minimal. This solution property induces full valuation bidding and increased bidder participation in the auction. Demand price solutions are obtained via bidding rules based on the Dutch auction procedure, or via a transportation problem formulation of the auction. Both solution methods yield the same auction outcome, which is unique with respect to the minimal commodity prices.

## APPENDIX

Here we present example solutions to the deterministic auction considered in sections 4A, B, and C. Solutions are presented for both single and multiple purchase auctions of dissimilar commodities according to (2)-(6), and for a single purchase resource constrained auction. These solutions correspond to the outcomes of sealed bid auctions, wherein bidders submit their full valuations,  $c_{ij}$ , of the offered commodities. In addition we present an equivalent solution procedure for progressive auctions. This procedure, to be described now, is an extended version of the modified Dutch Auction of Vickrey [12].

A. By way of introduction we recall the progressive auction described by Vickrey for a *single* object. In this case the price starts at an arbitrarily high value. As it is lowered participants submit bids at whatever price desired. The price continues downward until the second highest bid has been placed.<sup>17</sup> The object is sold to the highest bidder but at a price equal to the second highest bid. This procedure provides no incentives on the part of bidders to bid other than their honest evaluations of the object.

In the multiple commodity auction, a vector of prices is announced at an arbitrarily high value. Each price, of course, corresponds to an object up for auction. The prices are then lowered in any manner whatever--one at a time, all at once or at different rates. As the price of an object drops, bids are placed on that object. Bidders are allowed to place bids on as many objects as they wish, regardless of the number they can ultimately purchase. Rational behavior dictates that a bid be placed on any object whose price,  $p_j$ , reaches the value of the object to the individual,  $c_{ij}$ . As soon as two bids on the same object have been made, the highest bidder is informed that he has been temporarily assigned that object at a price equal to the second highest bidder. The price stops dropping at that time. No knowledge of objects assigned is provided to anyone but the highest bidder. At that time the highest bidder essentially knows that a minimum value for his earned surplus will be  $w_i = c_{ij} - p_j$ . As the bidding continues, a bidder may reach a point where he has the highest bid of at least two bids on  $g_i + 1$  objects, only  $g_i$  of which he can obtain. He will, therefore, retract his bid on the object which gives him the smallest earned surplus. He will also retract bids on other objects not yet awarded to him which would not at their present prices increase his earned surplus. This

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<sup>17</sup>No bidder knows if another has bid. This could be accomplished by electronic signalling devices that register on a board viewed only by the auctioneer.

act could be done automatically without interaction from the bidder. The prices on those objects with retracted bids will then start dropping once again until a new second bid is received. The retraction of bids is the necessary modification needed in the solution of multiple commodity auctions. Because bids may be retracted on objects, bidders have incentives to bid even when it is known that a higher bidder exists. Note that since the price stops as soon as two bids have been made on an object and starts only when one is retracted, there are at most two bidders on each object. When there are exactly two bids on each object, the auction stops. At this point it is easy to verify that constraints (2a)-(2d) and (2e') are satisfied. Ties in this process must be broken by some device, random or otherwise. Multiple bidders at a price are maintained in a queue which is drawn upon as bids are retracted. It should be noted that in the case of a tie in this situation one bidder is awarded the object, while the other determines the price. Because of the tie the earned surplus on this object is zero for the winning bid, and both bidders are therefore indifferent about the award.

As in single object auctions, there is no incentive to deviate from full valuation bidding. Not retracting bids or submitting false bids only serves to keep some objects at a higher than minimum price. This can only induce certain bidders to bid higher on still other objects so that in the long run all prices will remain high. In the case of many ties at a particular price, submitting a slightly higher price may assure the bidder of purchasing the object but at a price where the bidder is indifferent to the award. This strategy yields a negative expected return to the bidder since it is possible to be the buyer of the object at a price in excess of value.

We now present solutions to an example problem. First a single purchase solution corresponding to (2), (3), (4) is given. Sealed bids equal to  $c_{ij}$  ( $i = 1, \dots, 4; j = 1, \dots, 3$ ) are shown below:

		j				
	i	10	12	14	0	3
		17	20	13	0	3
$c_{ij} =$		14	16	15	0	3
		12	10	3	0	3
		4	4	4	0	

The right-hand sides of (4b)-(4d) are shown around the edge of the C matrix. Each participant may purchase at most one item.

The transportation problem solution yields: tight constraints are circled, the optimal values of  $X_{ij}$  are displayed in the upper left-hand corner of each cell.

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		j				
	i	1	2	3	4	5
$c_{ij} =$	1	10	12	14	0	3
	2	17	20	13	0	3
	3	14	16	15	0	3
	4	12	10	3	0	3
		4	4	4	0	

The right-hand sides of (4b)-(4d) are shown around the edge of the C matrix. Each participant may purchase at most one item.

The transportation problem solution yields: tight constraints are circled, the optimal values of  $X_{ij}$  are displayed in the upper left-hand corner of each cell.

$p_j =$	12	14	13	0
$w_i =$	1	10	12	14
	6	17	20	13
	2	14	16	15
	0	12	10	3
				0

To determine the set BUY assign first the singleton column 4 to participant 4. Then assign the new singleton in column 1, created by the removal of row 4, to participant 3. This process continues until the set BUY is determined:  
 BUY =  $\{(1,3); (2,2); (3,1); (4,4)\}$ .

We now solve the problem via the modified Dutch auction framework. Start the price vector at  $p = (25,25,25)$ . To reduce confusion assume that the prices fall one at a time. The following bidding progression results:

- $p = (17,25,25)$  Participant 2 bids 17 for object 1.
- $p = (\overline{14},25,25)$  Participant 3 bids 14 for object 1: setting its price. Participant 2 is awarded object 1 and achieves an earned surplus of 3.  $P_1$  Falls
- $p = (\overline{14},17,25)$  Participant 2 bids  $17 = 20 - 3$  for object 2.
- $p = (14,\overline{16},25)$  Participant 3 bids 16 for object 2: setting its price. Participant 2 is awarded the object who retracts his bid on object 1 and accepts object 2 with an earned surplus of 4.  $P_2$  Falls
- $p = (14,\overline{16},15)$  Participant 3 bids 15 for object 3.
- $p = (14,16,\overline{14})$  Participant 1 bids 14 for object 3: setting its price. Object 3 is awarded to participant 3 whose earned surplus is 1. He therefore retracts his bids on objects 1 and 2.  $P_3$  Falls
- $p = (13,16,\overline{14})$  Participant 3 bids 13 for object 1. Participant 2 bids 13 for object 1. The price is set at 13 and awarded to either 2 or 3 who are *both* indifferent at this point. We assume that neither participant accepts the object.  $P_1$  Falls
- $p = (13,\overline{15},\overline{14})$  Participant 3 bids 15 for object 2. This sets the price for the object and increases participant 2's earned surplus to 5.  $P_2$  Falls
- $p = (13,\overline{15},\overline{14})$  Participant 3 bids 13 for object 1.
- $p = (\overline{12},15,14)$  Participants 4 and 2 bid 12 for object 1, setting its price. Object 1 is awarded to participant 3 whose earned surplus is now 2. He therefore retracts his bid on objects 2 and 3.  $P_1$  Falls

$p = (\overline{12}, \overline{14}, 14)$  Participant 3 bids 14 for object 2. The object is still awarded to participant 2 whose earned surplus is now 6. Participant 2 retracts his bid for object 1. (This price remains fixed since participant 4 also bid 12.)  $P_2$  Falls

$p = (\overline{12}, \overline{14}, \overline{13})$  Participant 3 bids 13 for object 3. The object is awarded to participant 1. All objects are awarded and the bidding stops.  $P_3$  Falls

The solution is seen to be equivalent to that derived from (2)-(4) above.

We now solve the original problem subject to the condition that each participant may purchase up to two objects. The progressive auction process is similar to the one described above and will not be given. Problem 5 is shown below, with the right-hand sides listed around the edge of the matrix. The solution is shown in the upper left-hand corner of each cell whose  $x_{ij} \neq 0$ .

10	12	14	0	2
17	20	13	0	2
14	16	15	0	2
12	10	3	0	2
1	1	1	5	

Problem (6) derived from the above solution is

$P_i =$	13	15	14	0	0	
$w_i =$	0	10	12	14	0	0
	4	17	0	13	0	0
	5	0	20	13	0	0
	1	14	16	15	0	0
	0	12	10	3	0	0

(bidder 2)

To locate the optimal assignment, either look for singleton columns or use the solution to (5);  $BUY = \{(1,4); (2,1); (2,2); (3,3)\}$ .

B. An example of a single purchase auction situation with bidder resource constraints is presented below:

	object			resource constraint	
	1	2	3		
bidder	1	6	2	3	3
	2	5	4	1	2
	3	3	4	2	2
	4	0	1	1	none

The bidding algorithm is listed below, all prices start at 10.

- $p = (3, 10, 10)$  Participant 1 bids 3 units for object 1. He cannot bid more because of his resource constraint.
- $p = (\bar{2}, 10, 10)$  Bidder 2 and 3 both bid 2 units for object 1 thus setting its price. One of the two bids sets the price. Participant 1 now has a guaranteed earned surplus of 4.  $P_1$  Falls
- $p = (\bar{2}, \bar{2}, 10)$  Participants 2 and 3 both bid 2 units for object 2. The tie is broken by some mechanism in favor of say 2. The price is set by bidder 3.  $P_2$  Falls
- $p = (\bar{2}, \bar{2}, 2)$  Participant 3 bids 2 units for object 3.
- $p = (\bar{2}, \bar{2}, \bar{1})$  Participant 4 bids 1 unit for object 3 and sets the price. All prices are now set.  $P_3$  Falls

The earned surpluses are,  $w = (4, 2, 10)$ . The amount of regret incurred by bidders is a function of the way in which ties are broken. In a sealed bid auction in which all bidder evaluations are known, ties could be broken so as to minimize total regret.

#### REFERENCES

- [1] Bolten, S. "Treasury Bill Auction Procedures: An Empirical Investigation." *Journal of Finance* (1971).
- [2] Brimmer, A. "Price Determination in the United States Bills Market." *Review of Economics and Statistics* (May 1962).
- [3] Friedman, M. "Price Determination in the United States Treasury Bill Market: A Comment." *Review of Economics and Statistics* (August 1963).
- [4] Goldstein, H. "The Friedman Proposal for Auctioning Treasury Bills." *Journal of Political Economy* (August 1962).
- [5] Griesmer, J. H.; M. Shubik; and R. E. Levitan. "Towards a Study of Bidding Processes--Games with Unknown Costs." *Naval Logistics Quarterly* (1967), pp. 415-433.
- [6] Koopmans, T., and M. Beckmann. "Assignment Problems and the Location of Economic Activities." *Econometrica* (January 1957).
- [7] McDonald, J. G., and B. C. Jacquillat. "Pricing of Initial Equity Issues: The French Sealed-Bid Auction." *Journal of Business* (January 1974).
- [8] Mangasarian, O. L. *Non-Linear Programming*. New York: McGraw Hill (1969).
- [9] Smith, V. L. "Experimental Studies of Discrimination Versus Competition in Sealed Bid Auction Markets." *Journal of Business* (January 1967).
- [10] \_\_\_\_\_. "Bidding Theory and the Treasury Bill Auction--Does Price Discrimination Increase Bill Prices." *Review of Economics and Statistics* (1961), pp. 141-146.
- [11] Stark, R. M. "Competitive Bidding: A Comprehensive Bibliography." *Journal of Operations Research*, pp. 484-490.
- [12] Vickrey, W. "Counterspeculation, Auctions and Competitive Sealed Tenders." *Journal of Finance* (1961), pp. 8-37.