RANDOM GEOMETRIC GRAPH DIAMETER IN THE UNIT BALL

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ABSTRACT. The *unit ball random geometric graph* $G = G_p^d(\lambda, n)$ has as its vertices n points distributed independently and uniformly in the unit ball in $\mathbb{R}^d$, with two vertices adjacent if and only if their ℓ_p-distance is at most λ. Like its cousin the Erdős-Rényi random graph, G has a *connectivity threshold*: an asymptotic value for λ in terms of n, above which G is connected and below which G is disconnected. In the connected zone, we determine upper and lower bounds for the graph diameter of G. Specifically, almost always, $\mathrm{diam}_p(\mathbf{B})(1 - o(1))/\lambda \le \mathrm{diam}(G) \le \mathrm{diam}_p(\mathbf{B})(1 + O((\ln\ln n/\ln n)^{1/d}))/\lambda$, where $\mathrm{diam}_p(\mathbf{B})$ is the ℓ_p-diameter of the unit ball $\mathbf{B}$. We employ a combination of methods from probabilistic combinatorics and stochastic geometry.

1. INTRODUCTION

A *random geometric graph* consists of a set of vertices distributed randomly over some metric space X, with two vertices joined by an edge if the distance between them is sufficiently small. This construction presents a natural alternative to the classical Erdős-Rényi random graph model, in which the presence of each edge is an independent event (see, e.g., [2]). The study of random geometric graphs is a relatively new area; the monograph [9] by M. Penrose is the current authority. In addition to their theoretical interest, random geometric graphs have many applications, including wireless communication networks; see, e.g., [4, 10, 12].

In this article, we study the *unit ball random geometric graph* $G = G_p^d(\lambda, n)$, defined as follows. Let d and n be positive integers, $\mathbf{B}$ the Euclidean unit ball in $\mathbb{R}^d$ centered at the origin, λ a positive real number, and $p \in [1, \infty]$ (that is, either $p \in [1, \infty)$ or $p = \infty$). Let V_n be a set of n points in $\mathbf{B}$, distributed independently and uniformly with respect to Lebesgue measure on $\mathbb{R}^d$. Then G is the graph with vertex set V_n, where two vertices $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ are adjacent if and only if $\|x - y\|_p \le \lambda$. (Thus the larger λ is, the more edges G has.) Here $\|\cdot\|_p$ is the ℓ_p-*metric* defined by

$$\|x - y\|_p = \begin{cases} \left(\sum_{i=1}^d |x_i - y_i|^p\right)^{1/p} & \text{for } p \in [1, \infty), \\ \max\{|x_i - y_i| \,:\, 1 \le i \le d\} & \text{for } p = \infty, \end{cases}$$

where the case $p = 2$ gives the standard Euclidean metric on $\mathbb{R}^d$.

When $d = 1$, G is known as a *random interval graph*. (Note that the value of p is immaterial when $d = 1$.) Random interval graphs have been studied extensively in the literature; the asymptotic distributions for the number of isolated vertices and the number of connected components were determined precisely by E. Godehardt and J. Jaworski [7]. The *random Euclidean unit disk graph* $G_2^2(\lambda, n)$ was studied by X. Jia and the first and third authors [5].

In the present article, we focus on the case $d \geq 2$ and $p \in [1, \infty]$, but also comment along the way on the special case $d = 1$. We are interested in the asymptotic behavior of the connectivity and graph diameter of G as $n \to \infty$ and $\lambda \to 0$. In fact, G has a *connectivity threshold*: roughly speaking, an expression for λ as a function of n, above which G is connected and below which G is disconnected. (This behavior is ubiquitous in the theory of the Erdős-Rényi random graph model; cf. [2].)

We are interested primarily in the combinatorial graph diameter of G, $\mathrm{diam}(G)$, above the connectivity threshold. Our results include

- a lower bound for $\mathrm{diam}(G)$ (Proposition 7 of §4);
- an "absolute" upper bound $\mathrm{diam}(G) < K/\lambda$, where K is a constant depending only on d (Theorem 8 of §5); and
- an asymptotically tight upper bound within a factor of the form $(1 + o(1))$ of the lower bound, the proof of which builds on the absolute upper bound (Theorem 10 of §6).

2. Definitions and Notation

As mentioned above, the main object of our study is the random geometric graph $G = G_p^d(\lambda, n)$, where $d \geq 1$ is the dimension of the ambient unit ball $\mathbf{B}$, $p \in [1, \infty]$ describes the metric, $\lambda > 0$ is the ℓ_p-distance determining adjacency, and n is the number of vertices. We will generally avoid repeating the constraints on the parameters.

The graph distance $d_G(x, y)$ between two vertices $x, y \in V_n$ is defined to be the length of the shortest path between x and y in G, or ∞ if there is no such path. The graph diameter of G is defined to be $\mathrm{diam}(G) := \max\{d_G(x, y) : x, y \in V_n\}$. This graph-theoretic quantity is not to be confused with the ℓ_p-diameter of a set $X \subseteq \mathbb{R}^d$, defined as $\mathrm{diam}_p(X) := \sup\{\|x - y\|_p : x, y \in X\}$. The ℓ_p-ball of radius r centered at $x \in \mathbb{R}^d$ is defined as

$$B_p^d(x, r) := \{y \in \mathbb{R}^d : \|x - y\|_p \leq r\},$$

while the ℓ_p-ball of radius r around a set $X \subseteq \mathbb{R}^d$ is $B_p^d(X, r) := \cup_{x \in X} B_p^d(x, r)$. The origin of $\mathbb{R}^d$ is $O := (0, \dots, 0)$; when the center of a ball is not explicitly given, we define $B_p^d(r) := B_p^d(O, r)$. Thus $\mathbf{B} = B_2^d(1)$. The ℓ_p-diameter of $\mathbf{B}$ is

$$\mathrm{diam}_p(\mathbf{B}) = \begin{cases} 2d^{1/p - 1/2} & \text{when } 1 \leq p \leq 2, \\ 2 & \text{when } 2 \leq p \leq \infty. \end{cases}$$

The distance $d_p(X, Y)$ between two sets $X, Y \subseteq \mathbb{R}^d$ is defined as $\inf\{\|x - y\|_p : x \in X, y \in Y\}$. The *boundary* ∂X of X is its closure minus its interior (in the usual topology on $\mathbb{R}^d$), and its *volume* $\mathrm{vol}(X)$ is its Lebesgue measure.

We will make frequent use of the quantity

$$(1) \qquad \alpha_p^d := \frac{\mathrm{vol}(B_p^d(r))}{\mathrm{vol}(B_2^d(r))} = \frac{\Gamma\left(\frac{p+1}{p}\right)^d \cdot \Gamma\left(\frac{2+d}{2}\right)}{\Gamma\left(\frac{3}{2}\right)^d \cdot \Gamma\left(\frac{p+d}{p}\right)},$$

where Γ is the usual gamma function (see, e.g., [11]). The calculation of α_p^d, along with the proofs of several other useful facts about ℓ_p-geometry, may be found in the Appendix at the end of the article.

We will say that the random graph G has a property P *almost always*, or *a.a.*, if

$$\lim_{n\to\infty} \Pr\left[G \text{ has property } P\right] = 1.$$

By the notation $f(n) = o(g(n))$ and $f(n) = O(g(n))$, we mean, respectively, $\lim_{n\to\infty} f(n)/g(n) = 0$ and $\limsup_{n\to\infty} f(n)/g(n) \leq c$, for some absolute non-negative constant c.

3. CONNECTIVITY THRESHOLDS

In order for $G = G_p^d(\lambda, n)$ to have finite diameter, it must be connected. There-fore, we seek a *connectivity threshold* – a lower bound on λ so that G is almost always connected. When $d = 1$, all ℓ_p-metrics are identical. For this case we now quote parts of Theorems 10 and 12 of [7], to which we refer the reader for their pre-cise determination of the asymptotic Poisson distributions of the number of isolated vertices and the number of connected components.

Theorem 1 (Godehardt, Jaworski). *Let $\lambda_1(n) = \frac{1}{n}(\log n + c + o(1))$ and $\lambda_2(n) = \frac{2}{n}(\log n + c + o(1))$, where c is a constant. Then*

$$\lim_{n\to\infty} \Pr[G_p^1(\lambda_1(n), n) \text{ has an isolated vertex}] \;=\; e^{-e^{-c}},$$

$$\lim_{n\to\infty} \Pr[G_p^1(\lambda_2(n), n) \text{ is connected}] \;=\; e^{-e^{-c}}.$$

In particular, by replacing c in Theorem 1 with a nonnegative sequence $\gamma(n) \to \infty$, almost always $G_p^1(\lambda_1(n), n)$ has no isolated vertices and $G_p^1(\lambda_2(n), n)$ is con-nected. The case $d = 1$ is exceptional in that the thresholds for having isolated vertices and for connectivity are separated.

For $d \geq 2$, we will use the fact that the connectivity threshold coincides with the threshold for the disappearance of isolated vertices, which follows from two theorems of M. Penrose. First we compute the threshold for isolated vertices, which is easier to calculate.

Proposition 2. *Let $d \geq 2$, let $p \in [1, \infty]$, and let $\alpha = \alpha_p^d$ be the constant of* (1). *Suppose $\gamma(n)$ is a nonnegative sequence such that $\lim_{n\to\infty} \gamma(n) \to \infty$, and that*

$$\lambda \;\geq\; \left(\frac{1}{\alpha n}\left(\frac{2(d-1)}{d}\ln n + \frac{2}{d}\ln\ln n + \gamma(n)\right)\right)^{1/d}.$$

Then, almost always, $G = G_p^d(\lambda, n)$ has no isolated vertices.

Proof. Let $V_n = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. For each vertex v_i, let A_i be the event that v_i is an isolated vertex, and let X_i be the indicator of A_i; that is, $X_i = 1$ if A_i occurs and 0 otherwise. Set $X = X_1 + X_2 + \cdots + X_n$. We will show that $\mathbb{E}[X] = o(1)$.

By definition, v_i is isolated if and only if there are no other vertices in $B_p^d(v_i, \lambda) \cap$ **B**. We condition $\Pr[A_i]$ on the ℓ_2-distance from v_i to the origin O. If $\|v_i - O\|_2 \in [0, 1 - d^{1/2}\lambda)$, then $B_p^d(v_i, \lambda) \subseteq \mathbf{B}$. Otherwise, if $\|v_i - O\|_2 \in (1 - d^{1/2}\lambda, 1]$, then the volume of $B_p^d(v_i, \lambda) \cap \mathbf{B}$ is not less than $\frac{1}{2}\mathrm{vol}(B_p^d(v_i, \lambda))(1 + O(\lambda))$. Hence

$$\Pr[A_i] \;\leq\; (1 - d^{1/2}\lambda)^d(1 - \alpha\lambda^d)^{n-1} +$$
$$(1 - (1 - d^{1/2}\lambda)^d)\left(1 - \frac{\alpha\lambda^d}{2}(1 + O(\lambda))\right)^{n-1}.$$

Using $1 - x = e^{-x}(1 + o(1))$ as $x \to 0$ and the binomial expansion, we have

$$\Pr[A_i] \leq (1 + o(1))e^{-\alpha n \lambda^d} + (dd^{1/2}\lambda + O(\lambda^2))(1 + o(1))e^{-\alpha n \lambda^d/2}.$$

The first term is $o(n^{-1})$ for $d \geq 2$. By linearity of expectation, $\mathbb{E}[X] = n \cdot \Pr[A_i]$, and so

$$\mathbb{E}[X] \leq o(1) + d^{3/2} n \lambda (1 + o(1)) n^{-1+1/d}(\ln n)^{-1/d} e^{-\gamma(n)/2}.$$

The second term is $o(1)$, and so $\Pr[X > 0] \leq \mathbb{E}[X] \leq o(1)$; that is, almost always, G has no isolated vertices. $\qquad\square$

The number of isolated vertices below the threshold is easy to compute in certain special cases. For example, if $p \in [1, \infty]$, $d = 2$, $\lambda = \sqrt{c \ln n / n}$, and $0 \leq c < \alpha^{-1}$, then a minor modification of [5, Theorem 1] yields $X = (1 + o(1))n^{1-\alpha c}$ almost always. In general, to determine the behavior of X more exactly would require complicated integrals that describe the volume of $B_p^d(v_i, \lambda) \cap \mathbf{B}$ near the boundary of $\mathbf{B}$ (cf. [9, Chapter 8]). For our purposes, it suffices to concentrate on the values of λ for which G has no isolated vertices.

For $d \geq 2$ and $p \in (1, \infty]$, the connectivity threshold for the unit-*cube* random geometric graph coincides with the threshold for lacking isolated vertices. We quote Penrose's theorem [8, Thm. 1.1] after some supporting definitions. Define the *unit cube geometric graph* $H = H_p^d(\lambda, n)$ analogously to $G = G_p^d(\lambda, n)$, except that its vertices are points in $[0, 1]^d$ rather than $\mathbf{B}$. For any nonnegative integer k, define

(2a) $\qquad \rho(H; \kappa \geq k + 1) = \min\{\lambda \mid H \text{ has vertex connectivity } \kappa \geq k + 1\},$

(2b) $\qquad \rho(H; \delta \geq k + 1) = \min\{\lambda \mid H \text{ has minimum degree } \delta \geq k + 1\}.$

Theorem 3 (Penrose). *Let $p \in (1, \infty]$ and let $k \geq 0$ be an integer. Then*

$$\lim_{n \to \infty} \Pr[\rho(H; \kappa \geq k + 1) = \rho(H; \delta \geq k + 1)] = 1.$$

When $k = 0$, Theorem 3 asserts that as λ increases (forcing more edges into the graph), almost always, H becomes connected simultaneously as the last isolated vertex disappears. In the proof of Theorem 3 in [8], Penrose shows that the limiting probability distributions for $\rho(H; \kappa \geq k + 1)$ and $\rho(H; \delta \geq k + 1)$ are the same. The proof requires only a series of geometric and probabilistic arguments which hold in the unit ball as well as in the unit cube (see, in particular, Sections 2 and 5 of [8]), so we have as an immediate corollary the following.

Corollary 4. *Let $d \geq 2$ and $p \in (1, \infty]$, and let $\lambda = \lambda(n)$ be sufficiently large so that, almost always, $G = G_p^d(\lambda(n), n)$ has no isolated vertices. Then, almost always, G is connected.*

We now consider the case that $d \geq 2$ and $p = 1$. Here Theorem 3 does not apply. However, we can appeal to two general results about the behavior of a random geometric graph in an ℓ_p-metric space whose boundary is a compact $(d-1)$-submanifold of $\mathbb{R}^d$. For such a graph, Theorem 7.2 of [9] provides a threshold for the disappearance of isolated vertices, and Theorem 13.7 provides a threshold for connectivity. Applying these results to G, with the thresholds for G defined as in (2a) and (2b), we obtain the following fact.

Proposition 5. *Let $d \geq 2$, $p \in [1, \infty]$, and $G = G_p^d(\lambda(n), n)$. Let $\alpha = \alpha_p^d$ be the constant of (1), and let $k \geq 0$ be an integer. Then, almost always,*

$$\lim_{n \to \infty} \left(\frac{n\alpha}{\log n} \rho(G; \kappa \geq k+1)^d \right) = \lim_{n \to \infty} \left(\frac{n\alpha}{\log n} \rho(G; \delta \geq k+1)^d \right) = \frac{2(d-1)}{d}.$$

We now collect the above results to present the connectivity thresholds that we will use in the rest of the paper.

Theorem 6. *Let $G = G_p^d(\lambda, n)$ and let $\alpha = \alpha_p^d$ be the constant of (1).*

    (i) *Suppose that $d \geq 2$, $p \in (1, \infty]$, $\gamma(n)$ is a nonnegative sequence such that $\gamma(n) \to \infty$ and*

$$\lambda \geq \left(\frac{1}{\alpha n} \left(\frac{2(d-1)}{d} \ln n + \frac{2}{d} \ln \ln n + \gamma(n) \right) \right)^{1/d}.$$

    Then, almost always, G is connected.

    (ii) *Suppose that $d \geq 2$, $p \in [1, \infty]$, and $\lambda = (c \ln n / n)^{1/d}$ for some constant $c > 0$. Then, almost always, G is connected if $c > (2(d-1)/(d\alpha))$, and disconnected if $c < (2(d-1)/(d\alpha))$.*

    (iii) *Suppose that $d = 1$, $p \in [1, \infty]$, and $\lambda = 2(\ln n + \gamma(n))/n$. Then, almost always, G is connected if $\gamma(n) \to \infty$ and disconnected if $\gamma(n) \to -\infty$.*

Assertion (i) follows from combining Proposition 2 with Corollary 4, and assertion (ii) is implied by Proposition 5. (When $p > 1$ and $d \geq 2$, the lower bound in (ii) is implied by the stronger bound in (i).) Assertion (iii) is implied by Theorem 1 and its accompanying remarks.

4. A LOWER BOUND FOR DIAMETER

When G is connected, **B** will usually contain two vertices whose ℓ_p-distance is (asymptotically) $\mathrm{diam}_p(\mathbf{B})$. Therefore, the diameter of G will almost always be at least $\mathrm{diam}_p(\mathbf{B})(1 - o(1))/\lambda$. The precise statement is as follows.

Proposition 7 (Diameter lower bound). *Let $d \geq 1$ and $p \in [1, \infty]$, and suppose that $\lambda = \lambda(n)$ is sufficiently large so that Theorem 6 guarantees that almost always, $G = G_p^d(\lambda, n)$ is connected. If $h = h(n)$ satisfies*

$$(3) \qquad\qquad \lim_{n \to \infty} h^{(d+1)/2} n = \infty,$$

then, almost always,

$$\mathrm{diam}(G) \geq \frac{1-h}{\lambda} \mathrm{diam}_p(\mathbf{B}) = \begin{cases} 2(1-h)d^{1/p-1/2}/\lambda & \text{if } p \leq 2, \\ 2(1-h)/\lambda & \text{if } p \geq 2. \end{cases}$$

Proof. Let $\pm a$ be a pair of antipodes of the unit ball **B**, chosen as in Figure 4, and let $\pm C$ be the spherical cap formed by slicing **B** with hyperplanes at distance h from $\pm a$ respectively, perpendicular to the line joining a and $-a$. Let A be the event that at least one of the two caps $\pm C$ contains no vertex of V_n. Then

$$\Pr[A] = 2 \Pr[C \cap V_n = \emptyset] = 2 \left(1 - \frac{\mathrm{vol}(C)}{\mathrm{vol}(\mathbf{B})} \right)^n \leq 2 \exp \left(-n \frac{\mathrm{vol}(C)}{\mathrm{vol}(\mathbf{B})} \right).$$

On the other hand, $\mathrm{vol}(C)/\mathrm{vol}(B) = O(h^{(d+1)/2})$ by (18) of §A.3, which together with the condition (3) on h implies that $\Pr[A] = o(1)$. That is, G almost always

contains a vertex in each of C and $-C$. The result now follows from the definition of G and the lower bound (19) on the ℓ_p-distance between C and $-C$. $\qquad\square$

Note that for all $d \geq 1$, h can be chosen to satisfy both (3) and $\lim_{n\to\infty} h/\lambda = 0$. Also, if the limit in (3) is a nonnegative constant, then $\lim_{n\to\infty} \Pr[A] > 0$; that is, vertices are not guaranteed in both caps. For the case $p = 2$, Proposition 7 can be strengthened by identifying a collection of mutually disjoint antipodal pairs of caps of height h and showing that, almost always, both caps in at least one pair contain a vertex. Such a collection corresponds to an antipodally symmetric spherical code (see [3]).

5. The absolute upper bound

In this section we prove that when G is connected, the graph distance $d_G(x,y)$ between two vertices $x, y \in V_n$ is at most $K\|x - y\|_p/\lambda$, where $K > 0$ is a constant independent of n and p, but dependent on d. As a consequence, $\mathrm{diam}(G) \leq K \mathrm{diam}_p(\mathbf{B})/\lambda$. This will not be strong enough to meet (asymptotically) the lower bound in Proposition 7, but does guarantee a short path between any pair of vertices. This fact will be used repeatedly in the proof of the tight upper bound in Theorem 10 of §6. It is sufficient to prove the following Theorem 8, since for any two points $x, y \in \mathbb{R}^d$, we have $\|x - y\|_2 \leq d^{1/2}\|x - y\|_p$.

Theorem 8. *Let $d \geq 2$, and suppose that $\lambda = \lambda(n)$ is sufficiently large so that Theorem 6 guarantees that almost always, $G = G_p^d(\lambda, n)$ is connected. Then for any two points $x, y \in V_n$ there exists a constant K independent of n and p such that as $n \to \infty$, almost always,*

$$d_G(x,y) \ \leq \ \frac{K\|x - y\|_2}{\lambda}.$$

The proof is based on Proposition 9 below. For any two vertices $x, y \in V_n$, let

$$T_{x,y}(k) = \left[\text{convex closure of } (B_2^d(x, k\lambda) \cup B_2^d(y, k\lambda))\right] \cap \mathbf{B}.$$

Thus $T_{x,y}$ is a "lozenge"-shaped region. Let $A_n(k)$ be the event that there exist two vertices $x, y \in V_n$ such that (i) at least one point is inside $B_2^d(O, 1 - (k + \sqrt{d})\lambda)$, and (ii) there is no path of G that lies in $T_{x,y}(k)$ and connects x and y. The proof of our next result uses ingredients from [9, p. 285], adapted and extended for our present purposes.

Proposition 9. *Under the same assumptions as in Theorem 8, there exists a constant $k_0 > 0$, such that for all $k > k_0$,*

$$\lim_{n\to\infty} \Pr[A_n(k)] = 0.$$

Proof. First, we cover the unit ball $\mathbf{B}$ with d-dimensional cubes, each of side length $\epsilon\lambda$, where $\epsilon = 1/(4d)$. Let L_d be the set of centers of these cubes, and for each $z \in L_d$, denote the closed cube centered at z by Q_z.

Suppose $A_n(k)$ occurs for a pair of vertices x, y; without loss of generality, suppose $y \in B_2^d(O, 1 - (k + \sqrt{d})\lambda)$. Abbreviate $T_{x,y}(k)$ by $T_{x,y}$.

Step 1. First we construct a connected subset $P \subseteq T_{x,y}$ such that

(i) $B_p^d(P, \lambda/4) \subseteq \mathbf{B}$;
(ii) $\mathrm{diam}_2(P) \geq (k - \sqrt{d})\lambda$; and
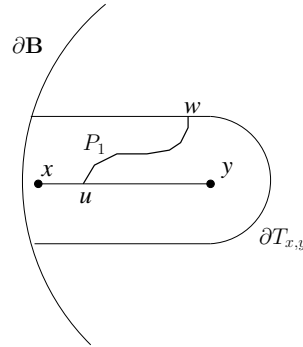(iii) $B_p^d(P, \lambda/4) \cap V_n = \emptyset$.

FIGURE 1. The frontier P_1 must intersect the line segment between vertices $x, y \in V_n$ at some point u, and must also satisfy $d_p(P_1, w) \leq \lambda/2$ for some point w on the boundary $\partial T_{x,y} \setminus \partial \mathbf{B}$.

Let $V_T = V_n \cap T_{x,y}$ be the set of vertices of G lying in $T_{x,y}$. Then $B_p^d(V_T, \lambda/2)$ is (topologically) disconnected with x and y lying in different connected components. Let D_x be the connected component of $B_p^d(V_T, \lambda/2)$ containing x. Let S be the closure of the connected component of $T_{x,y} \setminus D_x$ containing y. Let T be the closure of $T_{x,y} \setminus S$, so that T contains x. Then both S and T are connected, and their union is $T_{x,y}$. The lozenge $T_{x,y}$ is simply connected, so it is unicoherent [9, Lemma 9.1]; by definition of unicoherence, since $T_{x,y}$ is the union of closed connected sets $S, T \subseteq T_{x,y}$, then $P_1 := S \cap T$ is connected. Since $x \in T$, $y \in S$, and P_1 separates x and y, any path in $T_{x,y}$ from x to y must pass through P_1. In particular, P_1 intersects the line segment joining x and y. Let u be one of the intersection points.

Next, we show that there is a point w on $\partial T_{x,y}$ such that $d_p(P_1, w) \leq \lambda/2$, and derive from this that $\mathrm{diam}_2(P_1) \geq (k - \sqrt{d}/2)\lambda$ provided that the ℓ_2-distance between u and w is at least $k\lambda$ (see Figure 1 for an illustration). To achieve this, we must avoid the case that w lies in the boundary of $\mathbf{B}$. To this end, let $C_1 = \partial \mathbf{B} \cap T_{x,y}$, and let $C_2 = \partial T_{x,y} \setminus C_1$. If $d_p(P_1, C_2) \geq \lambda/2$, then C_2 must be a subset either of S or of T; without loss of generality, assume S. Then y is disconnected from x in G, which happens with probability tending to zero by Theorem 6. Hence, almost always, $d_p(P_1, C_2) < \lambda/2$. It follows that there is a point $w \in C_2$ such that $d_p(P_1, w) \leq \lambda/2$. Furthermore, $B_p^d(P_1, \lambda/2) \cap V_n = \emptyset$, by definition of P_1 as the intersection of S and T.

As constructed, P_1 may be too close to the boundary of $\mathbf{B}$ so that some cube Q_z intersecting P_1 might not lie entirely inside $\mathbf{B}$. To overcome this, we let P be obtained from P_1 by moving every point toward O by $\lambda/4$ under the transformation $x \rightarrow x - (\lambda/4)(x/\|x\|_2)$. Then P is connected, and $\mathrm{diam}_2(P) \geq \mathrm{diam}_2(P_1) - \lambda/2 \geq (k - \sqrt{d})\lambda$; that is P satisfies conditions (i–iii).

Step 2. We now show that when k is large enough, the probability

$$\Pr[Q_z \cap V_n = \emptyset \text{ for every } Q_z \subseteq B_p^d(P, \lambda/4)]$$

tends to zero. Let ω be the set of points $z \in L_d$ such that $Q_z \cap P \neq \emptyset$. Since P is connected, ω is a ∗-connected subset of L_d; that is, the union of the corresponding
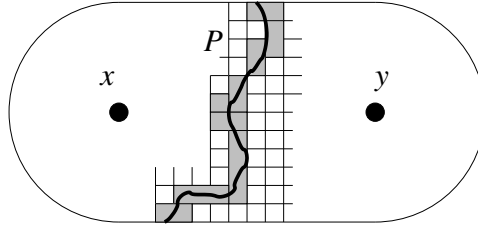
FIGURE 2. Two vertices $x, y \in V_n$ which are not connected by any path in $T_{x,y}(k)$, and the "frontier" P separating them, when $d = 2$. The gray squares are the $*$-connected subset ω intersecting P of the set of squares covering $\mathbf{B}$.

set of cubes is (topologically) connected (see Figure 2). For each $z \in \omega$, we have $Q_z \cap P \neq \emptyset$ and $\epsilon \leq 1/(4d^{1/p})$; hence $Q_z \subseteq B_p^d(P, \lambda/4)$. By considering the ℓ_2-diameter of P, we see that ω contains at least $4\sqrt{d}(k - \sqrt{d})$ points. Hence we have a $*$-connected subset $\omega \subseteq L_d$ with cardinality at least $4\sqrt{d}(k - \sqrt{d})$ such that $Q_z \cap V_n = \emptyset$ for all $z \in \omega$. We show that the probability of such an event is $o(1)$.

Let $\mathcal{C}_{d,i}$ denote the collection of $*$-connected sets of $\omega \subseteq L_d$ of cardinality i. It is known that the number of $*$-connected subsets of $\mathbb{Z}^d$ of cardinality i containing the origin is at most $2^{3^d i}$ (see, for example, [9, Lemma 9.3]). Since $|L_d| \leq (2/(\epsilon\lambda))^d$, we have $\mathcal{C}_{d,i} \leq (2/(\epsilon\lambda))^d 2^{3^d i} \leq 2^d(\epsilon\lambda)^{-d} e^{3^d i}$. Therefore

$$
\begin{aligned}
\Pr[A_n(k)] \quad &\leq \quad \sum_{i \geq 4\sqrt{d}(k-\sqrt{d})} \sum_{\omega \in \mathcal{C}_{d,i}} \Pr[V_n \cap (\cup_{z \in \omega} Q_z) = \emptyset] \\
&\leq \quad \sum_{i \geq 4\sqrt{d}(k-\sqrt{d})} 2^d(\epsilon\lambda)^{-d} \exp(3^d i)\left(1 - \frac{i}{\mathrm{vol}(\mathbf{B})}(\epsilon\lambda)^d\right)^n \\
(4) \quad &\leq \quad \sum_{i \geq 4\sqrt{d}(k-\sqrt{d})} cn \exp(-i\epsilon^d(d-1)\ln n/(d\alpha\,\mathrm{vol}(\mathbf{B}))) \\
(5) \quad &= \quad O\left(n^{1 - \left(4\sqrt{d}(k-\sqrt{d})(d-1)\right)/\left(d\alpha\,\mathrm{vol}(\mathbf{B})(4d)^d\right)}\right),
\end{aligned}
$$

where c is a constant and $\alpha = \alpha_p^d$ is the constant of (1). To justify inequality (4), when n is sufficiently large, we have $3^d < \epsilon^d(d-1)\ln n/(d\alpha\,\mathrm{vol}(\mathbf{B}))$. The order bound in (5) is immediate by geometric series, and the resulting quantity is $o(1)$ provided $k > \sqrt{d} + d\alpha\,\mathrm{vol}(\mathbf{B})(4d)^d/(4\sqrt{d}(d-1))$, which proves the existence of k_0 in the proposition. □

Proof of Theorem 8. Fix $k > k_0$ as in Proposition 9. Let x and y be two vertices in V_n with $\|x - y\|_p > \lambda$. If at least one of x, y lies in $B_2^d(O, 1 - (k + \sqrt{d})\lambda)$, then, almost always, there is a path of G connecting x and y in $T_{x,y}$. Suppose the shortest path between x and y in $T_{x,y}$ has length g. Then the ℓ_p-balls of radius $\lambda/2$ around every other vertex in the path must be pairwise disjoint, and each must lie inside the convex closure of $B_2^d(x, (k + \sqrt{d}/2)\lambda) \cup B_2^d(y, (k + \sqrt{d}/2)\lambda)$. By comparing the

volume of the ℓ_p-balls of radius $\lambda/2$ to the volume of $T_{x,y}$, we obtain

$$
\begin{aligned}
\left\lfloor \frac{g}{2} \right\rfloor \mathrm{vol}\left(B_p^d(x, \lambda/2)\right) \;\le\; & \mathrm{vol}\left(B_2^d(x, (k+\sqrt{d}/2)\lambda)\right) \\
& + d_2(x,y) \cdot \mathrm{vol}\left(B_2^{d-1}(x, (k+\sqrt{d}/2)\lambda)\right),
\end{aligned}
$$

which implies that $g \le K_1 + K_2 d_2(x,y)/\lambda \le (K_1\sqrt{d} + K_2)d_2(x,y)/\lambda$, where K_1 and K_2 are constants independent of n and p.

If both x and y lie outside $B_2^d(O, 1 - (k+\sqrt{d})\lambda)$, then we can travel from x to an intermediate vertex x_1 just inside $B(O, 1 - (k+\sqrt{d})\lambda)$ via a path of bounded length, and then on to y. To this end, let $r = \max\{(\alpha_p^d)^{1/d}\sqrt{d}, \sqrt{d}\}$, and let $E_n(k)$ be the event that there is a vertex $z \in V_n$ such that $z \notin B(O, 1 - (k+\sqrt{d})\lambda)$ and $V_n \cap (B(O, 1 - (k+\sqrt{d})\lambda) \cap B(z, (k+\sqrt{d}+2r)\lambda)) = \emptyset$. Then

$$
\Pr[E_n(k)] \le n\left(1 - (1 - (k+\sqrt{d})\lambda)^d\right)(1 - (\sqrt{d}\lambda)^d)^n = o(1).
$$

Applying this observation with $z = x$, we can find a point $x_1 \in V_n$ inside $B(O, 1 - (k+\sqrt{d})\lambda) \cap B(x, (k+\sqrt{d}+2r)\lambda)$. By the preceding argument we can first travel from x to x_1 in $Kd_2(x, x_1)/\lambda$ steps, and then from x_1 to y in $Kd_2(x_1, y)/\lambda$ steps. The total length of the path is no more than $K(d_2(x,y) + 2d_2(x,x_1))/\lambda$. Theorem 8 follows from the fact that $d_2(x, x_1) \le (k + \sqrt{d} + 2r)\lambda$. $\qquad\square$

We briefly discuss the case that $d = 1$, so that $\mathbf{B}$ is the interval $[-1, 1] \subset \mathbb{R}$. Suppose that $\lambda = \lambda(n)$ is sufficiently large so that Theorem 6 guarantees that, almost always, $G_p^1(\lambda, n)$ is connected. For any two vertices x, y, the shortest path between them clearly consists of a strictly increasing set of vertices $x = x_0 < x_1 < x_2 < \cdots < x_{d_G(x,y)} = y$. Moreover, the balls $B(x_0, \lambda/2), B(x_2, \lambda/2), B(x_4, \lambda/2), \ldots$ must be pairwise disjoint (else some x_i is redundant). Hence $|x - y| \ge \lceil d_G(x,y)/2 \rceil \lambda$, and from this it is not hard to deduce that $d_G(x, y) \le 2|x - y|/\lambda$.

6. THE ASYMPTOTICALLY TIGHT UPPER BOUND

In this section, we improve the upper bound in Theorem 8, reducing the constant K to $\mathrm{diam}_p(\mathbf{B})$ (asymptotically). Our main result is as follows:

Theorem 10. *Let $d \ge 2$ and $p \in [1, \infty]$, and suppose that $\lambda = \lambda(n)$ is sufficiently large so that Theorem 6 guarantees that almost always, $G = G_p^d(\lambda, n)$ is connected. Then as $n \to \infty$, almost always,*

$$
\mathrm{diam}(G) \le \begin{cases} \left(2d^{1/p-1/2} + O\left((\ln\ln n/\ln n)^{1/d}\right)\right)/\lambda & \text{when } 1 \le p \le 2, \\ \left(2 + O\left((\ln\ln n/\ln n)^{1/d}\right)\right)/\lambda & \text{when } 2 \le p \le \infty. \end{cases}
$$

That is, almost always, $\mathrm{diam}(G) \le \mathrm{diam}_p(\mathbf{B})(1 + O((\ln\ln n/\ln n)^{1/d}))/\lambda$.

The proof uses the geometric ingredients of *pins* and *pincushions*. A pin consists of a collection of evenly spaced, overlapping ℓ_p-balls whose centers lie on a diameter of the Euclidean unit d-ball $\mathbf{B}$. By making suitable choices for the geometry, we can ensure that each intersection of consecutive balls contains a vertex in V_n, so that the pin provides a "highway" through G. Having done this, we construct a pincushion so that every point of $\mathbf{B}$ is reasonably close to an ℓ_p-ball in one of its constituent pins. The following definitions are illustrated in Figure 3.
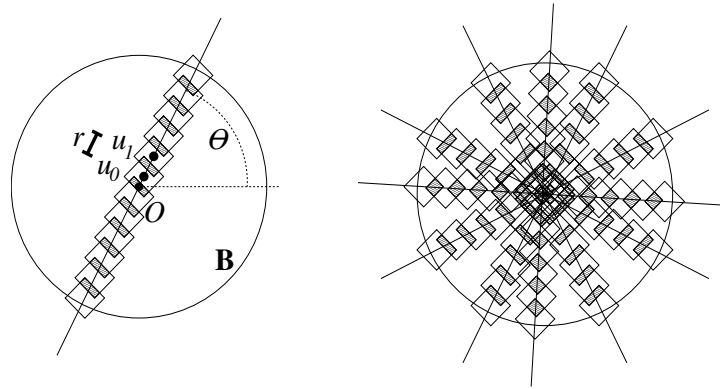
FIGURE 3. **(a)** A pin in the unit circle in $\mathbb{R}^2$, with angle θ. Here $p = 1$, so that the ℓ_p-balls are diamonds, and r is the ℓ_2-distance between consecutive centers (such as u_0 and u_1) of ℓ_1-balls. The petals are the shaded regions. **(b)** A pincushion consisting of several pins.

Definition 11 (Pins). Fix $d \geq 2$, $p \in [1, \infty]$, $\theta \in (-\pi/2, \pi/2]$, $\underline{\phi} = (\phi_3, \ldots, \phi_d) \in [0, \pi/2]^d$, and $\lambda, r > 0$. For $m \in \mathbb{Z}$, put

$$u_m = u_m(r, \theta) = \left(\frac{r}{2} + rm \right) \cdot (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

where

$$x_1 = \cos\theta \prod_{i=3}^{d} \sin\phi_i, \quad x_2 = \sin\theta \prod_{i=3}^{d} \sin\phi_i, \quad x_j = \cos\phi_j \prod_{i=j+1}^{d} \sin\phi_i \text{ for } 3 \leq j \leq d.$$

The corresponding *pin* $U(d, p, \theta, \underline{\phi}, r, \lambda)$ consists of the points $\{u_m : m \in \mathbb{Z}\} \cap \mathbf{B}$, together with a collection of ℓ_p-balls of radius $\lambda/2$, one centered at each point u_m. Note that the total number of ℓ_p-balls is $1 + 2\lfloor \frac{1}{r} \rfloor$.

Definition 12 (Pincushions). Fix $d \geq 2$, $1 \leq p \leq \infty$, and $\sigma \in \mathbb{Z}^+$. The corresponding *pincushion* (with parameters d, p, σ, r, λ) is the set of $(2\sigma)^{d-1}$ pins

$$\mathcal{U} := \left\{ U(d, p, \theta, \underline{\phi}, r, \lambda) \ : \ \theta, \phi_i \in \left\{ 0, \frac{\pi}{2\sigma}, \frac{2\pi}{2\sigma}, \ldots, \frac{(2\sigma - 1)\pi}{2\sigma} \right\} \right\}.$$

Definition 13 (Petals). Let U be a pin. A *petal* is the region of intersection of two overlapping ℓ_p-balls on U (the shaded regions in Figure 3). A petal is *nonempty* if it contains a vertex of V_n.

The probability that a petal is nonempty depends on its volume, which depends in turn on the parameters of the corresponding pin. Certainly, we must choose r so that the petal has positive volume: for example, it suffices to take $r \leq (\text{diam}_2(\mathbf{B})/\text{diam}_p(\mathbf{B}))\lambda$. Unfortunately, the volume is difficult to calculate exactly. Even finding the minimum volume over all angles, that is,

$$\xi = \xi_p^d(r, \lambda/2) := \inf\{\text{vol}\left(B_p^d(x, \lambda/2) \cap B_p^d(y, \lambda/2) \right) \ : \ x, y \in \mathbf{B}, \ \|x - y\|_2 = r\}.$$

requires integrals that are not easily evaluated (although for fixed d and r, it is certainly true that $\xi = \Theta(\lambda^d)$). The easiest way to find a lower bound for ξ is to inscribe another ℓ_p-ball in the petal.

Lemma 14. *Let $d \geq 1$, $p \in [1, \infty]$, $\lambda > 0$ and $0 \leq r \leq (\mathrm{diam}_p(\mathbf{B})/\mathrm{diam}_2(\mathbf{B}))\lambda$. Then, for all $x, y \in \mathbf{B}$ with $\|x - y\|_2 = r$,*

$$B_2^d \left(\tfrac{x+y}{2}, r' \right) \subseteq B_p^d(x, \lambda/2) \cap B_p^d(y, \lambda/2),$$

where

$$r' = \begin{cases} \frac{\lambda}{2} d^{1/2 - 1/p} - \frac{r}{2} & \text{when } 1 \leq p \leq 2, \\ \frac{\lambda}{2} - \frac{r}{2} & \text{when } 2 \leq p \leq \infty. \end{cases}$$

In particular,

$$\frac{\xi}{\mathrm{vol}(\mathbf{B})} \geq \begin{cases} \left(\frac{\lambda}{2} d^{1/2 - 1/p} - \frac{r}{2} \right)^d & \text{when } 1 \leq p \leq 2, \\ \left(\frac{\lambda}{2} - \frac{r}{2} \right)^d & \text{when } 2 \leq p \leq \infty. \end{cases}$$

Proof. Let $z \in B_2^d(\frac{x+y}{2}, r')$. By the triangle inequality,

$$\|z - x\|_2 \leq \left\|z - \tfrac{x+y}{2}\right\|_2 + \left\|\tfrac{x+y}{2} - x\right\|_2 \leq r' + \tfrac{r}{2},$$

which, together with (17), implies that $\|z - x\|_p \leq \lambda/2$. That is, $z \in B_p^d(x, \lambda/2)$. The same argument implies that $z \in B_p^d(y, \lambda/2)$. The bound on ξ is then a simple application of (14). □

For the remainder of this section, we work with the pincushion $\mathcal{U}$ defined by

(6a) $$\sigma = \sigma(n) = \lfloor (\ln n)^{1/d} \rfloor$$

and

(6b) $$r = r(n) = \begin{cases} \lambda d^{1/2 - 1/p}(1 - \rho(n)) & \text{when } 1 \leq p \leq 2, \\ \lambda(1 - \rho(n)) & \text{when } 2 \leq p \leq \infty, \end{cases}$$

where

(6c) $$\rho = \rho(n) = \begin{cases} 2d^{1/p - 1/2}(\ln \ln n / \ln n)^{1/d} & \text{when } 1 \leq p \leq 2, \\ 2(\ln \ln n / \ln n)^{1/d} & \text{when } 2 \leq p \leq \infty. \end{cases}$$

Let $\tau_U = \tau_U(n)$ be the number of empty petals along the pin $U \in \mathcal{U}$, and define

$$\tau = \tau(n) = \max\{\tau_U(n) \ : \ U \in \mathcal{U}\}.$$

We first calculate an upper bound on τ.

Lemma 15. *With the assumptions of Theorem 10, and the parameters σ, r, ρ as just defined, almost always,*

$$\tau \leq \sigma^{(d-1)/2} \frac{2}{r} \exp \left(\frac{-n\xi}{\mathrm{vol}(\mathbf{B})} \right).$$

Proof. Denote the right-hand side of the desired inequality by T. By linearity of expectation,

$$\Pr[\tau \geq T] \leq \mathbb{E}[|\{U : \tau_U \geq T\}|]$$

(7) $$\leq (2\sigma)^{d-1} \cdot \Pr[\tau_{U^*} \geq T],$$

where U^* is chosen so as to maximize $\Pr[\tau_U \geq T]$. Let X_i be the indicator random variable of the event that the ith petal of U^* is empty, and let $X = \sum_i X_i$. Now U^* contains at most $2/r$ petals, so by linearity of expectation,

$$(8) \qquad \mathbb{E}[X] \; \leq \; \frac{2}{r} \left(1 - \frac{\xi}{\mathrm{vol}(\mathbf{B})} \right)^n \; \leq \; \frac{2}{r} \exp \left(\frac{-n\xi}{\mathrm{vol}(\mathbf{B})} \right).$$

Also,

$$(9) \qquad \mathrm{var}[X] \; \leq \; \mathbb{E}[X] + \sum_{i \neq j} \mathrm{cov}(X_i, X_j),$$

where the covariance $\mathrm{cov}(X_i, X_j)$ for $i \neq j$ is

$$\begin{aligned} \mathrm{cov}(X_i, X_j) \; &= \; \Pr[X_i X_j = 1] - \Pr[X_i = 1] \cdot \Pr[X_j = 1] \\ &= \; \left(1 - \frac{2\xi}{\mathrm{vol}(\mathbf{B})} \right)^n - \left(1 - \frac{\xi}{\mathrm{vol}(\mathbf{B})} \right)^{2n} \\ (10) \qquad &\leq \; o(1) \cdot \exp \left(\frac{-2n\xi}{\mathrm{vol}(\mathbf{B})} \right). \end{aligned}$$

Combining (8), (9) and (10) gives

$$(11) \qquad \mathrm{var}[X] \leq \frac{2}{r} \exp \left(\frac{-n\xi}{\mathrm{vol}(\mathbf{B})} \right) + \frac{4}{r^2}\, o(1) \cdot \exp \left(\frac{-2n\xi}{\mathrm{vol}(\mathbf{B})} \right).$$

By Chebyshev's inequality (cf. [1]) and the bounds for $\mathbb{E}[X]$ and $\mathrm{var}[X]$ in (8) and (11),

$$\begin{aligned} \Pr\left[X \geq T \right] \; &\leq \; \Pr\left[|X - \mathbb{E}[X]| \geq T - \mathbb{E}[X] \right] \\ &\leq \; \frac{\mathrm{var}[X]}{(T - \mathbb{E}[X])^2} \; = \; o\left(\frac{1}{\sigma^{d-1}} \right). \end{aligned}$$

Now $\Pr[\tau_U \geq T] = \Pr[X \geq T]$. Therefore, substituting this last bound into (7) gives $\Pr[\tau \geq T] = o(1)$, which implies the desired result. $\qquad \square$

We can now prove the main result of this section.

Proof of Theorem 10. Let $x, y \in V_n$. We will find vertices x_1 and y_1 near x and y respectively, belonging to petals of the pincushion $\mathcal{U}$. From each of x_1, y_1, we walk along the appropriate pin to points x_2, y_2 near the origin and belonging to petals on the same pin as x_1 and y_1, respectively. We will then use Theorem 8 to construct a path from x_2 to y_2, as well as any "detours" needed in case there are missing edges in the paths along the pins. Without loss of generality, we may assume that

$$(12) \qquad \|x\|_2, \|y\|_2 \; \geq \; (\tau + 3/2)r.$$

The justification for this is deferred until the end of the proof.

Let R_x be the (possibly empty) petal nearest to x, and let U_x be the pin containing R_x. When n is sufficiently large, the distance from x to R_x is at most $d\pi/2\sigma$. By definition of τ, there is another vertex $x_1 \in V_n$, which also lies in a petal on U_x, but is closer to the origin, so that $\|x - x_1\|_2 \leq d\pi/2\sigma + r(\tau + 1)$. Repeat these constructions for y to obtain an analogous vertex y_1. Then

$$\|x - x_1\|_2, \|y - y_1\|_2 \; \leq \; \frac{d\pi}{2\sigma} + r(\tau + 1),$$

and, by Theorem 8,

$$d_G(x, x_1),\ d_G(y, y_1)\ \leq\ \left(\frac{d\pi}{2\sigma} + r(\tau + 1)\right)\frac{K}{\lambda}.$$

By definition of τ, there is a vertex x_2 that belongs to a petal R_{x_2} on U_x (indeed, lying on the same side of O along U_x) with $R_{x_2} \subseteq B_2^d(O, r(\tau + 3/2))$. In the worst case, all empty petals in U_x occur non-consecutively between x_1 and x_2, so for n large enough, we have

$$
\begin{aligned}
d_G(x_1, x_2)\ &\leq\ \frac{\|x_1\|_2}{r} + \frac{2Kr\tau}{\lambda} \\
&\leq\ \frac{\|x\|_2}{r} + \frac{d\pi}{2\sigma r} + \frac{2Kr\tau}{\lambda}.
\end{aligned}
$$

The same construction goes through if we replace the x's with y's. Moreover, $\|x_2 - y_2\|_2 \leq (2\tau + 3)r$, so the shortest path in G between x_2 and y_2 satisfies

$$d_G(x_2, y_2) \leq \frac{(2\tau + 3)rK}{\lambda}.$$

Concatenating all the above paths, we find that

$$
\begin{aligned}
d_G(x, y)\ &\leq\ d_G(x, x_1) + d_G(x_1, x_2) + d_G(x_2, y_2) + d_G(y_2, y_1) + d_G(y_1, y) \\
&\leq\ \frac{\|x\|_2 + \|y\|_2}{r} + \left(\frac{d\pi}{\sigma} + (8\tau + 5)r\right)\frac{K}{\lambda} + \frac{d\pi}{\sigma r} \\
(13)\qquad &\leq\ \frac{\mathrm{diam}_p(\mathbf{B})}{\mathrm{diam}_2(\mathbf{B})}\frac{\|x\|_2 + \|y\|_2 + O(\rho)}{\lambda} + O(\tau) + O\left(\frac{(\ln n)^{-1/d}}{\lambda}\right).
\end{aligned}
$$

By the definitions of r and ρ given in (6b) and (6c), and the bounds on ξ and τ (Lemmas 14 and 15), it follows that

$$\frac{\xi}{\mathrm{vol}(\mathbf{B})}\ \geq\ \lambda^d\frac{\ln\ln n}{\ln n}\ \geq\ \frac{\ln\ln n}{n},$$

where the second inequality follows from the assumption that λ is above the threshold for connectivity in Theorem 6. Therefore

$$
\begin{aligned}
\tau\ &\leq\ \frac{2\sigma^{(d-1)/2}}{r}\exp\left(\frac{-n\xi}{\mathrm{vol}(\mathbf{B})}\right) \\
&\leq\ \frac{4}{\lambda}(\ln n)^{(d-1)/(2d)}(\ln n)^{-1} = \frac{O\left((\ln n)^{-(d+1)/(2d)}\right)}{\lambda} = \frac{o(\rho)}{\lambda}.
\end{aligned}
$$

Plugging this bound for τ into (13) gives

$$d_G(x, y) \leq \frac{\mathrm{diam}_p(\mathbf{B})}{\mathrm{diam}_2(\mathbf{B})}\frac{\|x\|_2 + \|y\|_2 + O(\rho)}{\lambda},$$

and the theorem follows.

We now explain the assumption (12). If $\|x\|_2, \|y\|_2 \leq (\tau + 3/2)r$, then by Theorem 8, $d_G(u, v) \leq (2\tau + 3)rK/\lambda = o(\rho)/\lambda$. On the other hand, if $\|x\|_2 \leq (\tau + 3/2)r \leq \|y\|_2$, then, almost always, there is a vertex $x_2 \in B_2^d(O, (\tau + 3/2)r)$ belonging to some petal. By the preceding argument, $d_G(x, y) \leq o(\rho/\lambda) + d_G(x_2, y) \leq (\mathrm{diam}_p(\mathbf{B})/\mathrm{diam}_2(\mathbf{B}))(\|y\|_2 + O(\rho))/\lambda$. $\qquad\square$

We conclude with two remarks. First, the proof of Theorem 10 can easily be adapted to the case $d = 1$ to obtain the upper bound $\mathrm{diam}(G_p^1(\lambda, n)) \leq (2 + O(\ln \ln n / \ln n))/\lambda$. (Note that when $d = 1$ a pincushion consists of just one pin.) Second, the technique of Theorem 10 can be extended to obtain the stronger result $d_G(x, y) \leq (\|x - y\|_p + O((\ln \ln n / \ln n)^{1/d}))/\lambda$, so that the graph distance *approximates* the ℓ_p-metric. Each pin is replaced by σ^{d-1} evenly spaced parallel pins. We can still bound τ by $o(\rho)/\lambda$, but now any two vertices x and y are close to the same pin, on which a short path from x to y is found. We refer the reader to [6] for details.

ACKNOWLEDGEMENTS

APPENDIX A. FACTS ABOUT ℓ_p- AND SPHERICAL GEOMETRY

In the body of the article, we used various facts about the Euclidean and ℓ_p-geometry of balls and spherical caps. None of these facts are difficult; however, for convenience we present them together here along with brief proofs.

A.1. **Volume of the ℓ_p-unit ball.** Fix $p \in [1, \infty]$ and an integer $d \geq 1$. Let $B_p^d(r)$ be the ℓ_p-ball centered at the origin in $\mathbb{R}^d$ with radius r:

$$B_p^d(r) = \left\{ x \in \mathbb{R}^d \; : \; \sum_{i=1}^d |x_i|^p \leq r^p \right\}.$$

We will show that the (d-dimensional) volume of $B_p^d(r)$ is

$$(14) \qquad \mathrm{vol}(B_p^d(r)) = \frac{(2r)^d \Gamma\left(\frac{p+1}{p}\right)^d}{\Gamma\left(\frac{p+d}{p}\right)},$$

where Γ is the usual gamma function [11]. For $d = 1$ this is trivial. For $d > 1$ we have

$$\mathrm{vol}(B_p^d(r)) = 2 \int_0^r \mathrm{vol}\left(B_p^{d-1}((r^p - x^p)^{1/p})\right) \, dx.$$

Make the substitution $u = x^p / r^p$, $x = r u^{1/p}$, $dx = (r/p) u^{(1-p)/p} \, du$. By induction on d, we obtain

$$\mathrm{vol}(B_p^d(r)) = \frac{2^d r^d \Gamma\left(\frac{p+1}{p}\right)^{d-1}}{p \Gamma\left(\frac{p+d-1}{p}\right)} \int_0^1 (1 - u)^{\frac{d-1}{p}} u^{\frac{1-p}{p}} \, du.$$

Evaluating this integral as in [11, §12.4] yields the desired formula (14). It follows that

$$(15) \qquad \alpha_p^d := \frac{\mathrm{vol}(B_p^d(r))}{\mathrm{vol}(B_2^d(r))} = \frac{\Gamma\left(\frac{p+1}{p}\right)^d \cdot \Gamma\left(\frac{2+d}{2}\right)}{\Gamma\left(\frac{3}{2}\right)^d \cdot \Gamma\left(\frac{p+d}{p}\right)}.$$

A.2. ℓ_p-antipodes on the Euclidean sphere. Let $\mathbf{B} = B_2^d(1)$ be the Euclidean unit ball, centered at the origin in $\mathbb{R}^d$, and let $p \neq 2$. We wish to calculate the ℓ_p-diameter of $\mathbf{B}$, that is,

$$\mathrm{diam}_p(\mathbf{B}) = \max\{\|x - y\|_p \; : \; x, y \in \mathbf{B}\}.$$

A pair of points of $\mathbf{B}$ at distance $\mathrm{diam}_p(\mathbf{B})$ are called ℓ_p-*antipodes*. If $d = 1$, then $\mathbf{B}$ is a line segment and the only antipodes are its endpoints. Of course, if $p = 2$, then the antipodes are the pairs $\pm x$ with $\|x\|_2 = 1$.

Suppose that $d > 1$. Every pair of antipodes x, y must satisfy $\|x\|_2 = \|y\|_2 = 1$. Without loss of generality, we may assume $x_i \geq 0 \geq y_i$ for every i. Using the method of Lagrange multipliers (with objective function $(\|x - y\|_p)^p = \sum_{i=1}^d (x_i - y_i)^p$), we find that for every i,

(16) $$p(x_i - y_i)^{p-1} = 2\lambda x_i = -2\mu y_i,$$

where λ and μ are nonzero constants. In particular $y_i = (-\lambda/\mu)x_i$ for every i, so $y = -x$. (That is, every pair of ℓ_p-antipodes on $\mathbf{B}$ is a pair of ℓ_2-antipodes.) Substituting for y_i in (16) gives

$$\nu x_i^{p-1} = x_i,$$

where ν is some nonzero constant. In particular, the set of coordinates $\{x_1, \ldots, x_d\}$ can contain at most one nonzero value, so either x is a coordinate unit vector or else $x_i = d^{-1/2}$ for every i. In the first case $\|(-x) - x\|_p = 2$, while in the second case $\|(-x) - x\|_p = 2d^{-1/2+1/p}$. In summary:

Proposition 16. *Let $d \geq 2$, and let $\mathbf{B} = B_2^d(1)$ be the Euclidean unit ball, centered at the origin in $\mathbb{R}^d$. Then the pairs of ℓ_p-antipodes on $\mathbf{B}$ are precisely the pairs $\{\pm x\}$ satisfying the following additional conditions:*

$$\begin{cases} |x_1| = \cdots = |x_d| = d^{-1/2} & \text{when } 1 \leq p < 2, \\ \|x\|_p = 1 & \text{when } p = 2, \\ x \text{ is a coordinate unit vector} & \text{when } 2 < p \leq \infty. \end{cases}$$

In particular, the ℓ_p-diameter of $\mathbf{B}$ is

(17) $$\mathrm{diam}_p(B) = \max\left(2, \, 2d^{1/p - 1/2}\right) = \begin{cases} 2d^{1/p - 1/2} & \text{for } 1 \leq p \leq 2, \\ 2 & \text{for } 2 \leq p \leq \infty. \end{cases}$$

A.3. Spherical caps. Let $B = B_2^d(r)$ be the Euclidean ball of radius r, centered at the origin in $\mathbb{R}^d$. Let C be a spherical cap of B of height h, with $0 \leq h \leq r$; for instance,

$$C = \{(x_1, \ldots, x_d) \in B : \; r - h \leq x_1 \leq r\}.$$

The volume of C can be determined exactly, but the precise formula is awkward for large d (one has to evaluate the integral $\int \sin^d \theta \, d\theta$). On the other hand, we can easily obtain a lower bound for $\mathrm{vol}(C)$ by inscribing in it a "hypercone" H of height h whose base is a $(d - 1)$-sphere of radius $s = \sqrt{r^2 - (r - h)^2} = \sqrt{2rh - h^2}$. For $r - h \leq x \leq r$, the cross-section of H at $x = x_1$ is $B_2^{d-1}(s(r-x)/h)$, so applying (14) gives

$$\mathrm{vol}(H) = \frac{2^{d-1}s^{d-1}\Gamma\left(\frac{3}{2}\right)^{d-1}}{h^{d-1}\Gamma\left(\frac{d+1}{2}\right)} \int_{r-h}^r (r - x)^{d-1} \, dx$$
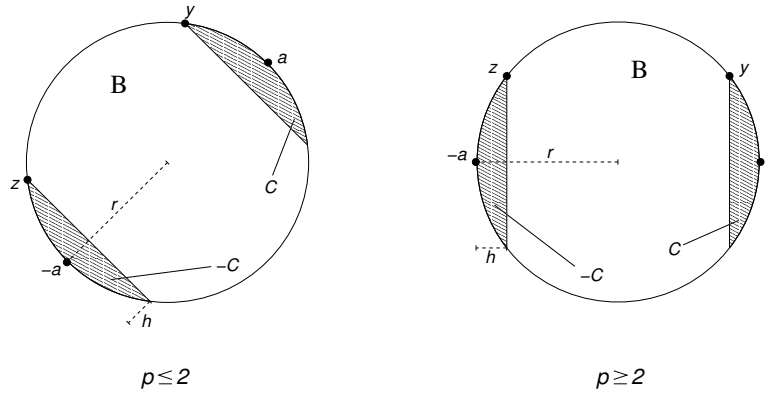
FIGURE 4. Antipodally aligned caps $\pm C$ centered at antipodes $\pm a$ of $B \subseteq \mathbb{R}^2$ for different values of p below and above 2.

$$= \frac{\pi(2r-h)^{(d-1)/2}h^{(d+1)/2}}{d \cdot \Gamma\left(\frac{d+1}{2}\right)}.$$

Since $2r - h \geq r$, we have the bound

$$(18) \qquad \text{vol}(C) \geq \frac{\pi r^{(d-1)/2}}{d \cdot \Gamma\left(\frac{d+1}{2}\right)} \cdot h^{(d+1)/2}.$$

A.4. **The ℓ_p-distance between opposite spherical caps.** By definition, the ℓ_p-distance between two sets $Y, Z \subseteq \mathbb{R}^d$ is

$$d_p(Y, Z) := \inf\{\|y - z\|_p \ : \ y \in Y, \ z \in Z\}.$$

Let $B = B_2^d(r)$, let $\pm a$ be a pair of ℓ_p-antipodes on B, and let $\pm C$ be the cap of height h centered at $\pm a$. Note that every pair y, z at minimum distance has displacement parallel to a (see Figure 4). (This can be verified by another easy Lagrange-multiplier calculation.) Therefore

$$(19) \qquad d_p(C, -C) \ = \ \frac{2(r-h)}{r}\|a\|_p \ = \ \begin{cases} 2(r-h)d^{1/p-1/2} & \text{for } 1 \leq p \leq 2, \\ 2(r-h) & \text{for } p \geq 2. \end{cases}$$

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