THE PARTITIONABILITY CONJECTURE

ART M. DUVAL, CAROLINE J. KLIVANS, AND JEREMY L. MARTIN

In 1979, Richard Stanley made the following conjecture: Every Cohen–Macaulay simplicial complex is partitionable. Motivated by questions in the theory of face numbers of simplicial complexes, the Partitionability Conjecture sought to connect a purely combinatorial condition (partitionability) with an algebraic condition (Cohen–Macaulayness). The algebraic combinatorics community widely believed the conjecture to be true, especially in light of related stronger conjectures and weaker partial results. Nevertheless, in a 2016 paper [DGKM16], the three of us (Art, Carly, and Jeremy), together with Jeremy's graduate student Bennet Goeckner, constructed an explicit counterexample. Here we tell the story of the significance and motivation behind the Partitionability Conjecture and its resolution. The key mathematical ingredients include relative simplicial complexes, nonshellable balls, and a surprise appearance by the pigeonhole principle. More broadly, the narrative of the Partitionability Conjecture highlights a general theme of modern algebraic combinatorics: to understand discrete structures through algebraic, geometric, and topological lenses.

1. History and motivation

As basic discrete building blocks, simplicial complexes arise throughout mathematics, whether as surface meshes, simplicial polytopes, or abstract models of multi-way relations. An *abstract simplicial complex* Δ is simply a family of subsets of a finite vertex set V that is closed under taking subsets: if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of Δ, which are called *faces* or *simplices*, admit a natural partial order by containment, and we typically regard a simplicial complex as equivalent to its face poset (partially ordered set of faces). The maximal faces are called *facets*. A simplicial complex can be realized topologically by representing each face σ by a geometric simplex of dimension $|\sigma| - 1$, as in Figure 1. For this reason, the standard convention is to write the dimension of Δ (that is, the largest dimension of a face of Δ) as $d - 1$. Here we are concerned with the problem of decomposing the face poset into *intervals*, i.e., subsets of the form $[\sigma, \tau] := \{\rho \in \Delta \mid \sigma \subseteq \rho \subseteq \tau\}$.

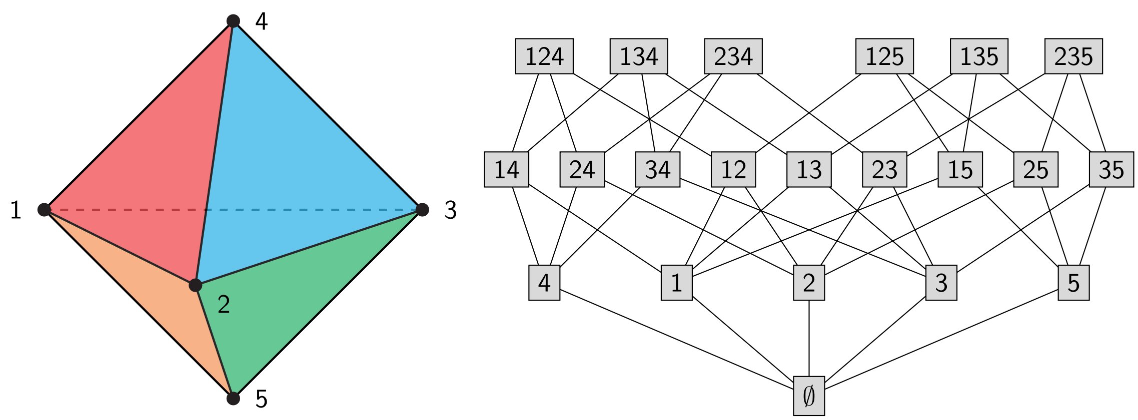

Figure 1. A simplicial complex and its face poset.

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The f-vector of a complex records the face numbers, dimension by dimension. Specifically, the f-vector is $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$ where f_i is the number of i-dimensional faces in Δ. Understanding the f-vectors of various classes of complexes is a major area of modern algebraic combinatorics; the standard source is [Sta96]. The Partitionability Conjecture began as an attempt to gain a combinatorial understanding of the face numbers of Cohen–Macaulay complexes.

The classical problem in this subject is to characterize the f-vectors of convex polytopes. For example, $(1, 5, 9, 6)$ is realized by the polytope in Figure 1, and it is not hard to convince yourself that $(1, 5, 9, 87)$ is impossible, but what about $(1, 15, 36, 12)$? What can be said about this combinatorial invariant of inherently geometric objects? This problem dates back at least to Euler, and remains unsolved in general, though many special cases are understood.

It often turns out to be more convenient to work with a linear transformation of the f-vector called the *h-vector* $h(\Delta) = (h_0, \ldots, h_d)$, defined by the identity

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}.$$

The formula can easily be inverted to express the f-vector in terms of the h-vector, so the two carry the same information. The h-vector is a key thread in our narrative, and we will see that it carries simultaneous interpretations in algebra, combinatorics, and topology. Many relations on the face numbers, particularly those arising in algebraic contexts, can be expressed naturally in terms of the h-vector. A spectacular example is the *Dehn-Sommerville equations*, which state that $h_k = h_{d-k}$ for every simplicial d-polytope. (This symmetry reflects Poincaré duality on the associated toric variety.) For instance, the bipyramid in Figure 1 has h-vector $(1, 2, 2, 1)$.

What do the h-numbers count? In general they need not even be positive, but for certain special classes such as *shellable* simplicial complexes they admit an elementary combinatorial interpretation. A shellable complex can be assembled facet by facet so that the new faces added at each step form an interval in the face poset; the number h_i counts the intervals of height $d - i$. The boundary complexes of convex polytopes are shellable, as proven by a beautiful geometric argument of Heinz Bruggesser and Peter Mani in 1970. Shellability of polytopes is an essential ingredient in a cornerstone of the theory of face numbers, Peter McMullen's *upper bound theorem*, which describes an upper bound that simultaneously maximizes all entries of the h-vectors of polytopes.

A weaker condition than shellability is *partitionability*, which first appears in the theses of Michael O. Ball and J. Scott Provan. Ball had been working on network reliability and Provan on diameters of polytopes.A partitionable complex can be decomposed into disjoint intervals, each one topped by a facet. That is, a partitioning matches each facet to one of its subfaces (possibly itself), namely the minimum element of the corresponding interval. Unlike a shelling, there is no restriction on how the intervals of a partitioning fit together. On the other hand, *when X is partitionable, the number h_i counts the intervals of height $d - i$*, just as for a shellable complex. (See Figure 2.) Hence the h-vectors of partitionable complexes have a simple combinatorial interpretation.

Constructible and *Cohen–Macaulay* simplicial complexes also have nonnegative h-vectors. Constructibility is a purely combinatorial recursive condition, while Cohen–Macaulayness arises in commutative algebra. Cohen–Macaulay *rings* have long enjoyed great importance in algebra and algebraic geometry; Mel Hochster is often quoted as saying that "life is really worth living" in a Cohen–Macaulay ring. Their significance in combinatorics, via what is now known as *Stanley-Reisner theory*, was established by Hochster, Gerald Reisner, and Richard Stanley in the early 1970s. In this theory, combinatorial questions about a complex are translated to algebraic questions about a ring. Specifically, the *Stanley-Reisner ring* (or *face ring*) of a simplicial complex Δ on the vertex set $\{1, 2, \ldots, n\}$ (over a field $\Bbbk$) is defined as $\Bbbk[\Delta] := \Bbbk[x_1, \ldots, x_n]/I_\Delta$, where I_Δ is the monomial ideal generated by non-faces of Δ. The construction is bijective: every quotient of a polynomial ring by a square-free monomial ideal gives rise to a simplicial complex. The two sides of the correspondence are tightly linked: for instance, the h-vector of Δ corresponds naturally to the Hilbert series of $\Bbbk[\Delta]$ via the formula

$$\sum_{i \geq 0} (\dim \Bbbk[\Delta]_i) t^i = \frac{\sum_j h_j(\Delta) t^j}{(1-t)^d},$$

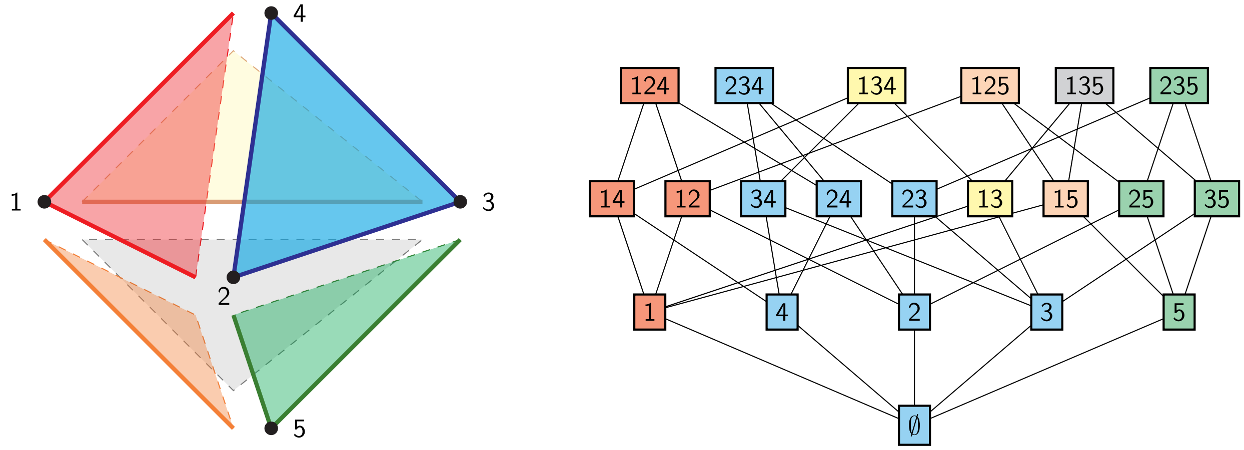

FIGURE 2. A shellable simplicial complex with h-vector $(1, 2, 2, 1)$ and the corresponding partitioning of the face poset into intervals.

where $\Bbbk[\Delta]_i$ denotes the ith graded piece of $\Bbbk[\Delta]$.

Two fundamental algebraic invariants of a commutative ring R are its (Krull) dimension and its depth. In all cases $\dim R \geq \operatorname{depth} R$, and if equality holds then the ring is *Cohen–Macaulay*. Likewise, a simplicial complex is Cohen–Macaulay if its Stanley-Reisner ring is Cohen–Macaulay. In the setting of Stanley-Reisner theory, the dimension of $\Bbbk[\Delta]$ is simply d, but depth is a subtler invariant of Δ, so it is not easy to translate this definition from algebra to combinatorics. Fortunately, Reisner found an equivalent criterion for Cohen–Macaulayness in terms of simplicial homology: Δ is Cohen–Macaulay if and only if for every face σ, the subcomplex $\operatorname{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset,\ \sigma \cup \tau \in \Delta\}$ has the homology type (over $\Bbbk$) of a wedge of $((d-1) - \dim \sigma)$-spheres. Reisner's criterion is the working definition of the Cohen–Macaulay property for many combinatorialists.

The Cohen–Macaulay property provided the algebraic bridge between topology and combinatorics in Stanley's celebrated extension of McMullen's upper bound theorem from polytopes to simplicial spheres. Specifically, Cohen–Macaulayness played the same role in bounding the h-vectors of spheres that shellability did for convex polytopes. Furthermore, the hierarchy of strict implications

$$\text{shellable} \quad \Longrightarrow \quad \text{constructible} \quad \Longrightarrow \quad \text{Cohen–Macaulay}$$

was well known since Hochster's 1973 work on face rings. (Of these properties, only Cohen–Macaulayness is topological, as proven by James Munkres in 1984.) On the other hand, Stanley proved in 1977 that the *h-vectors* of Cohen–Macaulay, shellable, and constructible complexes are precisely the same. The Partitionability Conjecture sought to explain this equality: perhaps partitionability sat at the base of the hierarchy above.

Conjecture 1 (The Partitionability Conjecture)**.** *Every Cohen–Macaulay simplicial complex is partitionable.*

The Partitionability Conjecture is one of many interrelated conjectures about the structure of simplicial complexes. Adriano Garsia made the same conjecture in 1980 in the more restricted setting of order complexes of Cohen–Macaulay posets. Motivated by the theory of algebraic shifting, Gil Kalai conjectured that *every* simplicial complex admits a more general decomposition into intervals. In 1993, Stanley conjectured that every *k-acyclic complex*—one for which $\operatorname{link}_\Delta(\sigma)$ is acyclic for every face σ of dimension less than k—admits a partitioning into intervals of rank at least $k+1$, with each interval topped by a face of dimension at least $\operatorname{depth} \Delta$. Each of Kalai's and Stanley's conjectures, if true, would imply the Partitionability Conjecture. Meanwhile, Masahiro Hachimori proved in 2000 that a certain strengthening of constructibility *does* in fact imply partitionability. The conjecture was widely believed to be true within the combinatorics community, and the works of Garsia, Kalai, Stanley, and Hachimori provided many tantalizing approaches, both combinatorial and algebraic.

The Partitionability Conjecture gained additional significance in the study of *Stanley depth*, a purely combinatorial counterpart to the depth of a graded module over a polynomial ring, introduced by Stanley in 1982. This invariant has attracted substantial attention in combinatorial commutative algebra over the last twenty years; the article [PSFTY09] by Mohammed Pournaki, Seyed A. Seyed Fakhari, Massoud Tousi, and Siamak Yassemi in the *Notices* is an excellent introduction. The focal point of this area has been Stanley's conjecture that the Stanley depth of every module is an upper bound for its (algebraic) depth. Jürgen Herzog, Ali Soleyman Jahan, and Yassemi proved in 2008 that Stanley's Depth Conjecture implies the Partitionability Conjecture.

On the other hand, as with so many areas of mathematics, combinatorics is full of surprises and the study of simplicial complexes is rife with counterintuitive examples. Shellability does not depend only on the topology of the underlying space: Mary Ellen Rudin famously constructed a nonshellable triangulation of the 3-dimensional ball in 1955. There are also nonshellable 3-spheres; the first was constructed in 1991 by William Lickorish. It can get even more wild: obstructions to shellability and constructibility can take the form of nontrivial knots embedded on a sphere. So perhaps the Partitionability Conjecture would also turn out to be false.

2. Building a counterexample

We started by trying to prove the Partitionability Conjecture, not to disprove it. Our previous joint work on simplicial and cellular trees, and follow-up work of Carly and Olivier Bernardi, had revealed unexpected connections to the Cohen–Macaulay condition and to Stanley's conjectures about decompositions of simplicial complexes. At first, we had hoped to use the theory of trees to approach partitionability. However, we eventually arrived at two turning points: key realizations that persuaded us to look for a counterexample instead of a proof.

When we first began to investigate partitionability, we wanted to experiment with a Cohen–Macaulay complex that is *not* shellable, because as we have seen, every shelling induces a partitioning. However, many Cohen–Macaulay complexes that arise in combinatorics, such as convex simplicial spheres and order complexes of certain posets, are naturally shellable. Rudin's nonshellable 3-ball would have met our requirements, but we chose the one constructed by Günter Ziegler in 1998 [Zie98], since it has fewer faces and is thus easier to work with. It has f-vector $(1, 10, 38, 50, 21)$, and is known to be partitionable.

Almost any approach to working with decompositions of simplicial complexes also requires working with *relative simplicial complexes*. A relative simplicial complex Q on vertex set V is a family of subsets of V that is convex with respect to the natural partial order by inclusion: if ρ and τ are faces of Q with $\rho \subseteq \sigma \subseteq \tau$, then σ is a face of Q as well. Equivalently, the face poset of Q is of the form $X \setminus A$, where X is a simplicial complex and A is a subcomplex (see Figure 3). The concept of a pair of spaces is familiar from algebraic topology; in particular, Q can be regarded as a combinatorial model of the quotient space X/A. The combinatorics of simplicial complexes, including Cohen–Macaulayness and partitionability, carries over well to the relative setting. Furthermore, the ability to remove subcomplexes makes it the right setting to study decompositions.

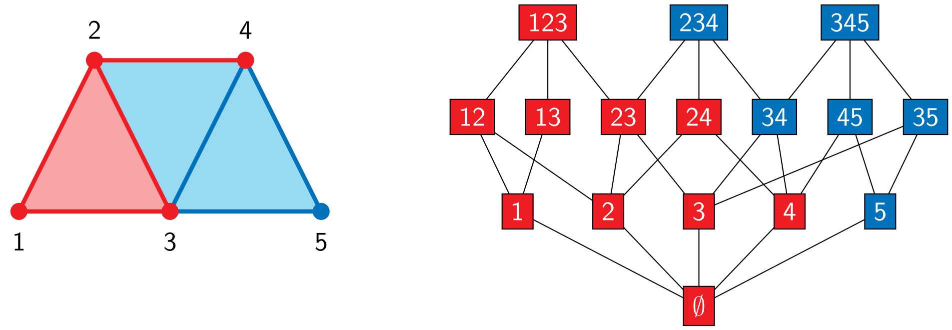

Figure 3. A simplicial complex as the union of a subcomplex and a relative complex.

Our first turning point arose from one of our experiments on Ziegler's 3-ball. We used a simple greedy algorithm to remove intervals one at a time, producing a sequence of smaller and smaller relative complexes, each of which remains Cohen–Macaulay (by a Mayer-Vietoris argument). In one trial, the final output was the relative complex Q_5 whose face poset is shown in Figure 4. It is not hard to check directly that Q_5 is non-partitionable. Therefore, *the Partitionability Conjecture is false for relative simplicial complexes.*

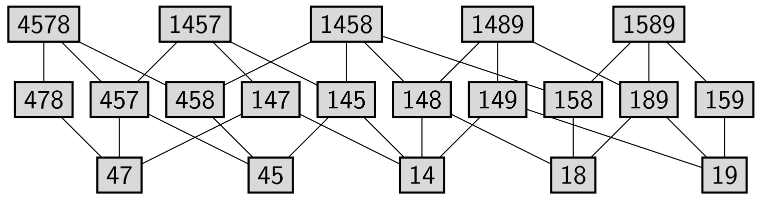

FIGURE 4. The non-partitionable Cohen–Macaulay relative simplicial complex Q_5 shows that the Partitionability Conjecture is false for relative simplicial complexes. The labeling of vertices follows Ziegler [Zie98].

We next wanted to find a general method for turning a relative counterexample $Q = (X, A)$, such as Q_5, into a non-relative counterexample. Our first idea was to construct a complex C_2 by gluing two isomorphic copies of X together along the common subcomplex A, as in Figure 5; this complex is Cohen–Macaulay (again, by a Mayer-Vietoris argument). The single extra copy of Q does not, however, create an obstruction to partitionability, since it may be possible to partition C_2 by pairing faces of A with facets in different copies of Q.

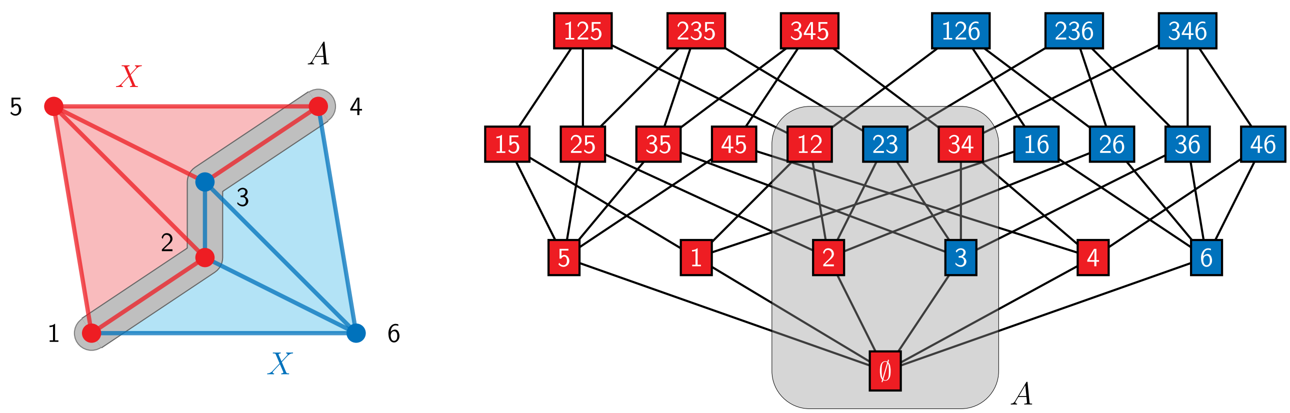

FIGURE 5. Gluing two isomorphic copies of X along A. The resulting complex is Cohen–Macaulay for appropriate X and A, but may still be partitionable, as in this example.

But now the pigeonhole principle comes into play. Construct a Cohen–Macaulay complex C_N by gluing N copies of X together along their common subcomplex A. Suppose that N is greater than the total number of faces of A. Then, in every partitioning of C_N, there must be at least one copy of Q whose facets are matched with faces in that same copy. That is, Q is partitionable! This contradiction implies that C_N cannot be partitionable. Here is a formal statement:

Theorem 2. *Let $Q = (X, A)$ be a relative complex such that*

(i) X and A are Cohen–Macaulay;
(ii) A has codimension at most 1;
(iii) every minimal face of Q is a vertex; and
(iv) Q is not partitionable.

Let k be the total number of faces of A, let $N > k$, and let $C = C_N$ be the simplicial complex constructed from N disjoint copies of X identified along the subcomplex A. Then C is Cohen–Macaulay and not partitionable.

Condition (iii) is a technical requirement to ensure that C is a simplicial complex, not merely a cell complex. For instance, the relative counterexample Q_5 does *not* satisfy Condition (iii).

It was not clear that Theorem 2 would produce an actual counterexample: finding a relative complex satisfying the conditions of the theorem might be just as difficult as proving or disproving Conjecture 1 through other means. But by considering the special case that A is a single facet, we realized that the conjecture implied a stronger version of itself: If X is a Cohen–Macaulay complex, then X has a partitioning including the interval $[\emptyset, \sigma]$ *for any facet* σ. This flexibility struck us as suspiciously strong: for example, the corresponding statement fails for shellability. Here was our second turning point.

Now we were determined to find a complex satisfying the conditions of Theorem 2. Once again, Ziegler's ball Z provided the answer. Consider the subcomplex B consisting of all faces supported on the vertex set $\{0, 2, 3, 4, 6, 7, 8\}$, so that the minimal faces of the relative complex $Q = Z/B$ are vertices 1, 5, and 9. We proved that B is Cohen–Macaulay, and that Q is not partitionable. These were exactly the ingredients we needed to construct a counterexample!

We can describe Q most simply as the relative complex (X, A), where X is the smallest simplicial complex containing Q. In fact X is a 3-ball with 10 vertices and 14 tetrahedra (see Figure 6), and A is a topological disk on the boundary of X, with f-vector $(1, 7, 11, 5)$. Therefore, gluing $(1 + 7 + 11 + 5) + 1 = 25$ copies of X along A produces a counterexample C_{25} to the Partitionability Conjecture. The complex C_{25} is Cohen–Macaulay (in fact, constructible) for the usual Mayer-Vietoris reasons, but it is non-partitionable by the pigeonhole argument.

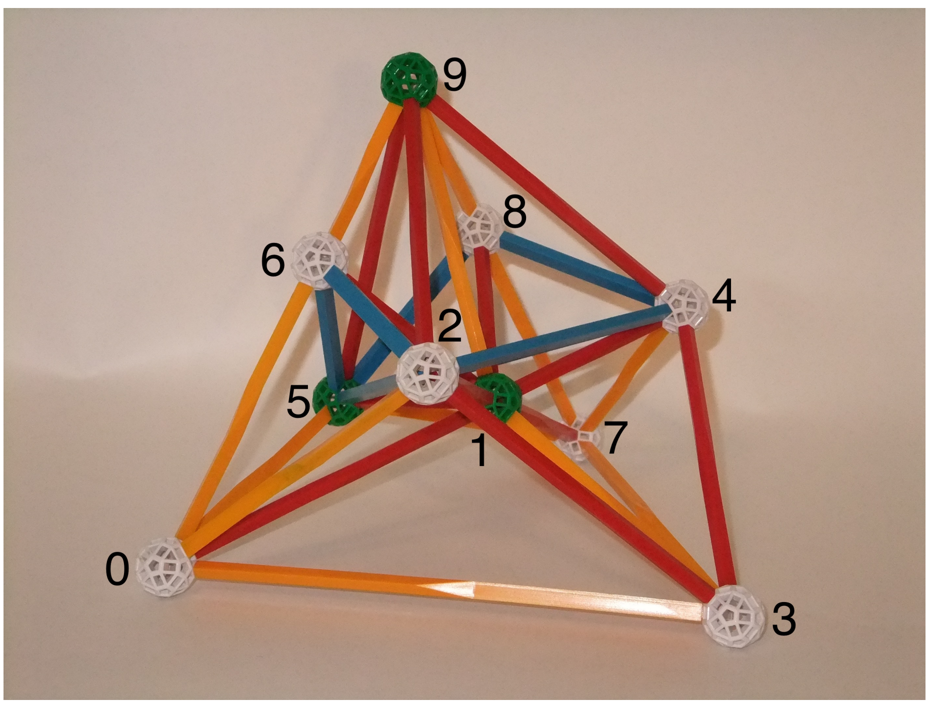

FIGURE 6. A model of X, the smallest simplicial complex containing Q. The subcomplex A consists of the five triangles without green vertices. (Photo: Jennifer Wagner)

In fact, the argument that Q is non-partitionable revealed that the full power of Theorem 2 was actually not necessary. Gluing just *three* copies of X together along A gives a non-partitionable complex C_3. The complex C_3 is the smallest counterexample we know: its f-vector is $f(A) + 3f(Q) = (1, 16, 71, 98, 42)$. It is contractible, but not homeomorphic to a ball. (See Figure 7.)

The counterexample disproves several stronger conjectures, as discussed above: that constructible complexes are partitionable; Kalai's conjecture about partitions indexed by algebraic shifting; and the Depth Conjecture. Yet the counterexample is not the end of the story, and many questions remain. For instance,

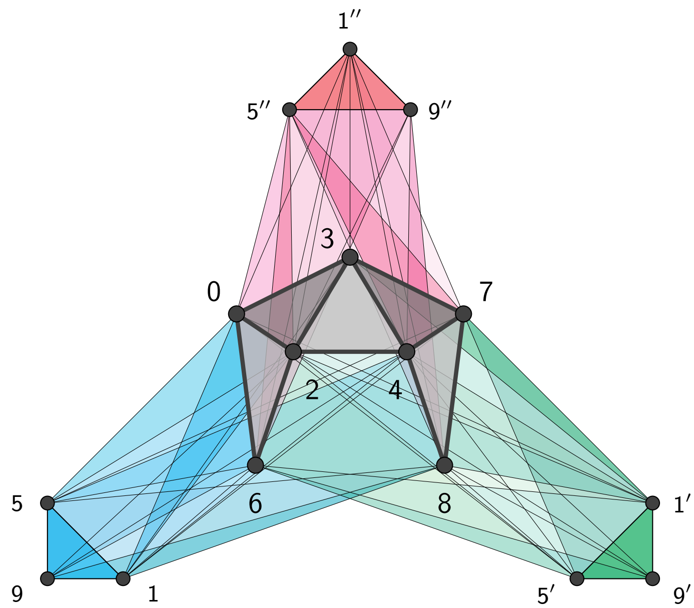

FIGURE 7. A partial representation of the counterexample C_3: three copies of the 3-ball X glued together along the 2-ball A. The figure shows all 16 vertices and 71 edges, but not all of the 98 triangles, nor any of the 42 tetrahedra.

does the Partitionability Conjecture hold in dimension two? Are simplicial balls partitionable? What about Garsia's version of the conjecture for Cohen–Macaulay posets? What are the consequences for the theory of Stanley depth? *What does the h-vector of a Cohen–Macaulay complex count?*

The story of the Partitionability Conjecture has many facets. Shellability, partitionability, constructibility, and the Cohen–Macaulay property come from different, but overlapping, areas of mathematics: combinatorics, commutative algebra, topology, and discrete geometry. The hierarchy of these structural properties turned out to be more complicated than we had anticipated, just as Rudin's and Ziegler's examples demonstrate that even the simplest spaces can have intricate combinatorics. Even though statements like the Partitionability Conjecture can seem too beautiful to be false, we should remember to keep our minds open about the mathematical unknown—the reality might be quite different, with its own unexpected beauty.

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