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given in terms of the field at the origin point P. Thus, we can calculate $\Gamma^i{}_{jk}(R)$[and by similar arguments $g_{ij}(R)$] provided a consistent set of Γ^i_{jk} exists at P. A consistent set of $\Gamma^i{}_{jk}$ at P must satisfy the requirement that the mixed derivatives[6] of all functions of $\Gamma^i{}_{jk}$ and g_{ij} be symmetric. These relations are just given by Eqs. (8) and (9). Thus, we conclude that local existence depends on being able to obtain solutions to (8) and (9) which, in fact, we have already found. A solution to (8) and (9) is given by (10), (11), and (12).

3. CONCLUSION

Thus, nontrivial solutions to (1) and (2) with $R^i{}_{jkl} \neq 0$ exist locally. Further investigations of the $\Gamma^i_{jk;l} = 0$, $g_{ij;k} = 0$ field theory appear elsewhere.[7]

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1 M. Muraskin, Ann. Phys. (N.Y.) **59**, 27 (1970). This reference gives background material for the present paper.

2 We shall make a few comments about the $\Gamma^i_{jk} = \Gamma^i{}_{kj}$ situation. For symmetric Γ^i_{jk}, we can make a general coordinate transformation so that $\Gamma^i_{jk} = 0$ at the origin. Then it might appear that $\Gamma^i_{jk} = 0$ at all points as a consequence of (1) and, thus, no nontrivial solutions would be possible for symmetric Γ^i_{jk}. However, this argument is not correct since Eq. (1) is not covariant under general coordinate transformations. Thus, the transformation that leads to $\Gamma^i_{jk} = 0$ at the origin also implies $\Gamma^i_{jk;l} \neq 0$.

3 See Sec. II for detailed discussion.

4 $R^1{}_{230} = - R^1{}_{203}$ is nonzero.

5 T. Apostol, *Mathematical Analysis* (Addison-Wesley, Reading, Mass., 1957), pp. 96, 123.

6 The problem of consistency when the field depends on a number of parameters as well as x, y, z, x^0 is discussed in L. Eisenhart, *Continuous Groups of Transformation* (Dover, New York, 1961), Chap. 1.

7 M. Muraskin and T. Clark, Ann. Phys. (N.Y.) **59**, 27 (1970). M. Muraskin, J. Math. Phys. **12**, 28 (1971); Intern. J. Theoret. Phys. **4**, 49 (1971).

A Class of Stationary Electromagnetic Vacuum Fields*

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It is shown how a new class of stationary electromagnetic vacuum fields can be generated from solutions of Laplace's equation. These fields are a stationary generalization of the static electromagnetic vacuum fields of Weyl, Majumdar, and Papapetrou, and are plausibly interpreted as exterior fields of static or steadily moving distributions of charged dust having numerically equal charge and mass densities.

1. INTRODUCTION

Coulomb's law and Newton's law of gravity are formally identical apart from a sign. Hence, classically, any unstressed distribution of matter can, if suitably charged, be maintained in neutral equilibrium under a balance between the gravitational attraction and electrical repulsion of its parts.

Indications that this obvious Newtonian fact has a relativistic analog first emerged when Weyl[1] obtained a particular class of static electromagnetic vacuum fields, later generalized by Majumdar[2] and Papapetrou[3] to remove Weyl's original restriction to axial symmetry, and further studied by Bonnor[4] and Synge.[5] The Papapetrou–Majumdar fields are to all appearances the external fields of static sources whose charge and mass are numerically equal (in relativistic units: $G = c = 1$). That they are indeed interpretable as external fields of static distributions of charged dust having equal charge and mass densities has been shown by Das,[6] who has examined the corresponding interior fields.

Astrophysical bodies are electrically neutral to a good approximation, and the Papapetrou–Majumdar solutions have up to now received little attention. It seems to us, however, that they can play a useful, if limited, astrophysical role in providing simple quasistatic analogues for complex dynamical processes like the disappearance of asymmetries in gravitational collapse or the collision of black holes. In reality, such a process always involves large kinetic energies and at present can only be handled by elaborate numerical integrations under the assumption of small departures from spherical symmetry.[7,8] However, for *charged* bodies in neutral equilibrium the process can be made arbitrarily slow, and the details easily followed as a sequence of stationary configurations. While this procedure prevents us from considering features of undeniable observational importance, such as the emission of gravitational waves, it is for that very reason ideally suited for isolating and elucidating certain basic issues of principle relating to the final phases of the process.

Some of these questions are pursued in detail elsewhere.[9] Our purpose here is to demonstrate that the Papapetrou–Majumdar class can be extended straightforwardly from the static to the stationary realm.

2. STATIONARY FIELDS

The metric of an arbitrary stationary field is conveniently expressed in the form[10]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f^{-1} \gamma_{mn} dx^m dx^n + f(\omega_m dx^m + dx^4)^2, \quad (1)$$

in which f, γ_{mn}, and ω_m are independent of the time coordinate x^4. The inverse of $g_{\mu\nu}$ is given by

$$g^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = - f \gamma^{mn} \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^n} + 2f \omega^m \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^4} + (f^{-1} - f\omega^2) \frac{\partial^2}{(\partial x^4)^2}, \quad (2)$$

where γ^{mn} is the 3×3 symmetric matrix inverse to γ_{mn}, $\omega^m = \gamma^{mn}\omega_n$ and $\omega^2 = \gamma^{mn}\omega_m\omega_n$. The determinants of $g_{\mu\nu}$ and γ_{mn} are related by

$$(-g)^{1/2} = f^{-1}\gamma^{1/2}. \tag{3}$$

The 3-vector ω_m in (1) is arbitrary up to an additive gradient $\partial_m \lambda(x^1, x^2, x^3)$, corresponding to the possibility of making arbitrary time translations $x^4 \to x^{4'} = x^4 - \lambda(x^1, x^2, x^3)$. However, we can derive from it an invariant "torsion vector"

$$f^{-2}\tau^m = -\gamma^{-1/2}\epsilon^{mpq}\partial_p\omega_q \quad \text{or} \quad f^{-2}\tau = -\operatorname{curl}\omega \tag{4}$$

in terms of a three-dimensional vector calculus employing $\gamma_{mn}dx^m dx^n$ as base metric.

We next consider a stationary electromagnetic field $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ in the space-time (1). The condition of time independence $\partial_4 A_\mu = 0$ yields for the "electric" components

$$F_{4n} = \partial_n A_4, \tag{5}$$

while the source-free Maxwell equations

$$\partial_n[(-g)^{1/2}\,{}^{(4)}F^{\mu n}] = 0 \tag{6}$$

for $\mu = m$ give the "magnetic" components

$${}^{(4)}F^{mn} = f\gamma^{-1/2}\epsilon^{mnp}\partial_p\Phi \tag{7}$$

in terms of a magnetic scalar potential Φ. All remaining components are then conveniently expressed in terms of these six; for example,

$${}^{(4)}F^{n4} = \omega_m\,{}^{(4)}F^{mn} + F_{4m}\gamma^{mn}, \tag{8}$$

an identity which follows readily from (1) or (2). Equation (6) with $\mu = 4$ now yields, on substituting (8), (7), (5) and (4),

$$\operatorname{div}(f^{-1}\nabla A_4) = -f^{-2}\,\tau\cdot\nabla\Phi. \tag{9}$$

Next, writing $F_{mn} (= \partial_n A_m - \partial_m A_n)$ in terms of (5) and (7) and expressing the cyclic identity $\epsilon^{mnp}\partial_p F_{mn} = 0$, we obtain

$$\operatorname{div}(f^{-1}\nabla\Phi) = f^{-2}\,\tau\cdot\nabla A_4. \tag{10}$$

If we now introduce[11] the complex scalar potential

$$\Psi = A_4 + i\Phi, \tag{11}$$

then (9) and (10) combine to give

$$\operatorname{div}(f^{-1}\nabla\Psi) = if^{-2}\,\tau\cdot\nabla\Psi. \tag{12}$$

We have thus reduced the entire set of Maxwell's equations to the single complex equation (12).

3. GRAVITATIONAL FIELD EQUATIONS

The Ricci tensor

$$R_{\mu\nu} = \partial_\mu\Gamma^\alpha_{\nu\alpha} - \partial_\alpha\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu} - \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu}$$

for the general stationary metric (1) is conveniently expressed in terms of a complex 3-vector $\mathbf{G}$, defined by

$$2f\mathbf{G} = \nabla f + i\tau. \tag{13}$$

Then[12]

$$-f^{-2}R_{44} = \operatorname{div}\mathbf{G} + (\mathbf{G}^* - \mathbf{G})\cdot\mathbf{G}, \tag{14a}$$

$$-2if^{-2}\,{}^{(4)}R^m_4 = \gamma^{-1/2}\epsilon^{mpq}(\partial_q G_p + G_p G_q^*), \tag{15a}$$

$$f^{-2}(\gamma_{pm}\gamma_{qn}\,{}^{(4)}R^{mn} - \gamma_{pq}R_{44}) = R_{pq}(\gamma) + G_p G_q^* + G_p^* G_q. \tag{16a}$$

Here, $R_{pq}(\gamma)$ denotes the Ricci tensor formed from the 3-metric $\gamma_{mn}dx^m dx^n$.

For the electromagnetic energy tensor

$$-4\pi\,T_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},$$

one derives from the formulas of the previous section

$$\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} = (\nabla\Phi)^2 - (\nabla A_4)^2,$$

$$8\pi f^{-1}T_{44} = (\nabla\Phi)^2 + (\nabla A_4)^2, \tag{14b}$$

$$4\pi f^{-1}\,{}^{(4)}T^m_4 = \gamma^{-1/2}\epsilon^{mpq}(\partial_p\Phi)(\partial_q A_4), \tag{15b}$$

$$-4\pi f^{-1}\,{}^{(4)}T^{mn} = (\partial^m\Phi)(\partial^n\Phi) + (\partial^m A_4)(\partial^n A_4) - \tfrac{1}{2}\gamma^{mn}[(\nabla\Phi)^2 + (\nabla A_4)^2] \tag{16b}$$

with $\partial^m = \gamma^{mn}\partial_n$.

We can now impose the Einstein field equations $R_{\mu\nu} = -8\pi\,T_{\mu\nu}$. From (15a), (15b), we find

$$\operatorname{curl}\tau = -4\nabla\Phi\times\nabla A_4 = i\operatorname{curl}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi),$$

so that the equation

$$\tau + i(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) = \nabla\psi \tag{17}$$

defines a real scalar ψ up to an additive constant.

We next define a complex function[11]

$$\mathcal{E} = f - \Psi\Psi^* + i\psi. \tag{18}$$

By virtue of (13) and (17),

$$f\mathbf{G} = \tfrac{1}{2}\nabla\mathcal{E} + \Psi^*\nabla\Psi. \tag{19}$$

Substituting (19) into the field equations (14a), (14b) and employing (12) leads to[11]

$$f\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot(\nabla\mathcal{E} + 2\Psi^*\nabla\Psi), \tag{20}$$

while (12) itself can be written

$$f\nabla^2\Psi = \nabla\Psi\cdot(\nabla\mathcal{E} + 2\Psi^*\nabla\Psi), \tag{21}$$

and we note from (18) that

$$f = \tfrac{1}{2}(\mathcal{E} + \mathcal{E}^*) + \Psi\Psi^*. \tag{22}$$

Finally the field equations (16a), (16b) reduce to

$$-f^2R_{mn}(\gamma) = \tfrac{1}{2}\mathcal{E}_{,(m}\mathcal{E}^*_{,n)} + \Psi\mathcal{E}_{,(m}\Psi^*_{,n)} + \Psi^*\mathcal{E}^*_{,(m}\Psi_{,n)} - (\mathcal{E} + \mathcal{E}^*)\Psi_{,(m}\Psi^*_{,n)}, \tag{23}$$

in which, for example,

$$2\mathcal{E}_{,(m}\mathcal{E}^*{}_{,n)} \equiv (\partial_m \mathcal{E})(\partial_n \mathcal{E}^*) + (\partial_n \mathcal{E})(\partial_m \mathcal{E}^*).$$

The complete system of electromagnetic and gravitational field equations for an arbitrary electromagnetic vacuum field are summed up in (20), (21), and (23).

4. GENERALIZED PAPAPETROU-MAJUMDAR SOLUTIONS

So far, our considerations have been quite general. We now examine whether solutions of the system (20), (21), and (23) exist for which the background metric $\gamma_{mn}\,dx^m dx^n$ is *flat*. In this case equations (23) [with $R_{mn}(\gamma)=0$] are satisfied if and only if there is a linear relation

$$\Psi = a + b\mathcal{E}, \quad \text{with } a^*b + ab^* = -\tfrac{1}{2}$$

(as one easily verifies, for example, by choosing $\mathcal{E} = x^1$ and $\mathcal{E}^* = x^2$ as coordinates). Both $\mathcal{E}$ and Ψ contain arbitrary additive constants, and it is convenient to adjust these so that $\mathcal{E} \to 1$ when $\Psi \to 0$. We thus obtain

$$\Psi = \tfrac{1}{2} e^{i\alpha}(1 - \mathcal{E}), \tag{24}$$

in which the arbitrary real constant α represents the "complexion" of the electromagnetic field. We can submit this field to any constant duality rotation without affecting the geometry.

If we now substitute (24) into (20) and (21), *both* reduce to

$$\nabla^2[(1+\mathcal{E})^{-1}] = 0 \tag{25}$$

which is Laplace's equation in Euclidean 3-space.

We conclude by summarizing the procedure for obtaining the complete field. (a) Write down a solution of (25) in terms of any convenient coordinates x^m. Suppose the Euclidean line element takes the form $\gamma_{mn}dx^m dx^n$ in these coordinates. (b) Obtain f, τ, and ω from the equations

$$f = \tfrac{1}{4}(1+\mathcal{E})(1+\mathcal{E}^*),$$
$$if^{-1}\,\tau = \nabla\{\ln[(1+\mathcal{E})/(1+\mathcal{E}^*)]\}, \quad \operatorname{curl}\boldsymbol{\omega} = -f^{-2}\tau. \tag{26}$$

The space-time metric is given by (1). (c) Obtain $\Psi = A_4 + i\Phi$ from (24). The electromagnetic field can be found from (5) and (7).

5. EXAMPLE: CHARGED KERR-LIKE SOLUTIONS

The Kerr-Newman solution with $m^2 = e^2$ corresponds to the simplest complex solution of (25). We choose

$$2/(1+\mathcal{E}) = 1 + m/R, \quad \text{with } R^2 = x^2 + y^2 + (z - ia)^2, \tag{27}$$

where a and m are real constants and x, y, z Cartesian coordinates. In terms of oblate spheroidal coordinates r, θ, ϕ defined by

$$x + iy = [(r-m)^2 + a^2]^{1/2} \sin\theta\, e^{i\phi}, z = (r-m)\cos\theta,$$

the Euclidean 3-metric becomes

$$\gamma_{mn}dx^m dx^n = [(r-m)^2 + a^2\cos^2\theta]\{dr^2/[(r-m)^2 + a^2] + d\theta^2\} + [(r-m)^2 + a^2]\sin^2\theta\, d\phi^2.$$

Further, we find

$$R = r - m - i\,a\cos\theta,$$
$$f = [(r-m)^2 + a^2\cos^2\theta]/(r^2 + a^2\cos^2\theta),$$
$$\Psi = e^{i\alpha}\, m/(r - i\,a\cos\theta),$$

and, after a somewhat lengthy calculation,

$$\omega_m dx^m = \{[(2mr - m^2)\,a\sin^2\theta]/[(r-m)^2 + a^2\cos^2\theta]\}\,d\phi.$$

Putting everything together, we recover the charged Kerr metric with $m^2 = e^2$ in its usual form.[13]

As a natural generalization of (27), one may consider

$$\frac{2}{1+\mathcal{E}} = 1 + \sum_{k=1}^{n} \frac{m_k}{R_k},$$

where $R_k^2 = (\mathbf{r} - \mathbf{c}_k)^2$, r is the Euclidean position vector, and $\mathbf{c}_k$ an arbitrary set of constant, complex vectors. The resulting metric will represent the field of a set of arbitrarily spinning, charged Kerr-like particles in neutral equilibrium. For the static analog of this solution, representing a set of Reissner-Nordström particles with $e_k = m_k$; see Ref. 5.

Note added in proof: The stationary extension of the Papapetrou-Majumdar solutions has since been obtained independently by Z. Perjés, Phys. Rev. Letters **27**, 1668 (1971).

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1 H. Weyl, Ann. Physik **54**, 117 (1917).
2 S. D. Majumdar, Phys. Rev. **72**, 390 (1947).
3 A. Papapetrou, Proc. Roy. Irish Acad. **A51**, 191 (1947).
4 W. B. Bonnor, Proc. Phys. Soc. (London) **A66**, 1451 (1953); **A67**, 225 (1954).
5 J. L. Synge, *Relativity, The General Theory* (North-Holland, Amsterdam, 1960), p. 367.
6 A. Das, Proc. Roy. Soc. (London) **A267**, 1 (1962).
7 V. de la Cruz, J. E. Chase, and W. Israel, Phys. Rev. Letters **24**, 423 (1970).
8 R. Price, Phys. Rev. (to be published)
9 W. Israel and G. A. Wilson (to be published). Also J. B. Hartle and S. W. Hawking (to be published), R. Ruffini (to be published).

10 Greek indices run from 1 to 4, Latin indices from 1 to 3. Lowering and raising of Latin indices is always carried out with γ_{mn} and its inverse γ^{mn} unless specifically noted by a left superscript 4. Thus, if $F_{\mu\nu}$ is a given covariant tensor, we write $F^{ab} = \gamma^{am}\gamma^{bn}F_{mn}$ and ${}^{(4)}F^{ab} = g^{a\mu}g^{b\nu}F_{\mu\nu}$.

11 Cf., for the special case of axial symmetry, F. J. Ernst, Phys. Rev. **168**, 1415 (1968), where the idea of a complex potential is first introduced. We have been informed that B. K. Harrison (1968, unpublished) has cast the stationary electromagnetic vacuum equations into a form similar to that given in Secs. 2 and 3. See also B. K. Harrison, J. Math. Phys. **9**, 1744 (1968). A recent publication by Ernst, J. Math. Phys. **12**, 2395 (1971) treats the general stationary vacuum case.
12 Z. Perjes, J. Math. Phys. **11**, 3383 (1970).
13 See, e.g., B. Carter, Phys. Rev. **174**, 1559 (1968).