QUOTIENTS OF METRIC SPACES

by

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INTRODUCTION

The question of when a given property of a topological space is preserved under mappings is one of the most familiar problems of general topology. Among the properties of greatest interest for general spaces is, without doubt, that of metrizability; metrizability always implies, in particular, a number of important special topological properties of the space in question (normality, regularity, paracompactness, etc.). To determine in general the conditions for preservation of metrizability under mappings appears to be a difficult problem. It may well be that in the class of arbitrary continuous mappings the problem has no meaningful solution. The purpose of this paper is to obtain conditions for the preservation of metrizability by quotient mappings and to study the properties of quotient spaces of metric spaces.

We will use "iff" as an abbreviation for "if and only if". If f is a function from X onto Y, we will write $f: X \twoheadrightarrow Y$.

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CHAPTER I. PRELIMINARIES

We begin by stating some basic definitions and theorems.

Definition 1.1: Let f be a function from a topological space X onto a set Y. Then the quotient topology for Y (relative to f and the topology of X) is the family $\mathcal{U} = \{U \subset Y: f^{-1}(U) \text{ is open in } X\}$. If Y has the quotient topology, then Y is called a quotient space and f a quotient map.

Since the inverse of an intersection (or union) of members of $\mathcal{U}$ is the intersection (union) of the inverses, $\mathcal{U}$ is indeed a topology for Y. If a subset U of Y is open in a topology relative to which f is continuous, then $f^{-1}(U)$ is open in X. Thus the quotient topology is the largest topology for Y such that the function f is continuous.

A subset B of Y is closed relative to the quotient topology iff $f^{-1}(Y - B) = X - f^{-1}(B)$ is open in X. Hence B is closed iff $f^{-1}(B)$ is closed.

Theorem 1.2: If f is a continuous function from the topological space $(X, \mathcal{T})$ onto the space $(Y, \mathcal{U})$ such that f is either open or closed, then $\mathcal{U}$ is the quotient topology.

Proof: Let f be an open map and let U be a subset of Y such that $f^{-1}(U)$ is open relative to $\mathcal{T}$. Then $U = f(f^{-1}(U))$ is open relative to $\mathcal{U}$. Consequently, if f is open, each set open relative to the quotient topology is open relative to $\mathcal{U}$, and the quotient topology is smaller than $\mathcal{U}$. If f is continuous as well as open, then since the quotient topology is the largest for which f is continuous, $\mathcal{U}$ is the quotient topology. To prove the theorem for a closed function f it is only necessary to replace "open" by "closed" in each of the preceding statements. Q.E.D.

If f is a continuous map of a topological space X onto a space Y, the continuity of any $g: Y \twoheadrightarrow Z$ implies that of $g\cdot f$. The characterizing property of quotient maps is that the converse is also true.

<u>Theorem 1.3</u>: Let f be a continuous map of a topological space X onto a space Y. Then f is a quotient map if and only if: for each topological space Z and each map $g: Y \twoheadrightarrow Z$, the continuity of $g\cdot f$ implies that of g.

<u>Proof</u>: Assume that f is a quotient map and that $g\cdot f$ is continuous. Let U be an open subset of Z. Then $(g\cdot f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X, so that $g^{-1}(U)$ is open in Y. Therefore g is continuous.

On the other hand, assume that the condition holds. Let Y' be the set Y with the quotient topology relative to f, and let $f': X \twoheadrightarrow Y'$ take the same values as f. Let $i: Y \twoheadrightarrow Y'$ be the identity map. Since $i\cdot f = f'$ is continuous, the condition assures that i is continuous. Since $i^{-1}\cdot f' = f$ is continuous, and f' is a quotient map, the first part of the proof shows that i^{-1} is continuous. Thus $i: Y \twoheadrightarrow Y'$ is a homeomorphism, and f is a quotient map. Q.E.D.

We now consider another way of looking at quotient spaces. Let f be a function from a topological space $(X, \mathcal{T})$ onto a space $(Y, \mathcal{Q})$, where $\mathcal{Q}$ is the quotient topology for Y. Define a relation R on X by xRy iff $f(x) = f(y)$ for x,y in X. Clearly, R is an equivalence relation; the equivalence classes of R are the sets $f^{-1}(y)$ with $y \in Y$. Let X/R be the family of equivalence classes, and let $R(x)$ be the equivalence class to which x belongs. Let p be the mapping of X onto X/R defined by $p(x) = R(x) = f^{-1}(f(x))$ for all $x \in X$, and give X/R the quotient topology relative to p. We will show that Y is homeomorphic to X/R.

Define a function g from X/R to Y by $g(R(x)) = f(x)$ for all $R(x) \in X/R$. Since $R(u) = R(x)$ implies $f(u) = f(x)$, g is well-defined. We have the following diagram, where $\mathcal{Q}'$ is the quotient topology for X/R:

$$\begin{tikzcd}
(X, \mathcal{T}) \arrow[r, "f"] \arrow[d, "p"'] & (Y, \mathcal{Q}) \\
(X/R, \mathcal{Q}') \arrow[ur, "g"'] &
\end{tikzcd}$$

Since f is onto Y, g also maps onto Y. If $g(R(x)) = g(R(u))$, then $f(x) = f(u)$ so that xRu and $R(x) = R(u)$; thus, g is one-to-one. Consequently, g^{-1} is a well-defined function. Since $g^{-1} \cdot f = p$ and p is continuous, Theorem 1.3 shows that g^{-1} is continuous. Finally, since $g \cdot p = f$ and f is continuous, Theorem 1.3 shows that g is continuous. Thus Y is homeomorphic to X/R.

These considerations lead naturally to the following example of a non-metrizable quotient space of a metric space.

Example 1.4: Let R be the real line with the usual topology and let Y be the space obtained from R be identifying the integers Z with 0. Then Y is not first countable (and so certainly not metrizable).

Proof: Let $f: R \to Y$ be the natural quotient map, and suppose $\{U_n: n = 1,2,\ldots\}$ is a countable local base of open neighborhoods of 0 in Y. Then $f^{-1}(U_n)$ is open in R and $Z \subset f^{-1}(U_n)$. Hence for each integer n there exists an $\epsilon_n > 0$ such that $N_{\epsilon_n}(n) \subset f^{-1}(U_n)$. Now $B = \bigcup_{n=1}^{\infty} N_{\epsilon_n/2}(n)$ is open in R and contains Z; it follows that $U = f(B)$ is open in Y and contains 0. But $U_n \not\subset U$ for each n, and consequently 0 cannot have a countable local base. Q.E.D.

We conclude the first chapter by proving two lemmas for later reference.

Lemma 1.5: In a first countable space X, $x \in \overline{M}$ iff there exists a sequence $\{x_n\}$ of points x_n in M such that $\{x_n\}$ converges to x.

Proof: Choose for x a decreasing local base of open neighborhoods $\{U_n: n = 1,2,\ldots\}$, $U_1 \supset U_2 \supset \ldots$. If $x \in \overline{M}$, then $U_n \cap M \neq \emptyset$ for all n. If $x_n \in U_n \cap M$, then $\{x_n\}$ converges to x. On the other hand, if $\{x_n\}$ converges to x with $x_n \in M$, then clearly $U_n \cap M \neq \emptyset$ for all n, and $x \in \overline{M}$. Q.E.D.

Lemma 1.6: Let E be a topological space and suppose $E = \bigcup_{\lambda} E_{\lambda}$, where each E_{λ} is an open metrizable subspace of E and the E_{λ}'s are pairwise disjoint. Then E is metrizable.

Proof: Let d_{λ} be a metric compatible with the subspace topology on E_{λ} and such that the diameter of E_{λ} is less than 1. Define a metric d for E by

$$d(x,y) = \begin{cases} d_{\lambda}(x,y), & \text{if } x \text{ and } y \text{ are in the same } E_{\lambda} \\ 1, & \text{otherwise.} \end{cases}$$

Since the E_{λ}'s are pairwise disjoint, d is well-defined. By considering cases, one easily shows that d is indeed a metric.

To show that every set open in the topology of E is open in the topology induced by d, let $x \in E_{\lambda} \subset E$ and let U be an open set containing x. Then $U \cap E_{\lambda}$ is open in E_{λ} and, consequently, there exists $0 < \epsilon < 1$ such that
$U \cap E_{\lambda} \supset \{y: d_{\lambda}(y,x) < \epsilon\} = \{y: d(y,x) < \epsilon\}$. Thus
$U \supset \{y: d(y,x) < \epsilon\}$.

To show that every set open in the topology induced by d is open in the topology of E, let $x \in E_{\lambda}$ and let $N_{\epsilon}(x) = \{y: d(y,x) < \epsilon\}$ be a metric neighborhood of x. We can assume $\epsilon < 1$. Thus
$N_{\epsilon}(x) = \{y: d_{\lambda}(y,x) < \epsilon\}$ contains a set U open in E_{λ}. Since E_{λ} is open in E, U is also open in E.

Thus the topology induced by d is compatible with the topology of E. Q.E.D.

CHAPTER II. SPACES IN WHICH SEQUENCES SUFFICE

In this chapter we show that quotients of metric spaces are among those spaces in which sequences are adequate for the description of many topological concepts.

Definition 2.1: A subset U of a topological space X is sequentially open iff each sequence in X converging to a point in U is eventually in U. A subset F of X is sequentially closed iff no sequence in F converges to a point not in F.

Clearly, any set open (closed) in X is sequentially open (sequentially closed) in X.

Definition 2.2: A topological space X is a sequential space iff every sequentially open set is open.

Thus, in sequential spaces the open sets are precisely the sequentially open sets.

Proposition 2.3: Every first countable space, and hence every metric space, is a sequential space.

Proof: Let U be sequentially open in the first countable space X, and suppose U is not open. Then there exists $x \in U - \text{Int}\, U$, where $\text{Int}\, U$ denotes the interior of U. Let $\{U_n : n = 1,2,\ldots\}$ be a local base of open neighborhoods of x such that $U_1 \supset U_2 \supset \ldots$. If $U_n \subset U$, then $x \in \text{Int}\, U$; consequently, for each integer n there exists $x_n \in U_n \cap (X-U)$. Clearly, $x_n \to x \in U$ but $\{x_n\}$ is not eventually in U, contradicting the assumption that U is sequentially open. Q.E.D.

<u>Proposition 2.4</u>: For any topological space X, (1) and (2) below are equivalent. If X is Hausdorff, then they are also equivalent to (3) and (4).

(1) X is a sequential space,

(2) Every sequentially closed set in X is closed.

(3) Every subset of X which intersects each convergent sequence in a closed set is closed. (We here use the term "convergent sequence" to mean the sequence plus all of its limit points.)

(4) Every subset of X which intersects each compact metric subspace of X in a closed set is closed.

<u>Proof</u>: (1) $\rightarrow$ (2). Let X be a sequential space and let F be sequentially closed in X. If $X - F$ is not sequentially open, there exists a sequence $\{x_n\}$ which converges to a point $x \in X - F$ but which is not eventually in $X - F$; hence for every integer N there exists $n_k \geq N$ such that $x_{n_k} \notin X - F$. Then $\{x_{n_k}\} \subseteq F$ but $x_{n_k} \rightarrow x \in X - F$, contradicting that F is sequentially closed. Consequently, $X - F$ is sequentially open and thus open. Therefore F is closed. The proof that (2) implies (1) is similar.

(2) $\rightarrow$ (3). Let X be T_2 and let F be a subset of X which intersects each convergent sequence in a closed set. If F is not sequentially closed, there exists a sequence $\{x_n\} \subseteq F$ which converges to $x \notin F$. Since X is T_2, x is the only limit point of $\{x_n\}$, so that $\{x_n\} \cup \{x\}$ is a convergent sequence. Then $x \in \overline{F \cap (\{x_n\} \cup \{x\})} - F \cap (\{x_n\} \cup \{x\})$, that is, $F \cap (\{x_n\} \cup \{x\})$

is not closed, a contradiction. So F is sequentially closed and thus closed.

(3) $\to$ (2). Let X be T_2 and let F be sequentially closed in X. Then F intersects each convergent sequence in a closed set. For suppose $x_n \to x$ but $F \cap (\{x_n\} \cup \{x\})$ is not closed. Since X is T_1, $F \cap (\{x_n\} \cup \{x\})$ is infinite, say $\{x_{n_k}\} \subseteq F$. Now $x_{n_k} \to x$ so that x is the only limit point of $F \cap (\{x_n\} \cup \{x\})$. Since $F \cap (\{x_n\} \cup \{x\})$ is not closed, $x \notin F$; thus F is not sequentially closed, a contradiction.

(3) $\to$ (4). For Hausdorff X, each convergent sequence is compact metric. For suppose $x_n \to x$, and let $A = \{x_n\} \cup \{x\}$. Since any set of an open covering of A that contains x contains all but at most finitely many elements of A, A is compact. We can define a metric on A as follows: $d(x_n, x_m) = |\frac{1}{n} - \frac{1}{m}|$ and $d(x, x_n) = \frac{1}{n}$.

(4) $\to$ (3). Let A be a subset of X which intersects each convergent sequence in a closed set. Then A intersects every compact metric subspace of X in a closed set. For suppose $M \subseteq X$ is compact metric and $A \cap M$ is not closed, say $x \in \overline{A \cap M} - A \cap M$. Since X is T_2, M is closed in X and consequently $\overline{A \cap M} \subseteq \overline{M} = M$; then, since M is first countable, there exists a sequence $\{x_n : n = 1, 2, \ldots\} \subseteq A \cap M$ such that $x_n \to x$. But $A \cap (\{x_n\} \cup \{x\}) = \{x_n\}$ is not closed, contradicting our assumption about A. Q.E.D.

<u>Lemma 2.5</u>: Let f be a quotient map of a sequential space X onto a topological space Y. Then Y is a sequential space.

Proof: Let $U \subseteq Y$ be sequentially open. To show that U is open in Y, we must show $f^{-1}(U)$ is open in X; since X is sequential, it suffices to show $f^{-1}(U)$ is sequentially open. So let $\{x_n: n = 1,2,\ldots\} \subseteq X$ converge to $x \in f^{-1}(U)$. Then, by the continuity of f, $f(x_n) \to f(x) \in U$; since U is sequentially open, $\{f(x_n)\}$ is eventually in U. But then $\{x_n\}$ is eventually in $f^{-1}(U)$. Q.E.D.

Definition 2.6: Let $\{X_\mu: \mu \in M\}$ be a family of topological spaces such that $X_\mu \cap X_{\mu'} = \emptyset$ if $\mu \neq \mu'$. Then the topological sum of the X_μ is the set $\Sigma = \cup\{X_\mu: \mu \in M\}$ with the following topology: $U \subseteq \Sigma$ is open in Σ iff $U \cap X_\mu$ is open in X_μ for every $\mu \in M$.

If the X_μ are not pairwise disjoint, we can construct homeomorphic spaces which are pairwise disjoint.

Theorem 2.7: Every sequential space X is a quotient of a topological sum of convergent sequences.

Proof: For each $x \in X$ and each sequence $s = \{s_n: n = 1,2,\ldots\}$ in X converging to x, let $S(s,x) = \{s_n: n = 1,2,\ldots\} \cup \{x\}$ be a topological space in which each s_n is a discrete point and $s_n \to x$ in $S(s,x)$. Let T be the disjoint topological sum of all possible $S(s,x)$, and define $f: T \to X$ by $f(t) = t$.

To show that f is continuous on T, we show that f is continuous on each $S(s,x)$. If $m \in S(s,x)$, then either $m = x$ or $m = s_n$ for some n. Suppose $m = x$ and let V be a neighborhood of $f(x) = x \in X$; since $s_n \to x$ in X there exists an integer N

such that $n \geq N$ implies $s_n \in V$; thus $U = \{x\} \cup \{s_i : i \geq N\}$ is a neighborhood of x such that $f(U) \subset V$, and hence f is continuous at x. Suppose $m = s_n$ for some n and let V be a neighborhood of $f(s_n)$; then $U = \{s_n\}$ is a neighborhood of s_n such that $f(U) \subset V$, and hence f is continuous at s_n.

To complete the proof that f is a quotient map, let U be a subset of X such that $f^{-1}(U)$ is open in T. To show that U is open in X, it suffices to show that U is sequentially open in X. If $x_0 \in U$ and $s = \{s_n : n = 1,2,\ldots\}$ converges to x_0, then $x_0 \in f^{-1}(U) \cap S(s,x_0)$ which is open in $S(s,x_0)$. Thus $\{s_n\}$, considered as a subset of $S(s,x_0)$, is eventually in $f^{-1}(U)$; consequently, $\{s_n\}$ is eventually in U. Q.E.D.

The following corollary shows that quotient spaces of metric spaces are precisely the sequential spaces.

Corollary 2.8: The following statements are equivalent.

(1) X is a sequential space.

(2) X is the quotient of a metric space.

(3) X is the quotient of a first countable space.

Proof: (1) $\rightarrow$ (2). Let X be a sequential space. Then by the theorem X is the quotient of a topological sum T of convergent sequences. Defining a metric on each $S(s,x)$ by $d(s_n, s_m) = |\frac{1}{n} - \frac{1}{m}|$ and $d(x, s_n) = \frac{1}{n}$, we obtain a topology compatible with the topology of $S(s,x)$. Then T is a metric space by Lemma 1.6.

(2) $\rightarrow$ (3). Every metric space is first countable.

(3) $\rightarrow$ (1). Suppose X is the quotient of the first countable space Y. By Proposition 2.3 Y is a sequential space. By Lemma 2.5 X is a sequential space. Q.E.D.

Corollary 2.9: Among T_2-spaces quotient spaces of metric spaces are precisely those spaces satisfying the following condition: a subset A of X is closed iff A intersects every compact metric subspace in a closed set.

Proof: The proof follows directly from Corollary 2.8 and Proposition 2.4. Q.E.D.

Definition 2.10: A topological space X is called a Fréchet space iff for every subset A of X $x \in \bar{A}$ iff there exists a sequence $\{x_n : n = 1,2,\dots\} \subseteq A$ such that $x_n \rightarrow x$.

By Lemma 1.5 every first countable space is a Fréchet space. Every Fréchet space is a sequential space, since every sequentially closed set in a Fréchet space is closed.

Definition 2.11: A continuous function $f: X \twoheadrightarrow Y$ is pseudo-open iff for any $y \in Y$ and for any open neighborhood U of $f^{-1}(y)$, $y \in \operatorname{Int} f(U)$.

Proposition 2.12: Every open (closed) continuous function is pseudo-open. Further, every pseudo-open map is a quotient map.

Proof: It is clear that every open continuous function is pseudo-open. Let f be a closed continuous mapping of X onto Y, let $y \in Y$, and let $f^{-1}(y) \subseteq U$ open in X. Then there exists an open set $V \supset \{y\}$ such that $f^{-1}(y) \subset f^{-1}(V) \subset U$. (See [4; page 86].) Thus $y \in V \subset f(U)$; since V is open, it follows that $y \in \text{Int } f(U)$ and that f is pseudo-open.

Let $f: X \twoheadrightarrow Y$ be pseudo-open. Let $U \subseteq Y$ be such that $f^{-1}(U)$ is open in X. Then $f^{-1}(y) \subseteq f^{-1}(U)$ for every $y \in U$; consequently, $y \in \text{Int } f(f^{-1}(U)) = \text{Int } U$ for every $y \in U$. Thus U is open and f is a quotient map. Q.E.D.

Since quotients of Fréchet spaces need not be Fréchet spaces, Lemma 2.5 does not have an analogue for Fréchet spaces. We have, however, the following theorem.

<u>Theorem 2.13</u>: If X and Y are T_2-spaces, X is a Fréchet space and $f: X \twoheadrightarrow Y$ is a quotient map, then Y is a Fréchet space iff f is pseudo-open.

<u>Proof</u>: Suppose that Y is a Fréchet space, $y \in Y$, and U is an open neighborhood of $f^{-1}(y)$. If $y \notin \text{Int } f(U)$, then $y \in \overline{Y - f(U)}$. Hence there is a sequence $\{y_n\} \subseteq Y - f(U)$ converging to y. Since Y is T_2, $\overline{\{y_n\}} = \{y_n\} \cup \{y\}$. If $F = f^{-1}(\{y_n\})$, then by the continuity of f $\overline{F} \subseteq f^{-1}(\overline{\{y_n\}}) = F \cup f^{-1}(y)$. But $f^{-1}(y) \subseteq U$ and $U \cap F = \emptyset$; hence $f^{-1}(y) \cap \overline{F} = \emptyset$ and $\overline{F} \subseteq F$. It follows that F is closed and thus that $X - F = f^{-1}(Y - \{y_n\})$ is open; therefore, since f is a quotient map, $Y - \{y_n\}$ is open, contradicting that $\{y_n\} \to y$. Hence $y \in \text{Int } f(U)$ and f is pseudo-open.

Assume f is pseudo-open, $M \subseteq Y$, and $y \in \overline{M}$. If $f^{-1}(y) \cap \overline{f^{-1}(M)} = \emptyset$, let $U = X - \overline{f^{-1}(M)}$; then

$y \in \mathrm{Int}\, f(U) \subseteq f(U) \subseteq Y - M$, contradicting $y \in \overline{M}$. Thus there is some $x_0 \in f^{-1}(y) \cap \overline{f^{-1}(M)}$. Since X is Fréchet, we can choose a sequence $\{x_n: n = 1,2,\ldots\} \subseteq f^{-1}(M)$ converging to x_0. Then $\{f(x_n): n = 1,2,\ldots\} \subseteq M$ and $f(x_n) \to y$ by the continuity of f. Thus Y is a Fréchet space. Q.E.D.

Proposition 2.14: The disjoint topological sum Σ of any family of Fréchet spaces $\{X_\mu: \mu \in M\}$ is a Fréchet space.

Proof: If $A \subseteq \Sigma$, then $\overline{A}^{\Sigma} = \bigcup_{\mu \in M} \overline{A \cap X_\mu}^{X_\mu}$ where $\overline{A}^{\Sigma}$ denotes the closure of the set A in the space Σ and $\overline{A \cap X_\mu}^{X_\mu}$ denotes the closure of the set $A \cap X_\mu$ in the subspace X_μ. The proposition now follows from the definition of a Fréchet space and the hypothesis that each X_μ is a Fréchet space. Q.E.D.

Theorem 2.15: Among T_2-spaces, Fréchet spaces are precisely the pseudo-open images of a topological sum of convergent sequences.

Proof: Let Y be a Fréchet space. By Theorem 2.7 Y is a quotient of a topological sum Σ of convergent sequences $S(s,x)$. Since each $S(s,x)$ is metrizable and is thus a Fréchet space, Proposition 2.14 shows that Σ is a Fréchet space. By Theorem 2.13 the quotient map is pseudo-open.

On the other hand, if Y is the pseudo-open image of a topological sum of convergent sequences, then Y is a Fréchet space by Theorem 2.13. Q.E.D.

CHAPTER III. QUOTIENTS OF SEPARABLE METRIC SPACES

In this chapter we characterize quotient spaces of separable metric spaces among the regular T_2-spaces. Except for Theorem 3.1 the results are due to E. A. Michael [10]. We need several preliminary results.

The weight of a topological space X is the least cardinal of a basis for the topology of X.

<u>Theorem 3.1</u>: (Ponomarev). Every first countable T_1-space is the open continuous image of a metric space of the same weight.

<u>Proof</u>: Let $\theta = \{\alpha\}$ have cardinality τ, and let $B_\tau = \{\psi: \psi = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots), \alpha_i \in \theta\}$. Define a metric on B_τ as follows: $d(\psi, \psi') = \frac{1}{k}$ where k is the least integer for which $\alpha_k \neq \alpha'_k$. An open neighborhood is $N(\psi, \frac{1}{k}) = N_{\alpha_1, \ldots, \alpha_k} = \{(\alpha'_1, \alpha'_2, \ldots): \alpha'_1 = \alpha_1, \ldots, \alpha'_k = \alpha_k\}$.

Let X be any first countable T_1-space of weight τ. Let $\mathcal{B} = \{U_\alpha: \alpha \in \theta\}$ be a basis of cardinality τ for X. We call $\psi = (\alpha_1, \alpha_2, \ldots) \in B_\tau$ "distinguished" if $U_{\alpha_1}, U_{\alpha_2}, \ldots$ form a base of some point $x \in X$. Let $W \subset B_\tau$ be the set of all distinguished points. Since X is a T_1-space, $\cap\{N_x: N_x \text{ is a neighborhood of } x\} = \{x\}$ for each $x \in X$; thus for every $\psi = (\alpha_1, \alpha_2, \ldots) \in W$ there exists a unique $x \in X$ such that $U_{\alpha_1}, U_{\alpha_2}, \ldots$ form a base at x, namely $\{x\} = \bigcap_{i=1}^{\infty} U_{\alpha_i}$. In this way we define a mapping $f: W \to X: \psi \to x$. Since X is first countable, f maps onto X.

To show that f is open, we will show that $f(W \cap N_{\alpha_1,\dots,\alpha_k}) = \bigcap_{i=1}^{k} U_{\alpha_i}$. Let $x \in \bigcap_{i=1}^{k} U_{\alpha_i}$, and add to $U_{\alpha_1},\dots,U_{\alpha_k}$ the neighborhoods $U_{\alpha_{k+1}}, U_{\alpha_{k+2}},\dots$ until you get a base at x. Let $\eta = (\alpha_1,\dots,\alpha_k,\alpha_{k+1},\dots)$. Then $f(\eta) = x$, so $f(W \cap N_{\alpha_1,\dots,\alpha_k}) \supseteq \bigcap_{i=1}^{k} U_{\alpha_i}$. On the other hand, it is clear from the definition of f that $f(W \cap N_{\alpha_1,\dots,\alpha_k}) \subseteq \bigcap_{i=1}^{k} U_{\alpha_i}$.

To see that f is continuous, let $\xi \in W$ and let V be an open neighborhood of $f(\xi)$. Since $\mathcal{B}$ is a basis, V is the union of members of $\mathcal{B}$, so $f(\xi) \in U_{\alpha_1}$, say. Then $\xi \in W \cap N_{\alpha_1}$ and $f(W \cap N_{\alpha_1}) \subseteq U_{\alpha_1} \subseteq V$. Hence f is continuous at ξ. Q.E.D.

<u>Corollary 3.2</u>: Every second countable T_0-space $(X,\mathcal{T})$ is a continuous image of a separable metric space.

<u>Proof</u>: Let $\mathcal{B}$ be a countable base of $(X,\mathcal{T})$, and let $\mathcal{B}' = \{C: C \in \mathcal{B}$ or $X - C \in \mathcal{B}\}$. Then, if $\mathcal{U}$ is the topology generated by $\mathcal{B}'$, $(X,\mathcal{U})$ is a T_1-space. A basis for $\mathcal{U}$ is the collection of sets which are finite intersections of members of $\mathcal{B}'$; since $\mathcal{B}'$ is countable, such a collection is also countable. Thus $(X,\mathcal{U})$ is second countable (so also first countable). By Theorem 3.1 $(X,\mathcal{U})$ is the continuous open image of a second countable metric space W under a mapping f. But $\mathcal{U} \supset \mathcal{T}$, so $(X,\mathcal{T})$ is the continuous image of W under f. Since W is second countable metric, W is separable. (See [4; page 187].) Q.E.D.

<u>Definition 3.3</u>: A <u>network</u> $\mathcal{F}$ for a topological space X is a collection of subsets of X such that any open set can be obtained

as a union of members of $\mathcal{F}$. A cosmic space is a regular T_2-space with a countable network.

The members of a network are not necessarily open subsets. Any base for the topology forms a network; so does the family of single points.

Definition 3.4: A collection $\mathcal{P}$ of subsets of a T_2-space X is a k-network for X iff, whenever $C \subseteq U$ with C compact and U open in X, then $C \subseteq P \subseteq U$ for some $P \in \mathcal{P}$. An $\aleph_0$-space is a regular T_2-space with a countable k-network.

Every k-network $\mathcal{P}$ is a network. For let $x \in U$ open; since $\{x\}$ is compact, there exists $P \in \mathcal{P}$ such that $x \in P \subseteq U$. Since this is true for every $x \in U$, U can be obtained as a union of members of $\mathcal{P}$. Thus every $\aleph_0$-space is a cosmic space.

Proposition 3.5: A subspace of an $\aleph_0$-space is an $\aleph_0$-space.

Proof: Let X be an $\aleph_0$-space with countable k-network $\mathcal{P}$. Let B be a subspace of X, and let $\mathcal{P}' = \{P \cap B\colon P \in \mathcal{P}\}$. Let $C \subset U$ with C compact and U open in B. Then C is also compact in X and $U = B \cap U'$ for some U' open in X. Hence there is an element P of $\mathcal{P}$ such that $C \subset P \subset U'$. Then $C = C \cap B \subset P \cap B \subset U' \cap B = U$. Therefore $\mathcal{P}'$ is a countable k-network for B. Subspaces of regular T_2-spaces are regular T_2-spaces, so B is an $\aleph_0$-space. Q.E.D.

Lemma 3.6: A separable metric space X is an $\aleph_0$-space.

Proof: Let $\mathscr{B} = \{B_n: n = 1,2,\ldots\}$ be a countable basis for X; without loss of generality we can assume $\mathscr{B}$ is closed under finite unions. We show that $\mathscr{B}$ is a countable k-network for X. Let $C \subset U$ with C compact and U open in X. Since X is regular, there exists $B_i \in \mathscr{B}$ such that $x \in B_i \subset U$ for every $x \in C$. Since C is compact, there exist finitely many B_i, say $B_1,\ldots,B_n$, such that $C \subset \bigcup_{i=1}^{n} B_i$. Since $\bigcup_{i=1}^{n} B_i \subset U$ and $\bigcup_{i=1}^{n} B_i \in \mathscr{B}$, the proof is complete. Q.E.D.

Definition 3.7: A Hausdorff space X is called a k-space iff a subset A of X is closed in X whenever $A \cap C$ is closed in C for every compact subset C of X.

Theorem 3.8: If X is a k-space with a countable k-network, then so is any Hausdorff quotient space Y of X.

Proof: A quotient space of a k-space is a k-space. (See [4; page 248].) Hence we need only show that Y has a countable k-network.

Let $\mathscr{P}$ be a countable k-network for X. The family of finite unions of elements of $\mathscr{P}$ is again countable and a k-network. Thus without loss of generality we may take $\mathscr{P}$ to be closed under finite unions. Let f be the quotient map of X onto Y, and let $\mathscr{R} = \{f(P): P \in \mathscr{P}\}$. We will show that $\mathscr{R}$ is a k-network for Y.

Let $C \subset U$ with C compact and U open in Y. Let $y \in U$ and let $x \in f^{-1}(y)$; since $\{x\}$ is compact and $f^{-1}(U)$ is open in X, there exists $P \in \mathscr{P}$ such that $\{x\} \subset P \subset f^{-1}(U)$ and therefore such that $y \in f(P) \subset U$. Thus, let $R_1, R_2, \ldots$ be an enumeration of the elements of $\mathscr{R}$ which are contained in U. Let

Let $R'_n = R_1 \cup \ldots \cup R_n$. We have $R'_n = f(P_1) \cup \ldots \cup f(P_n) = f(P_1 \cup \ldots \cup P_n) = f(P'_n)$ for some $P'_n \in \mathcal{P}$. Hence for every n $R'_n \in \mathcal{R}$ and $R'_n \subset U$, so it suffices to show that $C \subset R'_n$ for some n.

Suppose not. Then for each n there exists $x_n \in C - R'_n$. Let $A = \{x_n : n = 1,2,\ldots\}$. Now C is covered by elements of $\mathcal{R}$ and $R'_{n+1} \supset R'_n$. Hence for all $k > n$, $x_k \in C - R'_n$. Therefore $A \cap R'_n$ is finite for all n. Also, given n there is a k such that $\{x_1,\ldots,x_n\} \subset R'_k$. Hence $x_k \neq x_j$ for $j = 1,2,\ldots,n$. Therefore A is infinite.

Since $A \subset C$ and C is compact, A has a limit point x. Then $E = A - \{x\}$ is not closed in Y. Since f is a quotient map, $f^{-1}(E)$ is not closed in X; since X is a k-space it follows that there exists a compact set $K \subset X$ such that $f^{-1}(E) \cap K$ is not closed in K. Hence $E \cap f(K)$ is not closed in Y (if it were closed in Y, $f^{-1}(E \cap f(K)) = f^{-1}(E) \cap f^{-1}f(K)$ would be closed in X, and then $f^{-1}(E) \cap K = [f^{-1}(E) \cap f^{-1}f(K)] \cap K$ would be closed in K). Since Y is a T_1-space, $E \cap f(K)$ and the larger set $A \cap f(K)$ are infinite. Now C is compact and hence closed in the Hausdorff space Y, so $f^{-1}(C) \cap K$ is compact. Hence there exists $P \in \mathcal{P}$ such that $f^{-1}(C) \cap K \subset P \subset f^{-1}(U)$.

Now $A \subset C$, so $A \cap f(K) \subset C \cap f(K) = f(f^{-1}(C) \cap K) \subset f(P)$ and thus $A \cap f(K) \subset A \cap f(P)$. But $A \cap f(K)$ is infinite, so that $A \cap f(P)$ is also infinite. Since $f(P) \subset U$, we have $f(P) = R_n$ for some n. Thus $A \cap R_n$ is infinite and consequently so is $A \cap R'_n$, a contradiction. Q.E.D.

Lemma 3.9: Let X be a regular T_2-space. Then the following statements are equivalent.

(1) X is cosmic.

(2) X is the continuous image of a second countable T_2-space.

(3) X is the continuous image of a separable metric space.

Proof: (1) $\rightarrow$ (2). Let $\mathcal{F}$ be a countable network for X. Let Y be the set X, topologized by taking $\mathcal{F}$ to be a sub-base. As in the proof of Corollary 3.2, Y is second countable.

To see that Y is T_2, let $x \neq y$ be elements of Y. Consider x and y as points of X; since X is T_2, there exist sets U and V open in X such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Then there exist $F_1, F_2 \in \mathcal{F}$ such that $x \in F_1$, $y \in F_2$, and $F_1 \cap F_2 = \emptyset$. But considered as subsets of Y, F_1 and F_2 are open.

The identity map from Y to X is continuous. For if U is open in X, $U = \cup F_\alpha$ for some sub-family of $\mathcal{F}$. Then $f^{-1}(U) = f^{-1}(\cup F_\alpha) = \cup f^{-1}(F_\alpha) = \cup F_\alpha$ is open in Y.

(2) $\rightarrow$ (3). Apply Corollary 3.2.

(3) $\rightarrow$ (1). Let Y be separable metric, and $f\colon Y \twoheadrightarrow X$ continuous. If $\mathcal{B}$ is a countable base for Y, then $\mathcal{F} = \{f(B)\colon B \in \mathcal{B}\}$ is a countable network for X. Q.E.D.

If X and Y are topological spaces, let $\mathcal{C}(X,Y)$ denote the space of all continuous functions from X to Y, with the compact-open topology. This is the topology which has a sub-base consisting of all sets $W(C,U) = \{f \in \mathcal{C}(X,Y)\colon f(C) \subset U\}$ with C compact in X and U open in Y.

<u>Lemma 3.10</u>: If X is a compact metric space and if Y is an $\aleph_0$-space, then $\mathcal{C}(X,Y)$ is cosmic.

<u>Proof</u>: Since Y is a regular T_2-space, $\mathcal{C}(X,Y)$ is a regular T_2-space. (See [4; page 258].) A compact metric space is separable metric. (See [14; page 158].) Thus by Lemma 3.6 X is an $\aleph_0$-space. Let $\mathcal{P}$ and $\mathcal{R}$ be countable k-networks for X and Y, respectively. The collection of all finite intersections of sets $W(\overline{P},R)$ with $P \in \mathcal{P}$ and $R \in \mathcal{R}$ is clearly countable; we will show that it is a network for $\mathcal{C}(X,Y)$.

Let $f \in W(C,U)$ where C is a compact subset of X and U is open in Y. Since C is closed in X, $f^{-1}(U)$ is open in X, and $C \subset f^{-1}(U)$, the normality of X implies that there exists an open set V such that $C \subset V \subset \overline{V} \subset f^{-1}(U)$. Then there exists $P \in \mathcal{P}$ such that $C \subset P \subset V$, and consequently $C \subset \overline{P} \subset f^{-1}(U)$. Since $\overline{P}$ is closed in the compact space X, $\overline{P}$ is compact. Thus $f(\overline{P})$ is compact and $f(\overline{P}) \subset U$. Therefore there exists $R \in \mathcal{R}$ such that $f(\overline{P}) \subset R \subset U$. But then $f \in W(\overline{P},R) \subset W(C,U)$. Since the sets $W(C,U)$ are a sub-basis for the compact-open topology, the proof is complete. Q.E.D.

<u>Definition 3.11</u>: Let X and Y be topological spaces. A continuous function $f\colon X \to Y$ is a <u>compact covering</u> iff every compact subset of Y is the image under f of some compact subset of X.

<u>Lemma 3.12</u>: If Y is a Hausdorff k-space, then any compact-covering map $f\colon X \twoheadrightarrow Y$ is a quotient map.

<u>Proof</u>: Suppose $B \subset Y$ is such that $f^{-1}(B)$ is closed in X. To show that B is closed in Y, it is sufficient to show that $B \cap C$ is compact for every compact subset C of Y.

Let C be a compact subset of Y. Then $C = f(K)$ for some compact subset K of X. Then $f^{-1}(B) \cap K$ is compact, and hence, by the continuity of f, so is its image $B \cap C$. Q.E.D.

If X is a T_2-space with topology $\mathcal{T}$, then let $k(X)$ denote the set X, topologized by calling a subset closed iff its intersection with every $\mathcal{T}$-compact subset of X is $\mathcal{T}$-compact.

<u>Proposition 3.13</u>: $k(X)$ is a Hausdorff k-space and its topology yields the same compact sets as X.

<u>Proof</u>: Since the intersection of a closed set and a compact set is compact, every set closed in X is closed in $k(X)$. Thus the topology of $k(X)$ contains the topology of X. It follows that $k(X)$ is Hausdorff and that every set compact in $k(X)$ is compact in X.

On the other hand, let A be compact in X. Then A is closed in $k(X)$. Let $\mathcal{F}$ be any collection of $k(X)$-closed subsets of A having the finite intersection property; to prove that A is compact in $k(X)$, it is sufficient to prove that $\cap \mathcal{F} \neq \emptyset$. But $F \cap A$ is compact in the T_2-space X for every $F \in \mathcal{F}$ and hence $F = F \cap A$ is closed in X for every $F \in \mathcal{F}$. Thus $\mathcal{F}$ is a collection of X-closed subsets of A; since A is compact in X, $\cap \mathcal{F} \neq \emptyset$.

To see that $k(X)$ is a k-space, let A be a subset of $k(X)$, let C be a compact subset of $k(X)$, and suppose $A \cap C$ is closed in C. Then $A \cap C$ is $k(X)$-compact and thus X-compact. By the definition of the topology of $k(X)$, A is closed in $k(X)$. Q.E.D.

It is not difficult to see that a T_2-space X is a k-space iff $X = k(X)$, that is, iff the topologies of X and $k(X)$ are equal.

Let X and Y be T_2-spaces and let $f\colon X \to Y$ be a function. Then $f_k\colon k(X) \to k(Y)$ denotes the function taking the same values as f but from and into the new topological spaces.

Lemma 3.14: If X and Y are T_2-spaces and if $f\colon X \twoheadrightarrow Y$ is continuous, then so is $f_k\colon k(X) \twoheadrightarrow k(Y)$.

Proof: Let B be closed in $k(Y)$. To show that $f_k^{-1}(B)$ is closed in $k(X)$, it is sufficient to show that $f^{-1}(B) \cap C$ is closed in X for every compact subset C of X.

Let C be a compact subset of X. Since f is continuous, $f(C)$ is compact in Y; thus $B \cap f(C)$ is compact in the T_2-space Y and hence closed in Y. Since C is closed in X, it follows that $f^{-1}(B) \cap C = f^{-1}(B \cap f(C)) \cap C$ is closed in X. Q.E.D.

Lemma 3.15: Every compact $\aleph_0$-space X is separable metrizable.

Proof: Let $\mathcal{P}$ be a countable k-network for X. By Urysohn's metrization theorem, it will suffice to show that $\{\text{Int } P\colon P \in \mathcal{P}\}$ is a base for X.

Suppose not. Then there is an $x \in X$ and an open subset U of X such that $x \in U$ and such that there is no $P \in \mathcal{P}$ with

$x \in \operatorname{Int} P \subset U$. Let $U_1, U_2, \ldots$ be a sequence of neighborhoods of x. Since X is regular, there exists an open subset V of X such that $x \in V \subset \overline{V} \subset U$; let $V_n = V \cap U_n$ for all n. Let $P_1, P_2, \ldots$ be an enumeration of the elements of $\mathcal{P}$ contained in U. Suppose $V_n \subset P_n$ for some n. Being the intersection of two neighborhoods of x, V_n is a neighborhood of x. Hence $x \in \operatorname{Int} P_n \subset U$, contradicting our assumptions. Thus we can pick an $x_n \in V_n - P_n$ for all n. Let $C = \{x_1, x_2, \ldots\}$. Since X is compact, $\overline{C}$ is compact. But $\overline{C} \subset \overline{V} \subset U$, so $\overline{C} \subset P_n$ for some n, which is impossible since $x_n \notin P_n$. Q.E.D.

We now have the machinery necessary to prove the following theorem. It has a corollary concerning quotient spaces of separable metric spaces.

Theorem 3.16: The following properties of a regular T_2-space are equivalent.

(1) X is an $\aleph_0$-space.

(2) X is the image, under a compact-covering map, of a separable metric space.

(3) $k(X)$ is the image, under a compact-covering quotient map, of a separable metric space.

(4) $k(X)$ is a quotient space of a separable metric space.

Proof: (1) $\rightarrow$ (2). Let K denote the Cantor set, and define $\phi: \mathcal{C}(K,X) \times K \rightarrow X$ by $\phi(f,t) = f(t)$. Since K is compact, ϕ is continuous. (See [9; page 223].) By Lemma 3.10 $\mathcal{C}(K,X)$ is cosmic. Then by Lemma 3.9 there exists a continuous function u from a separable metric space S onto $\mathcal{C}(K,X)$. Since K is compact metric, K is separable. Thus $S \times K$ is a separable metric space. (See [4; pages 175 and 191].) Define $\psi\colon S \times K \to X$ by $\psi(s,t) = \phi(u(s),t)$. Since ϕ and u are continuous, so is ψ. We will show that ψ is a compact-covering map.

Let C be a non-empty compact subset of X. By Proposition 3.5 and Lemma 3.15 C is metrizable. Thus there exists $f \in \mathcal{C}(K,X)$ such that $f(K) = C$. (See [9; pages 165-6].) Pick $s \in S$ such that $u(s) = f$; we can do this since u maps onto $\mathcal{C}(K,X)$. By Tychonoff's theorem, $\{s\} \times K$ is a compact subset of $S \times K$. Now $\psi(\{s\} \times K) = C$. (If $C = \{x\}$, then this argument also shows that ψ maps onto X).

(2) $\to$ (3). Let $f\colon M \twoheadrightarrow X$ be a compact-covering map, with M separable metric. M is a T_2 k-space. (See [4; page 248].) Thus $M = k(M)$. By Lemma 3.14 $f_k\colon M \twoheadrightarrow k(X)$ is continuous. Since $k(X)$ and X have the same compact sets (see Proposition 3.13), f_k is a compact-covering map along with f. By Lemma 3.12 f_k is also a quotient map.

(3) $\to$ (4). Obvious.

(4) $\to$ (1). Suppose $k(X)$ is a quotient space of a separable metric space M. By Lemma 3.6 M is an $\aleph_0$-space, so Theorem 3.8 shows that $k(X)$ has a countable k-network. Since X has the same compact sets as $k(X)$ and every set open in X is open in $k(X)$, X also has a countable k-network. Thus X is an $\aleph_0$-space. Q.E.D.

The following corollary characterizes quotient spaces of separable metric spaces.

Corollary 3.17: Let X be a regular T_2-space. Then the following statements are equivalent.

(1) X is an $\aleph_0$-space and a k-space.

(2) X is a quotient space of a separable metric space M.

Proof: (1) $\rightarrow$ (2). This follows from (1) $\rightarrow$ (4) of Theorem 3.16, since, when X is a T_2 k-space, $k(X) = X$.

(2) $\rightarrow$ (1). Since M is a k-space and a quotient space of a k-space is a k-space, X is a k-space. By Lemma 3.6 M is an $\aleph_0$-space. Theorem 3.8 then shows that X is an $\aleph_0$-space. Q.E.D.

CHAPTER IV. GENERAL QUOTIENT SPACES

In this chapter we obtain criteria for metrizability of an arbitrary quotient space of a metrizable space. Our first main result is due to A. Arhangel'skiĭ [1]. Of fundamental importance is the following definition.

Definition 4.1: A mapping $f: X \twoheadrightarrow Y$ of a metrizable space X onto a topological space Y is called regular iff there exists a metric d compatible with the topology of X such that for each open subset G of Y and each point $y \in G$ there exists a neighborhood U of y such that $d(f^{-1}(U), X - f^{-1}(G)) > 0$.

We shall also need the following two definitions.

Definition 4.2: Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ be a covering of a space Y. For any $B \subset Y$, the set $\cup\{U_\alpha : U_\alpha \cap B \neq \emptyset\}$ is called the star of B with respect to $\mathcal{U}$, and is denoted by $St(B, \mathcal{U})$.

Definition 4.3: Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ be an open covering of Y. A sequence $\{\mathcal{V}_n : n = 1,2,\ldots\}$ of open coverings is called locally starring for $\mathcal{U}$ if for each $y \in Y$ there exists a neighborhood V of y and an integer n such that $St(V, \mathcal{V}_n) \subset$ some U_α.

We will make use of the following metrization theorem, also due to Arhangel'skiĭ.

Theorem 4.4: If Y is a T_1-space and if there exists one sequence $\{\mathcal{V}_n : n = 1,2,\ldots\}$ of open coverings that is locally starring for every open covering, then Y is metrizable.

Proof: The proof may be found in [4; pages 196-7].

Our goal is the next theorem.

Theorem 4.5: A Hausdorff quotient space of a metrizable space is metrizable if and only if the corresponding mapping is regular.

We first prove two lemmas. If X is a metric space with metric d, define $N_\epsilon(B) = \{x: d(x,B) < \epsilon\}$ for each $B \subset X$ and each $\epsilon > 0$.

Lemma 4.6: Let $f: X \twoheadrightarrow Y$ be a mapping of a metric space X onto a T_1-space Y which satisfies the following two conditions:

(1) f is regular.

(2) $y \in \operatorname{Int} f[N_\epsilon(f^{-1}(y))]$ for any point $y \in Y$ and number $\epsilon > 0$.

Then Y is metrizable.

Proof: Let $\mathcal{U}_n$, $n = 1,2,\ldots,$ consist of all open sets in X whose diameter is at most $\frac{1}{n}$. Let $\mathcal{V}_n$ denote, for any n, the family of sets in Y consisting of the interiors of the stars of the points of the latter space with respect to the set of images of members of the covering $\mathcal{U}_n$: $\mathcal{V}_n = \{\operatorname{Int} St(y, f(\mathcal{U}_n)): y \in Y\}$. We assert that $\{\mathcal{V}_n: n = 1,2,\ldots\}$ is a sequence of open coverings of Y that is locally starring for every open covering of Y.

First of all, the $\mathcal{V}_n$ are coverings of Y. Note that

$$\begin{aligned} St(y, f(\mathcal{U}_n)) &= \cup\{f(U) : U \in \mathcal{U}_n \text{ and } f(U) \cap \{y\} \neq \emptyset\} \\ &= \cup\{f(U) : U \in \mathcal{U}_n \text{ and } U \cap f^{-1}(y) \neq \emptyset\} \\ &= f(\cup\{U : U \in \mathcal{U}_n \text{ and } U \cap f^{-1}(y) \neq \emptyset\}) \\ &= f[St(f^{-1}(y), \mathcal{U}_n)] \ . \end{aligned}$$

Therefore $\operatorname{Int} St(y, f(\mathcal{U}_n)) = \operatorname{Int} f[St(f^{-1}(y), \mathcal{U}_n)] \ni y$ by condition (2).

Now let y and N be an arbitrary point and a neighborhood thereof in Y. Since f is regular, there exist neighborhoods N_1 and N_2 of y such that $d(f^{-1}(N_1), X - f^{-1}(N)) > 0$ and $d(f^{-1}(N_2), X - f^{-1}(N_1)) > 0$. Let r be the smaller of the two numbers on the left hand sides of these inequalities, and take integer M such that $\frac{1}{M} < r$. We shall now show that $St(N_2, \mathcal{V}_M) \subset N$.

Consider any point $y_0 \in St(N_2, \mathcal{V}_M) = \cup\{V : V \in \mathcal{V}_M$ and $V \cap N_2 \neq \emptyset\} = \cup\{V : V = \operatorname{Int} St(p, f(\mathcal{U}_M))$ for some $p \in Y$ and $V \cap N_2 \neq \emptyset\}$. Let $p \in Y$ be such that

1') $y_0 \in \operatorname{Int} \cup\{f(U) : U \in \mathcal{U}_M$ and $f(U) \cap \{p\} \neq \emptyset\}$
$\subset \cup\{f(U) : U \in \mathcal{U}_M$ and $p \in f(U)\}$

2') $\emptyset \neq N_2 \cap \operatorname{Int} \cup\{f(U) : U \in \mathcal{U}_M$ and $f(U) \cap \{p\} \neq \emptyset\}$
$\subset N_2 \cap \cup\{f(U) : U \in \mathcal{U}_M$ and $p \in f(U)\}$.

By 1') $y_0 \in f(G^M_{\alpha_1})$ for some $G^M_{\alpha_1} \in \mathcal{U}_M$ such that $p \in f(G^M_{\alpha_1})$. By 2') $N_2 \cap f(G^M_{\alpha_2}) \neq \emptyset$ for some $G^M_{\alpha_2} \in \mathcal{U}_M$ such that $p \in f(G^M_{\alpha_2})$. That is, $\mathcal{U}_M$ contains two members $G^M_{\alpha_1}$ and $G^M_{\alpha_2}$

which satisfy simultaneously the following relations:

1") $f(G^M_{\alpha_2}) \cap N_2 \neq \emptyset$

2") $f(G^M_{\alpha_1}) \cap f(G^M_{\alpha_2}) \neq \emptyset$

3") $y_0 \in f(G^M_{\alpha_1})$.

By 1"), $G^M_{\alpha_2} \cap f^{-1}(N_2) \neq \emptyset$ and consequently, since diam $G^M_{\alpha_2} \leq \frac{1}{M} < r \leq d(f^{-1}(N_2), X - f^{-1}(N_1))$, we have $G^M_{\alpha_2} \subset f^{-1}(N_1)$. Hence $f(G^M_{\alpha_2}) \subset N_1$, so that $f(G^M_{\alpha_1}) \cap f(G^M_{\alpha_2}) \subset N_1$. By 2"), we may write $f(G^M_{\alpha_1}) \cap N_1 \neq \emptyset$, or, equivalently, $G^M_{\alpha_1} \cap f^{-1}(N_1) \neq \emptyset$. Since diam $G^M_{\alpha_1} \leq \frac{1}{M} < r \leq d(f^{-1}(N_1), X - f^{-1}(N))$, it follows that $G^M_{\alpha_1} \subset f^{-1}(N)$, so that $f(G^M_{\alpha_1}) \subset N$. Consequently, $y_0 \in N$ by 3").

Since y_0 was an arbitrary point in $St(N_2, \mathcal{V}_M)$, we have shown that $St(N_2, \mathcal{V}_M) \subset N$. Thus, the family $\{\mathcal{V}_n: n = 1,2,\ldots\}$ is locally starring for every open covering of Y. By Theorem 4.4 Y is metrizable. Q.E.D.

Lemma 4.7: Let $f: X \twoheadrightarrow Y$ be a regular quotient mapping of a metric space X onto a T_2-space Y. Then if a set $M \subseteq X$ satisfies the condition $f^{-1}f(M) = M$, the set $f^{-1}f(\overline{M})$ is closed.

Proof: Assuming the contrary, consider a point $x_0 \in \overline{f^{-1}f(\overline{M})} - f^{-1}f(\overline{M})$. Let $y_0 = f(x_0)$. Then $f^{-1}(y_0) \cap f^{-1}f(\overline{M}) = \emptyset$ and consequently $f^{-1}(y_0) \cap \overline{M} = \emptyset$.

Let us say that a point $x \in M$ is ϵ-accessible from x_0 if for some $y \in Y$ we have $N_\epsilon(f^{-1}(y)) \supset \{x_0\} \cup \{x\}$. We assert that for any $\epsilon > 0$, there exists a point in M which is ϵ-accessible from x_0. Since $x_0 \in \overline{f^{-1}f(\bar{M})}$, there exists a sequence in $f^{-1}f(\bar{M})$ converging to x_0; take $x_1 \in f^{-1}f(\bar{M})$ such that $d(x_0,x_1) < \epsilon$. Let $y_1 = f(x_1)$. If $f^{-1}(y_1) \cap \bar{M} = \emptyset$, then $y_1 \notin f(\bar{M})$ and $x_1 \notin f^{-1}f(\bar{M})$, a contradiction. Thus there is an $x_2 \in \bar{M}$ such that $f(x_2) = y_1$. In turn, the set M contains a point x_3 such that $d(x_3,x_2) < \epsilon$. Then $N_\epsilon(f^{-1}(y_1)) \supset \{x_0\} \cup \{x_3\}$, so that x_3 is ϵ-accessible from x_0.

For every n, pick a point x_n in M which is $\frac{1}{n}$-accessible from x_0. We will first show that $\{f(x_n)\}$ converges to y_0. Let U be an arbitrary neighborhood of y_0. Pick a neighborhood U_1 of y_0 such that $d(f^{-1}(U_1),X - f^{-1}(U)) > 0$. Then also $d(f^{-1}(y_0),X - f^{-1}(U_1)) > 0$. Now let N be an integer such that

$$\frac{1}{N} < \min\{d(f^{-1}(U_1),X - f^{-1}(U)),d(f^{-1}(y_0),X - f^{-1}(U_1))\} \quad .$$

Then $f(x_n) \in U$ for $n > N$. Indeed, consider the point y_n for which $N_{1/n}(f^{-1}(y_n)) \supset \{x_0\} \cup \{x_n\}$. Since $d(x_0,f^{-1}(y_n)) < \frac{1}{n}$, there exists $z_n \in f^{-1}(y_n)$ such that $d(x_0,z_n) < \frac{1}{n}$. If $z_n \in X - f^{-1}(U_1)$, then $d(f^{-1}(y_0),X - f^{-1}(U_1)) \leq d(x_0,z_n) < \frac{1}{n} < \frac{1}{N}$, a contradiction; thus $z_n \in f^{-1}(U_1) \cap f^{-1}(y_n)$. It follows that $y_n \in U_1$. Hence $f^{-1}(y_n) \subset f^{-1}(U_1)$ and $\{x_n\} \subset N_{1/n}(f^{-1}(y_n)) \subset N_{1/n}(f^{-1}(U_1))$. But then the inequality $\frac{1}{n} < \frac{1}{N} < d(f^{-1}(U_1),X - f^{-1}(U))$ implies that $x_n \in f^{-1}(u)$ and hence $f(x_n) \in U$. This proves that the sequence $\{f(x_n)\}$ converges to y_0.

Since Y is Hausdorff, y_0 is the unique limit point of the set $\{f(x_n)\}$; hence $P = \{f(x_n)\} \cup \{y_0\}$ is closed in Y.

Therefore, the set $Q = f^{-1}(P) \cap \bar{M}$ is closed in X. Since $f^{-1}f(M) = M$ and $x_n \in M$ for all n, we have that $f^{-1}(\{f(x_n)\}) \subset M$. Thus

$$\begin{aligned} Q &= [f^{-1}(\{f(x_n)\}) \cup f^{-1}(y_0)] \cap \bar{M} \\ &= [f^{-1}(\{f(x_n)\}) \cap \bar{M}] \cup [f^{-1}(y_0) \cap \bar{M}] \\ &= f^{-1}(\{f(x_n)\}) \ , \end{aligned}$$

where Q is closed in X, whereas $\{f(x_n)\}$ is a non-closed set in Y. This contradicts the assumption that f is a quotient mapping. Q.E.D.

Proof of Theorem 4.5: To prove sufficiency, let $f: X \twoheadrightarrow Y$ be a regular quotient mapping of a metric space X onto a T_2-space Y. We will show that f satisfies condition 2) of Lemma 4.6. Let y be any point of Y and let U be any open set in X containing $f^{-1}(y)$. Let $L = Y - f(U)$ and $M = f^{-1}(L)$. Then $f^{-1}f(M) = M$ and $U \cap M = \emptyset$. Hence $f^{-1}(y) \cap \bar{M} = \emptyset$, that is, $y \notin f(\bar{M})$. By Lemma 4.7 $f^{-1}f(\bar{M})$ is closed in X; this means, since f is a quotient mapping, that $f(\bar{M})$ is closed in Y. But $y \in Y - f(\bar{M}) \subset Y - f(M) = Y - L = f(U)$, so that $y \in \operatorname{Int} f(U)$.

To prove necessity, we will show that every continuous mapping f of a metrizable space X onto a metrizable space Y is regular with respect to some metric on X. Suppose that d_x, d_y are metrics compatible with the topologies of X,Y, respectively, and define a metric d on X by

$$d(u,v) = d_x(u,v) + d_y(f(u),f(v)) \quad \text{for} \quad u,v \in X \ .$$

Then d is compatible with the topology of X. Since, clearly, $N_\epsilon^d(u) \subset N_\epsilon^{d_x}(u)$, we have that $\mathcal{T}(d_x) \subset \mathcal{T}(d)$, where, for example, $\mathcal{T}(d)$ is the topology in X determined by the metric d. To see that $\mathcal{T}(d) \subset \mathcal{T}(d_x)$, let $\epsilon > 0$ and $u \in X$ be given; by the continuity of f choose $0 < \delta < \epsilon$ such that $d_x(u,v) < \delta$ implies that $d_y(f(u),f(v)) < \epsilon$. Then, clearly, $N_\delta^{d_x}(u) \subset N_{2\epsilon}^d(u)$. Thus $\mathcal{T}(d_x) = \mathcal{T}(d)$. To see that $f:(X,d) \twoheadrightarrow (Y,d_y)$ is regular, let G be an open subset of Y and $y \in G$. Let $\epsilon = \frac{1}{2} d_y(y,Y-G) > 0$ and $U = N_\epsilon^{d_y}(y)$. Then $U \subset G$ and $d_y(U,Y-G) \geq \epsilon$. Therefore, $d(f^{-1}(U),X - f^{-1}(G)) > 0$; for, if $u \in f^{-1}(U)$ and $v \in X - f^{-1}(G)$, we have

$$\begin{aligned} d(u,v) &= d_x(u,v) + d_y(f(u),f(v)) \\ &\geq d_y(f(u),f(v)) \\ &\geq d_y(U,Y-G) \\ &\geq \epsilon > 0 \ . \end{aligned}$$

Thus $f:(X,d) \twoheadrightarrow (Y,d_y)$ is regular. Q.E.D.

In the next theorem, due to Himmelberg, we work with pseudo-metric spaces and obtain an interesting explicit definition of a pseudo-metric for the quotient space.

<u>Theorem 418</u>: Let f be a function from a pseudo-metrizable space X onto a topological space Y, and suppose that Y has the quotient topology relative to f. Then the following assertions are equivalent.

1) Y is pseudo-metrizable.

2) There exists a pseudo-metric d compatible with the topology of X and a subbase $\mathscr{S}$ for the topology of Y such that for each $G \in \mathscr{S}$ there exists a set $\{\epsilon(y): y \in G\}$ of positive real numbers satisfying:

(i) $N_{\epsilon(y)}(f^{-1}(y)) \subset f^{-1}(G)$ for all $y \in G$, and

(ii) $d(f^{-1}(y), f^{-1}(z)) \geq \epsilon(y) - \epsilon(z)$ for all $y, z \in G$.

3) There exists a pseudo-metric d compatible with the topology of X such that the topology of Y is compatible with the pseudo-metric ρ defined by

$$\rho(y,z) = \inf \Sigma_{i=1}^{n} d(f^{-1}(y_{i-1}), f^{-1}(y_i)) ,$$

where $y, z \in Y$, $y_i \in Y$ for all $0 \leq i \leq n$, and the infimum is taken over all finite chains $y = y_0$, $y_1, \ldots, y_n = z$.

<u>Proof</u>: (1) $\rightarrow$ (2). Suppose that d_x, d_y are pseudo-metrics compatible with the topologies of X, Y, respectively, and that d_y is bounded. Let $\mathscr{S}$ be any subbase for the topology of Y and define d as in the proof of Theorem 4.5. If G is any proper open subset of Y, let $\epsilon(y) = d_y(y, Y-G) > 0$ for each $y \in G$. To verify (i), let $x \in X$ be such that $d(x, f^{-1}(y)) < \epsilon(y)$. Then there exists $z \in f^{-1}(y)$ such that $d(x,z) < \epsilon(y)$; consequently, $d_y(f(x), y) = d_y(f(x), f(z)) < \epsilon(y)$, that is, $f(x) \in G$ and $x \in f^{-1}(G)$. To verify (ii), let $y, z \in G$ and let $m \in f^{-1}(y)$, $n \in f^{-1}(z)$. Then

$$\begin{aligned} d(m,n) &= d_X(m,n) + d_Y(y,z) \\ &\geq d_Y(y,z) \\ &\geq d_Y(y,Y\text{-}G) - d_Y(z,Y\text{-}G) \\ &= \epsilon(y) - \epsilon(z) \quad . \end{aligned}$$

Thus $d(f^{-1}(y),f^{-1}(z)) \geq \epsilon(y) - \epsilon(z)$.

(3) → (1). Trivial.

(2) → (3). Let $d, \mathscr{S}$ be a pseudo-metric and subbase, respectively, as given by (2). Define ρ as in (3). It is a trivial matter to verify that ρ is a pseudo-metric. Moreover, $f\colon X \twoheadrightarrow Y$ is continuous if Y has the topology defined by ρ, since

$$\rho(f(u),f(v)) \leq d(f^{-1}[f(u)],f^{-1}[f(v)]) \leq d(u,v) \quad ,$$

whenever $u,v \in X$. Thus all that remains to be shown is that the topology defined by ρ is larger than (and therefore equal to) the quotient topology on Y. To do this it is sufficient to show that each member of the subbase $\mathscr{S}$ is open relative to ρ. So let $G \in \mathscr{S}$ and let $\{\epsilon(y)\colon y \in G\}$ be as in (2). Then $\rho(y,z) < \epsilon(y)$ implies that $z \in G$. For suppose $\rho(y,z) < \epsilon(y)$ and choose δ such that $\rho(y,z) < \delta < \epsilon(y)$. Then there exists a finite chain $y = y_0,y_1,\ldots,y_n = z$ of points of Y such that

$$(*) \qquad \begin{aligned} \Sigma_{i=1}^{n}\, d(f^{-1}(y_{i-1}),f^{-1}(y_i)) &< \rho(y,z) + (\epsilon(y) - \delta) \\ &= (\rho(y,z) - \delta) + \epsilon(y) \\ &< \epsilon(y) \quad . \end{aligned}$$

In particular, $d(f^{-1}(y), f^{-1}(y_1)) < \epsilon(y)$. This means that $d(f^{-1}(y), u) < \epsilon(y)$ for some $u \in f^{-1}(y_1)$. Then (i) of (2) implies that $u \in f^{-1}(G)$ and $y_1 \in G$.

Now apply (ii) of (2) to (*) to obtain

$$\begin{aligned} \Sigma_{i=2}^{n} d(f^{-1}(y_{i-1}), f^{-1}(y_i)) &< \epsilon(y) - d(f^{-1}(y), f^{-1}(y_1)) \\ &\leq \epsilon(y) - \epsilon(y) + \epsilon(y_1) \\ &= \epsilon(y_1). \end{aligned}$$

Thus, by repeating the argument following the inequality (*), we deduce that $y_1, y_2, \ldots, y_n = z$ all belong to G. In particular $z \in G$. We have thus proved that G is open relative to ρ. Q.E.D.

CHAPTER V. CLOSED QUOTIENT MAPS

In this chapter we will give a strong metrization theorem proved independently by Morita and Hanai (11) and A. H. Stone (13). Part of the proof that we provide is due to C. J. Himmelberg. We need the following version of Frink's metrization theorem (6); its proof is the same as Frink's.

<u>Theorem 5.1</u>: Y is pseudo-metrizable iff for each $y \in Y$ there exists a sequence $\{W_n(y)\}$ of open subsets of Y such that

1) $W_1(y) \supset W_2(y) \ldots,$
2) $\{W_n(y)\}$ is a local base at y, and
3) given n and y, there exists m such that $W_m(z) \subset W_n(y)$ if $W_m(z)$ meets $W_m(y)$.

We will denote the boundary of a subset A of a topological space X by BdA. Recall that A is closed in X iff $BdA \subset A$ and that A is open in X iff $A \cap BdA = \emptyset$. Also recall that the boundary of any set is closed.

<u>Lemma 5.2</u>: Let f be a function from a pseudo-metrizable space X onto a topological space Y such that $Bdf^{-1}(y)$ is compact for each $y \in Y$. Then there exists a closed subset A of X such that $f(A) = Y$ and $\overline{f^{-1}(y) \cap A}$ is compact for each $y \in Y$.

<u>Proof</u>: For each $y \in Y$, let

$$L(y) = \begin{cases} \text{Int } f^{-1}(y), & \text{if } f^{-1}(y) \cap Bdf^{-1}(y) \neq \emptyset \\ f^{-1}(y) - \overline{\{p_y\}}, & \text{if } f^{-1}(y) \cap Bdf^{-1}(y) = \emptyset \end{cases},$$

where p_y is an arbitrary point of $f^{-1}(y) \neq \emptyset$. Note that if $f^{-1}(y) \cap \mathrm{Bd}f^{-1}(y) = \emptyset$, then $f^{-1}(y) - \overline{\{p_y\}}$ is the set difference of an open set and a closed set and hence is open.

Let $L = \cup\{L(y): y \in Y\}$ and $A = X-L$. Then L is open, A is closed, and $f(A) = Y$.

If $f^{-1}(y) \cap \mathrm{Bd}f^{-1}(y) \neq \emptyset$, then $f^{-1}(y) \cap A \subset \mathrm{Bd}f^{-1}(y)$; since $\mathrm{Bd}f^{-1}(y)$ is closed and compact and since closed subsets of compact sets are compact, it follows that $\overline{f^{-1}(y) \cap A}$ is compact. If $f^{-1}(y) \cap \mathrm{Bd}f^{-1}(y) = \emptyset$, then $f^{-1}(y) \cap A \subset \overline{\{p_y\}}$; since X is pseudo-metrizable, $\overline{\{p_y\}}$ is compact and we again obtain that $\overline{f^{-1}(y) \cap A}$ is compact. Q.E.D.

<u>Lemma 5.3</u>: Let f be a closed continuous mapping of a pseudo-metrizable space X onto a topological space Y such that

1) each open subset of Y contains the closure of each of its points, and
2) each $\overline{f^{-1}(y)}$ is compact.

Then Y is pseudo-metrizable.

<u>Proof</u>: Let a pseudo-metric d for X be given. For each $y \in Y$ and each integer $n > 0$ define

$$V_n(y) = \cup\{f^{-1}(z): f^{-1}(z) \subset N_{1/n}(\overline{f^{-1}(y)})\}$$

and

$$W_n(y) = f(V_n(y)) = Y - f[X - N_{1/n}(\overline{f^{-1}(y)})] .$$

Since f is a closed mapping, $W_n(y)$ is open; then, by the continuity of f, $V_n(y) = f^{-1}f(V_n(y)) = f^{-1}(W_n(y))$ is also open.

Moreover, $\overline{f^{-1}(z)} \subset V_n(y)$ whenever $f^{-1}(z)$ meets $V_n(y)$. For suppose $f^{-1}(z) \cap V_n(y) \neq \emptyset$; then $z \in W_n(y)$ and consequently, by (1), $\overline{\{z\}} \subset W_n(y)$. Applying the continuity of f, we obtain $\overline{f^{-1}(z)} \subset f^{-1}(\overline{\{z\}}) \subset f^{-1}(W_n(y)) = V_n(y)$.

We now verify (1), (2), and (3) of Theorem 5.1.

(1). This is trivial.

(2). Let G be an open subset of Y and let $y \in G$. Then $\overline{f^{-1}(y)} \subset f^{-1}(\overline{\{y\}}) \subset f^{-1}(G)$, and, since $\overline{f^{-1}(y)}$ is compact and $f^{-1}(G)$ is open, there exists n such that $N_{1/n}(\overline{f^{-1}(y)}) \subset f^{-1}(G)$. Hence $V_n(y) \subset f^{-1}(G)$, and $y \in W_n(y) \subset G$.

(3). Suppose n and y are given and choose m such that

a) $m > 2n$,

b) $d(\overline{f^{-1}(y)}, X - V_{2n}(y)) > \frac{2}{m}$.

(Recall that $V_{2n}(y)$ is open and $\overline{f^{-1}(y)} \subset V_{2n}(y)$.)

Now suppose $W_m(z)$ meets $W_m(y)$, say each contains $w = f(u)$ with $u \in V_m(z) \cap V_m(y)$. Then

$$\begin{aligned} d(\overline{f^{-1}(z)}, \overline{f^{-1}(y)}) &\leq d(\overline{f^{-1}(z)}, u) + d(u, \overline{f^{-1}(y)}) \\ &< \frac{1}{m} + \frac{1}{m} = \frac{2}{m} , \end{aligned}$$

and consequently $d(f^{-1}(z), \overline{f^{-1}(y)}) < \frac{2}{m}$. Then there exists a $p \in f^{-1}(z)$ such that $d(p, \overline{f^{-1}(y)}) < \frac{2}{m}$; it follows from (b) that $p \in f^{-1}(z) \cap V_{2n}(y)$. Hence

$$\overline{f^{-1}(z)} \subset V_{2n}(y) \subset N_{1/2n}(\overline{f^{-1}(y)}) \quad .$$

Now suppose $x \in V_m(z)$. Then $x \in N_{1/n}(\overline{f^{-1}(y)})$, since $d(x, \overline{f^{-1}(z)}) < \frac{1}{m} < \frac{1}{2n}$ and $\overline{f^{-1}(z)} \subset N_{1/2n}(\overline{f^{-1}(y)})$. It follows that

$V_m(z) \subset V_n(y)$, and consequently that $W_m(z) \subset W_n(y)$. Q.E.D.

Theorem 5.4: If X is pseudo-metrizable, Y is T_1 or regular, f is a closed continuous mapping of X onto Y, and each $\mathrm{Bd}f^{-1}(y)$ is compact, then Y is pseudo-metrizable. (In fact, instead of using T_1 or regular for Y, we use only the assumption that each open subset of Y contains the closure of each of its points.)

Proof: By Lemma 5.2 let A be a closed subset of X such that $f(A) = Y$ and $\overline{f^{-1}(y) \cap A}$ is compact for each $y \in Y$. If $g = f|A$, then g is onto Y and $\overline{g^{-1}(y)}$ is compact. To see that $g: A \twoheadrightarrow Y$ is closed, let F be a closed subset of A. Since A is closed in X, F is closed in X; then, since f is a closed mapping, $g(F) = f(F)$ is closed in Y. By Lemma 5.3 Y is pseudo-metrizable. Q.E.D.

The following is Stone's theorem.

Theorem 5.5: Let f be a closed continuous mapping of a metric space X onto a topological space Y. Then the following statements are all equivalent.

(1) Y satisfies the first countability axiom.

(2) $\mathrm{Bd}f^{-1}(y)$ is compact for each $y \in Y$.

(3) Y is metrizable.

Proof: Note that Y is in any case a T_1-space, for each $y \in Y$ is of the form $f(x)$ where $\{x\}$ is closed. Thus (2) $\rightarrow$ (3) follows from Theorem 5.4. That (3) $\rightarrow$ (1) is trivial.

Assuming that (1) is true, let $\{W_n(y): n = 1,2,\ldots\}$ be a countable basis of open neighborhoods of $y \in Y$. Let $F_y = f^{-1}(y)$. If BdF_y is not compact, there is a sequence $\{x_n: n = 1,2,\ldots\}$ of points of BdF_y having no cluster point in BdF_y; since BdF_y is closed, $\{x_n\}$ has no cluster point in X. Now Y is a T_1-space and f is continuous, so F_y is closed; consequently, $x_n \in BdF_y \subset F_y \subset f^{-1}(W_n(y))$. But $f^{-1}(W_n(y))$ is open, and hence there exists $0 < \epsilon < \frac{1}{n}$ such that $N_\epsilon(x_n) \subset f^{-1}(W_n(y))$. It follows that there exists $y_n \in X - F_y$ such that $y_n \in f^{-1}(W_n(y))$ and $d(x_n,y_n) < \frac{1}{n}$, d denoting the distance in X. Let $P = \{y_n\}$; P is closed, since the sequence $\{y_n\}$ has no cluster point in X (else the sequence $\{x_n\}$ would). Hence $Q = f(P)$ must be closed in Y. Since $y_n \notin F_y$, $y \notin Q$; yet $y \in \overline{Q}$ since $W_n(y)$ meets Q in $f(y_n)$, and this contradicts the closedness of Q. Q.E.D.

In some cases the condition that f be closed follows from the other hypotheses. This is so, for example, if X is compact and Y is Hausdorff. Another example will be given in the next theorem, also due to Stone.

Definition 5.6: A decomposition of a set X is a pairwise disjoint family $\mathcal{D}$ of non-empty subsets of X whose union is X. The projection of X onto the decomposition $\mathcal{D}$ is the function P whose value at x is the unique member of $\mathcal{D}$ to which x belongs.

Definition 5.7: A decomposition $\mathcal{D}$ of a topological space X is upper semi-continuous iff for each D in $\mathcal{D}$ and each open set U containing D there is an open set V such that $D \subset V$ and every D' meeting V is contained in U.

Using an equivalent definition, Kelley proves the following.

Lemma 5.8: A decomposition $\mathcal{D}$ of a topological space X is upper semi-continuous if and only if the projection P of X onto $\mathcal{D}$ is closed, where $\mathcal{D}$ has the quotient topology relative to P.

Proof: See [9; page 99]. Q.E.D.

Definition 5.9: A topological space X is locally peripherally compact if every point has arbitrarily small neighborhoods with compact boundaries. (Every 0-dimensional space is locally peripherally compact.)

Theorem 5.10: Let f be a quotient mapping of a locally peripherally compact metric space X onto a Hausdorff space Y such that $Bd(f^{-1}(y))$ is compact. Then f is closed, and consequently Y is metrizable.

Proof: As before, we write $F_y = f^{-1}(y)$. We will prove that the decomposition $\{F_y: y \in Y\}$ of X is upper semi-continuous. Let $y \in Y$ and let U be an open subset of X containing F_y. Since X is locally peripherally compact, there exists, for every $x \in BdF_y \subset F_y$, an open set U_x containing x such that $U_x \subset U$ and BdU_x is compact. Since BdF_y is compact, we can cover BdF_y by a finite number of open sets $U_1,\ldots,U_m$ such that $U_i \subset U$ and BdU_i is compact $(1 \leq i \leq m)$. Let $V = U_1 \cup \ldots \cup U_m \cup \text{Int } F_y$; then V is open, $F_y \subset V \subset U$, and the set BdV, being a closed subset of the compact set $BdU_1 \cup \ldots \cup BdU_m \cup BdF_y$, is compact. Let $W_n = N_{1/n}(BdF_y) \cup \text{Int } F_y$, an open set containing F_y; since $V \subset U$, it is enough to prove that, for some n, every F_q meeting W_n is contained in V.

Suppose this is false; then, for each n, we obtain a $q_n \in Y$ such that F_{q_n} meets both W_n and $X - V$. Since F_{q_n} meets W_n, either F_{q_n} contains a point of $\text{Int } F_y$ or F_{q_n} contains a point of $N_{1/n}(BdF_y)$. If $F_{q_n} = f^{-1}(q_n)$ contains a point x of $\text{Int } F_y$, then $q_n = f(x) = y$; since $q_n \neq y$ (for $F_y \subset V$), this is impossible. Thus F_{q_n} contains a point y_n of $N_{1/n}(BdF_y)$, and there exists $x_n \in BdF_y$ such that $d(y_n,x_n) < \frac{1}{n}$. Now $BdF_y \subset F_y \subset V$, BdF_y is compact, and $X - V$ is closed; it follows that there is an N such that $d(BdF_y,X-V) > \frac{1}{n}$ for all $n > N$. Thus $W_n \subset V$ for all $n > N$. Hence F_{q_n} meets both V and $X - V$ $(n > N)$; since F_{q_n} is connected, it must meet BdV, say in z_n $(n > N)$.

Since BdV and BdF_y are sequentially compact, $z_{n'} \to z \in BdV$ and $x_{n'} \to x \in BdF_y$ for a suitable subsequence of values n' of n. Then also $y_{n'} \to x$. Now $z \notin F_y$, since V contains F_y and is disjoint from BdV. Let $f(z) = q$; then $q \in Y - \{y\}$ and $z \in F_q$. Since Y is a Hausdorff space, there exist open sets B_1 and B_2 such that $q \in B_1$, $y \in B_2$, and $B_1 \cap B_2 = \emptyset$; then $M_1 = f^{-1}(B_1)$ and $M_2 = f^{-1}(B_2)$ are disjoint open sets such that $M_1 \supset F_q$ and $M_2 \supset F_y$. Since $z \in M_1$ and $x \in M_2$, there exists n' sufficiently large that $z_{n'} \in M_1$ and $y_{n'} \in M_2$; by our choice of $\{y_n\}$ and

$\{z_n\}$ it follows that $F_{q_n'}$ meets, and so is clearly contained in, both M_1 and M_2. But this contradicts the connectedness of $F_{q_n'}$.

This proves the decomposition upper semi-continuous; the mapping f is therefore closed by Lemma 5.8. Y is metrizable by Theorem 5.5. Q.E.D.

Duda (3) has proved an interesting complement to Theorem 5.5 in the case that X is also locally compact. Roughly, it states that if one "throws away" from Y all of the "bad" points (points y such that the boundary of $f^{-1}(y)$ is not compact), then the result is still metrizable. We shall use the following definitions in the proof.

Definition 5.11: A subset S of a topological space X is said to be a scattered set if every subset of S is a closed subset of X. An inverse set of a mapping f of X onto Y is any subset A of X for which $A = f^{-1}f(A)$.

Definition 5.12: Let $\{A_n\}$ be a sequence of subsets of a topological space X. Then

(1) $x \in \liminf A_n$ iff there exists $a_n \in A_n$ for all n such that $a_n \to x$.

(2) $x \in \limsup A_n$ iff there exists a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ and $a_{n_k} \in A_{n_k}$ such that $a_{n_k} \to x$.

Clearly, $\liminf A_n \subset \limsup A_n$.

Recall that a Hausdorff space X is locally compact iff each point has a neighborhood whose closure is compact. Equivalently, X is locally compact iff for each compact subset C of X and each open subset U containing C, there exists an open set V such that $\overline{V}$ is compact and $C \subset V \subset \overline{V} \subset U$. (See [4; page 238].)

Lemma 5.13: Let f be a closed continuous mapping of a locally compact metric space X onto a topological space Y. If F is the union of the point inverses which are not compact, then

(1) F is closed in X;

(2) for an arbitrary compact set K in X, only a finite number of non-compact point inverses can intersect K;

(3) any inverse set A contained in F is a closed subset of X.

Proof: Note that, as in the proof of Theorem 5.5, Y is a T_1-space. Since f is continuous, $f^{-1}(y)$ is closed in X for every $y \in Y$.

(1). If there are no compact point inverses, then $F = X$ and F is closed. Suppose that $f^{-1}(y)$ is a compact point inverse. Since X is locally compact, there exists an open set U containing $f^{-1}(y)$ such that $\overline{U}$ is compact. Since f is closed and continuous, the set $f^{-1}f(X-U)$ is closed; its complement is an open inverse set V containing $f^{-1}(y)$ and contained in U.

To prove (1), it suffices to show that $X - F$ is an open set. Let $x \in X - F$, say $x \in f^{-1}(y)$, where $f^{-1}(y)$ is compact; choose V as above. Since V is open, there exists an open set N such that $x \in N \subset V$. If $N \subset f^{-1}(y) \subset X - F$, then, clearly, $X - F$ is open. If $N \not\subset f^{-1}(y)$, let $f^{-1}(m)$ be any point inverse such that $N \cap f^{-1}(m) \neq \emptyset$. Then $f^{-1}(m) \subset V \subset U \subset \overline{U}$, since V is an inverse set and $N \subset V$; it follows that $f^{-1}(m)$ is a closed subset of the compact set $\overline{U}$ and is therefore compact. Thus N intersects only compact point inverses, so that $N \subset X - F$ and, again, $X - F$ is open.

(2). Suppose K is a compact set in X and $\{f^{-1}(y_n)\}$ is a sequence of distinct non-compact point inverses, each of which intersects K in a non-empty set. Let $x_n \in f^{-1}(y_n) \cap K$ for all n; since K is sequentially compact, the sequence $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ which converges, say to x. By replacing $\{f^{-1}(y_n)\}$ by $\{f^{-1}(y_{n_k})\}$ if necessary, we can thus assume $\liminf f^{-1}(y_n) \neq \emptyset$. Letting $L = \limsup f^{-1}(y_n)$, it follows that L is in some point inverse. For let $p \in \liminf f^{-1}(y_n)$, say $p_n \to p$ where $p_n \in f^{-1}(y_n)$ for all n; then, by the continuity of f, $y_n \to f(p)$. It is easy to show that $L \subset f^{-1}(f(p))$. By removing a term of the sequence $\{f^{-1}(y_n)\}$ if necessary, we can further assume that L is not in any one of the $f^{-1}(y_n)$.

Suppose L is a compact set. Then there is an open set W containing L such that $\overline{W}$ is compact. Suppose $f^{-1}(y_n) \subset \overline{W}$ for some n; since $f^{-1}(y_n)$ is closed in X and therefore closed in $\overline{W}$, it follows that $f^{-1}(y_n)$ is compact, a contradiction to our assumptions. Thus each $f^{-1}(y_n)$ has at least one point x_n not in $\overline{W}$. If $H = \bigcup_{n=1}^{\infty} x_n$ is not closed, there exists a sequence $\{x_{n_k}\}$ in H which converges to a point $x \in X - H$. Since $x_{n_k} \in f^{-1}(y_{n_k})$, $x \in L \subset W$. Now W is open, consequently there exists a neighborhood of x containing only points of W and hence not containing any x_{n_k} (since $x_{n_k} \notin \overline{W}$), a contradiction to $x_{n_k} \to x$. Thus $\bigcup_{n=1}^{\infty} x_n$

is a closed set. Since f is closed and continuous, it follows that $f^{-1}f(\bigcup_{n=1}^{\infty} x_n) = \bigcup_{n=1}^{\infty} f^{-1}(y_n)$ is closed. On the other hand, $\bigcup_{n=1}^{\infty} f^{-1}(y_n)$ is not closed. For let $a \in \lim\inf f^{-1}(y_n) \subset L$; then there exists $a_n \in f^{-1}(y_n)$ for all n such that $a_n \to a$. But since L is in some point inverse and L is not in any one of the $f^{-1}(y_n)$, $a \notin \bigcup_{n=1}^{\infty} f^{-1}(y_n)$; consequently $\bigcup_{n=1}^{\infty} f^{-1}(y_n)$ is not closed. Thus L is not compact.

If L is not compact, it contains an infinite sequence of distinct points z_n such that $\lim\sup z_n = \emptyset$. There is an $f^{-1}(y_{n_1})$ and a point x_1 of $f^{-1}(y_{n_1})$ such that $d(z_1,x_1) < 1$, an $f^{-1}(y_{n_2})$ and a point x_2 of $f^{-1}(y_{n_2})$ such that $d(z_2,x_2) < \frac{1}{2}$, and, in general, an $f^{-1}(y_{n_k})$ and a point x_k of $f^{-1}(y_{n_k})$ such that $d(z_k,x_k) < \frac{1}{k}$. The sequence $\{x_k\}$ must have $\lim\sup x_k = \emptyset$, otherwise $\lim\sup z_k \neq \emptyset$. Thus $\bigcup_{k=1}^{\infty} x_k$ is a closed set; as before, it follows that $\bigcup_{k=1}^{\infty} f^{-1}(y_{n_k})$ is closed. On the other hand, $\bigcup_{k=1}^{\infty} f^{-1}(y_{n_k})$ is not closed. For let $a \in \lim\inf f^{-1}(y_n) \subset L$; then there exists $a_n \in f^{-1}(y_n)$ for all n such that $a_n \to a$. In particular, there exists $a_{n_k} \in f^{-1}(y_{n_k})$ such that $a_{n_k} \to a$. But, as before, $a \notin \bigcup_{k=1}^{\infty} f^{-1}(y_{n_k})$.

Thus L is neither compact nor non-compact. This contradiction establishes (2).

(3). Let A be an inverse set contained in F. If A is not closed, then there is a sequence $\{x_n\}$ of distinct points of A converging to a point x not in A. Fix the integer i; then only finitely many x_n are in $f^{-1}f(x_i)$. For suppose $x_{n_k} \in f^{-1}f(x_i)$, $k = 1,2,\ldots$. Since $x_i \in A$, we have $f^{-1}f(x_i) \subset f^{-1}f(A) = A$. But $x_n \to x$ and consequently $x_{n_k} \to x \notin A$; this contradicts the fact that $f^{-1}f(x_i)$ is closed. Thus, for any fixed i, there exist only finitely many x_n such that $f(x_n) = f(x_i)$. Choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ as follows:

let $x_{n_1} = x_1$,

let n_2 be the smallest integer such that $f(x_{n_2}) \neq f(x_1)$,

$\vdots$

let n_k be the smallest integer such that $f(x_{n_k}) \neq f(x_{n_j})$,

$$j = 1,2,\ldots,k-1.$$

$\vdots$

Then $x_{n_k} \to x$ and $f(x_{n_i}) \neq f(x_{n_j})$ for $i \neq j$. Replacing the sequence $\{x_n\}$ by the sequence $\{x_{n_k}\}$ if necessary, we can thus assume that $f^{-1}f(x_i) \cap f^{-1}f(x_j) = \emptyset$ if $i \neq j$.

Now $x_n \to x$ yields that the set $K = \{x_n: n = 1,2,\ldots\} \cup \{x\}$ is compact. Since $f^{-1}f(x_n) \subset A \subset F$ for all n, the definition of the set F implies that each $f^{-1}f(x_n)$ is non-compact. But $f^{-1}f(x_n)$ meets K in x_n. Thus an infinite number of distinct non-compact point inverses meet K, contradicting (2). Therefore A is closed. Q.E.D.

Theorem 5.14: Let f be a closed continuous mapping of a locally compact metric space X onto a topological space Y. If S is the set of all y in Y such that the boundary of $f^{-1}(y)$ is not compact, then S is a scattered set and $Y - S$ is a locally compact metric space. Moreover, if X is separable, then S is countable.

Proof: Let B be any subset of S. Then $f^{-1}(B) \subset F$, where F is as defined in the preceding lemma. For let $x \in f^{-1}(B)$; then $f(x) \in B \subset S$, and, consequently, $\mathrm{Bd}f^{-1}f(x)$ is not compact. Since $f^{-1}f(x)$ is closed, $\mathrm{Bd}f^{-1}f(x) \subset f^{-1}f(x)$. It follows that $f^{-1}f(x)$ is not compact and, thus, that $x \in f^{-1}f(x) \subset F$. Then by (3) of Lemma 5.13 we have that $f^{-1}(B)$ is closed in X; since f is closed, $ff^{-1}(B) = B$ is closed in Y. Therefore S is a scattered set.

The set $Y - S$ is the continuous image of $X - f^{-1}(S)$ under f and the mapping f restricted to $X - f^{-1}(S) = f^{-1}(Y-S)$ is a closed mapping of $X - f^{-1}(S)$ onto $Y - S$, since the restriction of a closed mapping to an inverse set is closed. By (2) $\rightarrow$ (3) of Theorem 5.5, we can say that $Y - S$ is metrizable.

To show $Y - S$ is locally compact, let $y \in Y - S$ and suppose that $\mathrm{Bd}f^{-1}(y) \neq \emptyset$. Since X is locally compact and $\mathrm{Bd}f^{-1}(y)$ is compact, there exists an open set W containing $\mathrm{Bd}f^{-1}(y)$ such that $\overline{W}$ is compact. Since $\emptyset \neq \mathrm{Bd}f^{-1}(y) \subset f^{-1}(y)$, we have that $y \in f(W)$. If y is not interior to $f(W)$, then there exists a sequence of points y_n of $(Y-S) - f(W)$ converging to y. Note that $y \neq y_n$ for all n. If $\limsup f^{-1}(y_n) = \emptyset$, then for every sequence $\{x_n\}$ such that $x_n \in f^{-1}(y_n)$, no subsequence of $\{x_n\}$ converges; thus the set $A = \{x_n : n = 1,2,\ldots\}$ is closed. But $f(A) = \{y_n : n = 1,2,\ldots\}$ is not closed, since $y \in \overline{f(A)} - f(A)$. This contradicts that f is a closed map. Hence $\limsup f^{-1}(y_n) \neq \emptyset$. Furthermore, $\limsup f^{-1}(y_n) \subset \mathrm{Bd}f^{-1}(y) = f^{-1}(y) \cap \overline{X - f^{-1}(y)}$. For let $x \in \limsup f^{-1}(y_n)$, say $a_{n_k} \to x$ with $a_{n_k} \in f^{-1}(y_{n_k})$. Then, by the continuity of f, $y_{n_k} \to f(x)$; but also $y_{n_k} \to y$, so that $f(x) = y$ and $x \in f^{-1}(y)$. Since $y \neq y_n$ for all n, we have that $f^{-1}(y_n) \subset X - f^{-1}(y)$ for every n; consequently, $a_{n_k} \in X - f^{-1}(y)$ for every k. It follows that $x \in \overline{X - f^{-1}(y)}$. Hence $\limsup f^{-1}(y_n) \subset \mathrm{Bd}f^{-1}(y)$.

Now infinitely many of the $f^{-1}(y_n)$ meet W. For, let $p \in \limsup f^{-1}(y_n)$; then there exists $p_{n_k} \in f^{-1}(y_{n_k})$ such that $p_{n_k} \to p$. Since $\limsup f^{-1}(y_n) \subset \mathrm{Bd}f^{-1}(y) \subset W$ and since W is open, W contains infinitely many of the p_{n_k}. It follows that infinitely many of the y_n are in $f(W)$. This gives a contradiction, hence y is interior to $f(W)$.

Since f is both closed and continuous and $\overline{W}$ is compact, $f(\overline{W})$ is closed and compact. Therefore we have that $y \in \mathrm{Int}\, f(W) \subset \overline{\mathrm{Int}\, f(W)} \subset \overline{f(W)} \subset \overline{f(\overline{W})} = f(\overline{W})$, and consequently $\overline{\mathrm{Int}\, f(W)}$ is compact. Thus $\mathrm{Int}\, f(W)$ is a neighborhood of y whose closure is compact.

If $\mathrm{Bd}f^{-1}(y) = \emptyset$, then $f^{-1}(y)$ is an open set. Now, by Theorem 1.2, f is a quotient map; consequently, $\{y\}$

is open. Since $Y - S$ is a T_1-space, $\overline{\{y\}} = \{y\}$. Thus $\{y\}$ is a neighborhood of y whose closure is compact. Thus $Y - S$ is locally compact.

If X is separable as well as metric, X is Lindelöf. Letting $\{V(x): x \in X\}$ be a covering of X by open sets whose closures are compact, we can extract a countable sub-covering of X, so that X is a countable union of compact sets. By (2) of Lemma 5.13 only a finite number of point inverses of points in S can meet any one of these compact sets. Thus $f^{-1}(S)$ is a countable union of single point inverses. Therefore S is countable. Q.E.D.

CHAPTER VI. OPEN QUOTIENT MAPS

In this chapter we prove a theorem of Stone concerning open continuous mappings. A space S is called locally separable if every point $x \in S$ has a separable neighborhood.

Theorem 6.1: If f is an open continuous mapping of a locally separable metric space S onto a regular Hausdorff space E, and if for each $p \in E$ the set $f^{-1}(p)$ is separable, then E is metrizable and locally separable.

In our proof we use the following theorem of Alexandroff; the proof that we give is due to David Kullman.

Theorem 6.2: If X is a locally separable metric space, then X can be expressed as a union of pairwise disjoint, open, separable subspaces.

Proof: Since an open subspace of a separable space is separable, we can assume without loss of generality that each $x \in X$ has an open separable neighborhood. Let $\mathcal{U}$ be a cover of X by open separable sets. Since X is paracompact, $\mathcal{U}$ has a precise, open, locally finite refinement, say $\mathcal{V} = \{V_U: U \in \mathcal{U}\}$. (See [4; pages 161-2].)

We first assert that each $V_U \in \mathcal{V}$ meets at most countably many $V_{U'} \in \mathcal{V}$. Fix V_U; since V_U is an open subset of U, V_U is separable, metrizable, and hence Lindelöf. Also, since $\mathcal{V}$ is locally finite, each point $x \in V_U$ has a neighborhood $N(x)$ which meets only finitely many $V_{U'}$. Thus V_U has a countable subcover

$N(x_1), N(x_2), \ldots,$ and, since each $N(x_i)$ meets only finitely many $V_{U'}$, we have that $V_U \subset \bigcup_{i=1}^{\infty} N(x_i)$ meets at most countable many $V_{U'}$.

Now define a relation R on X by xRy iff there exist $U_1, \ldots, U_n \in \mathcal{U}$ such that $x \in V_{U_1}$, $y \in V_{U_n}$, and $V_{U_i} \cap V_{U_{i+1}} \neq \emptyset$ for $i = 1,2,\ldots,n-1$. Clearly, R is an equivalence relation, so let $\{X_\alpha \colon \alpha \in \mathcal{A}\}$ be the equivalence classes defined by R. The X_α's are pairwise disjoint. To show that X_α is open, let $x \in X_\alpha$ and $x \in V_U$. Then $V_U \subset X_\alpha$, so that X_α can be expressed as a union of open sets V_U.

We next show that the union of a countable, locally finite family of separable sets is separable. For, let $\{U_n \colon n = 1,2,\ldots\}$ be a countable, locally finite family of separable sets. Then each U_n contains a countable dense subset D_n. The family of sets $\{D_n \colon n = 1,2,\ldots\}$ is also locally finite. Therefore $\overline{\bigcup_n D_n} = \bigcup_n \overline{D_n} = \bigcup_n U_n$, so $\bigcup_n D_n$ is dense in $\bigcup_n U_n$. Also $\bigcup_n D_n$ is countable, so $\bigcup_n U_n$ is separable.

Finally we show that each X_α is separable. Fix X_α and $V_U \subset X_\alpha$. Let

$$
\begin{aligned}
G_1 &= \bigcup\{V_{U'} \in \mathcal{V} \colon V_{U'} \cap V_U \neq \emptyset\} \\
&= \{x \in X \colon xRz, \text{ for some } z \in V_U, \text{ by a "chain" of at most 2 elements}\} \\
G_2 &= \bigcup\{V_{U'} \in \mathcal{V} \colon V_{U'} \cap G_1 \neq \emptyset\} \\
&= \{x \in X \colon xRz, \text{ for some } z \in V_{U'}, \text{ by a "chain" of at most 3 elements}\} \\
&\ \ \vdots
\end{aligned}
$$

Clearly, $X_\alpha = G_1 \cup G_2 \cup \ldots$. By our first assertion, each G_i is a countable union of sets $V_{U'} \in \mathcal{V}$, so X_α is also a countable union of sets $V_{U'}$. But these $V_{U'}$ make up a locally finite family of separable sets, so X_α is separable. Q.E.D.

Lemma 6.3: Theorem 6.1 is true if S is separable, and E is then separable.

Proof: If S is a separable metric space, then S is second countable. If $\{U_n: n = 1,2,\ldots\}$ is a countable basis for S, then $\{f(U_n): n = 1,2,\ldots\}$ is a countable basis for E. For, let G be an open subset of E. Since f is continuous, $f^{-1}(G)$ is open in S; thus $f^{-1}(G) = \bigcup_{i=1}^{\infty} U_{m_i}$ and $G = \bigcup_{i=1}^{\infty} f(U_{m_i})$. Also, each $f(U_n)$ is open, since f is an open map. That is, E is second countable. The Lemma follows by Urysohn's metrization theorem. Q.E.D.

Lemma 6.4: If $f(S) = E$, where f is open and each $f^{-1}(p)$ is separable, then for every separable subset Y of E, $f^{-1}(Y)$ is separable.

Proof: Let $Q = \{q_m: m = 1,2,\ldots\}$ be a countable subset of Y such that $\bar{Q} \supset Y$, and for each m let X_m be a countable dense subset of $f^{-1}(q_m)$. Write $X = \cup X_m$. X is countable and $\bar{X} \supset f^{-1}(Y)$. For, let $z \in f^{-1}(Y)$ and let U be an open neighborhood of z. Then $f(U)$ is an open neighborhood of $f(z) \in Y \cup \bar{Q}$. Consequently, $f(U)$ contains some q_m, say $q_m = f(x)$, where $x \in U$. Then $x \in f^{-1}(q_m) \subset \overline{X_m}$, and therefore $U \cap X_m \neq \emptyset$. Thus

$U \cap X \neq \emptyset$ and $z \in \bar{X}$. Q.E.D.

Proof of Theorem 6.1: By Theorem 6.2, S can be expressed as a union of pairwise disjoint nonempty open sets S_λ, each of which is separable. Write $S_\lambda \sim S_\mu$ to mean that there exists a finite sequence $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_k = \mu$ such that each set $f(S_{\lambda_{i-1}})$ meets $f(S_{\lambda_i})$, $i = 1,2,\ldots,k$. It is easily verified that $\sim$ is an equivalence relation. Let the union of the S_μ's equivalent to S_λ be T_λ; thus T_λ is open, and T_λ and $T_{\lambda'}$ are either identical or disjoint. Further, T_λ is an inverse set. Since $f^{-1}(f(T_\lambda)) \supset T_\lambda$, it is sufficient to show that whenever $f^{-1}(p)$ meets T_λ, we have $f^{-1}(p) \subset T_\lambda$. So suppose $f^{-1}(p)$ meets T_λ and let $y \in f^{-1}(p) \cap S_\mu$, where $S_\mu \sim S_\lambda$. Note that $p \in f(S_\mu)$. Let $x \in f^{-1}(p)$; then $x \in S_\nu$ for some ν and hence $p \in f(S_\nu)$. Consequently, $f(S_\mu)$ meets $f(S_\nu)$ and thus $S_\nu \sim S_\mu$. Therefore $S_\nu \sim S_\lambda$ and $x \in T_\lambda$.

It follows that the distinct sets $f(T_\lambda)$ are disjoint and open, and they cover E. To prove the theorem, it will suffice to prove that each $f(T_\lambda)$ is separable metric. (See Lemma 1.6.) By Lemma 6.3 it suffices to prove that each T_λ is separable.

Now let T_λ^n denote the union of those sets S_μ which can be reached from S_λ in at most n steps -- that is, for which there is a sequence $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_k = \mu$, of the type used to define $\sim$, with $k \leq n$. Clearly, $T_\lambda = T_\lambda^0 \cup T_\lambda^1 \cup \ldots$, and it is enough to prove that each T_λ^n is separable. Suppose this is true for one particular value of n. Then T_λ^{n+1} consists of T_λ^n together with those sets S_μ for which $f(S_\mu)$ meets $f(T_\lambda^n)$ -- that is, for which S_μ meets $f^{-1}(f(T_\lambda^n))$. There are only countably many such sets S_μ. For, by hypothesis, T_λ^n is separable; hence, by the continuity of f, so is $f(T_\lambda^n)$, and Lemma 6.4 now shows that $f^{-1}(f(T_\lambda^n))$ is also separable. It follows that $f^{-1}(f(T_\lambda^n))$ can meet at most countably many of the disjoint open sets S_μ. (See [14; page 115].) Thus T_λ^{n+1} is a union of countably many separable sets, and is again separable. Since $T_\lambda^0 = S_\lambda$ and is separable, the separability of T_λ^n follows for all n. Q.E.D.

CHAPTER VII. OPEN AND CLOSED QUOTIENT MAPS

Lemma 7.1: An open continuous image $Y = f(X)$ of a first countable space X is first countable.

Proof: Let $y \in Y$ where $y = f(x)$. Since X is first countable, there is a countable basis of open neighborhoods $\{U_n: n = 1,2,\ldots\}$ for x; since f is open, $V_n = f(U_n)$ is open. Then $\{V_n: n = 1,2,\ldots\}$ is a countable basis of open neighborhoods of y. For, if G is any open neighborhood of y, $f^{-1}(G)$ is an open neighborhood of x (by continuity of f) and therefore contains a U_n; thus $G = ff^{-1}(G) \supset f(U_n) = V_n$. Q.E.D.

By combining Theorem 5.5 with Lemma 7.1 we obtain immediately that an open, closed, continuous image of a metric space is metrizable. This result was originally proved by Balanchandran (2), who, however, gave an explicit metric for the image space. In this chapter we shall give his proof of the aforementioned result. We first prove two lemmas and give a definition.

Lemma 7.2: Let f be an open map of a first countable space X onto a topological space Y. Then, if $y_n \to y$ in Y and $x \in f^{-1}(y)$, we can choose a point $x_n \in f^{-1}(y_n)$ $(n = 1,2,\ldots)$ such that $x_n \to x$ in X.

Proof: Let $H_1 \supset H_2 \supset \ldots$ be a decreasing basis of open neighborhoods of x. If any $H_j (j = 1,2,\ldots)$ be disjoint with an infinite number of $f^{-1}(y_n)$, say $f^{-1}(y_{n_i})$ $(i = 1,2,\ldots)$, then

$y_{n_i} \notin f(H_j)$ $(i = 1,2,\ldots)$. Since f is open and H_j is an open neighborhood of x, $f(H_j)$ is an open neighborhood of y and hence contains all the y_{n_i} for sufficiently large i (since $y_{n_i} \to y$). This contradiction shows that every H_j must meet all $f^{-1}(y_n)$ from a certain stage onwards. So for each j there is a least integer n_j such that $H_j \cap f^{-1}(y_n) \neq \emptyset$ for $n \geq n_j$.

We shall now show how to choose the sequence $\{x_n\}$. For each integer $n < n_1$, choose as x_n any point in $f^{-1}(y_n)$; for $n = n_1$, choose as x_n any point in $H_1 \cap f^{-1}(y_{n_1})$. Next for arbitrary $n > n_1$, choose as x_n any point in $H_m \cap f^{-1}(y_n)$, where the integer m (depending on n) is determined as follows: set $m = n$ if $H_n \cap f^{-1}(y_n) \neq \emptyset$; otherwise set $m = k$, the largest integer $< n$ such that $H_k \cap f^{-1}(y_n) \neq \emptyset$ (then $n_{k+1} > n \geq n_k$). Thereby we obtain a sequence $\{x_n\}$ which is such that (each) H_j contains all x_n for $n \geq \max(j,n_j)$. It follows that $x_n \to x$. Q.E.D.

<u>Definition 7.3</u>: If X,Y are non-null closed subsets of a metric space R with distance function d, the Hausdorff distance $\rho(X,Y)$ is defined to be the greater of the two numbers

$$\sup_{x \in X} d(x,Y) \qquad \text{and} \qquad \sup_{y \in Y} d(y,X)$$

where $d(x,Y) = \inf_{y \in Y} d(x,y)$.

<u>Lemma 7.4</u>: If $\rho(X,X_n) \to 0$, then for each $x \in X$ we can choose a sequence $\{x_n\}$ such that $x_n \in X_n$ for all n and such that $x_n \to x$.

Proof: Let $\epsilon > 0$ be fixed. By definition of $d(x,X_n)$ there exists $x_n \in X_n$ with $d(x,x_n) < d(x,X_n) + \frac{\epsilon}{n}$ for any n. Since $d(x,X_n) \leq \rho(X,X_n)$ and $\rho(X,X_n) \to 0$, it follows that $d(x,x_n) \to 0$. Q.E.D.

The metric space R is said to be totally bounded if for every positive ϵ, the open cover of R by all spheres of radius ϵ includes a finite subcovering. It can be shown that R is totally bounded iff every sequence of its points contains a Cauchy subsequence. (See [15; page 236].)

Theorem 7.5: Let f be an open, closed, and continuous map of a metric space R onto a topological space R^*. Then R^* is a metric space with the metric $d^*(x^*,y^*) = \rho(f^{-1}(x^*),f^{-1}(y^*))$ for all x^*,y^* in R^*.

Proof: By Lemma 7.1 R^* is first countable. Since R is T_1 and f is closed, R^* is T_1. Let d be the metric of R. If d is not bounded, we can define an equivalent metric σ for R by $\sigma(x,y) = \min\{1,d(x,y)\}$; thus we may assume the metric d of R to be bounded. Introduce in (the set) R^* a metric d^* as follows: $d^*(x^*,y^*) = \rho(X,Y)$, where $X = f^{-1}(x^*)$, $Y = f^{-1}(y^*)$. (Since R^* is T_1 and f is continuous, X and Y are closed in R.)

We shall now show that the given topology of R^* is the same as the one induced by the metric d^*. Since R^* is first countable it suffices, in view of Lemma 1.5, to show that a sequence $x_n^* \to x^*$ in the given topology of R^* if and only if $x_n^* \to x^*$ under the metric d^*, i.e., $d^*(x_n^*, x^*) \to 0$ as $n \to \infty$.

First let $x_n^* \to x^*$ in R^*. If $d^*(x_n^*, x^*) \not\to 0$, then $d^*(x_n^*, x^*) = \rho(X_n, X) > \epsilon$ for some $\epsilon > 0$ and for infinitely many integers n, where $X_n = f^{-1}(x_n^*)$, $X = f^{-1}(x^*)$. Thus for infinitely many n either

$$\sup_{y \in X} d(y, X_n) > \epsilon$$

or

$$\sup_{x_n \in X_n} d(x_n, X) > \epsilon \quad .$$

That is, for infinitely many n either

(1) $d(y_n, X_n) > \epsilon$, $y_n \in X$, or

(2) $d(x_n, X) > \epsilon$, $x_n \in X_n$.

By replacing $\{x_n^*\}$ by a suitable subsequence (which converges also to x^*) we can suppose without loss of generality that one of these possibilities happens for all values of n.

Case I. Let $d(y_n, X_n) > \epsilon > 0$, $n = 1, 2, \ldots$.

If the set $Y = \{y_n\}$ is totally bounded, then $\{y_n\}$ contains a Cauchy subsequence $\{y_{n'}\}$ which can be so chosen that $d(y_{m'}, y_{n'}) < \frac{\epsilon}{2}$ for any m', n'. Then for any fixed m' and all n'

$$\text{(A)} \qquad \begin{aligned} d(y_{m'}, X_{n'}) &\geq d(y_{n'}, X_{n'}) - d(y_{n'}, y_{m'}) \\ &> \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2} \quad . \end{aligned}$$

But, since $\{x_{n'}^*\}$ is a subsequence of $\{x_n^*\}$ and $x_n^* \to x^*$, we have $x_{n'}^* \to x^*$. Since $y_{m'} \in X = f^{-1}(x^*)$, Lemma 7.2 shows that points

$x_{n'} \in X_{n'}$ can be so selected that $x_{n'} \to y_{m'}$ for any fixed m', a contradiction to inequality (A).

If the set $Y = \{y_n\}$ is not totally bounded, there exists some subsequence $\{y_{n'}\}$ and a $\delta > 0$ with $d(y_{m'}, y_{n'}) > 2\delta$ for any m', n' such that $m' \neq n'$. Since f is open, $x_n^* \to x^*$, and $y_{n'} \in X = f^{-1}(x^*)$ $(n = 1,2,\ldots)$, for each n' we can choose by Lemma 7.2 an n'' such that

$$d(y_{n'}, x_{n''}) < \frac{\delta}{2} \;, \qquad x_{n''} \in X_{n''} = f^{-1}(x^*_{n''}) \;.$$

Then

$$\begin{aligned} d(x_{m''}, x_{n''}) &\geq d(y_{m'}, y_{n'}) - d(y_{m'}, x_{m''}) - d(y_{n'}, x_{n''}) \\ &> 2\delta - \frac{\delta}{2} - \frac{\delta}{2} = \delta \;, \qquad m'' \neq n'' \;. \end{aligned}$$

Thus $X'' = \{x_{n''}\}$ has no limit points and hence is closed in R; since f is closed, $f(X'')$ is closed in R^*. On the other hand, note that our assumption that $d(y_{n''}, X_{n''}) > 0$ implies that $X_{n''} \neq X$ and hence that $x^*_{n''} \neq x^*$; since $f(X'') = \{x^*_{n''}\}$ and $x_n^* \to x^*$ (and so also $x^*_{n''} \to x^*$), it follows that $f(X'')$ cannot be closed, so that we again have a contradiction.

The assumption $d(y_n, X_n) > \epsilon$ is therefore untenable.

Case II. Let $d(x_n, X) > \epsilon > 0$ with $x_n \in X_n$.

If we write $X' = \{x_n\}$, then it is clear that $\overline{X'} \cap X = \emptyset$. If $f(\overline{X'}) \cap f(X) \neq \emptyset$, then $\overline{X'} \cap X = \overline{X'} \cap f^{-1}f(X) \neq \emptyset$, a contradiction; thus $f(\overline{X'}) \cap f(X) = \emptyset$. But since f is closed and continuous we have that $f(\overline{X'}) = \overline{f(X')}$ and hence $\overline{f(X')} \cap f(X) = \emptyset$. (See [4; page 87].) On the other hand,

$f(X) = x^*$ belongs to $\overline{f(X')}$ since $f(X') = \{x_n^*\}$ and $x_n^* \to x^*$. This contradiction shows that the assumption $d(x_n, X) > \epsilon$ is also untenable. Hence, $d^*(x_n^*, x^*) \to 0$.

Conversely, if $d^*(x_n^*, x^*) = \rho(X_n, X) \to 0$ and $x \in X$, then Lemma 7.4 shows that points $x_n \in X_n$ can be found such that $x_n \to x$. It follows by the continuity of f that $x_n^* \to x^*$. Q.E.D.

CHAPTER VIII. ANOTHER RESULT

We conclude by proving the following theorem, in which the mapping f can in fact be neither open nor closed.

Theorem 8.1: If f is a quotient mapping of a locally compact separable metric space S onto a first countable T_2-space E, then E is a locally compact separable metric space.

We shall need the following lemmas.

Lemma 8.2: Let f be a quotient mapping of a topological space S onto a T_2-space E with a countable local base at $p \in E$, and let $\{U_n: n = 1,2,\ldots\}$ be an increasing sequence of open subsets of S such that $\bigcup_{n=1}^{\infty} U_n \supset f^{-1}(p)$. Then, for some n, $p \in \operatorname{Int} f(U_n)$.

Proof: Let $\{W_n: n = 1,2,\ldots\}$ be a basis of neighborhoods of p; we may suppose $W_1 \supset W_2 \supset \ldots$. Without loss of generality, we may assume that each U_n meets $f^{-1}(p)$. Then $p \in f(U_n)$ for all n. We show that, for some n, $W_n \subset f(U_n)$.

Suppose not; then, for each n, there is a point $q_n \in W_n - f(U_n)$. Let $Q = \{q_n: n = 1,2,\ldots\}$ and let $X = f^{-1}(Q) = \bigcup_{n=1}^{\infty} \{f^{-1}(q_n): n = 1,2,\ldots\}$. Since $q_n \to p$, we have that $p \in \overline{Q}$. But $p \notin Q$, and, consequently, Q is not closed. Since f is a quotient map, X is not closed and there exists a point $x \in \overline{X} - X$. Then, by the continuity of f, $f(x) \in f(\overline{X}) \subset \overline{f(X)} = \overline{Q}$, while $f(x) \notin Q$, that is, $f(x) \in \overline{Q} - Q$.

Since $q_n \to p$ and E is Hausdorff, p is the only limit point of Q; consequently, $f(x) = p$. Hence $x \in f^{-1}(p)$, so $x \in U_n$ for some n. Since E is a T_1-space and f is continuous, $f^{-1}(q_m)$ is closed in S for every m; consequently, $\cup\{f^{-1}(q_m): 1 \leq m \leq n\}$ is closed. Then, since U_n is open in S, $N = U_n - \cup\{f^{-1}(q_m): 1 \leq m \leq n\}$ is an open set containing x. Since $\{U_n\}$ is an increasing sequence, the sequence $\{f(U_n)\}$ is also increasing; consequently, $q_{n+i} \notin f(U_n)$ and $f^{-1}(q_{n+i}) \cap U_n = \emptyset$ for all $i \geq 1$. It follows that N is disjoint from X, contradicting $x \in \overline{X}$. Q.E.D.

A set is σ-compact if it is the union of countably many compact sets.

Lemma 8.3: If f is a quotient mapping of a locally compact topological space S onto a first countable Hausdorff space E and if $\mathrm{Bd}f^{-1}(p)$ is σ-compact for each $p \in E$, then E is locally compact.

Proof: By hypothesis, given $p \in E$, we can write $\mathrm{Bd}f^{-1}(p) = \bigcup_{n=1}^{\infty} K_n$, where K_n is compact. Since S is locally compact, K_n can be covered by finitely many open sets with compact closures; in this way we obtain open sets $G_n \supset K_n$ such that $\overline{G_n}$ is compact and $G_1 \subset G_2 \subset \ldots$. Applying Lemma 8.2 to $U_n = G_n \cup \mathrm{Int}\, f^{-1}(p)$, we have $p \in \mathrm{Int}\, f(U_n)$ for some n. If $\mathrm{Bd}f^{-1}(p) \neq \emptyset$, then $K_r \neq \emptyset$ for some r; choose $m = \max\{n,r\}$. Then $G_m \supset G_r \supset K_r$ and, hence, G_m meets $\mathrm{Bd}f^{-1}(p)$; since

$\mathrm{Bdf}^{-1}(p) \subset f^{-1}(p)$, G_m actually meets $f^{-1}(p)$. Thus $p \in f(G_m)$, and so $f(U_m) = f(G_m)$. Then $p \in \mathrm{Int}\, f(U_n) \subset \mathrm{Int}\, f(U_m) = \mathrm{Int}\, f(G_m) \subset f(\overline{G_m})$, which is compact and closed in the Hausdorff space E. That is, $f(\overline{G_m})$ is a neighborhood of p whose closure is compact.

If $\mathrm{Bdf}^{-1}(p) = \emptyset$, then, as in Theorem 5.14, $\{p\}$ is a compact neighborhood of p. Q.E.D.

Lemma 8.4: If f is a quotient mapping of a second countable space S onto a first countable Hausdorff space E, then E is second countable.

Proof: Let $\mathcal{B} = \{B_m : m = 1,2,\ldots\}$ be a countable base of open sets of S; we prove E has a base of the form $\{\mathrm{Int}\, f(U)\}$ where U is a finite union of sets of $\mathcal{B}$. Given any $p \in E$ and any open set G containing p, we have $f^{-1}(p) \subset f^{-1}(G)$. Since f is continuous, $f^{-1}(G)$ is open and consequently can be written as the union of members of $\mathcal{B}$. Thus $f^{-1}(p)$ can be covered by a sequence of sets $B_{m_1}, B_{m_2}, \ldots$, of $\mathcal{B}$, all satisfying $B_{m_i} \subset f^{-1}(G)$. Write $U_n = B_{m_1} \cup \ldots \cup B_{m_n}$; by Lemma 8.2 we have $p \in \mathrm{Int}\, f(U_n)$ for some n, where $f(U_n) \subset G$. Q.E.D.

Lemma 8.5: A subspace A of a locally compact space S is locally compact if it is of the form $V \cap F$, where V is open in S and F is closed in S.

Proof: Given $a \in A$, choose a set U open in S such that $\overline{U}$ is compact and $a \in U \subset \overline{U} \subset V$. Then $U \cap A$ is a neighborhood of a in A. The closure of this neighborhood in A is

$\bar{U} \cap A = \bar{U} \cap (V \cap F) = \bar{U} \cap F$, which is a set closed in $\bar{U}$, and consequently is compact. Q.E.D.

Lemma 8.6: If S is second countable, then every subspace of S is separable.

Proof: Let M be a subset of S and let $\{U_i : i = 1,2,\ldots\}$ be a countable basis for S. Then $\{M \cap U_i : i = 1,2,\ldots\}$ is a countable basis for M. Choose a $y_i \in M \cap U_i$ for each i. The set $\{y_i : i = 1,2,\ldots\}$ is dense in M since each set open in M is a union of the $M \cap U_i$. Q.E.D.

Proof of Theorem 8.1: Since S is separable metric, S is second countable; by Lemma 8.6 each $\mathrm{Bd}f^{-1}(p)$ is separable metric. Since S is locally compact and $\mathrm{Bd}f^{-1}(p) = \mathrm{Bd}f^{-1}(p) \cap S$ is the intersection of a closed set and an open set, each $\mathrm{Bd}f^{-1}(p)$ is a locally compact space (Lemma 8.5). Then each $\mathrm{Bd}f^{-1}(p)$ is a Lindelöf locally compact space and consequently is σ-compact. By Lemma 8.3 E is a locally compact T_2-space, and therefore E is regular. Lemma 8.4 shows that E is second countable. It follows by Urysohn's metrization theorem that E is metrizable. Q.E.D.

As another immediate consequence of Lemma 8.4 we have the following corollary.

Corollary 8.7: If f is a quotient mapping of a separable metric space S onto a regular Hausdorff space E satisfying the first axiom of countability, then E is separable metric.

Proof: Apply Lemma 8.4 and Urysohn's metrization theorem. Q.E.D.

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