Bifurcation Analysis of Zellner's Marshallian Macroeconomic Model

By

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Abstract:

The Marshallian Macroeconomic Model (MMM) developed by Veloce and Zellner (1985) provides a novel way to study sectoral dynamics of an economy in the presence of a dynamic entry/exit equation. Later extended by Zellner and Israilevich (2005) to include interactions between households, production firms and the government, this model exhibits very interesting dynamical behavior of key economic variables such as the sales, number of firms and prices at the aggregate as well the disaggregated level. Zellner and Israilevich (2005) show that such dynamical behavior can range from smooth convergence or damped oscillatory convergence to equilibrium to "booms and busts" typical of chaotic systems depending on the choice of parameter values. Under these observations we have undertaken the task of examining more closely the change in the qualitative properties of the long-run equilibrium in a special nested case of the two sector MMM under variation of parameter values. We show the possibility of stable solutions and an oscillatory convergence to the long-run equilibrium and are able to offer a plausible explanation of such behavior based on price and income elasticity parameters. Additionally we detect the presence of codim-1 Hopf bifurcations in this model when we vary either the sector one entry/exit parameter or the tax rate.

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Contents

1 Introduction **1**

1.1 History and development of the MMM 3

1.1.1 The SEMTSA approach 3

1.1.2 Advantages of disaggregated data 5

1.1.3 Entry and exit in competitive models 6

2 Virtues of Continuous Time Econometric models **14**

3 Non-linear Dynamics and Bifurcation Analysis in Economics **18**

3.1 Non-linear Dynamics 18

3.2 Bifurcation in Economic Models 20

3.3 Some Concepts and Definitions 25

4 The Marshallian Macroeconomic Model **33**

4.1 The One Sector Marshallian Macro Model without the Government and Money in Continuous time 33

4.2 The n-Sector Marshallian Macroeconomic Model in Continuous Time 42

4.2.1 The Firm 43

4.2.2 Output Demand 45
4.2.3 Factor Markets 46
4.2.4 Government 51
4.2.5 Price Aggregates and Sales Aggregate 52

5 A Special Nested Case of the MMM 53

5.1 The Model 53
5.1.1 The Firm 54
5.1.2 Output Demand 55
5.1.3 Factor Markets 56
5.1.4 Government 58
5.1.5 Sales and Price Aggregates 60
5.2 Solving the model 61

6 Stability and Dynamics 62

7 Bifurcation Analysis of MMM 66

7.1 Calibration 66
7.2 Bifurcation Analysis 67
7.2.1 Hopf Bifurcation with Sector One Entry/Exit Parameter 69
7.2.2 Hopf Bifurcation with Income Tax Rate 72

8 Conclusion and Future Research 75

Appendices 77

A Deriving the Dynamic Equations 77

B Calibration Parameters for Bifurcation with F_1 **82**

C Calibration Parameters for Bifurcation with T^s **83**

Chapter 1

Introduction

In my dissertation, I propose to provide a close examination of the dynamics in the Marshallian Macroeconomic Model (MMM) initiated by Veloce and Zellner (1985) in a one sector model and later extended by Zellner and Israilevich (2005) to a multi-sector model with the government and monetary sector (see Ngoie and Zellner (2008) for an application of the multi-sector model to evaluate policy experiments in South Africa). The advantage of MMM is that it is a convenient way to incorporate information from disaggregation of different sectors. For instance, explicit modeling of entry and exit behavior of firms within sectors can explain the dynamics of output and prices within sectors and their effects on aggregate output and prices. In econometric models that consider only aggregate data such rich information is typically lost. In the above mentioned models, the authors describe how the dynamics of the solution can be affected by the parameters of the model and furthermore how a discrete approximation to the solution could exhibit even chaotic behavior. In my study of these models, I intend to examine this dependence of the solution dynamics on model parameters and identify theoretically feasible parameter regions that could lead to bifurcations within the model.

This thesis is divided into the following chapters. Chapter 1 traces the history and development of the MMM with an emphasis on the role of the SEMTSA approach, the role of disaggregation and the role of entry/exit equations in competitive models. This chapter also contains a brief review of literature on models of entry/exit behavior of firms. This is followed by a discussion on the advantages of using continuous time econometric models as opposed to discrete time in Chapter 2. A survey of continuous time macroeconometric models is also provided in this chapter. Following this, the development and use non-linear dynamics and bifurcation analysis in economics is discussed in Chapter 3. Chapter 4 discuss the one-sector MMM and the n-sector generalized version of MMM. In Chapter 5, I present the special nested case of the MMM in continuous time along with an outline of the solution procedure for obtaining the long-run equilibrium of this model. Section 6 draws a comparison between the one sector MMM and the special nested case we consider in terms of stability and nature of disequilibrium dynamics. Chapter 7 includes a discussion of the calibrated parameter values that we have chosen for our bifurcation analysis and presents the main results of this thesis. Chapter 8 concludes with directions for future research.

1.1 History and development of the MMM

1.1.1 The SEMTSA approach

Before we delve deep into the model and its dynamics it is imperative that we trace the origin, the cause and sequence of development of such a model which has been shown to do very well with respect to econometric estimation, forecasting precision and being rationalizable in terms of existing economic theory. (See Zellner and Chen (2001) for an implementation of the MMM for forecasting output growth rates in 11 US industrial sectors.)

One of most important reasons for the use of the MMM can be traced back to the seminal works of Zellner and Palm (1974, 1975, 2004), Palm (1976, 1977, 1983) and Zellner (1997, 2004) along with contributions from Garcia-Ferrer, Highfield, Palm and Zellner (1987), Hong (1989) and Min (1992), in the development and application of the SEMTSA (Structural Econometric Modeling, Time-Series Analysis) approach for the purpose of econometric model building and checking existing dynamic econometric models. The superiority of such an approach when compared with large statistical models like the VAR and other popular large multi-equation stochastic structural models lies in the fact that theoretical models can be used to derive and thus justify the use of empirical models based on SEMTSA. Thus such an approach to econometric modeling provides the much needed tools that help in analyzing and investigating empirical macroeconomic phenomena using "...a structural or causal model rather than just an empirical, statistical forecasting model that does not explain outcomes or possible causal relations very well..." as Zellner states in Zellner (2002).

The starting point of the SEMTSA approach involves the use of simple models in the form of dynamic equations for individual variables. Once such a model is formulated, it is tested with past data and forecasting exercises using the model is undertaken. If the model responds reasonably well to these tests, then such variables are used to form a multivariate model which in turn is again put to test in terms of how well it does with respect to explaining variations in past data and forecasting. This process is followed by a continuous effort to improve the performance of the model.

The first variable chosen as a candidate for SEMTSA was growth rate of real GDP for nine industrialized countries by Zellner and his co-authors in Garcia-Ferrer, Highfield, Palm and Zellner (1987). They observed that with the use of an autoregressive leading indicator (ARLI) model and later an autoregressive leading indicator world income (ARLI/WI) model for eighteen industrialized countries combined with the use of the Bayesian shrinkage technique (see Zellner (1997), Zellner and Min (1999) and Zellner and Palm (2004)) produced improved forecast precision and a remarkable success in forecasting 70% of the 211 unique turning points.

The next step in the SEMTSA approach was to rationalize these ARLI and ARLI/WI forecasting equations for eighteen industrialized countries in terms of economic theory. In fact several researchers were able to derive these empirical forecasting relations from theoretical models. For instance, Zellner (2000) was able to derive them from an aggregate demand and supply model, Hong (1989) from a Hicksian IS-LM macroeconomic model and Min (1992) from generalized real business cycle model. However it was observed that the root mean square error of the model forecasts of annual real GDP growth rates from these SEMTSA models were still high. This could be due to the fact that the theoretical models justifying the ARLI and ARLI/WI given above do not consider important phenomena like industrial sectors exhibiting different cyclical properties, entry and exit of firms, sector linkages, etc. Thus in order to improve the accuracy of forecasts Zellner proposed the use of sector-wise disaggregated data and the use of a Marshallian competitive model for each sector.

1.1.2 Advantages of disaggregated data

In all such works involving the MMM we notice the role of sectoral disaggregation which automatically brings the following question to mind. How beneficial is disaggregation and how does one judge the improvement in forecasting abilities of these models? Zellner and Israilevich (2005) summarize some of the advantages of considering disaggregated data. Firstly, different sectors exhibit different seasonal, cyclical and trend behaviors and predicting such behavior is itself of interest. In addition each sector is subject to both sector-specific as well as aggregate variable effects. Furthermore, given that sector specific relations have errors that have differing variances that are correlated across sectors, it is possible to combine joint estimation and prediction methods with Stein-like shrinkage techniques to improve sectoral and aggregate parameter estimates and prediction precision.

Furthermore, Zellner and Tobias (2000), showed that the use of disaggregated data in their ARLI model definitely helped in terms of better forecasts given that their model exhibited lower root mean squared error (RMSE) and mean average error (MAE). Also disaggregation provides the modeler with greater number of observations which when coupled with a good model specification can yield better forecasts. In Ngoie and Zellner (2008), the authors point out that in the presence of output growth rate differentials across sectors, as was the case of South Africa, the use of aggregate data would fail to capture and thus hinder the analysis of detailed policy shocks such as Thatcher-like policy reforms implemented in South Africa. In addition to this, since labor, capital and technology play different roles in different sectors it is important that one uses sector-wise disaggregated data for each sector which will help forecast sectoral growth rates more accurately and capture the seasonal, cyclical and trend like behavior more consistently. We refer the reader to the Zellner and Chen (2001) paper for a graphical illustration of the disparity in the sectoral growth rates of 11 different sectors of the US economy.

1.1.3 Entry and exit in competitive models

As pointed out by Veloce and Zellner (1985), most empirical models for competitive markets treat the number of firms as constant, a plausible assumption for the short run but not for long run. There have been relatively few models that have actually examined the dynamics of typical demand-supply-entry (DSE) models of industries. However, it is possible that firm entry and exit behavior could affect the dynamics of sectoral outputs and hence have aggregate effects. Veloce and Zellner (1985) developed a barebones MMM where they estimated the aggregate supply and entry/exit equations for the Canadian furniture industry and showed that the introduction of the number of firms as a variable in the model improved the explanining power of the model and accuracy of estimates (for instance, the supply elasticity with respect to wage rate was positive when the number of firms was not included as an explanatory variable). Similar models for different sectors (like agriculture, services, mining, construction, etc) of the US economy were formulated with the aim of obtaining disaggregate sectoral forecasts that could be summed across sectors to give aggregate forecasts (see Zellner and Chen (2001)). In a more recent paper Zellner and Israilevich (2005), extend the MMM to multiple sectors and include the government and monetary sectors. As expected, their simulation experiments indicate the importance of entry/exit behavior in determining the dynamics of sectoral output.

There is an extensive literature on entry/exit behavior in the field of industrial organization and several of these studies include macroeconomic variables as determinants of entry/exit decisions. I will now give a brief survey of several models of entry/exit to outline some of the important considerations for firms in their entry or exit decisions. Before outlining some of the earlier works on entry/exit models, I will draw attention to the definition of entry as suggested by Mansfield (1961). Mansfield (1961) defined entry in three ways. The first concept defines entry as the net change in the number of firms in an industry. The second concept defines entry as the extent to which firms establish themselves in an industry by either construction of new plants or by acquiring existing firms. The third concept deals with the gross measure of entry that takes account of the number of firms that enter with new plants regardless of the number of firms that did away with their plant during that period. Using the first two concepts, Mansfield (1961) studied the quantitative effects of factors such as capital requirement and profitability on the rate of entry in an industry and the effect of successful innovation on a firm's growth rate. By testing for Gibrat's law of proportionate effect, he concluded that smaller firms have relatively higher death rates while those firms that do survive tend to exhibit higher and more variable growth rates than larger firms. Peltzman (1965) touched upon the issue of entry through his empirical study on factors such as licensing requirements from government agency that affect entry in the US commercial banking sector. However in contrast to the definitions used by Mansfield (1961), Peltzman (1965) looked at the formation of a new bank (applicable to other firms) as the investment in new capital in the industry.

In Orr (1974), a model of entry is formulated and estimated to study the determinants of entry. The study involves the use of data on 71 three-digit industries of the Canadian manufacturing sector. In analyzing the cross-sectional difference in entry across these industries, the author considers entry barriers as determinants of entry rather than profit rates. This is an improvement over past works (Comaner and Wilson (1967) and Miller (1969)) where profit rate is regressed on factors that deter entry. A problem with using profit rates instead of an entry variable is the error in calculating profit rates which can occur due to difference in treatment of depreciation across firms, the treatment of advertising expenditure and expenditure on research and development as current expense rather than as a depreciable investment and not accounting for human capital and other forms of intangible capital. After considering the factors (such as past profit rate, past growth rate of industry output, capital requirements, advertising intensity etc.) that may explain entry, the author concludes that capital requirements and advertising intensity are significant barriers to entry while past profit rates and industry growth rate have weak but positive effect on entry. It is worth noting here that the MMM considers the difference between current profit and long-run equilibrium profits as a determinant of entry which is more in line with competitive models.

Berck and Perloff (1984) studied a dynamic model of open access fishery defined by equations for the evolution of fish stock and entry-exit of boats from the fishing industry over time. It is indeed very interesting how the authors formulate the entry-exit behavior of boats and compare between the two cases where agents are assumed to have either perfect foresight or adaptive expectations. In both these cases entry is proportional to present value of expected profits. In particular with adaptive expectations expected profits are set equal to current profits while under the assumption of rational expectations, the present value of expected profits equals present value of realized profits. It is worth mentioning that in the MMM, the long-run profits are determinants of entry and ideally this should be the expected long-run profits. Different ways to model the formation of expectations can result in different kind of dynamics and is still an open area for research.

Chetty and Heckman (1986) present a dynamic model of a competitive industry with entry and exit of firms and propose a method for estimating the lag structure of output and factor demands based on economic theory. Moreover, the production units are treated as heterogeneous in terms of their effciency of factor utilization. With this model they were able to derive an explicit empirically tractable lag structure for industry demand and supply based on economic theory and these lag structures were dependent on economic history and yet estimable. Thus this made it possible to avoid the use of purely statistical time series that do not adequately identify lag structures. The construction of aggregates depends on economic conditions which in turn determine entry and exit of firms in the industry. While this model looks only at within industry dynamics, the approach of the MMM is an attempt to incorporate the dynamics within the industry (or sectors) which in turn determine the dynamics of aggregate output and price.

In Hopenhayn (1992), a dynamic stochastic competitive equilibrium model is analyzed that can account for entry-exit endogenously. Each firm faces uncertainty in the form of a firm specific productivitiy shock. The decision to exit the industry depends on these shocks. To enter the industry, potential entrants require an unrecoverable investment. The MMM does suppose that there is such an entry cost but does not explicitly model it. However it is flexible enough to allow these costs to be time-varying and model explicitly.

In Siegfried and Evans (1994), the following definitions of entry and exit are considered. Entry occurs if a firm starts producing a new product not produced before or sells an existing product in a new geographic location. Diversification by an already existing firm as well as new business start-ups are also considered as entry. Exit is said to occur if a firm stops producing a product or discontinues sales in a particular geographic market. Given that entry and exit affect competition in a market as well as encourage innovation and change, this paper provides a survey of past empirical work on factors that encourage and impede both entry and exit. Among factors that serve as incentives for entry were expected profitability (see Hirschey (1981), Duetsch (1984), Chappell, Kimenyi and Mayer (1990)) and market growth rate (see Orr (1974), Duetsch (1975)). Profitability was seen to have a positive effect on net entry based on a cross-sectional study of US manufacturing industries while market growth rate (measured by past growth rate of industrial sales revenue) was also seen to have a positive effect on net entry as reported in the aforementioned papers. Previous studies on both structural barriers (such as absolute cost barriers, scale economies and multi-plant operations) as well as behavioral barriers (such as limit pricing, excess capacity, signing of long term contracts with customers etc) were reported in this paper. Through various studies on scale economies as an entry deterrent, it seemed that this effect was ambiguous and although scale economies do serve as a barrier to large scale entry, the empirical evidence is not overwhelming. Most empirical work showed that excess capacity did not prove to be an entry deterrent (see Highfield and Smiley (1987), Yip (1982), Hilke (1984)). Masson and Shaanan (1986) did however find evidence of lower entry volume in markets with excess capacity based on a study of 26 US manufacturing industries. In the MMM, the supposition of decreasing returns to scale rules out excess capacity.

In Ilmakunnas and Topi (1999), several microeconomic and macroeconomic influences on entry and exit are studied empirically with reference to the the case of the Finnish manufacturing industry. Panel data for three digit industries covering the six year period between 1988-1993 is used for this study given that this period includes both high growth as well as recessions making it especially suitable for taking a look at various macroeconomic factors that may determine entry and exit of firms. Based on the premise that potential entrants decide to enter or not by looking at future profits, the authors consider a model where entry (measured by number firms entering the industry) is a function of the difference between expected profit and the cost of entry along with such factors as supply of entrepreneurs and credit supply. The cost of entry in turn depends on entry barriers (such as scale economies) and cost of finance. It is assumed that expected profits depend on past profits, growth and size of the firm as well as macroeconomic factors such as change in real interest rate, change in real GDP and changes in real exchange rate. They conclude that past profits did not explain entry while scale economies had a negative impact on entry. Macroeconomic variables such as real interest rate had a negative effect on entry when real exchange rate was included while there was a positive relation between GDP growth and entry. The entry/exit equation in the MMM is flexible enough to include macro variables like aggregate price level, factor price index and interest rates. However the entry equation needs to be appropriately modified for these variables. The interesting aspect of the MMM is that it accommodates the inter-linkage between entry/exit across different sectors and the macroeconomic variables.

The dynamic entry/exit equation considered in this model relates the decision of a firm to enter or exit to the difference between the long-run aggregate industry profitability to the current aggregate profit in the sector. In the original 2 sector MMM with government and money markets included, Zellner and Israilevich (2005) conduct simulation exercises to explore dynamics of key variables in the model. They show that it is possible to have dynamics ranging from smooth convergence to equilibrium, oscillations or even "booms and busts" which is consistent with chaotic models. This wide array of dynamics gives rise to a need for a more detailed inspection of the qualitative properties of the model. In this paper I choose the entry/exit parameter of sector one and the tax rate to investigate their individual effects on the sales in both sectors.

Chapter 2

Virtues of Continuous Time Econometric models

The complete one sector and n-sector MMM was originally developed in discrete time models. Zellner and Israilevich (2005) mention that whether the economy is best modeled in continuous or discrete time is an issue that would require more theoretical and empirical analysis. I however will consider the equivalent continuous time versions of these models for my analysis. To justify the use of continuous time, I will list some of the virtues of continuous time econometric models as enumerated by Bergstrom (1996). Most macroeconomic variables are measured at regular discrete intervals like quarterly or annually. However given that these variables adjust at much shorter random intervals as a result of economic agents making uncoordinated decisions at different points of time, it is best to consider continuous time models. These models are able to better account for the interaction of variables during the unit observation period. Furthermore, economic theory provides information on the particular interactions of these variables. If the sample size is small it is important that one uses all this information for the purpose of estimation. This again can be accomplished using continuous time models. A second advantage relates to the ability of continuous time models to represent a causal system. In causal chain models each variable responds to stimulus provided by a proper subset only of the other variables of the model even though all variables interact during the unit observation period. Causal chain models are able to take account of the a priori information regarding the causal orderings of variables. For instance in a linear differential equations model, most coefficients are restricted to zero and the causal orderings are represented by the zero restriction patterns. The assumption that the variables can be arranged as such a causal chain is not dependent on any economic theory but only on the modeler's knowledge of information available to agents at diferent points in time. For example, consider the case of aggregate consumer expenditure on a particular day. In this case variables known to the consumer (such as the personal income, personal assets and prices for that particular day) will affect expenditure. Variables such as exports, imports or investments for that day will not affect expenditure. The use of this information can reduce variance of parameter estimates but to do this efficiently, one would need to use continuous time models.

The standard estimation procedures for discrete time models treat stock and flow variables in the same manner thus leading to bias due to specification error. Estimation procedures for continuous time models are able to distinguish between stock and flow variables. Discrete time models are not flexible enough since the form of any particular model will depend on the unit observation period and this is a drawback due to the different types of data available. However, continuous time models are not affected by this drawback as they do not depend on the observation period. This is an advantage for econometricians who generally work on available data rather than choose the observation period. Moreover even if variables are observable at discrete intervals of time, continuous time models can be used to generate continuous time paths for such variables.

In what follows I will outline a brief suvey of continuous time macroeconomic models that have been in existence since the early 1970's. The list is not exhaustive by any means and and for the sake of brevity I will mention only some of the papers. For a more detailed survey, refer to Bergstrom (1996) from which this survey is taken. The use of continuous time models to analyze macroeconomic phenomenon from an empirical standpoint started in the 1970's. Two of the earliest works in this direction were the disequilibrium adjustment model of the UK financial markets by Wymer (1973) and the three equation model of the US business cycle by Hillinger, Bennett and Benoit (1973). This was followed by a dynamic disequilibrium neoclassical growth model for the UK economy due to Bergstrom and Wymer (1976). Two other significant models for the UK economy were due to Jonson (1976) and Knight and Wymer (1978).

Following the tradition of Bergstrom and Wymer (1976), economy wide models for other countries were formulated. Notable papers in this direction include the continuous time models for the Australian economy by Jonson, Moses and Wymer (1977) and for the Italian economy by Gandolfo and Padoan (1982, 1984, 1987, 1990). Among other macroeconometric models that were developed were models for the Italian economy by Tulio (1981) and Fusari (1990), models for Canada by Knight and Mathieson (1979), for Germany by Kirkpatrick (1987), for the US by Donaghy (1993), for Sweden by Sjoo (1993), for New Zealand by Bailey, Hall and Phillips (1987). The 1976 model by Bergstrom and Wymer

(1976) for the UK economy was re-formulated as a system of second order differential equations along with certain other modifications such as a more elaborate financial sector in Bergstrom, Nowman and Wymer (1992) and estimated using Gaussian method.

Other areas in which continuous time macroeconometric models have been used include those for investigation of business cycles in Germany by Hillinger and Schuler (1978) and in the US by Hillinger and Reiter (1992) as well as models for the UK financial sector by Wymer (1973), model for the Eurodollar market by Knight (1977) and a global adjustment model of exchange rate and interest rates by Richard (1980).

Chapter 3

Non-linear Dynamics and Bifurcation Analysis in Economics

3.1 Non-linear Dynamics

The fact that economic phenomena is not neccessarily linear and attempts at incorporating this non-linearity into economic modeling is certainly not a recent development. In fact early works aiming at the development of non-linear economic models dates back to the period between 1930-1950. See Perona (2005) for a comprehensive survey of the use of non-linear dynamics in macroeconomic models. It is particularly important to mention the work of Kaldor (1940) who understood the need for introducing non-linearities to model the mechanics of the functioning of an economy. His attempts at this follow from Kalecki's work (Kalecki (1935, 1937)) on explaining the origin of trade cycles. Having chosen a linear investment decision, Kalecki ended up with a higher order linear system and was faced with the dilemma of trying to explain how a linear system exhibiting damped oscillations could account for sustained cyclical processes. In this context it is particularly worthwhile mentioning Richard Goodwin who was the first economist to come up with an explicit mathematical non-linear model of the trade cycle. Some of his works include the non-linear accelerator model (Goodwin (1951)), and his celebrated article “A Growth Cycle” (Goodwin (1967)) where he used a predator-prey model which led to the emergence of a limit cycle.

In the 1980’s there was a resurgence of interest in the use of non-linear dynamics in macroeconomics. Grandmont (1985) showed that it was possible for even the most classical dynamic general equilibrium macroeconomic models to demonstrate stable solutions or more complex solutions in the form of cycles or chaos. The reason behind such disparate behavior was not a difference in structure of the model but rather the fact that the parameter space of such models was stratified into subsets or bifurcation regions each of which supported a very different kind of dynamics. As Barnett (2000) pointed out, it is possible for economists having different policy views as represented by different parameter values to agree on structurally similar models which is in contrast to earlier beliefs that models with different structures translated to different policy views. Additionally, once modelers accept the presence of non-linearity, empirical research on differing parametric values can take precedence over the need to find alternative structural macroeconomic specification.

Based on the book by Gandolfo (2009) we know continuous time dynamical systems can be represented by systems of differential equations. There are two ways to analyze these differential equations. The first approach involves a quantitative way in which one focuses on finding explicit analytical solutions to these equations or finding approximations to the equations using power series.

However in many cases it is not possible to find a solution even if existence and uniqueness theorems ensure that a solution exists. Thus it is important to adopt the second approach or the qualitative approach. This approach involves examining phase plots, Liapunov's second method etc. to study the properties of the solutions of differential equations without actually knowing these explicit analytical solutions. The qualitative approach is very important in economic dynamics since in a lot of situations we do not know the exact form of the functions involved in the model. However economic theory does tell us some of the qualitative properties such as signs of partial derivatives.

3.2 Bifurcation in Economic Models

In this paper I will be focussing on the phenomenon of bifurcation which forms an integral part of qualitative approach to studying dynamical systems. Informally, we say that a system has undergone a bifurcation if a small, smooth change in parameter value(s) produces a sudden topological change in the nature of the singular points and trajectories of the system. In empirical economic models it is common to estimate parameter values and provide appropriate confidence intervals for such estimates. If the bifurcation or critical value of the parameter lies inside the confidence interval, the system can lose structural stability. Thus it is important that the modeler is aware of the existence of such boundaries particularly if analysis of policy implications is the issue at hand. Also, if these bifurcation regions happen to intersect the feasible parameter space (based on economic theory), then this could have serious implications for theoretical models of economic dynamics as well.

Bifurcations can be both local and global. Local bifurcations can be studied through analysis of changes in the local stability properties of equilibria or periodic orbits of a system when parameters cross a critical value. The basic technique for examining local bifurcations involves linearizing a non-linear system around its equilibrim since in general non-linear systems behave in the same manner as linear systems in a close neighborhood of the equilibrium. This objective is achieved by obtaining and evaluating the Jacobian matrix of the original system at the equilibrium and then studying the eigenvalues of the Jacobian. At a bifurcation point the number of equilibrium may change, there maybe changes in their stability and/or changes in the nature of orbits near the equilibrium.

Examples of local bifurcations include saddle-node (fold) bifurcation, transcritical bifurcation, pitch-fork bifurcation, period-doubling (flip) bifurcation (occurs only in discrete time system) and Hopf bifurcation. Formal definitions of all these concepts are provided in the following subsection. All definitions and related concepts are from Kuznetsov (2004) and Barnett and He (2004).

Global bifurcations occur when larger invariant sets of the system collide with each other or with the equilibrium of the system. Such bifurcations cannot be detected through local linearization of the original system and I will not be examining this phenomenon for my present purpose.

Examining the existence of bifurcations in economic models has important consequences for theoretical and empirical model building in economics. This section provides a brief survey of the literature of bifurcation detections in economic models. Boldrin and Woodford (1989) give an extensive survey of developments in dynamic general equilibrium theory and conditions under which endogenous fluctuations are possible. In their survey they mention that deterministic dynamical systems can generate chaotic dynamics. They also point out that even though a stable model gives the best fit within the class of models considered, it is not proof that the true model is stable (one that generates endogenous cycles). Their survey covers several papers that show the existence of limit cycles and Hopf bifurcations. The possibility that general equilibrium models could exhibit chaotic behavior was developed by Benhabib and Day (1982) and Grandmont (1985).

Torre (1977) studied the structural stability of a continuous time Keynesian model using bifurcation analysis. Torre analytically constructs bifurcation boundaries to identify those areas of possible Hopf bifurcations within the theoretically feasible parameter space. Benhabib and Nishimura (1979) study a multisector optimal growth model and use bifurcation analysis to show that the optimal growth path (steady state) becomes a closed orbit for some values of the discount rate within the theoretical feasible region. Nishimura and Takahashi (1992) consider a multisector neoclassical optimal growth model and show that for a given discount factor, Hopf bifurcations can happen based on the factor intensity within each sector. We refer the reader to the survey by Boldrin and Woodford (1989) for more details on work in this direction until the late 1980s. Bala (1997) studies the continuous time tatonnement process for a two agent and two commodity exchange economy and finds that varying a parameter determining the dominance of income effect over substitution effect results in a pitch-fork bifurcation, i.e. the equilibrium looses local stability while two new locally stable equilibria appear.

More recent work on detecting bifurcations in macroeconomic models have been undertaken by Barnett and his co-authors, some of which is listed below. Barnett and Chen (1988) and Barnett, Gallant, Hinich, Jungeilges, Kaplan and Jensen (1997), among others, have tested for chaos and for other forms of non-linearity in univariate time series. Barnett and He (2001) provide report on a competition among competing empirical tests for non-linearity and chaos in data and they find that the literature in this area still has several unresolved problems. Barnett and He (2002) analyze the Bergstrom, Nowman and Wymer (1992) dynamic continuous time macroeconometric model of UK to determine if stabilization policy would indeed result in stability. They found that the dynamic properties of the model is sensitive to parameter changes and show the existence of transcritical and Hopf bifurcations for different policy parameter. They also numerically construct bifurcation boundaries that intersect with the statistical confidence regions provided by Bergstrom, Nowman and Wymer (1992). In conclusion they determine that policies based just on reasonable economic intuition could prove to be counterproductive. Furthermore, Barnett and He (2004, 2010) analyzed the Leeper and Sims (1994) Euler equations model for the US and found the existence of singularity bifurcations within the empirical parameter space. This type of bifurcation has not been found before in economics and is more common in engineering. Barnett and Duzhak (2008, 2010) recently found the presence of period doubling and Hopf bifurcations in New Keynesian models.

There has also been some work in studying the sensitivity of model dynamics to parameter choices in dynamic microeconomic models in the industrial organization literature. Some of these are particularly relevant for models that consider dynamic entry and exit of firms. For instance, Ferreira and Braga (2005), study the effects of markup variability in Cournot model with free entry. They show that with a positive markup (some degree of market power) there may be indeterminacy of equilibrium and also Hopf bifurcations may emerge depending on the parameters of the model.

3.3 Some Concepts and Definitions

In this section I will outline some of the concepts and definitions used in this dissertation. All definitions and concepts have been taken from Kuznetsov (2004) and Barnett and He (2004).

Consider a continuous time dynamical system defined by

$$\dot{x} = f(x), x \in \Re^n$$

where f is smooth. Let $x_0 = 0$ be an equilibrium of the system and let A denote the Jacobian matrix $\frac{df}{dx}$ evaluated at x_0. Let n_-, n_0 and n_+ be the numbers of eigenvalues of A (counting multiplicities) with negative, zero and positive real parts respectively.

Definition 3.3.1 *An equilibrium is called hyperbolic if $n_0 = 0$, that is if there are no eigenvalues on the imaginary axis.*

Consider two dynamical systems:

$$\dot{x} = f(x, \alpha), x \in \Re^n, \alpha \in \Re^m \tag{3.3.1}$$

and

$$\dot{y} = g(y, \beta), y \in \Re^n, \beta \in \Re^m \tag{3.3.2}$$

with smooth right hand sides and the same number of variables and parameters.

Definition 3.3.2 *Dynamical system (4.1) is topologically equivalent to dynamical system (4.2) if*

1. *there exists a homeomorphism of the parameter space* $p : \Re^m \to \Re^m, \beta = p(\alpha)$

2. *there is a parameter-dependent homeomorphism of the phase space* $h_\alpha : \Re^n \to \Re^n, y = h_\alpha(x)$, *mapping orbits of the system (4.2) at parameter values* $\beta = p(\alpha)$, *preserving the direction of time.*

Definition 3.3.3 *The appearance of a topologically non-equivalent phase portrait under variation of parameters is called a bifurcation.*

Sufficiently small perturbations of parameters do not lead to changes in structural stability of a hyperbolic equilibrium. Thus bifurcation of equilibrium takes place only at non-hyperbolic points.

Consider a continuous time dynamical system that depends on parameters represented as

$$\dot{x} = f(x, \alpha), x \in \Re^n, \alpha \in \Re^m \tag{3.3.3}$$

where x represents phase variables and α represents parameters respectively.

Definition 3.3.4 *The codimension of a bifurcation in system (4.3) is the difference between the dimension of the parameter space and the dimension of the corresponding bifurcation boundary.*

A more practical definition of codimension as in Kuznetsov (2004) is the number of independent conditions determining the bifurcation boundary.

Definition 3.3.5 *A transcritical bifurcation occurs when a system has non-hyperbolic equilibrium with a geometrically simple zero eigenvalue at the bifurcation point and when additional transversality conditions are also satisfied.*

Consider a one dimensional dynamical system

$$Dx = f(x, \theta) \tag{3.3.4}$$

For the one dimensional system as (4.4) the transversality conditions for a transcritical bifurcation at $(x, \theta) = (0, 0)$ are:

- $f(0,0) = f_x(0,0) = 0$
- $f_\theta(0,0) = 0$
- $f_{xx}(0,0) \neq 0$
- $f_{\theta x}^2 - f_{xx} f_{\theta\theta}(0,0) > 0$

The canonical form for a one-dimensional system exhibiting transcritical bifurcation is given by

$$\dot{x} = \theta x - x^2$$

. Notice that for the parameter $\theta \neq 0$ there are two solutions for this dynamical system given by $x^* = 0$ which is stable for $\theta < 0$ and unstable for $\theta > 0$ and $x^* = \theta$ which is stable for $\theta > 0$ and unstable for $\theta < 0$. Thus at exactly the critical value $\theta = 0$ there is an exchange of stability between the two equilibria of the system and we have a transcritical bifurcation. This phenomenon is demonstrated in the following bifurcation diagram for the canonical form.

The bifurcation diagram illustrates the behavior of the equilibrium of the system as a function of the bifurcation parameter. The solid lines represent the stable arms of the equilibrium while the dotted line represents the unstable arm with the exchange of equilibrium occuring at $\theta = 0$.

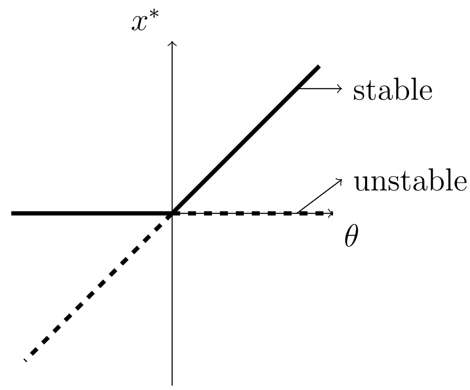


Figure 3.1: Diagram for Transcritical Bifurcation

Definition 3.3.6 *Consider a one-variable, one-parameter differential equation*

$$Dx = f(x, \theta) \tag{3.3.5}$$

Suppose that there exists an equilibrium x^* and a parameter value θ^* such that (x^*, θ^*) satisfies the following conditions:

- $\frac{\partial f(x,\theta^*)}{\partial x} \,|_{x=x^*} = 0$
- $\frac{\partial^3 f(x,\theta^*)}{\partial x^3} \,|_{x=x^*} \neq 0$
- $\frac{\partial^2 f(x,\theta)}{\partial x \partial \theta} \,|_{x=x^*,\theta=\theta^*} \neq 0$

Then (x^*, θ^*) is a pitchfork bifurcation point.

The canonical form for a system demonstrating pitch-fork bifurcation is given by

$$\dot{x} = \theta x - x^3$$

. As can be seen this system has three solutions for $\theta \neq 0$. For all $\theta > 0$, we have two solutions $x^* = \pm\sqrt{\theta}$ being stable and $x^* = 0$ being unstable. For $\theta < 0$, the system has one solution given by $x^* = 0$ which is stable. Thus at the critical value $\theta = 0$, a pitch-fork bifurcation occurs where equilibria appear and disappear in pairs. The figure below demonstrates this kind of bifurcation for the normal form of a system exhibiting pitch-fork bifurcation.

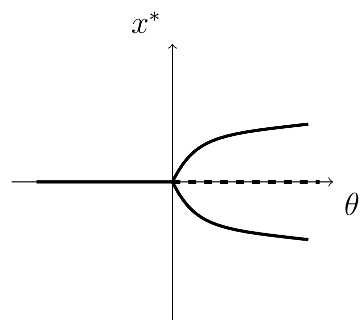


Figure 3.2: Diagram for Pitchfork Bifurcation

For all $\theta < 0$, there is only one stable equilibrium denoted by the solid line. At the bifurcation point, two additional equilibria appear with both being stable for all $\theta > 0$ while the $x^* = 0$ equilibrium becomes unstable. This is called a super-critical pitch-fork bifurcation.

Definition 3.3.7 *Consider a general one dimensional system*

$$Dx = f(x, \theta) \tag{3.3.6}$$

Let x^* be a non-hyperbolic equilibrium and let θ^* be the corresponding parameter, so that (x^*, θ^*) satisfies

- $\frac{\partial f(x,\theta^*)}{\partial x} \mid_{x=x^*} = 0$
- $f(x^*, \theta^*) = 0$

Then the transversality conditions for saddle-node bifurcations are

- $\frac{\partial f(x,\theta)}{\partial \theta} \mid_{x=x^*,\theta=\theta^*} \neq 0$
- $\frac{\partial^2 f(x,\theta)}{\partial x^2} \mid_{x=x^*,\theta=\theta^*} \neq 0$

The simplest one-dimensional system that can exhibit saddle-node bifurcation is represented by the following dynamical equation,

$$\dot{x} = \theta - x^2$$

. This system has two solutions for all $\theta > 0$ given by $x* = \sqrt{\theta}$ which is stable while the solution $x^* = -\sqrt{\theta}$ which is unstable. There are no soluions when the parameter $\theta < 0$. Thus exactly at the critical parameter value $\theta = 0$ a saddle-node bifurcation occurs where two equilibria come together, collide and annihilate each other. The bifurcation diagram below demonstrates the saddle-node bifurcation for the canonical system described above. In case of a discrete time dynamical system this kind of bifurcation is commomly referred to as a fold bifurcation.

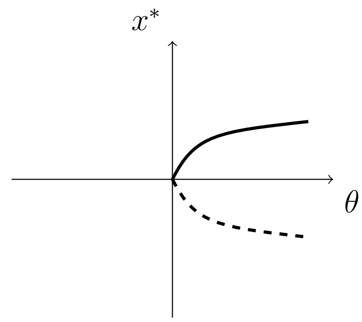


Figure 3.3: Diagram for Saddle-node Bifurcation

Definition 3.3.8 *The bifurcation corresponding to the presence of*

$$\lambda_{1,2} = \pm i\omega_0, \text{ with } \omega_0 > 0,$$

is called a Hopf (or Andronov-Hopf) bifurcation.

Here $\lambda_{1,2}$ are the complex conjugate eigenvalues of the continuous time dynamical system.

The canonical form of a two dimensional dynamical system exhibiting Hopf bifurcation is given by

$$\dot{x} = -y + x(\theta - (x^2 + y^2))$$

$$\dot{y} = x + y(\theta - (x^2 + y^2))$$

This system has an equilibrium at $(x^*, y^*) = (0, 0)$.The Jacobian of this system has the complex conjugates $\theta \pm i$ as its eigenvalues. At the critical parameter value $\theta = 0$, the roots become purely imaginary resulting in a Hopf bifurcation. The bifurcation diagram below depicts the occurence of a Hopf point for the canonical system described above. As θ increases through the value zero, the previously stable solution at the origin loses stability and a new stable solution called the limit cycle emerges.

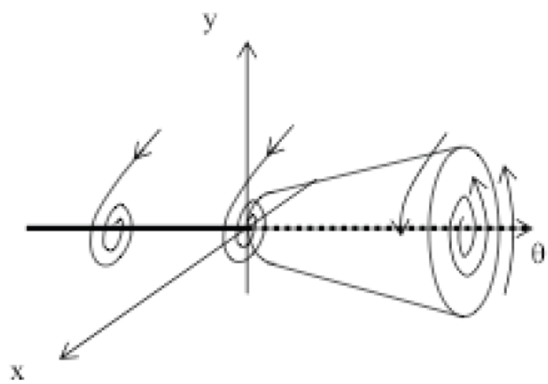


Figure 3.4: Diagram for Hopf Bifurcation.

Note that unlike for the three codimension one bifurcations, namely fold, transcritical and pitchfork, Hopf bifurcation requires at least a 2×2 system. Hopf bifurcation is the commonly encountered bifurcation in economic models. Optimal growth models as well as overlapping generations models have been seen to exhibit Hopf bifurcations.

Chapter 4

The Marshallian Macroeconomic Model

4.1 The One Sector Marshallian Macro Model without the Government and Money in Continuous time

In what follows I will outline the extended version of the Marshallian Macro Model (MMM) as formulated by Veloce and Zellner (1985). The current model is an improvement over the former given that Zellner and his co-author Israilevich introduce factor markets into the original three equation deterministic demand, supply and entry (DSE) model and obtain a complete seven equation model for a one sector Marshallian industry.

It is assumed that at any point of time t, there exists $N = N(t)$ identical, competitive, profit-maximizing firms in operation. Firms produce a single homogenous product using two factors of production, labor $\mathcal{L}$ and capital $\mathcal{K}$. A typical firm has a Cobb-Douglas production function

$$q = A^{*}\mathcal{L}^{\alpha}\mathcal{K}^{\beta}$$

where $0 < \alpha, \beta < 1$ and $A^{*} = A^{*}(t) = A_N(t)A_L(t)A_K(t)$. Here $A_N(t)$ is a neutral technological change factor, $A_L(t)$ is a labor augmenting factor and $A_K(t)$ is a capital augmenting factor. These factors are introduced to take account of any qualitative changes in the firm's inputs, namely labor and capital. The production function exhibits decreasing returns to scale (captured by $\alpha + \beta < 1$)with respect to labor and capital which could arise as a result of exclusion of such factors as entrepreneurial skills from the model.

Each firm maximizes profit π by choosing inputs $\mathcal{L}$ and $\mathcal{K}$ given price of the product p and the nominal wage rate $w = w(t)$ for labor and nominal rent $r = r(t)$ for capital services. Thus each firm's optimization problem is given as

$$\max_{\mathcal{L},\mathcal{K}} \pi = p(A^* \mathcal{L}^\alpha \mathcal{K}^\beta) - w\mathcal{L} - r\mathcal{K}$$

If q is the profit maximizing output then $s = pq$ is the sales for the firm. The aggregate nominal sales supply function $S = Npq$ for this industry is obtained by summing over all firms and is given by

$$S = NAp^{1/\theta} w^{-\alpha/\theta} r^{-\beta/\theta}$$

where $A = A^{*1/\theta}$ and $0 < \theta = 1 - \alpha - \beta < 1$.

The above equation is expressed in terms of growth rates by logging and differentiating it with respect to time to obtain

$$\frac{\dot{S}}{S} = \frac{\dot{N}}{N} + \frac{\dot{A}}{A} + (\frac{1}{\theta})\frac{\dot{p}}{p} - (\frac{\alpha}{\theta})\frac{\dot{w}}{w} - (\frac{\beta}{\theta})\frac{\dot{r}}{r} \tag{4.1.1}$$

Notice here that in the absence of entry and exit from this sector i.e $\frac{\dot{N}}{N} = 0$ and no technological change i.e $\frac{\dot{A}}{A} = 0$, an equi-proportionate change in product price as well as the factor prices has no effect on real sales i.e $\frac{\dot{S}}{s} - \frac{\dot{p}}{p} = 0$.

The industry aggregate output demand function is assumed to be given by

$$Q = Bp^{-\eta} Y^{\eta_s} H^{\eta_h} x_1^{\eta_1} x_2^{\eta_2} ... x_d^{\eta_d}$$

where B is a constant, p denotes product price, Y is the nominal disposable income and x_i is the ith determinant of demand, for $i = 1, 2, ...d$. x_i includes such factors as money balances or demand trends. On multiplying the aggregate output demand function by price p we obtain the aggregate industry nominal sales (expenditure)

$$\frac{\dot{S}}{S} = (1-\eta)\frac{\dot{p}}{p} + \eta_s \frac{\dot{S}}{S} + \eta_h \frac{\dot{H}}{H} + \sum_{i=1}^{d} \eta_i \frac{\dot{x}}{x} \tag{4.1.2}$$

Notice here that in the absence of government and thus taxes, nominal disposable income can be replaced by nominal sales S. Here $\eta > 0$ is the own price elasticity, η_s is the income elasticity, η_h is the elasticity with respect to number of households and η_i represents elasticity of demand with respect to the other demand shift factors. Here again with $\eta_s = \eta$ implying absence of money illusion, everything else remaining unchanged, an equi-proportionate change in prices and nominal income will leave real demand unaffected.

The following equation governs the entry/exit behavior for firms in the industry. $\Pi = \theta S$ is current nominal industry aggregate profit while F_e is the equilibrium profit at time t taking account of discounted entry costs. Here γ' and $\gamma = \gamma'\theta$ are both functions of time and both assumed positive. γ is the speed of adjustment coefficient. Given that we expect $\gamma > 0$, we can interprete the entry/exit equation as follows. A positive departure from equilibrium profits F_e will attract new firms into the industry while a negative departure from F_e

will induce firms to leave the industry as predicted by Marshall. The larger the value of γ the faster is the adjustment to the equilibrium level of profit.

$$\frac{\dot{N}}{N} = \gamma^{'}(\Pi - F_e) = \gamma(S - F) \tag{4.1.3}$$

From the first order conditions of profit maximization the optimal aggregate demand for labor and capital are found to be $L = \frac{N\alpha pq}{w} = \frac{\alpha S}{w}$ and $K = \frac{N\beta pq}{r} = \frac{\beta S}{r}$ respectively. Again on taking log and differentiating with respect to time the following factor demand equations for this sector are obtained. Equation 4.1.4 gives the aggregate demand for labor L while Equation 4.1.5 gives the aggregate demand for capital K.

$$\frac{\dot{L}}{L} = \frac{\dot{S}}{S} - \frac{\dot{w}}{w} \tag{4.1.4}$$

$$\frac{\dot{K}}{K} = \frac{\dot{S}}{S} - \frac{\dot{r}}{r} \tag{4.1.5}$$

It is assumed that the labor supply takes the following form

$$L = D(w/p)^{\delta}(Y/p)^{\delta_s}H^{\delta_h}z_1^{\delta_1}z_2^{\delta_2}...z_l^{\delta_l}$$

As we can see labor supply depends on real wage w/p, real disposable income Y/p, the number of households H and certain other "supply shifters" given by z_i, $i = 1, 2, ...l$. Here again the nominal disposable income Y will be replaced by nominal sales S. δ is the elasticity of labor supply with respect to real wage w/p, δ_s is the elasticity of labor supply with respect to real income Y/p, δ_h is the elasticity with respect to number of households H and δ_i is the elasticity of labor supply with respect to the ith determinant of labor supply for $i = 1, 2, ...l$.

On logging and differentiating with respect to time the following labor supply function in terms of growth rates is obtained

$$\frac{\dot{L}}{L} = \delta(\frac{\dot{w}}{w} - \frac{\dot{p}}{p}) + \delta_s(\frac{\dot{S}}{S} - \frac{\dot{p}}{p}) + \delta_h\frac{\dot{H}}{H} + \sum_{i=1}^{l}\delta_i\frac{\dot{z}_i}{z_i} \tag{4.1.6}$$

Similarly for the capital supply it is assumed to be of the following form

$$K = E(r/p)^{\phi}(Y/p)^{\phi_s}H^{\phi_h}v_1^{\phi_1}v_2^{\phi_2}...v_k^{\phi_k}$$

As can be seen capital supply depends on real rental rate r/p, real disposable income Y/p, the number of households H and other supply shifters denoted by v_i, $i = 1, 2, ...k$. Again the exponents of these determinants turn out to be elasticities of capital supply with respect to the determinants.

Logging and differentiating with respect to time the following capital supply function is obtained

$$\frac{\dot{K}}{K} = \phi(\frac{\dot{r}}{r} - \frac{\dot{p}}{p}) + \phi_s(\frac{\dot{S}}{S} - \frac{\dot{p}}{p}) + \phi_h\frac{\dot{H}}{H} + \sum_{i=1}^{k}\phi_i\frac{\dot{v}_i}{v_i} \tag{4.1.7}$$

Equation 4.1.1 through Equation 4.1.7 define the Marshallian Macroeconomic Model with one sector in the absence of government and money markets. There are seven endogenous variables N, L, K, p, w, r and S while the variables H, A^*, F_e, x, z and v are assumed to be exogenously determined. The model can be solved analytically for the reduced form equation for $\frac{\dot{S}}{S}$ given by

$$\frac{\dot{S}}{S} = a(S - F) + bg \tag{4.1.8}$$

where $a = \frac{\gamma}{1-f}$ and $b = \frac{1}{1-f}$ are parameters and g is a linear function of the rates of change of the exogenous varibales A, x_i, z_i and v_i. For explicit expression of g, a and b refer to Zellner and Israilevich (2005).

Note that if a, b, g and F are constants, then Equation 4.1.8 is the differential equation for the logistic function. However if $g = g(t)$ is a function of time, then Equation 4.1.8 will be a particular form of Bernoulli differential equation as noted in Veloce and Zellner (1985). The logistic Equation 4.1.8 can be expressed as

$$\frac{dS}{dt} = k_1 S(1 - (\frac{k_2}{k_1})S) \qquad (4.1.9)$$

where $k_1 = \frac{g-\gamma F}{1-f}$ and $k_2 = \frac{-\gamma}{1-f}$. It can be seen that Equation 4.1.9 has two equilibrium values at $S = 0$ and $S = k_1/k_2$. The solution $S = k_1/k_2$ is unstable for positive values of k_1 and k_2. The nature of the solutions to Equation 4.1.9 have been discussed in great detail in Veloce and Zellner (1985) and Zellner and Israilevich (2005). If the parameters in Equation 4.1.9 are constants, then there cannot be cyclical movements. However if these parameters are allowed to vary then the solution can be quite variable. In some cases where it is possible to have discrete lags in Equation 4.1.9, we will end up with mixed differential-difference equation that can again produce cyclical solutions.

Further Zellner and Israilevich (2005) provide some discrete approximations to Equation 4.1.9 which can exhibit quite different dynamics in the form of oscillatory movements even with constant parameters. Two such discrete approximations considered by the authors are provided below:

$$S_{t+1} - S_t = k_1 S_t(1 - (\frac{k_2}{k_1})S_t) \qquad (4.1.10)$$

$$\ln S_{t+1} - \ln S_t = k_1(1 - (\frac{k_2}{k_1})S_t) \quad (4.1.11)$$

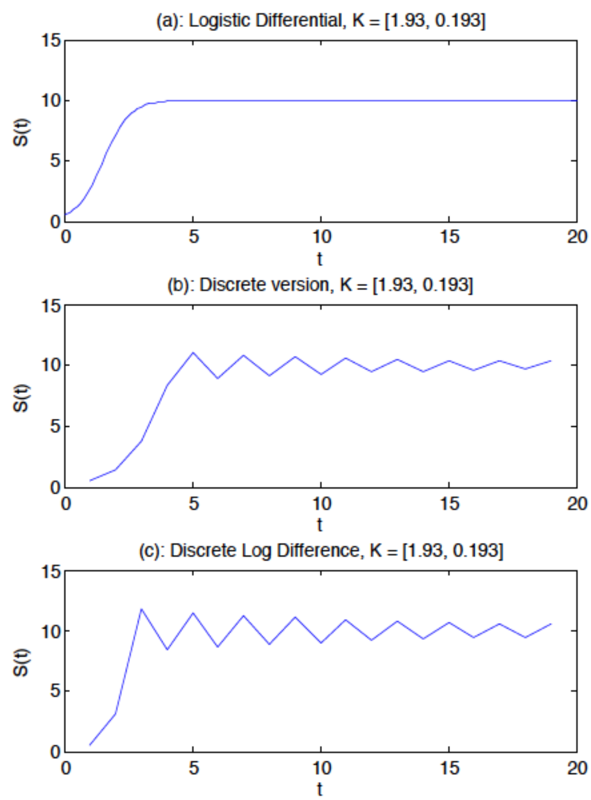


Figure 4.1: One Sector Model (from Zellner and Israilevich (2005)).

Note that the dynamics of Equation 4.1.9 is determined by the parameters k_1 and k_2. As I have shown in Figure 4.3 for this differential equation at $k_2 = 0$ there is only one solution $S = 0$. For all other k_2 there are two solutions ($S = 0$ and $S = \frac{k_1}{k_2}$). Computationally in Figure 4.2 we see that there is a branching point at $(S, k_1) = (0, 0)$. Moreover from the phase plots in Figure 4.3 we can see that as k_1 changes from positive to negative the nature (stability) of the two equilibria changes. That is there is exchange of stability of the two equilibria at $k_1 = 0$. This is otherwise known as a transcritical bifurcation.

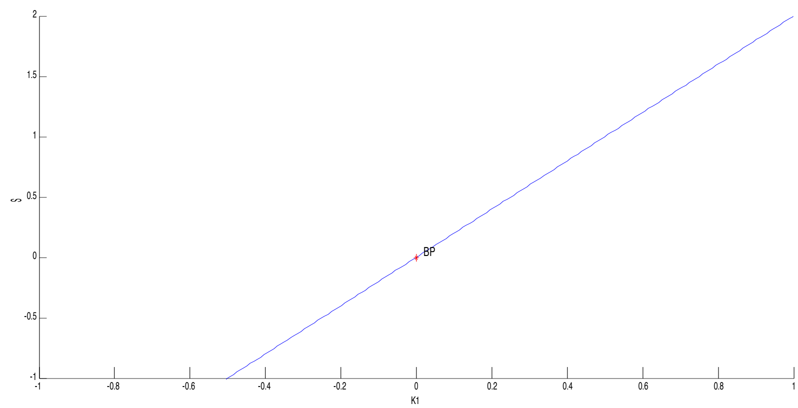


Figure 4.2: Branching Point.

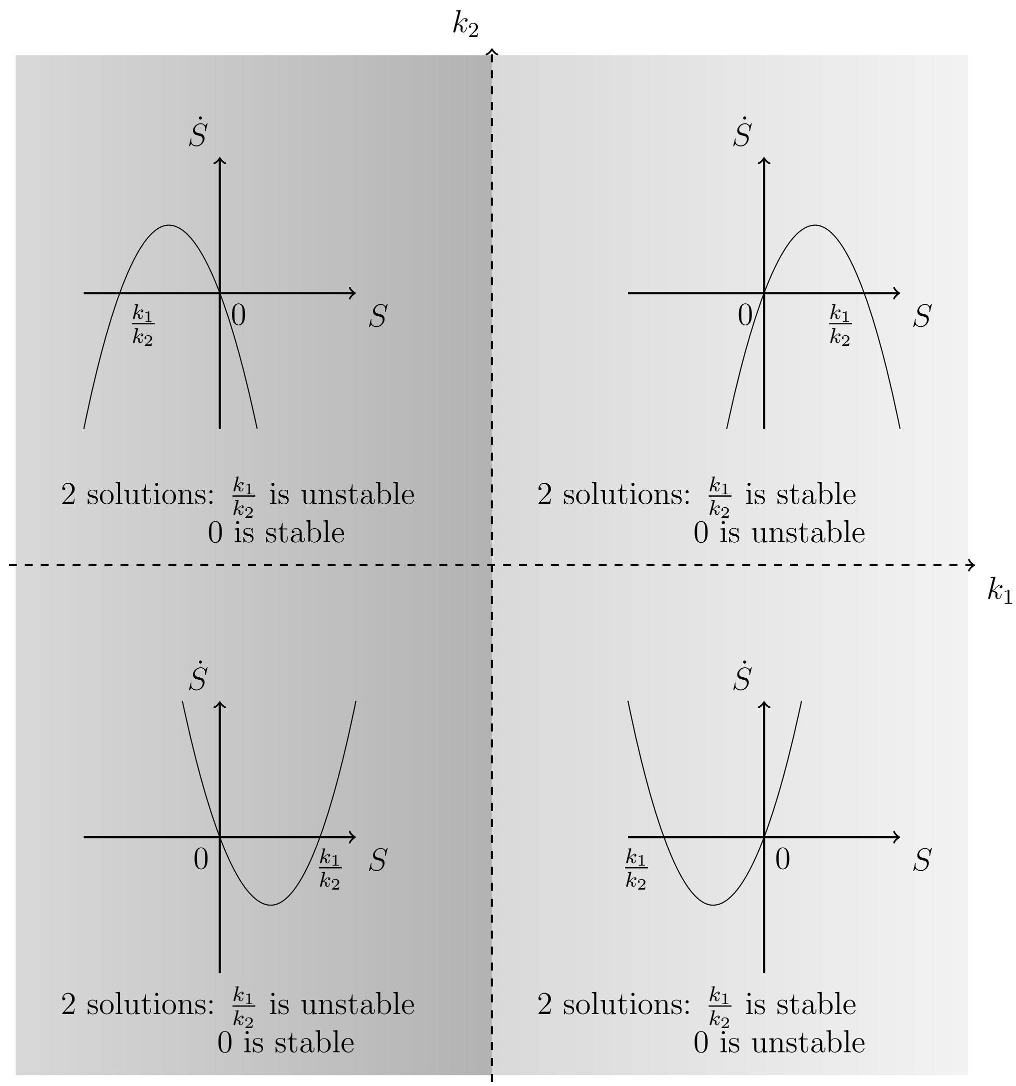


Figure 4.3: Transcritical bifurcation at $k_1 = 0$

4.2 The n-Sector Marshallian Macroeconomic Model in Continuous Time

In this section I will present the generalized n sector MMM due to Zellner and Israilevich (2005). The original model by Zellner and Israilevich (2005) was formulated in discrete time. However in what follows I will present the continuous time equivalent version of the MMM. This model is an extension of the MMM due to Veloce and Zellner (1985) which allows for interactions among the various sectors of the economy like consumers, producers and the government. Here each of the n-sectors is modelled as a Marshallian competitive industry with an aggregate demand function for its output, an aggregate output supply function, an entry/exit equation and aggregate input demand functions. The government plays a crucial role in this model as a purchaser of final goods and services in the output market, inputs in the factor market and by collecting taxes from firms. Additionally it is assumed that government expenditure influences firm level productivity through an effect on technological change factor in each sector.

I will adopt the following notations for specifying the rate of change and the growth rate of any variable X_i at time t. Let $\dot{X}_i = \frac{dX_i}{dt}$ be the rate of change of X_i at time t and $\hat{X}_i = \frac{\dot{X}_i}{X_i}$ be the growth rate of X_i. For the sake of convenience I will omit the subscript t in the subsequent sections.

4.2.1 The Firm

Output Supply

There are N_i identical firms in each sector i, $i = 1..N_i$ facing a Cobb-Douglas type production function

$$q_i = A_i^* \mathcal{L}_i^{\alpha_i} \mathcal{K}_i^{\beta_i} \mathcal{M}_i^{\lambda_i} \tag{4.2.1}$$

where $0 < \alpha_i, \beta_i, \lambda_i < 1$. q_i is the output produced by firm i and $A_i^* = A_i^* = A_N^*.A_L^*.A_K^*$ is the product of a neutral technological change factor and labor and capital augmentation factors. These factors are introduced to take account of any qualitative changes in the firms' inputs. $\mathcal{L}$ (resp. $\mathcal{K}_i$ and $\mathcal{M}_i$) is the labor demand (resp. capital demand and demand for monetary services) for firm i. α_i (resp. β_i and λ_i) denotes the output elasticity of labor (resp. capital and monetary services). Zellner assumes that the production function exhibits decreasing returns to scale with respect to the inputs to accomodate for the exclusion of certain factors such as entrepreneurial skills from the model.

A typical firm maximizes profit π_i by choosing optimal amounts of $\mathcal{L}_i$, $\mathcal{K}_i$ and $\mathcal{M}_i$ given the nominal wage rate w, nominal rental rate r, nominal interest rate τ and price of good i, P_i. Thus each firm's optimization problem is given by

$$\max_{\mathcal{L}_i, \mathcal{K}_i, \mathcal{M}_i} \pi_i = P_i(A^* \mathcal{L}_i^{\alpha_i} \mathcal{K}_i^{\beta_i} \mathcal{M}_i^{\lambda_i}) - w\mathcal{L}_i - r\mathcal{K}_i - \tau\mathcal{M}_i$$

The aggregate nominal profit-maximizing output supply S_i of sector i is N_i times the nominal profit-maximizing supply of each firm within the sector. This is given by $S_i = N_i A P^{\frac{1}{\theta_i}} w^{\frac{-\alpha_i}{\theta_i}} r^{\frac{-\beta_i}{\theta_i}} \tau^{\frac{-\lambda_i}{\theta_i}}$ where $0 < \theta_i = 1 - \alpha_i - \beta_i - \lambda_i < 1$ and $A = A_i^{*\frac{1}{\theta_i}}$.

Taking log and differentiating the aggregate nominal sales supply function with respect to time gives the growth rate of aggregate nominal sales supply function for sector i as

$$\hat{S}_i = \hat{N}_i + \hat{A}_i + \frac{1}{\theta_i}\hat{P}_i - \frac{\alpha_i}{\theta_i}\hat{w} - \frac{\beta_i}{\theta_i}\hat{r} - \frac{\lambda_i}{\theta_i}\hat{\tau} \tag{4.2.2}$$

Entry/Exit

$$\hat{N}_i = \gamma_i \left[\frac{\theta_i S_i (1 - T^c)}{N_i P} - \frac{F_i}{P^I} \right] \tag{4.2.3}$$

Equation (2.3) is the dynamic entry/exit equation for sector i. The first term $\theta_i S_i (1 - T_i^c)$ is the aggregate after tax profit with T_i^c being the corporate profit tax rate for this sector. The after tax profit is deflated by the output price index P since a monetary sector is included in this model and potential entrants will base their decision to enter on real profit. This gives us the real aggregate after tax profit. When divided by N_i, we obtain real average after tax profit. $\frac{F_i}{P^I}$ is a long run equilibrium profit after taking account of discounted entry cost deflated by input price index P^I. γ_i, assumed to be positive is the time varying speed of adjustment parameter. We can thus interprete the entry/exit equation in the following way. A positive departure from long run equilibrium profit will induce new firms to enter the industry while a negative departure will cause existing firms to leave the industry.

Technology

Equation (2.4) captures the effect of changes in real government expenditure $\frac{G}{P}$ on firm level productivity.

$$\hat{A}_i = \omega_{ig}(\hat{G} - \hat{P}) + \sum_{j=1}^{B} \omega_{ij}\hat{b}_{ij} \tag{4.2.4}$$

ω_{ig} is the elasiticity of technological change factor with respect to growth rate of real government expenditure. This could be a result of government expenditure in the form of provision of public services, infrastructure and R&D. b_{ij} accounts for all other technological shift factors with ω_{ij}, $j = 1..B$, being the respective elasticity.

4.2.2 Output Demand

The aggregate output demand for each sector i is assumed to be given exogenously. In terms of growth rates the following equation shows the nominal aggregate demand for good i as a weighted sum of growth rates of demands from the government and households with weights $\frac{G_i}{S_i}$ and $(1 - \frac{G_i}{S_i})$ respectively.

$$\hat{S}_i = \frac{G_i}{S_i}\hat{G}_i + (1 - \frac{G_i}{S_i})\left[(1-\eta_{ii})\hat{p}_i + \eta_{ij}\hat{p}_j + \Psi_{is}(\hat{S} + \hat{T}^{s'}) + \Psi_{im}\hat{M}_h + \sum_{1}^{d} \Psi_j \hat{x}_j\right] \tag{4.2.5}$$

G_i is the nominal government expenditure on good i. It is assumed by the authors that G_i grows at same rate as aggregate government expenditure G such that $\hat{G}_i = \hat{G}$. The second term in equation (2.5) represents the growth rate of aggregate nominal demand for good i from households. η_{ii} is the own price elasticity while η_{ij} is the cross-price elasticity of good i with respect to good j. Ψ_{is} represents the income elasticity of demand with respect to aggregate disposable income $S(1 - T^{s'})$ with $T^{s'} = (1 - T^s)$, T^s being the income tax rate. Household demand for goods also depends on demand for real money balances M_h with Ψ_{im} denoting the elasticity of demand for good i with respect to demand for monetary balances. Additionally, the variables x_j, $j = 1..d$ are other demand shift variables with Ψ_{ij} being the respective demand elasticity.

4.2.3 Factor Markets

Labor Market

Optimal demand for labor for is obtained from individual firm optimization in each sector i and is given by $\mathcal{L}_i = \frac{\alpha_i P q_i}{w}$. On multiplying this by price P_i, the optimal aggregate demand for sector i is obtained as $L_i = \frac{\alpha_i N_i P q_i}{w} = \frac{\alpha_i S_i}{w}$. On logging and differentiating this demand with respect to time, the following equation is obtained for growth rate of L_i

$$\hat{L}_i = \hat{S}_i - \hat{w} \tag{4.2.6}$$

with $i = 1...n$.

Similarly, it is assumed that the government's optimal demand for labor is given by $L_g = \frac{G_L}{w}$. However it is assumed in this model that aggregate nominal government expenditure G grows at the same rate as each component of the expenditure. Thus we have $\hat{G} = \hat{G}_L$ such that in terms of growth rates, the aggregate real government expenditure is given by

$$\hat{L}_g = \hat{G} - \hat{w} \tag{4.2.7}$$

The supply of labor in the economy is given exogenously as

$$L = \left(\frac{w}{P}\right)^{\delta} \left(\frac{S}{P}\right)^{\delta_s} H^{\delta_h} \prod_{i=}^{l} \delta_i z_i$$

As we can see here, supply of labor depends on real wage $\frac{w}{P}$, the real income $\frac{S}{P}$, the number of households H and some other supply shifters denoted by z_i. In terms of growth rates, the supply of labor can be represented as

$$\hat{L} = \delta(\hat{w} - \hat{p}) + \delta_s(\hat{S} - \hat{P}) + \delta_h \hat{H} + \sum_{i=1}^{l} \delta_i \hat{z}_i \tag{4.2.8}$$

Here δ is the elasticity of labor supply with respect to real wage, δ_s is the elasticity of labor supply with respect to real income, δ_h is the elasticity with respect to the number of households and δ_i is the elasticity with respect to variable z_i.

The equilibrium wage rate w can be solved for by equating labor supply to aggregate demand for labor. The following equation shows the market clearing condition in terms of growth rates.

$$\hat{L} = \sum_{i=1}^{n} \frac{L_i}{L} \hat{L}_i + \frac{L_g}{L} \hat{L}_g \tag{4.2.9}$$

The left hand side of equation is the weighted sum of growth rates of demand for labor from the two sectors and the government.

Capital Market

Similarly in the case of the capital market, optimal demand for capital is obtained as a result of firm level optimization and is given by $\mathcal{K}_i = \frac{\beta_i P q_i}{r}$. Summing over all firms in sector i, the aggregate demand for capital is found to be $K_i = \frac{\beta_i N_i P_i q_i}{r} = \frac{\beta_i S_i}{r}$. On logging and differentiating the aggregate optimal firm demand for capital, demand for capital in terms of growth rate is obtained as

$$\hat{K}_i = \hat{S}_i - \hat{r} \tag{4.2.10}$$

with $i = 1...n$

Government's optimal demand for capital is assumed to be given by $K_g = \frac{G_K}{r}$ where G_K denotes the nominal capital demand. As mentioned earlier in the case of labor market, $\hat{G} = \hat{G}_K$ such that in terms of growth rates, the aggregate real government expenditure on capital is given by

$$\hat{K}_g = \hat{G} - \hat{r} \tag{4.2.11}$$

The supply of capital in the economy is given exogenously as

$$K = \left(\frac{r}{P}\right)^{\phi} \left(\frac{S}{P}\right)^{\phi_s} H^{\phi_h} \prod_{i=}^{k} \phi_i v_i$$

Supply of capital depends on real rental rate $\frac{r}{P}$, the real income $\frac{S}{P}$, the number of households H and some other supply shifters denoted by v_i. In terms of growth rates, the supply of capital can be represented as

$$\hat{K} = \phi(\hat{r} - \hat{p}) + \phi_s(\hat{S} - \hat{P}) + \phi_h \hat{H} + \sum_{i=1}^{k} \phi_i \hat{v}_i \tag{4.2.12}$$

Here ϕ is the elasticity of capital supply with respect to real wage, ϕ_s is the elasticity of capital supply with respect to real income, ϕ_h is the elasticity with respect to the number of households and ϕ_i is the elasticity with respect to variable v_i.

The equilibrium rental rate r can be solved for by equating capital supply to aggregate demand for capital. The following equation shows the market clearing condition in terms of growth rates.

$$\hat{K} = \sum_{i=1}^{n} \frac{K_i}{K}\hat{K}_i + \frac{K_g}{K}\hat{K}_g \tag{4.2.13}$$

The left hand side of equation is the weighted sum of growth rates of demand for capital from the two sectors and the government.

Money Market

The demand for real balances arises from the two sectors, households as well as the government. In the case of firms, optimal demand for monetary services is obtained from profit maximization conditions. The optimal firm level demand for monetary service in the ith sector is $\mathcal{M}_i = \frac{\lambda_i P_i q_i}{\tau}$. On aggregating over all firms in this sector,the aggregate demand is obtained as $M_i = \frac{\lambda_i N_i P_i q_i}{\tau} = \frac{\lambda_i S_i}{\tau}$. When expressed in terms of growth rates, the aggregate real demand for monetary services is given by

$$\hat{M}_i = \hat{S}_i - \hat{\tau} \tag{4.2.14}$$

with $i = 1...n$.

In the case of government,the demand for real balances is assumed to be given exogenously. Thus in terms of growth rates, the demand for real balances is given by

$$\hat{M}_g = \hat{G} - \hat{\tau} \tag{4.2.15}$$

Households demand for money is assumed to be given exogenously by

$$M_h = \left(\frac{\tau}{P}\right)^{\mu} \left(\frac{S}{P}\right)^{\mu_s} H^{\mu_h} \prod_{i=1}^{m} y_i^{\mu_i}$$

Thus aggregate demand for real balances by households depends on real interest rate $\frac{\tau}{P}$, real income $\frac{S}{P}$, the number of households H and any other demand shifters denoted by variable y_i. In terms of growth rates the aggregate demand can be represented as

$$\hat{M}_h = \mu(\hat{\tau} - \hat{P}) + \mu_s(\hat{S} - \hat{P}) + \mu_h \hat{H} + \sum_{i=1}^{m} \mu_i \hat{y}_i \tag{4.2.16}$$

Here μ represents elasticity of demand with respect to real interest rate, μ_s represents elasticity with respect to real income, μ_h represents elasticity with respect to number of households while μ_i represents the demand elasticity with respect to any other demand shifters.

The nominal supply of money M_0 is given exogenously. Thus in terms of growth rates, the supply of real balances $\frac{M_0}{P}$ is given by

$$\hat{M} = \hat{M}_0 - \hat{P} \tag{4.2.17}$$

Equation(2.18) shows the market clearing condition in the monetary sector in terms of growth rates.

$$\hat{M} = \sum_{i=1}^{n} \frac{M_i}{M} \hat{M}_i + \frac{M_h}{M} \hat{M}_h + \frac{M_g}{M} \hat{M}_g \tag{4.2.18}$$

The left hand side of Equation (2.18) shows the growth rate of aggregate demand for money in the economy as a weighted sum of growth rates of demand from the various sectors of the economy. By equating this to the growth rate of supply of real balances, it is possible to solve for the equilibrium rate of interest τ.

4.2.4 Government

Government Expenditure

A deficit/surplus term D is defined as $D = \frac{G}{R}$ and fixed exogenously in the MMM. Thus in growth rate terms we have,

$$\hat{G} = \hat{R} + \hat{D} \tag{4.2.19}$$

The government revenue consists of income tax at the rate T^s and corporate profit tax imposed on both sectors at the rate T^c. In growth rate terms we have

Government Revenue

$$\hat{R} = \sum_{i=1}^{n} \frac{S_i T_i^*}{R} (\hat{S}_i + \hat{T}_i^*) \tag{4.2.20}$$

4.2.5 Price Aggregates and Sales Aggregate

Equation (2.21) denotes the aggregate factor price index.

$$\hat{P}^I = \frac{wL\hat{w} + rK\hat{r} + \tau M\hat{\tau}}{wL + rK + \tau M} \tag{4.2.21}$$

Equation (2.22) denotes the aggregate product price index.

$$\hat{P} = \sum_{i=1}^{n} \frac{S_i}{S} \hat{P}_i \tag{4.2.22}$$

Equation (2.23) denotes the aggregate nominal sales index.

$$\hat{S} = \sum_{i=1}^{n} \frac{S_i}{S} \hat{S}_i \tag{4.2.23}$$

Chapter 5

A Special Nested Case of the MMM

5.1 The Model

In this section I will consider a special nested case of the MMM to investigate the presence of local bifurcations. As before, the economy is made up of two production sectors, consumers and the government. However this being a special case of the original MMM, there are several simplifying assumptions made to make it possible to obtain closed form solutions for the steady states. For now we have excluded the monetary sector from the model, excluded corporate profits tax and assumed that there is a single uniform income tax rate imposed on all households in the economy. Additionally we have assumed that the technological change factor A_i for each sector i exhibits zero growth. The following section presents the MMM under the above assumptions.

5.1.1 The Firm

Output Supply

There are N_i identical firms in the ith sector each facing a Cobb-Douglas type production function,

$$q_i = A_i^* \mathcal{L}_i^{\alpha_i} \mathcal{K}_i^{\beta_i} \tag{5.1.1}$$

with $0 < \alpha_i, \beta_i < 1$ and $0 < \theta_i = 1 - \alpha_i - \beta_i < 1$. A_i^* is the product of a neutral technological change and labor and capital augmentation factors, which is assumed in this paper to be a constant.

The aggregate nominal profit-maximizing output supply of each sector i, is the number of firms in the sector, N_i times the nominal profit-maximizing supply of each firm within that sector. In terms of growth rates, this can be expressed as

$$\hat{S}_i = \hat{N}_i + \frac{1}{\theta_i}\hat{P}_i - \frac{\alpha_i}{\theta_i}\hat{w} - \frac{\beta_i}{\theta_i}\hat{r} \tag{5.1.2}$$

Entry/Exit

I consider the simplest form of the entry/exit equation proposed by Zellner and Israilevich Zellner and Israilevich (2005)

$$\hat{N}_i = \gamma_i[\theta_i S_i - F_i] \tag{5.1.3}$$

For our analysis, just as in the examples in Zellner and Israilevich (2005), we will consider γ_i to be time invariant. γ_i is the speed of adjustment coefficient for sector i. Given that γ_i is assumed to be positive we can interpret the entry/exit equations as follows. A positive departure from equilibrium profits F_i^e will attract new firms into the industry while a negative departure will induce firms to leave the industry. The larger the value of γ_i, the faster will be this adjustment.

5.1.2 Output Demand

Since both the government and households demand goods from the two sectors, the total demand for goods in the ith sector, $i = 1, 2$, is sum of the demands from the government and the aggregate demand from households. As in Zellner and Israilevich (2005) these demands are given exogenously but some of the 'other factors' determining household demand are omitted for simplicity. The aggregate demand is thus given by,

$$S_i = G_i + P_i^{1-\eta_{ii}} P_j^{\eta_{ij}} (S(1 - T^s))^{\eta_{is}}$$

where G_i is the nominal government expenditure in sector i, $S = S_1 + S_2$ is the total income (ouput), T^s is the tax rate, η_{ii} is the own price elasticity, η_{ij} is the cross price elasticity and η_{is} is the income elasticity.

As in Zellner-Israilevich, it is assumed that G_i, $i = 1, 2$ grows at the same rate as aggregate nominal government expenditure G. Expressed in terms of growth rates, the aggregate demand for goods in each sector is the weighted sum of growth rates of demand from the government and households and is given by

$$\hat{S}_i = g_i \hat{G} + (1 - g_i)[(1 - \eta_{ii})\hat{P}_i + \eta_{ij}\hat{P}_j + \eta_{is}(\hat{S} + \hat{T}^{s'})] \qquad (5.1.4)$$

5.1.3 Factor Markets

Labor Market

Given the Cobb-Douglas technologies, the aggregate profit-maximizing labor demand from sector i is $L_i = \frac{\alpha_i S_i}{w}$. In terms of growth rates this turns out to be

$$\hat{L}_i = \hat{S}_i - \hat{w} \tag{5.1.5}$$

The government demand for labor is assumed to be exogenously given by $L_g = \frac{G_L}{w}$. Expressed in terms of growth rates, this demand is

$$\hat{L}_g = \hat{G_L} - \hat{w} \tag{5.1.6}$$

where G_L is the government's nominal demand for labor.

Thus gorwth rate of aggregate demand from labor from firms and government is found to be the weighted sum of gorwth rates of individual demands given below as

$$\frac{L_1}{L}\hat{L_1} + \frac{L_2}{L}\hat{L_2} + \frac{L_g}{L}\hat{L_g} = l_1\hat{L_1} + l_2\hat{L_2} + l_g\hat{L_g} \tag{5.1.7}$$

The explicit dependence of the weights l_i, $i = 1, 2, g$ on S_1 and S_2 is given in A. The supply of labor is assumed to be exogenous as in Zellner and Israilevich (2005),

$$L = \left(\frac{w}{P}\right)^{\delta} \left(\frac{S}{P}\right)^{\delta_s}$$

Here again we leave out the 'other factors' that affect labor supply for simplicity.

In terms of growth rates the labor supply is given as

$$\hat{L} = \delta(\hat{w} - \hat{P}) + \delta_s(\hat{S} - \hat{P}) \tag{5.1.8}$$

with δ (resp. δ_s) is the elasticity of labor supply with respect to real wage (resp. real income).

Capital Market

Similarly in the capital market, the aggregate demand from sector i is $K_i = \frac{\beta_i S_i}{r}$ while the demand for capital from government is given exogenously as $K_g = \frac{G_K}{r}$. In terms of growth rates these demands can be expressed as

$$\hat{K}_i = \hat{S}_i - \hat{r} \tag{5.1.9}$$

and

$$\hat{K}_g = \hat{G}_K - \hat{r} \tag{5.1.10}$$

respectively. As before we assume $\hat{G_K} = \hat{G}$. Thus the growth rate of aggregate capital demand is found to be the weighted sum of growth rates of demand from firms and the government as follows

$$\frac{K_1}{K}\hat{K}_1 + \frac{K_2}{K}\hat{K}_2 + \frac{K_g}{K}\hat{K} = k_1\hat{K}_1 + k_2\hat{K}_2 + k_g\hat{K}_g \tag{5.1.11}$$

The explicit dependence of k_i, $i = 1, 2, g$ on S_i is given in the appendix.

The supply of capital is given exogenously by

$$K = \left(\frac{r}{P}\right)^{\phi}\left(\frac{S}{P}\right)^{\phi_s}$$

, where ϕ (resp. ϕ_s) are the elasiticities of capital supply with respect to real rental rate (resp. real income). In terms of growth rates, the supply of capital turns out to be

$$\hat{K} = \phi(r - P) + \phi_s(S - P) \tag{5.1.12}$$

'Other factors' that may affect the supply of capital have been omitted for the sake of simplicity.

5.1.4 Government

Revenue

The government imposes a single uniform tax at the rate T^s on output/income. Corporate profit taxes are excluded in this special nested case for ease of analysis. The tax revenue collected by the government is thus given by

$$R = T^s \cdot S$$

. In terms of growth rates, the revenue equation can be written as

$$\hat{R} = \hat{T}^s + \hat{S} \tag{5.1.13}$$

Expenditure

Government spending includes expenditure on not only final output from the two sectors but on inputs as well. Thus total npminal government expenditure G is given by

$$G = G_1 + G_2 + G_L + G_K$$

where G_i with $i = 1, 2, L, K$ denotes the respective expenditures. Zellner and Israilevich assume that each component of G grows at the same rate as G itself. Thus assumeption is accommodated in the model in the following way.

$$G_i = \zeta_i G \tag{5.1.14}$$

Here ζ_i, $i = 1, 2, L, K$ denotes the fraction of total government expenditure in each market such that $\sum_i \zeta_i = 1$. In terms of growth rates, this assumption implies

$$\hat{G}_i = \hat{G} \tag{5.1.15}$$

Government Budget

Zellner and Israilevich (2005) further assume that there is an exogenously determined deficit/surplus D, defined as the government expenditures as a percentage of revenues, i.e. D = G/R. Thus the budget equation of the government in terms of growth rates is

$$\hat{G} = \hat{D} + \hat{R} = \hat{D} + \hat{T}^s + \hat{S} \tag{5.1.16}$$

5.1.5 Sales and Price Aggregates

Given that aggregate sales in the economy is $S = S_1 + S_2$, the growth rate of the sales aggregate is defined as

$$\hat{S} = s_1\hat{S}_1 + s_2\hat{S}_2 \tag{5.1.17}$$

Similarly, the growth rate of the aggregate price index P is defined as

$$\hat{P} = s_1\hat{P}_1 + s_2\hat{P}_2 \tag{5.1.18}$$

where $s_i = \frac{S_i}{S}$.

5.2 Solving the model

The above model is solved using market clearing conditions in all markets i.e output and factor markets and incorporating the government's budget equation. The complete solution procedure is outlined in Appendix A. We are able to reduce all these equations to yield the following system of dynamic equations that govern the behavior of S_1 and S_2.

$$\begin{bmatrix} \dot{S}_1 \\ \dot{S}_2 \end{bmatrix} = \boldsymbol{\mathcal{F}}(S_1, S_2; \mathbf{\Omega}) = (\boldsymbol{\mathcal{H}}(S_1, S_2; \mathbf{\Omega}))^{(-1)} \boldsymbol{\mathcal{D}}(S_1, S_2; \mathbf{\Omega}). \tag{5.2.1}$$

The explicit form of the non-linear functions in $\boldsymbol{\mathcal{F}}$ (i.e. the matrix $\boldsymbol{\mathcal{H}}$ and the vector $\boldsymbol{\mathcal{D}}$) can be found in Appendix A. $\mathbf{\Omega}$ is the vector of all structural parameters. The assumed values for these parameters are given in Appendix B.

It is easy to see that one of the equilibria can be obtained by setting

$$\boldsymbol{\mathcal{D}}(S_1, S_2; \mathbf{\Omega}) = 0$$

and solving for S_1 and S_2. This solution is particularly relevant for economic purposes as this is the value of (S_1, S_2) at which there is no further entry or exit (under the assumption the government deficit and tax rate do not change). Further examination of the entry/exit equation yields that the values of S_1 and S_2 at which the system is at equilibrium is given by

$$S_1 = \frac{1}{\theta_1} F_1 \tag{5.2.2}$$

$$S_2 = \frac{1}{\theta_2} F_2. \tag{5.2.3}$$

Chapter 6

Stability and Dynamics

Unlike the continuous one sector version of the MMM, the two sector model can exhibit oscillatory behavior for various parameter settings. To see why this may happen we first consider the dynamics in the one sector model. In the one sector model the dynamics of S is described by the following equation (which is the one sector version of Equation 5.2.1), Veloce and Zellner (1985),

$$\dot{S} = aS(S - F).$$

Here the stationary solution $S = F$ is stable if $a < 0$, which in Veloce and Zellner (1985) is true if and only if the price elasticity $\eta < 1$ (i.e. inelastic demand). This result can be understood by the following argument adapted from Veloce and Zellner (1985). Suppose $\theta S > F$ then with current profitability greater than future profitability firms will enter causing the market supply to increase resulting in a lower price. This drop in price will result in a lower aggregate sales S (due the inelasticity of demand) decreasing the difference between the current profitability, θS and equilibrium profitability, F. In a continuous time model this process will result in a monotonic path to equilibrium. However, if demand is elastic ($\eta > 1$) then the solution is unstable and any deviation from the equilibrium will result in divergence.

In a multisector model we need to consider the effect of cross price and income elasticities along with own price elasticity. The interesting features arising from the analysis of the two sector model in this regards is two fold:

1. Even when the two sectors have elastic demand (own price elasticity greater than 1), the solution may be stable.

2. The path to the long run equilibrium may not be monotonic i.e. it may depict oscillatory behavior.

To understand why these results may obtain consider the following argument when Sector 1 is out of equilibrium. This argument is very general and can be directly adapted to the case of Sector 2. Suppose the two sectors produce normal goods that are substitutes and consider a situation where Sector 1 is out of equilibrium say, $S_1 > \frac{1}{\theta_1}F_1$ and Sector 2 is at equilibrium i.e. $S_2 = \frac{1}{\theta_2}F_2$. Then current profitability being higher than equilibrium profitability in Sector 1 will result in entry and hence an increase in industry supply which in turn causes a drop in Sector 1 price, P_1. This decline in P_1 will affect industry sales through two channels:

1. Since demand is elastic the decline in price will cause industry sales, S_1 to increase. Even though this may seem destabilizing there may be a cross price and aggregate income effect that may offset this potentially destabilizing effect as described in the following point.

2. The decline in P_1 will cause a decrease in Sector 2 demand (the goods being substitutes) causing a decline in Sector 2 price, P_2 and quantity, Q_2 and hence a decline in Sector 2 sales, S_2. If this decline in S_2 is greater in magnitude than the increase in S_1 given in the previous point then aggregate sales (or nominal income) $S = S_1 + S_2$ will decline. Since the goods are normal goods this decline in S will result in S_1 declining.

It is thus clear that there are two opposing effects on S_1 when Sector 1 profitability is greater than equilibrium profitability. Thus if the second effect dominates the former then S_1 will decrease bringing it closer to the equilibrium level $\frac{1}{\theta_1}F_1$ given in Equation 5.2.2. Of course, this would imply even though there are more firms in Sector 1 now they are each producing less given the cross price and income effects stemming from changes in Sector 2. Note that the above analysis will be valid even if S_2 is not initially at the equilibrium level.

Now notice that the decline in P_1 has caused a decrease in Sector 2 demand and hence a decline in S_2 pushing Sector 2 sales below the equilibrium level $\frac{1}{\theta_2}F_2$. We can adapt the previous analysis and apply it to Sector 2 sales being below the equilibrium level which will ultimately result in S_2 increasing.

The above explanation indicates an oscillatory convergence to equilibrium. It is worth mentioning that the above delicate mechanism depends crucially on the cross price and income elasticities and the magnitude of shifts in demand and supply in each sector. Thus it is definitely possible that these shifts are not sufficient and may result in the solution being unstable. Also in all of the above analysis the elasticity parameters need to be consistent with values of the other parameters in production, input markets, entry/exit equations and government policy. If these parameters were to change, for instance a change in the government policy like taxes, then it is very likely that the economy may go from cyclical convergence to stable cycles or even explosive behavior.

The main objective of this thesis is to examine the nature of this dynamic behavior when these other parameters vary. The technique used involves the search for bifurcation points/boundaries within the theoretically feasible parameter space of this model. The behavior of the model can be drastically different depending on which side of the boundary the true parameter lies.

Chapter 7

Bifurcation Analysis of MMM

7.1 Calibration

The MMM does not contain definitive parameter estimates due to non-availablity of appropriate data for all the variables in the model. In this section I will discuss some of the calibrated parameter values that were chosen for the bifurcation analysis. The exogenously fixed deficit/surplus term was defined as the ratio of government expenditure and government revenue. I considered the annual government current receipts and current expenditure for the period 1929 to 2009 for the US economy from the St.Louis FRED. This ratio calculated for each year for this entire period ranged from 0.702703 (for 1929) to 2.105263 (for 1931) with an average of 1.139. We choose $D = 1.2$(respectively $D = 1.4$) for our bifurcation analysis with respect to F_1(respectively T^s). To justify a zero growth rate for D, we calculated the growth rate of D for this entire period. The average growth rate turns out to be -0.004087071.

The model takes account of only income tax. The value of income tax rate considered is 25%. The US federal income tax rate ranges from 10% to 35%

depending on the marital status and other factors of the individual filing the tax.

The future aggregate profitability parameter in the entry/exit equation for sector 2 is fixed at $F_2 = 2$ which is the same as considered by Zellner and Israilevich (2005).

7.2 Bifurcation Analysis

A bifurcation refers to a qualitative change in the dynamics of a system due to small changes in control parameter value(s).

Consider the dynamical system:

$$\dot{\mathbf{x}} = \boldsymbol{f}(\mathbf{x}, \boldsymbol{\alpha}), \mathbf{x} \in \mathbb{R}^n, \boldsymbol{\alpha} \in \mathbb{R}^m \tag{7.2.1}$$

with smooth right hand side.

In the System 7.2.1, with $n = 2$, denote the Jacobian of $\boldsymbol{f}$ with respect to $\mathbf{x}$ by $\mathbf{J}_f$, its trace by $tr(\mathbf{J}_f)$ and its determinant by $Det(\mathbf{J}_f)$. Let $\omega = \sqrt{Det(\mathbf{J}_f(\boldsymbol{\alpha})}$. Then the eigenvalues of $\mathbf{J}_f$ are given by

$$\lambda_{1,2} = \frac{1}{2}\left(tr(\mathbf{J}_f) \pm \sqrt{(tr(\mathbf{J}_f))^2 - 4Det(\mathbf{J}_f)}\right).$$

Consider the following two dimensional system depending on a single parameter $\alpha \in \mathbb{R}$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sigma(x_1^2 + x_2^2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $\sigma = sign(l_1(\alpha))$ is the sign of the first Lyapunov coefficient. This is the normal form for the Hopf bifurcation and σ determines the type of Hopf bifurcation, i.e. supercritical or subcritical. This system has an equilibrium at $(x_1, x_2) = (0, 0)$ and when $\alpha = 0$ has a pair of purely imaginary eigenvalues. Thus this system has a Hopf bifurcation at $\alpha = 0$. If $\sigma > 0$ (*resp.* $\sigma < 0$) then there is a supercritical (resp. subcritical) Hopf bifurcation at $\alpha = 0$, i.e. there are stable (resp. unstable) limit cycles bifurcating from the equilibrium for $\alpha > 0$ (*resp.* $\alpha < 0$).

Consider any generic system as in System 7.2.1 with $n = 2$ and $m = 1$, with an equilibrium $(x_1, x_2) = (0, 0)$ and satisfying the Hopf bifurcation conditions at $\alpha = 0$. We can use the following two theorems in Kuznetsov (2004) to rewrite the system in the normal form and do all the necessary analysis for a Hopf bifurcation.

Theorem 7.2.1 *Suppose that System 7.2.1 (with $n = 2$ and $m = 1$) with smooth f, has for all sufficiently small $|\alpha|$ the equilibrium $(x_1, x_2) = (0, 0)$ with eigenvalues*

$$\lambda_{1,2} = \mu(\alpha) \pm i\omega(\alpha)$$

where $\mu(0) = \frac{1}{2}(tr(\mathbf{J}_f(0))) = 0, \omega_0 = \left(\sqrt{Det(\mathbf{J}_f(0))}\right) > 0.$

Let the following conditions be satisfied:

1. *(Non-degeneracy Condition) $l_1(0) \neq 0$, where l_1 is the first Lyapunov coefficient;*

2. *(Transversality Condition) $\mu'(0) \neq 0$.*

Then, there are invertible co-ordinate and parameter changes and time reparam-

eterization transforming System 7.2.1 into

$$\frac{d}{d\tau}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \beta & -1 \\ 1 & \beta \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \pm (y_1^2 + y_2^2)\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + O(||y||^4).$$

Theorem 7.2.2 *Any generic system 7.2.1 (with $n = 2$ and $m = 1$) having at $\alpha = 0$ the equilibrium $(x_1, x_2) = (0, 0)$ with eigenvalues*

$$\lambda_{1,2}(0) = \pm i\omega(0), \qquad \omega(0) > 0$$

is topologically equivalent to one of the following normal forms:

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \beta & -1 \\ 1 & \beta \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \pm (y_1^2 + y_2^2)\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

If System 7.2.1 (with $n = 2$ and $m = 1$) has a solution $(x_1, x_2) \neq (0, 0)$ and the Hopf bifurcation conditions hold at some critical $\alpha_0 \neq 0$, i.e. $tr(\mathbf{J}_f(\alpha_0)) = 0$ and $Det(\mathbf{J}_f(\alpha_0)) > 0$, we can still apply the above Hopf bifurcation analysis to this situation, keeping in mind the direction of change in α and the corresponding change in the nature of the equilibrium (stable to unstable) and conclude about the type of limit cycles (stable or unstable cycles) based on the sign of the first Lyapunov coefficient $l_1(\alpha_0)$.

7.2.1 Hopf Bifurcation with Sector One Entry/Exit Parameter

In order to analyze a codim-1 Hopf bifurcation for the System A.0.7, we first look for the value of (S_1, S_2) and the bifurcation parameter (F_1) at which the following conditions hold simultaneously:

$$\mathcal{F}_1(S_1, S_2, F_1) = 0 \tag{7.2.2}$$

$$\mathcal{F}_2(S_1, S_2, F_1) = 0 \tag{7.2.3}$$

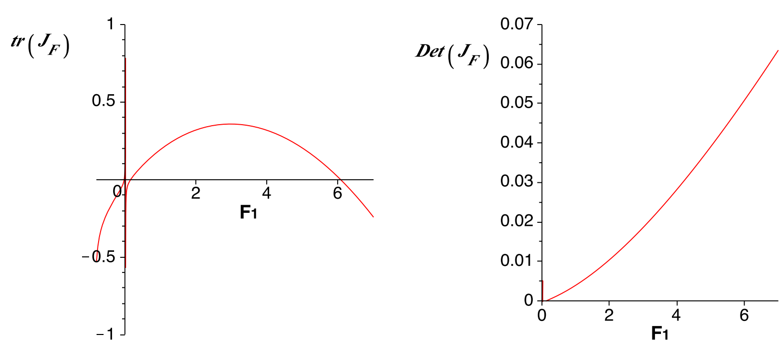
$$tr(\mathbf{J}_f(S_1, S_2, F_1)) = 0 \tag{7.2.4}$$

$$det(\mathbf{J}_f(S_1, S_2, F_1)) > 0 \tag{7.2.5}$$

Note that since all the parameters in $\mathbf{\Omega}$ are fixed, except F_1, the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ explicitly depend on F_1. It is easy to verify that one of the solutions to Equation 7.2.2 and Equation 7.2.3 is

$$S_1 = 5F_1 \text{ and } S_2 = 10.$$

Using this solution in the trace and determinant of $\mathbf{J}_f$ we obtain the following plots.



(a) Trace of J_F **(b) Determinant of J_F**

Figure 7.1: Trace and Determinant of J_f

In Figure 7.1(a) we notice that $tr(\mathbf{J}_f) > 0$ for $0 < F_1 < 6.070386762$ and $tr(\mathbf{J}_f) < 0$ for $F_1 > 6.070386762$. Denote $F_H = 6.070386762$. Since the eigenvalues of $\mathbf{J}_f$ can be written as

$$\lambda_{1,2} = \frac{1}{2}\left(tr(\mathbf{J}_f) \pm \sqrt{(tr(\mathbf{J}_f))^2 - 4Det(\mathbf{J}_f)}\right),$$

we can conclude that this solution is unstable for $0 < F_1 < F_H$ and stable for $F_1 > F_H$. There is a change of stability at the critical parameter value $F_1 = F_H$. Also at this critical parameter value, from Figure 7.1(b) we see that the the determinant is positive, implying that the eigenvalues are purely imaginary. Moreover at this critical parameter value we find that the transversality condition and non-degeneracy conditions from Theorem 10.2 are satisfied since

$$\frac{d}{dF_1}\left(tr(\mathbf{J}_f(F_1))\right)\Big|_{F_H} = 0.1626547356 \neq 0 \tag{7.2.6}$$

$$l_1(F_1)\Big|_{F_H} = -5.171543710 * 10^9 < 0. \tag{7.2.7}$$

Thus we can conclude that a unique and stable limit cycle bifurcates from the equilibrium via a Hopf bifurcation for $F_1 < F_H$.

The following two figures depict the paths of S_1 and S_2 and the phase space for arbitrary starting values and different values of F_1. In Figure 7.2(a) and Figure 7.2(b) we see that S_1 and S_2 converge to the equilibrium values which is also evident in the phase space in Figure 7.2(c). Once the parameter F_1 crosses the critical value and becomes less than F_H we can see from Figure 7.3 (a) and 7.3(b) that S_1 and S_2 have a cyclical path around the equilibrium. From the phase space in Figure 7.3(c) we see that S_1 and S_2 will continue on this cyclical path.

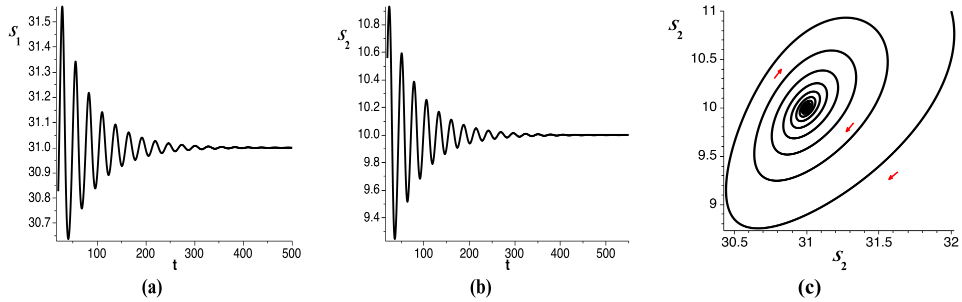


Figure 7.2: Path of S_1 and S_2 for $F_1 = 6.2 > F_H$

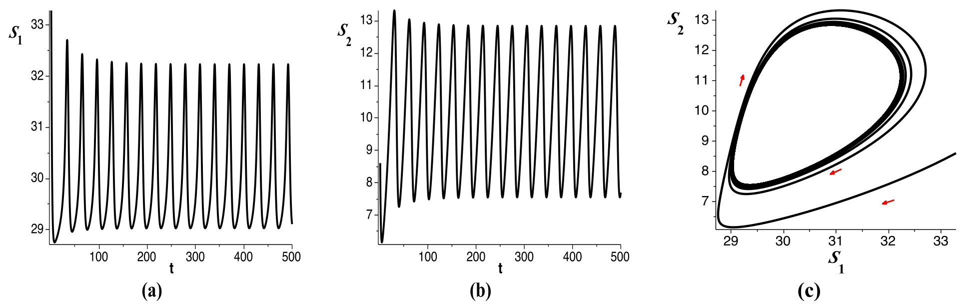


Figure 7.3: Path of S_1 and S_2 for $F_1 = 6 < F_H$

7.2.2 Hopf Bifurcation with Income Tax Rate

A Hopf bifurcation analysis is also conducted with respect to the tax rate T^s. The following graphs depict $tr(\mathbf{J}_f(S_1, S_2, T^s))$ and determinant $Det(\mathbf{J}_f)$ as functions of the bifurcation parameter T^s. In Figure 7.4, the trace crosses the positive axis at $T^s = 0.26$ while the determinant is positive for this value as well. Denote the value $T^s = 2.6$ as T^s_H. For all values of $T^s > T^s_H$, the trace is negative implying a stable equilibrium while for $T^s < T^s_H$, the equilibrium is unstable.Thus at exactly T^s_H, the system loses stability via a Hopf bifurcation and a limit cycle emerges. Figures 7.5 and 7.6 illustrate the time paths of S_1, S_2 and the phase space plots for these variables.

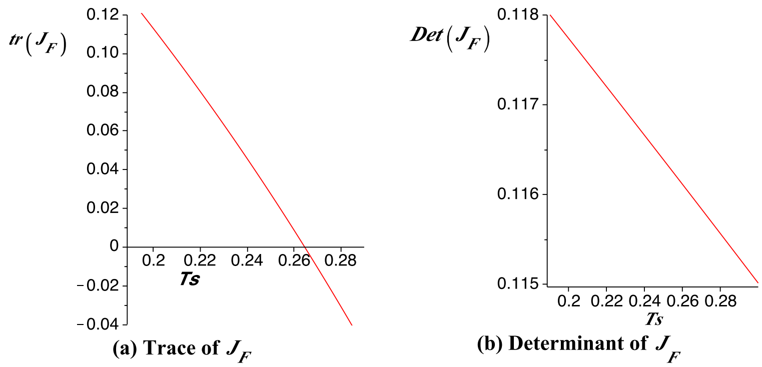


Figure 7.4: Trace and Determinant of J_f

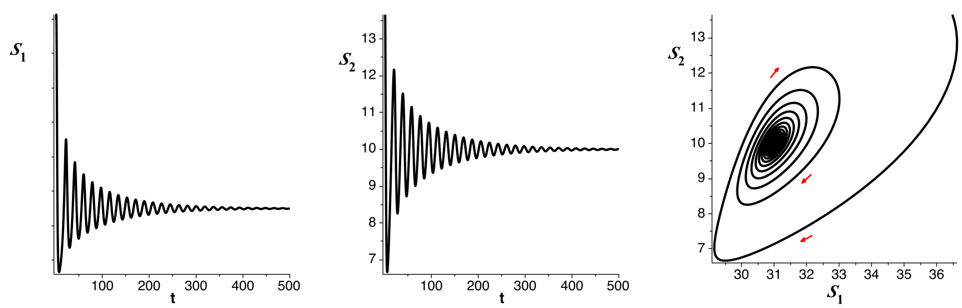


Figure 7.5: Path of S_1 and S_2 for $T_s = 0.275 > T_H^s$

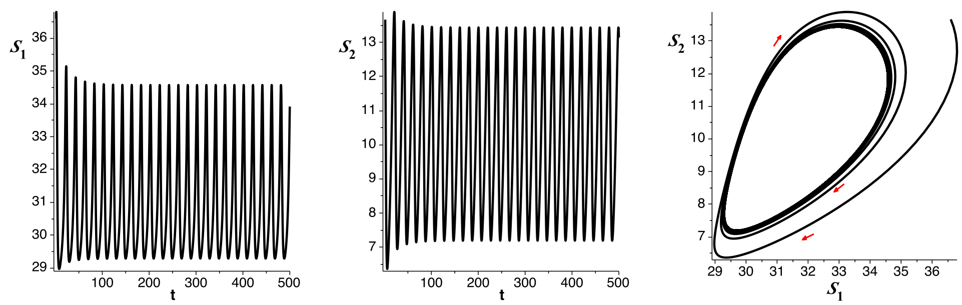


Figure 7.6: Path of S_1 and S_2 for $T_s = 0.25 < T_H^s$

Chapter 8

Conclusion and Future Research

The MMM provides a novel way to model economic activity in an economy disaggregated by industrial sectors. This micro-based model analyzes the dynamics of variables at the aggregate as well as the disaggregate levels in the presence of interactions among households, firms, the government and monetary sector. In my dissertation, I have chosen to investigate the dynamical properties of a special nested case of the MMM with an emphasis on detecting local bifurcations with respect to the entry/exit parameter F_1 and the tax rate T^s. For the sake of analytical simplicity, the model under consideration excludes the monetary sector.

Bifurcation analysis of the parameter space of the MMM indicated the presence of Hopf points. Two very important facts emerge from this analysis. First, it is possible for the two sector MMM to exhibit a stable long-run equilibrium even if demand is elastic. Second, convergence to the long-run equilibrium will be oscillatory. Both these phenomena can be explained using the dynamic entry/exit equations via which the entire adjustment takes place. This re-emphasizes the importance of a dynamic entry/exit equation in models of this class.

There are several avenues that need to be explored from here. Zellner and Israilevich (2005) pointed out that the possibilities of improving this bare-bones MMM are vast. Modifying the entry/exit equations to incorporate expectations, introduction of dynamic optimization, other market structures and demand and production relations are just a few of these possibilities. We are confident from our analysis that incorporation of any of these features would yield very interesting dynamical behavior with the possibility of other kinds of bifurcation. As a future research agenda we would like to introduce the money market and examine the possibility of a singularity-induced bifurcation with respect to government policy parameters.

Appendix A

Deriving the Dynamic Equations

Firstly we recall that the government's budget equation in growth rates is

$$\hat{G} = \hat{R} + \hat{D} = \hat{D} + \hat{T}^s + s_1\hat{S}_1 + s_2\hat{S}_2 \tag{A.0.1}$$

Using the assumption that each component of the government expenditure grows at the same rate total government expenditure the output demand functions in terms of growth rates an be expressed as:

$$\hat{S}_i = g_i\hat{G} + (1 - g_i)[(1 - \eta_{ii})\hat{P}_i + \eta_{ij}\hat{P}_j + \eta_{is}(\hat{S} + \hat{T}^{s'})] \tag{A.0.2}$$

In this expression we also renamed taxes by $T^{s'} = (1 - T^s)$ and thus $\hat{T}^{s'} = \frac{-T^s}{(1-T^s)}\hat{T}^s$. The weight g_i can be shown to explicitly depend on the level variables S_1 and S_2 to get

$$g_i = \frac{G_i}{S_i} = \frac{\zeta_i DT^s(S_1 + S_2)}{S_1}.$$

We can now express the system of demand equations given by Equation A.0.2 in matrices and solve for growth rate of prices as follows.

$$\boldsymbol{\mathcal{N}}\begin{bmatrix}\hat{P}_1\\ \hat{P}_2\end{bmatrix} = \boldsymbol{\mathcal{A}}\begin{bmatrix}\hat{S}_1\\ \hat{S}_2\end{bmatrix} + \begin{bmatrix}C_1\\ C_2\end{bmatrix} \tag{A.0.3}$$

Where

$$\boldsymbol{\mathcal{N}} = \begin{bmatrix} (1-\eta_{11}) & \eta_{12} \\ \eta_{21} & (1-\eta_{22}) \end{bmatrix},$$

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} \frac{1}{1-g_1} + [(1-\eta_{s_1}) - \frac{1}{1-g_1}]s_1 & (1-\eta_{s_1} - \frac{1}{1-g_1})s_2 \\ (1-\eta_{s_2} - \frac{1}{1-g_2})s_1 & \frac{1}{1-g_2} + [(1-\eta_{s_2}) - \frac{1}{1-g_2}]s_2 \end{bmatrix},$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} (1+\eta_{s_1}\frac{T^s}{1-T^s} - \frac{1}{1-g_1})\hat{T}^s - \frac{g_1}{1-g_1}\hat{D} \\ (1+\eta_{s_2}\frac{T^s}{1-T^s} - \frac{1}{1-g_2})\hat{T}^s - \frac{g_1}{1-g_1}\hat{D} \end{bmatrix}.$$

Assumption A.0.1 *We assume that the cross price elasticities are such that $\boldsymbol{\mathcal{N}}$ is invertible.*

We denote $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{N}}^{-1}$ and $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{A}}$ to obtain the following:

$$\begin{bmatrix} \hat{P}_1 \\ \hat{P}_2 \end{bmatrix} = \boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{A}} \begin{bmatrix} \hat{S}_1 \\ \hat{S}_2 \end{bmatrix} + \boldsymbol{\mathcal{P}} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \boldsymbol{\mathcal{B}} \begin{bmatrix} \hat{S}_1 \\ \hat{S}_2 \end{bmatrix} + \boldsymbol{\mathcal{P}} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{A.0.4}$$

Moving to the factor markets, the growth rate of aggregate profit maximizing labor demand from each sector i is $\hat{L}_i = \hat{S}_i - \hat{w}$ and the labor demand from the government is $\hat{L}_g = \hat{G} - \hat{w}$. Since the total demand for labor is the sum of sectoral demand and the government demand for labor the of growth rate of the total demand for labor is given by the following weighted sum

$$\frac{L_1}{L}\hat{L}_1 + \frac{L_2}{L}\hat{L}_2 + \frac{L_g}{L}\hat{L}_g = l_1\hat{L}_1 + l_2\hat{L}_2 + l_g\hat{L}_g$$

where the weights are determined by the level variables as follows

$$l_i = \frac{L_i}{L} = \frac{\alpha_i S_i}{\alpha_1 S_1 + \alpha_2 S_2 + \zeta_l D T^s (S_1 + S_2)}$$
$$l_g = \frac{L_g}{L} = \frac{\zeta_l D T^s (S_1 + S_2)}{\alpha_1 S_1 + \alpha_2 S_2 + \zeta_l D T^s (S_1 + S_2)}.$$

Equating the growth rates of labor demand and labor supply we can solve for growth rate of equilibrium wage rate

$$\hat{w} = \frac{1}{1+\delta}\Bigg\{(l_1 + (l_g - \delta_s)s_1)\hat{S}_1 + (l_2 + (l_g - \delta_s)s_2)\hat{S}_2$$
$$+(\delta + \delta_s)(s_1\hat{P}_1 + s_2\hat{P}_2) + l_g(\hat{D} + \hat{T}^s))\Bigg\}.$$

Working through the capital market equations as in the labor market equations we have the growth rate of aggregate capital demand as

$$\hat{K} = \frac{K_1}{K}\hat{K}_1 + \frac{K_2}{K}\hat{K}_2 + \frac{K_g}{K}\hat{K} = k_1\hat{K}_1 + k_2\hat{K}_2 + k_g\hat{K}_g$$

where the weights are determined by the level variables as follows

$$k_i = \frac{K_i}{K} = \frac{\beta_i S_i}{\beta_1 S_1 + \beta_2 S_2 + \zeta_k D T^s(S_1 + S_2)}$$
$$k_g = \frac{K_g}{K} = \frac{\zeta_k D T^s(S_1 + S_2)}{\beta_1 S_1 + \beta_2 S_2 + \zeta_k D T^s(S_1 + S_2)}.$$

Equating the growth rates of capital demand and capital supply we can solve for growth rate of equilibrium rental rate

$$\hat{r} = \frac{1}{1+\phi}\Bigg\{(k_1 + (k_g - \phi_s)s_1)\hat{S}_1 + (k_2 + (k_g - \phi_s)s_2)\hat{S}_2$$
$$+(\phi + \phi_s)(s_1\hat{P}_1 + s_2\hat{P}_2) + k_g(\hat{D} + \hat{T}^s)\Bigg\}$$

Now we can substitute $\hat{N}_i, \hat{w}, \hat{r}$ and $\hat{P}_i$ into the output supply (growth rate) equations to find $\dot{S}_1$ and $\dot{S}_2$. Let us define

$$
\begin{aligned}
\boldsymbol{\mathcal{H}} &= \Bigg\{ \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} \\
&- \begin{bmatrix} \frac{(S_1+S_2)-\left(\frac{\alpha_1(\delta+\delta_s)}{(1+\delta)}+\frac{\beta_1(\phi+\phi_s)}{(1+\phi)}\right)S_1}{(S_1+S_2)} & \frac{-\left(\frac{\alpha_1(\delta+\delta_s)}{(1+\delta)}+\frac{\beta_1(\phi+\phi_s)}{(1+\phi)}\right)S_2}{(S_1+S_2)} \\ \frac{-\left(\frac{\alpha_2(\delta+\delta_s)}{(1+\delta)}+\frac{\beta_2(\phi+\phi_s)}{(1+\phi)}\right)S_1}{(S_1+S_2)} & \frac{(S_1+S_2)-\left(\frac{\alpha_2(\delta+\delta_s)}{(1+\delta)}+\frac{\beta_2(\phi+\phi_s)}{(1+\phi)}\right)S_2}{(S_1+S_2)} \end{bmatrix} \\
&\times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \\
&+ \begin{bmatrix} \frac{\alpha_1\left(l_1(S_1+S_2)+(l_g-\delta_s)S_1\right)}{(1+\delta)(S_1+S_2)} & \frac{\alpha_1\left(l_2(S_1+S_2)+(l_g-\delta_s)S_2\right)}{(1+\delta)(S_1+S_2)} \\ \frac{\alpha_2\left(l_1(S_1+S_2)+(l_g-\delta_s)S_1\right)}{(1+\delta)(S_1+S_2)} & \frac{\alpha_2\left(l_2(S_1+S_2)+(l_g-\delta_s)S_2\right)}{(1+\delta)(S_1+S_2)} \end{bmatrix} \\
&+ \begin{bmatrix} \frac{\beta_1\left(k_1(S_1+S_2)+(k_g-\phi_s)S_1\right)}{(1+\phi)(S_1+S_2)} & \frac{\beta_1\left(k_2(S_1+S_2)+(k_g-\phi_s)S_2\right)}{(1+\phi)(S_1+S_2)} \\ \frac{\beta_2\left(k_1(S_1+S_2)+(k_g-\phi_s)S_1\right)}{(1+\phi)(S_1+S_2)} & \frac{\beta_2\left(k_2(S_1+S_2)+(k_g-\phi_s)S_2\right)}{(1+\phi)(S_1+S_2)} \end{bmatrix} \Bigg\} \times \begin{bmatrix} \frac{1}{S_1} & 0 \\ 0 & \frac{1}{S_2} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\mathcal{D}} &= \begin{bmatrix} \theta_1\gamma_1\big(\theta_1 S_1 - F_1\big) \\ \theta_2\gamma_2\big(\theta_2 S_2 - F_2\big) \end{bmatrix} \\
&- \begin{bmatrix} \frac{\alpha_1 l_g(\hat{D}+\hat{T}^s)}{1+\delta} + \frac{\beta_1 k_g(\hat{D}+\hat{T}^s)}{1+\phi} \\ \frac{\alpha_2 l_g(\hat{D}+\hat{T}^s)}{1+\delta} + \frac{\beta_2 k_g(\hat{D}+\hat{T}^s)}{1+\phi} \end{bmatrix} \\
&+ \begin{bmatrix} \frac{(S_1+S_2)-\left(\frac{\alpha_1(\delta+\delta_s)}{(1+\delta)}+\frac{\beta_1(\phi+\phi_s)}{(1+\phi)}\right)S_1}{(S_1+S_2)} & \frac{-\left(\frac{\alpha_1(\delta+\delta_s)}{(1+\delta)}+\frac{\beta_1(\phi+\phi_s)}{(1+\phi)}\right)S_2}{(S_1+S_2)} \\ \frac{-\left(\frac{\alpha_2(\delta+\delta_s)}{(1+\delta)}+\frac{\beta_2(\phi+\phi_s)}{(1+\phi)}\right)S_1}{(S_1+S_2)} & \frac{(S_1+S_2)-\left(\frac{\alpha_2(\delta+\delta_s)}{(1+\delta)}+\frac{\beta_2(\phi+\phi_s)}{(1+\phi)}\right)S_2}{(S_1+S_2)} \end{bmatrix} \\
&\times \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}
\end{aligned}
$$

In the representation above, the expansions for the matrices $\boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{P}}$ and the elements C_1, C_2 are described in Equation A.0.4. Our final dynamic equations

$\dot{S}_1$ and $\dot{S}_2$ can now be written as the following system of dynamic equations

$$\boldsymbol{\mathcal{H}}(S_1, S_2; \boldsymbol{\Omega}) \begin{bmatrix} \dot{S}_1 \\ \dot{S}_2 \end{bmatrix} = \boldsymbol{\mathcal{D}}(S_1, S_2; \boldsymbol{\Omega}) \tag{A.0.5}$$

where $\boldsymbol{\Omega}$ is the vector of all structural parameters of the model.

If $\boldsymbol{\mathcal{H}}(S_1, S_2; \boldsymbol{\Omega})$ is invertible then we can further reduce this to the following system of ordinary differential equations (ODEs).

$$\begin{bmatrix} \dot{S}_1 \\ \dot{S}_2 \end{bmatrix} = (\boldsymbol{\mathcal{H}}(S_1, S_2; \boldsymbol{\Omega}))^{(-1)} \boldsymbol{\mathcal{D}}(S_1, S_2; \boldsymbol{\Omega}). \tag{A.0.6}$$

$$\begin{bmatrix} \dot{S}_1 \\ \dot{S}_2 \end{bmatrix} = \boldsymbol{\mathcal{F}}(S_1, S_2; \boldsymbol{\Omega}) = \begin{bmatrix} \mathcal{F}_1(S_1, S_2; \boldsymbol{\Omega}) \\ \mathcal{F}_2(S_1, S_2; \boldsymbol{\Omega}) \end{bmatrix} \tag{A.0.7}$$

Appendix B

Calibration Parameters for Bifurcation with F_1

In the following table BIF indicates the bifurcation parameter.

Table B.1: Parameterizations ($\mathbf{\Omega}$)

Production		Entry/Exit		Government		Elasticities	
α_1	0.6	γ_1	0.5	D	1.2	η_{11}	2
β_1	0.2	γ_2	0.1	$\hat{D}$	0	η_{12}	1
θ_1	0.2	F_1	BIF	T^s	0.25	η_{21}	2
α_2	0.2	F_2	2	$\hat{T}^s$	0	η_{22}	2
β_2	0.6			ζ_1	0.2	δ	1
θ_2	0.2			ζ_2	0.2	δ_s	-1
				ζ_l	0.4	ϕ	1
				ζ_k	0.2	ϕ_s	-1
						η_{1s}	1
						η_{2s}	1

Appendix C

Calibration Parameters for Bifurcation with T^s

In the following table BIF indicates the bifurcation parameter.

Table C.1: Parameterizations ($\mathbf{\Omega}$)

Production		Entry/Exit		Government		Elasticities	
α_1	0.6	γ_1	0.55	D	1.4	η_{11}	2
β_1	0.2	γ_2	0.2	$\hat{D}$	0	η_{12}	1
θ_1	0.2	F_1	6.2	T^s	BIF	η_{21}	2
α_2	0.2	F_2	2	$\hat{T}^s$	0	η_{22}	2
β_2	0.6			ζ_1	0.2	δ	1
θ_2	0.2			ζ_2	0.2	δ_s	-1
				ζ_l	0.4	ϕ	1
				ζ_k	0.2	ϕ_s	-1
						η_{1s}	1
						η_{2s}	1

Bibliography

Bailey, P.W., Hall, V.B. and Phillips, P.C.B. (1987) A model of output, employment, capital formation and inflation in G. Gandolfo and F. Marzano, ed. *Keynesian Theory Planning Models and Quantitative Economics: Essays in Memory of Vittorio Marrama*, Giuffre, Milano.

Bala, V. (1997), A pitchfork bifurcation in the tatonnement process. *Economic Theory*, 10, 521-530.

Barnett, W.A. (2000), Perspective on the current state of macroeconomic theory. *In W.A. Barnett and A. Serletis (eds.), The Theory of Monetary Aggregation*, North Holland, Amsterdam, 600-601.

Barnett, W.A. and Chen, P. (1988), The aggregation theoretic monetary aggregates are chaotic and have strange attractors: An econometric application of mathematical chaos. *In W.A. Barnett, E. R. Berndt, & H. White (eds.), Dynamic Econometric Modeling*, Cambridge University, New York, 199-246.

Barnett, W.A. and Duzhak, E.A. (2008), Non-robust dynamic inferences from macroeconometric models: Bifurcation Stratification of Confidence Regions. *Physica A*, 387, 3817-3825.

Barnett, W.A. and Duzhak, E.A. (2010), Empirical assessment of bifurcation regions within New Keynesian models. *Economic Theory*, 45(1),99-128.

Barnett, W.A. and He, Y. (2001), Nonlinearity, chaos, and bifurcation: A competition and an experiment. *Economic theory, dynamics and markets: Essays in Honor of Ryuzo Sato*, Dordrecht, Kluwer, 167-187.

Barnett, W.A. and He, Y. (2002), Stabilization policy as bifurcation selection: Would stabilization policy work if the economy really were unstable? *Macroeconomic Dynamics*, 6, 713-747.

Barnett, W.A. and He, Y. (2004), Bifurcations in macroeconomic models. *In: Dowrick, S., Pitchford, R., Turnovsky, S. (eds.) Economic Growth and Macroeconomic Dynamics: Recent Developments in Economic Theory*, Cambridge University Press, Cambridge, 95-112.

Barnett, W.A. and He, Y. (2010), Existence of singularity bifurcation in an Euler-equations model of the United States economy: Grandmont was right. *Economic Modelling*, special issue edited by Arnold Zellner, Lawrence Klein and Nicholas Tavlas in honor of PAVB Swamy, 27(6), 1345-1354.

Barnett, W.A., Gallant, R., Hinich, M.J., Jungeilges, J., Kaplan, T. D and Jensen, M (1997), A single-blind controlled competition among tests for nonlinearity and chaos. *Journal of Econometrics*, 82(1), 157-192.

Benhabib, J. and Day, R.H. (1982), A characterization of erratic dynamics in the overlapping generations model. *Journal Economic Dynamics Control*, 4, 37 - 55.

Benhabib, J. and Nishimura, K. (1979), The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal economic growth. *Journal of Economic Theory*, 21(3), 421–444.

Berck, P. and Perloff, J.M. (March 1984), An open-access fishery with rational expectations. *Econometrica*, 52(2), 489-506.

Bergstrom, A.R. (1996), Survey of continuous time econometrics *In W.A. Barnett, G. Gandolfo, and C. Hillinger, ed.,. Dynamic Disequilibrium Modeling*, Cambridge University Press, Cambridge, 3-26.

Bergstrom, A.R. and Wymer, C.R. (1976), A model of disequilibrium neoclassic growth and its application to the United Kingdom. *In A.R. Bergstrom (ed.), Statistical Inference in Continuous Time Economic Models*, North Holland, Amsterdam.

Bergstrom, A.R., Nowman, K.B. and Wymer, C.R. (1992) Gaussian estimation of a second order continuous time macroeconometric model of the UK. *Economic Modelling*, 9, 313-351.

Boldrin, M. and Woodford, M. (1989), Equilibrium models displaying endogenous fluctuations and chaos: a survey. *Journal of Monetary Economics*, 25, 189-222.

Chappell, W.F., Kimenyi, M.S. and Mayer, W.J. (1990) A Poisson probability model of entry and market structure with an application to the US industries during 1972-77. *Southern Economic Journal*, 56, 918–927.

Chetty, V.K. and Heckman, J.J. (1986), A dynamic model of aggregate output supply, factor demand and entry and exit for a competitive industry with heterogeneous plants. *Journal of Econometrics*, 33(1-2), 237-262.

Comaner, W.S. and Wilson, T.A. (1967), Advertising, market structure and performance. *The Review of Economics and Statistics*, 49(4), 423-440.

Donaghy, K.P. (1993), A continuous time model of the United States economy in G. Gandolfo, ed., *Continuous Time Econometrics*, Chapman and Hall, London.

Duetsch, L.L. (1975), Structure, performance and the net rate of entry into manufacturing industries. *Southern Economic Journal*, 41, 450-456.

Duetsch, L.L. (1984), Entry and extent of multiplant operations. *The Journal of Industrial Economics*, 32, 477-488.

Ferreira, R.D. and Braga, T.L. (2005), Non-linear endogenous fluctuations with free entry and variable markups. *Journal of Economic Dynamics & Control*, 29, 847-871.

Fusari, A. (1990) A macrodynamic model centred on the interrelationships between the public and private sectors of the Italian economy. *New York Economic Review*, 20, 32-58.

Gandolfo, G. (2009) Economic dynamics: Methods and models, North-Holland, Amsterdam.

Gandolfo, G. and Padoan, P.C. (1982), Policy simulations with a continuous time macrodynamic model of the Italian Economy: A preliminary analysis. *Journal of Economic Dynamics and Control*, 4, 205-224.

Gandolfo, G. and Padoan, P.C. (1984), A disequilibrium model of real and financial accumulation in an open economy. Springer Verlag, Berlin.

Gandolfo, G. and Padoan, P.C. (1987), The mark V version of the Italian continuous time model. Instituto di Economia della Facolta di Scienze Economiche e Bancarie.

Gandolfo, G. and Padoan, P.C. (1990), The Italian continuous time model, theory and empirical results. *Economic Modelling*, 7, 91-132.

Garcia-Ferrer, A. and Highfield, R. and Palm, F.C. and Zellner, A. (1987), Macroeconomic forecasting using pooled international data. *Journal of Business and Economic Statistics*, 5:53-67.

Goodwin, R. (1951), The nonlinear accelerator and the persistence of business cycles. *Econometrica*, 19(1), 1-17.

Goodwin, R. (1967), A growth cycle. *Econometrica*, 19(1), 1-17.

Grandmont, J.-M. (1985), On endogenous competitive business cycles. *Econometrica*, 53(5), 995-1045.

Highfield, R. and Smiley, R. (1987), New business starts and economic activity. *International Journal of Industrial Organization*, 5, 51-66.

Hilke, J.C. (1984), Excess capacity and entry: Some empirical evidence. *The Journal of Industrial Economics*, 33, 233-240.

Hillinger, C. and Reiter, M. (1992) The quantitative and qualitative explanation of macroeconomic investment and production cycles in C. Hillinger, ed., *Cyclical Growth in Market and Planned Economies*, Clarendon Press, Oxford.

Hillinger, C. and Schuler, K.W. (1978), Cyclical fluctuations in the German economy: A continuous time econometric model . *Munich Social Science Review*, 75–85.

Hillinger, C., Bennett, J. and Benoit, M.L. (1973), Cyclical fluctuations in the US economy. *Dynamic Modelling and Control of National Economics*, (IEEE Conference Publication no.101).

Hirschey, M. (1981), The effect of advertising on industrial mobility, 1947-1972. *The Journal of Business*, 54, 329-339.

Hong, C. (1989), Forecasting real output growth rates and cyclical properties of models: A Bayesian approach. PhD. Dissertation, University of Chicago.

Hopenhayn, H. A. (1992), Entry, exit, and firm dynamics in long run equilibrium. *Econometrica*, 60, 1127-1150.

Ilmakunnas, P. and Topi, J. (1999), Microeconomic and macroeconomic influences on entry and exit of firms. *Review of Industrial Organization*, 15, 283-301.

Jonson, P. D. (1976), Money and economic activity in the open economy: The United Kingdom 1880-1970. *Journal of Political Economy*, 84, 979-1012.

Jonson, P. D., Moses, E.R. and Wymer, C.R. (1977), The RBA 76 model of the Australian economy in W.E.Norton, ed., *Conference in Apllied Economic Research*, Reserve Bank of Australia, Sydney.

Kaldor, N. (1940), A model of trade cycle. *Economic Journal*, 50(197), 78-92.

Kalecki, M. (1935), A macrodynamic theory of business cycles. *Econometrica*, 3(3), 327-44.

Kalecki, M. (1937), A theory of the business cycle. *Review of Economic Studies*, 4(2), 77-97.

Kirkpatrick, G. (1987), Employment, growth and economic policy: An econometric model of Germany. Tubingen, Mohr.

Knight, M.D. (1977) Euro dollars, capital mobility and the monetary approach to balance of payments. *Economica*, 44, 1-21.

Knight, M.D. and Mathieson, D.J. (1979) Model of an industrial country under fixed and flexible exchange rates in J.S. Bhandari and B.H. Putnam, eds., *Economic Interdependence and Flexible Exchange Rates*, MIT Press, Cambridge, MA.

Knight, M.D. and Wymer, C.R. (1978), A macroeconomic model of the United Kingdom. *IMF Staff Papers*, 25, 742-748.

Kuznetsov, Y. (2004), Elements of applied bifurcation theory, Springer Verlag, New York.

Leeper, E. and Sims, C. (1994) Toward a modern macro model usable for policy analysis. *NBER Macroeconomics Annual, New York: National Bureau of Economic Research*, 81-117.

Mansfield, E. (1961), Entry, gibrat's law, innovation, and the growth of firms. *Cowles Foundation Discussion Papers 126, Cowles Foundation*, Yale University.

Masson, R.T. and Shaanan, J. (1986), Excess capacity and limit pricing: An empirical test. *Economica*, 53, 365-378.

Miller, R.A. (1969) Market structure and industrial performance:Relation of profit rates to concentration, advertising intensity and diversity. *Journal of Industrial Economics*, 104-118.

Min, C. (1992), Economic analysis and forecasting of international growth rates using Bayesian techniques. PhD. Dissertation, University of Chicago.

Ngoie, J.K. and Zellner, A. (2008), The use of a Marshallian macroeconomic model for policy evaluation: Case of South Africa. Working Paper, University of Chicago.

Nishimura, K. and Takahashi, H. (1992), Factor intensity and Hopf bifurcations. *In Feichtinger G (Eds.), Dynamic Economic Models and Optimal Control* 135-149.

Orr, D. (1974), The determinants of entry: A sudy of the Canadian manufacturing industries. *The Review of Economics and Statistics*, 56(1), 58-66.

Palm, F.C. (1976), Testing the dynamic specification of an econometric model with an application to Belgian data *European Economic Review*, 8, 269-289.

Palm, F.C. (1977), On univariate time series methods and simaltaneous equation econometric models. *Journal of Econometrics*, 5, 379–388.

Palm, F.C. (1983), Structural economtric modeling and time series analysis: An integrated approach. *In A.Zellner (ed.), Applied Time Series Analysis of Econometric Data*, Bureau of Census, U.S. Department of Commerce, Washington, DC, 192-223.

Peltzman, S. (1965), Entry in commercial banking. *Journal of Law and Economics*, 8, 11-50.

Perona, E. (2005), Birth and early history of nonlinear dynamics in economics. *Revista de Economia y Estadistica*, 43.

Richard, D. (1980) A global adjustment model of the exchange and interest rates. *In D. Bigman and T. Taya, ed.,The Functioning of Floating Exchange Rates: Theory, Evidence and Policy Implications*, Ballinger, Cambridge, MA.

Siegfried, J. J. and Evans, L. B. (1994), Empirical studies of entry and exit: A survey of the evidence. *Journal of Law and Economics*, 9, 121-155.

Sjoo, B. (1993), CONTIMOS- A continuous-time econometric model for Sweden based on monthly data *In G. Gandolfo, ed., Continuous Time Econometrics*, Chapman and Hall, London.

Torre, V. (1977), Existence of limit cycles and control in complete Keynesian system by theory of bifurcations. *Econometrica*, 45, 1457-1466.

Tulio, G. (1981), Demand management and exchange rate policy: The Italian experience. *IMF Staff Papers*, 28, 80-117.

Veloce, W. and Zellner, A. (1985), Entry and empirical demand and supply analysis for competitive industries. *Journal of Econometrics*, 30(1-2), 459-471.

Wymer, C.R. (1973), A continuous disequilibrium adjustment model of the United Kingdom financial markets. *In A.A. Powell and R.A. Williams, ed., Economic Studies of Macro and Monetary Relations*, North Holland, Amsterdam.

Yip, G. (1982), Barriers to entry: A corporate strategy perspective. Lexington Books, Lexington, MA.

Zellner, A. (1997), Bayesian analysis in econometrics and statistics, Cheltenham: Edward Elgar.

Zellner, A. (2000), Bayesian and non-Bayesian approaches to scientific modelling and inference in economics and econometrics. *Korean Journal of Money & Finance*, 5(2), 11-56.

Zellner, A. (2002), Econometric and statistical data mining, prediction and policy-making. *Working Paper, University of Chicago.*

Zellner, A. (2004), Statistics, econometrics and forecasting, Cambridge University Press, Cambridge, UK.

Zellner, A. and Chen, B. (2001), Bayesian modeling of economies and data requirements. *Macroeconomic Dynamics*, 5, 673-700.

Zellner, A. and Israilevich, G. (2005), Marshallian macroeconomic model: A progress report. *Macroeconomic Dynamics*, 9(02), 220-243.

Zellner, A. and Min, C. (1999) Forecasting turning points in countries growth rates: A response to Milton Friedman. *Journal of Econometrics*, 40, 183-202.

Zellner, A. and Palm, F.C. (1974), Time series analysis and simultaneous equation econometric models. *Journal of Econometrics*, 2, 17-54.

Zellner, A. and Palm, F.C. (1975), Time series analysis of structural monetary models of the U.S. economy. *Sankya Ser.C*, 37, 12-56.

Zellner, A. and Palm, F.C. (2004), The Structural econometric modeling, time series analysis (SEMTSA) approach, Cambridge University Press, Cambridge, UK.

Zellner, A. and Tobias, J (2000), A note on aggregation, disaggregation and forecasting. *Journal of Forecasting*, 19, 457-469.