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Article

Fractional Diffusion in Gaussian Noisy Environment

Guannan Hu and Yaozhong Hu *

Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA;
E-Mail: g040h641@ku.edu

* Author to whom correspondence should be addressed; E-Mail: yhu@ku.edu.

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Abstract: We study the fractional diffusion in a Gaussian noisy environment as described by the fractional order stochastic heat equations of the following form: $D_t^{(\alpha)}u(t,x) = Bu + u \cdot \dot{W}^H$, where $D_t^{(\alpha)}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$ with respect to the time variable t, B is a second order elliptic operator with respect to the space variable $x \in \mathbb{R}^d$ and $\dot{W}^H$ a time homogeneous fractional Gaussian noise of Hurst parameter $H = (H_1, \cdots, H_d)$. We obtain conditions satisfied by α and H, so that the square integrable solution u exists uniquely.

Keywords: fractional derivative; fractional order stochastic heat equation; mild solution; time homogeneous fractional Gaussian noise; stochastic integral of the Itô type; multiple integral of the Itô type; chaos expansion; Fox's *H*-function; Green's functions

MSC classifications: 26A33; 60H15; 60H05; 35K40; 35R60

1. Introduction

In recent years, there have been a great amount of works on anomalous diffusions in the study of biophysics, and so on (see, for example, [1–4], to mention just a few). In mathematics, some of these anomalous diffusions (such as sub-diffusions) can be described by the so-called fractional order diffusion processes. As for the term "fractional order diffusion", one has to distinguish two completely different types. One is the equation of the form $\partial_t u(t,x) = -(-\Delta)^\alpha u(t,x)$, where $t \geq 0$, $x \in \mathbb{R}^d$, $\alpha \in (0,1)$ is a positive number $\partial_t = \frac{\partial}{\partial t}$ and $\Delta = \sum_{i=1}^d \partial_{x_i}^2$ is the Laplacian. This equation is not associated with the anomalous diffusion. Instead, it is associated with the so-called stable process (or, in general, the Lévy process), which has jumps. Another equation is of the form $D_t^{(\alpha)}u(t,x) = \Delta u(t,x)$, where $D_t^{(\alpha)}$ is the Caputo fractional derivative with respect to t. It is also possible to use the Riemann–Liouville fractional derivative instead of the Caputo one (see [5] for the study of various fractional derivatives). This equation is relevant to the anomalous diffusion that we mentioned and has been studied by a number of researchers. Let us mention a few recent publications concerning the applications of subdiffusive fractional equations. The work in [3] studied the applications to the transport in biological cells. The work in [6,7] studied the fractional chemotaxis diffusion equation. The work in [4,8] studied the morphogen gradient formation. Anomalous electrodiffusion in nerve cells is studied in [9]. The work in [10,11] studied subdiffusive transport equations; it was argued that it is unlikely that a Caputo form of a transport equation can be derived from a chemotaxis model on the lattice, and the use of the Riemann–Liouville-type equation was strongly advocated if the anomalous exponent α is space dependent.

If one considers the anomalous diffusion in a random environment, then this naturally leads to the study of a fractional order stochastic partial differential equation of the form $D_t^{(\alpha)}u(t,x) = Bu(t,x) + u(t,x)\dot{W}(t,x)$, where B is a second order differential operator, including the Laplacian as a special example, and $\dot{W}$ is a noise. In this paper, we shall study this fractional order stochastic partial differential equation when $\dot{W}(t,x) = \dot{W}^H(x)$ is a time homogeneous fractional Gaussian noise of Hurst parameter $H = (H_1, \cdots, H_d)$. Mainly, we shall find a relation between α and H, such that the solution to the above equation has a unique square integrable solution.

If α is formally set to one, then the above stochastic partial differential equation has been studied in [12]. Therefore, our work can be considered as an extension of the work [12] to the case of fractional diffusion (in Gaussian noisy environment). Let us also mention that when we formally set $\alpha = 1$, and we recover one of the main results in [12] (see Remark 3 in Section 2 below). Thus, our condition (2.10) given below is also optimal.

Here is the organization of the paper. The main result of the paper is stated in Section 2. In our proof, we need to use the properties of the two fundamental solutions (Green's functions) $Z(t,x,\xi)$ and $Y(t,x,\xi)$ associated with the equation $D_t^{(\alpha)}u(t,x) = Bu(t,x)$, which is represented by Fox's ***H***-function. We shall recall some most relevant results on the ***H***-function and Green's function $Z(t,x,\xi)$ and $Y(t,x,\xi)$ in Section 3. A number of preparatory lemmas are needed to prove our main result, and they are presented in Section 4. Finally, Section 5 is devoted to the proof of our main theorem.

2. Main Result

Let:

$$B = \sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j} + c(x)$$

be a uniformly elliptic second-order differential operator with bounded continuous real-valued coefficients. Let u_0 be a given bounded continuous function (locally Hölder continuous if $d > 1$). Let $\{W^H(x), x \in \mathbb{R}^d\}$ be a time homogeneous (time-independent) fractional Brownian field on some probability space $(\Omega, \mathcal{F}, P)$ (like elsewhere in probability theory, we omit the dependence of $W^H(x) = W^H(x, \omega)$ on $\omega \in \Omega$). Namely, the stochastic process $\{W^H(x), x \in \mathbb{R}^d\}$ is a (multi-parameter) Gaussian process with mean zero, and its covariance is given by:

$$\mathbb{E}\left(W^H(x)W^H(y)\right) = \prod_{i=1}^{d} R_{H_i}(x_i, y_i)\,, \tag{2.1}$$

where $H_1, \cdots, H_d$ are some real numbers in the interval $(0, 1)$. Due to some technical difficulty, we assume that $H_i > 1/2$ for all $i = 1, 2, \cdots, d$. The symbol $\mathbb{E}$ denotes the expectation on $(\Omega, \mathcal{F}, P)$ and:

$$R_{H_i}(x_i, y_i) = \frac{1}{2}\left(|x_i|^{2H_i} + |y_i|^{2H_i} - |x_i - y_i|^{2H_i}\right)\,, \quad \forall\, x_i, y_i \in \mathbb{R}$$

is the covariance function of a fractional Brownian motion of Hurst parameter H_i.

Throughout this paper, we fix an arbitrary parameter $\alpha \in (0, 1)$ and a finite time horizon $T \in (0, \infty)$. We study the following stochastic partial differential equation of fractional order:

$$\begin{cases} D_t^{(\alpha)} u(t, x) = \boldsymbol{B} u(t, x) + u(t, x) \cdot \dot{W}^H(x), & t \in (0, T], \quad x \in \mathbb{R}^d\,; \\ u(0, x) = u_0(x)\,, \end{cases} \tag{2.2}$$

where:

$$D_t^{(\alpha)} u(t, x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\alpha} u(\tau, x)d\tau - t^{-\alpha} u(0, x)\right]$$

is the Caputo fractional derivative (see, e.g., [5]) and $\dot{W}^H(x) = \frac{\partial^d}{\partial x_1 \cdots \partial x_d} W^H(x)$ is the distributional derivative (generalized derivative) of W^H, called fractional Brownian noise.

Our objective is to obtain condition on α and H, such that the above equation has a unique solution. However, since W^H is not differentiable or since $\dot{W}^H(x)$ does not exist as an ordinary function, we have to describe under what sense a random field $\left\{u(t, x), t \geq 0, x \in \mathbb{R}^d\right\}$ is a solution to the above Equation (2.2).

To motivate our definition of the solution, let us consider the following (deterministic) partial differential equation of fractional order with the term $u(t, x) \cdot \dot{W}^H(x)$ in Equation (2.2) replaced by $f(t, x)$:

$$\begin{cases} D_t^{(\alpha)} \tilde{u}(t, x) = \boldsymbol{B} \tilde{u}(t, x) + f(t, x), & t \in (0, T], \quad x \in \mathbb{R}^d\,; \\ \tilde{u}(0, x) = u_0(x)\,, \end{cases} \tag{2.3}$$

where the function f is bounded and jointly continuous in (t, x) and locally Hölder continuous in x.

In [13], it is proven that there are two Green's functions $\left\{Z(t, x, \xi)\,,\ Y(t, x, \xi)\,, 0 < t \leq T\,, x, \xi \in \mathbb{R}^d\right\}$, such that the solution to the Cauchy problem Equation (2.3) is given by:

$$\tilde{u}(t, x) = \int_{\mathbb{R}^d} Z(t, x, \xi)u_0(\xi)d\xi + \int_0^t ds \int_{\mathbb{R}^d} Y(t-s, x, y)f(s, y)dy. \tag{2.4}$$

In general, there is no explicit form for the two Green's functions $\{Z(t, x, \xi)\,,\ Y(t, x, \xi)\}$. However, their constructions and properties are known (see [13–15] and the references therein). We shall recall some needed results in the next section.

From the classical solution expression Equation (2.4), we expect that the solution $u(t,x)$ to Equation (2.2) satisfies formally:

$$u(t,x)=\int_{\mathbb{R}^d} Z(t,x,\xi)u_0(\xi)d\xi+\int_0^t ds\int_{\mathbb{R}^d} Y(t-s,x,y)u(s,y)\dot{W}^H(y)dy\,.$$

The above formal integral $\int_0^t ds\int_{\mathbb{R}^d} Y(t-s,x,y)u(s,y)\dot{W}^H(y)dy$ can be defined by Itô–Skorohod stochastic integral $\int_{\mathbb{R}^d}\left[\int_0^t Y(t-s,x,y)u(s,y)ds\right]W^H(dy)$, as given in [12].

Now, we can give the following definition.

Definition 1. *A random field* $\left\{u(t,x)\,,0\le t\le T\,,x\in\mathbb{R}^d\right\}$ *is called a mild solution to the Equation (2.2) if:*

(1) $u(t,x)$ *is jointly measurable in* $t\in[0,T]$ *and* $x\in\mathbb{R}^d$*;*

(2) $\forall (t,x)\in[0,T]\times\mathbb{R}^d$, $\int_0^t\int_{\mathbb{R}^d} Y(t-s,x,y)u(s,y)dsW^H(dy)$ *is well defined in* $\mathcal{L}^2=L^2(\Omega,\mathcal{F},P)$*;*

(3) The following holds in $\mathcal{L}^2$*:*

$$u(t,x)=\int_{\mathbb{R}^d} Z(t,x,\xi)u_0(\xi)d\xi+\int_0^t\int_{\mathbb{R}^d} Y(t-s,x,y)u(s,y)W^H(dy)ds. \tag{2.5}$$

Let us return to the discussion of the two Green's functions $\{Z(t,x,\xi)\,,\ Y(t,x,\xi)\}$. If $\alpha=1$, namely, if $D_t^{(\alpha)}$ in Equation (2.3) is replaced by ∂_t and $B=\Delta:=\sum_{i=1}^d \partial_{x_i}^2$, then:

$$Z(t,x,\xi)=Y(t,x,\xi)=(4\pi t)^{-d/2}\exp\left\{-\frac{|x-\xi|^2}{4t}\right\}. \tag{2.6}$$

In this case, the stochastic partial differential equation of the form:

$$\frac{\partial u(t,x)}{\partial t}=\Delta u(t,x)+u\cdot\dot{W}^H(x),\qquad x\in\mathbb{R}^d, \tag{2.7}$$

was studied in [12]. The mild solution to the above Equation (2.7) is proven to exist uniquely under conditions:

$$H_i>1/2\,,\quad i=1,\cdots,d\qquad\text{and}\qquad \sum_{i=1}^d H_i>d-1\,. \tag{2.8}$$

The main result of this paper is to extend the above result in [12] to our Equation (2.2).

Theorem 2. *Let the coefficients* $a_{ij}(x)$, $b_i(x)\,,i,j=1,\cdots,d\,,$ *be bounded and Hölder continuous with exponent* γ*.*

Let $a_{ij}(x)$ *be uniformly elliptic. Namely, there is a constant* $a_0\in(0,\infty)$*, such that:*

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j\ge a_0|\xi|^2\quad\forall\ \xi=(\xi_1,\cdots,\xi_d)\in\mathbb{R}^d\,.$$

Let u_0 be bounded continuous (and locally Hölder continuous if $d > 1$). Assume:

$$H_i > \begin{cases} \frac{1}{2} & \text{if } d = 1,2,3,4 \\ 1 - \frac{2}{d} - \frac{\gamma}{2d} & \text{if } d \geq 5 \end{cases} \tag{2.9}$$

and:

$$\sum_{i=1}^{d} H_i > d - 2 + \frac{1}{\alpha} . \tag{2.10}$$

Then, the mild solution to (2.2) exists uniquely in $L^2(\Omega, \mathcal{F}, P)$.

Remark 3. *(i) If α is formally set to one and B $= \Delta$, then $H_i > 1/2$ implies Condition (5.8). Thus, Condition (2.10) is the same as Condition (2.8) (which is a condition given in [12]). Therefore, in some sense, our condition (2.10) is optimal.*

(ii) Since $H_i < 1$ for all $i = 1, 2, \cdots, d$, the condition is possible only when $\alpha > 1/2$.

3. Green's Functions Z and Y

3.1. Fox's H-Function

We shall use the *H*-function to express the Green's functions Z and Y in Definition 1. In this subsection, we recall some results about the *H*-function and the two Green's functions. We shall follow the presentation in [16] (see also [13] and the references therein).

Definition 4. *Let m, n, p, q be integers, such that $0 \leq m \leq q, 0 \leq n \leq p$. Let $a_i, b_i \in \mathbb{C}$ be complex numbers, and let α_j, β_j be positive numbers, $i = 1, 2, \cdots, p; j = 1, 2, \cdots, q$. Let the set of poles of the gamma functions $\Gamma(b_j + \beta_j s)$ not intersect with that of the gamma functions $\Gamma(1 - a_i - \alpha_i s)$, namely,*

$$\left\{ b_{jl} = \frac{-b_j - l}{\beta_j}, l = 0, 1, \cdots \right\} \bigcap \left\{ a_{ik} = \frac{1 - a_i + k}{\alpha_i}, k = 0, 1, \cdots \right\} = \emptyset$$

for all $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$. The H-function:

$$H_{pq}^{mn}(z) \equiv H_{pq}^{mn} \left[z \middle| \begin{array}{ccc} (a_1, \alpha_1) & \cdots & (a_p, \alpha_p) \\ (b_1, \beta_1) & \cdots & (b_q, \beta_q) \end{array} \right]$$

is defined by the following integral:

$$H_{pq}^{mn}(z) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_j + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)} z^{-s} ds \,, \quad z \in \mathbb{C} \,, \tag{3.1}$$

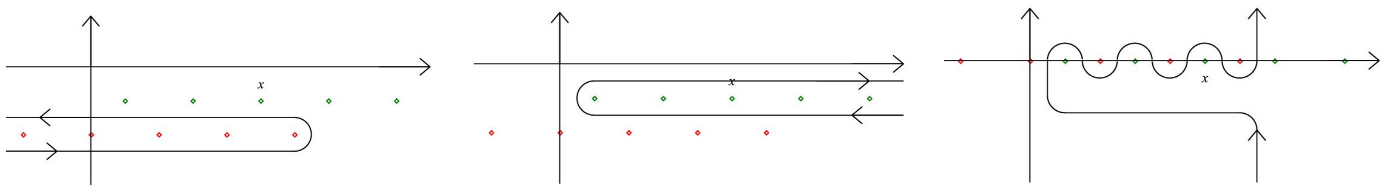
where an empty product in Equation (3.1) means one and L in Equation (3.1) is the infinite contour, which separates all of the points b_{jl} to the left and all the points a_{ik} to the right of L. Moreover, L has one of the following forms:

Case 1. $L = L_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

Case 2. $L = L_{+\infty}$ *is a right loop situated in a horizontal strip starting at point* $\infty + i\phi_1$ *and terminating at point* $\infty + i\phi_2$ *for some* $-\infty < \phi_1 < \phi_2 < \infty$*;*

Case 3. $L = L_{i\gamma\infty}$ *is a contour starting at point* $\gamma - i\infty$ *and terminating at point* $\gamma + i\infty$ *for some* $\gamma \in (-\infty, \infty)$.

To illustrate L, we give the following graphs.



Case 1 Case 2 Case 3

The integral Equation (3.1) exists when $\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geq 0$ (see [16], Theorem 1.1).

Example 5. *To compare with the classical case* $\alpha = 1$*, we consider the case* $m = 2$*,* $n = 0$*,* $p = 1$*,* $q = 2$*,* $a_1 = \alpha_1 = b_2 = \beta_1 = \beta_2 = 1$ *and* $b_1 = \frac{d}{2}$*. Let* $L = L_{-\infty}$*. Then, we have:*

$$\begin{aligned}
H_{12}^{20}\left[z\left|\begin{array}{cc} (1,1) & \\ (\frac{d}{2},1), & (1,1) \end{array}\right.\right] &= \frac{1}{2\pi i}\int_L \frac{\Gamma(\frac{d}{2}+s)\Gamma(1+s)}{\Gamma(1+s)} z^{-s} ds \\
&= \frac{1}{2\pi i}\int_L \Gamma(\frac{d}{2}+s) z^{-s} ds \\
&= \sum_{v=0}^{\infty} \lim_{s\to-(\frac{d}{2}+v)} (s+\frac{d}{2}+v)\Gamma(\frac{d}{2}+s) z^{-s} \\
&= \sum_{v=0}^{\infty} \lim_{s\to-(\frac{d}{2}+v)} \frac{\Gamma(v+\frac{d}{2}+s+1)}{(s+\frac{d}{2}+v-1)\cdots(s+\frac{d}{2})} z^{-s} \\
&= \sum_{v=0}^{\infty} z^{d/2}(-1)^v \frac{1}{v!} z^v \\
&= z^{d/2}\exp(-z)\,. \qquad (3.2)
\end{aligned}$$

3.2. Green's Functions Z and Y When B Has Constant Coefficients

In this subsection, let us consider Z and Y when the operator B in Equation (2.2) has the following form:

$$B = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

where the matrix $A = (a_{ij})$ is positive definite. In this case, Z and Y (we call them Z_0 and Y_0 to distinguish from the general coefficient case) are given as follows.

$$Z_0(t,x) = \frac{\pi^{-d/2}}{(\det A)^{1/2}} \left[\sum_{i,j=1}^{d} A^{(ij)} x_i x_j\right]^{-d/2}$$

$$\times H_{12}^{20}\left[\frac{1}{4}t^{-\alpha}\sum_{i,j=1}^{d}A^{(ij)}x_ix_j\,\middle|\,\begin{matrix}(1,\alpha) & \\ (\frac{d}{2},1), & (1,1)\end{matrix}\right],$$

where $(A^{(ij)}) = A^{-1}$ and

$$Y_0(t,x) = \frac{\pi^{-d/2}}{(\det A)^{1/2}}\left[\sum_{i,j=1}^{d}A^{(ij)}x_ix_j\right]^{-d/2}t^{\alpha-1}$$

$$\times H_{12}^{20}\left[\frac{1}{4}t^{-\alpha}\sum_{i,j=1}^{d}A^{(ij)}x_ix_j\,\middle|\,\begin{matrix}(\alpha,\alpha) & \\ (\frac{d}{2},1), & (1,1)\end{matrix}\right].$$

It is easy to see that for the constant coefficients, both of the Green's functions are homogeneous in time and space. Namely,

$$Z_0(t,x,\xi) = Z_0(t,x-\xi)\,, \quad Y_0(t,x,\xi) = Y_0(t,x-\xi)\,.$$

In particular, when $\alpha = 1$, it is easy to see from the above expression and the explicit form Equation (3.2) of $H_{12}^{20}(z)$ that:

$$Z_0(t,x,\xi) = Y_0(t,x,\xi) = (4\pi)^{-d/2}\det(A)^{-1/2}\exp\left\{-\frac{\sum_{i,j=1}^{d}A^{(ij)}(x_i-\xi_i)(x_j-\xi_j)}{4t}\right\},$$

which reduces to Equation (2.6) when $A = I$ is the identity matrix.

With the above expressions for Z_0 and Y_0 and the properties of the *H*-function, one can obtain the following estimates.

Proposition 6. *Denote:*

$$p(t,x) = \exp\left(-\sigma t^{-\frac{\alpha}{2-\alpha}}|x|^{\frac{2}{2-\alpha}}\right), \quad t>0\,,\ x\in\mathbb{R}^d\,, \tag{3.3}$$

where and in what follows the positive constants C and σ are generic, which may be different in different occurrences. Then, we have the following estimates:

$$|Z_0(t,x)| \le \begin{cases} Ct^{-\frac{\alpha}{2}}p(t,x) & \textit{when} \quad d=1 \\ Ct^{-\alpha}[|\log\frac{|x|^2}{t^\alpha}|+1]p(t,x) & \textit{when} \quad d=2 \\ Ct^{-\alpha}|x|^{2-d}p(t,x) & \textit{when} \quad d\ge 3\,, \end{cases} \tag{3.4}$$

where, for instance, $|Z_0(t,x)| \le Ct^{-\frac{\alpha}{2}}p(t,x)$ means that there are positive constant C and positive constant σ, such that the above inequality holds.

Proof. Denote $R = |x|^2/t^\alpha$. From [13], Proposition 1, it follows that when $R \le 1$, we have:

$$|Z_0(t,x)| \le \begin{cases} Ct^{-\frac{\alpha}{2}} & \text{when} \quad d=1 \\ Ct^{-\alpha}[|\log\frac{|x|^2}{t^\alpha}|+1] & \text{when} \quad d=2 \\ Ct^{-\alpha}|x|^{2-d} & \text{when} \quad d\ge 3\,. \end{cases}$$

Since when $R \leq 1$, $p(t,x)$ is bounded from below. This proves the inequality Equation (3.4) when $R \leq 1$.

When $R > 1$, then by [13], Proposition 1, we have $|Z_0(t,x)| \leq Ct^{-\frac{\alpha d}{2}} p(t,x)$. It is clear that this implies the inequality Equation (3.4) when $d = 1$ and $d = 2$. Now, we assume that $d \geq 3$. We have:

$$\begin{aligned} |Z_0(t,x)| &\leq Ct^{-\frac{\alpha d}{2}} p(t,x) \leq Ct^{-\alpha}|x|^{2-d} \left(\frac{|x|^2}{t^\alpha}\right)^{\frac{d}{2}-1} p(t,x) \\ &\leq Ct^{-\alpha}|x|^{2-d} p(t,x)\,, \end{aligned}$$

where we used the fact that $\left(\frac{|x|^2}{t^\alpha}\right)^{\frac{d}{2}-1} p(t,x) \leq p(t,x)$ for a different σ in the later $p(t,x)$. □

Similarly, we can use [13], Proposition 2 (for the $d = 1$ case), and [13], Section 4.2 (for the $d \geq 2$ case), to obtain the following estimates for $Y_0(t,x)$.

Proposition 7. *We follow the same notation $p(t,x)$ as defined by Equation (3.3). We have:*

(i) When $d = 1$, we have the following estimates:

$$|Y_0(t,x)| \leq \begin{cases} Ct^{\frac{\alpha}{2}-1} p(t,x) & \text{when} \quad t^{-\alpha}|x|^2 \geq 1 \\ Ct^{\frac{\alpha}{2}-1} & \text{when} \quad t^{-\alpha}|x|^2 \leq 1. \end{cases} \tag{3.5}$$

(ii) When $d \geq 2$, we have the following estimates:

$$|Y_0(t,x)| \leq \begin{cases} Ct^{-1} p(t,x) & \text{when} \quad d = 2 \\ Ct^{-\frac{\alpha}{2}-1} p(t,x) & \text{when} \quad d = 3 \\ Ct^{-\alpha-1}[|\log \frac{|x|^2}{t^\alpha}| + 1] p(t,x) & \text{when} \quad d = 4 \\ Ct^{-\alpha-1}|x|^{4-d} p(t,x) & \text{when} \quad d \geq 5\,. \end{cases} \tag{3.6}$$

3.3. Green's Functions Z and Y in the General Coefficient Case

If the coefficients of B are not constant, then the Green's functions Z and Y are more complicated and may be obtained by a method similar to the Levi parametrix for the parabolic equations.

Denote:

$$\begin{aligned} M(t,x,\xi) &= \sum_{i,j=1}^{d} [a_{ij}(x) - a_{ij}(\xi)] \frac{\partial^2}{\partial x_i \partial x_j} Z_0(t, x-\xi, \xi) \\ &\quad + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} Z_0(t, x-\xi, \xi) + c(x) Z_0(t, x-\xi, \xi) \\ K(t,x,\xi) &= \sum_{i,j=1}^{d} [a_{ij}(x) - a_{ij}(\xi)] \frac{\partial^2}{\partial x_i \partial x_j} Y_0(t, x-\xi, \xi) \\ &\quad + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} Y_0(t, x-\xi, \xi) + c(x) Y_0(t, x-\xi, \xi)\,. \end{aligned}$$

Let $Q(s,y,\xi)$ and $\Phi(s,y,\xi)$ be defined by:

$$\begin{aligned} Q(t,x,\xi) &= M(t,x,\xi) + \int_0^t ds \int_{\mathbb{R}^d} K(t-s,x,y)Q(s,y,\xi)dy\,; \\ \Phi(t,x,\xi) &= K(t,x,\xi) + \int_0^t ds \int_{\mathbb{R}^d} K(t-s,x,y)\Phi(s,y,\xi)dy\,. \end{aligned}$$

The following proposition is proven in [13] (see the Section 2.2 of that paper).

Proposition 8. *Let the coefficients $a_{ij}(x)$ and $b_i(x)$ satisfy the conditions in Theorem 2. Recall that γ is the Hölder exponent of the coefficients with respect to the spatial variable x. Then, the Green's functions $\{Z(t,x,\xi), Y(t,x,\xi)\}$ have the following form:*

$$Z(t,x,\xi) = Z_0(t,x-\xi,\xi) + V_Z(t,x,\xi);$$

$$Y(t,x,\xi) = Y_0(t,x-\xi,\xi) + V_Y(t,x,\xi), \tag{3.7}$$

where

$$\begin{aligned} V_Z(t,x,\xi) &= \int_0^t ds \int_{\mathbb{R}^d} Y_0(t-s,x,y)Q(s,y,\xi)dy; \\ V_Y(t,x,\xi) &= \int_0^t ds \int_{\mathbb{R}^d} Y_0(t-s,x,y)\Phi(s,y,\xi)dy. \end{aligned}$$

Moreover, the function $V_Z(t,x,\xi), V_Y(t,x,\xi)$ satisfies the following estimates.

$$|V_Z(t,x,\xi)| \le \begin{cases} Ct^{(\gamma-1)\frac{\alpha}{2}}p(t,x-\xi)\,, & \text{when } d=1\,; \\ Ct^{\frac{\gamma\alpha}{2}-\alpha}p(t,x-\xi)\,, & \text{when } d=2\,; \\ Ct^{\frac{\gamma_0\alpha}{2}-\alpha}|x-\xi|^{2-d+\gamma-\gamma_0}p(t,x-\xi)\,, & \text{when } d=3 \text{ or } d\ge 5\,; \\ Ct^{(\gamma-\gamma_0)\frac{\alpha}{2}-\alpha}|x-\xi|^{-2+\gamma-2\gamma_0}p(t,x-\xi)\,, & \text{when } d=4 \end{cases} \tag{3.8}$$

and:

$$|V_Y(t,x,\xi)| \le \begin{cases} Ct^{\alpha-1+(\gamma-1)\frac{\alpha}{2}}p(t,x-\xi)\,, & \text{when } d=1\,; \\ Ct^{\frac{\gamma\alpha}{2}-1}p(t,x-\xi)\,, & \text{when } d=2\,; \\ Ct^{(\gamma_0+\gamma)\frac{\alpha}{4}-1}|x-\xi|^{2-d+(\gamma-\gamma_0)/2}p(t,x-\xi)\,, & \text{when } d=3 \text{ or } d\ge 5\,; \\ Ct^{(\gamma-\gamma_0)\frac{\alpha}{4}-1}|x-\xi|^{-2+\gamma-2\gamma_0}p(t,x-\xi)\,, & \text{when } d=4\,. \end{cases} \tag{3.9}$$

Here, γ_0 is any number, such that $0<\gamma_0<\gamma$, and in the case $d\ge 3$, the constant C depends on γ_0.

4. Auxiliary Lemmas

To prove our main theorem, we need to dominate certain multiple integrals involving $Y(t,x,\xi)$ and $Z(t,x,\xi)$. Since both $Y(t,x,\xi)$ and $Z(t,x,\xi)$ are complicated, we shall first bound them by $p(t,x-\xi)$ from the estimations of $|Y_0(t,x,\xi)|$ and $|V_Y(t,x,\xi)|$. More precisely, we have the following bounds for $Y(t,x,\xi)$.

Lemma 9. *Let* $x \in \mathbb{R}^d, t \in (0,T]$. *Then:*

$$|Y(t,x,\xi)| \leq \begin{cases} Ct^{-1+\frac{\alpha}{2}}p(t,x-\xi), & d=1; \\ Ct^{-1}p(t,x-\xi), & d=2; \\ Ct^{-(\gamma-2\gamma_0)\frac{\alpha}{2}-1}|x-\xi|^{-2+\gamma-2\gamma_0}p(t,x-\xi), & d=4; \\ Ct^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1}|x-\xi|^{2-d+(\gamma-\gamma_0)/2}p(t,x-\xi), & d=3 \text{ or } d\geq 5. \end{cases} \tag{4.1}$$

Proof. We shall prove the lemma by the above different cases. First, when $d=1$, by Proposition 7, we have:

$$|Y_0(t,x-\xi,\xi)| \quad \leq \quad \begin{cases} Ct^{\frac{\alpha}{2}-1}p(t,x-\xi), & t^{-\alpha}|x-\xi|^2 \geq 1; \\ Ct^{\frac{\alpha}{2}-1}, & t^{-\alpha}|x-\xi|^2 \leq 1. \end{cases}$$

If $t^{-\alpha}|x-\xi|^2 \leq 1$, then:

$$|Y_0(t,x-\xi,\xi)| \leq Ct^{-1+\frac{\alpha}{2}} \cdot \frac{p(x,t)}{e^{-\sigma}} \leq Ct^{\frac{\alpha}{2}-1}p(t,x-\xi).$$

Therefore:

$$\begin{aligned} |Y(t,x,\xi)| &\leq |Y_0(t,x-\xi,\xi)| + |V_Y(t,x,\xi)| \\ &\leq Ct^{\alpha-1+(\gamma-1)\frac{\alpha}{2}}p(t,x-\xi) + Ct^{-1+\frac{\alpha}{2}}p(t,x-\xi) \\ &\leq Ct^{-1+\frac{\alpha}{2}}p(t,x-\xi)\,. \end{aligned}$$

Now, we consider the case $d=2$. From the following inequalities:

$$\begin{aligned} |V_Y(t,x,\xi)| &\leq Ct^{\gamma\frac{\alpha}{2}-1}p(t,x-\xi)\,; \\ |Y_0(t,x-\xi,\xi)| &\leq Ct^{-1}p(t,x-\xi) \end{aligned}$$

we have easily:

$$|Y(t,x,\xi)| \leq |Y_0(t,x-\xi,\xi)| + |V_Y(t,x,\xi)| \leq Ct^{-1}p(t,x-\xi)\,.$$

We are going to prove the lemma when $d=3$. From Proposition 7, we have:

$$\begin{aligned} |Y_0(t,x-\xi,\xi)| &\leq Ct^{-\frac{\alpha}{2}-1}p(t,x-\xi) \\ &= Ct^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1}|x-\xi|^{-1+(\gamma-\gamma_0)/2}\left|\frac{x-\xi}{t^{\frac{\alpha}{2}}}\right|^{1-(\gamma-\gamma_0)/2}p(t,x-\xi) \\ &\leq Ct^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1}|x-\xi|^{-1+(\gamma-\gamma_0)/2}p(t,x-\xi)\,. \end{aligned}$$

Combining this inequality with Proposition 8, we obtain:

$$|Y(t,x,\xi)| \leq Ct^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1}|x-\xi|^{-1+(\gamma-\gamma_0)/2}p(t,x-\xi)\,.$$

We turn to consider the case $d=4$. Proposition 7 yields that for any $\theta>0$, the following holds true:

$$\begin{aligned} |Y_0(t,x-\xi,\xi)| &\leq Ct^{-\alpha-1}\left[\left(\frac{|x-\xi|^2}{t^\alpha}\right)^\theta + \left(\frac{t^\alpha}{|x-\xi|^2}\right)^\theta\right]p(t,x-\xi)\,; \\ &= Ct^{-\alpha-1}\left(\frac{t^\alpha}{|x-\xi|^2}\right)^\theta\left[\left(\frac{|x-\xi|^2}{t^\alpha}\right)^{2\theta}+1\right]p(t,x-\xi)\,. \end{aligned}$$

If $\frac{|x-\xi|^2}{t^\alpha} > 1$, then:

$$\left[\left(\frac{|x-\xi|^2}{t^\alpha}\right)^{2\theta} + 1\right] p(t, x-\xi) \leq 2\left(\frac{|x-\xi|^2}{t^\alpha}\right)^{2\theta} p(t, x-\xi) \leq Cp(t, x-\xi)\,.$$

As a consequence, we have:

$$|Y_0(t, x-\xi, \xi)| \leq Ct^{-\alpha-1}\left(\frac{t^\alpha}{|x-\xi|^2}\right)^{\theta} p(t, x-\xi)\,.$$

If $\frac{|x-\xi|^2}{t^\alpha} \leq 1$, then the above inequality is obviously true. Now, we can choose $\theta > 0$, such that $-2\theta \geq (-2+\gamma-2\gamma_0)$. Thus, we have:

$$\begin{aligned} |Y_0(t, x-\xi, \xi)| &= Ct^{-\alpha-1+\alpha\theta+(-2\theta-(-2+\gamma-2\gamma_0))\frac{\alpha}{2}} |x-\xi|^{-2+\gamma-2\gamma_0} \\ &\quad \cdot \left(\frac{|x-\xi|}{t^{\frac{\alpha}{2}}}\right)^{-2\theta-(-2+\gamma-2\gamma_0)} p(t, x-\xi) \\ &\leq Ct^{-(\gamma-2\gamma_0)\frac{\alpha}{2}-1} \cdot |x-\xi|^{-2+\gamma-2\gamma_0} p(t, x-\xi)\,. \end{aligned}$$

Combining the above inequality with Proposition 8, we have:

$$\begin{aligned} |Y(t, x, \xi)| &\leq Ct^{-(\gamma-2\gamma_0)\frac{\alpha}{2}-1} |x-\xi|^{-2+\gamma-2\gamma_0} p(t, x-\xi) \\ &\quad + Ct^{(\gamma_0+\gamma)\frac{\alpha}{4}-1} |x-\xi|^{-2+\gamma-2\gamma_0} p(t, x-\xi) \\ &\leq Ct^{-(\gamma-2\gamma_0)\frac{\alpha}{2}-1} |x-\xi|^{-2+\gamma-2\gamma_0} p(t, x-\xi) \end{aligned}$$

since $-(\gamma-2\gamma_0)\frac{\alpha}{2} - 1 \leq (\gamma_0+\gamma)\frac{\alpha}{4} - 1$.

Finally, we consider the case $d \geq 5$. From the estimates: $|Y_0(t, x-\xi, \xi)| \leq Ct^{-\alpha-1}|x-\xi|^{4-d} p(t, x-\xi)$ we obtain:

$$\begin{aligned} |Y_0(t, x-\xi, \xi)| &\leq Ct^{-(\gamma_0+\gamma)\frac{\alpha}{4}-1} |x-\xi|^{2-d+(\gamma-\gamma_0)/2} \left|\frac{x-\xi}{t^{\frac{\alpha}{2}}}\right|^{2-(\gamma-\gamma_0)/2} p(t, x-\xi) \\ &\leq t^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1} |x-\xi|^{2-d+(\gamma-\gamma_0)/2} p(t, x-\xi)\,. \end{aligned}$$

Therefore, we have:

$$|Y(t, x, \xi)| \leq Ct^{-(\gamma-\gamma_0)\frac{\alpha}{4}-1} |x-\xi|^{2-d+(\gamma-\gamma_0)/2} p(t, x-\xi)\,.$$

The proposition is then proven. □

The bound Equation (4.1) will greatly help to simplify our estimation of the multiple integrals that we are going to encounter. However, when the dimension d is greater than or equal to two, the multiple integrals are still complicated to estimate, and our main technique is to reduce the computation to one dimensional. This means that we shall further bound the right-hand side of the inequality Equation (4.1) by the product of functions of one variable. Before doing so, we denote the exponents of t and $|x-\xi|$ in Equation (4.1) by ζ_d and κ_d. Namely, we denote:

$$\zeta_d = \begin{cases} -1 + \frac{\alpha}{2}, & d = 1; \\ -1, & d = 2; \\ -(\gamma - 2\gamma_0)\frac{\alpha}{2} - 1, & d = 4; \\ -(\gamma - \gamma_0)\frac{\alpha}{4} - 1, & d = 3 \text{ or } d \geq 5\,. \end{cases} \tag{4.2}$$

and:

$$\kappa_d = \begin{cases} 0, & d = 1, 2; \\ -2 + \gamma - 2\gamma_0, & d = 4; \\ 2 - d + (\gamma - \gamma_0)/2, & d = 3 \text{ or } d \geq 5\,. \end{cases} \tag{4.3}$$

From now on, we shall exclusively use $p(t, x) = \exp\big(- \sigma t^{-\frac{\alpha}{2-\alpha}} |x|^{\frac{2}{2-\alpha}} \big)$ to denote a function of one variable. However, the constant σ may be different in different appearances of $p(t, x)$ (for notational simplicity, we omit the explicit dependence on σ of $p(t, x)$).

With these notation, Lemma 9 yields:

Lemma 10. *The following bound holds true for the Green's function Y:*

$$|Y(t, x, \xi)| \leq C \prod_{i=1}^{d} t^{\zeta_d/d} |x_i - \xi_i|^{\kappa_d/d} p(t, x_i - \xi_i)\,. \tag{4.4}$$

Proof. It is easy to see that:

$$|x| = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2} \geq \max_{1 \leq i \leq d} |x_i| \geq \prod_{i=1}^{d} |x_i|^{\frac{1}{d}}\,.$$

Thus, for any positive number $\alpha > 0$, $|x|^{-\alpha} \leq \prod_{i=1}^{d} |x_i|^{-\frac{\alpha}{d}}$.

On the other hand,

$$\begin{aligned} |x|^{\frac{2}{2-\alpha}} &= \left[\sum_{i=1}^{d} |x_i|^2 \right]^{\frac{1}{2-\alpha}} \geq \left[\max_{1 \leq i \leq d} |x_i|^2 \right]^{\frac{1}{2-\alpha}} \\ &= \max_{1 \leq i \leq d} |x_i|^{\frac{2}{2-\alpha}} \geq \frac{1}{d} \sum_{i=1}^{d} |x_i|^{\frac{2}{2-\alpha}}. \end{aligned}$$

Combining the above with Equation (4.1) yields Equation (4.4), since the exponents in $|x - \xi|$ in Equation (4.1) are negative. □

Lemma 11. *Let $-1 < \beta \leq 0, x \in \mathbb{R}$. Then, there is a constant C, dependent on σ, α and β, but independent of ξ and s, such that:*

$$\sup_{\xi \in \mathbb{R}} \int_{\mathbb{R}} |x|^{\beta} p(s, x - \xi) dx \leq C s^{\frac{\alpha\beta}{2} + \frac{\alpha}{2}}\,.$$

Proof. Making the substitution $x = y s^{\frac{\alpha}{2}}$, we obtain:

$$\begin{aligned} \int_{\mathbb{R}} |x|^{\beta} p(s, x - \xi) dx &= s^{\frac{\alpha\beta}{2} + \frac{\alpha}{2}} \int_{\mathbb{R}} |y|^{\beta} \cdot \exp\left(- \sigma \left| y - \frac{\xi}{s^{\frac{\alpha}{2}}} \right|^{\frac{2}{2-\alpha}} \right) dy \\ &\leq s^{\frac{\alpha\beta}{2} + \frac{\alpha}{2}} \left(\int_{|y| \leq 1} |y|^{\beta} dy + \int_{\mathbb{R}} \exp\left(- \sigma \left| y - \frac{\xi}{s^{\frac{\alpha}{2}}} \right|^{\frac{2}{2-\alpha}} \right) dy \right) \\ &\leq C s^{\frac{\alpha\beta}{2} + \frac{\alpha}{2}} \end{aligned}$$

since the two integrals inside the parenthesis are finite (and independent of s and ξ). □

The following is a slight extension of the above lemma.

Lemma 12. *There is a constant C, dependent on σ, α and β, but independent of ξ and s, such that:*

$$\sup_{\xi\in\mathbb{R}}\int_{\mathbb{R}} |x|^{\beta}\left|\log|x|\right| p(s,x-\xi)dx \le Cs^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}}\left[1+|\log s|\right].$$

Proof. We shall follow the same idea as in the proof of Lemma 11. Making the substitution $x = ys^{\frac{\alpha}{2}}$, we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}} |x|^{\beta}\left|\log|x|\right| p(s,x-\xi)dx \\
&\le Cs^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}}\int_{\mathbb{R}} |y|^{\beta}\left[|\log|y||+|\log s|\right]\cdot\exp\left(-\sigma\left|y-\frac{\xi}{s^{\frac{\alpha}{2}}}\right|^{\frac{2}{2-\alpha}}\right)dy \\
&\le Cs^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}}(1+|\log s|)\left(\int_{|y|\le e}|y|^{\beta}|\log|y||dy+\int_{\mathbb{R}}\exp\left(-\sigma\left|y-\frac{\xi}{s^{\frac{\alpha}{2}}}\right|^{\frac{2}{2-\alpha}}\right)dy\right) \\
&\le Cs^{\frac{\alpha\beta}{2}+\frac{\alpha}{2}}(1+|\log s|).
\end{aligned}$$

This proves the lemma. □

Lemma 13. *Let θ_1 and θ_2 satisfy $-1<\theta_1<0$ and $-1<\theta_2\le 0$. Then, for any $\rho_1,\tau_2\in\mathbb{R}, \rho_1\neq\tau_2$,*

(i) If $\theta_1+\theta_2=-1$, then:

$$\int_{\mathbb{R}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1 \le C+C|\log(\rho_2-\tau_1)|.$$

(ii) If $\theta_1+\theta_2<-1$, then:

$$\int_{\mathbb{R}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1 \le C|\rho_2-\tau_1|^{1+\theta_1+\theta_2}.$$

Proof. Without loss of generality, we suppose $\tau_1\le\rho_2$. We divide the integral domain into four intervals.

$$\begin{aligned}
&\int_{\mathbb{R}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1 \\
=\;&\int_{-\infty}^{\frac{3\tau_1-\rho_2}{2}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1 \\
&+\int_{\frac{3\tau_1-\rho_2}{2}}^{\frac{\tau_1+\rho_2}{2}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1 \\
&+\int_{\frac{\tau_1+\rho_2}{2}}^{\frac{3\rho_2-\tau_1}{2}}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1 \\
&+\int_{\frac{3\rho_2-\tau_1}{2}}^{\infty}|\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}p(s_2-s_1,\rho_2-\rho_1)d\rho_1 \\
=:\;&I_1+I_2+I_3+I_4.
\end{aligned}$$

Let us consider I_2 first. When $\rho_1 \in \left[\dfrac{3\tau_1 - \rho_2}{2}, \dfrac{\tau_1 + \rho_2}{2}\right]$, we have $|\rho_2 - \rho_1| \geq \frac{\rho_2 - \tau_1}{2}$. Noticing $p(s_2 - s_1, \rho_2 - \rho_1) \leq 1$, we have the following estimate for I_2:

$$\begin{aligned} I_2 &\leq \left(\frac{\rho_2 - \tau_1}{2}\right)^{\theta_2} \int_{\frac{3\tau_1 - \rho_2}{2}}^{\frac{\tau_1 + \rho_2}{2}} |\rho_1 - \tau_1|^{\theta_1} d\rho_1 \\ &\leq \left(\frac{\rho_2 - \tau_1}{2}\right)^{\theta_2} \left[\int_{\tau_1}^{\frac{\tau_1 + \rho_2}{2}} (\rho_1 - \tau_1)^{\theta_1} d\rho_1 + \int_{\frac{3\tau_1 - \rho_2}{2}}^{\tau_1} (\tau_1 - \rho_1)^{\theta_1} d\rho_1\right] \\ &= C\bigg(\rho_2 - \tau_1\bigg)^{1+\theta_1+\theta_2}. \end{aligned}$$

With the same argument, we have:

$$I_3 \leq C\bigg(\rho_2 - \tau_1\bigg)^{1+\theta_1+\theta_2}.$$

Now, we study I_1. The term I_4 can be analyzed in a similar way. Since $\rho_1 < \frac{3\tau_1 - \rho_2}{2} < \tau_1 < \rho_2$, we have:

$$I_1 \leq \int_{-\infty}^{\frac{3\tau_1 - \rho_2}{2}} (\tau_1 - \rho_1)^{\theta_1+\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 .$$

To estimate the above integral, we divide our estimation into three cases.

Case (i): $\theta_1 + \theta_2 < -1$.

In this case, we bound $p(s_2 - s_1, \rho_2 - \rho_1)$ by 1. Thus, we have:

$$I_1 \leq \int_{-\infty}^{\frac{3\tau_1 - \rho_2}{2}} (\tau_1 - \rho_1)^{\theta_1+\theta_2} d\rho_1 = \frac{1}{1 + \theta_1 + \theta_2}\left(\frac{\rho_2 - \tau_1}{2}\right)^{1+\theta_1+\theta_2}.$$

Case (ii): $\theta_1 + \theta_2 = -1$, $\frac{\rho_2 - \tau_1}{2} \geq 1$.

In this case, we have $\frac{3\tau_1 - \rho_2}{2} \leq \tau_1 - 1$. Thus, we have:

$$\begin{aligned} I_1 &\leq \int_{-\infty}^{\tau_1 - 1} (\tau_1 - \rho_1)^{-1} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \\ &\leq \int_{-\infty}^{\tau_1 - 1} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \\ &\leq \int_{-\infty}^{\infty} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \end{aligned}$$

which is bounded when s_1 and s_2 are in a bounded domain.

Case (iii): $\theta_1 + \theta_2 = -1$, $\frac{\rho_2 - \tau_1}{2} < 1$.

In this case, we divide the integral into two intervals as follows.

$$\begin{aligned} I_1 &= \int_{-\infty}^{\frac{3\tau_1 - \rho_2}{2}} (\tau_1 - \rho_1)^{\theta_1+\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \\ &\leq \int_{-\infty}^{\tau_1 - 1} (\tau_1 - \rho_1)^{-1} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 + \int_{\tau_1 - 1}^{\frac{3\tau_1 - \rho_2}{2}} (\tau_1 - \rho_1)^{-1} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \\ &\leq C + \int_{\tau_1 - 1}^{\frac{3\tau_1 - \rho_2}{2}} (\tau_1 - \rho_1)^{-1} d\rho_1 \\ &\leq C + C|\ln(\rho_2 - \tau_1)| . \end{aligned}$$

Similar argument works for I_4. Combining the estimates for I_k, $k = 1, 2, 3, 4$ yields the lemma. □

Lemma 14. *Let θ_1 and θ_2 satisfy $-1 < \theta_1 < 0, -1 < \theta_2 \le 0$ and $\theta_1 + 2\theta_2 > -2$. Let $0 \le r_1 < r_2 \le T$ and $0 \le s_1 < s_2 \le T$. Then, for any $\rho_1, \tau_2 \in \mathbb{R}, \rho_1 \ne \tau_2$, we have:*

$$\int_{\mathbb{R}^2} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} |\tau_2 - \tau_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) p(r_2 - r_1, \tau_2 - \tau_1) d\rho_1 d\tau_1$$
$$\le \begin{cases} C(s_2 - s_1)^{\frac{\alpha(\theta_1+\theta_2+1)}{2}} (r_2 - r_1)^{\frac{\alpha(\theta_2+1)}{2}}, & \theta_1 + \theta_2 > -1; \\ C(r_2 - r_1)^{\frac{\alpha(\theta_1+2\theta_2+2)}{2}}, & \theta_1 + \theta_2 < -1; \\ C(r_2 - r_1)^{\frac{\alpha(\theta_2+1)}{2}} \left[1 + |\log(r_2 - r_1)|\right], & \theta_1 + \theta_2 = -1. \end{cases} \tag{4.5}$$

Proof. First, we write:

$$\begin{aligned} I &:= \int_{\mathbb{R}^2} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} |\tau_2 - \tau_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) p(r_2 - r_1, \tau_2 - \tau_1) d\rho_1 d\tau_1 \\ &= \int_{\mathbb{R}} f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) |\tau_2 - \tau_1|^{\theta_2} p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1 \,, \end{aligned} \tag{4.6}$$

where:

$$f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) = \int_{\mathbb{R}} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \,.$$

We divide the situation into three cases.

Case (i): $\theta_1 + \theta_2 > -1$.

In this case, we apply the Hölder's inequality to obtain:

$$\begin{aligned} f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) &\le \left\{ \int_{\mathbb{R}} |\rho_1 - \tau_1|^{\theta_1+\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \right\}^{\frac{\theta_1}{\theta_1+\theta_2}} \\ &\quad \cdot \left\{ \int_{\mathbb{R}} |\rho_2 - \rho_1|^{\theta_1+\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) d\rho_1 \right\}^{\frac{\theta_2}{\theta_1+\theta_2}} \\ &\le C(s_2 - s_1)^{\frac{\alpha(\theta_1+\theta_2)}{2} + \frac{\alpha}{2}} \,, \end{aligned} \tag{4.7}$$

where the last inequality follows from Lemma 11. Substituting the above estimate Equation (4.7) into Equation (4.6), we have:

$$\begin{aligned} I &= \int_{\mathbb{R}} f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) |\tau_2 - \tau_1|^{\theta_2} p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1 \\ &\le C(s_2 - s_1)^{\frac{\alpha(\theta_1+\theta_2)}{2} + \frac{\alpha}{2}} \int_{\mathbb{R}} |\tau_2 - \tau_1|^{\theta_2} p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1 \,. \end{aligned}$$

Using Lemma 11, again we have,

$$I \le C(s_2 - s_1)^{\frac{\alpha(\theta_1+\theta_2)}{2} + \frac{\alpha}{2}} (r_2 - r_1)^{\frac{\alpha\theta_2}{2} + \frac{\alpha}{2}} \,.$$

Case (ii): $\theta_1 + \theta_2 < -1$.

In this case, from Lemma 13, Part (ii), it follows:

$$f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) \le C|\rho_2 - \tau_1|^{\theta_1+\theta_2+1} \,.$$

Hence, we have:

$$I \le C \int_{\mathbb{R}} |\rho_2 - \tau_1|^{\theta_1+\theta_2+1} |\tau_2 - \tau_1|^{\theta_2} p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1 \,.$$

Now, since from the condition of the lemma, $\theta_1 + 2\theta_2 + 1 > -1$, we can use Hölder's inequality, such as in the inequality (4.7) in Case (i), to obtain:

$$\begin{aligned} I &\le C \int_{\mathbb{R}} |\rho_2 - \tau_1|^{\theta_1+\theta_2+1} |\tau_2 - \tau_1|^{\theta_2} p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1 \\ &\le C(r_2 - r_1)^{\frac{\alpha(\theta_1+2\theta_2)}{2}+\alpha} \,. \end{aligned}$$

Case (iii): $\theta_1 + \theta_2 = -1$.

In this case, we first use Lemma 13, Part (i), to obtain:

$$f(\tau_1, \rho_2, s_1, s_2, \theta_1, \theta_2) \le C\left[1 + |\log|\rho_2 - \tau_1||\right] \,.$$

Thus, using Lemma 12, we have:

$$\begin{aligned} I &\le C \int_{\mathbb{R}} \{1 + |\log|\rho_2 - \tau_1||\} |\tau_2 - \tau_1|^{\theta_2} p(r_2 - r_1, \tau_2 - \tau_1) d\tau_1 \\ &\le C(r_2 - r_1)^{\frac{\alpha(\theta_2+1)}{2}} \left[1 + |\log|r_2 - r_1||\right] \,. \end{aligned}$$

The lemma is then proven. □

Corollary 15. *Let θ_1 and θ_2 satisfy $-1 < \theta_1 < 0, -1 < \theta_2 \le 0$ and $\theta_1 + 2\theta_2 > -2$. Let $0 \le r_1 < r_2 \le T$ and $0 \le s_1 < s_2 \le T$. Then, for any $\rho_1, \tau_2 \in \mathbb{R}$, $\rho_1 \ne \tau_2$, we have:*

$$\begin{aligned} &\int_{\mathbb{R}^2} |\rho_1 - \tau_1|^{\theta_1} |\rho_2 - \rho_1|^{\theta_2} |\tau_2 - \tau_1|^{\theta_2} p(s_2 - s_1, \rho_2 - \rho_1) p(r_2 - r_1, \tau_2 - \tau_1) d\rho_1 d\tau_1 \\ \le &\begin{cases} C(s_2 - s_1)^{\frac{\alpha(\theta_1+2\theta_2+2)}{4}} (r_2 - r_1)^{\frac{\alpha(\theta_1+2\theta_2+2)}{4}} \,; & \theta_1 + \theta_2 \ne -1 \\ C(s_2 - s_1)^{\frac{\alpha(\theta_2+1)}{4}} (r_2 - r_1)^{\frac{\alpha(\theta_2+1)}{4}} \left[1 + |\log(r_2 - r_1)| + |\log(s_2 - s_1)|\right] \,; & \theta_1 + \theta_2 = -1 \,. \end{cases} \end{aligned} \tag{4.8}$$

Proof. Consider first the case $\theta_1 + \theta_2 < -1$. Denote the integral on the left-hand side of Equation (4.8) by I. Then, the inequality Equation (4.8) implies:

$$I \le C(r_2 - r_1)^{\frac{\alpha(\theta_1+2\theta_2)}{2}+\alpha} \,.$$

In the same way, we have:

$$I \le C(s_2 - s_1)^{\frac{\alpha(\theta_1+2\theta_2)}{2}+\alpha} \,.$$

Now, we use the fact that if three numbers satisfying $a \le b$ and $a \le c$, then $a = a^{1/2} a^{1/2} \le b^{1/2} c^{1/2}$.

$$I \le C(r_2 - r_1)^{\frac{\alpha(\theta_1+2\theta_2)}{4}+\alpha/2} (s_2 - s_1)^{\frac{\alpha(\theta_1+2\theta_2)}{4}+\alpha/2}$$

which simplifies to Equation (4.8). Exactly the same argument can be applied to the case $\theta_1 + \theta_2 = -1$ and the case $\theta_1 + \theta_2 > -1$. Thus, the inequality Equation (4.7) implies Equation (4.8). □

Lemma 16. *Let* $p_1, \cdots, p_n > 0$. *Then for any* $T > 0$,

$$\int_{0\le s_1<\cdots<s_n\le T} (s_n - s_{n-1})^{p_n-1} \cdots (s_2 - s_1)^{p_2-1} s_1^{p_1-1} ds = \frac{T^n \prod_{k=1}^n \Gamma(p_k)}{\Gamma(p_1 + \cdots + p_n + 1)} . \tag{4.9}$$

Proof. This is well known. For example, it is a straightforward consequence of Formula 4.634 of [17] with some obvious transformations. □

Lemma 17. *Assume that* u_0 *is bounded. Then:*

$$\sup_{x\in\mathbb{R}} \int_{\mathbb{R}^d} Z(t, x, \xi) u_0(\xi) d\xi \le C .$$

Proof. We use $Z(t, x, \xi) = Z_0(t, x - \xi, \xi) + V_Z(t, x, \xi)$. Since u_0 is bounded,

$$\left| \int_{\mathbb{R}^d} Z_0(t, x, \xi) u_0(\xi) d\xi \right| \le C \int_{\mathbb{R}^d} |Z_0(t, x, \xi)| d\xi$$

which is bounded by the estimates in Equation (3.4) and a substitution $\xi = x + t^{\frac{\alpha}{2}} y$. In fact, we have, for example, when $d \ge 3$,

$$\int_{\mathbb{R}^d} |Z_0(t, x - \xi)| d\xi \le C \int_{\mathbb{R}^d} t^{-\alpha} t^{\frac{(2-d)\alpha}{2}} t^{\frac{d\alpha}{2}} |y|^{2-d} \exp\{-\sigma |y|^{\frac{2}{2-\alpha}}\} dy \le C t^{1-\alpha} \le C.$$

Similarly, using the estimation for $V_Z(t, x, \xi)$ given in Proposition 8, we can bound $\int_{\mathbb{R}^d} |V_Z(t, x, \xi)| d\xi$ by a constant. In fact, for example, when $d = 3$, we have:

$$\int_{\mathbb{R}^d} |V_Z(t, x, \xi)| d\xi \le C t^{\frac{\gamma_0 \alpha}{2} - \alpha} \int_{\mathbb{R}^d} t^{\frac{3\alpha}{2}} t^{\frac{(\gamma-\gamma_0-1)\alpha}{2}} |y|^{\gamma-\gamma_0-1} \exp\{-\sigma |y|^{\frac{2}{2-\alpha}}\} dy \le C t^{\frac{\gamma\alpha}{2}} \le C.$$

The other dimension cases can be dealt with the same way. □

5. Proof of the Main Theorem 2

Change t to s and x to y, and the Equation (2.5) for the mild solution becomes:

$$u(s, y) = \int_{\mathbb{R}^d} Z(s, y, \xi) u_0(\xi) d\xi + \int_0^s \int_{\mathbb{R}^d} Y(s - r, y, z) u(r, z) W^H(dz) dr .$$

Substituting the above into Equation (2.5), we have:

$$\begin{aligned} u(t, x) &= \int_{\mathbb{R}^d} Z(t, x, \xi) u_0(\xi) d\xi + \int_0^t \int_{\mathbb{R}^{2d}} Y(t - s, x, y) Z(s, y, \xi) u_0(\xi) d\xi W^H(dy) ds \\ &\quad + \int_0^t \int_0^s \int_{\mathbb{R}^{2d}} Y(t - s, x, y) Y(s - r, y, z) u(r, z) W^H(dz) dr W^H(dy) ds . \end{aligned}$$

We continue to iterate this procedure to obtain:

$$u(t, x) = \sum_{n=0}^{\infty} \Psi_n(t, x) , \tag{5.1}$$

where Ψ_n satisfies the following recursive relation:

$$\begin{aligned}
\Psi_0(t,x) &= \int_{\mathbb{R}^d} Z(t,x,\xi)u_0(\xi)d\xi \\
\Psi_{n+1}(t,x) &= \int_0^t \int_{\mathbb{R}^d} Y(t-s,x,y)\Psi_n(s,y)W^H(dy)ds\,, \quad n=0,1,2,\cdots
\end{aligned}$$

To write down the explicit expression for the expansion (5.1), we denote:

$$f_n(t,x;x_1,\cdots,x_n) = \int_{T_n}\int_{\mathbb{R}^d} Y(t-s_n,x,x_n)\cdots Y(s_2-s_1,x_2,x_1)Z(s_1,x_1,\xi)u_0(\xi)d\xi d\mathbf{s}\,, \tag{5.2}$$

where:

$$T_n = 0 \le s_1 < s_1 < \cdots < s_1 \le t \quad \text{and} \quad d\mathbf{s} = ds_1 ds_2 \cdots ds_n\,.$$

With these notations, we see from the above iteration procedure that:

$$\begin{aligned}
\Psi_n(t,x) &= I_n(\tilde{f}_n(t,x)) \\
&= \int_{\mathbb{R}^{nd}} f_n(t,x;x_1,\cdots,x_n)W^H(dx_1)W^H(dx_2)\cdots W^H(dx_n) \\
&= \int_{\mathbb{R}^{nd}} \tilde{f}_n(t,x;x_1,\cdots,x_n)W^H(dx_1)W^H(dx_2)\cdots W^H(dx_n)\,,
\end{aligned} \tag{5.3}$$

where I_n denotes the multiple Itô-type integral with respect to $W(x)$ (see [12]) and $\tilde{f}_n(t,x;x_1,\cdots,x_n)$ is the symmetrization of $f_n(t,x;x_1,\cdots,x_n)$ with respect to $x_1,\cdots,x_n$:

$$\tilde{f}_n(t,x;x_1,\cdots,x_n) = \frac{1}{n!}\sum_{i_1,\cdots,i_n\in\sigma(n)} f_n(t,x;x_{i_1},\cdots,x_{i_n})\,,$$

where $\sigma(n)$ denotes the set of permutations of $(1,2,\cdots,n)$.

The Expansion (5.1) with the explicit Expression (5.3) for Ψ_n is called the chaos expansion of the solution.

If Equation (2.2) has a square integrable solution, then it has a chaos expansion according to a general theorem of Itô. From the above iteration procedure, it is easy to see that this chaos expansion of the solution is given uniquely by Equations (5.1)–(5.3). This is the uniqueness.

If we can show that the series Equation (5.1) is convergent in $L^2(\Omega,\mathcal{F},P)$, then it is easy to verify that $u(t,x)$ defined by Equations (5.1)–(5.3) satisfies Equation (2.5). Thus, the existence of the solution to Equation (2.2) is solved, and the explicit form of the solution is also given (by Equations (5.1)–(5.3)). We refer to [12] for more detail.

Thus, our remaining task is to prove that the series defined by Equation (5.1) is convergent in $L^2(\Omega,\mathcal{F},P)$. To this end, we need to use the lemmas that we proved in the previous section.

Let now $u(t,x)$ be defined by Equations (5.1)–(5.3). Then, we have:

$$\begin{aligned}
\mathbb{E}[u(t,x)^2] &= \sum_{n=0}^{\infty}\mathbb{E}\left[I_n(\tilde{f}_n(t,x))\right]^2 \\
&= \sum_{n=0}^{\infty} n!\langle \tilde{f}_n,\tilde{f}_n\rangle_H \\
&\le \sum_{n=0}^{\infty} n!\langle f_n,f_n\rangle_H\,,
\end{aligned} \tag{5.4}$$

where:

$$\langle f_n, f_n\rangle_H = \int_{\mathbb{R}^{2nd}} \prod_{i=1}^{n} \varphi_H(u_i, v_i) f_n(u_1, \cdots, u_n) f_n(v_1, \cdots, v_n) du_1 dv_1 du_2 dv_2 \cdots du_n dv_n \tag{5.5}$$

and the last inequality follows from Hölder inequality. Here, and in the remaining part of the paper, we use the following notations:

$$\begin{aligned} u_i &= (u_{i1}, \cdots, u_{id}), \quad du_i = du_{i1} \cdots du_{id}, \quad i = 1, 2, \cdots, n; \\ \varphi_H(u_i, v_i) &= \prod_{j=1}^{d} \varphi_{H_j}(u_{ij}, v_{ij}) = \prod_{j=1}^{d} H_j(2H_j - 1)|u_{ij} - v_{ij}|^{2H_j - 2}. \end{aligned}$$

We use the idea in [12] to estimate each term $\Theta_n(t, x) = n!\langle f_n, f_n\rangle_H$ in the series (5.4). By the defining formula (5.2) for f_n, we have:

$$\begin{aligned} \Theta_n(t, x) &= n! \int_{T_n^2} \int_{\mathbb{R}^{2nd+2}} \prod_{i=1}^{n} \varphi_H(\xi_i - \eta_i)\, Y(t - s_n, x, \xi_n) \cdots Y(s_2 - s_1, \xi_2, \xi_1) \\ &\quad \cdot \int_{\mathbb{R}^d} Z(s_1, \xi_1, \xi_0) u_0(\xi_0) d\xi_0 \cdot Y(t - r_n, x, \eta_n) \cdots Y(r_2 - r_1, \eta_2, \eta_1) \\ &\quad \cdot \int_{\mathbb{R}^d} Z(r_1, \eta_1, \eta_0) u_0(\eta_0) d\eta_0 d\xi d\eta d\mathbf{s} d\mathbf{r}. \end{aligned}$$

Application of Lemma 17 to the above integral yields:

$$\begin{aligned} \Theta_n(t, x) &\leq Cn! \int_{T_n^2} \int_{\mathbb{R}^{2nd}} \prod_{i=1}^{n} \varphi_H(\xi_i - \eta_i) Y(t - s_n, x, \xi_n) \cdots Y(s_2 - s_1, \xi_2, \xi_1) \\ &\quad \cdot Y(t - r_n, x, \eta_n) \cdots Y(r_2 - r_1, \eta_2, \eta_1) d\xi d\eta d\mathbf{s} d\mathbf{r}. \end{aligned}$$

Using Lemma 10 for the above integral, we have:

$$\Theta_n(t, x) \leq C^n n! \int_{T_n^2} \prod_{i=1}^{d} \Theta_{i,n}(t, x_i, \mathbf{s}, \mathbf{r}) d\mathbf{s} d\mathbf{r}, \tag{5.6}$$

where:

$$\begin{aligned} \Theta_{i,n}(t, x_i, \mathbf{s}, \mathbf{r}) &= \int_{\mathbb{R}^{2n}} \left\{ \prod_{k=1}^{n} \varphi_{H_i}(\rho_k - \tau_k) \right\} |t - s_n|^{\frac{\zeta_d}{d}} |x_i - \rho_n|^{\frac{\kappa_d}{d}} p(t - s_n, x_i - \rho_n) \\ &\quad \cdots |s_2 - s_1|^{\frac{\zeta_d}{d}} |\rho_2 - \rho_1|^{\frac{\kappa_d}{d}} p(s_2 - s_1, \rho_2 - \rho_1) \\ &\quad \cdot |t - r_n|^{\frac{\zeta_d}{d}} |x_i - \tau_n|^{\frac{\kappa_d}{d}} p(t - r_n, x_i - \tau_n) \cdots |r_2 - r_1|^{\frac{\zeta_d}{d}} \\ &\quad \cdot |\tau_2 - \tau_1|^{\frac{\kappa_d}{d}} p(r_2 - r_1, \tau_2 - \tau_1) d\rho d\tau. \end{aligned}$$

Here, we use the notation $\rho_k = \xi_{ki}$ and $\tau_k = \eta_{ki}$, $k = 1, \cdots, n$. The quantity $\Theta_{i,n}$ can be written as:

$$\begin{aligned} \Theta_{i,n}(t, x_i, \mathbf{s}, \mathbf{r}) &= |t - s_n|^{\frac{\zeta_d}{d}} |t - r_n|^{\frac{\zeta_d}{d}} \cdots |s_2 - s_1|^{\frac{\zeta_d}{d}} |r_2 - r_1|^{\frac{\zeta_d}{d}} \\ &\quad \cdot \int_{\mathbb{R}^{2n}} \left\{ \prod_{k=1}^{n} \varphi_{H_i}(\rho_k - \tau_k) \right\} |x_i - \rho_n|^{\frac{\kappa_d}{d}} p(t - s_n, x_i - \rho_n) \\ &\quad \cdot |x_i - \tau_n|^{\frac{\kappa_d}{d}} p(t - r_n, x_i - \tau_n) \cdots |\rho_2 - \rho_1|^{\frac{\kappa_d}{d}} p(s_2 - s_1, \rho_2 - \rho_1) \\ &\quad \cdots |\tau_2 - \tau_1|^{\frac{\kappa_d}{d}} p(r_2 - r_1, \tau_2 - \tau_1) d\rho d\tau. \end{aligned} \tag{5.7}$$

From the definition Equation (4.3) of κ_d, we see easily that $\frac{\kappa_d}{d} > -1$. We assume:

$$2H_i + \frac{2\kappa_d}{d} > 0 \,. \tag{5.8}$$

Under the above condition, we can apply Corollary 15 with $\theta_1 = 2H_i - 2 > -1$, $\theta_2 = \frac{\kappa_d}{d} > -1$ to the integration $d\rho_1 d\tau_1$ in Expression (5.7) (Condition (5.8) implies that $\theta_1 + 2\theta_2 > -2$). Then, when $\theta_1 + \theta_2 \neq -1$, we have:

$$\begin{aligned}\Theta_{i,n}(t, x_i, \mathbf{s}, \mathbf{r}) \;\le\; & C|t-s_n|^{\frac{\zeta_d}{d}}|t-r_n|^{\frac{\zeta_d}{d}}\cdots|s_3-s_2|^{\frac{\zeta_d}{d}}|r_3-r_2|^{\frac{\zeta_d}{d}} \\ & \cdot|s_2-s_1|^{\frac{\zeta_d}{d}+\frac{H_id+\kappa_d}{2d}\alpha}|r_2-r_1|^{\frac{\zeta_d}{d}+\frac{H_id+\kappa_d}{2d}\alpha} \\ & \cdot\int_{\mathbb{R}^{2n-2}}\left\{\prod_{k=2}^n \varphi_{H_i}(\rho_k-\tau_k)\right\}|x_i-\rho_n|^{\frac{\kappa_d}{d}}p(t-s_n, x_i-\rho_n) \\ & \cdot|x_i-\tau_n|^{\frac{\kappa_d}{d}}p(t-r_n, x_i-\tau_n)\cdots|\rho_3-\rho_2|^{\frac{\kappa_d}{d}}p(s_3-s_2, \rho_3-\rho_2) \\ & \cdots|\tau_3-\tau_3|^{\frac{\kappa_d}{d}}p(r_3-r_2, \tau_3-\tau_2)d\rho_n\cdots d\rho_2 d\tau_n\cdots d\tau_2 \,.\end{aligned}$$

Repeatedly applying this argument, we obtain:

$$\Theta_{i,n}(t, x_i, \mathbf{s}, \mathbf{r}) \;\le\; C^n\prod_{k=1}^n|t_{k+1}-t_k|^{\ell_i}|s_{k+1}-s_k|^{\ell_i}\,, \tag{5.9}$$

where we recall the convention that $t_{n+1} = t$ and $s_{n+1} = s$ and where:

$$\ell_i = \frac{\zeta_d}{d} + \frac{H_id+\kappa_d}{2d}\alpha\,.$$

Substituting the above estimate of $\Theta_{i,n}$ into the expression for Θ_n, we have:

$$\begin{aligned}\Theta_n(t,x) \;\le\; & C^n\int_{T_n^2}\prod_{k=1}^n(s_{k+1}-s_k)^{\ell}(r_{k+1}-r_k)^{\ell}d\mathbf{s}d\mathbf{r} \\ = \; & C^n\left[\int_{T_n}\prod_{k=1}^n(s_{k+1}-s_k)^{\ell}d\mathbf{s}\right]^2 \,,\end{aligned}$$

where:

$$\ell = \sum_{i=1}^d \ell_i = \zeta_d + \frac{|H|\alpha}{2} + \frac{\kappa_d\alpha}{2} \quad \text{with} \quad |H| = \sum_{i=1}^d H_i\,.$$

Now, we apply Lemma 16 to obtain:

$$\begin{aligned}\Theta_n(t,x) \;\le\; & C^n\left[\frac{\Gamma(\ell+1)}{\Gamma(n(\ell+1))}\right]^2 \\ \le\; & \frac{C^n}{\Gamma(2n(\ell+1))}\,.\end{aligned}$$

This estimate combined with Equation (5.4) proves that if:

$$2(\ell+1) > 1\,, \tag{5.10}$$

then $\sum_{n=0}^\infty \Theta_n(t,x)$ is bounded, which implies that the series (5.1) is convergent in $L^2(\Omega, \mathcal{F}, P)$.

Now, we analyze the above condition (5.10). By the definition of ℓ, this condition can be written as:

$$\ell = \zeta_d + \frac{|H|\alpha}{2} + \frac{\kappa_d \alpha}{2} > -1/2\,.$$

or:

$$|H| > -\frac{1}{\alpha} - \kappa_d - \frac{2\zeta_d}{\alpha}\,. \tag{5.11}$$

Using the definitions of κ_d and ζ_d defined by Equations (4.2) and (4.3), we see that the right-hand side of Equation (5.11) is:

$$\begin{cases} -\frac{1}{\alpha} - 0 - \frac{2}{\alpha}\left(-1+\frac{\alpha}{2}\right) = \frac{1}{\alpha} - 1 & \text{when } \ d=1 \\ -\frac{1}{\alpha} - 0 - \frac{2}{\alpha}\left(-1\right) = \frac{1}{\alpha} & \text{when } \ d=2 \\ -\frac{1}{\alpha} + (2-\gamma+2\gamma_0) + \frac{2}{\alpha}\left((\gamma-2\gamma_0)\frac{\alpha}{2}+1\right) = \frac{1}{\alpha}+2 & \text{when } \ d=4 \\ -\frac{1}{\alpha} - 2 + d - \frac{\gamma-\gamma_0}{2} + \frac{2}{\alpha}\left((\gamma-\gamma_0)\frac{\alpha}{4}+1\right) = \frac{1}{\alpha} - 2 + d & \text{when } \ d=3 \text{ or } d \geq 5\,. \end{cases}$$

Summarizing the above computations, we obtain that Condition (5.11) or Condition (5.10) is equivalent to:

$$\sum_{i=1}^{d} H_i > d - 2 + \frac{1}{\alpha}\,. \tag{5.12}$$

When $\theta_1 + \theta_2 = -1$, Corollary 15 implies that, for any $\varepsilon > 0$,

$$\begin{aligned} &\int_{\mathbb{R}^2} |\rho_1-\tau_1|^{\theta_1}|\rho_2-\rho_1|^{\theta_2}|\tau_2-\tau_1|^{\theta_2} p(s_2-s_1,\rho_2-\rho_1)p(r_2-r_1,\tau_2-\tau_1)d\rho_1 d\tau_1 \\ \leq\ & C(s_2-s_1)^{\frac{\alpha(\theta_2+1+\varepsilon)}{4}}(r_2-r_1)^{\frac{\alpha(\theta_2+1++\varepsilon)}{4}}\,. \end{aligned}$$

Now, we can follow the above same argument to obtain that if:

$$2(\ell+1) > 1\,, \tag{5.13}$$

where $\ell = \frac{d\epsilon+\kappa_d+d}{4}\alpha$, then $\sum_{n=0}^{\infty} \Theta_n(t,x)$ is convergent in $L^2(\Omega, \mathcal{F}, P)$. In the same way as in the case $\theta_1+\theta_2 \neq -1$, we can show that Condition (5.12) implies Equation (5.13).

Now, we consider Condition (5.8). From the definition Equation (4.3) of κ_d, we see that when $d = 1,2,3,4$, $H_i > 1/2$ implies Equation (5.8). When $d \geq 5$, then Condition (5.8) is implied by the following:

$$H_i > 1 - \frac{2}{d} - \frac{\gamma}{2d}$$

by choosing γ_0 sufficiently small. Theorem 2 is then proven. □.

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Conflicts of Interest

The authors declare no conflict of interest.

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