A General Stochastic Volatility Model on VIX Options

By

Yanhao Cui

Submitted to the Department of Mathematics and the
Graduate Faculty of the University of Kansas
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

Committee members

Terry Soo, Chairperson

Yaozhong Hu, Co-Chair

Xuemin Tu

Zhipeng Liu

Jie Han

Date defended: Dec 13, 2019

The Dissertation Committee for Yanhao Cui certifies
that this is the approved version of the following dissertation :

A General Stochastic Volatility Model on VIX Options

Terry Soo, Chairperson

Date approved: Dec 13, 2019

Abstract

In this dissertation, we study a general stochastic volatility model for the VIX options (Chicago Board Options Exchange) volatility index, which is a stochastic differential equation with 8 unknown parameters. It originated from a nested stochastic model based on several known models in the paper [7], stochastic volatility models and the Pricing of VIX Options. To estimate the parameters in these models from the real financial data a commonly used approach is the Generalized Method of Moments of Hansen (1982). We will study the model in more generality and we shall provide a completely different parameter estimation technique using the ergodic theory.

Since our equation is more general and new and since our equation is singular in the sense it does not satisfy the global Lipschitz condition, we shall first study the existence, uniqueness and positivity of the solution of the SDE, in which Feller's test will be used to calculate a criteria of all parameters such that the SDE has a unique and positive weak solution. The positivity property of the solution is crucial, since volatility is always positive.

Then, we use the strong large law of numbers theorems given e.g. in [4] to give the region for the parameters to live in order that the model is ergodic. In important condition for the ergodicity is the positive recurrency. We give verifiable condition on the parameters so that process is positive recurrent. This results also provide ways to calculate the invariant distribution (limiting distribution).

The next step is to provide a theoretical methodology of parameter estimation. Simulation process will be introduced with giving an example for each case. In the future study, I will work on testing the model using numerical schemes.

Keywords: Stochastic volatility model, VIX options, Feller's test, ergodicity, parameter estimation.

Acknowledgements

I would like to express my sincere gratitude to my advisor, Professor Yaozhong Hu and his wife Ms. Jun Fu. Prof. Hu has been completely supportive throughout my entire research studies. I deeply appreciated his patience and generosity in guiding me. And during some of my most difficult times staying in Kansas, they gave me the moral support and the freedom I needed to move on. Without them, I would not make my way to graduation. Their kindness and generosity made my life a lot easier in Lawrence.

Additionally, I would like to thank Professor Terry Soo for advising me besides Prof. Hu. He has given me all good suggestions, understanding and encouragement on my way to graduation, especially when I have a full-time job and a baby to take care. Thanks for not giving me up in any situation. Without him, I would not be able to complete my research.

Moreover, I thank Professor Xuemin Tu, Zhipeng Liu and Jie Han for taking part in the committee.

I am grateful to the whole faculty and staff of Department of Mathematics at University of Kansas for their support over the last few years. In particularly I would like to thank Lori Springs for diverting time from her busy schedule to talk to me, both academically and non- academically, whenever I stopped by her office.

I appreciate the support, motivation and help of my managers from work, Dale Bruegeman and Susan Lacey, for their fully understanding and generosity. Without their trust and encouragement, I can never balance my work and study to complete my dissertation.

Thanks are due to my friends Yan Meng, Qi Feng for staying connected from China. I also thank my friends Suchen Kang and Wenjun Ma for the good times that we had spent together in Lawrence.

I thank my family, who have cherished with me every great moment and supported me whenever I needed it. Finally, to my husband, Kun, who has been with me throughout my graduate studies; to my son, Nolan, who is the best gift for us. Thanks for loving and supporting me unconditionally during the best and worst moments.

Contents

Chapter 1

Introduction

1.1 Introduction and Background

1.1.1 Introduction of Volatility Index (VIX)

In today's stock market, there are many different players, including stockbrokers, traders, stock analysts, portfolio managers and investment bankers. They are all interested in looking for various financial instruments to decrease the volatility of their portfolios values. Volatility in Finance is the degree of a trading price series over time as measured by standard deviation of logrithmic returns. In Finance, the unit of volatility is a percentage, and they usually use percent per annum. If one would like to know how the stock market is performing, they could look at an index of stocks for either the whole market or for a specific segment of the market. Here index is the measurement of how the whole stock market changes. In the United States, there are many kinds of various indices, each made of a different pool of stocks. The major indices in financial market in the United States are Dow Jones Industrial Average, NASDAQ Composite Index, Russell 2000, and Standard and Poor's 500 (S&P 500). The S&P 500 consists of the largest 500 capitalization stocks traded in the United States. Thus, available financial tools to measure the volatility of those stock market indices become very important for investors all over the world.

Among the past 25 years in the United States, one of the most popular indicator of the whole stock market volatility is the CBOE (the Chicago Board Options Exchange) volatility index. CBOE volatility index is shorted for VIX. VIX is also the first successful volatility index. It indicates the stock market's expectation of 30-day volatility. VIX offers a measurement of the implied volatility. The implied volatility is an estimation of a security's price, of options on the S&P 500 index from eight different SPX option series within a maturity of thirty days. This volatility that is meant to be forward looking is calculated for both call options and put options. It is widely used and applied in measuring the market risk. VIX is also famous for the name of "investor fear gauge". Because when it is during the severe market movement and dramatic market turmoil, the VIX tends to increase, while as in the time of market is bullish, the VIX index remains in a low and gently changing level.

VIX Option uses the CBOE volatility index as the underlying asset, and it is one type of non-equity option. VIX option is the first exchange-traded option that enable investors to trade market volatility. It is a very useful new financial instrument of VIX option offered to investors for hedging their portfolios against unexpected and sudden sharp stock market changes directly on the S&P 500 index, and also for speculating forward looking volatility movements. When traders think that there might be increasing market volatility, they are able to make profit by purchasing VIX call options as a method of hedging instead of purchasing the normal index options, because a bearish market usually comes along with a large increase in market. So VIX option becomes a particularly interesting and a useful financial tool to incorporate various kinds of advanced strategies for professional investors in recent years. Although VIX is relatively new in the stock market, it has a rich history of development.

1.1.2 History of VIX development

VIX was first introduced in 1993 by Whaley in [22]. It was originally a weighted measure of the implied volatility of eight S&P100 at-the-money put and call options. In the beginning of VIX construction process, the underlying security market is OEX options that trades on the Chicago Board Options Exchange. OEX is the ticker symbol or the stock symbol to idendify S&P 100 index options. OEX option market is the most active and highly liquid index option market in the United States. At any given time, the volatility index stands for the implied volatility of a hypothetical at-the-money OEX option within 30 days to the expiration. Under the assumption that there exists a futures contract on volatility with that the current index level equals to the futures price, and based on a lognormal volatility process, Whaley himself uses the Black Scholes formula (See Appendix B) in [11] to obtain the fair price of the volatility options.

Since Whaley introduced the VIX from 1993, an increasing amount of research activities has emerged on the topic of pricing of VIX options. In some literature for example in Harvey and Whaley's [6], it is empirically proven that the volatility, V, is mean-reverting. That is to say, the implied volatility V, over time, will move back or return to its average historical levels. However, stock and option prices typically do not have such a mean reversion property, like a stock can go up and keep going up, and it would not go back to its average price over any specific period of time. This is also another reason why investors and traders care much about the volatility so that more and more literature are about the valuation of volatility. In 1996, Grunbichler and Longstaff in the article [3] evaluated volatility futures and options by a mean-reverting stochastic volatility process with a square-root diffusion term:

$$dV = (a - bV)dt + c\sqrt{V}dZ, \quad b > 0$$

assuming that the volatility risk premium was proportional to volatility risk. In the above equation and in the remaining part of this dissertation, dZ refers to the Itô "differential" of the Wiener process or Standard Brownie Motion Process Z.

In 1999, Demeterfi et al. described how volatility swaps work and derives pricing and hedging equations for them in his most influential paper [14]. Demeterfi et al. replicated the payoff of a variance swap pretty well by a static portfolio of ordinary European calls and puts on the price of the underlying asset. With utilizing a portfolio of options in which weights are inversely proportional to their strikes it realizes the replication of a log contract. Ever since 2003, the construction of the VIX started to use the simple valuation process Demeterfi stated in [14].

Later in 2000, Detemple and Osakwa examined the valuation of European and American-style volatility options based on a general equilibrium stochastic volatility framework under the volatility model in [12]:

$$dV = (a - bV)dt + cV^{\gamma}dZ. \tag{1.1}$$

Valuation formulas have been derived in the following four specific cases depending on the parameters a, γ, b, and c:

(1) Geometric Brownian Motion (GMBP), $a = 0, \gamma = 1$,

$$dV = -bVdt + cVdZ.$$

(2) Mean-Reverting Gaussian (MRGP), $\gamma = 0$,

$$dV = (a - bV)dt + cVdZ.$$

(3) Mean-Reverting Square-Root (MRSRP), $\gamma = 0.5, c^2 = 4a$,

$$dV = (\frac{c^2}{4} - bV)dt + c\sqrt{V}dZ.$$

(4) Mean-Reverting Log Process (MRLP),

$$d(\ln V) = (A - B\ln V)dt + CdZ.$$

The solutions of the above four volatility processes are also summarized and their distributional properties are listed in [12]. There are a growing body of literature focusing on this area in the following couple of years. Little and Pant in [21] developed a finite difference approach in an extended Black-Scholes framework to evaluate variance swaps assumed that local volatility is a known function of time and underlying asset price. In 2004, Howison, Rafailidis and Rasmussen considered the pricing of a certain amount of volatility derivatives, including volatility and variance swaps and swaptions in the paper [17] and generated closed solutions for volatility-average and variance swaps under a range of diffusion and jump-diffusion models for volatility. One of the models belongs to (1.1) as followed:

$$dV = (a - bV)dt + cV dZ$$

and

$$dV = (a - bV)dt + c\sqrt{V} dZ.$$

Then in 2005, under the assumption that returns are pure jump processes with independent increments, Carr priced volatility options in [16] by models able to capture the observed variation of market prices of vanilla stock options across strike and maturity. In [9] in 2006, within a jump diffusion asset model, in order to price some specific discretely sampled volatility derivatives, Windcliff, Forsyth and Vetzal provided the solution for a partial integro-differential equation in a numerical approach. In 2006, Buehler provided a general framework for modeling a joint market of stock price and derived a term structure of variance swaps in a Heath-Jarrow-Morton (HJM) arbitrage conditions in [5]. Then in 2008 Sepp developed an analytical method to price and hedge options on the realized variance within the Heston stochastic variance model under the assumption that there are jumps in asset returns and variance in [20]. In 2009, Albaness and Mijatovic in [2] proposed a new numerical method called spectral methods to develop a pricing framework such that European options, forward-starts, options on the realized variance and options on the VIX can be handled at the same time.

Since then, an enormous amount of research articles emerged in the area of volatility options pricing under stochastic volatility models like what was stated in (1.1) because of special properties stochastic volatility models hold (See Appendix A). One of the very famous stochastic volatility model is Heston's model proposed in 1993 in [10] which has the following form.
Let $S = V^2$, the following model is a mean-reverting square-root process:

$$dS = (a - bS)dt + c\sqrt{S}dZ. \tag{1.2}$$

1.2 A General Model

1.2.1 Model Inspiration

All the models included in the previous section are particular cases of the model introduced in Goard, Mazur and Mathew's paper [7], Stochastic Volatility Models and the Pricing of VIX Options. And this is the inspiration of this dissertation. In the paper [7], except the models mentioned above, it also included the popular 3/2-model:

$$dV = (aV + bV^2)dt + cV^{\frac{3}{2}}dZ. \tag{1.3}$$

There are some distinguished characteristics of the model (1.3). With the diffusion term having a relatively higher exponent of $\frac{3}{2}$ which is larger than any model stated in the history of development, it reduces the volatility's heteroskedasticity. Also, compared to other models possessing a linear drift coefficient, the model (1.3) with a nonlinear drift coefficient is able to behave good non-linear mean-reversion activity in a long-term view. With these features, the volatility changes would be more stable as if ever since a big volatility hit, it would also decrease fast, on the other hand, when it is experiencing small volatilities, it would take longer time to go up. It also provided the condition of (1.3) to keep the volatility V always positive. This paper tested empirically the ability to capture the dynamics of the VIX of those stochastic volatility models separately. Although in the paper [7], all of the models tested are nested within the large model:

$$dV = (c_1 + \frac{c_2}{V} + c_3 V \ln V + c_4 V + c_5 V^2)dt + kV^{\gamma} dZ. \tag{1.4}$$

The testing process is still based on individual models mentioned previously. It was found in the inspiration paper that the value of γ is a significant characteristic of distinguishing the volatility models in various categories and $\frac{3}{2}$ is the unconstrained estimate.
In this dissertation, inspired by the paper [7], a more generalized model with one more item $\ln V$ in the drift coefficient is introduced, and all the research is based on the new model. We look at the new model as a whole thing instead of only reserching on special known cases with assuming some of the parameters equal to zero. The preconditions of the parameters will be calculated such that the volatility model exists unique, positive solution and the model will be ergodic. Numerical methods will be introduced related to parameters estimation and simulation. Simulation for case one will be shown as an example. More testing will be carried out in the future studies.

1.2.2 Model Build-up

As is stated in the previous subsection, based on all the literatures mentioned, a new and more general stochastic volatility model is proposed as the following:

$$dV = (\frac{\rho_1}{V} + \rho_2 + \rho_3 \ln V + \rho_4 V + \rho_5 V \ln V + \rho_6 V^2)dt + \rho_0 V^{\gamma} dZ. \tag{1.5}$$

In this model, the drift coefficient is:

$$\mu(V) = \frac{\rho_1}{V} + \rho_2 + \rho_3 \ln V + \rho_4 V + \rho_5 V \ln V + \rho_6 V^2, \tag{1.6}$$

and the diffusion coefficient is:

$$\sigma(V) = \rho_0 V^{\gamma}. \tag{1.7}$$

It not only covers all the models in the drift coefficient existing in the literatures, but also a newly added term $\ln V$ is introduced to make the drift coefficient to exhibit a better nonlinear mean-reverting behavior when the volatility is higher than its long-run mean. In the next chapter, the existence, uniqueness and positivity of the solution of the model (1.5) will be given.

Chapter 2

Existence, Uniqueness and Positivity of the Solution

2.1 Weak Solution

2.1.1 Weak Solution up to an Explosion

In this subsection, the definitive results of week solutions of the time-homogeneous stochastic differential equation in 1-dimension are presented. These results are mainly taken from Engelbert and Schmidt's book. There are a certain number of textbooks providing a numerous amount of knowledge of the solution of SDE theories. Here I will use [13] as the main reference for the most of the results in this Chapter.

Consider the stochastic differential equation as followed,

$$dX_t = \mu(X_t)dt + \sigma(X_t)dZ_t, \tag{2.1}$$

where $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are Borel-measurable coefficients. Solutions of equation (2.1) may not exist globally, but only up to an "explosion time" S. We first recall uniqueness of solution for one-dimensional ODEs with Lipschitz-continuous condition.

Theorem 2.1 (Theorem 2.5 in [13]). *Suppose that the coefficients $\mu(x)$ and $\sigma(x)$ in* (2.1) *are locally Lipschitz-continuous; i.e., for every integer $n \geq 1$ there exists a constant $K_n > 0$ such that for $\|x\| \leq n$ and $\|y\| \leq n$:*

$$\|\mu(x) - \mu(y)\| + \|\sigma(x) - \sigma(y)\| \leq K_n \|x - y\|.$$

Then equation (2.1) *has a unique solution up to a positive random time $\tau > 0$ a.s..*

Example 2.2. Consider the stochastic differential equation,

$$X_t = 1 + \int_0^t X_s^2 \, ds.$$

The solution of this equation is $X_t = \frac{1}{1-t}$. By theorem 2.5 in the book [13] that is stated above, the solution is unique with Lipschitz continuous conditions satisfied.
With $\mu(x) = x^2$, $|\mu(x) - \mu(y)| = |x^2 - y^2| = |(x+y)(x-y)|$. For every integer $n \geq 1$, there exists a constant $K_n = 2n > 0$ such that,for every $|x| \leq n$ and $|y| \leq n$:

$$|(x+y)(x-y)| \leq 2n|x-y| = K_n|x-y|.$$

Example 2.2 indicates that even for ordinary differential equation, it is not enough to ensure the global solution when only a local Lipschitz condition is satisfied. The solution $\frac{1}{1-t}$ in the above example "explodes" as $t \uparrow 1$. This motivates us to introduce the concept of ”explosion".

Based on the model we built in Chapter 1, we shall only look at the one-dimensional case. Then we recall the definition of the weak solution up to an explosion time on the whole real line:

Definition 2.3. A weak solution up to an explosion time of equation (2.1) is a triple (X, Z), $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}$, where

i. $(\Omega, \mathcal{F}, P)$ is a probability space, and $\{\mathcal{F}_t\}$ is a filtration of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions;

ii. $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a continuous, adapted, $\mathbb{R} \cup [0,\infty]$-value process with $|X_0| < \infty$ a.s., and $\{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard, 1-dimensional Brownian motion;

iii. with

$$S_n := \inf\{t \geq 0 : |X_t| \geq n\}, \tag{2.2}$$

we have for all $n \geq 1$

$$P\Big[\int_0^{t\wedge S_n} \{|\mu(X_s)| + \sigma^2(X_s)\}\, ds < \infty\Big] = 1;\ \forall\, 0 \leq t < \infty, \tag{2.3}$$

and for $\forall\, 0 \leq t < \infty$,

$$P[X_{t\wedge S_n} = X_0 + \int_0^t \mu(X_s)\mathbb{1}_{\{s\leq S_n\}}\, ds + \int_0^t \sigma(X_s)\mathbb{1}_{\{s\leq S_n\}}\, dZ_s] = 1. \tag{2.4}$$

We refer to

$$S = \lim_{n\to\infty} S_n, \tag{2.5}$$

as the explosion time of X.

2.1.2 Positivity of Weak Solution

Under the circumstance of this dissertation in the finance context, volatility must be nonnegative over the timeline. Hence, rather than working on the process taking values on the entire real line as we have mentioned, we look into the weak solution on the positive interval $I = (0,\infty)$ to ensure the process is always positive.

Definition 2.4. A weak solution in the interval $I = (0,\infty)$ of equation (2.1) is a triple (X,Z), $(\Omega,\mathcal{F},P)$, $\{\mathcal{F}_t\}$, where

i. $(\Omega,\mathcal{F},P)$ is a probability space, and $\{\mathcal{F}_t\}$ is a filtration of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions (the filtered space is complete and filtration is right-continuous);

ii. $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ is a continuous, adapted, $[0,\infty]$-value process with $X_0 \in I$ a.s., and $\{Z_t, \mathcal{F}_t; 0 \le t < \infty\}$ is a standard, 1-dimensional Brownian motion;

iii. Let $\{a_n\}$ and $\{b_n\}$ be two strictly monotone sequences satisfying $0 < a_n < b_n < \infty$, $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = \infty$, and

$$S_n := \inf\{t \ge 0 : X_t \notin (a_n, b_n)\}, n \ge 1,$$

we have for all $n \ge 1$,

$$P(\int_0^{t\wedge S_n} \{|\mu(X_s)| + \sigma^2(X_s)\}\, ds < \infty) = 1;\ \ \forall\, 0 \le t < \infty,$$

and $\forall\, 0 \le t < \infty$,

$$P(X_{t\wedge S_n} = X_0 + \int_0^t \mu(X_s)\mathbb{1}_{\{s\le S_n\}}\, ds + \int_0^t \sigma(X_s)\mathbb{1}_{\{s\le S_n\}}\, dZ_s) = 1.$$

We refer to

$$S = \inf\{t \ge 0 : X_t \notin (0,\infty)\} = \lim_{n\to\infty} S_n, \tag{2.6}$$

as the exit time from I.

2.2 Feller's Test for Explosions

2.2.1 Precondition

Assume that the coefficients $\sigma : I \to \mathbb{R}$, $\mu : I \to \mathbb{R}$ satisfy nondegeneracy and local integrability:

$$\sigma^2(x) > 0; \forall x \in I, \tag{ND}$$

$$\forall x \in I, \exists \varepsilon > 0 \text{ such that} \int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|\mu(y)|}{\sigma^2(y)}\, dy < \infty. \tag{LI}$$

Fix a number $c \in I$, and we define the scale function $p(x)$ as:

$$p(x) = \int_c^x \exp\left\{-2\int_c^\xi \frac{\mu(\zeta)\,d\zeta}{\sigma^2(\zeta)}\right\} d\xi, \quad x \in I \text{ and } c \in I. \tag{2.7}$$

With $p'(x) = \exp\{-2\int_c^x \frac{\mu(\zeta)d\zeta}{\sigma^2(\zeta)}\} > 0$, the scale function $p(x)$ has a continuous, strictly positive derivative, and $p(x)$ is strictly increasing over I, hence the one-to-one function $p(x)$ maps I onto $(p(0), p(\infty))$. We will be using this scale function in Chapter 3 when discussing the ergodicity of the process and the invariant probability. Also we define the function $v(x)$ which will be used in Feller's Test as:

$$v(x) = \int_c^x p'(y) \int_c^y \frac{2dz}{p'(z)\sigma^2(z)}\, dy. \tag{2.8}$$

Lemma 2.5. With the scale function $p(x)$ and $v(x)$ defined above, we have the following implications,

$$p(\infty-) = \infty \Rightarrow v(\infty-) = \infty, \tag{2.9}$$

$$p(0+) = -\infty \Rightarrow v(0+) = \infty. \tag{2.10}$$

Proof. For any $\varepsilon > 0$, and $x \in [c+\varepsilon, \infty)$, we have

$$\begin{aligned} v(x) &= \int_c^x p'(y) \int_c^y \frac{2dz}{p'(z)\sigma^2(z)}\, dy \\ &\geq \int_c^x p'(y)\, dy \int_c^{c+\varepsilon} \frac{2}{p'(z)\sigma^2(z)}\, dz \\ &= [p(x) - p(c)] \int_c^{c+\varepsilon} \frac{2}{p'(z)\sigma^2(z)}\, dz. \end{aligned}$$

Hence, $p(\infty-) = \infty$ implies $v(\infty-) = \infty$. Equation (2.10) can be proved in a similar argument. □

2.2.2 Feller's Test

Explosions in one dimensional ODEs is quite common to see. Let $u(t)$ be the solution of

$$\dot{u} = b(u), \quad u(0) = x_0.$$

If $b(\cdot) > 0$, then there exists a finite time T such that $\lim_{t \to T} u(t) = +\infty$ if and only if $\int_c^\infty \frac{1}{b(u)} du < +\infty$. In this case, we also have an explicit formula for the explosion time T, which can be written as,

$$T = \int_{x_0}^{\infty} \frac{1}{b(u)} du. \tag{2.11}$$

(See [8])

On the other hand, it is much more complicated in the case of a stochastic differential equation. We may not find an exact formula for the time of the explosion as it happens in (2.11). However, the Feller's Test for explosions provides us a precise and concise criteria to determine, in terms of μ and σ whether solutions explode with probability zero, positive or one.

Feller's Test for explosions is stated in [13] as:

Theorem 2.6 (Feller's (1952) Test for explosion, Theorem 5.29, [13])**.** *Assume that the nondegeneracy* (ND) *and local integrability* (LI) *hold, and let* (X, Z), $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}$ *be a weak solution in* $I = (0, \infty)$ *of* (2.1) *with nonrandom initial condition* $X_0 = x \in I$*. Then* $P(S = \infty) = 1$ *or* $P(S = \infty) < 1$*, according to whether* $v(0+) = v(\infty-) = \infty$ *or not, where* $S = \inf\{t \geq 0 : X_t \notin (0, \infty)\}$.

2.2.3 Uniqueness

A mild modification of Theorem 5.15 in [13] will give us a criteria to the uniqueness of the positive solution of a stochastic differential equation. From the following theorem, we will note that if a positive solution exist, the uniqueness of the positive solution can be easily determined with some specific conditions.

Theorem 2.7. (Uniqueness) *Assume that* σ^{-2} *is locally integrable at every point in I, and condition* (ND) *and* (LI) *hold. Then for every initial distribution* μ*, the equation* (2.1) *has a weak solution up to an explosion time, and this solution is unique in the sense of probability law.*

2.3 Feller's Test Applied to the Model

Now we have computable conditions that the equation model (1.5) will have unique positive solutions, and we will apply these conditions to the model. The most difficult part is we have many parameters to create too many cases. To avoid calculating each case separately, we will introduce a Lemma to help simpify the process. Based on the stochastic differential equation model (1.5) that we have built up at the beginning, we start to let $I = (0, \infty)$, and fix a constant $c = 1$. Actually the choice of this constant c does not make any difference if it is chosen from I. Thus $p(x)$ becomes to be,

$$p(x) = \int_1^x \exp\left\{-2\int_1^\xi \frac{\mu(\zeta)\,d\zeta}{\sigma^2(\zeta)}\right\} d\xi. \tag{2.12}$$

By the Lemma 2.5, we can see a straight and transparent relationship between $p(0+)$ and $v(0+)$, $p(\infty-)$ and $v(\infty-)$. Moreover, since $p(x)$ is much less complicated to calculate explicitly than $v(x)$, we will determine the ranges of those eight parameters defined in the model (1.5) with the usage of Feller's Test, thus to attain a sufficient condition such that positive solution to the model (1.5) exists.

First, to satisfy both (ND) and (LI), we just need $\rho_0 \neq 0$. If $\rho_0 \neq 0$, we have,

$$\sigma^2(x) = \rho_0^2 x^{2\gamma} > 0, \quad \forall x \in I,$$

and $\forall x \in I$, $\exists\, \varepsilon > 0$ such that

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1+\rho_1 y^{-1}+\rho_2+\rho_3 \ln y+\rho_4 y+\rho_5 y\ln y+\rho_6 y^2}{\rho_0^2 y^{2\gamma}}\, dy < \infty.$$

Then, we would like to determine values of parameters in (1.5) by satisfying $p(0+) = -\infty$ and $p(\infty-) = \infty$ using Lemma 2.5. Based on the expression of $p(x)$, and with that both drift coefficient (1.6) and diffusion coefficient (1.7) in the model (1.5) are algebraic expressions, it should not be challenging to calculate $p(0-)$ and $p(\infty-)$. However, there would be an enormous amount of workloads if we compute them one by one. Also, each parameter can be zero or nonzero, and it would affect the choice of dominant terms to make $p(0+) = -\infty$ and $p(\infty-) = \infty$. So we will introduce the following proposition to better present the conditions and corresponding results.

Proposition 2.8. Define two functions $f(x)$ and $h(x)$ on $I = (0,\infty)$ with $\rho \neq 0$ and $\rho_0 \neq 0$ as:

$$f(x) = \int_1^x \exp\left\{-2\int_1^\xi \frac{\rho}{\rho_0^2}\zeta^{\alpha-2\gamma}\,d\zeta\right\}d\xi, \tag{2.13}$$

$$h(x) = \int_1^x \exp\left\{-2\int_1^\xi \frac{\rho}{\rho_0^2}\zeta^{\alpha-2\gamma}\ln\zeta\,d\zeta\right\}d\xi. \tag{2.14}$$

We have,

1. If $\alpha - 2\gamma > -1$, or $\gamma < \dfrac{\alpha+1}{2}$, then
   (a) $\rho < 0$ if and only if $f(\infty-) = h(\infty-) = \infty$;
   (b) $f(0+)$ and $h(0+)$ are always finite for any ρ.
2. If $\alpha - 2\gamma < -1$, or $\gamma > \dfrac{\alpha+1}{2}$, then
   (a) $\rho \neq 0$ if and only if $f(\infty-) = h(\infty-) = \infty$;
   (b) $\rho > 0$ if and only if $f(0+) = -\infty$;
   (c) $\rho < 0$ if and only if $h(0+) = -\infty$.
3. If $\alpha - 2\gamma = -1$, or $\gamma = \dfrac{\alpha+1}{2}$, then
   (a) $\rho \leq \dfrac{\rho_0^2}{2}$ if and only if $f(\infty-) = \infty$;
   (b) $\rho \geq \dfrac{\rho_0^2}{2}$ if and only if $f(0+) = -\infty$;
   (c) $\rho < 0$ if and only if $h(\infty-) = \infty$;
   (d) $\rho < 0$ if and only if $h(0+) = -\infty$.

Proof. If $\alpha - 2\gamma \neq -1$, or $\gamma \neq \dfrac{\alpha+1}{2}$, we have

$$\begin{aligned}
f(\infty-) &= \lim_{a\to\infty}\int_1^a \exp\Big\{-2\int_1^\xi \tfrac{\rho}{\rho_0^2}\zeta^{\alpha-2\gamma}d\zeta\Big\}d\xi \\
&= \lim_{a\to\infty}\int_1^a \exp\Big\{-\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\zeta^{\alpha-2\gamma+1}\Big|_1^\xi\Big\}d\xi \\
&= k_1\lim_{a\to\infty}\int_1^a \exp\Big\{-\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\Big\}d\xi \quad (k_1>0).
\end{aligned}$$

Similarly, we can have

$$\begin{aligned}
f(0+) &= k_1\lim_{b\to 0+}\int_1^b \exp\Big\{-\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\Big\} \\
&= -k_1\lim_{b\to 0+}\int_b^1 \exp\Big\{-\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\Big\}d\xi \quad (k_1>0).
\end{aligned}$$

$$\begin{aligned}
h(\infty-) &= \lim_{a\to\infty}\int_1^a \exp\Big\{-2\int_1^\xi \tfrac{\rho}{\rho_0^2}\zeta^{\alpha-2\gamma}\ln\zeta\, d\zeta\Big\}d\xi \\
&= \lim_{a\to\infty}\int_1^a \exp\Big\{-2\int_1^\xi \tfrac{\rho}{\rho_0^2(\alpha-2\gamma+1)}\ln\zeta\, d(\zeta^{\alpha-2\gamma+1})\Big\}d\xi \\
&= \lim_{a\to\infty}\int_1^a \exp\Big\{-\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}[\zeta^{\alpha-2\gamma+1}\ln\zeta\Big|_1^\xi - \int_1^\xi \zeta^{\alpha-2\gamma}d\zeta]\Big\}d\xi \\
&= \lim_{a\to\infty}\int_1^a \exp\Big\{-\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}[\xi^{\alpha-2\gamma+1}\ln\xi - \frac{1}{\alpha-2\gamma+1}(\xi^{\alpha-2\gamma+1}-1)]\Big\}d\xi \\
&= k_2\lim_{a\to\infty}\int_1^a \exp\Big\{-\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi - \frac{1}{\alpha-2\gamma+1})\Big\}d\xi \quad (k_2>0).
\end{aligned}$$

and

$$\begin{aligned}
h(0+) &= \lim_{b\to 0+}\int_1^b \exp\Big\{-2\int_1^\xi \tfrac{\rho}{\rho_0^2}\zeta^{\alpha-2\gamma}\ln\zeta\, d\zeta\Big\}d\xi \\
&= -k_2\lim_{b\to 0+}\int_b^1 \exp\Big\{-\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi - \frac{1}{\alpha-2\gamma+1})\Big\}d\xi \quad (k_2>0).
\end{aligned}$$

With $\alpha - 2\gamma \neq -1$, $f(x)$ and $h(x)$ have the same conditions of ρ to make both $f(\infty-)$ and $h(\infty-)$ equal to ∞, and opposite conditions of ρ to make $f(0+)$ and $h(0+)$ equal to $-\infty$ because when $\xi \to \infty-$, $\ln\xi > 0$ and when $\xi \to 0^+$, $\ln\xi < 0$.

Now, let us split this case into 2 situations. When $\alpha - 2\gamma > -1$, $-\frac{2}{\rho_0^2(\alpha-2\gamma+1)} < 0$. Then if and only if $\rho < 0$ we will have $f(\infty-) = h(\infty-) = \infty$.

In fact, if $\rho < 0$, $-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)} > 0$, and $\lim_{\xi\to\infty}(\ln\xi - \frac{1}{\alpha-2\gamma+1}) = \infty$, then we have both

$$\lim_{\xi\to\infty}\exp\left\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\right\} = \infty$$

and

$$\lim_{\xi\to\infty}\exp\left\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi - \frac{1}{\alpha-2\gamma+1})\right\} = \infty.$$

Then $f(\infty-) = h(\infty-) = \infty$.

If $\rho > 0$, $-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)} < 0$, and $\lim_{\xi\to\infty}(\ln\xi - \frac{1}{\alpha-2\gamma+1}) = \infty$, there must exist some finite number $s > 1$, such that over the interval (s,∞), we have that

$$\exp\left\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\right\} < \frac{1}{\xi^2}$$

and

$$\exp\left\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi - \frac{1}{\alpha-2\gamma+1})\right\} < \frac{1}{\xi^2}.$$

Of course $\lim_{a\to\infty}\int_s^a \frac{1}{\xi^2}\,d\xi < \infty$. Therefore if $f(\infty-) = h(\infty-) = \infty$, we must need $\rho < 0$.

However, while $\alpha - 2\gamma > -1$, $f(0+)$ and $h(0+)$ are both finite thus cannot reach $-\infty$ for any ρ because over the bounded interval $(0,1)$

$$\exp\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\} < \infty,$$

and

$$\exp\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi - \frac{1}{\alpha-2\gamma+1})\} < \infty.$$

On the other hand, when $\alpha - 2\gamma < -1$, or $\alpha - 2\gamma + 1 < 0$, let's first consider the values of

$f(\infty-)$ and $h(\infty-)$.

$$\lim_{\xi\to\infty}\frac{-2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}=\lim_{\xi\to\infty}\frac{-2\rho}{\rho_0^2(\alpha-2\gamma+1)\xi^{2\gamma-\alpha-1}}=0,$$

then we have

$$\lim_{\xi\to\infty}\exp\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\}=1.$$

Therefore, no matter what non-zero value ρ is, we have

$$f(\infty-)=k_1\lim_{a\to\infty}\int_1^a\exp\Big\{-\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\Big\}d\xi=\infty.$$

Similarly, when ξ goes up to positive infinity,

$$\lim_{\xi\to\infty}\exp\Big\{-\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi-\frac{1}{\alpha-2\gamma+1})\Big\}=1,$$

then $h(\infty-)=\infty$ no matter what non-zero value ρ is.

Now let's consider the values of $f(0+)$ and $h(0+)$, with $\alpha-2\gamma<-1$, or $\alpha-2\gamma+1<0$, we will show that if and only if $\rho>0$, $f(0+)=-\infty$. If $\rho>0$, then $-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}>0$, hence

$$\lim_{\xi\to0^+}\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\}=\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)\xi^{\alpha-2\gamma+1}}\}=\infty,$$

then

$$\lim_{\xi\to0^+}\exp\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\}=\infty,$$

hence $f(0+)=-\infty$. On the other hand, if $\rho<0$,

$$\lim_{\xi\to0^+}\exp\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\}=0,$$

then $f(0+)$ is finite which is not the case we are looking for. Therefore, if and only if $\rho>0$,

$f(0+) = -\infty$.

Similarly, if and only if $\rho < 0$, with $(\ln\xi - \frac{1}{\alpha-2\gamma+1}) < 0$ when $0 < \xi < e^{\frac{1}{\alpha-2\gamma+1}} < 1$,

$$-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi - \frac{1}{\alpha-2\gamma+1}) > 0,$$

then

$$\lim_{\xi\to 0^+}\exp\{-\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi - \frac{1}{\alpha-2\gamma+1})\} = \infty,$$

hence $h(0+) = -\infty$. Clearly we can see that $f(0+)$ and $f(0+)$ have opposite condition to make them $-\infty$ as we discussed in the beginning of the proof.

If $\alpha - 2\gamma = -1$, or $\gamma = \dfrac{\alpha+1}{2}$, we have

$$\begin{aligned}
f(\infty-) &= \lim_{a\to\infty}\int_1^a \exp\Big\{-2\int_1^\xi \tfrac{\rho}{\rho_0^2}\zeta^{-1}\,d\zeta\Big\}\,d\xi \\
&= \lim_{a\to\infty}\int_1^a \exp\Big\{-\tfrac{2\rho}{\rho_0^2}\ln\zeta\Big|_1^\xi\Big\}\,d\xi \\
&= \lim_{a\to\infty}\int_1^a \exp\Big\{-\tfrac{2\rho}{\rho_0^2}\ln\xi\Big\}\,d\xi \\
&= \lim_{a\to\infty}\int_1^a \xi^{-\frac{2\rho}{\rho_0^2}}\,d\xi.
\end{aligned}$$

$$\begin{aligned}
h(\infty-) &= \lim_{a\to\infty}\int_1^a \exp\Big\{-2\int_1^\xi \tfrac{\rho}{\rho_0^2}\zeta^{-1}\ln\zeta\,d\zeta\Big\}\,d\xi \\
&= \lim_{a\to\infty}\int_1^a \exp\Big\{-\tfrac{\rho}{\rho_0^2}\big(\ln\zeta\big)^2\Big|_1^\xi\Big\}\,d\xi \\
&= \lim_{a\to\infty}\int_1^a \exp\Big\{-\tfrac{\rho}{\rho_0^2}\big(\ln\xi\big)^2\Big\}\,d\xi.
\end{aligned}$$

Similarly, we get

$$f(0+) = \lim_{b\to 0+}\int_1^b \xi^{-\frac{2\rho}{\rho_0^2}}\,d\xi,$$

$$h(0+) = \lim_{b\to 0+}\int_1^b \exp\Big\{-\tfrac{\rho}{\rho_0^2}\big(\ln\xi\big)^2\Big\}\,d\xi.$$

Therefore, if and only if $\rho \le \dfrac{\rho_0^2}{2}$ we have that $f(\infty-) = \infty$ while $\rho \ge \dfrac{\rho_0^2}{2}$ if and only if $f(0+) = -\infty$.

And for $h(\infty-) = \infty$ and $h(0+) = -\infty$, they both require $\rho < 0$. $\square$

To apply Feller's Test, we need to satisfy the condition that $v(0+) = v(\infty-) = \infty$. However, $v(x)$ is much more complicated to calculated than $p(x)$ and we have Lemma 2.5 to use, so we can find the values for the coefficients such that $p(\infty-) = \infty$ and $p(0+) = -\infty$ are satisfied. For the convenience of notation, let's assume that

$$\mu(V) = \rho_1 V^{\alpha_1} + \rho_2 V^{\alpha_2} + \rho_3 V^{\alpha_3} \ln V + \rho_4 V^{\alpha_4} + \rho_5 V^{\alpha_5} \ln V + \rho_6 V^{\alpha_6}, \tag{2.15}$$

where $\alpha_1 = -1$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 1$, $\alpha_5 = 1$, $\alpha_6 = 2$ and obviously α_i is increasing.
In the order of α_1 to α_6, we name the first nonzero parameter ρ_m, $m = 1, \cdots, 6$, and the last nonzero we name it ρ_M, $M = 1, \cdots, 6$. It is obvious that $m \leq M$ and $\alpha_m \leq \alpha_M$. Based on Proposition 2.8 the algebraic property of $p(x)$, to make $p(\infty-) = \infty$, since the term with higher indicater number is easier to explode in the infinity, we just need

$$f_M(\infty-) = \lim_{a \to \infty} \int_1^a \exp\left\{-2 \int_1^\xi \frac{\rho_M}{\rho_0^2} \zeta^{\alpha - 2\gamma} d\zeta \right\} d\xi = \infty$$

or

$$h_M(\infty-) = \lim_{a \to \infty} \int_1^a \exp\left\{-2 \int_1^\xi \frac{\rho_M}{\rho_0^2} \zeta^{\alpha - 2\gamma} \ln \zeta \, d\zeta \right\} d\xi = \infty.$$

It depends on $M = 3$ or 5 or not to decide to use f_M or h_M. Similarly, to make $p(0+) = -\infty$, we need $f_m(0+) = -\infty$ or $h_m(0+) = -\infty$.

From Proposition 2.8 part 1.(b), we can see that $\alpha_m - 2\gamma \leq -1$ must be satisfied. This leads us to divide the problem to 5 cases. Now we use Proposition 2.8 to discuss the values of parameters in model (1.5) case by case. Before we start it, we need to clarify that $\rho_0 \neq 0$ for all cases.

2.3.1 Case 1

If $\alpha_m - 2\gamma < -1$ or $\gamma > \dfrac{\alpha_m+1}{2}$, and $\alpha_M - 2\gamma < -1$ or $\gamma > \dfrac{\alpha_M+1}{2}$, with $\alpha_m \leq \alpha_M$, that is to say if $\gamma > \dfrac{\alpha_M+1}{2}$, by Proposition 2.8 part 2 we have

$$\boxed{\text{When } m = 3 \text{ or } 5, \text{ then } \rho_m < 0,\ \rho_M \neq 0.} \tag{2.16}$$

$$\boxed{\text{When } m \neq 3 \text{ and } 5, \text{ then } \rho_m > 0,\ \rho_M \neq 0.} \tag{2.17}$$

Here $\rho_M \neq 0$ is naturally satisfied because ρ_M is the last nonzero term.

2.3.2 Case 2

If $\alpha_m - 2\gamma < -1$ or $\gamma > \dfrac{\alpha_m+1}{2}$, and $\alpha_M - 2\gamma > -1$ or $\gamma < \dfrac{\alpha_M+1}{2}$, that is to say if $\boxed{\dfrac{\alpha_m+1}{2} < \gamma < \dfrac{\alpha_M+1}{2}}$ where obviously $\alpha_m < \alpha_M$, by Proposition 2.8 part 1.(a) and part 2.(b) and (c) we have

$$\boxed{\text{When } m = 3 \text{ or } 5, \text{ then } \rho_m < 0,\ \rho_M < 0.} \tag{2.18}$$

$$\boxed{\text{When } m \neq 3 \text{ and } 5, \text{ then } \rho_m > 0,\ \rho_M < 0.} \tag{2.19}$$

2.3.3 Case 3

If $\alpha_m - 2\gamma < -1$ or $\gamma > \dfrac{\alpha_m+1}{2}$, and $\alpha_M - 2\gamma = -1$ or $\gamma = \dfrac{\alpha_M+1}{2}$, that is to say if $\gamma = \dfrac{\alpha_M+1}{2} > \dfrac{\alpha_m+1}{2}$, by Proposition 2.8 part 2.(b) & (c) and part 3.(a) & (c) we have

$$\boxed{\text{When } M \neq 3 \text{ and } 5,\ m = 3 \text{ or } 5, \text{ then } \rho_M \leq \frac{\rho_0^2}{2},\ \rho_m < 0.} \tag{2.20}$$

$$\boxed{\text{When } M = 3 \text{ or } 5,\ m \neq 3 \text{ and } 5, \text{ then } \rho_M < 0,\ \rho_m > 0.} \tag{2.21}$$

$$\boxed{\text{When } M \neq 3 \text{ and } 5,\ m \neq 3 \text{ and } 5, \text{ then } \rho_M \leq \frac{\rho_0^2}{2},\ \rho_m > 0.} \tag{2.22}$$

$$\boxed{\text{When } M = 5,\ m = 3,\ \text{then } \rho_M = \rho_5 < 0,\ \rho_m = \rho_3 < 0.} \tag{2.23}$$

2.3.4 Case 4

If $\alpha_m - 2\gamma = -1$ or $\gamma = \dfrac{\alpha_m + 1}{2}$, and $\alpha_M - 2\gamma > -1$ or $\gamma < \dfrac{\alpha_M + 1}{2}$, that is to say if $\gamma = \dfrac{\alpha_m + 1}{2} < \dfrac{\alpha_M + 1}{2}$, by Proposition 2.8 part 1.(a) and part 3.(b) & (d) we have

$$\boxed{\text{When } m \neq 3 \text{ and } 5,\ \text{then } \rho_m \geq \tfrac{\rho_0^2}{2},\ \rho_M < 0.} \tag{2.24}$$

$$\boxed{\text{When } m = 3 \text{ or } 5,\ \text{then } \rho_m < 0,\ \rho_M < 0.} \tag{2.25}$$

2.3.5 Case 5

If $\alpha_m - 2\gamma = -1$ or $\gamma = \dfrac{\alpha_m + 1}{2}$, and $\alpha_M - 2\gamma = -1$ or $\gamma = \dfrac{\alpha_M + 1}{2}$, that is to say if $\gamma = \dfrac{\alpha_m + 1}{2} = \dfrac{\alpha_M + 1}{2}$ where $\alpha_m = \alpha_M$, and $m \leq M$. In this case, we have 2 subcases, $m = M$ or $m < M$. When $m < M$, there would only be 2 cases, either $m = 2, M = 3$ or $m = 4, M = 5$, then by Proposition 2.8 part 4 we have

$$\boxed{\text{When } m = M \neq 3 \text{ and } 5,\ \text{then } \rho_m = \rho_M = \tfrac{\rho_0^2}{2}.} \tag{2.26}$$

$$\boxed{\text{When } m = M = 3 \text{ or } 5,\ \text{then } \rho_m = \rho_M < 0.} \tag{2.27}$$

$$\boxed{\text{When } m < M,\ \text{then } \rho_m \geq \tfrac{\rho_0^2}{2},\ \rho_M =< 0.} \tag{2.28}$$

2.3.6 Conclusion

There are 5 cases above in which positive solution of the model exists. Note again that we have the prerequisite $\rho_0 \neq 0$ for all the 5 cases. To make the positive solution unique, by therom 2.7, it is easy to see that since σ^{-2} is locally integrable in I, hence unique positive solution exists for the model.

Chapter 3

Ergodicity of the General Model (1.5)

3.1 Ergodicity of a Diffusion Process

As Rabi and Edward defined in Chapter V of [4],

Definition 3.1. A one-dimensional unrestricted diffusion is defined as a Markov process in continuous time with state space $S = (a, b)$, where $-\infty \leq a < b \leq \infty$, having continuous sample path.

As we stated in Chapter 1, the drift coefficient $\mu(V_t)$ is defined as (1.6), and the diffusion coefficient is defined as (1.7). We have figured out that under the conditions of 5 cases in Chapter 2, the stochastic volatility model we introduce in (1.5) has a unique weak solution on the interval $I = (0, \infty)$. The final goal of this dissertation is to technically provide a methodology of the parameter estimation of the generated model (1.5) and the estimation method will be stated in Chapter 4. In this process, the stationary distribution or invariant distribution or invariant measure will be needed. We will need to define some more items before we start to calculate the invariant distribution.

3.1.1 Recurrence

We will start with the definition of recurrence of the process in [4].

First, let $\{X_t^x : t \geq 0\}$ be a diffusion: on the interval $S = (a, b)$, with drift and diffusion coefficients

$\mu(x)$ and $\sigma(x)$, starting at x. Then we write

$$\rho_{xy} = P_x(\{X_t\} \text{ ever reaches } y), \quad (x, y \in S).$$

Definition 3.2. (From [4] Definition 9.1) A state y is recurrent if $\rho_{xy} = 1$ for all $x \in S$ such that $\rho_{yx} > 0$, and is transient otherwise. If all states in S are recurrent, then the diffusion $\{X_t\}$ is said to be recurrent.

From the definition, it is extremely hard to determine whether a given diffusion $\{X_t\}$ is recurrent or not. In [4], a calculation method is introduced in a corollary to determine whether and when a given diffusion is recurrent. Here we will use the scale function we defined in Chapter 2 and a speed function will also be needed.

Let us recall what is stated in Chapter 2, fix a number $c \in S$, the scale function $p(x)$ is defined as:

$$p(x) = \int_c^x \exp\left\{-2\int_c^{\xi} \frac{\mu(\zeta)\,d\zeta}{\sigma^2(\zeta)}\right\} d\xi, \quad x \in S \text{ and } c \in S.$$

Then we define the speed function $m(x)$,

$$m(x) = \int_c^x \frac{2}{\sigma^2(\xi)} \exp\left\{2\int_c^{\xi} \frac{\mu(\zeta)\,d\zeta}{\sigma^2(\zeta)}\right\} d\xi, \quad x \in S \text{ and } c \in S. \tag{3.1}$$

Corollary 3.3. (From [4] Corollary 9.3) A diffusion $\{X_t\}$ on $S = (a, b)$ with coefficients $\mu(x)$, $\sigma^2(x)$ is recurrent if and only if

$$p(a) = -\infty \text{ and } p(b) = \infty.$$

We apply this Corollary 3.3 to our model (1.5), first through Chapter 2 we have that the unique weak solution $\{V_t :, t \geq 0\}$ is a diffusion process on $S = I = (0, \infty)$ with the following coefficients: drift coefficient,

$$\mu(x) = \frac{\rho_1}{x} + \rho_2 + \rho_3 \ln x + \rho_4 x + \rho_5 x lnx + \rho_6 x^2;$$

diffusion coefficient,

$$\sigma^2(x) = \rho_0^2 x^{2\gamma}.$$

We have found the 5 cases when $p(0+) = -\infty$ and $p(\infty-) = \infty$ that satisfy Corollary 3.3. So the stochastic process $\{V_t\}$ is actually recurrent on I. This Corollary 3.3 can be another view to find the requirements such that the solution of the model 1.5 exists. However, recurrence is not enough to show ergodicity of the process. Then we will introduce positive recurrence which will give us the sufficient criteria.

3.1.2 Positive Recurrence

First, for a diffusion $\{X_t\}$, let τ_y denote the first passage time to a state y, and we write

$$\tau_y = \inf\{t \geq 0 : X_t = y\}.$$

Definition 3.4. (From [4] Definition 10.1) A diffusion $\{X_t\}$ on $S = (a,b)$ is positive recurrent if

$$E_x \tau_y < \infty \text{ for all } x, y \in S.$$

A recurrent diffusion that is not positive recurrent is null recurrent.

It is easy to see that the positive recurrence of a diffusion can imply its recurrence, but not vice versa. The following Proposition in [4] provides a straightforward criteria to determine whether a diffusion is positive recurrent or not, which is easy to use.

Proposition 3.5. (From [4] Proposition 10.2) Suppose $S = (a,b)$. Then the diffusion $\{X_t\}$ is positive recurrent, if and only if

$$p(a) = -\infty, \quad m(a) > -\infty, \tag{3.2}$$

and

$$p(b) = \infty, \quad m(b) < \infty. \tag{3.3}$$

3.1.3 Invariant Distribution and the Strong Law of Large Numbers

In Proposition 3.5, the $p(x)$ and $m(x)$ are the scale function and the speed function which we defined previously. Also, it is obviously that our five cases generated in Chapter 2 satisfy the criteria for the scale function if we apply the Proposition 3.5 to our model (1.5). We will calculate the requirements of all the parameters ρ_0, ρ_1, ρ_2, ρ_3, ρ_4, ρ_5, ρ_6 and γ such that the conditions for the speed function in Prop 3.5 will be satisfied. Here we will first see the relationship between the ergodicity of the diffusion and positive recurrence. In other words, positive recurrence is a necessary condition for the Strong Law of Large Numbers of a diffusion. We will utilize a theorem in the book [4] to illustrate this relationship. We are needing the ergodicity of the Model 1.5 to do the parameter estimation and process simulation in the following part of this dissertation.

Theorem 3.6. *(From Theorem 12.2 in* [4]*) Suppose that the diffusion $\{X_t\}$ is positive recurrent on $S = (a, b)$, then we have the following results.*

1. *Then there exists a unique invariant distribution $\pi(dx)$.*

2. *For every real-valued f such that*

$$\int_S |f(x)|\pi(dx) < \infty,$$

the Strong Law of Large Numbers holds, i.e., with probability 1,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X_s)ds = \int_S f(x)\pi(dx),$$

no matter what the initial distribution may be.

3. The invariant measure is the normalized speed measure,

$$\pi(x) = \frac{m'(x)}{m(b) - m(a)}.$$

Combining Proposition 3.5 and Theorem 3.6, it gives us a transparent method that is easy to follow. It makes the process of calculation related to the Model (1.5) clear to see. Moreover, it provides us a nice formula that we can use to calculate the invariant distribution or invariant measure that is vital in the parameter estimation process. When we come to the step of calculating invariant distribution, we actually will use a method provided in Samuel and Howard's [18]. This also gives us an idea of how the how this formula is derived and applied on our Model (1.5). This will be seen in the last portion of this Chapter.

3.2 Strong Law of Large Numbers applied to the Model 1.5

3.2.1 Positive recurrence for the Model

With Theorem 3.6, we can easily see the idea of how we can put restrictions on all parameters such that the model (1.5) is ergodic, and such that we can do parameter estimation and simulation. Based on all the work we have done in Chapter 2, actually, the conditions for recurrence have been already satisfied. Now we need to find the criteria of the parameters such that our model 1.5 is positive recurrent, then we can use the Strong Law of Large Numbers to do the parameter estimation.

When it comes to the model (1.5), we need to assume that $S = (a,b) = I = (0,\infty)$ and $c = 1$ in both the scale function $p(x)$ and the speed function $m(x)$. Due to the property of the drift coefficient $\mu(x)$ and the diffusion coefficient $\sigma^2(x)$, we just need to satisfy that the determinant term in the integral would meet the conditions in Theorem 3.6. Similar to what we did to find the 5 cases, we will still start with a proposition to simplify the calculation process. The only difference here is all the calculation and results will be based on the model (1.5) having solution or the model is recurrent. In the other words, we will assume we have the precondition that 5 cases are satisfied. So we are generating a bigger Proposition than 2.8 such that both requirements related to the scale function and speed function would be satisfied.

Proposition 3.7. Based on the conditions and results satisfied in Proposition 2.8, assume that:

$$f(x) = \int_1^x \exp\left\{-2\int_1^\xi \frac{\rho}{\rho_0^2}\zeta^{\alpha-2\gamma}\,d\zeta\right\}d\xi, \tag{3.4}$$

$$h(x) = \int_1^x \exp\left\{-2\int_1^\xi \frac{\rho}{\rho_0^2}\zeta^{\alpha-2\gamma}\ln\zeta\,d\zeta\right\}d\xi. \tag{3.5}$$

Here we define two functions $f_1(x)$ and $h_1(x)$ on $I=(0,\infty)$ with $\rho_0 \neq 0$ and $\rho \neq 0$ as:

$$f_1(x) = \int_1^x \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\left\{2\int_1^\xi \frac{\rho}{\rho_0^2}\zeta^{\alpha-2\gamma}\,d\zeta\right\}d\xi, \tag{3.6}$$

$$h_1(x) = \int_1^x \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\left\{2\int_1^\xi \frac{\rho}{\rho_0^2}\zeta^{\alpha-2\gamma}\ln\zeta\,d\zeta\right\}d\xi. \tag{3.7}$$

Then we have,

1. If $\alpha-2\gamma > -1$, or $\gamma < \dfrac{\alpha+1}{2}$, then
   (a) $\rho < 0$ if and only if $f(\infty-) = h(\infty-) = \infty$, $f_1(\infty-) < \infty$, and $h_1(\infty-) < \infty$;
   (b) $f(0+)$ and $h(0+)$ are always finite for any ρ. So there is no need to consider $f_1(0+)$ and $h_1(0+)$.
2. If $\alpha-2\gamma < -1$, or $\gamma > \dfrac{\alpha+1}{2}$, then
   (a) $\gamma > \frac{1}{2}$ if and only if $f(\infty-) = h(\infty-) = \infty$, $f_1(\infty-) < \infty$, and $h_1(\infty-) < \infty$;

(b) $\rho > 0$ if and only if $f(0+) = -\infty$ and $f_1(0+) > -\infty$;

(c) $\rho < 0$ if and only if $h(0+) = -\infty$ and $h_1(0+) > -\infty$.

3. If $\alpha - 2\gamma = -1$, or $\gamma = \dfrac{\alpha+1}{2}$, then

(a) $\rho < \dfrac{\rho_0^2}{2}$ and $\gamma \geq 1$ if and only if $f(\infty-) = \infty$ and $f_1(\infty-) < \infty$;
$\rho < \dfrac{\rho_0^2(2\gamma-1)}{2}$ and $\frac{1}{2} < \gamma < 1$ if and only if $f(\infty-) = \infty$ and $f_1(\infty-) < \infty$

(b) $\rho \geq \dfrac{\rho_0^2}{2}$ and $\frac{1}{2} < \gamma < 1$ if and only if $f(0+) = -\infty$ and $f_1(\infty-) > \infty$;
$\rho > \dfrac{\rho_0^2(2\gamma-1)}{2}$ and $\gamma \geq 1$ if and only if $f(0+) = -\infty$ and $f_1(\infty-) > \infty$;

(c) $\rho < 0$ if and only if $h(\infty-) = \infty$ and $h_1(\infty-) < \infty$;

(d) $\rho < 0$ if and only if $h(0+) = -\infty$ and $h(0+) > -\infty$.

Proof. We would not restate any proof that has been shown in Proposition 2.8.
If $\alpha - 2\gamma \neq -1$, or $\gamma \neq \dfrac{\alpha+1}{2}$, we have

$$\begin{aligned}
f_1(\infty-) &= \lim_{a\to\infty} \int_1^a \frac{2}{\rho_0^2 \xi^{2\gamma}} \exp\Big\{2\int_1^{\xi} \tfrac{\rho}{\rho_0^2} \zeta^{\alpha-2\gamma} d\zeta\Big\} d\xi \\
&= \lim_{a\to\infty} \int_1^a \frac{2}{\rho_0^2 \xi^{2\gamma}} \exp\Big\{\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)} \zeta^{\alpha-2\gamma+1}\Big|_1^{\xi}\Big\} d\xi \\
&= \lim_{a\to\infty} \int_1^a \frac{2k_1}{\rho_0^2 \xi^{2\gamma}} \exp\Big\{\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)} \xi^{\alpha-2\gamma+1}\Big\} d\xi \quad (k_1 > 0).
\end{aligned}$$

Similarly, we can have

$$\begin{aligned}
f_1(0+) &= k_1 \lim_{b\to 0+} \int_1^b \frac{2}{\rho_0^2 \xi^{2\gamma}} \exp\Big\{\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)} \xi^{\alpha-2\gamma+1}\Big\} \\
&= -\lim_{b\to 0+} \int_b^1 \frac{2k_1}{\rho_0^2 \xi^{2\gamma}} \exp\Big\{\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)} \xi^{\alpha-2\gamma+1}\Big\} d\xi \quad (k_1 > 0).
\end{aligned}$$

$$
\begin{aligned}
h_1(\infty-) &= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{2\int_1^\xi \tfrac{\rho}{\rho_0^2}\zeta^{\alpha-2\gamma}\ln\zeta\, d\zeta\Big\}\,d\xi \\
&= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{2\int_1^\xi \tfrac{\rho}{\rho_0^2(\alpha-2\gamma+1)}\ln\zeta\, d(\zeta^{\alpha-2\gamma+1})\Big\}\,d\xi \\
&= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}[\zeta^{\alpha-2\gamma+1}\ln\zeta\Big|_1^\xi-\int_1^\xi \zeta^{\alpha-2\gamma}d\zeta]\Big\}\,d\xi \\
&= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}[\xi^{\alpha-2\gamma+1}\ln\xi-\frac{1}{\alpha-2\gamma+1}(\xi^{\alpha-2\gamma+1}-1)]\Big\}\,d\xi \\
&= \lim_{a\to\infty}\int_1^a \frac{2k_2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi-\frac{1}{\alpha-2\gamma+1})\Big\}\,d\xi \quad (k_2>0),
\end{aligned}
$$

and

$$
\begin{aligned}
h_1(0+) &= \lim_{b\to 0+}\int_1^b \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{2\int_1^\xi \tfrac{\rho}{\rho_0^2}\zeta^{\alpha-2\gamma}\ln\zeta\, d\zeta\Big\}\,d\xi \\
&= -\lim_{b\to 0+}\int_b^1 \frac{2k_2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi-\frac{1}{\alpha-2\gamma+1})\Big\}\,d\xi \quad (k_2>0).
\end{aligned}
$$

Now, let us split this case into 2 situations. When $\alpha-2\gamma>-1$, there is no need to look at $f_1(0+)$ and $h_1(0+)$. In order to satisfy the related scale function condition, first we need $\rho<0$. Then $\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}<0$. It is easy to see that $f_1(\infty-)<\infty$. Since with $\rho<0$, and

$$
\exp\Big\{\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi-\frac{1}{\alpha-2\gamma+1})\Big\}<\exp\Big\{\tfrac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\Big\},
$$

Then we also have $h_1(\infty-)<\infty$.

On the other hand, when $\alpha-2\gamma<-1$, with Proposition 2.8 case 2, we know $f(\infty-)=h(\infty-)=\infty$ for any $\rho\neq 0$.

However, over the interval $(1,\infty)$, we have

$$
\lim_{x\to\infty}\exp\{\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\}=1.
$$

If we define

$$l(x) = \exp\{\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\},$$

then

$$l'(x) = \frac{2\rho}{\rho_0^2}\xi^{\alpha-2\gamma}\exp\{\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\}.$$

Obviously, $l(x)$ is either strictly increasing or decreasing over $(1,\infty)$. There exist a finite number $M > 0$ such that $1 < l(x) < M$ on $(1,\infty)$, where $M > \exp\{\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\}$. With $l(x)$ is always finite in $f_1(\infty-)$, then it is easy to see that if and only if $2\gamma > 1$ or $\gamma > \frac{1}{2}$, we will have $f_1(\infty-) < \infty$. Similarly, we could see $\gamma < \frac{1}{2}$ is also the condition for $h_1(\infty-) < \infty$.

Meanwhile over the interval $(0,1)$, from 2.8 we know that $\rho > 0$ if and only if $f(0+) = -\infty$. Then we will first see when $\rho > 0$ what would make $f_1(0+) > -\infty$. If $\rho > 0$, we have that

$$\frac{2k_1}{\rho_0^2\xi^{2\gamma}}\lim_{x\to 0^+}\exp\{\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}\} = 0,$$

hence $f_1(0+) < -\infty$.

Similarly also, we have that $\rho < 0$ if and only if $h(0+) = -\infty$, then we only need to look at $h_1(0+)$. When $\rho < 0$, we have that

$$\frac{2k_2}{\rho_0^2\xi^{2\gamma}}\lim_{x\to 0^+}\exp\{\frac{2\rho}{\rho_0^2(\alpha-2\gamma+1)}\xi^{\alpha-2\gamma+1}(\ln\xi - \frac{1}{\alpha-2\gamma+1})\} = 0,$$

hence $h_1(0+) = -\infty$ as we expected without any additional condition needed.

If $\alpha - 2\gamma = -1$, or $\gamma = \dfrac{\alpha+1}{2}$, we have

$$\begin{aligned}
f_1(\infty-) &= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{2\int_1^\xi \tfrac{\rho}{\rho_0^2}\zeta^{-1}\,d\zeta\Big\}\,d\xi\\
&= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{\tfrac{2\rho}{\rho_0^2}\ln\zeta\Big|_1^\xi\Big\}\,d\xi\\
&= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{\tfrac{2\rho}{\rho_0^2}\ln\xi\Big\}\,d\xi\\
&= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2\xi^{2\gamma}}\xi^{\frac{2\rho}{\rho_0^2}}\,d\xi\\
&= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2}\xi^{\frac{2\rho}{\rho_0^2}-2\gamma}d\xi.
\end{aligned}$$

$$\begin{aligned}
h(\infty-) &= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{2\int_1^\xi \tfrac{\rho}{\rho_0^2}\zeta^{-1}\ln\zeta\,d\zeta\Big\}\,d\xi\\
&= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{\tfrac{\rho}{\rho_0^2}\big(\ln\zeta\big)^2\Big|_1^\xi\Big\}\,d\xi\\
&= \lim_{a\to\infty}\int_1^a \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{\tfrac{\rho}{\rho_0^2}\big(\ln\xi\big)^2\Big\}\,d\xi.
\end{aligned}$$

Similarly, we get

$$f(0+) = \lim_{b\to 0+}\int_1^b \frac{2}{\rho_0^2}\xi^{\frac{2\rho}{\rho_0^2}-2\gamma}d\xi,$$

$$h(0+) = \lim_{b\to 0+}\int_1^b \frac{2}{\rho_0^2\xi^{2\gamma}}\exp\Big\{\tfrac{\rho}{\rho_0^2}\big(\ln\xi\big)^2\Big\}\,d\xi.$$

We have known that if and only if $\rho \le \dfrac{\rho_0^2}{2}$ we have that $f(\infty-) = \infty$. For $f_1(\infty-)$, we want that

$$\frac{2\rho}{\rho_0^2} - 2\gamma < -1.$$

Then based on $\rho \le \frac{\rho_0^2}{2}$ and $f(\infty-) = \infty$, if $\frac{1}{2} < \gamma < 1$ then $\rho < \frac{\rho_0^2(2\gamma-1)}{2}$ implies $f_1(\infty-) < \infty$, and if $\gamma \ge 1$ then $\rho < \frac{\rho_0^2}{2}$ implies $f_1(\infty-) < \infty$.

Meanwhile, since $\rho \ge \dfrac{\rho_0^2}{2}$ if and only if $f(0+) = -\infty$. Based on this result, if $\frac{1}{2} < \gamma < 1$ then $\rho \ge \frac{\rho_0^2}{2}$ implies $f_1(0+) > -\infty$, and if $\gamma \ge 1$ then $\rho > \frac{\rho_0^2(2\gamma-1)}{2}$ implies $f_1(0+) > -\infty$.

It is easy to see that for $h(\infty-) = \infty$ and $h(0+) = -\infty$, they both require $\rho < 0$ from 2.8, the condition remains the same for $h(\infty-) < \infty$ and $h(0+) > -\infty$. $\square$

As we have a new proposition we will update the 5 cases to satisfy Theorem 3.6. Here we will use the same set of notations as we used in Chapter 2 when we generated those 5 cases. We will still assume that:

$$\mu(V) = \rho_1 V^{\alpha_1} + \rho_2 V^{\alpha_2} + \rho_3 V^{\alpha_3} \ln V + \rho_4 V^{\alpha_4} + \rho_5 V^{\alpha_5} \ln V + \rho_6 V^{\alpha_6}, \tag{3.8}$$

where $\alpha_1 = -1$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 1$, $\alpha_5 = 1$, $\alpha_6 = 2$ and obviously α_i is increasing.
From α_1 to α_6, ρ_m, $m = 1, \cdots, 6$, will still represent the first nonzero parameter and ρ_M, $M = 1, \cdots, 6$ stands for the last nonzero parameter. Based on the property of equation 3.8, if a parameter is nonzero then the term is nonzero on $I = (0, \infty)$. It is obvious that $m \leq M$ and $\alpha_m \leq \alpha_M$.

3.2.2 Case 1

If $\alpha_m - 2\gamma < -1$ or $\gamma > \dfrac{\alpha_m + 1}{2}$, and $\alpha_M - 2\gamma < -1$ or $\gamma > \dfrac{\alpha_M + 1}{2}$, with $\alpha_m \leq \alpha_M$, that is to say if $\gamma > \dfrac{\alpha_M + 1}{2}$, by Proposition 3.7 part 2 we have

$$\boxed{\text{When } m = 3 \text{ or } 5, \text{ then } \rho_m < 0, \ \rho_M \neq 0 \text{ and } M \geq 2.} \tag{3.9}$$

$$\boxed{\text{When } m \neq 3 \text{ and } 5, \text{ then } \rho_m > 0, \ \rho_M \neq 0 \text{ and } M \geq 2.} \tag{3.10}$$

Here $\rho_M \neq 0$ is naturally satisfied because ρ_M is the last nonzero term. From Proposition 3.7 part 2 (a) we know that under case 1 we need $\gamma > \frac{1}{2}$. With the precondition that $\gamma > \dfrac{\alpha_M + 1}{2}$ then $\gamma > \max\{\frac{1}{2}, \frac{\alpha_M+1}{2}\}$. Since α_M can be only chosen from -1, 0, 1 and 2, then it is easy to see that $M \geq 2$.

3.2.3 Case 2

If $\alpha_m - 2\gamma < -1$ or $\gamma > \dfrac{\alpha_m + 1}{2}$, and $\alpha_M - 2\gamma > -1$ or $\gamma < \dfrac{\alpha_M + 1}{2}$, that is to say if $\boxed{\dfrac{\alpha_m + 1}{2} < \gamma < \dfrac{\alpha_M + 1}{2}}$ where obviously $\alpha_m < \alpha_M$, by Proposition 3.7 part 1.(a) and part 2.(b) and (c) we have

$$\boxed{\text{When } m = 3 \text{ or } 5, \text{ then } \rho_m < 0,\ \rho_M < 0.} \tag{3.11}$$

$$\boxed{\text{When } m \neq 3 \text{ and } 5, \text{ then } \rho_m > 0,\ \rho_M < 0.} \tag{3.12}$$

This one has no difference with the first version of 5 cases because the conditions in Proposition 2.8 and 3.7 are not changed.

3.2.4 Case 3

If $\alpha_m - 2\gamma < -1$ or $\gamma > \dfrac{\alpha_m + 1}{2}$, and $\alpha_M - 2\gamma = -1$ or $\gamma = \dfrac{\alpha_M + 1}{2}$, that is to say if $\gamma = \dfrac{\alpha_M + 1}{2} > \dfrac{\alpha_m + 1}{2}$, by Proposition 2.8 part 2.(b) & (c) and part 3.(a) & (c) we have

$$\boxed{\text{When } M \neq 3 \text{ and } 5,\ m = 3 \text{ or } 5, \text{ then } \rho_M < \frac{\rho_0^2}{2},\ \rho_m < 0 \text{ and } M = 4 \text{ or } 6.} \tag{3.13}$$

$$\boxed{\text{When } M = 3 \text{ or } 5,\ m \neq 3 \text{ and } 5, \text{ then } \rho_M < 0,\ \rho_m > 0.} \tag{3.14}$$

$$\boxed{\text{When } M \neq 3 \text{ and } 5,\ m \neq 3 \text{ and } 5, \text{ then } \rho_M \leq \frac{\rho_0^2}{2},\ \rho_m > 0 \text{ and } M = 4 \text{ or } 6.} \tag{3.15}$$

$$\boxed{\text{When } M = 5,\ m = 3, \text{ then } \rho_M = \rho_5 < 0,\ \rho_m = \rho_3 < 0.} \tag{3.16}$$

For the case 3.13, since $m = 3$ or 5, and $M \neq 3$ and 5, then M can only be 4 or 6. With the precondition that $\gamma = \dfrac{\alpha_M + 1}{2}$, we can make sure that $\gamma \geq 1$, hence $\rho_M \leq \frac{\rho_0^2}{2}$ and there is no need to consider the case when $\frac{1}{2} < \gamma < 1$.

For the case 3.16, since $M \neq 3$ and 5, then $\gamma = \frac{1}{2}$ or 1. From Proposition 3.7, we need to drop the case when $\gamma = \frac{1}{2}$. Then $M = 4$ or 6.

There is no changes for the other 2 cases.

3.2.5 Case 4

If $\alpha_m - 2\gamma = -1$ or $\gamma = \dfrac{\alpha_m + 1}{2}$, and $\alpha_M - 2\gamma > -1$ or $\gamma < \dfrac{\alpha_M + 1}{2}$, that is to say if $\gamma = \dfrac{\alpha_m + 1}{2} < \dfrac{\alpha_M + 1}{2}$, by Proposition 3.7 part 1.(a) and part 3.(b) & (d) we have

$$\boxed{\text{When } m \neq 3 \text{ and } 5, \text{ then } \rho_m > \tfrac{\rho_0^2}{2},\ \rho_M < 0 \text{ and } m = 4.} \tag{3.17}$$

$$\boxed{\text{When } m = 3 \text{ or } 5, \text{ then } \rho_m < 0,\ \rho_M < 0.} \tag{3.18}$$

In the case 3.17, $\gamma = \dfrac{\alpha_m + 1}{2}$, by Proposition 3.7 part 3 (b), the only choice for γ is $\gamma = 1$. Since m cannot be 3 and 5, also as the first nonzero number, m cannot be 6, then what is left is the case $m = 4$.

3.2.6 Case 5

If $\alpha_m - 2\gamma = -1$ or $\gamma = \dfrac{\alpha_m + 1}{2}$, and $\alpha_M - 2\gamma = -1$ or $\gamma = \dfrac{\alpha_M + 1}{2}$, that is to say if $\gamma = \dfrac{\alpha_m + 1}{2} = \dfrac{\alpha_M + 1}{2}$ where $\alpha_m = \alpha_M$, and $m \leq M$. In this case, we have 2 subcases, $m = M$ or $m < M$. When $m < M$, there would only be 2 cases, either $m = 2, M = 3$ or $m = 4, M = 5$, then by Proposition 3.7 part 4 we have

$$\boxed{\text{When } m = M = 3 \text{ or } 5, \text{ then } \rho_m = \rho_M < 0.} \tag{3.19}$$

$$\boxed{\text{When } m < M, \text{ then } \rho_m > \tfrac{\rho_0^2}{2},\ \rho_M < 0 \text{ and } m = 4,\ M = 5.} \tag{3.20}$$

There would be no need to consider when $m = M \neq 3$ and 5 because in this case, to satisfy $\gamma > \frac{1}{2}$, M can only be 4 or 6. In this case, $\gamma \geq 1$ There is a contradiction between $\rho_m = \rho_m < \frac{\rho_0^2}{2}$ and $\rho_m = \rho_m > \frac{\rho_0^2(2\gamma-1)}{2}$ where $\gamma \geq 1$

For the case 3.20, there is no need to take a look at the situation when $m = 2$ and $M = 3$. Because if $m = 2$, we need to apply Proposition 3.7 part 3 (b) where $\gamma > \frac{1}{2}$ which leads to a contradiction. So there is only one possibility left, $m = 4$ and $M = 5$ which is easy to analyze.

The updated 5 cases will ensure our model (1.5) to satisfy the condition in the Theorem 3.6. That is to say, under these 5 cases, there exists an invariant measure, so the Strong Law of Large Numbers can be applied in the process of parameter estimation and model simulation. Now we will see how we can calculate the stationary distribution or invariant distribution. We will use Samuel and Howard's [18] to show the calculation methodology and finally we can see it is the same thing with the formula given in the Theorem 3.6.

3.3 Invariant Distribution for the Model

3.3.1 General Calculation of the Stationary Distribution

Generally saying, it is not very easy to determine whether a stationary distribution exists or not for a given process. However in our model, we have calculated out when the invariant measure exists and what does the formula looks like. Here Samuel and Howard's [18] provided a general calculation method under the assumption of an invariant measure or stationary distribution exists just like the situation in our model (1.5). We will first take a look at this method and then in next subsection we will apply this method to our model to show that it will come up with the same result as the formula given in the Theorem 3.6.

If it exists, a stationary density $\psi(y)$ necessarily satisfies

$$\psi(y) = \int \psi(x) p(t,x,y)\, dx, \quad \text{for all } t > 0. \tag{3.21}$$

where $p(t,x,y)$ is transition probability. In Samuel and Howard's [18] the authors deduce that $\psi(y)$ satisfies

$$0 = \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(y)\psi(y)] - \frac{\partial}{\partial y}[\mu(y)\psi(y)]. \tag{3.22}$$

In fact, If we took the derivative with respect to t for the above equation (3.21),

$$\frac{\partial \psi(y)}{\partial t} = \int \psi(x) \frac{\partial p(t,x,y)}{\partial t} dx, \quad \text{for all } t > 0. \tag{3.23}$$

Using the backward equation as followed:

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2(x)\frac{\partial^2 p}{\partial x^2} + \mu(x)\frac{\partial p}{\partial x}.$$

We will have that,

$$\frac{\partial \psi(y)}{\partial t} = \int \psi(x)\{\frac{1}{2}\sigma^2(x)\frac{\partial^2 p}{\partial x^2} + \mu(x)\frac{\partial p}{\partial x}\} dx.$$

Applying integration by parts we will have,

$$\begin{aligned}\frac{\partial \psi(y)}{\partial t} &= \int \psi(x)\{\frac{1}{2}\sigma^2(x)\frac{\partial^2 p}{\partial x^2}\} dx + \int \{\mu(x)\frac{\partial p}{\partial x}\} dx \\ &= \int \{\frac{\partial^2}{\partial x^2}(\frac{\sigma^2(x)}{2}\psi(x)) - \frac{\partial}{\partial x}(\mu(x)\psi(x))\} p \, dx.\end{aligned}$$

Then by the definition of the (3.21) we will get (3.22).

Integrating (3.22) gives

$$\frac{d}{dy}\left[\frac{\sigma^2(y)}{2}\psi(y)\right] - \mu(y)\psi(y) = \tfrac{1}{2}C_1, \tag{3.24}$$

where C_1 is a constant. Multiplying by the integrating factor

$$s(y) = \exp\left\{-\int^y \left[\frac{2\mu(\xi)}{\sigma^2(\xi)}\right] d\xi\right\}.$$

We can rewrite (3.24) in the compact form $\frac{d}{dy}[s(y)\sigma^2(y)\psi(y)] = C_1 s(y)$. Let another integration $S(x) = \int^x s(y)\, dy$, and $S(x)$ is actually p(x), the scale function (3.1.1) we defined in the beginning.

It gives

$$
\begin{aligned}
\psi(x) &= C_1 \frac{S(x)}{s(x)\sigma^2(x)} + C_2 \frac{1}{s(x)\sigma^2(x)} \\
&= w(x)[C_1 S(x) + C_2].
\end{aligned} \tag{3.25}
$$

Actually the $w(x)$ here has a close relation with the scale function $m(x)$ with

$$
m(x) = \int 2w(x)\, dx.
$$

The constants are determined to guarantee the constraints $\psi \geq 0$ on (l, r) and $\int_l^r \psi(\xi)\, d\xi = 1$. If this is possible then a stationary density exists and otherwise not.

3.3.2 Calculating the Invariant Measure in the Model

In order for $\psi(x)$ to be non-negative, we need both $w(x) \geq 0$ and $C_1 S(x) + C_2 \geq 0$. With $w(x) = \frac{1}{s(x)\sigma^2(x)}$, since $s(x) = \exp\left\{-\int^x \left[\frac{2\mu(\xi)}{\sigma^2(\xi)}\right] d\xi\right\} > 0$ and $\sigma(x) > 0$, then $w(x) > 0$ obviously.
In the model (1.5), we have $\sigma^2(\xi) = \rho_0^2 \xi^{2\gamma}$ and $\mu(\xi) = \dfrac{\rho_1}{\xi} + \rho_2 + \rho_3 \ln \xi + \rho_4 \xi + \rho_5 \xi \ln \xi + \rho_6 \xi^2$.
To ensure that for $x \to 0$,

$$
\begin{aligned}
C_1 S(x) + C_2 &= C_1 \int_1^x s(y) dy + C_2 \\
&= C_1 \int_1^x \exp\left\{-\int_1^y \left[\frac{2\mu(\xi)}{\sigma^2(\xi)}\right] d\xi\right\} dy \\
&\geq 0,
\end{aligned}
$$

we need $C_1 = 0$. It follows that the unique stationary measure for the model (1.5) is

$$\begin{aligned}\psi(x) &= \frac{C_2}{s(x)\sigma^2(x)}\\ &= \frac{C_2}{\exp\left\{-\int_1^x\left[\frac{2\mu(\xi)}{\sigma^2(\xi)}\right]d\xi\right\}\sigma^2(x)}\\ &= \frac{C_2\exp\left\{-2\int_1^x\left(\rho_1\xi^{-1-2\gamma}+\rho_2\xi^{-2\gamma}+\rho_3\xi^{-2\gamma}\ln\xi+\rho_4\xi^{1-2\gamma}+\rho_5\xi^{1-2\gamma}\ln\xi+\rho_6\xi^{2-2\gamma}\right)d\xi\right\}}{\rho_0^2x^{2\gamma}},\end{aligned} \tag{3.26}$$

where $C_2 > 0$. In fact, we can calculate out C_2 by utilizing the scale function $m(x)$. We will need $\int_l^r \psi(\xi)\,d\xi = 1$, then we have

$$\begin{aligned}\int_l^r \psi(x)\,dx &= \int_l^r \frac{C_2}{s(x)\sigma^2(x)}\,dx\\ &= \frac{C_2}{2}\int_l^r \frac{2}{s(x)\sigma^2(x)}\,dx\\ &= \frac{C_2}{2}(m(r)-m(l))\\ &= 1.\end{aligned}$$

Then $C_2 = \frac{2}{m(r)-m(l)}$. We can also rewrite (3.26) to be

$$\psi(x) = \frac{2}{(m(r)-m(l))s(x)\sigma^2(x)} = \frac{m'(x)}{m(r)-m(l)},$$

where is the same result as it shows in the Theorem 3.6.

Then we calculate out $\psi(x)$ in 5 cases.

1. If $\gamma = 0$,

$$\begin{aligned}\psi(x) \;\;=\;\; & \frac{C_2}{\rho_0^2}\exp\Big\{2\big[\rho_1\ln x+\rho_2 x+\rho_3 x(\ln x-1)\\ & +\frac{\rho_4}{2}x^2+\frac{\rho_5}{4}x^2(2\ln x-1)+\frac{\rho_6}{3}x^3\big]\Big\}.\end{aligned} \tag{3.27}$$

2. If $\gamma = \frac{1}{2}$,

$$\begin{aligned}\psi(x) &= \frac{C_2}{\rho_0^2 x}\exp\Big\{2\big[-\rho_1 x^{-1} + \rho_2 \ln x + \frac{\rho_3}{2}(\ln x)^2 \\ &\quad + \rho_4 x + \rho_5 x(\ln x - 1) + \frac{\rho_6}{2}x^2\big]\Big\}. \qquad (3.28)\end{aligned}$$

3. If $\gamma = 1$,

$$\begin{aligned}\psi(x) &= \frac{C_2}{\rho_0^2 x^2}\exp\Big\{2\big[-\frac{\rho_1}{2}x^{-2} - \rho_2 x^{-1} - \rho_3 x^{-1}(\ln x + 1) \\ &\quad + \rho_4 \ln x + \frac{\rho_5}{2}(\ln x)^2 + \rho_6 x\big]\Big\}. \qquad (3.29)\end{aligned}$$

4. If $\gamma = \frac{3}{2}$,

$$\begin{aligned}\psi(x) &= \frac{C_2}{\rho_0^2 x^3}\exp\Big\{2\big[-\frac{\rho_1}{3}x^{-3} - \frac{\rho_2}{2}x^{-2} - \frac{\rho_3}{4}x^{-2}(2\ln x + 1) \\ &\quad - \rho_4 x^{-1} - \rho_5 x^{-1}(\ln x + 1) + \rho_6 \ln x\big]\Big\}. \qquad (3.30)\end{aligned}$$

5. If $\gamma \neq 0, \frac{1}{2}, 1, \frac{3}{2}$,

$$\begin{aligned}\psi(x) &= \frac{C_2}{\rho_0^2 x^{2\gamma}}\exp\Big\{2\big[\frac{\rho_1}{-2\gamma}x^{-2\gamma} + \frac{\rho_2}{1-2\gamma}x^{1-2\gamma} \\ &\quad + \frac{\rho_3}{(1-2\gamma)^2}x^{1-2\gamma}\big[(1-2\gamma)\ln x - 1\big] + \frac{\rho_4}{2-2\gamma}x^{2-2\gamma} \\ &\quad + \frac{\rho_5}{(2-2\gamma)^2}x^{2-2\gamma}\big[(2-2\gamma)\ln x - 1\big] + \frac{\rho_6}{3-2\gamma}x^{3-2\gamma}\big]\Big\}. \qquad (3.31)\end{aligned}$$

$$(3.32)$$

The C_2 is the above formulas is given by

$$C_2 = \frac{2}{m(r) - m(l)} = \frac{2}{m(\infty-) - m(0+)}.$$

This result also shows us why we need the scale function $m(x)$ to be bounded in the two boundaries. And actually if we combine the result with the updated 5 cases we calculated for the model to be ergodic, we need to drop the first case when $\gamma = 0$. Although looking complicated, it actually gives us a relatively clear form of what the invariant measure is, which is also easier to calculate using numerical methods.

Chapter 4

Parameter Estimation

4.1 Introduction

Generally saying, in the real world, for a stochastic differential equation problem-solving issue, there does not exist an analytical solution. An exceptional case could be the widely applied Black-Scholes formula in Finance, which is briefly introduced in Appendix B.. It has an explicitly analytical solution since the underlying process is a Geometric Brownian Motion.

As a result, when it comes to a more complicated SDE problem, just like our model (1.5) , numerical methods will be necessary to estimate the solution. Here the main idea in the numerical integration scheme of solving SDE is first discretization and then simulation. Usually in a SDE with all coefficients of each term in drift term and diffusion term are known and fixed, discretization and then simulation with a finite number of paths should be able to achieve a good estimation of the solution, however in the models there are some unknown parameters, we should do parameter estimation first and then do simulation to see whether the estimated values can perform a decent convergence or not.

4.2 GMM method

There are various numerical methods in parameter estimation methods in different papers related to stochastic differential equation, especially when positive solutions are expected in Finance area. Like in the inspiration paper [7], to estimate parameters in the continuous-time based model in the paper with utilizing the discrete-time econometric specification. The technique applied in this paper to estimate the parameters and do comparison between models is the Generalized Method of Moments (GMM) of Hansen raised in 1982.

A lot of authors chose to use the GMM method to do parameter estimation for their continuous-time stochastic differential equation models based on different type of underlying including but not limited to temperature indices used in weather derivatives and interest rates in stock and market. Chan et al. who worked on interest rate models estimation in 1992 has proved that there is no assumption needed of the distribution of the changes in volatility to be stationary and ergodic. Moreover, it is also shown that by Chan et al. that though the noises are conditionally heteroskedastic, the standard errors for Generalized Method of Moments estimator are consistent.

4.3 A New Estimation Methodology using Theorem 3.6

Unlike the inspiration paper [7], we have proved and calculated the assumptions and requirements for those parameters such that our model (1.5) will be ergodic. Then we can utilize this good property to do parameter estimation.

As is stated in Theorem 3.6, based on our model setting up in the beginning of the dissertation, we have that $\{V_t\}$ is positive recurrent on $(0, +\infty)$ under those 5 cases stated in Section 3.2, then we have,

- There must exist a unique invariant distribution $\psi(dx)$.

- For every real valued function f such that

$$\int_S |f(x)|\psi(dx) < \infty,$$

  and with probability 1,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X_s)ds = \int_S f(x)\psi(dx),$$

  no matter what the initial distribution may be.

- The invariant measure $\psi(x)$ is calculated as in Section 3.3

As we stated in the Introduction part of this Chapter, discretization is the first step as followed, let us assume that a closed positive interval $[c,d]$ in $(0,\infty)$. Assign a grid of points with high frequency and equal distance,

$$c = t_0 < t_1 < t_2 < \cdots < t_n = d.$$

Approximate x values are

$$V_{t_0} = w_0, V_{t_1} = w_1, \ldots, V_{t_n} = w_n.$$

They will be determined at the specific given time t points. The SDE initial value problem will be,

$$dV_t = \mu(V_t)dt + \sigma(V_t)dZ_t,$$

$$V_c = w_0,$$

where

$$\mu(V_t) = \frac{\rho_1}{V_t} + \rho_2 + \rho_3 \ln V_t + \rho_4 V_t + \rho_5 V_t \ln V_t + \rho_6 V_t^2, \tag{4.1}$$

and :

$$\sigma(V_t) = \rho_0 V_t^{\gamma}. \tag{4.2}$$

Then for every real valued function f such that

$$\int |f(x)|\pi(dx) \approx \frac{1}{n}\sum_{k=1}^{n}|f(w_k)|,$$

with n large enough, we have that

$$\frac{1}{n}\sum_{k=1}^{n} f(V_{t_k}) = \frac{1}{n}\sum_{k=1}^{n} f(w_k) \approx \int f(x)\pi(dx), \tag{4.3}$$

where $Ef(V_\infty) = \int f(x)\pi(dx) < \infty$.

New Estimation Methodology

- Assumed there are M $(M \leq 8)$ non-zero parameters among the 8 parameters of ρ_0, ρ_1, ρ_2, ρ_3, ρ_4, ρ_5, ρ_6 and γ in the SDE model (1.5), and assign initial values for the M non-zero parameters which satisfies the conditions according to the 5 cases in Chapter 3 to generate a simulated SDE $\{V_t\}$, where $\{V_{t_k} = w_k\}$. For the numerical purpose, more requirements will be added in the next Chapter;

- Find M real valued functions $f_1, \cdots, f_M$, then we can generate the following equation system with M unknown parameters and M equations,

  $$\int |f_m(x)|\pi(dx) = \frac{1}{n}\sum_{k=1}^{n}|f_m(w_k)|,$$

  where $m = 1, \cdots, M$;

- Test the estimated parameters until passing equation (4.3)

Compared to the other previously used parameter estimation methodology, which tests the SDE models directly without adding any criteria, this new estimation method has a lot preparation calculation to provide pre-conditions, which could help narrow down the choice of parameters to ensure the ergodicity of the SDE and the invariant measure existence. With all the requirements set up at the beginning of the parameters, the initial guess of the parameter values will be more efficient for simulation. This could also provide another thought for people who are interested in doing parameter estimations for SDE models.

Chapter 5

Simulation of the model

In Chapter 4, we have seen, theoretically, the main idea of approaching the parameter estimation and solution convergence using numerical methods. To start this process, we will do simulation first. In this Chapter, we will see some numerical method we can use to simulate the parameters in the model (1.5). In the paper [19], a number of methods of calculating numerical solution of stochastic differential equations in Finance were introduced.

5.1 Explicit Numerical Methods

5.1.1 Euler-Maruyama Method

We will start with the very basic and the most familiar stochastic integration numerical, the Euler-Maruyama method, also called the Euler method. This was also mentioned in the Diploma Thesis of Christian Kahl's [1].

$$w_0 = V_c,$$

$$w_{i+1} = w_i + \mu(w_i)\Delta t_{i+1} + \sigma(w_i)\Delta Z_{i+1},$$

where

$$\Delta t_{i+1} = t_{i+1} - t_i,$$

$$\Delta Z_{i+1} = Z(t_{i+1}) - Z(t_i).$$

The vital question is how to model the Wienner process ΔZ_i. In Timothy's paper [19], each random number ΔZ_i is estimated as

$$\Delta Z_i = z_i \sqrt{\Delta t_i},$$

where z_i is chosen from $N(0,1)$ which is the standard random variable that is normally distributed with mean 0 and standard deviation 1. In a couple of examples introduced in the paper [19], this method might not be fast enough compared to the other method like the following method Milstein method.

5.1.2 Milstein Method

The Milstein stochastic integration scheme may have a faster convergence as followed

$$w_0 = V_c,$$

$$w_{i+1} = w_i + \mu(w_i)\Delta t_i + \sigma(w_i)\Delta Z_i + \frac{1}{2}\sigma(w_i)\sigma'(w_i)(\Delta Z_i^2 - \Delta t_i). \tag{5.1}$$

Here the same method will be used to estimate ΔZ_i. Note also that the Milstein Method is the same with the Euler-Maruyama Method if the diffusion coefficient $\sigma(x)$ is a constant. In the paper [19], Timothy shows that Milstein method converges faster than Euler-Maruyama method on the Black Scholes stochastic differential equation.

The two basic numerical stochastic integration schemes are both explicit. However, they might not work well on preserving positivity and averting instabilities due to stiffness. Under our stochastic volatility model, positivity is a requirement. Although we have found the criteria for which the model (1.5) will have unique and positive weak solution on $(0,\infty)$ theoretically, when it comes to numerical schemes, positivity might not be preserved if choosing improper stochastic integration methods. Christian Kahl stated a general result of numerical positivity of Milstein method. To start with this result, we need to know the definition of finite and external life defined in [1].

Definition 5.1. Let $\{X_t\}$ be a stochastic process with

$$P(\{X_t > 0 \text{ for all } t\}) = 1.$$

Then the stochastic integration scheme has an eternal life time if

$$P(\{X_{n+1} > 0 | X_n > 0\}) = 1.$$

Otherwise it has a finite life time.

Christian Kahl also stated and proved in [1] that Euler method has a finite life time for all stochastic differential equations. However, he provided a theorem for Milstein method as a general result of when the stochastic integration scheme possesses an eternal life time.

Theorem 5.2. *(Theorem 4.7 in* [1]*)*

Based on the Milstein method in (5.1)*, it has an eternal life time if the following conditions are true:*

$$\sigma(x)\sigma'(x) > 0, \tag{5.2}$$

$$x > \frac{\sigma(x)}{2\sigma'(x)}, \tag{5.3}$$

$$\Delta t_i < \frac{2x\sigma'(x) - \sigma(x)}{(\sigma(x)\sigma'(x) - 2\mu(x))\sigma'(x)}. \tag{5.4}$$

The last property is needed only if

$$(\sigma(x)\sigma'(x) - 2\mu(x))\sigma'(x) > 0$$

To simulate our model (1.5) using Milstein method, we would like to apply Theorem 5.2 to get a better estimation range for γ.

As it is in the model (1.5), we know that,

$$\mu(x) = \frac{\rho_1}{x} + \rho_2 + \rho_3 \ln x + \rho_4 x + \rho_5 x \ln x + \rho_6 x^2,$$

$$\sigma(x) = \rho_0 x^\gamma.$$

By Theorem (5.2) condition (5.2), we have

$$\begin{aligned}\sigma(x)\sigma'(x) &= \rho_0 x^\gamma \rho_0 \gamma x^{\gamma-1} \\ &= \rho_0^2 \gamma x^{2\gamma-1} \\ &> 0.\end{aligned}$$

With volatility to be always positive, then $x > 0$. Obviously we can see that $\rho_0^2 > 0$. Then we need $\gamma > 0$ to satisfy condition (5.2).

For condition (5.3), we have that

$$\begin{aligned}x &> \frac{\sigma(x)}{2\sigma'(x)} \\ &= \frac{\rho_0 x^\gamma}{2\rho_0 \gamma x^{\gamma-1}} \\ &= \frac{x}{2\gamma}.\end{aligned}$$

With $x > 0$, then we will need $\gamma > \frac{1}{2}$ to satisfy the condition (5.3).

The last condition (5.4) gives a criteria of the choice of time step Δt_i includes more complexity and cannot be explicitly solved like condition (5.2) and (5.3).

$$\begin{aligned}\Delta t_i &< \frac{2x\sigma'(x) - \sigma(x)}{(\sigma(x)\sigma'(x) - 2\mu(x))\sigma'(x)} \\ &= \frac{2\rho_0 \gamma x^\gamma - \rho_0 x^\gamma}{(\rho_0^2 \gamma x^{2\gamma-1} - 2\mu(x))\rho_0 \gamma x^{\gamma-1}} \\ &= \frac{(2\gamma-1)x}{\rho_0^2 \gamma x^{2\gamma-1} - 2\gamma\mu(x)}.\end{aligned}$$

Since there are all other parameters except ρ_0 and γ including in $\mu(x)$, then we have to consider this condition when we are in the numerical test of the model based on different cases we generalized in Chapter 3. And we also need to keep in mind that this only applies to the case when the denominator is positive. If we could find a set of parameters such that the bottom is negative, then we do not need to worry about the choice of time step Δt_i.

5.2 Implicit Numerical Methods

Now we have seen basic explicit numerical stochastic integration schemes which might have issues when it comes to the ability of preserving positivity and maintaining stability through a lot of related literatures. In the previous section, we have seen explicit Euler method and Milstein method. Accordingly in this section, we will take a look at Implicit Euler and Implicit Milstein method.

5.2.1 Implicit Euler

The stochastic integration scheme for Implicit Euler scheme is as followed:

$$w_0 = V_c,$$

$$w_{i+1} = w_i + \mu(w_{i+1})\Delta t_{i+1} + \sigma(w_i)\Delta Z_{i+1},$$

where Δt_{i+1} and ΔZ_{i+1} are defined same as it is in the explicit Euler method. The implicit Euler method varies from the explicit Euler method mainly just on the drift coefficient changing to the next integration step, and it comes to be an issue of solving nonlinear equation systems. We can imagine that this approach will be very expensive and may lead to a lower accuracy of the simulation.

5.2.2 Implicit Milstein Method

Similar to Implicit Euler method, if we change the drift coefficient to the next integration step, we will have a relative Implicit Milstein method as followed:

$$w_{i+1} = w_i + \mu(w_{i+1})\Delta t_i + \sigma(w_i)\Delta Z_i + \frac{1}{2}\sigma(w_i)\sigma'(w_i)(\Delta Z_i^2 - \Delta t_i).$$

After seeing the numerical methods above theoretically, we will do Case one as an example for simulation of using Milstein Method.

5.3 One Simulation Example Based on Case 1

In the previous Chapters, we are in a comfortable level theoretically on the existence, uniqueness and positivity of the stochastic volatility model's solution. We will show an example of simulation based on case 1. The other cases will be similar.

We have known that $\gamma > \frac{1}{2}$ from Theorem (5.2). Recall Case 1, based on $\gamma > \frac{\alpha_M+1}{2}$, let's assume $\gamma = 1$, then M can only be chosen from 1, 2, 3,. We have 2 subcases under Case 1, and here we can only apply the second subcase with:

$$m \neq 3 \text{ and } \rho_m > 0,\ \ \rho_M \neq 0,\ \ M \geq 2.$$

Following the subcondition above, let's assume $\gamma = 1$, $\rho_1 = \rho_m = 1$, $\rho_2 = 1$, $\rho_3 = \rho_M = 1$. Then,

$$\mu(x) = \frac{1}{x} + 1 + \ln x,$$

$$\sigma(x) = x.$$

From condition (5.4) in Theorem (5.2), we have the criteria for the time steps Δt_i,

$$\Delta t_i < \frac{(2\gamma-1)x}{\rho_0^2 \gamma x^{2\gamma-1} - 2\gamma\mu(x)} = \frac{x}{x - 2(\frac{1}{x} + 1 + \ln x)}$$

Assume that the control function of Δt_i above is $f(x) = \dfrac{x}{x-2(\frac{1}{x}+1+\ln x)}$, and we will prove that $f(x) \geq 1$.

Proof.

$$
\begin{aligned}
f(x) &= \frac{x}{x-2(\frac{1}{x}+1+\ln x)} \\
&= \frac{1}{1-2(\frac{1}{x^2}+\frac{1}{x}+\frac{\ln x}{x})}.
\end{aligned}
$$

Now assume that $g(x) = 1-2(\frac{1}{x^2}+\frac{1}{x}+\frac{\ln x}{x})$, then

$$
\begin{aligned}
g(x) &= 1-2(\frac{1}{x^2}+\frac{1}{x}+\frac{\ln x}{x}) \\
&= 1-\frac{2}{x^2}-\frac{2}{x}-\frac{2\ln x}{x} \\
&\leq 1-\frac{2}{x^2}-\frac{2\ln x}{x} \\
&= 1-\frac{2}{x^2}(1+x\ln x).
\end{aligned}
$$

Let $h(x) = 1+x\ln x$, then $h'(x) = \ln x+1$. Solve the equation $h'(x) = 0$, and we get when $x = \frac{1}{e}$, $h(x)$ will reach the minimum of $h_{min}(x) = h(\frac{1}{e}) = 1-\frac{1}{e}$.

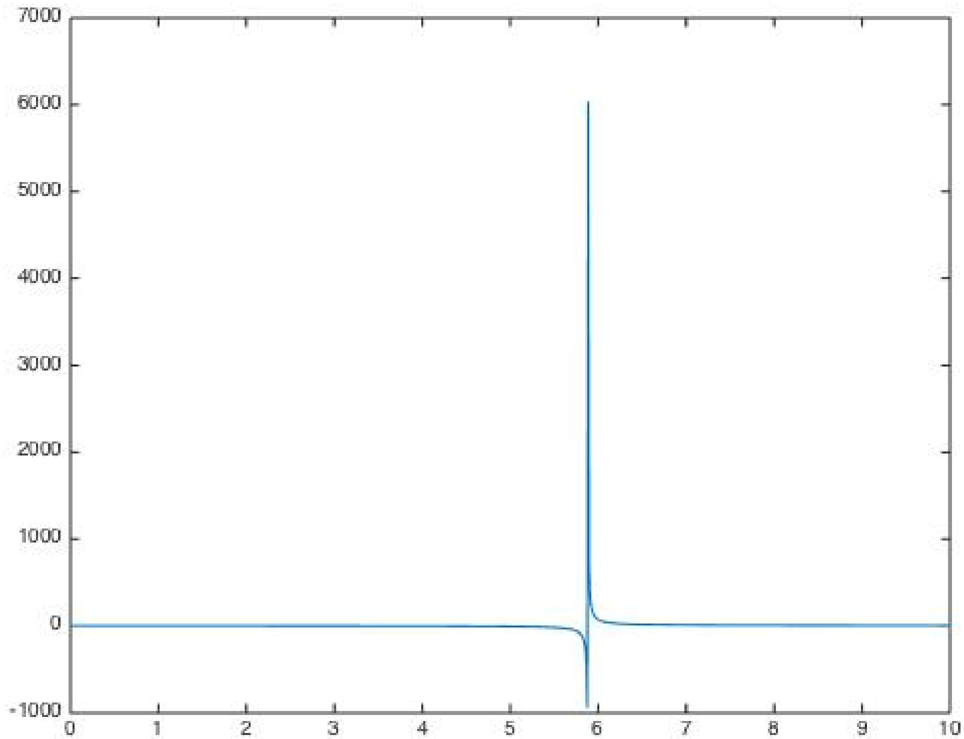
Then we have

$$
\begin{aligned}
g(x) &\leq 1-\frac{2}{x^2}(1+x\ln x) \\
&= 1-\frac{2}{x^2}h(x) \\
&\leq 1-\frac{2}{x^2}(1-\frac{1}{e}) \\
&= 1.
\end{aligned}
$$

Therefore, the control function $f(x)$ of Δt_i is always less than or equals to 1.

□

On the other hand, we can see the range of Δt_i from a graphing view. Here is a graph plotted in Matlab showing the activity of the control function of Δt_i. There is no need to consider this restriction is the bottom is negative and we can easily see that we could ignore the graph under 0. and for those positive values, it is obvious that the control function is always bigger than 1. So if $\Delta t_i < 1$, condition (5.4) would be satisfied. Let's assume $\Delta t_i = 0.1$.



Recall the explicit Milstein's Method,

$$w_0 = V_c$$

$$w_{i+1} = w_i + 0.1(\frac{1}{w_i} + 1 + \ln w_i) + w_i \Delta Z_i + \frac{1}{2} w_i (\Delta Z_i^2 - 0.1)$$

Then we program this following the matlab code below:

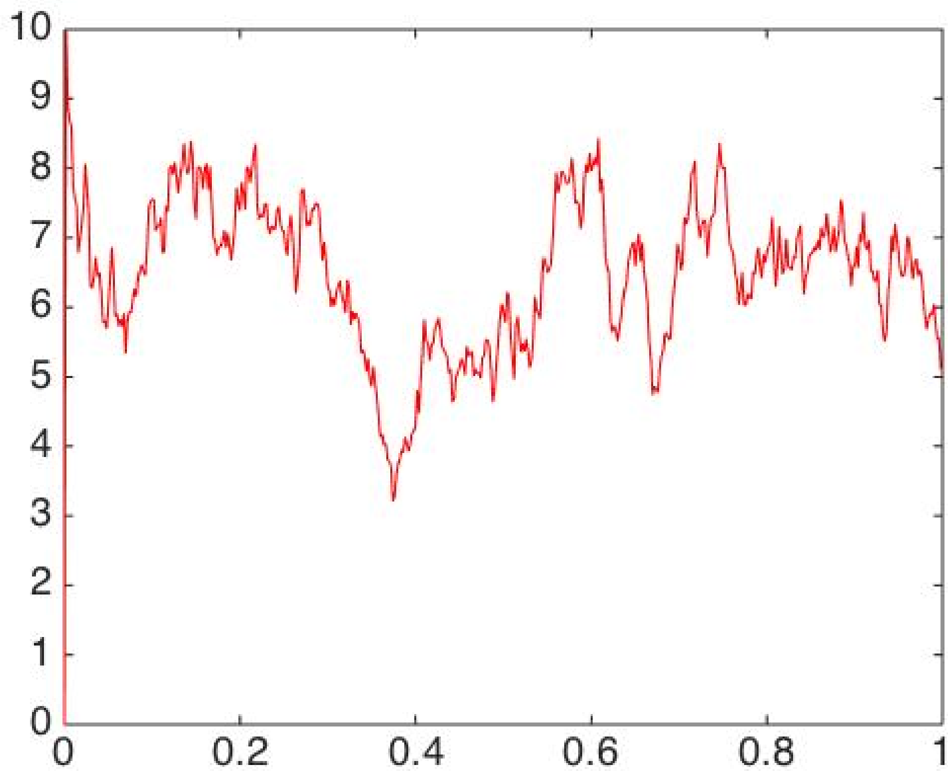
```
%case_1
T=1;N=500; dt=T/N;
dW=zeros(1,N);                          %Preallocate arrays ...
dW(1)=sqrt(dt)*randn;
```

```
W=zeros(1,N);                              %for efficiency
W(1)=10;

for j=2:N
    dW(j)=sqrt(dt)*randn;                  %General increament
    W(j)=W(j-1)+0.1*(1/W(j-1)+1+log(W(j-1)))
              + W(j-1)*dW(j)+0.5*W(j-1)*((dW(j))^2-0.1);
end

plot(0:dt:T,[0,W],'r-')                    %Plot W against t
```

Then we get the following graph of the simulation for case 1 under the condition that $\gamma = 1, \rho_1 = \rho_m = 1, \rho_3 = \rho_M = 1$. From the simulation we can see that all the volatilities are above 0



1.jpg

5.4 Future Study

In this dissertation, we mainly focused on and discussed a new parameter estimation methodology theoretically based on a one-dimension stochastic volatility model (1.5) without doing much numerical testing. In future studies, I would like to test the methodology numerically in matlab and do analysis on the convergence of the estimated parameters.

Appendix A

Stochastic Analysis

The stochastic analysis is a widely used mathematical tool in real world for modeling stochastic processes based on Ito's calculus. A thorough and complete understanding of Stochastic analysis requires comprehensive knowledge in higher level probability theory and stochastic process. Stochastic models are quite fundamental and indispensable in a lot fields including but not limited to Finance, Economics, Biology, Geographic, Energy and Chemistry, especially for Finance in stock market area. Most of this appendix is from [15] and [13].

A.1 Probability Theory

Definition A.1. A **σ-algebra** on a space Ω is a class $\mathcal{F}$ of subsets of Ω such that the following properties hold:

1. $\emptyset \in \mathcal{F}$;
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;
3. If $A_1, A_2, \cdots \in \mathcal{F}$, then the countable union $\bigcup_{i \in N} A_i \in \mathcal{F}$.

That is to say the σ-algebra $\mathcal{F}$ is closed under complements and countable intersection.

We call $(\Omega, \mathcal{F})$ a **measurable space**.

Definition A.2. A **measurable map** $X : (\Omega, \mathcal{F}) \to (E, \varepsilon)$, where both $(\Omega, \mathcal{F})$ and (E, ε) are measurable spaces, is a map from Ω to E s.t. for any set $B \in \varepsilon$,

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

A real valued **random variable** on $(\Omega, \mathcal{F})$ is a measurable map from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \varepsilon)$, where ε is a Borel σ-algebra.

Definition A.3. A **probability measure** $\mathbb{P}$ is defined as on the measurable space $(\Omega, \mathcal{F})$ a measurable map from $\mathcal{F}$ to the closed interval $[0,1]$ such that the following properties hold:

1. $\mathbb{P}(\Omega) = 1$,
2. For any $A_1, A_2, \cdots \in \mathcal{F}$ which is a countable family of disjoint sets $A_i \in \mathcal{F}$,

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

$(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

A.2 Stochastic Process

Definition A.4. On a probability space $(\Omega, \mathcal{F}, P)$, a **Stochastic Process** $\{X_t\}_{t\geq 0}$ is a family of random variables $X_t : \Omega \to \mathbb{R}^n$. If fix $t \geq 0$, $\omega \to X_t(\omega)$ is a random variable. If $\omega \in \Omega$ is fixed, then $t \to X_t(\omega)$ is a mapping from $[0, \infty)$ to $\mathbb{R}^n$, and this is the so-called path of a stochastic process.

Definition A.5. A **filtration F** = $\{\mathcal{F}_t\}_{t\geq 0}$ is a family of σ-algebras $\mathcal{F}_t$ on one probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that: $\{\mathcal{F}_t\}_{t\leq 0}$ is non-decreasing, i.e., for any $0 \leq s \leq t$, $\{\mathcal{F}_t\} \subseteq \{\mathcal{F}_s\} \subseteq \{\mathcal{F}\}$.
Usually we assume that the filtration satisfies **Usual condition**, which is as followed:

1. **F** is right-continuous, that is, $\{\mathcal{F}_t\} = \cap_{s>t}\{\mathcal{F}_s\}$;
2. **F** is completed by null sets, i.e., any subset of a zero probability set is $\{\mathcal{F}_0\}$ measurable.

A **Filtered Probability Space** $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbf{F}$ that satisfies the usual conditions.

Definition A.6. A real value process X is **progressively measurable** with a filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$ if $\forall t > 0$, $(\omega, s) \to X_s(\omega)$ mapping from $\Omega \times [0,t]$ to $\mathbb{R}$ is $\mathcal{F}_t \times \mathcal{B}([0,t])$ - measurable.

Definition A.7. A process X on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ is **F-adapted** if for any $t \geq 0$, the random variable X_t is $\mathcal{F}_t$ measurable.

One of the most significant Stochastic Process is Brownian Motion, or the so-called Winner Process we mentioned in the beginning of the dissertation.

Definition A.8. We say $\{W_t\}$ a **Brownian Motion** if the following properties hold:

1. W_t is an $\mathbf{F}_t$ adapted process
2. $W_0 = 0$
3. $W_t - W_s$ is independent of the filtration $\mathbf{F}_s$ for any $0 \leq s \leq t$
4. $W_t - W_s$ is normally distributed with $N(0, t-s)$ for any $0 \leq s \leq t$

A.3 Stochastic Integral

Here we assume that the processes satisfy the usual conditions.

Definition A.9. If 2 $\mathcal{F}_t$-adapted processes $X = \{X_t; 0 \leq t < \infty\}$ and $Y = \{Y_t; 0 \leq t < \infty\}$ satisfies

$$X_t(\omega) = Y_t(\omega); \quad \mu_M - a.e.,$$

where the measure $\mu_M(A) = E\int_0^\infty \mathbb{1}_A(t,\omega) d < M >_t (\omega)$ is on $([0,\infty) \times \Omega, \mathcal{B}([0,\infty) \otimes \mathcal{F}))$, , then X and Y are **equivalent**.

Definition A.10. We define that for a measurable and $\mathcal{F}_t$-adapted processes X, $[X]_T$ is the L^2**-norm** of the process X, where $[X]_T^2 = E\int_0^T X_t^2 d<M>_t$ and $E\int_0^T X_t^2 d<M>_t < \infty$.

Denote that $\mathcal{L}^*$ is the set of equivalent class of the progressively measurable processes such that $[X]_T$ is finite for $\forall T > 0$.

Definition A.11. On a measurable space $(\Omega, \mathcal{F})$ with a filtration $\mathbf{F}$, a **Stopping Time** T is a random variable such that the event $\{T \leq t\} = \{\omega; T(\omega) \leq t\} \in \mathbf{F}$ for every $t \geq 0$.

Definition A.12. The process $\{X_t; 0 \leq t < \infty\}$ with a filtration $\mathbf{F}$ is a **martingale** if we have $E(X_t|\mathcal{F}_s) = X_s$ for all $0 \leq s < t < \infty$. We say $\{X_t\}$ is a **submartingale** or a **supermartingale** if $E(X_t|\mathcal{F}_s) \geq X_s$ or $E(X_t|\mathcal{F}_s) \leq X_s$ respectively under the same conditions.

Definition A.13. If $X = \{X_t; 0 \leq t < \infty\}$ is a right-continuous and $\mathcal{F}_t$-adapted martingale, then X is said to be **square-integrable** with $EX_t^2 < \infty$, $\forall t \geq 0$. Moreover, if $X_0 = 0$ a.s., then we denote $X \in \mathcal{M}_2$. We write $X \in \mathcal{M}_2^c$ if X is continuous.

Then we can define stochastic integral as followed.

Definition A.14. Suppose that $X \in \mathcal{L}^*$ and M is a martingale with $M \in \mathcal{M}_2^c$, then $I(X)$ is the **stochastic integral** if it satisfies the following conditions:

1. $I(X) = \{I_t(X); 0 \leq t < \infty\}$ is a unique, square integrable martingale which is $\mathcal{F}_t$-adapted.

2. $I_t(X) = \int_0^t X_s \, dM_s; \, 0 \leq t < \infty$

3. For $\forall\{X^n\}_n^\infty$ a sequence in $\mathcal{L}_0$ converging to X, $\lim_{n\to\infty} ||I(X^{(n)}) - I(X)|| = 0$

One of the most famous result in stochastic integral is the It$\hat{o}$ Rule or It$\hat{o}$ formula.

Theorem A.15. *If $f \in C^2$ is a real value function on $\mathbb{R}$ and $X = \{X_t\}$ be a continuous semimartingale with the decomposition $X_t = X_0 + M_t + B_t$, where $M = \{M_t\}$ is a continuous local martingale with $M_0 = 0$, and $B = \{B_t\}$ where $B_t = A_t^+ - A_t^-$, A_t is continuous, nondecreasing $\mathcal{F}_t$-adapted processes and $A_0^\pm = 0$ a.s. in probability, then the following formula is true a.s. in probability.*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dM_s + \int_0^t f'(X_s) dB_s + \frac{1}{2}\int_0^t f''(X_s) d<M>_s$$

Appendix B

Stock Market

This dissertation is based on stock and equity market. Naturally, Stochastic calculus is the most significant and straightforward tool to model the stock prices and other financial index in financial area. This portion of appendix will give a brief introduction of very basic concepts and idea in financial market. First, we should assume in the financial market, stocks or asses are traded in continuous time. We will use the following assumptions and notations,

- The stock prices S^i, $i = 1, \cdots, d$ for d assets, where S^i's are semimartingale $\mathcal{F}_t$-adapted.
- Denote the stock price of some underlying asset at a specific time t S_t
- Denote T as the maturity of a stock
- Denote the fixed number K as the strike price
- Assume the assets are riskless. We call those risky assets securities.
- Denote the interest risk-free rate r
- Denote the volatility σ

The famous Black and Scholes model is a one important and special case of diffusion models. Under the historical probability, the Black and Sholes model describes the dynamics of the price of the underlying asset S_t at specific continuous time $0 \leq t < \infty$ as followed

$$dS_t = S_t(\mu dt + \sigma dW_t),$$

which has a explicit solution that most diffusion models do not have,

$$S_t = S_0 \exp(\mu t + \sigma W_t - \frac{\sigma^2}{2} t)$$

where the risk free interest rate r, the trend or drift μ, and the volatility σ are constant.

In a variety of financial derivatives, European Call and Put Options are the most representative ones.

Definition B.1. A **European Call Option** is an option for the right to buy a stock or an index at a certain price, strick K on a specific time, maturity T. The price of a call is that the buyer of the call will pay to the seller when $t = 0$.

Theorem B.2. *(Black and Schole Formula)*
Suppose that the price of a risky asset follows the dynamics of the Black and Schole's model: $dS_t = S_t(\mu dt + \sigma dW_t)$*, then at a specific time t, with maturity T and strike K, the value* $C(S_t, t, T, r, \sigma, K)$ *of a European call option is*

$$C(S_t, t, T, r, \sigma, K) = S_t \psi(d_1) - K e^{-r(T-t)} \psi(d_2),$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

In the Black and Scholes formula above, the volatility σ is fixed with different maturity and strikes. However, since the prices of the European call options are observable, then we define the implied volatility as follows

Definition B.3. The **Implied Volatility** is defined as the solution σ' of the following equation, with C^{obs} as the observed price of the European call option, S as the stock price, T as the maturity, K as the strike,

$$C(S,t,T,r,\sigma',K) = C^{obs}$$

References

[1] *Positive numerical integration of stochastic differential equations*, University of Wuppertal, 2004.

[2] *Spectral methods for volatility derivatives*, Quantit. Finance **9** (2009), no. 6, 663–692.

[3] Francis A. Longstaff Andreas Grunbichler, *Valuing futures and options on volatility*, Journal of Banking and Finance 20 **985** (1996), no. 1001.

[4] Rabi N. Bhattacharya and Edward C. Waymire, *Stochastic processes with applications*, A Wiley-Interscience Publication, 1990.

[5] Hans Buehler, *Consistent variance curve models*, Finance Stoch. **10** (2006), no. 2, 178–203.

[6] Robert E. Whaley Campbell R. Harvey, *Market volatility prediction and the efficiency of the s & p 100 index option market*, Journal of Financial Economics **31** (1992), no. 1, 43–73.

[7] Joanna Goard and Mathew Mazur, *Stochastic volatility models and the pricing of vix options*, Math. Finance **23** (2013), no. 3, 439–458. MR3070371

[8] Rossi Julio Groisman Pablo, *Explosion time in stochastic differential equations with small diffusion*, Electron. J. Differental Equations **140** (2007), no. 9.

[9] P.A. Forsyth H. Windcliff and K. R. Vetzal, *Pricing methods and hedging strategies for volatility derivatives*, J. Bank. Finance **30** (2006), 409–431.

[10] Steven L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, The Review of Financial Studies **6** (1993), no. 2, 327–343.

[11] John C. Hull, *Fundamentals of futures and options markets* (2002).

[12] Carlton Osakwe Jerome Detemple, *The valuation of volatility options*, European FInance Rev **4** (2000), 21–50.

[13] Ioannis Karatzas and Steven E. Shreve, *Brownian motion and stochastic calculus*, Second, Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991. MR1121940

[14] Michael Kamal Kresimir Demeterfi Emanuel Derman and Joseph Zou, *A guide to volatility and variance swaps*, The Journal of Derivatives **6** (1999), no. 4, 9–34.

[15] Marc Chesney Monique Jeanblanc Marc Yor, *Mathematical methods for financial markets*, Springer Finance Textbook, 2009.

[16] Dilip B. Madan Peter Carr Helyette Geman and Marc Yor, *Pricing options on realized variance*, Finance Stoch. **9** (2005), 453–475.

[17] Henrik Rasmussen Sam Howison Avraam Rafailidis, *On the pricing and hedging of volatility derivatives*, App. Math. Finance **11** (2004), 317–346.

[18] Howard M. Taylor Samuel Karlin, *A second course in stochastic process*, 1981.

[19] Timothy Sauer, *Numerical solution of stochastic differential equations in finance*, Handbook of Computational Finance, 2011.

[20] Artur Sepp, *Pricing options on realized variance in heston model with jumps in returns and volatility*, J. Computat. Finance **11** (2008), no. 4, 33–70.

[21] Vijay Pant Thomas Little, *A finite-difference method for the valuation of variance swaps*, J. Computat. Finance **5** (2001), no. 1, 81–103.

[22] Robert. E. Whaley, *Derivatives on market volatility: Hedging tools long overdue*, J. Derivat **1** (1993), 71–84.