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ON THE GENERATION OF STABLE KERR FREQUENCY COMBS IN THE LUGIATO–LEFEVER MODEL OF PERIODIC OPTICAL WAVEGUIDES*

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Abstract. We consider the Lugiato–Lefever (LL) model of optical fibers. We construct a two parameter family of steady state solutions, i.e., Kerr frequency combs, for small pumping parameter $h > 0$ and the correspondingly (and necessarily) small detuning parameter, $\alpha > 0$. These are $O(1)$ waves, as they are constructed as a bifurcation from the standard dnoidal solutions of the cubic nonlinear Schrödinger equation. We identify the spectrally stable ones, and more precisely, we show that the spectrum of the linearized operator contains the eigenvalues $0, -2\alpha$, while the rest of it is a subset of $\{\mu : \Re\mu = -\alpha\}$. This is in line with the expectations for effectively damped Hamiltonian systems, such as the LL model.

Key words. Lugiato–Lefever, Kerr frequency combs, periodic waveguides, stability

AMS subject classifications. 35Q55, 35P10, 42B37, 42B35

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1. Introduction.

Optical frequency combs are a series of precisely spaced, sharp spectral lines, which provide a new level of capabilities in the field of precision measurements as well as many other exciting applications. The discovery that these Kerr combs can be generated by a special class of microresonators, called whispering gallery mode (WGM) resonators, has recently renewed physicists' interest in this already active area of research [2, 12, 13, 16, 14].

Mathematically, this is modeled by a variant of the Lugiato–Lefever equation, introduced in 1987 in [13]. The equation, which is a version of the nonlinear Schrödinger equation that includes driving, damping and detuning, was shown to be the appropriate spatiotemporal model for Kerr-comb generation in whispering-gallery-mode resonators (see [3]). There are numerous papers dealing with the model derivation, as well as further reductions to dimensionless variables (see, for example, [15, 7, 14] among others). The model equation, considered in [5, 6] as well as [17, 18], is given by

$$\psi_t + i\beta\psi_{xx} + (\gamma + i\delta)\psi - i|\psi|^2\psi = F. \tag{1.1}$$

Here, one distinguishes between the cases $\beta > 0$ and $\beta < 0$, the former one being the case of standard dispersion, whereas the latter is referred to as anomalous dispersion (see [5, 6] for further discussion). The other parameters have distinct physical meaning, which is explained below, for a slightly more specific version of the model.

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In this work, we shall be concerned with the model with anomalous dispersion $\beta < 0$. In addition, it is convenient, after several equivalent scaling transformations, to reduce to an equivalent format, which works better for our purposes. More precisely,

$$\begin{cases} iu_t + u_{xx} - u + 2|u|^2 u = -i\alpha u - h, t \geq 0, -T \leq x \leq T, \\ u(t,-T) = u(t,T), u'(t,-T) = u'(t,T) \end{cases} \tag{1.2}$$

shall be referred to as the Lugiato–Lefever (LL) equation. Here, u denotes the field envelope, a (complex-valued) function, t is the normalized time, and x is the azimuthal coordinate, while $\alpha > 0$ is the detuning/damping parameter and the normalized pumping strength parameter is $h > 0$. It is important to state that T is a fixed parameter, which shall be kept fixed throughout.

We are interested in time independent solutions, that is, frequency/Kerr combs $u(t,x) = \varphi(x)$ and their stability, as solutions of the full time dependent problem (1.2). These satisfy the time-independent equation

$$-\varphi'' + \varphi - 2|\varphi|^2\varphi = i\alpha\varphi + h, -T \leq x \leq T, \tag{1.3}$$

where φ satisfies the periodic boundary conditions.

A few words about the range of the parameters. Physically, it is preferred that the pumping parameter h be small. In fact, the case $h = 0$ is used by many authors as a bifurcation point to construct such waves, starting with a "good" solution at $h = 0$. On the other hand, in a recent paper [14], the authors have studied the relationship between α and h, which supports the existence of Kerr combs. The case $\alpha = 0$ offers another useful starting point for bifurcation analysis, when $h \neq 0$. This point of view is explored via formal methods in [16], by using the approach in [1], in the related context of the forced nonlinear Schrödinger equation (NLS) model. Similar construction was carried out in the periodic case in [19], since one can write an explicit solution in the case $\alpha = 0$ in terms of Jacobi elliptic functions.

In the periodic setting, there are numerous recent developments as to the existence (and, subsequently, stability) of periodic solitary waves, which are close to constants, both in cases of standard and anomalous dispersion. In [19, 18, 17, 5, 6], the authors have studied stationary solutions of (1.1), close to constants. More precisely, in [5, 6], the close to constant periodic solutions have been constructed, by means of bifurcations close to points of Turing instabilities. In [17] the authors proved asymptotic stability of close to constant solutions, given their spectral stability.

In the whole line case, two families of explicit solutions were explicitly found in [1] for the case $\alpha = 0$. Their spectral stability was also discussed there, in the setting of forced NLS—one family was found to be unstable, whereas the stability of the other family was left as an open question. To the best of our knowledge, no solutions of (1.3) have been constructed, when $\alpha \neq 0, h \neq 0$, so this seems to be an interesting direction for future research.

Our goal in this paper is to explore the existence and the stability properties of the solutions of (1.3), in the physically relevant regime $0 < h << 1$. Mathematically (and also from a physical perspective), it turns out that α is also necessarily a small parameter; in fact $\alpha \sim h$. In addition, we are looking for large solutions close to the standard dnoidal solutions for NLS, with $h = \alpha = 0$, as these are well-known in both the theoretical context and also easily physically realizable. Most importantly, we are interested in such solutions that are dynamically stable as solutions of (1.2). We achieve all of these goals, by first constructing a family of such solutions, as long as the necessary conditions on α, to be established below, are met. Next, we provide an explicit characterization of their spectral stability; in fact, we provide a fairly explicit description of spectrum of the linearized operator, which should be useful in further studies of its semigroup properties. We postpone these considerations for a future publication.

1.1. Construction of stationary solutions. It will be important to understand the behavior of the solutions of (1.2) in the case when one of the parameters is zero. This is interesting in itself, but it will also give us important clues as to what is important (and reasonable to expect) in the case of interest $0 < \alpha, h << 1$.

PROPOSITION 1 ($h = 0$ does not support stationary solutions). *The equation*

$$\varphi'' - \varphi + 2|\varphi|^2\varphi + i\alpha\varphi = 0, -T \le x \le T \tag{1.4}$$

does not have *nontrivial periodic classical solutions* φ_α.

In the case $h > 0$, $\alpha = 0$, one looks for spatially periodic, time-independent solutions of (1.2), $u = \varphi(x)$. That is, we look to solve the equation

$$\varphi'' - \varphi + 2\varphi^3 = -h, -T \le x \le T. \tag{1.5}$$

PROPOSITION 2 (the stationary waves for $\alpha = 0, h > 0$). *There exists* $h_0 > 0$*, so that for each* $h : 0 < h < h_0$*, there is a one parameter family of solutions* $\varphi_{c,h}$ *of* (1.5)*, where* c *is a real-parameter, so that the quartic equation*

$$z^4 - z^2 + 2hz + c = 0 \tag{1.6}$$

has four different real roots. We have the following explicit formula for it:

$$\varphi(x) = \frac{\zeta_4(\zeta_3 - \zeta_1) + \zeta_1(\zeta_4 - \zeta_3)sn^2(\frac{x}{\sqrt{g}}, \kappa)}{(\zeta_3 - \zeta_1) + (\zeta_4 - \zeta_3)sn^2(\frac{x}{\sqrt{g}}, \kappa)}, \tag{1.7}$$

where $\zeta_1 < \zeta_2 < \zeta_3 < \zeta_4$ *are the roots of the quartic equation* (1.6) *and*

$$\kappa^2 = \frac{(\zeta_4 - \zeta_3)(\zeta_2 - \zeta_1)}{(\zeta_4 - \zeta_2)(\zeta_3 - \zeta_1)}, \quad g = \frac{2}{\sqrt{(\zeta_4 - \zeta_2)(\zeta_3 - \zeta_1)}}.$$

The function φ *is* $2T$ *periodic, with* $T = T_{h,c} = \sqrt{g}K(\kappa)$*, which is a continuous function of* c, h*. Finally, for small enough* h*, there are* $2T$ *periodic solutions with period satisfying*

$$T \in \left(2^{-3/4}\pi + O(h), 2^{-\frac{11}{4}}\frac{\pi}{h} + O(1))\right). \tag{1.8}$$

Remarks.

1. In the case $h = 0$, one sees that the parameter c must range within $(0, \frac{1}{4})$ and we may explicitly compute the roots as follows:

$$\xi_4 = -\xi_1 = \sqrt{\frac{1 + \sqrt{1 - 4c}}{2}}, \quad \xi_3 = -\xi_2 = \sqrt{\frac{1 - \sqrt{1 - 4c}}{2}}.$$

After some algebraic manipulations, the solution for $h = 0$, φ_0, turns out to be a rescaling of the standard dnoidal solution of the cubic NLS, parametrized by c, to account for all possible periods $T = T(c) \in (2^{-3/4}\pi, \infty)$.

2. For small $h \neq 0$, one needs to require $c \in (0, \frac{1}{4}) + O(h)$. In addition, a better parametrization is via $m = \min_{-T \leq x \leq T} \varphi(x)$, where $m \in (0, \frac{1}{\sqrt{2}})$. More precisely, one can write

(1.9)
$$\zeta_1 = -\sqrt{1-m^2} + O(h),\ \zeta_2 = -m + O(h),\ \zeta_3 = m,\ \zeta_4 = \sqrt{1-m^2} + O(h).$$

We now turn to a necessary condition for the existence of the waves, when both parameters are nonzero, $h, \alpha > 0$. Our first result in this direction states that if h is small and φ is an $O(1)$ solution, then $\alpha = O(h)$.

PROPOSITION 3. *Let $0 < h << 1$. Assume that* (1.3) *has a solution φ. Then*

$$\alpha \leq h \frac{\sqrt{2T}}{\|\varphi\|^{-1}_{L^2[-T,T]}}.$$

For the proof, take a dot product with $\varphi = \varphi_1 + i\varphi_2$ in (1.3). Then, since its left-hand side is real, taking imaginary parts results in

$$\alpha \|\varphi\|^2 = h \int_{-T}^{T} \varphi_2(x) dx \leq h\sqrt{2T}\|\varphi\|.$$

Thus, if $\|\varphi\|_{L^2[-T,T]} = O(1)$ and $T = O(1)$, we have that $\alpha = O(h)$.

Now that we know that $\alpha = O(h)$, we take the ansatz $\alpha = \alpha_0 h$ per Proposition 3. Our next result describes the existence of waves for $0 < h << 1$, $\alpha = \alpha_0 h$, which only holds for a specific range of values α_0.

In order to state the result, fix the period $2T$ and for small enough $h > 0$, denote φ_h to be the solution produced in Proposition 2. Indeed, as we will show in the proof of Proposition 2, for a fixed and sufficiently small h and T inside the prescribed range, (1.8), there is a $c = c(h)$, for which there is a solution φ_h, with the prescribed period. Note that for $h = 0$, one obtains the classical dnoidal solutions, φ_0, as discussed earlier. With that, introduce the self-adjoint operators $\mathcal{L}_\pm$, with domains $H^2[-T,T]$,

$$\begin{aligned}
\mathcal{L}_{+,h} &= -\partial_x^2 + 1 - 6\varphi_h^2, \\
\mathcal{L}_{-,h} &= -\partial_x^2 + 1 - 2\varphi_h^2, \\
L_\pm &:= \mathcal{L}_{\pm,0},
\end{aligned}$$

which will be important for our arguments in what follows. The next result is the main existence result of the paper. Note that all implicit constants will depend on the fixed period T.

THEOREM 1. *Let $\alpha_0 : 0 < \alpha_0 < \frac{\langle 1, \varphi_0 \rangle}{\|\varphi_0\|^2}$. There exists $h_0 = h_0(\alpha_0) > 0$, so that for every $h : 0 < h < h_0$ and $\alpha := \alpha_0 h$, there exists a stationary solution $\varphi_\alpha = \varphi_{\alpha,1} + i\varphi_{\alpha,2}$ of* (1.3) *and there is the following Taylor expansion formula for it:*

$$\varphi_{\alpha,1} = \left(a_0 + \frac{b_0}{2} h D_2^0 + O(h^2)\right)\varphi_0 + h\Psi_1^0 + O_{\{\varphi_0\}^\perp}(h^2), \tag{1.10}$$

$$\varphi_{\alpha,2} = \left(b_0 - \frac{a_0}{2} h D_2^0 + O(h^2)\right)\varphi_0 + h\Psi_2^0 + O_{\{\varphi_0\}^\perp}(h^2), \tag{1.11}$$

where $q = O_{\{\varphi_0\}^\perp}(h^2)$ denotes a function, with $q \perp \varphi_0 : \|q\|_{L^2[-T,T]} \leq Ch^2$. In *addition, there are the following relations:*

$$a_0 = \sigma_0 \frac{\|\varphi_0\|^2}{\langle 1, \varphi_0 \rangle}, \quad b_0 = \alpha_0 \frac{\|\varphi_0\|^2}{\langle 1, \varphi_0 \rangle}, \sigma_0 = \pm \sqrt{\frac{\langle 1, \varphi_0 \rangle^2}{\|\varphi_0\|^4} - \alpha_0^2};$$
$$D_2^0 = 8 \frac{\langle \varphi_0^2 L_+^{-1}[1], L_-^{-1}[b_0 - \alpha_0 \varphi_0] \rangle}{\langle 1, \varphi_0 \rangle};$$
$$\Psi_1^0 = a_0^2 L_+^{-1}[1] + b_0 L_-^{-1}[b_0 - \alpha_0 \varphi_0];$$
$$\Psi_2^0 = a_0 b_0 L_+^{-1}[1] - a_0 L_-^{-1}[b_0 - \alpha_0 \varphi_0].$$

Remarks.

1. Note that there are two solutions constructed in Theorem 1—one for $\sigma_0 > 0$ and another for $\sigma_0 < 0$.
2. Note that by Proposition 5 below, $\mathrm{Ker}[L_-] = span[\varphi_0], \mathrm{Ker}[L_+] = span[\varphi_0']$. Therefore, the expression $L_-^{-1}[b_0 - \alpha_0 \varphi_0]$ is well-defined, since by the definition of b_0, we have that $b_0 - \alpha_0 \varphi_0 \perp \mathrm{Ker}[L_-]$. Similarly, $1 \perp \mathrm{Ker}[L_+]$, whence $L_+^{-1}[1]$ is well-defined.
3. The theorem applies under the more general ansatz $\alpha = \alpha_0 h + O(h^2)$. In fact, since its statement is of first order in h, the proof in this more general case goes without any changes or modifications.

1.2. Stability of the stationary solutions. Before we discuss our stability results, let us emphasize that all of them are regarding spectral stability with respect to co-periodic perturbations—that is, the perturbations are taken to be $2T$ periodic. The question for stability in the spaces with more general periods $2nT, n = 2, \ldots$ or more generally with localized perturbations is undoubtedly interesting and highly nontrivial, but it falls outside the scope of this paper.

Our first result is regarding the instability of φ_h for $\alpha = 0$.

PROPOSITION 4. *Let $\alpha = 0$. Then, for all sufficiently small values of $h : 0 < h << 1$, the waves* (1.7) *are spectrally unstable solutions of* (1.2)*, with respect to co-periodic perturbations. Moreover, the instability presents itself in the linearized operator by a single real unstable eigenmode.*

The proof of Proposition 4 is presented in section 2.4 below. It is a direct consequence of the spectral properties of the operators $\mathcal{L}_{\pm,h}$, presented in Lemma 1 below, and the instability index count.

Regarding the waves constructed in Theorem 2.1, we have the following complete characterization.

THEOREM 2. *Let $h, \alpha_0, \varphi_\alpha$ be as in Theorem* 1*. Then, φ_α is spectrally stable with respect to co-periodic perturbations* if and only if

$$\sigma_0 = -\sqrt{\frac{\langle 1, \varphi_0 \rangle^2}{\|\varphi_0\|^4} - \alpha_0^2}.$$

In addition, in the stable case, the spectrum of the full linearized operator has two real simple eigenvalues 0 *and* -2α*, and the rest of the spectrum is on the vertical line* $\{\mu : \Re\mu = -\alpha\}$.

In the spectrally unstable case, which occurs for $\sigma_0 = \sqrt{\frac{\langle 1,\varphi_0\rangle^2}{\|\varphi_0\|^4} - \alpha_0^2}$, *there is a single real unstable eigenvalue in the form* $\mu_0\sqrt{h} + O(h)$, *where*[1]

$$\mu_0 = \sqrt{\frac{\sigma_0\|\varphi_0\|^2}{-\langle L_+^{-1}\varphi_0, \varphi_0\rangle}} > 0.$$

We briefly explain our approach toward the proof of Theorem 2. It turns out, rather unsurprisingly, that the linearized problem looks like a damped version of the linearized NLS. This is manifested in the form of the linearized operator, $\mathcal{J}\mathcal{N}_h - \alpha$, with $\mathcal{J}^* = -\mathcal{J}, \mathcal{N}_h^* = \mathcal{N}_h$ (see (4.1) below). Note that this looks like a standard Hamiltonian linearized operator, moved α units to the left. Since the translational symmetry persists even when both $\alpha \neq 0, h \neq 0$, the system keeps its zero eigenvalue due to the translational invariance. More concretely, a simple differentiation in the profile equation yields $(\mathcal{J}\mathcal{N}_h - \alpha)\binom{\varphi'_{\alpha,1}}{\varphi'_{\alpha,2}} = 0$, while Hamiltonian symmetry dictates another eigenvalue at -2α. These are the movements of the algebraic multiplicity two eigenvalue at zero (with one eigenvector and one adjoint), present at $\alpha = 0$, due to translational invariance. These arguments account for the translational eigenvalue(s) for the case $0 < h << 1, \alpha = \alpha_0 h$.

For the original Schrödinger problem, with $\alpha = h = 0$, there is a modulational invariance, i.e., $u \to e^{i\theta}u$ preserves solutions. In its linearized operator, one finds another pair of eigenvalues at zero, again with one eigenvector and one adjoint. Once $h > 0$, modulational invariance is broken, but as we are still close to the problem for $h = 0$, this double eigenvalue is expected to move close by. The main focus is then on the movement of this pair of eigenvalues.

We show that there are two scenarios—for the wave φ_α with $\sigma_0 > 0$, the modulational eigenvalue splits into a pair of positive and negative real eigenvalues, of order $O(\sqrt{h})$, so it presents itself as instability even after taking into account the damping, which moves the spectrum to the left $\alpha = \alpha_0 h$ units, thus still resulting in instability since $\sqrt{h} >> h$.

In the other case $\sigma_0 < 0$, the multiplicity two eigenvalue at zero for $h = 0$ splits into a pair of two marginally stable eigenvalues, with *negative Krein signature*. At the same time, the self-adjoint operator $\mathcal{N}_0$ has initially only one negative eigenvalue and an eigenvalue at zero, generated by the modulational invariance, and in fact $\mathcal{N}_0\binom{\varphi_0}{0} = 0$. After we turn on the $h > 0$, we show that the zero eigenvalue moves to the left, creating a second negative eigenvalue for $\mathcal{N}_h$, so $n(\mathcal{N}_h) = 2$. This presents an interesting stability configuration—while it has two negative (potentially unstable) directions for $\mathcal{N}_h$, we encounter two marginally stable eigenvalues with negative Krein signature, which allows us to conclude spectral stability, by instability index count.

Let us mention that an approach similar to the one offered here definitely fails in the case of periods $2nT, n = 2, \ldots$. Even for the standard case of the cubic NLS, the stability of dnoidal waves in the spaces $L^2[-nT, nT]$ [4, 8] uses additional structure, like higher order conservation laws, available only for this specific model. This is unlikely to work in this case, so we leave this question for future investigation.

The plan of the paper is as follows. In section 2, we start with the details of the construction for the case $\alpha = 0, h > 0$ and the nonexistence of stationary waves for $h = 0, \alpha > 0$. Then, we present the spectral properties of the linearized operators

[1]Note that by Proposition 5 below, the expression under the square root is positive, since $\langle L_+^{-1}\varphi_0, \varphi_0\rangle < 0$.

$L_\pm$—most of them are well-known from previous investigations, but a few are not and are necessary for our arguments.[2] Next, we extend some of these results by perturbative methods to the waves φ_h and the operators $\mathcal{L}_{\pm,h}$. This allows us to show rigorously the instability of the waves arising for $\alpha = 0$, namely, φ_h, and we can in fact compute its single unstable mode to its leading order $O(\sqrt{h})$. In section 3, we proceed with the construction of the waves in the case $0 < h << 1, \alpha = \alpha_0 h$.

We perform first an informal analysis of the problem. In particular, we show solvability of an appropriate *formally* linearized system, which then allows us to set the nonlinear problem in a way so that the implicit function theorem applies rigorously—to establish the desired existence of φ_α. In section 4, we study the linearized stability of the waves φ_α, as explained above.

2. Preliminaries.

We first present the proofs of Propositions 1 and 2.

2.1. The stationary waves for $h = 0, \alpha > 0$ and $h > 0, \alpha = 0$.

Proof of Proposition 1. Let $\varphi_\alpha = \varphi_{\alpha,1} + i\varphi_{\alpha,2}$ be a solution of (1.4). Then, we have

$$\left| \begin{array}{l} \varphi''_{\alpha,1} - \varphi_{\alpha,1} + 2(\varphi^2_{\alpha,1} + \varphi^2_{\alpha,2})\varphi_{\alpha,1} - \alpha\varphi_{\alpha,2} = 0, \\ \\ \varphi''_{\alpha,2} - \varphi_{\alpha,2} + 2(\varphi^2_{\alpha,1} + \varphi^2_{\alpha,2})\varphi_{\alpha,2} + \alpha\varphi_{\alpha,1} = 0. \end{array} \right. \tag{2.1}$$

Denoting the second order self-adjoint differential operator $L := -\partial_x^2 + 1 - 2(\varphi^2_{\alpha,1} + \varphi^2_{\alpha,2})$, we see that (2.1) is a relationship in the form

$$\left| \begin{array}{l} L[\varphi_{\alpha,1}] = -\alpha\varphi_{\alpha,2}, \\ L[\varphi_{\alpha,2}] = \alpha\varphi_{\alpha,1}. \end{array} \right.$$

Thus, applying L to to first equation, we obtain $L^2[\varphi_{\alpha,1}] = -\alpha^2\varphi_{\alpha,1}$. By taking a dot product with $\varphi_{\alpha,1}$, we obtain

$$0 \leq \|L\varphi_{\alpha,1}\|^2 = \langle L^2[\varphi_{\alpha,1}], \varphi_{\alpha,1}\rangle = -\alpha^2\|\varphi_{\alpha,1}\|^2 < 0,$$

which is a contradiction. ∎

Proof of Proposition 2. We integrate once (1.5) to get

$$\varphi'^2 = -\varphi^4 + \varphi^2 - 2h\varphi - c, \tag{2.2}$$

where c is a constant of integration. Recall that our interest is in the regime $0 < h << 1$. We demand that $\zeta_1 < \zeta_2 < \zeta_3 < \zeta_4$ are four real roots of the polynomial $z^4 - z^2 + 2hz + c$. Then, we rewrite (2.2) in the form

$$\varphi'^2 = (\zeta_4 - \varphi)(\varphi - \zeta_1)(\varphi - \zeta_2)(\varphi - \zeta_3). \tag{2.3}$$

The solution of (2.3) is given by (1.7), where

$$\left| \begin{array}{l} \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 = 0, \\ \zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_1\zeta_4 + \zeta_2\zeta_3 + \zeta_2\zeta_4 + \zeta_3\zeta_4 = -1, \\ \zeta_1\zeta_2\zeta_3 + \zeta_1\zeta_2\zeta_4 + \zeta_2\zeta_3\zeta_4 + \zeta_1\zeta_3\zeta_4 = -2h, \\ \zeta_1\zeta_2\zeta_3\zeta_4 = c. \end{array} \right. \tag{2.4}$$

[2]Some of the more technical calculations are actually left for the appendix; see Proposition 8.

These are the solutions that we shall be interested in. These solutions have been found in the Lugiato–Lefever context in [1] and [16] (see also [15]). In the whole line case, the explicit formulas appear, for example, in [1].

The current construction gives us a parametrization in terms of c, h. We now comment on the range of c, for which the condition that the polynomial $z^4 - z^2 + 2hz + c$ has four different and real roots. At least for $h = 0$, this is easy to characterize. Namely, the quartic has four real roots exactly when $c \in (0, \frac{1}{4})$. Then, for $0 < h << 1$, we clearly must require that $c \in (0, \frac{1}{4})$ within an error of $O(h)$.

For future purposes, however, it will be beneficial to parametrize the waves in terms of a different set of parameters m, h, where $m = \min_{-T \le x \le T} \varphi(x)$. In fact, m is exactly the root φ_3 above, since the explicit solution φ varies in the interval $[\varphi_3, \varphi_4]$, and hence $c = -m^4 + m^2 - 2hm$.

We proceed as follows—set $\varphi = m + \psi$ in (2.2), whence we require that $\psi \ge 0$ and we obtain the following equation for ψ:

$$(\psi')^2 = \psi[-\psi^3 - 4m\psi^2 + (1 - 6m^2)\psi + (2m - 4m^3 - 2h)]. \tag{2.5}$$

In order for such ψ to exist, we clearly need $(2m - 4m^3 - 2h) > 0$, that is, $m \in (0, \frac{1}{\sqrt{2}})$ within $O(h)$. Note that this is consistent with the relations $c \in (0, \frac{1}{4})$ and $c = -m^4 + m^2$ within $O(h)$. In addition, the polynomial $z \to -z^3 - 4mz^2 + (1 - 6m^2)z + (2m - 4m^3 - 2h)$ has a positive root—denote the smallest positive root by ψ_1. In this case there is unique solution to the equation

$$\psi' = -\sqrt{\psi[-\psi^3 - 4m\psi^2 + (1 - 6m^2)\psi + (2m - 4m^3 - 2h)]}, -T \le x \le T, \tag{2.6}$$

which satisfies the following:

- ψ is even, decaying in $[0, T]$ (and so $\psi'(0) = 0$),
- $\psi(0) = \psi_4 - \psi_3$, $\psi(T) = 0$,

since $\psi(0) = \varphi(0) - m = \psi_4 - \psi_3$. Now, it is much easier to parametrize the roots $\zeta_1, \dots, \zeta_4$, which will be useful in what follows. Take again $h = 0$, and then the final result will be within $O(h)$. We have the equation

$$z^4 - z^2 = -c = m^4 - m^2.$$

This has solutions $z_{1,2} = \pm m, z_{3,4} = \pm\sqrt{1 - m^2}$. By the restriction, $m \in (0, \frac{1}{\sqrt{2}})$, we have that $\sqrt{1 - m^2} > m$, whence we arrive at (1.9).

We now compute the range of the $T = T_{c,h}$. As it is a continuous function of c, h, it will cover an interval, so we aim at computing its endpoints, modulo errors $O(h)$, as h will be small in the applications. It is worth rewriting an equivalent definition of the roots, in order to discuss the asymptotics of κ, g, which enter the formula for the period. We have

$$\zeta_4 = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - c - 2h\zeta_4}}, \quad \zeta_3 = \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - c - 2h\zeta_3}},$$
$$\zeta_2 = -\sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - c - 2h\zeta_2}}, \quad \zeta_1 = -\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - c - 2h\zeta_1}}.$$

Clearly, as $c \to \frac{1}{4}$, we have that $\xi_4, \xi_3 \to \frac{1}{\sqrt{2}} + O(h)$, $\xi_2, \xi_1 \to -\frac{1}{\sqrt{2}} + O(h)$, so $\kappa \to O(h), g \to \sqrt{2} + O(h)$. So, using that $\lim_{\kappa \to 0} K(\kappa) = \frac{\pi}{2}$, we conclude that in this limit, $\lim_{c\to\frac14} T_{c,h} = 2^{-3/4}\pi + O(h)$. For the other limit, as $c \to 0$ and small enough h, write

$$1-\kappa^2 = \frac{(\zeta_4-\zeta_1)(\zeta_3-\zeta_2)}{(\zeta_4-\zeta_2)(\zeta_3-\zeta_1)}.$$

Note that $(\zeta_4-\zeta_1) = 2+O(h), \zeta_4-\zeta_2 = 1+O(h), \zeta_3-\zeta_1 = 1+O(h)$, so $g = 2+O(h)$.

Note, however, that

$$\zeta_3 = \sqrt{\frac12 - \sqrt{\frac14 - c - 2h\zeta_3}} = \sqrt{\frac{c+2h\zeta_3}{\frac12+\sqrt{\frac14-c-2h\zeta_3}}} \geq \sqrt{c+2h\zeta_3}.$$

In addition, $\zeta_3 - \zeta_2 \geq \zeta_3$. Thus, the expression $\zeta_3-\zeta_2$ is minimized, exactly at $c = 0, \zeta_2 = 0$, in which case $\zeta_3(0,h) = 2h + O(h^2)$. Plugging this in the formula $T_{0,h} = \sqrt{g}K(\kappa(0,h))$, we obtain

$$T_{0,h} = (2^{1/4} + O(h))K(1-4h+O(h^2)).$$

Note that since $\lim_{x\to 0+} xK(1-x) = \frac{\pi}{2}$, we conclude that $T_{0,h} = 2^{-\frac{11}{4}}\frac{\pi}{h}+O(1)$. Thus, we see that the period sweeps the interval $T \in (2^{-3/4}\pi + O(h), 2^{-\frac{11}{4}}\frac{\pi}{h} + O(1))$. ∎

We now give the basic spectral properties of the linearized operators associated with φ_h.

2.2. Spectral properties. Before listing these properties, let us state them in the easier case $h = 0$, of which we bifurcate as $h \neq 0$.

PROPOSITION 5. *The linearized operators $L_\pm$ satisfy the following spectral properties:*

- $L_- \geq 0$ *with* $L_-[\varphi_0] = 0$, $L_-|_{\{\varphi_0\}^\perp} > 0$,
- $n(L_+) = 1$, $L_+[\varphi_0'] = 0$ *and* 0 *is a simple eigenvalue for* L_+.

In addition, the following two relations hold:

$$\langle \mathcal{L}_{+,0}^{-1}\varphi_0, \varphi_0\rangle < 0, \quad \langle \mathcal{L}_{+,0}^{-1}[1], \varphi_0\rangle = 0.$$

Remark. The condition $\langle \mathcal{L}_{+,0}^{-1}\varphi_0, \varphi_0\rangle < 0$ is equivalent to the stability of the wave φ_0, in the context of the periodic NLS problem (1.2), with $h = 0$.

Proof. Note that φ_0 satisfies (1.2) with $h = 0$, and hence $L_-[\varphi_0] = 0$. Since $\varphi_0 > 0$, it follows by the Sturm–Liouville theory that $L_- \geq 0$ and 0 is the bottom of the spectrum and $\mathcal{L}_{-,0}|_{\{\varphi_0\}^\perp} > 0$.

Next, we show the properties of L_+. By differentiating the profile equation (1.5), $L_+[\varphi_0'] = 0$, that is, 0 is an eigenvalue for L_+. Since

$$\langle L_+\varphi, \varphi\rangle = -4\int_{-T}^{T} \varphi_0^4(x)dx < 0,$$

it follows that L_+ has a negative eigenvalue. Since φ_0' has exactly one change of sign, it follows that there is a unique simple negative eigenvalue, so $n(L_+) = 1$.

Finally, it remains to show that $\mathrm{Ker}[L_+] = span\{\varphi_0'\}$. In order to do that, we will show that the second independent solution of the equation $L_+[g] = 0$ does not belong to the space $L^2_{per.}[-T,T]$. Normally, such a solution g can be written down by the reduction of order formula as follows:

$$g(x) = \varphi_0'(x)\int_a^x \frac{1}{(\varphi_0'(y))^2}dy.$$

The problem is that such a formula blows up whenever the interval of integration contains 0. So, we use an alternative description of the eigenfunction, due to Rofe-Beketov (see [20, Exercise 5.11, p. 154]),

$$g(x) = \varphi_0'(x) \int_0^x \frac{(2-6\varphi_0^2(y))((\varphi_0'(y))^2 - (\varphi_0''(y))^2)}{((\varphi_0'(y))^2 + (\varphi_0''(y))^2))^2} dy - \frac{\varphi_0''(x)}{(\varphi_0'(x))^2 + (\varphi_0''(x))^2}.$$

This function is well-defined and satisfies $\mathcal{L}_{+,0}[g] = 0$. In order to show that the eigenvalue at zero is simple, it suffices to prove that g is not $2T$ periodic. Clearly, the second part of the formula in g is $2T$ periodic, so we concentrate on showing that the first piece, $g_1(x)$, is nonperiodic. In fact, $g_1(-T) = g_1(T)$, since $\varphi_0'(T) = \varphi_0'(-T) = 0$.

We show that in fact $g_1'(-T) \neq g_1'(T)$. Since, $\varphi_0''(-T) = \varphi_0''(T) \neq 0$, it suffices to show that

$$\int_0^T \frac{(2-6\varphi_0^2(y))((\varphi_0'(y))^2 - (\varphi_0''(y))^2)}{((\varphi_0'(y))^2 + (\varphi_0''(y))^2))^2} dy \neq \int_0^{-T} \frac{(2-6\varphi_0^2(y))((\varphi_0'(y))^2 - (\varphi_0''(y))^2)}{((\varphi_0'(y))^2 + (\varphi_0''(y))^2))^2} dy.$$

Since the integrand is even, this is equivalent to

$$\int_0^T \frac{(2-6\varphi_0^2(y))((\varphi_0'(y))^2 - (\varphi_0''(y))^2)}{((\varphi_0'(y))^2 + (\varphi_0''(y))^2))^2} dy \neq 0. \tag{2.7}$$

We postpone the verification of (2.7) and the proofs of $\langle \mathcal{L}_{+,0}^{-1}\varphi_0, \varphi_0\rangle < 0$ and $\langle \mathcal{L}_{+,0}^{-1}\varphi_0, 1\rangle = 0$ to the appendix. The computations are somewhat long and technical, but otherwise standard. ∎

We now continue with our investigation of the behavior of $\mathcal{L}_\pm$ when $0 < h << 1$. By a simple differentiation of (1.5), we still obtain, even for $h \neq 0$, $\mathcal{L}_{+,h}[\varphi'] = 0$, so 0 is still an eigenvalue. This is of course due to the translational invariance, which is preserved even after adding h.

2.3. The waves φ_h and their linearized operators $\mathcal{L}_{\pm,h}$. In order to set the stage for our later considerations, it is helpful to observe that given the relations (1.9),

$$\varphi_h = \varphi_0 + O(h),\ \mathcal{L}_{\pm,h} = L_{\pm,0} + O_{B(L^2)}(h),\ \lambda_j(\mathcal{L}_{\pm,h}) = \lambda_j(L_\pm) + O(h),$$

where we have used the notation $\lambda_0(\mathcal{L}) \leq \lambda_1(\mathcal{L}) \leq \dots$ to enumerate the eigenvalues of a self-adjoint operator $\mathcal{L}$ bounded from below, in an increasing order. In particular, it follows that $\lambda_0(\mathcal{L}_{+,h}) = \lambda_0(L_+) + O(h) < 0$ for small values of h, whereas $\lambda_1(\mathcal{L}_{+,h}) = 0$, while $\lambda_2(\mathcal{L}_{+,h}) = \lambda_2(L_+) + O(h) > 0$. Thus, the structure of the spectrum for $\mathcal{L}_{+,h}$ is the same as L_+ as described in Proposition 5. In particular, the operator $\mathcal{L}_{+,h}$ has a one dimensional kernel, spanned by φ_h', and it is hence invertible on the subspace of even functions.

Our next result concerns the structure of $\mathcal{L}_{-,h}$ when $h \neq 0$. Note that the modulational invariance is lost after the addition of h, which is why the zero eigenvalue for L_- is expected to move away from zero once we turn on the h parameter. Let us record the formula $\mathcal{L}_{-,h}\varphi_h = h$, which is just a restatement of (1.5). We have the following lemma.

LEMMA 1. *There exists $h_0 > 0$, so that for all $|h| < h_0$, we have the following formulas:*

$$\varphi_h = \varphi_0 + hL_+^{-1}[1] + O(h^2), \tag{2.8}$$

$$\lambda_0(\mathcal{L}_{-,h}) = \frac{\int_0^T \varphi_0(x)dx}{\int_0^T \varphi_0^2(x)dx} h + O(h^2), \tag{2.9}$$

$$\tilde{\varphi}_h = \varphi_0 + L_-^{-1}\left[4\varphi_0^2 L_+^{-1}[1] + \frac{\int_0^T \varphi_0(x)dx}{\int_0^T \varphi_0^2(x)dx}\varphi_0\right] h + O(h^2), \tag{2.10}$$

where[3] $\tilde{\varphi}_h : \mathcal{L}_{-,h}[\tilde{\varphi}_h] = \lambda_0(\mathcal{L}_{-,h})\tilde{\varphi}_h$ *is the ground state of* $\mathcal{L}_{-,h}$. *In particular,* $\mathcal{L}_{-,h} > 0$ *for* $0 < h << 1$.

Remark. A simple perturbation argument shows that $\lambda_1(\mathcal{L}_{-,h}) = \lambda_1(L_-) + O(h)$, which is well-separated from zero.

Proof. By differentiating with respect to h the profile equation, we obtain $\mathcal{L}_{+,h}[\partial_h\varphi_h] = 1$. As a consequence, since we know $\mathrm{Ker}[\mathcal{L}_{+,h}] = span[\varphi_h']$ (and hence $1 \perp \mathrm{Ker}[\mathcal{L}_{+,h}]$),

$$\partial_h\varphi_h = \mathcal{L}_{+,h}^{-1}[1] + \delta\varphi_h'$$

for some δ. We claim $\delta = 0$. Indeed, we know that φ_h is an even function, and so is $\partial_h\varphi_h$. Clearly $\mathcal{L}_{+,h}$ (and its inverse on $\mathrm{Ker}[\mathcal{L}_{+,h}]^\perp$) acts invariantly on the even subspace, so $\mathcal{L}_{+,h}^{-1}[1]$ is even as well. Thus, the odd piece $\delta\varphi_h'$ is actually zero, whence $\delta = 0$. Thus,

$$\varphi_h = \varphi_0 + h\mathcal{L}_{+,h}^{-1}[1] + O(h^2). \tag{2.11}$$

Next, since L_- has a simple eigenvalue at zero, $L_{-,h}$ has a single eigenvalue close to zero, in the form $\lambda_0(\mathcal{L}_{-,h}) = ah + O(h^2)$. Say the corresponding eigenfunction is in the form $\varphi_0 + hz$, $z \in H^2[-T, T]$. Thus,

$$\mathcal{L}_{-,h}[\varphi_0 + hz] = ah(\varphi_0 + hz). \tag{2.12}$$

However, by (2.11),

$$\begin{aligned}\mathcal{L}_{-,h} &= -\partial_x^2 + 1 - 2\varphi_h^2 = -\partial_x^2 + 1 - 2\varphi_0^2 - 4h\mathcal{L}_{+,h}^{-1}[1]\varphi^0 + O(h^2)\\ &= L_-5 - 4h\varphi_0\mathcal{L}_{+,h}^{-1}[1] + O(h^2).\end{aligned}$$

By taking the first order in h terms in (2.12), we obtain

$$L_-[z] = 4\varphi_0^2\mathcal{L}_{+,h}^{-1}[1] + a\varphi_0.$$

Now, take dot product with φ_0. Note that since $1 \perp \mathrm{Ker}[\mathcal{L}_{+,h}]$, we have that $\mathcal{L}_{+,h}^{-1}[1] = L_+^{-1}[1] + O(h)$. Since $\langle L_-[z], \varphi_0\rangle = \langle z, L_-[\varphi_0]\rangle = 0$, we obtain the relation

$$a\|\varphi_0\|^2 + 4\langle\varphi_0^3, L_+^{-1}[1]\rangle = 0.$$

Thus, $a = -\frac{4}{\|\varphi_0\|^2}\langle L_+^{-1}[\varphi_0^3], 1\rangle$. However, the profile equation can be rewritten as

$$L_+[\varphi_0] = -4\varphi_0^3,$$

[3]The quantity $\left(4\varphi_0^2 L_+^{-1}[1] + \frac{\int_0^T \varphi_0(x)dx}{\int_0^T \varphi_0^2(x)dx}\varphi_0\right) \perp \varphi_0$, whence L_-^{-1} is well-defined.

whence $L_+^{-1}[\varphi_0^3] = -\frac{1}{4}\varphi_0 + \delta\varphi_0'$ and

$$a = \frac{\int_{-T}^{T} \varphi_0(x)dx}{\int_{-T}^{T} \varphi_0^2(x)dx} > 0.$$

Also,

$$z = \mathcal{L}_{-,0}^{-1}\left[4\varphi_0^2 L_+^{-1}[1] + \frac{\int_{-T}^{T} \varphi_0(x)dx}{\int_{-T}^{T} \varphi_0^2(x)dx}\varphi_0\right] + O(h),$$

which is (2.10). □

Next, we linearize about φ_h. Let $u(t,x) = \varphi_h(x) + v(t,x)$, where v is a complex-valued function. Plugging this in (1.2) and ignoring the contributions of all terms in the form $O(v^2)$, we obtain

$$\begin{aligned} &-v_{2t} + v_{1xx} - v_1 + 6\varphi_h^2 v_1 = 0, \\ &v_{1t} + v_{2xx} - v_2 + 2\varphi_h^2 v_2 = 0. \end{aligned}$$

This is clearly in the form

$$\mathcal{J}\begin{pmatrix} \mathcal{L}_{+,h} & 0 \\ 0 & \mathcal{L}_{-,h} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_{1t} \\ v_{2t} \end{pmatrix},$$

where $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Introduce $\mathcal{L}_h := \begin{pmatrix} \mathcal{L}_{+,h} & 0 \\ 0 & \mathcal{L}_{-,h} \end{pmatrix}$. Taking the ansatz $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to e^{\lambda t}\begin{pmatrix} llv_1 \\ v_2 \end{pmatrix}$,

$$\mathcal{J}\mathcal{L}_h \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{2.13}$$

Thus, the stability of the wave φ_h is determined from the eigenvalue problem (2.13). Following the usual notions of spectral stability, we say that the wave is spectrally stable if (2.13) has no nontrivial solutions (that is, $\vec{v} \neq 0$), $(\lambda, \vec{v}) : \vec{v} \in H^2[-T,T]$ with $\Re\lambda > 0$.

2.4. Proof of Proposition 4. As an immediate consequence of the results of Lemma 1, we can conclude the instability for the eigenvalue problem (2.13). Indeed, we have that $n(\mathcal{L}_h) = n(\mathcal{L}_{+,h}) + n(\mathcal{L}_{-,h}) = 1$, while $\mathrm{Ker}[\mathcal{L}_{-,h}] = \{0\}$, $\mathrm{Ker}[\mathcal{L}_{+,h}] = span[\partial_x\varphi_h]$. In addition, since $\langle \mathcal{L}_{-,h}^{-1}[\partial_x\varphi^h], \partial_x\varphi^h\rangle > 0$, by the positivity of $\mathcal{L}_{-,h}$ (whence $\mathcal{L}_{-,h}^{-1} > 0$), we conclude $n(D) = n(\langle \mathcal{L}_{-,h}^{-1}[\partial_x\varphi_h], \partial_x\varphi_h\rangle) = 0$. By the instability index counting theory, we conclude that the eigenvalue problem (1.9) has a single real unstable eigenvalue for all small values of h. This completes the proof of Proposition 4.

2.5. A precise asymptotic for the unstable eigenvalue. For the purposes of the analysis of the full problem (that is, with $h \neq 0, \alpha \neq 0$), we need to compute the unstable eigenvalue of the eigenvalue problem (2.13), at least to leading order in h.

To this end, for the spectral analysis of (2.13), we are looking to find the pair $\lambda = 0, \vec{v} = \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix}$, which solves (2.13) for $h = 0$. In other words, we claim that the instability established in Proposition 4 is due to the bifurcation of the zero eigenvalue, present at $h = 0$ and corresponding to the modulational invariance. Due to the Hamiltonian symmetries, this multiplicity two eigenvalue at zero splits into one positive and one negative eigenvalues, once $h > 0$.

Multiplying (2.13) by $\mathcal{J}$ and taking the ansatz $v_2 = \varphi_h + hz, z \in L^2_{even}[-T,T]$ (and observing that $\mathcal{L}_{+,h}$ is invertible on the even subspace), we obtain

$$\mathcal{L}_{-,h}[\varphi_h + hz] = -\lambda^2 \mathcal{L}_{+,h}^{-1}[\varphi_h + hz]. \tag{2.14}$$

Taking into account $\mathcal{L}_{-,h}[\varphi_h] = h$ and $\mathcal{L}_{+,h}^{-1}[\varphi_h] = L_+^{-1}[\varphi_0] + O(h)$, we arrive at

$$h(1 + \mathcal{L}_{-,h}[z]) = -\lambda^2[L_+^{-1}[\varphi_0] + F(h,z)], \quad F(h,z) = O(h) + O(z). \tag{2.15}$$

It becomes clear that the ansatz for the eigenvalue λ must be in the form $\lambda = a\sqrt{h} + O(h)$, whence by taking dot product of (2.15) with φ_h, and taking only $O(h)$ terms

$$-a^2 \langle L_+^{-1}\varphi_0, \varphi_0\rangle = \int_{-T}^{T} \varphi_0(x)dx.$$

Recalling $\langle L_+^{-1}\varphi_0, \varphi_0\rangle < 0$, we derive the formula

$$a = \sqrt{\frac{\int_{-T}^{T} \varphi_0(x)dx}{-\langle L_+^{-1}\varphi_0, \varphi_0\rangle}} > 0.$$

Furthermore, (2.15) is solvable for small h, by the inverse function theorem. In this way, we have rigorously shown the following, more precise and quantitative, version of Proposition 4.

PROPOSITION 6. *There exists $h_0 > 0$, so that for all $h : 0 < h < h_0$, the eigenvalue problem* (2.13) *has the unstable eigenvalue in the form*

$$\lambda_h = \sqrt{\frac{\int_{-T}^{T} \varphi_0(x)dx}{-\langle L_+^{-1}\varphi_0, \varphi_0\rangle}}\sqrt{h} + O(h).$$

Beyond this point λ_h, the rest of the spectrum is stable. In fact,

$$\sigma(\mathcal{J}\mathcal{L}_h) \setminus \{\lambda_h, -\lambda_h\} \subset i\mathbf{R}.$$

3. The construction of the waves for $0 < h << 1, 0 < \alpha << 1$. We now proceed with the construction of the waves in the regime where both parameters h, α are turned on. We henceforth assume $h > 0$. In addition, we wish to keep the solutions in the even class.

Let $\varphi_\alpha(x) = \varphi_{\alpha,1} + i\varphi_{\alpha,2}$ be a solution of (1.2). That is,

$$\left| \begin{array}{l} \varphi_{\alpha,1}'' - \varphi_{\alpha,1} + 2(\varphi_{\alpha,1}^2 + \varphi_{\alpha,2}^2)\varphi_{\alpha,1} = \alpha\varphi_{\alpha,2} - h, \\ \\ \varphi_{\alpha,2}'' - \varphi_{\alpha,2} + 2(\varphi_{\alpha,1}^2 + \varphi_{\alpha,2}^2)\varphi_{\alpha,2} = -\alpha\varphi_{\alpha,1}. \end{array} \right. \tag{3.1}$$

For a more symmetric formulation, introduce

$$\tilde{\varphi}_1 := \varphi_{\alpha,1} + \varphi_{\alpha,2}, \tilde{\varphi}_2 := \varphi_{\alpha,1} - \varphi_{\alpha,2}.$$

We have the equations

$$\left| \begin{array}{l} \tilde{\varphi}_1'' - \tilde{\varphi}_1 + (\tilde{\varphi}_1^2 + \tilde{\varphi}_2^2)\tilde{\varphi}_1 = -\alpha\tilde{\varphi}_2 - h, \\ \\ \tilde{\varphi}_2'' - \tilde{\varphi}_2 + (\tilde{\varphi}_1^2 + \tilde{\varphi}_2^2)\tilde{\varphi}_2 = \alpha\tilde{\varphi}_1 - h. \end{array} \right. \tag{3.2}$$

Introducing the operator $\tilde{\mathcal{L}} = -\partial_x^2 + 1 - (\tilde{\varphi}_1^2 + \tilde{\varphi}_2^2) = -\partial_x^2 + 1 - 2(\varphi_{\alpha,1}^2 + \varphi_{\alpha,2}^2)$, we can rewrite the previous relations in the form $\tilde{\mathcal{L}}[\tilde{\varphi}_1] = \alpha\tilde{\varphi}_2 + h, \tilde{\mathcal{L}}[\tilde{\varphi}_2] = h - \alpha\tilde{\varphi}_1$. Apply $\tilde{\mathcal{L}}$ to the first equation. We obtain

$$\left| \begin{array}{l} (\tilde{\mathcal{L}}^2 + \alpha^2)[\varphi_{\alpha,1}] = h\tilde{\mathcal{L}}[1], \\ (\tilde{\mathcal{L}}^2 + \alpha^2)[\varphi_{\alpha,2}] = \alpha h. \end{array} \right. \tag{3.3}$$

It is now useful to perform some analysis in the regime $h << 1$. If $\tilde{\mathcal{L}}$ does not have any eigenvalues close to zero, that is, $\tilde{\mathcal{L}}^2 \geq \delta^2, \delta = O(1)$, we will have from (3.3) that $\varphi_{\alpha,1} = O(h), \varphi_{\alpha,2} = O(h)$, whence we have $\tilde{\mathcal{L}} = -\partial_x^2 + 1 + O(h)$. In this case, one can show that (3.3) has (small) solutions, given approximately by

$$\left| \begin{array}{l} \varphi_{\alpha,1} = h(-\partial_x^2 + 1)^{-1}[1] + O(h^2), \\ \varphi_{\alpha,2} = \alpha h(-\partial_x^2 + 1)^{-2}[1] + O(h^2). \end{array} \right. \tag{3.4}$$

So, we have shown the following.

PROPOSITION 7 (existence of small solutions). *There exists $h_0 > 0$, so that for all $0 < h < h_0, \alpha > 0$ there exists a solution of* (3.3)*, in the form* (3.4)*.*

Unfortunately, these solutions are not very useful from a practical point of view, since they are small. On the other hand, one can show that they are spectrally stable in a straightforward manner. Indeed, this follows once we recall the observation above that $\tilde{\mathcal{L}} = -\partial_x^2 + 1 + O(h)$ is a strictly positive operator. Then, it is easy to conclude the stability of the linearization around the solutions (3.4).

3.1. $O(1)$ solutions of (3.2)—An informal analysis of the profile equation. As we have mentioned above, we shall use h as a small parameter, by taking $\alpha := \alpha_0 h, \alpha_0 = O(1)$. Next, we assume that (3.2) (or equivalently (3.3)) has a solution. In addition, we model $\tilde{\mathcal{L}}$ to be a small perturbation of L_-. In particular, it has a small and simple eigenvalue close to zero. This is needed in order to produce $O(1)$ solutions of (3.3). Denote the small eigenvalue by $\sigma_h = \sigma_0 h + O(h^2)$, with a corresponding eigenfunction $\varphi_h = \varphi_0 + O(h)$. In addition, the next eigenvalue is positive and order $O(1)$.

By projecting (3.3) onto φ_h and its complementary subspace $\{\varphi_h\}^\perp$, we arrive at the formula

$$\varphi_{\alpha,1} = \frac{h\sigma}{\sigma^2 + \alpha^2} \frac{\langle 1, \varphi_h \rangle}{\|\varphi_h\|^2} \varphi_h + q_1, \quad q_1 = O(h), q_1 \in \{\varphi_h\}^\perp, \tag{3.5}$$

$$\varphi_{\alpha,2} = \frac{\alpha h}{\sigma^2 + \alpha^2} \frac{\langle 1, \varphi_h \rangle}{\|\varphi_h\|^2} \varphi_h + q_2, \quad q_2 = O(h^2), q_2 \in \{\varphi_h\}^\perp. \tag{3.6}$$

One can in principle continue with the construction of $\varphi_{\alpha,1}, \varphi_{\alpha,2}$ based on (3.5) and (3.6), but it becomes hard to keep track of the expansion of σ_h in powers of h. Instead, we will pass to the known waves φ_0, since we have a good understanding of the operator L_-. More precisely, we take the ansatz

$$\varphi_{\alpha,1} = (a_0 + a_1 h + O(h^2))\varphi_0 + h\Psi_1, \quad \Psi_1 \perp \varphi_0, \tag{3.7}$$

$$\varphi_{\alpha,2} = (b_0 + b_1 h + O(h^2))\varphi_0 + h\Psi_2, \quad \Psi_2 \perp \varphi_0. \tag{3.8}$$

Comparing the expansions (3.5) with (3.7) (and (3.6) with (3.8), respectively), we have the formula

$$a_0 = \frac{\sigma_0}{\sigma_0^2+\alpha_0^2}\frac{\langle 1,\varphi_0\rangle}{\|\varphi_0\|^2}, \tag{3.9}$$

$$b_0 = \frac{\alpha_0}{\sigma_0^2+\alpha_0^2}\frac{\langle 1,\varphi_0\rangle}{\|\varphi_0\|^2}. \tag{3.10}$$

Next, using the form of the operator $\tilde{\mathcal{L}}$, we have

$$\tilde{\mathcal{L}} = -\partial_x^2 + 1 - 2(\varphi_{\alpha,1}^2 + \varphi_{\alpha,2}^2) = -\partial_x^2 + 1 - 2(a_0^2+b_0^2)\varphi_0^2 + O(h).$$

Since we expect that $\tilde{\mathcal{L}}$ is a perturbation of L_-, we must require $a_0^2+b_0^2 = 1$. This, together with (3.9) and (3.10), implies that σ_0 is completely determined by α_0 and in fact,

$$\sigma_0^2+\alpha_0^2 = \frac{\langle 1,\varphi_0\rangle^2}{\|\varphi_0\|^4}. \tag{3.11}$$

We can rewrite (3.2) equivalently as follows:

$$(\tilde{\mathcal{L}} - i\alpha)[i(\varphi_{\alpha,1}+\varphi_{\alpha,2}) + (\varphi_{\alpha,1}-\varphi_{\alpha,2})] = h(1+i). \tag{3.12}$$

Denoting $q := i(\varphi_{\alpha,1}+\varphi_{\alpha,2}) + (\varphi_{\alpha,1}-\varphi_{\alpha,2})$, we can write $\tilde{\mathcal{L}} = -\partial_x^2 + 1 - |q|^2$. In addition, q has the representation

$$q = (c_0 + c_1 h + O(h^2))\varphi_0 + h\Psi, \quad \Psi \perp \varphi_0, \tag{3.13}$$

where clearly $c_0 = i(a_0+b_0) + (a_0-b_0)$ can be expressed in terms of α_0. For example, $|c_0|^2 = 2(a_0^2+b_0^2) = 2$. Compute

$$|q|^2 = |c_0|^2\varphi_0^2 + 2h\varphi_0^2\Re[c_0\bar{c_1}] + h\varphi_0[c_0\bar{\Psi} + \bar{c_0}\Psi] + O(h^2) = 2\varphi_0^2 + hV_\Psi + O(h^2),$$

upon introducing $V_\Psi := 2\varphi_0^2\Re[c_0\bar{c_1}] + \varphi_0[c_0\bar{\Psi} + \bar{c_0}\Psi]$. It follows that

$$\tilde{\mathcal{L}} = -\partial_x^2 + 1 - |q|^2 = L_- - hV_\Psi + O(h^2),$$

whence (3.12) becomes

$$(L_- - h(V_\Psi + i\alpha_0) + O(h^2))((c_0 + c_1 h + O(h^2))\varphi_0 + h\Psi) = h(1+i). \tag{3.14}$$

In order to resolve this equation, we need to go in powers of h. The terms with power h^0 are clearly absent, due to $L_-[\varphi_0] = 0$, which is just the profile equation. For the first order in h terms, we have the equation

$$L_-\Psi - (V_\Psi + i\alpha_0)c_0\varphi_0 = 1 + i. \tag{3.15}$$

Taking (3.15), and its complex conjugate, and in addition the form of V_Ψ and $|c_0|^2 = 2$, we arrive at the system

$$\begin{pmatrix} -\partial_x^2 + 1 - 4\varphi_0^2 & -c_0^2\varphi_0^2 \\ -\bar{c}_0^2\varphi_0^2 & -\partial_x^2 + 1 - 4\varphi_0^2 \end{pmatrix}\begin{pmatrix}\Psi \\ \bar{\Psi}\end{pmatrix} = \begin{pmatrix} 1 + i + i\alpha_0 c_0\varphi_0 + 2\varphi_0^3 c_0\Re[c_0\bar{c_1}] \\ 1 - i - i\alpha_0\bar{c}_0\varphi_0 + 2\varphi_0^3\bar{c}_0\Re[c_0\bar{c_1}]\end{pmatrix}. \tag{3.16}$$

Diagonalizing the system leads to the equations

$$L_+[\bar{c}_0\Psi + c_0\bar{\Psi}] = c_0 + \bar{c}_0 + i(\bar{c}_0 - c_0) + 8\varphi_0^3\Re[c_0\bar{c_1}], \tag{3.17}$$

$$L_-[-\bar{c}_0\Psi + c_0\bar{\Psi}] = c_0 - \bar{c}_0 - i(c_0 + \bar{c}_0) - 4i\alpha_0\varphi_0. \tag{3.18}$$

Note that one solvability condition for (3.18) is exactly $\langle c_0-\bar{c}_0-i(c_0+\bar{c}_0)-4i\alpha_0\varphi_0, \varphi_0\rangle = 0$. Elementary computations show that this is equivalent to $b_0\langle 1, \varphi_0\rangle = \alpha_0\|\varphi_0\|^2$, which is exactly the relation (3.10) and (3.11).

The other relation is that since $\Psi \perp \varphi_0$, we need to have $\bar{c}_0\Psi + c_0\bar{\Psi} \in \{\varphi_0\}^\perp$. This imposes the relation, from (3.17), $L_+^{-1}[c_0+\bar{c}_0+i(\bar{c}_0-c_0)+8\varphi_0^3\Re[c_0\bar{c_1}]] \in \{\varphi_0\}^\perp$ or equivalently

$$[c_0+\bar{c}_0+i(\bar{c}_0-c_0)]\langle L_+^{-1}[1], \varphi_0\rangle + 8\Re[c_0\bar{c_1}]\langle L_+^{-1}[\varphi_0^3], \varphi_0\rangle = 0.$$

In fact, since $\langle L_+^{-1}[1], \varphi_0\rangle = 0$ and $\langle L_+^{-1}[\varphi_0^3], \varphi_0\rangle = -\frac{1}{4}\langle \varphi_0, \varphi_0\rangle \neq 0$, it follows that $\Re[c_0\bar{c_1}] = 0$. It even looks as if we have one degree of freedom, since c_1 is complex valued (and hence two parameters are involved). In the actual nonlinear problem, however, we need to involve a higher order solvability condition for (3.18), which will finally yield the right number of equations.

3.2. Solutions of (3.2)—Rigorous construction. We now set up the full nonlinear problem (3.2), with $0 < h << 1$, in the equivalent formulation (3.12). More precisely, armed with the results from our informal analysis, we set the unknown function

$$q = \varphi_{\alpha,1} - \varphi_{\alpha,2} + i(\varphi_{\alpha,1} + \varphi_{\alpha,2}) \in L^2_{per.}[-T, T]$$

in the form

$$q = (c + dh)\varphi_0 + h\Psi, \Psi \perp \varphi_0, \quad c = a_0 - b_0 + i(a_0 + b_0),$$

and a_0, b_0 are given by (3.9), (3.10), and (3.11) in terms of α_0. Note that $|c|^2 = 2$. Also, in accordance with (3.11), we require $\alpha_0 : 0 < \alpha_0 < \frac{\langle 1, \varphi_0\rangle}{\|\varphi_0\|^2}$.

Now that we have set q (and in particular a_0, b_0), we are looking for a scalar function $d = d(h)$ and a function $\Psi(h) \in \{\varphi_0\}^\perp$, so that (3.12) holds. We compute

$$|q|^2 = 2\varphi_0^2 + hV + h^2[|d|^2\varphi_0^2 + \varphi_0(d\bar{\Psi} + \bar{d}\Psi) + |\Psi|^2],$$

where

$$V = 2\varphi_0^2\Re[c\bar{d}] + 2\varphi_0\Re[c\bar{\Psi}]$$

is a real-valued function as before. Introduce the *real-valued function*

$$G = G(d, \Psi) = |d|^2\varphi_0^2 + \varphi_0(d\bar{\Psi} + \bar{d}\Psi) + |\Psi|^2.$$

We thus have a formula for $\tilde{\mathcal{L}}$ as follows:

$$\tilde{\mathcal{L}} = -\partial_x^2 + 1 - |q|^2 = -\partial_x^2 + 1 - 2\varphi_0^2 - hV - h^2G = L_- - hV - h^2G.$$

Plugging this into (3.12), we obtain the following relation:

$$(L_- - h(V + i\alpha_0) - h^2G)[(c + dh)\varphi_0 + h\Psi] = h(1 + i). \tag{3.19}$$

After some algebraic manipulations, we obtain

$$L_-\Psi - c(V + i\alpha_0)\varphi_0 - (1 + i) - h[(V + i\alpha_0)(d\varphi_0 + \Psi) + c\varphi_0G] = h^2G(d, \Psi)\Psi. \tag{3.20}$$

Similar to the derivation of (3.16), we take (3.20) and its complex conjugate to obtain the following *nonlinear* in h system of equations:

$$\begin{pmatrix} -\partial_x^2 + 1 - 4\varphi_0^2 & -c^2\varphi_0^2 \\ -\bar{c}^2\varphi_0^2 & -\partial_x^2 + 1 - 4\varphi_0^2 \end{pmatrix}\begin{pmatrix} \Psi \\ \bar{\Psi} \end{pmatrix} = \begin{pmatrix} 1 + i + i\alpha_0c\varphi_0 + 2\varphi_0^3c\Re[c\bar{d}] \\ 1 - i - i\alpha_0\bar{c}\varphi_0 + 2\varphi_0^3\bar{c}\Re[c\bar{d}] \end{pmatrix}$$
$$+ h\begin{pmatrix} (V + i\alpha_0)(d\varphi_0 + \Psi) + c\varphi_0G \\ (V - i\alpha_0)(\bar{d}\varphi_0 + \bar{\Psi}) + \bar{c}\varphi_0G \end{pmatrix} + h^2\begin{pmatrix} G(d, \Psi)\Psi \\ G(d, \Psi)\bar{\Psi} \end{pmatrix}.$$

Diagonalizing yields the equivalent equations

$$L_+[\bar{c}\Psi + c\bar{\Psi}] = 4a_0 + 8\varphi_0^3\Re[c\bar{d}] + hE_1(h,d,\Psi) \tag{3.21}$$

$$\begin{aligned} L_-[-\bar{c}\Psi + c\bar{\Psi}] &= 4i(b_0 - \alpha_0\varphi_0) + 2ih[V\varphi_0\Im[c\bar{d}] \\ &\quad + V\Im[c\bar{\Psi}] - \alpha_0\varphi_0\Re[c\bar{d}] - \alpha_0\Re[c\bar{\Psi}]] + h^2E_2, \end{aligned} \tag{3.22}$$

where E_1, E_2 are smooth functions of the respective arguments. This is the system that we need to solve—that is, the goal is to find a neighborhood $(0,h_0)$, so that for every $h \in (0,h_0)$, there is a scalar function $d = d(h)$ and a function $\Psi = \Psi(h) \in \{\varphi_0\}^\perp$, so that the pair satisfies the previous two relations.

To that end, we shall use the implicit function theorem. It is clear that it is more convenient to introduce two real variables[4] $D_1 := \Re[c\bar{d}], D_2 := \Im[c\bar{d}]$. Clearly, the system requires some solvability conditions. We have already established that with our choice of c, we have that

$$c - \bar{c} - i(c+\bar{c}) - 4i\alpha_0\varphi_0 \perp \varphi_0.$$

So, from (3.22), we need to require

$$\begin{aligned} 0 &= \langle V\varphi_0\Im(c\bar{d}) + V\Im[c\bar{\Psi}] - \alpha_0\varphi_0\Re[c\bar{d}] - \alpha_0\Re[c\bar{\Psi}], \varphi_0\rangle + O(h) \\ &= \langle D_2V\varphi_0 - D_1\alpha_0\varphi_0 + V\Im[c\bar{\Psi}], \varphi_0\rangle + O(h). \end{aligned}$$

In the last identity, we used that $\Psi \perp \varphi_0$, whence by the reality of φ_0, we have that $\bar{\Psi} \perp \varphi_0$ as well (and thus any linear combination of $\Psi, \bar{\Psi}$ is perpendicular to φ_0). Thus, we end up requiring

$$\langle D_2V\varphi_0 - D_1\alpha_0\varphi_0 + V\Im[c\bar{\Psi}], \varphi_0\rangle + O(h) = 0. \tag{3.23}$$

Since $\bar{c}\Psi + c\bar{\Psi} \perp \varphi_0$, we need to have

$$0 = 4a_0\langle L_+^{-1}[1], \varphi_0\rangle + 8D_1\langle L_+^{-1}[\varphi_0^3], 1\rangle + O(h).$$

Recalling that $\langle L_+^{-1}[1], \varphi_0\rangle = 0$ and $L_+[\varphi_0] = -4\varphi_0^3$, whence $L_+^{-1}[\varphi_0^3] = -\frac{1}{4}\varphi_0$ and the previous relation reads

$$-2D_1\langle\varphi_0, 1\rangle + O(h) = 0. \tag{3.24}$$

The analysis so far allows us to solve the system (3.21), (3.22) for $h = 0$. Namely, from (3.24), we infer that

$$D_1^0 = 0. \tag{3.25}$$

The next step is to find Ψ^0, from (3.21) and (3.22), at $h = 0$. Inverting L_+ in (3.21) and L_- in (3.22) and taking the difference, and taking into account that $\Re[c\bar{d}] = D_1^0 + O(h) = O(h)$, we obtain[5]

$$\Psi^0 = \frac{4a_0L_+^{-1}[1] - 4iL_-^{-1}[b_0 - \alpha_0\varphi_0]}{2\bar{c}} = ca_0L_+^{-1}[1] - icL_-^{-1}[b_0 - \alpha_0\varphi_0]. \tag{3.26}$$

Note that $\Psi^0 \perp \varphi_0$ (as it should be), since $L_+^{-1}[1] \perp \varphi_0$, and $Image[L_-^{-1}] \perp \varphi_0$.

[4]Recall that c is already fixed in terms of α_0, so finding D_1, D_2 is akin to finding the complex number d.

[5]Recall that $b_0 - \alpha_0\varphi_0 \perp \varphi_0$, so taking L_-^{-1} is justified. Similarly, with the definition of D_1^0 in (3.25), taking L_+^{-1} is justified as well.

Finally, we use (3.23) to determine D_2^0. We obtain the formula

$$D_2^0\langle V^0\varphi_0, \varphi_0\rangle = -\langle \Im[c\bar{\Psi}_0], V^0\varphi_0\rangle. \tag{3.27}$$

We clearly need to compute $\Re[c\bar{\Psi}_0], \Im[c\bar{\Psi}_0]$. We have from (3.26),

$$\Re[c\bar{\Psi}_0] = 2a_0 L_+^{-1}[1],$$
$$\Im[c\bar{\Psi}_0] = 2L_-^{-1}[b_0 - \alpha_0\varphi_0].$$

According to its definition

$$V^0 = V(0, \Psi_0) = 2\varphi_0^2 D_1^0 + \varphi_0[c\bar{\Psi}^0 + \bar{c}\Psi^0] = 2\varphi_0\Re[c\bar{\Psi}_0] = 2a_0\varphi_0 L_+^{-1}[1].$$

Consequently, since $L_+^{-1}[\varphi_0^3] = -\frac{1}{4}\varphi_0$,

$$\langle V^0\varphi_0, \varphi_0\rangle = 4a_0\langle \varphi_0^3, L_+^{-1}[1]\rangle = -a_0\langle 1, \varphi_0\rangle. \tag{3.28}$$

Finally,

$$\langle \Im[c\bar{\Psi}_0], V^0\varphi_0\rangle = 8a_0\langle \varphi_0^2 L_+^{-1}[1], L_-^{-1}[b_0 - \alpha_0\varphi_0]\rangle.$$

From (3.27), we deduce

$$D_2^0 = 8\frac{\langle \varphi_0^2 L_+^{-1}[1], L_-^{-1}[b_0 - \alpha_0\varphi_0]\rangle}{\langle 1, \varphi_0\rangle}. \tag{3.29}$$

To recapitulate, we have determined, in (3.26), together with D_1^0, D_2^0 as determined above, the unique solutions of (3.21) and (3.22), when $h = 0$. We now set up the implicit function argument, which will work in a neighborhood of the solution $h = 0$, Ψ^0, given by (3.26), and D_1^0, D_2^0.

First, we set the solvability condition arising in (3.22), namely, the scalar function[6]

$$\begin{aligned} Q_1(h; \Psi, D_1, D_2) &= 2i\langle D_2 V(D_1, \Psi)\varphi_0 + V\Im[c\bar{\Psi}] - \alpha_0\varphi_0 D_1 - \alpha_0\Im[c\bar{\Psi}], \varphi_0\rangle \\ &+ h\langle E_2(h, \Psi, D_1, D_2), \varphi_0\rangle = 2i\langle D_2 V(D_1, \Psi)\varphi_0 + V(D_1, \Psi)\Im[c\bar{\Psi}] - \alpha_0\varphi_0 D_1, \varphi_0\rangle \\ &+ h\langle E_2(h, \Psi, D_1, D_2), \varphi_0\rangle, \end{aligned}$$

where $V(D_1, \Psi) = 2D_1\varphi_0^2 + \varphi_0(\bar{c}\Psi + c\bar{\Psi})$. The other function is constructed as follows: apply L_+^{-1} in (3.21) and L_-^{-1} in (3.22) (once we make sure that the right-hand side is orthogonal to φ_0). After subtracting and simplifying,

$$\begin{aligned} Q_2(h; \Psi, D_1, D_2) &= 2\bar{c}\Psi - \left[4a_0 L_+^{-1}[1] - 2D_1\varphi_0 + hL_+^{-1}[E_1(h; \Psi, D_1, D_2)]\right] \\ &+ 4iL_-^{-1}[b_0 - \alpha_0\varphi_0] + L_-^{-1}[P_{\{\varphi_0\}^\perp}[2ih(D_2 V(D_1, \Psi)\varphi_0 - D_1\alpha_0\varphi_0 + V(D_1, \Psi)\Im[c\bar{\Psi}] \\ &\qquad - \alpha_0\Im[c\bar{\Psi}]) + h^2 E_2(h; \Psi, D_1, D_2)]]. \end{aligned}$$

Note that the projection $P_{\{\varphi_0\}^\perp}$ becomes irrelevant, once we impose the condition $Q_1(h; \Psi, D_1, D_2) = 0$! On the other hand, we need it in the definition of Q_2 to keep it well-defined, even when $Q_1(h; \Psi, D_1, D_2) = 0$ is not enforced. We now consider

$$(Q_1, Q_2)(h; \Psi, D_1, D_2) : \mathbf{R} \times \{\varphi_0\}^\perp \times \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times L^2_{per.}[-T, T]$$

[6]Here we use again that $\langle \Im[c\bar{\Psi}], \varphi_0\rangle = 0$.

and we would like to solve

$$\left| \begin{array}{l} Q_1(h; \Psi, D_1, D_2) = 0, \\ Q_2(h; \Psi, D_1, D_2) = 0. \end{array} \right. \tag{3.30}$$

We again note that if one obtains solutions to (3.30), the projection $P_{\{\varphi_0\}^\perp}$ becomes irrelevant and the system $Q_1 = Q_2 = 0$ becomes equivalent to the system (3.21) and (3.22). Observe that by our earlier considerations, for $h = 0$, we have a solution that is

$$\left| \begin{array}{l} Q_1(0; \Psi^0, 0, D_2^0) = 0, \\ Q_2(0; \Psi^0, 0, D_2^0) = 0, \end{array} \right.$$

where Q_2^0 is given in (3.29). Our construction of the family $\Psi(h), D_1(h), D_2(h)$ in a neighborhood of $(0, h_0)$ will follow, once we can verify that

$$d(Q_1, Q_2)(0; \Psi^0, 0, D_2^0)[\cdot, \cdot, \cdot] : \{\varphi_0\}^\perp \times \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times L^2_{per.}[-T, T]$$

is an isomorphism. That is, for every $\chi \in L^2_{per.}[-T, T]$ and $z \in \mathbf{R}$, there must be unique solution $\psi \in \{\varphi_0\}^\perp, d_1 \in \mathbf{R}, d_2 \in \mathbf{R}$ of the linear system

$$\begin{aligned} dQ_1(0; \Psi^0, 0, D_2^0)[\psi, d_1, d_2] = z, \\ dQ_2(0; \Psi^0, 0, D_2^0)[\psi, d_1, d_2] = \chi \end{aligned}$$

so that the linear mapping $(\chi, z) \to (\psi(\chi, z), d_1(\chi, z), d_2(\chi, z))$ is continuous.

First, we compute $dQ_2(0; \Psi^0, 0, D_2^0)[\psi, d_1, d_2] = 2\bar{c}\psi + 2d_1\varphi_0$. In order to prepare the calculation for $dQ_1(0; \Psi^0, 0, D_2^0)[\psi, d_1, d_2]$, observe that

$$\begin{aligned} V(D_1, \Psi) &= 2D_1\varphi_0^2 + 2\varphi_0 \Re[c\bar{\Psi}], \\ dV(0, \Psi^0)(d_1, \psi) &= 2d_1\varphi_0^2 + 2\varphi_0 \Re[c\bar{\psi}]. \end{aligned}$$

Consequently,

$$\begin{aligned} dQ_1(0; \Psi^0, 0, D_2^0)[\psi, d_1, d_2] = 2i\langle d_2 V^0 \varphi_0 + D_2^0(2d_1\varphi_0^2 + 2\varphi_0\Re[c\bar{\psi}]), \varphi_0\rangle \\ + 2i\langle(2d_1\varphi_0^2 + 2\varphi_0\Re[c\bar{\psi}])\Im[c\bar{\Psi}_0], \varphi_0\rangle + 2i\langle V^0 \Im[c\bar{\psi}], \varphi_0\rangle - 2i\alpha_0 d_1 \|\varphi_0\|^2. \end{aligned}$$

Now, the equation $\chi = dQ_2(0; \Psi^0, 0, D_2^0)[\psi, d_1, d_2]$ has the form

$$\chi = dQ_2(0; \Psi^0, 0, D_2^0)[\psi, d_1, d_2] = 2\bar{c}\psi + 2d_1\varphi_0.$$

It clearly has the unique solution

$$d_1 = \frac{\langle \chi, \varphi_0 \rangle}{2\|\varphi_0\|^2}, \psi = \frac{1}{2\bar{c}}(\chi - 2d_1\varphi_0) \in \{\varphi_0\}^\perp.$$

Plugging the expressions for d_1 and ψ into the equation $dQ_1 = z$ produces a linear equation for d_2, once we take a dot product with φ_0. More precisely,

$$\begin{aligned} d_2\langle V^0\varphi_0, \varphi_0\rangle = \frac{z}{2i} - \langle D_2^0(2d_1\varphi_0^2 + 2\varphi_0\Re[c\bar{\psi}]), \varphi_0\rangle - \langle(2d_1\varphi_0^2 + 2\varphi_0\Re[c\bar{\psi}])\Im[c\bar{\Psi}_0], \varphi_0\rangle \\ + \alpha_0 d_1\|\varphi_0\|^2 - \langle V^0 \Im[c\bar{\psi}], \varphi_0\rangle, \end{aligned}$$

which also has a unique solution, provided the coefficient in front of it, $\langle V^0\varphi_0, \varphi_0\rangle \neq 0$, is nonzero. But we have already verfied that (see (3.28)). We also see that the solution d_2 depends linearly, through a nice formula on χ, z. It follows that the mapping

$(\chi, z) \to (\psi, d_1, d_2)$ is indeed an isomorphism, in the sense specified above. The implicit function theorem thus applies and we have constructed the solutions.

It remains to verify the formulas (1.10) and (1.11). First, we have that $c\bar{d} = D_1^0 + iD_2^0 + O(h) = iD_2^0 + O(h)$, whence

$$d^0 = -\frac{ic}{2}D_2^0 = \frac{a_0 + b_0}{2}D_2^0 + i\frac{b_0 - a_0}{2}D_2^0. \tag{3.31}$$

Thus, starting with the relation $\varphi_{\alpha,1} - \varphi_{\alpha,2} + i(\varphi_{\alpha,1} - \varphi_{\alpha,2}) = q = (c + dh)\varphi_0 + h\Psi$, we deduce

$$\varphi_{\alpha,1} = \left(a_0 + \frac{b_0}{2}hD_2^0\right)\varphi_0 + h(a_0^2 L_+^{-1}[1] + b_0 L_-^{-1}[b_0 - \alpha_0\varphi_0]) + O(h^2),$$
$$\varphi_{\alpha,2} = \left(b_0 - \frac{a_0}{2}hD_2^0\right)\varphi_0 + h(a_0 b_0 L_+^{-1}[1] - a_0 L_-^{-1}[b_0 - \alpha_0\varphi_0]) + O(h^2),$$

which is the final claim in Theorem 1.

4. Stability analysis for the waves. The linearization of (1.2) around the solution φ_α from Theorem 1 is constructed as follows. Set $u = \varphi_\alpha + v, \varphi_\alpha = \varphi_{\alpha,1} + i\varphi_{\alpha,2}, v = v_1 + iv_2$. After ignoring $O(|v|^2)$ terms (and keeping in mind that $\alpha = \alpha_0 h$), we obtain the following system:

$$\begin{pmatrix} -\partial_t v_2 \\ \partial_t v_1 \end{pmatrix} = \begin{pmatrix} -\partial_x^2 + 1 - (6\varphi_{\alpha,1}^2 + 2\varphi_{\alpha,2}^2) & -4\varphi_{\alpha,1}\varphi_{\alpha,2} \\ -4\varphi_{\alpha,1}\varphi_{\alpha,2} & -\partial_x^2 + 1 - (2\varphi_{\alpha,1}^2 + 6\varphi_{\alpha,2}^2) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \alpha\begin{pmatrix} v_2 \\ -v_1 \end{pmatrix}.$$

Introduce the self-adjoint operator (with domain $H^2[-T, T] \times H^2[-T, T]$)

$$\mathcal{N}_h := \begin{pmatrix} -\partial_x^2 + 1 - (6\varphi_{\alpha,1}^2 + 2\varphi_{\alpha,2}^2) & -4\varphi_{\alpha,1}\varphi_{\alpha,2} \\ -4\varphi_{\alpha,1}\varphi_{\alpha,2} & -\partial_x^2 + 1 - (2\varphi_{\alpha,1}^2 + 6\varphi_{\alpha,2}^2) \end{pmatrix}.$$

In the eigenvalue ansatz, $v_j(t, \cdot) \to e^{\lambda t}z_j(\cdot)$, the problem becomes

$$\mathcal{J}\mathcal{N}_h\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (\lambda + \alpha)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{4.1}$$

Introducing $\mu := \lambda + \alpha$, note that (4.1) is a Hamiltonian eigenvalue problem in the form $\mathcal{J}\mathcal{N}_h\vec{z} = \mu\vec{z}$, enjoying all the symmetries afforded by the Hamiltonian structure. Let us record it as

$$\mathcal{J}\mathcal{N}_h\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \mu\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{4.2}$$

In addition, $\lambda = 0$ and $z_1 = \varphi'_{\alpha,1}, z_2 = \varphi'_{\alpha,2}$ is an eigenvalue (of algebraic multiplicity two) for (4.1), in accordance with the translational invariance of the system (1.2).

Our task here is a bit unusual in that we need to make a good distinction between (4.1) and (4.2). More precisely, our goal is to find conditions for (or actually characterize) the waves that are stable, or equivalently, we need to ensure that the eigenvalue problem (4.1) and λ satisfy $\Re\lambda \le 0$. In terms of μ, the stability is equivalent to $\Re\mu \le \alpha$. Here and below, we use the instability index theory developed for eigenvalue problems in the form (4.2), which among other things counts eigenvalues with positive real parts for (4.2). Let us reiterate again that the existence of those does not necessarily mean instability for (4.1), unless $\Re\mu > \alpha = \alpha_0 h$. In fact, we have already one "instability" for (4.2), namely, an eigenvalue $\mu = \alpha$ with e-vector $\binom{\varphi'_{\alpha,1}}{\varphi'_{\alpha,2}}$.

To this end, we need to track the evolution of the eigenvalues at zero, as we turn on h. Before we get on with this task, we need a few preparatory calculations. We need to compute

$$\mathcal{W} := \begin{pmatrix} 6\varphi_{\alpha,1}^2 + 2\varphi_{\alpha,2}^2 & 4\varphi_{\alpha,1}\varphi_{\alpha,2} \\ 4\varphi_{\alpha,1}\varphi_{\alpha,2} & 2\varphi_{\alpha,1}^2 + 6\varphi_{\alpha,2}^2 \end{pmatrix}$$

in powers of h. Expanding in orders of h, we obtain

$$\mathcal{W} = \varphi_0^2 \begin{pmatrix} 2+4a_0^2 & 4a_0b_0 \\ 4a_0b_0 & 2+4b_0^2 \end{pmatrix}$$
$$+ 2h\varphi_0 \begin{pmatrix} 6a_0\Psi_1^0 + 2b_0\Psi_2^0 + 2a_0b_0D_2^0 & 2a_0\Psi_2^0 + 2b_0\Psi_1^0 + (b_0^2 - a_0^2)D_2^0 \\ 2a_0\Psi_2^0 + 2b_0\Psi_1^0 + (b_0^2-a_0^2)D_2^0 & 2a_0\Psi_1^0 + 6b_0\Psi_2^0 - 2a_0b_0D_2^0 \end{pmatrix} + O(h^2).$$

Diagonalizing the matrix

$$\begin{pmatrix} 2+4a_0^2 & 4a_0b_0 \\ 4a_0b_0 & 2+4b_0^2 \end{pmatrix} = S \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} S^{-1}$$

via

$$S = \begin{pmatrix} a_0 & -b_0 \\ b_0 & a_0 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} a_0 & b_0 \\ -b_0 & a_0 \end{pmatrix}$$

leads to the representation

$$\mathcal{W} = S\left[\varphi_0^2\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} + 2h\varphi_0[\Psi_1^0\begin{pmatrix} 6a_0 & -2b_0 \\ -2b_0 & 2a_0 \end{pmatrix} + \Psi_2^0\begin{pmatrix} 6b_0 & 2a_0 \\ 2a_0 & 2b_0 \end{pmatrix} - D_2^0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}]\right] S^{-1} + O(h^2).$$

Upon the introduction of the new variables

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = S^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

and since $S^{-1}\mathcal{J}S = \mathcal{J}$, we can rewrite the eigenvalue problem (4.2) in the form

$$\mathcal{J}\mathcal{M}_h\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \mu\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \tag{4.3}$$

where

$$\mathcal{M}_h = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}$$
$$- 2h\varphi_0\left[\Psi_1^0\begin{pmatrix} 6a_0 & -2b_0 \\ -2b_0 & 2a_0 \end{pmatrix} + \Psi_2^0\begin{pmatrix} 6b_0 & 2a_0 \\ 2a_0 & 2b_0 \end{pmatrix} - D_2^0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] + O(h^2).$$

Note that $\mathcal{N}_h = S\mathcal{M}_hS^{-1}$ and $\mathcal{J}\mathcal{N}_h = S\mathcal{J}\mathcal{M}_hS^{-1}$, so the spectra of the full linearized operator, $\mathcal{J}\mathcal{N}_h$, is equivalent to $\mathcal{J}\mathcal{M}_h$. Also, $\sigma(\mathcal{N}_h)$ is equivalent to $\sigma(\mathcal{M}_h)$.

Since the two problems are equivalent, we note that the form (4.3) of the eigenvalue problem is more suggestive of our approach. For $h = 0$, we have two dimensional Ker$[\mathcal{M}_0]$, spanned by the vectors is[7] $\binom{\varphi_0'}{0}$ and $\binom{0}{\varphi_0}$. We need to see what the evolution of the modulational eigenvalue is as $h : 0 < h << 1$, i.e., the one corresponding to the eigenvector $\binom{0}{\varphi_0}$. This is because, by index counting theory, the instability can only appear in the even subspace of the problem. Also, we can clearly consider $\mathcal{M}_h$ instead of $\mathcal{L}_h$ as the two operators are similar through the matrix S.

[7]Both vectors have one additional generalized eigenvector, so an algebraic multiplicity four at zero.

4.1. Tracking the modulational eigenvalue for $\mathcal{N}_h$ $0 < h << 1$. Since $\mathcal{N}_h$ is similarlily equivalent to $\mathcal{M}_h$, we might as well consider $\mathcal{M}_h$ instead. We set up the following ansatz for the eigenvalue problem for $\mathcal{M}_h$:

$$\mathcal{M}_h \begin{pmatrix} hp_1 \\ \varphi_0 + hp_2 \end{pmatrix} = \sigma h \begin{pmatrix} hp_1 \\ \varphi_0 + hp_2 \end{pmatrix}. \tag{4.4}$$

Using the precise form of $\mathcal{M}_h$, to the leading order h, we have

$$\begin{aligned} &L_+ p_1 + 2\varphi_0^2(2b_0\Psi_1^0 - 2a_0\Psi_2^0 - D_2^0) = 0, \\ &L_- p_2 - 4\varphi_0^2(a_0\Psi_1^0 + b_0\Psi_2^0) = \sigma\varphi_0. \end{aligned}$$

The first equation is resolvable, since $2\varphi_0^2(2b_0\Psi_1^0 - 2a_0\Psi_2^0 - D_2^0)$ is even and hence perpendicular to $\mathrm{Ker}[L_+] = span[\varphi_0']$. The solvability condition for the second one, $\sigma\varphi_0 + 4\varphi_0^2(a_0\Psi_1^0 + b_0\Psi_2^0) \perp \varphi_0$, is what yields the formula for σ_0, namely,

$$\sigma_0 = -4\frac{\langle \varphi_0^2(a_0\Psi_1^0 + b_0\Psi_2^0), \varphi_0\rangle}{\|\varphi_0\|^2} = -4a_0\frac{\langle L_+^{-1}[1], \varphi_0^3\rangle}{\|\varphi_0\|^2} = a_0\frac{\langle 1, \varphi_0\rangle}{\|\varphi_0\|^2}, \tag{4.5}$$

since $a_0\Psi_1^0 + b_0\Psi_2^0 = a_0 L_+^{-1}[1]$ and $L_+^{-1}[\varphi_0^3] = -\frac{1}{4}\varphi_0$.

In conclusion, the modulational eigenvalue at zero for $h = 0$ has moved to the left for $a_0 < 0$, while it moves to the right for $a_0 > 0$.

4.2. Tracking the modulational eigenvalue for $\mathcal{J}\mathcal{N}_h$ $0 < h << 1$. Taking cues from the proof of Proposition 6, we take the following ansatz for the former modulation eigenvalue at zero and its corresponding eigenvector $\left(\begin{smallmatrix} 0 \\ \varphi_0 \end{smallmatrix}\right)$—we take $\mu = \mu_0\sqrt{h}$ and $\left(\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \sqrt{h}q_1 \\ \varphi_0 + hq_2 \end{smallmatrix}\right)$. Plugging this in (4.3), we obtain, after some elementary algebraic manipulations,

$$\begin{pmatrix} L_+ + O(h) & O(h) \\ O(h) & L_- - 4h\varphi_0(a_0\Psi_1^0 + b_0\Psi_2^0) \end{pmatrix} \begin{pmatrix} \sqrt{h}q_1 \\ \varphi_0 + hq_2 \end{pmatrix} = \mu_0\sqrt{h} \begin{pmatrix} -\varphi_0 - hq_2 \\ \sqrt{h}q_1 \end{pmatrix}.$$

Resolving the first equation, to its leading order $\sqrt{h}$, yields the relation $L_+ q_1 = -\mu_0\varphi_0$, or, since L_+ is invertible on φ_0,

$$q_1 = -\mu_0 L_+^{-1}[\varphi_0] + O(h). \tag{4.6}$$

In the second equation, the leading order is h, whence we get the equation

$$L_- q_2 - 4\varphi_0^2(a_0\Psi_1^0 + b_0\Psi_2^0) = \mu_0 q_1 = -\mu_0^2 L_+^{-1}[\varphi_0]. \tag{4.7}$$

This equation is solvable, provided we ensure $4\varphi_0^2(a_0\Psi_1^0 + b_0\Psi_2^0) - \mu_0^2 L_+^{-1}[\varphi_0] \perp \varphi_0$, whence

$$q_2 = L_-^{-1}[4\varphi_0^2(a_0\Psi_1^0 + b_0\Psi_2^0) - \mu_0^2 L_+^{-1}[\varphi_0]].$$

Thus, we have located the former modulational invariance eigenvalue. Namely, it is $\mu_0\sqrt{h}$, where μ_0 ensures $4\varphi_0^2(a_0\Psi_1^0 + b_0\Psi_2^0) - \mu_0^2 L_+^{-1}[\varphi_0] \perp \varphi_0$. Equivalently

$$\mu_0^2 = 4\frac{\langle \varphi_0^2(a_0\Psi_1^0 + b_0\Psi_2^0), \varphi_0\rangle}{\langle L_+^{-1}\varphi_0, \varphi_0\rangle}. \tag{4.8}$$

It remains to compute this last expression. We have

$$\mu_0^2 = 4\frac{\langle \varphi_0^3, a_0\Psi_1^0 + b_0\Psi_2^0\rangle}{\langle L_+^{-1}\varphi_0, \varphi_0\rangle} = 4\frac{a_0}{\langle L_+^{-1}\varphi_0, \varphi_0\rangle}\langle \varphi_0^3, L_+^{-1}[1]\rangle = -a_0\frac{\langle \varphi_0, 1\rangle}{\langle L_+^{-1}\varphi_0, \varphi_0\rangle}.$$

Since $\langle L_+^{-1}\varphi_0, \varphi_0\rangle < 0$, we have that $sgn(\mu_0^2) = sgn(a_0)$. In other words, if $a_0 > 0$, we have the instable mode

$$\mu_0 = \sqrt{\frac{a_0\langle\varphi_0, 1\rangle}{-\langle L_+^{-1}\varphi_0, \varphi_0\rangle}}\sqrt{h} + O(h), \tag{4.9}$$

while for $a_0 < 0$, we have a marginally stable pairs of eigenvalues

$$\pm i\left[\sqrt{\frac{a_0\langle\varphi_0, 1\rangle}{\langle L_+^{-1}\varphi_0, \varphi_0\rangle}}\sqrt{h} + O(h)\right].$$

4.3. Stable and unstable eigenvalues: Putting it together. Before we proceed with our arguments, let us discuss our findings so far.

In the case $a_0 > 0$ (or equivalently $\sigma_0 > 0$), we have an unstable eigenvalue in the form $\mu_0\sqrt{h} + O(h)$, where μ_0 is real and determined from (4.9).

The case where μ_0 is purely imaginary is more complicated and it needs extra arguments, based on our earlier computations of the sign of the eigenvalues and the index theory. We deal with it below. Henceforth, assume $a_0 < 0$. Note that both the even and odd subspaces are invariant under the action of $\mathcal{N}_h$ and $\mathcal{J}\mathcal{N}_h$, so we will consider them separately.

4.3.1. Spectral analysis on the even subspace. In this case, we have established the emergence, from the modulational eigenvalue at $h = 0$ (which has algebraic multiplicity two), of a pair of marginally stable eigenvalues $i[\pm\mu_0\sqrt{h}+O(h)]$. We now compute the Krein index of this marginally stable pair of eigenvalues[8] $\pm i\sqrt{\mu_0}\sqrt{h} + O(h)$.

For a simple pair of eigenvalues, the Krein index coincides with the sign of the expression $\langle\Re\left(\begin{smallmatrix}\sqrt{h}q_1\\ \varphi_0+hq_2\end{smallmatrix}\right), \mathcal{M}_h\Re\left(\begin{smallmatrix}\sqrt{h}q_1\\ \varphi_0+hq_2\end{smallmatrix}\right)\rangle$ (see [10, p. 267]). To this end, realizing from (4.6) that q_1 is purely imaginary to the leading order (in the case under consideration),

$$\begin{aligned}\left\langle\Re\begin{pmatrix}\sqrt{h}q_1\\ \varphi_0+hq_2\end{pmatrix}, \mathcal{M}_h\Re\begin{pmatrix}\sqrt{h}q_1\\ \varphi_0+hq_2\end{pmatrix}\right\rangle &= -4h\langle\varphi_0, \varphi_0^2(a_0\Psi_1^0 + b_0\Psi_2^0)\rangle + O(h^{3/2})\\ = -4a_0h\langle\varphi_0^3, L_+^{-1}[1]\rangle + O(h^{3/2}) &= a_0h\langle\varphi_0, 1\rangle + O(h^{3/2}).\end{aligned}$$

It follows that for $a_0 < 0$, the problem has a pair of marginally stable eigenvalues $\pm i[\sqrt{\mu_0}\sqrt{h} + O(h)]$ with a negative Krein signature.

Recall that $n(\mathcal{N}_0) = n(\mathcal{M}_0) = 1$, which arises in the even subspace. Moreover, $\dim(\mathrm{Ker}[\mathcal{N}_0]) = \dim(\mathrm{Ker}[\mathcal{M}_0]) = 2$, but recall that one of them occurs in the even subspace, namely, $\left(\begin{smallmatrix}0\\ \varphi_0\end{smallmatrix}\right)$, while the other occurs in the odd subspace, namely, $\left(\begin{smallmatrix}\varphi_0'\\ 0\end{smallmatrix}\right)$. In order to alleviate the notation, for a self-adjoint, bounded from below operator M, acting invariantly on the even and odd subspaces, we introduce $M_{even/odd} := M|_{even/oddsubspace}$ and then $n_{even}(M) = n(M_{even})$, and similarly for $n_{odd}(M)$.

We claim that for small enough $h > 0$, $n_{even}(\mathcal{N}_h) = 2$, due to (4.5). Indeed, $n_{even}(\mathcal{N}_0) = n_{even}(L_+) = 1$, according to Proposition 5—this $O(1)$ eigenvalue remains negative, after the perturbation. In addition, as we have shown in (4.5), the modulational eigenvalue at zero for $h = 0$ has become negative. Thus, all potentially negative eigenvalues in the even subspace are accounted for. (Recall the translational

[8] Which is of course relevant computation, only if $b_0 > a_0$.

eigenvalues appear when one considers the action on the odd subspace.) By the index counting formulas

$$
\begin{aligned}
0 \leq n_{unstable,even}(\mathcal{J}\mathcal{N}_h) &\leq n(\mathcal{N}_h) - \#\{\lambda \in i\mathbf{R} : \lambda \text{ has negative Krein signature}\} \\
&\leq n_{even}(\mathcal{N}_h) - 2 = 0,
\end{aligned}
$$

since we have already verified that there is at least one pair of purely imaginary eigenvalues, $\pm i\sqrt{\mu_0}\sqrt{h} + O(h)$ with negative Krein signatures (and even eigenfunctions!). Thus, $n_{unstable,even}(\mathcal{J}\mathcal{N}_h) = 0$, proving that no instability can occur in the even subspace. By the Hamiltonian symmetry considerations, $\sigma(\mathcal{J}\mathcal{N}_h) \subset i\mathbf{R}$ and $\sigma(\mathcal{J}\mathcal{N}_h - \alpha) \subset -\alpha + i\mathbf{R}$. This concludes the analysis on the even subspace, and we have stability there, for $a_0 < 0$.

4.3.2. Spectral analysis on the odd subspace. For the odd subspace, the index count is somewhat simpler. Recall there is the "unstable" eigenvalue at $\mu = \alpha = \alpha_0 h$ for the eigenvalue problem (4.3), which corresponds to the translational invariance; see (4.2) and the discussion immediately after. So, this implies that $n_{unstable,odd}(\mathcal{J}\mathcal{N}_h) \geq 1$. On the other hand, initially $n_{odd}(\mathcal{N}_0) = n_{odd}(\mathcal{M}_0) = 0$. So, $n_{odd}(\mathcal{N}_h) \leq 1$ if the translational eigenvalue has moved to the left, or if it has moved to the right, then[9] $n_{odd}(\mathcal{N}_h) = 0$. Applying index counting theory, however, we have that on the other hand $n_{odd}(\mathcal{N}_h) \geq n_{unstable,odd}(\mathcal{J}\mathcal{N}_h)$, whence

$$1 \leq n_{unstable,odd}(\mathcal{J}\mathcal{N}_h) \leq n_{odd}(\mathcal{N}_h) \leq 1.$$

In particular, $1 = n_{unstable,odd}(\mathcal{J}\mathcal{N}_h) = n_{odd}(\mathcal{N}_h)$, and so the translational eigenvalue has moved to the left to become a negative eigenvalue for $\mathcal{N}_{h,odd}$. In addition, the "unstable eigenvalue" is exactly the one computed explicitly, namely, $\mu = \alpha$, and there are no other unstable eigenvalues in the odd subspace. Note that by the Hamiltonian symmetry the eigenvalues of (4.3) are symmetric with respect to the imaginary axes, so there is another, stable one at $\mu = -\alpha$. We mention that there is a lot of point spectrum on the imaginary axes as well, since the problem is of periodic nature and the resolvents are compact operators, forcing the discreteness of the spectrum.

In terms of the original spectral variables, by combining the conclusions for the even and odd subspaces, we obtain that if $\lambda \in \sigma(\mathcal{J}\mathcal{N}_h - \alpha)$, then

$$\lambda = \mu - \alpha \subset \{\{\alpha\} \cup \{-\alpha\} \cup \{\lambda : \Re\lambda = 0\}\} - \alpha = \{0\} \cup \{-2\alpha\} \cup \{\lambda : \Re\lambda = -\alpha\}.$$

This is exactly the statement of Theorem 2 and its proof is now complete.

Appendix A. Tracking the translational eigenvalue for $\mathcal{N}_h$ $0 < h << 1$. Again, we work with $\mathcal{M}_h$, since $\mathcal{N}_h$ and $\mathcal{M}_h$ are equivalent. We set up the following ansatz for the eigenvalue problem for $\mathcal{M}_h$:

$$\mathcal{M}_h \begin{pmatrix} \varphi_0' + hp_1 \\ hp_2 \end{pmatrix} = \sigma h \begin{pmatrix} \varphi_0' + hp_1 \\ hp_2 \end{pmatrix}. \tag{A.1}$$

In other words,

$$\begin{pmatrix} L_+ - 12h\varphi_0(a_0\Psi_1^0 + b_0\Psi_2^0) & O(h) \\ 2h\varphi_0(2b_0\Psi_1^0 - 2a_0\Psi_2^0 - D_2^0) & L_- + O(h) \end{pmatrix} \begin{pmatrix} \varphi_0' + hp_1 \\ hp_2 \end{pmatrix} = \sigma h \begin{pmatrix} \varphi_0' + hp_1 \\ hp_2 \end{pmatrix}.$$

[9]See the appendix for a calculation involving the direction of the move, which turns out to be inconclusive to its leading order, so this calculation is not sufficient to establish $n(\mathcal{N}_h) = 1$.

To the leading order h, we have the equations

$$L_+p_1 - 12\varphi_0(a_0\Psi_1^0 + b_0\Psi_2^0)\varphi_0' = \sigma\varphi_0',$$
$$L_-p_2 + 2\varphi_0(2b_0\Psi_1^0 - 2a_0\Psi_2^0 - D_2^0)\varphi_0' = 0.$$

The second equation always has a solution as $2\varphi_0(2b_0\Psi_1^0 - 2a_0\Psi_2^0 - D_2^0)\varphi_0'$ is an odd function, so it is perpendicular to $\mathrm{Ker}[L_-] = span[\varphi_0]$.

The first equation requires the solvability condition $\sigma\varphi_0' + 12\varphi_0(a_0\Psi_1^0 + b_0\Psi_2^0)\varphi_0' \perp \varphi_0'$. Noting that $a_0\Psi_1^0 + b_0\Psi_2^0 = a_0 L_+^{-1}[1]$, we derive the formula for σ_0,

$$\sigma_0 = -12\frac{\langle\varphi_0\varphi_0'(a_0\Psi_1^0 + b_0\Psi_2^0), \varphi_0'\rangle}{\|\varphi_0'\|^2} = -12\frac{a_0}{\|\varphi_0'\|^2}\langle L_+^{-1}[1], \varphi_0(\varphi_0')^2\rangle. \tag{A.2}$$

This quantity can be computed explicitly; in fact $\langle L_+^{-1}[1], \varphi_0(\varphi_0')^2\rangle = 0$, see Proposition 8 below. So $\sigma_0 = 0$. Thus, our analysis is not precise enough[10] to compute explicitly the next order term. As we have shown above, with index counting calculations, it turns out that this eigenvalue must have moved to the left (of order $o(h)$) to become a negative one.

Appendix B. Computation of the relevant quantities involving L_+^{-1}.

PROPOSITION 8. *In the setup of Proposition* 5*, we have the following formulas:*

$$\langle L_+^{-1}\varphi_0, \varphi_0\rangle < 0, \tag{B.1}$$
$$\langle L_+^{-1}[1], \varphi_0\rangle = 0, \tag{B.2}$$
$$\langle L_+^{-1}[1], \varphi_0(\varphi_0')^2\rangle = 0, \tag{B.3}$$
$$\int_0^T \frac{(2 - 6\varphi_0^2(y))((\varphi_0'(y))^2 - (\varphi_0''(y))^2)}{((\varphi_0'(y))^2 + (\varphi_0''(y))^2)^2} dy \neq 0. \tag{B.4}$$

Proof. For the proof of (B.1), recall the basic properties at $h = 0$. In this case (1.5) has a solution

$$\varphi_0(x) = \alpha dn(\alpha x, \kappa),$$

where $\kappa^2 = \frac{2\alpha^2 - 1}{\alpha^2}$. Also $L_+ = -\partial_x^2 + 1 - 2(\varphi_0)^2$. Since $L_+\varphi_0' = 0$, the function

$$\psi(x) = \varphi_0'(x)\int^x \frac{1}{\varphi_0'^2(s)} ds, \quad \begin{vmatrix} \varphi_0' & \psi \\ \varphi_0'' & \psi' \end{vmatrix} = 1$$

is also solution of $L_+\psi = 0$. Formally, since φ_0' has zeros using the identities

$$\frac{1}{cn^2(y,\kappa)} = \frac{1}{dn(y,\kappa)}\frac{\partial}{\partial_y}\frac{sn(x,\kappa)}{cn(y,\kappa)}, \quad \frac{1}{sn^2(y,\kappa)} = -\frac{1}{dn(y,\kappa)}\frac{\partial}{\partial_y}\frac{cn(x,\kappa)}{sn(y,\kappa)}$$

and integrating by parts we get

$$\psi(x) = \frac{1}{\alpha^2\kappa^2}\left[\frac{1 - 2sn^2(\alpha x,\kappa)}{dn(\alpha x,\kappa)} - \alpha\kappa^2 sn(\alpha x,\kappa)cn(\alpha x,\kappa)\int_0^x \frac{1 - 2sn^2(\alpha s,\kappa)}{dn^2(\alpha s,\kappa)} ds\right].$$

[10]We do not have the precise asymptotic expressions of order $O(h^2)$ above, although this is in principle possible, after heavy computations.

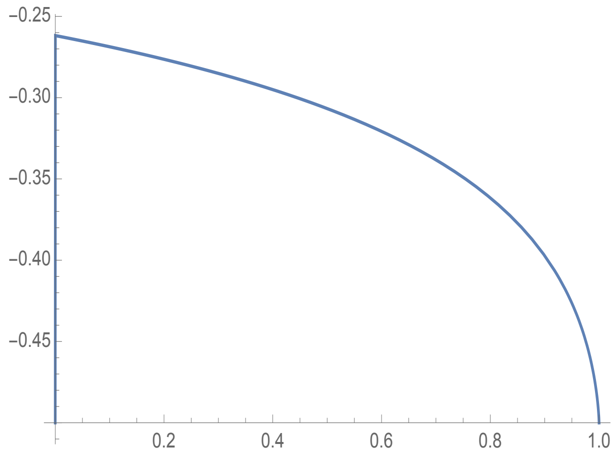

FIG. 1. *The function* $k \to \frac{E^2(\kappa)-(1-\kappa^2)K^2(\kappa)}{2[2(1-\kappa^2)K(\kappa)-(2-\kappa^2)E(\kappa)]}$.

Thus, we may construct the Green function

$$L_+^{-1} f = \varphi_0' \int_0^x \psi(s) f(s) ds - \psi(s) \int_0^x \varphi_0'(s) f(s) s + C_f \psi(x),$$

where C_f is chosen such that $L_{+,0}^{-1} f$ is periodic with the same period as $\varphi_0(x)$. After integrating by parts, we get

$$\langle L_+^{-1}\varphi_0, \varphi_0\rangle = -\langle \varphi_0^3, \psi\rangle + \frac{\varphi_0^2(T)+\varphi_0^2}{2}\langle \varphi_0, \psi\rangle + C_{\varphi_0}\langle \varphi_0, \psi\rangle. \tag{B.5}$$

We have

$$\begin{aligned} &\langle \varphi_0, \psi\rangle = \tfrac{1}{\alpha^2\kappa^2}[E(\kappa)-K(\kappa)], \\ &\langle \varphi_0^3, \psi\rangle = \tfrac{1}{2\kappa^2}[(2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)], \\ &C_{\varphi_0} = -\tfrac{\varphi_0''(T)}{2\psi(T)}\langle \varphi_0, \psi\rangle + \tfrac{\varphi_0^2(T)-\varphi_0^2(0)}{2}. \end{aligned} \tag{B.6}$$

With this finally we get

$$\langle L_+^{-1}\varphi_0, \varphi_0\rangle = \frac{E^2(\kappa)-(1-\kappa^2)K^2(\kappa)}{2[2(1-\kappa^2)K(\kappa)-(2-\kappa^2)E(\kappa)]} < 0.$$

See Figure 1.

For the proof of (B.2), we have

$$L_+^{-1}[1] = \varphi_0' \int_0^x \psi(s) ds - \psi(x) \int_0^x \varphi_0'(s) ds,$$

and

$$\langle L_+^{-1}[1], \varphi_0\rangle = \frac{\varphi_0^2(T)}{2}\int_{-T}^T \psi(x)dx - \frac{3}{2}\int_{-T}^T \psi(x)\varphi_0^2(x)dx + [C_1+\varphi_0(0)]\int_{-T}^T \psi(x)\varphi_0(x)dx,$$

where

$$C_1 = \varphi_0(T) - \varphi(0) - \frac{\varphi_0''(T)}{\psi'(T)} \int_0^T \psi(x)dx.$$

Using that $\frac{d}{dx} dn(x) = -\kappa^2 sn(x)cn(x)$ and integrating be parts, we get

$$\begin{aligned}
&\int_{-T}^{T} \psi(x)\varphi_0(x)dx \\
&= \frac{2}{\kappa^2}\left[\frac{2}{3}\int_0^T dn(\alpha x)(1-2sn^2(\alpha x))dx + \frac{1}{3}dn^3(\alpha T)\int_0^T \frac{1-2sn^2(\alpha x)}{dn^2(\alpha x)}dx\right] \\
&= \frac{2}{3\kappa^2}dn^3(K)\int_0^T \frac{1-2sn^2(\alpha x)}{dn^2(\alpha x)}dx.
\end{aligned}$$

Similarly, integrating by parts

$$\int_{-T}^{T} \psi(x)dx = \frac{2dn(\alpha T)}{\alpha^2\kappa^2}\int_0^T \frac{1-2sn^2(\alpha x)}{dn^2(\alpha x)}dx.$$

Thus,

$$\frac{\varphi_0^2(T)}{2}\int_{-T}^{T}\psi(x)dx - \frac{3}{2}\int_{-T}^{T}\psi(x)\varphi_0^2(x)dx = 0.$$

Using that $\varphi_0''(T) = \alpha^3\kappa^2\sqrt{1-\kappa^2}$, and $\psi'(T) = \sqrt{1-\kappa^2}\int_0^T \frac{1-2sn^2(\alpha x)}{dn^2(\alpha x)}dx$, we get

$$C_1 + \varphi_0(0) = \varphi_0(T) - \frac{\varphi_0''(T)}{\psi'(T)}\int_0^T \psi(x) = 0.$$

For the proof of (B.3),

$$\begin{aligned}
\langle L_+^{-1}[1], \varphi_0(x)(\varphi_0'(x))^2\rangle &= \int_{-T}^{T}\varphi_0(x)(\varphi_0'(x))^3\int_0^x \psi(s)dsdx - \int_{-T}^{T}\psi(x)\varphi_0^2(x)(\varphi_0'(x))^2dx \\
&\quad + (C_1 + \varphi_0(0))\int_{-T}^{T}\psi(x)\varphi_0(x)(\varphi_0'(x))^2dx.
\end{aligned}$$

Again integrating by parts, we get

$$\int_0^x \psi(s)ds = \frac{dn(\alpha x)}{\alpha^2\kappa^2}\int_0^x \frac{1-2sn^2(\alpha s)}{dn^2(\alpha s)}ds,$$

and

$$\begin{aligned}
&\int_{-T}^{T}\varphi_0(x)(\varphi_0'(x))^3\int_0^x \psi(s)dsdx - \int_{-T}^{T}\psi(x)\varphi_0^2(x)(\varphi_0'(x))^2dx \\
&= \int_{-T}^{T}\varphi_0(x)(\varphi_0'(x))^2\left[\varphi_0'(x)\int_0^x \psi(s)ds - \psi(x)\varphi_0(x)\right]dx \\
&= -\alpha^3\kappa^2\int_0^{K(\kappa)}(1-2sn^2(x))sn^2(x)cn^2(x)dn(x)dx = 0.
\end{aligned}$$

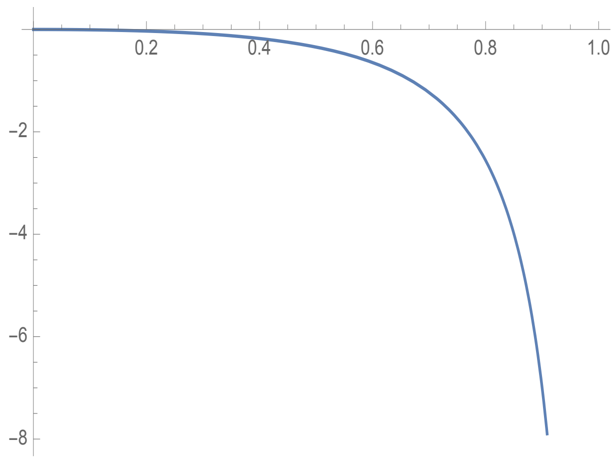

FIG. 2. *The function* $k \to 2K(k) - \frac{(2-k^2)}{1-k^2}E(k)$.

Now, using that $\kappa^2 = \frac{2\alpha^2-1}{\alpha^2}$, we get

$$\int_0^T \frac{(2-6\varphi_0^2(y))((\varphi_0'(y))^2-(\varphi_0''(y))^2)}{((\varphi_0'(y))^2+(\varphi_0''(y))^2))^2}dy$$
$$= \int_0^{K(\kappa)} \frac{\left(2-\frac{6}{2-\kappa^2}dn^2(y)\right)\left(sn^2(y)cn^2(y)-\frac{1}{2-\kappa^2}[cn^2(y)-sn^2(y)]^2dn^2(y)\right)}{\left[sn^2(y)cn^2(y)+\frac{1}{2-\kappa^2}[cn^2(y)-sn^2(y)]^2dn^2(y)\right]^2}dy.$$

Using Mathematica, we are able to compute this last expression explicitly

$$\int_0^T \frac{(2-6\varphi_0^2(y))((\varphi_0'(y))^2-(\varphi_0''(y))^2)}{((\varphi_0'(y))^2+(\varphi_0''(y))^2))^2}dy = 2K(k) - \frac{(2-k^2)}{1-k^2}E(k) < 0,$$

as is clear from Figure 2. ∎

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