A POSITIVITY PHENOMENON IN ELSER'S GAUSSIAN-CLUSTER PERCOLATION MODEL

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ABSTRACT. Veit Elser proposed a random graph model for percolation in which physical dimension appears as a parameter. Studying this model combinatorially leads naturally to the consideration of numerical graph invariants which we call *Elser numbers* $\mathsf{els}_k(G)$, where G is a connected graph and k a nonnegative integer. Elser had proven that $\mathsf{els}_1(G) = 0$ for all G. By interpreting the Elser numbers as Euler characteristics of appropriate simplicial complexes called *nucleus complexes*, we prove that for all graphs G, they are nonpositive when $k = 0$ and nonnegative for $k \geqslant 2$. The last result confirms a conjecture of Elser. Furthermore, we give necessary and sufficient conditions, in terms of the 2-connected structure of G, for the nonvanishing of the Elser numbers.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a connected undirected graph and $k \geqslant 0$ an integer. A **nucleus** of G is a connected subgraph $N \subseteq G$ such that $V(N)$ is a **vertex cover**; that is, every edge of G has at least one endpoint in $V(N)$. Let $\mathcal{N}(G)$ denote the set of all nuclei of G. The $\boldsymbol{k}^{\mathbf{th}}$ **Elser number** of G is

$$\mathsf{els}_k(G) = (-1)^{|V(G)|+1} \sum_{N \in \mathcal{N}(G)} (-1)^{|E(N)|} |V(N)|^k \tag{1}$$

This invariant was introduced by Veit Elser [Els84], who conjectured [Els10] that $\mathsf{els}_k(G) \geqslant 0$ for all graphs G and integers $k \geqslant 2$. In this paper, we answer completely the question of when $\mathsf{els}_k(G)$ is positive, negative or zero.

Theorem 1.1. *Let G be a connected graph with at least two vertices. Then:*

(a) $\mathsf{els}_0(G) \leqslant 0$.
(b) $\mathsf{els}_1(G) = 0$.
(c) $\mathsf{els}_k(G) \geqslant 0$ *for all integers $k \geqslant 2$. That is, Elser's conjecture holds.*

Part (b) is [Els84, Theorem 2]. Theorem 1.1 extends [Els84, Theorem 2] to all k. We also extend the previous result: a characterization of strict positivity of the Elser numbers.

Theorem 1.2. *Let G be a connected simple graph.*

(a) If G is 2-connected, then $\mathsf{els}_0(G) < 0$, $\mathsf{els}_1(G) = 0$, and $\mathsf{els}_k(G) > 0$ for all $k \geqslant 2$.
(b) Otherwise, $\mathsf{els}_k(G) \neq 0$ if and only if $k \geqslant \ell$, where $\ell \geqslant 2$ is the number of leaves in the block-cutpoint tree of G (that is, the number of 2-connected components of G that contain exactly one cut-vertex).

Before describing the methods of proof, we describe the motivation behind Elser's conjecture, which arises in percolation theory. Roughly speaking, percolation models a physical medium by a random graph Γ, often taken to be a subgraph of $\mathbb{Z}^2$ or some other periodic lattice. Vertices or edges occur independently with some fixed probability, corresponding to the presence or absence of atoms or bonds between them, and the permeability of the medium is modeled by the component structure of the graph. For an overview

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of percolation theory, see the excellent expository article by Kesten [Kes06]. In many percolation models, the ambient graph (such as $\mathbb{Z}^2$) controls the combinatorics so strongly that one cannot consider physical dimension as a parameter of the model, but must study different dimensions as separate problems.

For this reason, Elser [Els84] proposed a percolation model in which dimension can be treated as a parameter, following work of Gaunt and Fischer [FG64] and Leibbrandt [Lei75]. Elser's model starts with a random geometric graph model consisting of a collection of N points uniformly distributed throughout a D-dimensional volume V. The edge between two points z_1, z_2 occurs with probability $\exp(-a\|z_1 - z_2\|^2)$, where a is some fixed constant. Let $n_k = n_k(a, V, z_1, \ldots, z_N)$ be the expected number of *k-clusters*, or connected components with k vertices. Using a property of Gaussian integrals due to Kirchhoff [Kir47], Elser expanded the generating function for the numbers n_k as

$$\mathsf{F}(x,y) = \sum_{k=1}^{\infty} y^k n_k = \sum_{m=1}^{\infty} \frac{x^{m-1}}{m!} \left(\sum_{G \in \mathscr{C}_m} \left(\frac{1}{\tau(G)} \right)^{D/2} W(G,y) \right)$$

[Els84, eqn. (6)], where $\mathscr{C}_m$ denotes the set of simple connected graphs on m labeled vertices; $\tau(G)$ the number of spanning trees of G; and

$$W(G,y) = \sum_{N \in \mathcal{N}(G)} (-1)^{|E(G)|-|E(N)|} y^{|V(N)|}$$

where as before $\mathcal{N}(G)$ is the set of nuclei of G. What we call the k^{th} Elser number equals $(yD_y)^k W(G,y)|_{y=1}$, where D_y means differentiation with respect to y.

We prove Elser's conjecture using techniques from topological combinatorics. Our general approach is to interpret the numbers $\mathsf{els}_k(G)$ as sums of reduced Euler characteristics $\tilde{\chi}(\Delta_U^G)$. Here Δ_U^G is a simplicial complex whose faces correspond to nuclei containing a specified set U of vertices. The precise formula is given by Theorem 3.4 below. While the topology of these simplicial complexes remains mysterious in most cases, it is nonetheless possible to establish a deletion/contraction-type recurrence for their reduced Euler characteristics (Theorem 5.2) and thus to determine precisely the sign of $\tilde{\chi}(\Delta_U^G)$, which turns out to be just $(-1)^{|E(G)|+|V(G)|}$ (Theorem 1.1). The upshot is that every summand in the expression for $\mathsf{els}_k(G)$ in Theorem 3.4 is nonnegative, proving Theorem 1.1.

The paper is structured as follows. In Section 2 we set up notation for graphs and simplicial complexes, and give basic definitions and facts about nuclei and Elser numbers. The proofs of the main theorems occupy Sections 3–7 of the paper. In Section 3, we construct the simplicial complexes Δ_U^G and prove the main theorems linking the Elser numbers to their Euler characteristics and establishing the deletion-contraction recurrence. Using these tools, we then prove Elser's conjecture for trees in Section 4 and for general graphs in Section 6. Theorem 1.2 is proved in Section 7. Its proof requires a purely graph-theoretic result (Theorem 7.2) that strengthens the standard result that every 2-connected graph has an ear decomposition, and may be of independent interest. Section 8 proves a monotonicity result: $\mathsf{els}_k(G) \geqslant \mathsf{els}_k(G/e) + \mathsf{els}_k(G\backslash e)$ for all G and e, with equality when $k = 0$. We conclude in Section 9 with observations and conjectures on the topology of nucleus complexes, which appear to have a rich structure.

2. Preliminaries

As a general reference for the graph theory necessary for this paper, we refer the reader to [Wes96, Sections 1.1, 1.3, 2.1, 4.1, 4.2]. Throughout, "graph" means "undirected graph." The vertices and edges of a graph are denoted by $V(G)$ and $E(G)$ respectively. For $A \subseteq E(G)$, we write $\overline{A} = E(G)\backslash A$ for the complement of A, if the ambient graph G is clear from context.

Two edges are **parallel** if they have the same pair of endpoints (or are both loops incident to the same vertex). We write $\mathrm{Par}(e)$ for the equivalence class of all edges parallel to e. The **deparallelization** $\mathrm{Dep}(G)$ is the graph obtained from G by identifying all edges in the same parallel class.

The **deletion** of an edge e from G is the graph $G\backslash e$ with vertex set $V(G)$ and edge set $E(G)\backslash\{e\}$. The **contraction** of e in G is the graph G/e obtained by removing e and identifying its endpoints v, w into a single vertex (denoted vw). A **minor** of G is a graph obtained by some sequence of deletions and contractions, i.e., of the form $G/C\backslash D$, where C, D are disjoint subsets of $E(G)$. Every $U \subseteq V(G)$ gives rise to a set

$U/e \subseteq V(G/e)$, for an edge $e = \{u, v\}$, defined by

$$U/e = \begin{cases} U & \text{if } v, w \notin U, \\ U \backslash \{v, w\} \cup \{vw\} & \text{otherwise.} \end{cases}$$

This notation can be iterated; if $C = \{e_1, \ldots, e_k\} \subseteq E(G)$, then we set $U/C = (U/e_1)/e_2/\cdots$; the order of contraction does not matter. If $H = G/C\backslash D$ is a minor of G, then we write $U[H]$ for U/C.

A **vertex cover** of G is a set $C \subseteq V(G)$ such that every edge $e \in E(G)$ has at least one endpoint in C. In particular, if G has a loop at vertex v, then every vertex cover of G must contain v. Notice that the vertex covers of $\mathrm{Dep}(G)$ are the same as those of G.

Definition 2.1. A **nucleus** of G is a connected subgraph N of G whose vertices $V(N)$ form a vertex cover of G. We denote the set of nuclei of G by $\mathcal{N}(G)$.

Note that Elser assumed that G is simple, which is most natural from a physical point of view; however, we do not make this assumption, since non-simple graphs will naturally arise.

Proposition 2.2. *Let $N \in \mathcal{N}(G)$. Let $C \subseteq V(G)$, and suppose $G\backslash C$ is disconnected. Then $V(N) \cap C \neq \varnothing$. In particular, $V(N)$ contains all cut vertices of G.*

Proof. Suppose $V(N) \cap C$ is empty. Since $V(N)$ is a vertex cover, $V(N)$ must contain all neighbors of vertices in C. In particular, N contains two vertices in different components of $G\backslash C$. But since $N = N\backslash C$, this implies N is not connected, a contradiction. $\square$

Example 2.3. The complete graph K_2 on two vertices has three nuclei: itself and its two one-vertex subgraphs. Therefore,

$$\mathsf{els}_k(K_2) = (-1)^{2+1} \sum_{N \in \mathcal{N}(K_2)} (-1)^{|E(N)|} |V(N)|^k = -1(1 + 1 - 2^k) = 2^k - 2.$$

Example 2.4. For many standard graphs, it is easy to determine their nuclei and Elser numbers.

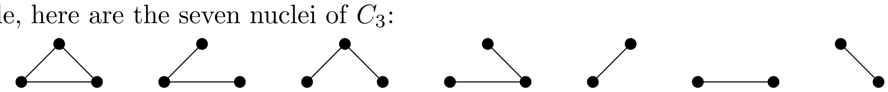
(a) Let T be a tree with $n \geqslant 3$ vertices. Then its nuclei are precisely the subgraphs obtained by deleting some set of leaf vertices. In particular, if T has ℓ leaves, then it has 2^ℓ nuclei. Moreover, if L is the set of leaf vertices in T, then

$$\mathsf{els}_k(T) = (-1)^{n+1} \sum_{J \subseteq L} (-1)^{n-|J|-1} |V(T)\backslash J|^k = \sum_{j=0}^{\ell} (-1)^{\ell+j} \binom{\ell}{j} (n-j)^k.$$

(b) As a special case, for $n \geqslant 3$, the n-vertex path P_n has four nuclei: itself and the paths obtained by deleting one or both endpoints. So:

$$\mathsf{els}_k(P_n) = n^k - 2(n-1)^k + (n-2)^k.$$

(c) The cycle graph C_n has precisely $2n + 1$ nuclei: itself, the n copies of P_n obtained by deleting a single edge, and the n copies of P_{n-1} obtained by deleting a single vertex and its two incident edges. For example, here are the seven nuclei of C_3:



Thus the Elser numbers are:

$$\begin{aligned} \mathsf{els}_k(C_n) &= (-1)^{n+1} \Big((-1)^n n^k + n \left((-1)^{n-1} n^k \right) + n \left((-1)^{n-2} (n-1)^k \right) \Big) \\ &= n(n-1) \left(n^{k-1} - (n-1)^{k-1} \right). \end{aligned}$$

When $n = 3$, this reduces to $\mathsf{els}_k(C_3) = 6\ (3^{k-1} - 2^{k-1})$.

We will study the nuclei of a graph using the language of simplicial complexes, which we now introduce. An **(abstract) simplicial complex** Δ on a set X is a collection of subsets of X such that

(i) $\varnothing \in \Delta$;
(ii) If $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$.

The elements of a simplicial complex are called its **faces**. A **subcomplex** Δ' of a complex Δ is a subcollection of Δ which satisfies (i) and (ii). A simplicial complex Δ is said to be a **cone** with **cone point** $\{x\} \in \Delta$ if for every face $\sigma \in \Delta$ we have $\sigma \cup \{x\} \in \Delta$. Note that every cone is contractible.

The **(reduced) Euler characteristic** of a simplicial complex Δ is

$$\tilde{\chi}(\Delta) = \sum_{\sigma\in\Delta} (-1)^{\dim\sigma} = \sum_{n\geqslant 0} (-1)^n \dim_{\mathbb{R}}(\tilde{H}_n(\Delta;\mathbb{R}))$$

where $\tilde{H}_n$ denotes reduced simplicial homology. Thus $\tilde{\chi}(\Delta) = 0$ if Δ is contractible (in particular, if Δ is a cone). For further details on simplicial homology, we refer the reader to [Hat02, Section 2.1].

3. Nucleus complexes

In this section, we study the ***U*-nucleus complexes** Δ_U^G of a graph G for $U \subseteq V(G)$, consisting of complements of nuclei whose vertex support contains U. The k^{th} Elser number of G may be written as a weighted sum of Euler characteristics of nucleus complexes. This reduces Elser's conjecture to understanding the Euler characteristics of nucleus complexes. The key result is a deletion-contraction recurrence for these Euler characteristics (Theorem 5.2).

Elser notes the following identity [Els84, Proof of Theorem 2] :

$$\begin{aligned}\mathsf{els}_1(G) &= (-1)^{|V(G)|+1} \sum_{N\in\mathcal{N}(G)} (-1)^{|E(N)|}|V(N)| \\ &= (-1)^{|V(G)|+1} \sum_{v\in V(G)} \sum_{\substack{N\in\mathcal{N}(G):\\ v\in V(N)}} (-1)^{|E(N)|}\end{aligned}$$

This identity allowed Elser to characterize $\mathsf{els}_1(G)$ for any G. We give a more general identity, which works for any $k \geqslant 0$. Let $\mathrm{Sur}(a, b)$ denote the number of surjections from a set of size a to a set of size b. (We adopt the conventions that $\mathrm{Sur}(a, b) = 0$ if exactly one of a, b is zero, and $\mathrm{Sur}(0, 0) = 1$.)

Theorem 3.1. *Let G be a graph and k a nonnegative integer. Then*

$$\mathsf{els}_k(G) = (-1)^{|E(G)|+|V(G)|+1} \sum_{U\subseteq V(G)} \mathrm{Sur}(k, |U|) \sum_{\substack{N\in\mathcal{N}(G):\\ U\subseteq V(N)}} (-1)^{|E(\overline{N})|}.$$

Proof. The term $|V(N)|^k$ counts functions $[k] \to V(N)$, and such a function is the same thing as a surjection from $[k]$ to some subset of $V(N)$. Therefore,

$$\begin{aligned}\mathsf{els}_k(G) &= (-1)^{|V(G)|+1} \sum_{N\in\mathcal{N}(G)} (-1)^{|E(N)|} \sum_{U\subseteq V(N)} \mathrm{Sur}(k, |U|) \\ &= (-1)^{|E(G)|+|V(G)|+1} \sum_{U\subseteq V(G)} \mathrm{Sur}(k, |U|) \sum_{\substack{N\in\mathcal{N}(G):\\ U\subseteq V(N)}} (-1)^{|E(\overline{N})|}.\end{aligned}$$ □

We find it convenient to rephrase Theorem 3.1 in terms of the Euler characteristics of certain simplicial complexes.

Definition 3.2. Let G be a connected graph with $|V(G)| \geqslant 3$, and let $U \subseteq V(G)$. The ***U*-nucleus complex of *G*** is the simplicial complex $\Delta_U^G = \{E(G)\backslash E(N) : \ N \in \mathcal{N}(G),\ V(N) \supseteq U\}$. The set Δ_U^G is a simplicial complex because every graph obtained by adding edges to a nucleus is also a nucleus.

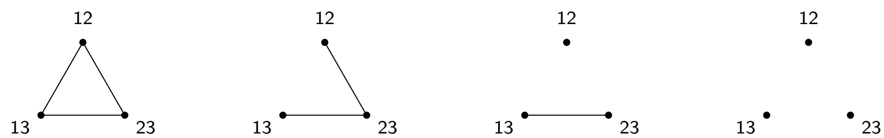
The case $|V(G)| = 2$ requires special treatment. Consider the graph $G = cK_2$ with two vertices v_1, v_2 and $c > 0$ parallel edges. The subtlety here is that cK_2 has two distinct nuclei with the same edge sets, namely the subgraphs N_1, N_2 with $V(N_i) = \{v_i\}$ and $E(N_i) = \varnothing$. Accordingly, we define $\Delta_{\varnothing}^{cK_2}$ to be the Δ-complex consisting of two $(c-1)$-dimensional simplices σ_1, σ_2 on vertex set $E(cK_2)$, glued along their boundaries. Each simplex σ_i should be regarded as recording the complement in $E(cK_2)$ of $E(N_i)$. While this construction is artificial, it is necessary to preserve the correspondence between nuclei of G and faces of $\Delta^G_\varnothing$. For $U \neq \varnothing$, we can define $\Delta^{cK_2}_U$ just as in Definition 3.2: in particular,

$$\Delta^{cK_2}_\varnothing \cong \mathbb{S}^{c-1}, \qquad \Delta^{cK_2}_{\{v_1\}} = \sigma_1, \qquad \Delta^{cK_2}_{\{v_2\}} = \sigma_2, \qquad \Delta^{cK_2}_{\{v_1,v_2\}} = \partial\sigma_1 = \partial\sigma_2 \cong \mathbb{S}^{c-2}, \tag{2}$$

(where $\mathbb{S}^k$ means the k-dimensional sphere), so that

$$\tilde{\chi}(\Delta^{cK_2}_\varnothing) = (-1)^{c-1}, \qquad \tilde{\chi}(\Delta^{cK_2}_{\{v_1\}}) = 0, \qquad \tilde{\chi}(\Delta^{cK_2}_{\{v_2\}}) = 0, \qquad \tilde{\chi}(\Delta^{cK_2}_{\{v_1,v_2\}}) = (-1)^c. \tag{3}$$

Example 3.3. Label the vertices of K_3 as 1, 2, 3 and its edges as 12, 13, 23. The nuclei of K_3 are shown in Example 2.4(c). Accordingly, its nucleus complexes $\Delta^{K_3}_U$ are as shown in the following figure. Up to isomorphism, the complex $\Delta^{K_3}_U$ depends only on $|U|$.



$U = \varnothing$	$U = \{1\}$	$U = \{1,2\}$	$U = \{1,2,3\}$
$\tilde{\chi}(\Delta^{K_3}_U) = -1$	$\tilde{\chi}(\Delta^{K_3}_U) = 0$	$\tilde{\chi}(\Delta^{K_3}_U) = 1$	$\tilde{\chi}(\Delta^{K_3}_U) = 2$

The inner sum over nuclei in Theorem 3.1 is just $-\tilde{\chi}(\Delta^G_U)$, so we can rewrite Theorem 3.1 to give a formula for Elser numbers in terms of Euler characteristics:

Theorem 3.4. *Let G be a graph and $k \geqslant 0$ an integer. Then*

$$\mathsf{els}_k(G) = (-1)^{|E(G)|+|V(G)|} \sum_{U \subseteq V(G)} \mathrm{Sur}(k, |U|)\, \tilde{\chi}(\Delta^G_U).$$

Nucleus complexes are well-behaved with respect to loops and cut-edges, at least at the level of Euler characteristic. Let $G \neq K_2$ be a graph and $e \in E(G)$ be a cut-edge. If one of the endpoints x of e has degree 1, we say that e is a **leaf edge (with leaf x)**.

Proposition 3.5. *Let G be a graph and $U \subseteq V(G)$.*

(a) If G has a loop ℓ, then Δ^G_U is a cone with cone point ℓ. In particular, $\tilde{\chi}(\Delta^G_U) = 0$.
(b) Let $D = \mathrm{Dep}(G)$. Then $\tilde{\chi}(\Delta^D_U) = (-1)^{|E(G)|-|E(D)|}\tilde{\chi}(\Delta^G_U)$.
(c) Let $G \neq K_2$ and let e be a cut-edge of G, and let $U \subseteq V(G)$. If e is a leaf edge with leaf x and $x \notin U$, then Δ^G_U is a cone. Otherwise, $\Delta^G_U = \Delta^{G/e}_{U/e}$.

Proof. (a) Let $s \in V(G)$ be the vertex incident to ℓ. The vertex set of every nucleus $N \in \mathcal{N}(G)$ must contain s, regardless of U. So for all $N \in \mathcal{N}(G)$, let N' be the subgraph of G induced by the edges $E(G) \cup \{e\}$. Then $N \in \mathcal{N}(G)$ and it follows that Δ^G_U is a cone with cone point ℓ. Since every cone is contractible, the Euler characteristic is zero.

(b) By induction, it suffices to show that if a, b are parallel edges in G and $G' = G - b$, then

$$\tilde{\chi}(\Delta^{G'}_U) = -\tilde{\chi}(\Delta^G_U)$$

for every $U \subseteq V(G)$. Let

$$\mathcal{A}_0 = \{A \in \Delta^G_U : \ a, b \notin A\}, \qquad \mathcal{A}_a = \{A \in \Delta^G_U : \ a \in A,\ b \notin A\},$$
$$\mathcal{A}_b = \{A \in \Delta^G_U : \ a \notin A,\ b \in A\}, \qquad \mathcal{A}_{ab} = \{A \in \Delta^G_U : \ a, b \in A\}.$$

Toggling a gives a bijection between $\mathcal{A}_b$ and $\mathcal{A}_{ab}$, so

$$\begin{aligned}\tilde{\chi}(\Delta_U^G) &= \sum_{A\in\mathcal{A}_0}(-1)^{E(G)-|A|}+\sum_{A\in\mathcal{A}_a}(-1)^{E(G)-|A|}+\sum_{A\in\mathcal{A}_b}(-1)^{E(G)-|A|}+\sum_{A\in\mathcal{A}_{ab}}(-1)^{E(G)-|A|}\\ &= \sum_{A\in\mathcal{A}_0}(-1)^{E(G)-|A|}+\sum_{A\in\mathcal{A}_a}(-1)^{E(G)-|A|}\\ &= \sum_{\substack{A\in\Delta_U^{G'}\\ a\notin A}}(-1)^{|E(G')|+1-|A|}+\sum_{\substack{A\in\Delta_U^{G'}\\ a\in A}}(-1)^{|E(G')|+1-|A|}\\ &= -\tilde{\chi}(\Delta_U^{G'}).\end{aligned}$$

(c) Suppose the cut-edge e is not a leaf edge. Then it must belong to every nucleus in G, because every vertex cover must include at least one vertex from each component of $G-e$. On the other hand, every nucleus N in G/e must include at least one edge in each cut-component of the fused vertex xy, and since N is connected we must have $xy\in V(N)$. Therefore, $\Delta_U^G=\Delta_{U/e}^{G/e}$ for all U.

Now, suppose that e is a leaf edge with leaf x, say $e=\{x,y\}$ with $\deg_G(x)=1$. Every nucleus must include y in its vertex set, and toggling e does not change whether an edge set is a nucleus. Therefore, if $x\notin U$, then Δ_U^G is a cone with cone point e, hence has reduced Euler characteristic 0. If $x\in U$ then every U-nucleus must include the edge e, so $\Delta_U^G=\Delta_{U/e}^{G/e}$. □

Corollary 3.6. *In the language of Elser numbers:*

(a) If G contains a loop, then $\mathsf{els}_k(G)=0$ for all k.
(b) For all G and k, $\mathsf{els}_k(G)=\mathsf{els}_k(\mathrm{Dep}(G))$.

4. Elser numbers for trees

Let T be a tree with $n\geqslant 3$ vertices. Let L be the set of leaf vertices in T, and let $\ell=|L|$. Recall from Example 2.4 that

$$\mathsf{els}_k(T)=(-1)^{n+1}\sum_{J\subseteq L}(-1)^{n-|L|+|J|-1}|(V(T)\backslash L)\cup J|^k=\sum_{j=0}^{\ell}(-1)^{\ell+j}\binom{\ell}{j}(n-\ell+j)^k.$$

This formula has the disadvantage that its sign is not obvious. On the other hand, we can use Theorem 3.4 to give an formula for $k\geqslant 1$ which is obviously nonnegative.

Proposition 4.1. *Let T be a tree with two or more vertices, let $U\subseteq V(T)$, and let L denote the set of leaves of T. Then:*

$$\tilde{\chi}(\Delta_U^T)=\begin{cases}1 & \text{if } T=K_2 \text{ and } |U|=0,\\ 0 & \text{if } T=K_2 \text{ and } |U|=1,\\ -1 & \text{if } T=K_2 \text{ and } |U|=2,\\ 0 & \text{if } T\neq K_2 \text{ and } L\nsubseteq U,\\ -1 & \text{if } T\neq K_2 \text{ and } L\subseteq U.\end{cases}$$

Proof. The first three cases are a restatement of (3). On the other hand, suppose that $|V(T)|\geqslant 3$. If $L\nsubseteq U$, Proposition 3.5 (c) implies Δ_U^T is a cone and therefore $\tilde{\chi}(\Delta_U^T)=0$. When $L\subseteq U$, the only connected subgraph of T containing L is T itself. Thus $\Delta_U^G=\{\varnothing\}$ and so the reduced Euler characteristic is -1. □

Ultimately we will reduce the general graph problem to the case of tree graphs, so Proposition 4.1 will be crucial for the proof of Elser's conjecture. We now give a formula for $\mathsf{els}_k(T)$ when T is a tree.

Corollary 4.2. *Let $k\geqslant 1$. Let T be a tree with n vertices and ℓ leaves. Then*

$$\mathsf{els}_k(T)=\sum_{i=0}^{n-\ell}\binom{n-\ell}{i}\mathrm{Sur}(k,\ell+i)$$

In particular, Elser's conjecture is true for trees. That is, $\mathsf{els}_k(T)\geqslant 0$.

Proof. Theorem 3.4 gives

$$\mathsf{els}_k(T) = (-1)^{|E(T)|+|V(T)|} \sum_{U \subseteq V(T)} \mathrm{Sur}(k, |U|)\ \tilde{\chi}(\Delta_U^T).$$

Since T is a tree, we have $|E(T)| + |V(T)| = 2|V(T)| - 1$ and thus

$$\begin{aligned}\mathsf{els}_k(T) &= -\sum_{U \subseteq V(T)} \mathrm{Sur}(k, |U|)\ \tilde{\chi}(\Delta_U^T) \\ &= \sum_{U \subseteq V(T)} \mathrm{Sur}(k, |U|)\ \left[-\tilde{\chi}(\Delta_U^T)\right].\end{aligned}$$

Let L denote the set of leaves of T. By Proposition 4.1, for $U \neq \varnothing$,

$$-\tilde{\chi}(\Delta_U^T) = \begin{cases} 0 & \text{if } L \nsubseteq U, \\ 1 & \text{if } L \subseteq U \end{cases}.$$

Then

$$\begin{aligned}\mathsf{els}_k(T) &= \sum_{\substack{U \subseteq V(T) \\ L \subseteq U}} \mathrm{Sur}(k, |U|) \\ &= \sum_{i=0}^{|V(T)|-|L|} \binom{|V(T)| - |L|}{i} \mathrm{Sur}(k, |L| + i).\end{aligned}$$

□

5. A deletion-contraction recurrence for nucleus complexes

In this section, we develop a deletion-contraction recurrence for the reduced Euler characteristic of nucleus complexes of an arbitrary connected graph G. In the general, the main technical tool is a simple bijection ψ relating the nucleus complexes of G, G/e, and $G \backslash e$. Special care must be taken for small graphs, because of the difficulty in defining the nucleus complex of cK_2.

Define a map $\psi : 2^{E(G)} \to 2^{E(G \backslash e)} \mathbin{\dot\cup} 2^{E(G/e)}$ as follows:

$$\psi(A) = \begin{cases} A \backslash e \subseteq E(G \backslash e) & \text{if } e \in A, \\ A \subseteq E(G/e) & \text{if } e \notin A. \end{cases}$$

Note that ψ sends complements of nuclei to complements of nuclei and ψ is a bijection with inverse given by

$$\psi^{-1}(B) = \begin{cases} B \cup e & \text{for } B \subseteq E(G \backslash e), \\ B & \text{for } B \subseteq E(G/e). \end{cases}$$

Notice that we have not assumed that G is a simple graph, only that e is not a loop. In particular, ψ is well-defined and a bijection even if there is another edge in G with the same endpoints as e. The key technical properties of ψ we will need are as follows.

Proposition 5.1. *Let G be a graph, let $e = xy$ be a non-loop edge of G, and let $U \subseteq V(G)$. Assume that either (i) $|V(G)| \geqslant 4$, or (ii) $|V(G)| = 3$ and $U \neq \varnothing$. Then:*

$$\psi\left(\Delta_U^G\right) \subseteq \Delta_U^{G \backslash e} \mathbin{\dot\cup} \Delta_{U/e}^{G/e}, \text{ and} \tag{4}$$

$$\{B \in \Delta_U^{G \backslash e} : \psi^{-1}(B) \notin \Delta_U^G\} = \{B \in \Delta_{U/e}^{G/e} : \psi^{-1}(B) \notin \Delta_U^G\}. \tag{5}$$

The assumption in the proposition avoids the difficulties in defining $\Delta_\varnothing^{cK_2}$, which can arise from contractions if $|V(G)| \geqslant 3$.

Proof. To prove (4), let $A \in \Delta_U^G$. If $e \in A$, then $\psi(A) = A \backslash e$ and so $E(G \backslash e) \backslash \psi(A) = E(G \backslash e) \backslash (A \backslash e) = E(G) \backslash A$ is a U-nucleus of G not containing e. On the other hand, if $e \notin A$ then $e \in E(G) \backslash A$ and $\psi(A) = A$, so

$$E(G \backslash e) \backslash \psi(A) = E(G \backslash e) \backslash A.$$

Since contraction preserves connectedness and the property of being a vertex cover, the set $E(G\backslash e)\backslash A$ is a $(U\backslash e)$-nucleus. Thus

$$\psi(A) \in \begin{cases} \Delta_U^{G\backslash e} & \text{if } e \in A \\ \Delta_U^{G/e} & \text{if } e \notin A, \end{cases}$$

which proves (4).

To prove (5), suppose that $B \subseteq E(G)\backslash e$, so that B can be regarded as a set of edges of any of G, $G\backslash e$, or G/e. Let $\hat{B} = E(G\backslash e)\backslash B = E(G)\backslash(B \cup e)$, let $W = V_G(\hat{B})$, and let $W' = V_{G/e}(\hat{B})$. Then the following conditions are equivalent:

(a) $B \in \Delta_U^{G\backslash e}$ and $\psi^{-1}(B) \notin \Delta_U^G$.
(b) $B \in \Delta_U^{G\backslash e}$ and $B \cup e \notin \Delta_U^G$.
(c) $\hat{B}$ is a U-nucleus of $G\backslash e$, but not a U-nucleus of G.
(d) $\hat{B}$ is connected in $G\backslash e$; $W \supseteq U$; and W is a vertex cover of $G\backslash e$, but not of G.
(e) $\hat{B}$ is connected in $G\backslash e$; $W \supseteq U$; W is a vertex cover of $G\backslash e$; and $x, y \notin W$.
(f) $\hat{B}$ is connected in G/e; $W' \supseteq U$; W'/e is a vertex cover of G/e; and $x, y \notin W'$.
(g) $\hat{B}$ is a U-nucleus of G/e and $\hat{B} \cup e$ is a disconnected subgraph of G.
(h) $\hat{B}$ is a U-nucleus of G/e and $\hat{B} \cup e$ is not a U-nucleus of G.
(i) $B \in \Delta_{U/e}^{G/e}$ and $B \notin \Delta_U^G$.
(j) $B \in \Delta_{U/e}^{G/e}$ and $\psi^{-1}(B) \notin \Delta_U^G$.

Most of these equivalences are self-explanatory. For (d) $\iff$ (e), first note that if $\hat{B}$ is connected in $G\backslash e$ then it is connected in G/e; on the other hand, if $x, y \notin W'$ and $\hat{B}$ is connected in G/e then $\hat{B}$ is connected in $G\backslash e$. For the equivalence (f) $\iff$ (g), the forward direction is immediate; the $\Longleftarrow$ direction follows from the observation that $V_G(\hat{B} \cup e)$ is a vertex cover, so $\hat{B} \cup e$ were connected then it would be a nucleus. □

Now we state and prove the main deletion/contraction recurrence.

Theorem 5.2. *Let G be an arbitrary connected graph with $|V(G)| \geqslant 2$. Let $e \in E(G)$ be neither a loop or cut-edge, and let $U \subseteq V(G)$. Then*

$$\tilde{\chi}(\Delta_U^G) = \tilde{\chi}(\Delta_{U/e}^{G/e}) - \tilde{\chi}(\Delta_U^{G\backslash e}).$$

Proof. If G has a loop, then the recurrence is trivially true by Proposition 3.5 (a).

If G has another edge parallel to e (so that contracting e produces a loop), then $\tilde{\chi}(\Delta_{U/e}^{G/e}) = 0$ and $\tilde{\chi}(\Delta_U^G) = -\tilde{\chi}(\Delta_U^{G\backslash e})$ by Proposition 3.5 (b), implying the recurrence.

If $n(G) = 2$ and G has no loop, then $G = cK_2$ for some $c \geqslant 1$. If $c = 1$ then no such edge e exists and the theorem is vacuously true. If $c > 1$ then G/e has a loop, so $\tilde{\chi}(\Delta_{U/e}^{G/e}) = 0$ by Proposition 3.5 (a), and the desired recurrence reduces to $\tilde{\chi}(\Delta_U^G) = -\tilde{\chi}(\Delta_U^{G\backslash e}) = 0$, which follows from (3).

One more case requires special handling. Suppose that $\operatorname{Dep}(G) = K_3$ and $U = \varnothing$ (so that Proposition 5.1 does not apply), and that no other edges are parallel to e. Let a and b be the sizes of the other two parallel classes; note that $a, b > 0$. Then $G/e = (a+b)K_2$ and $G - e$ is a graph whose deparallelization is a 3-vertex path. By Proposition 3.5 (b) together with Proposition 4.1 and Example 2.4 (c), we have

$$\tilde{\chi}(\Delta_\varnothing^G) = (-1)^{a+b-1}, \qquad \tilde{\chi}(\Delta_\varnothing^{G/e}) = (-1)^{a+b-1}, \qquad \tilde{\chi}(\Delta_\varnothing^{G\backslash e}) = 0,$$

so the desired recurrence is satisfied.

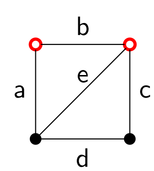
In all other cases, the pair G, U satisfies the hypothesis of Proposition 5.1, so

$$
\begin{aligned}
\tilde{\chi}(\Delta_U^G) &= \sum_{A \in \Delta_U^G} (-1)^{|A|+1} \\
&= \sum_{\substack{A \in \Delta_U^G: \\ e \notin A}} (-1)^{|A|+1} + \sum_{\substack{A \in \Delta_U^G: \\ e \in A}} (-1)^{|A|+1} \\
&= \sum_{\substack{A \in \Delta_U^G: \\ e \notin A}} (-1)^{|\psi(A)|+1} - \sum_{\substack{A \in \Delta_U^G: \\ e \in A}} (-1)^{|\psi(A)|+1} \\
&= \sum_{\substack{B \in \Delta_{U/e}^{G/e}: \\ \psi^{-1}(B) \in \Delta_U^G}} (-1)^{|B|+1} - \sum_{\substack{B \in \Delta_U^{G \setminus e}: \\ \psi^{-1}(B) \in \Delta_U^G}} (-1)^{|B|+1} && \text{(by (4))} \\
&= \left(\sum_{B \in \Delta_{U/e}^{G/e}} (-1)^{|B|+1} - \sum_{\substack{B \in \Delta_{U/e}^{G/e}: \\ \psi^{-1}(B) \notin \Delta_U^G}} (-1)^{|B|+1} \right) - \left(\sum_{B \in \Delta_U^{G \setminus e}} (-1)^{|B|+1} - \sum_{\substack{B \in \Delta_U^{G \setminus e}: \\ \psi^{-1}(B) \notin \Delta_U^G}} (-1)^{|B|+1} \right) \\
&= \sum_{B \in \Delta_{U/e}^{G/e}} (-1)^{|B|+1} - \sum_{B \in \Delta_U^{G \setminus e}} (-1)^{|B|+1} && \text{(by (5))} \\
&= \tilde{\chi}(\Delta_{U/e}^{G/e}) - \tilde{\chi}(\Delta_U^{G \setminus e}). && \square
\end{aligned}
$$

6. Proof of Theorem 1.1

In this section, we combine Theorem 3.4, Proposition 4.1, and Proposition 5.2 (the deletion-contraction recurrence) to prove Elser's conjecture for all connected graphs. The idea is to repeatedly apply Proposition 5.2 to edges that are neither loops nor cut-edges, so as to write $\tilde{\chi}(\Delta_U^G)$ as a signed sum of expressions $\tilde{\chi}(\Delta_{U_i}^{T_i})$. It will turn out that the signs in this sum are all the same, which will imply immediately that the Elser numbers $\mathsf{els}_k(G)$ are positive for all $k \geqslant 2$. This computation can be recorded by a binary tree, which we call a **restricted deletion/contraction tree**, or RDCT. To illustrate this idea, we begin with an example.

Example 6.1. Let G be the graph shown below, with the subset $U \subseteq V(G)$ indicated by hollow red circles.



We can calculate $\tilde{\chi}(\Delta_U^G)$ by repeated applications of Theorem 5.2. One possible set of minors of G obtained from the recurrence is recorded by the RDCT $\mathscr{B}$ shown in Figure 1. The non-leaf nodes of $\mathscr{B}$ are the minors H with no loops and at least one non-cut-edge s; the left and right children are $H \setminus s$ and H/s respectively. The identity of s in each case should be clear from the diagram. The vertex sets $U[H]$ is indicated by hollow red circles; observe that changing the original subset $U \subseteq V(G)$ would change the sets $U[H]$, but not the graphs H themselves. The recurrence stops when it reaches a graph T that is either a tree, in which case Theorem 5.2 does not apply, or has a loop, so that $\tilde{\chi}(\Delta_U^T) = 0$ for all U by Proposition 3.5(1). These graphs are precisely the leaves of $\mathscr{B}$.

For the tree shown in Figure 1, we obtain

$$\tilde{\chi}(\Delta_U^G) = \sum_{i=1}^{7} \tilde{\chi}(\Delta_{U_i}^{T_i})$$

where the T_i are the leaves of $\mathscr{B}$ and $U_i = U[T_i]$. The graphs T_3, T_6, T_7 have loops and the summands therefore vanish. In the other cases, by Theorem 5.2, each deletion changes the sign of the Euler characteristic and each contraction preserves the sign. So the sign of the i^{th} term is $(-1)^{d_i}$, where d_i is the number of edge deletions required to obtain T_i from G. In this case, $d_i = 2$ for every i, so all the signs are positive; as we will shortly see, this is not an accident.

There are many other possibilities for the RDCT $\mathscr{B}$, depending on which non-cut-edge is chosen at each branch. Nevertheless, any RDCT for G can be used to compute $\tilde{\chi}(\Delta_U^G)$ for any U, by giving rise to an equation of the form

$$\tilde{\chi}(\Delta_U^G) = \sum_{i=1}^{s} \varepsilon_i \; \tilde{\chi}(\Delta_{U[T_i]}^{T_i}) \tag{6}$$

where $\{T_1, \ldots, T_s\}$ is a family of tree minors of G, all with at least two vertices and $\varepsilon_i \in \{\pm 1\}$ for each i. The left-hand side is clearly independent of the choice of RDCT; we will see another non-obvious invariant of all RDCTs in Proposition 7.1. In all cases, $|U[T_i]| \leqslant |U|$, and $U[T_i] = \varnothing$ if and only if $U = \varnothing$. Moreover, each T_i has at least two vertices because K_1 cannot be obtained from a larger simple graph by deleting or

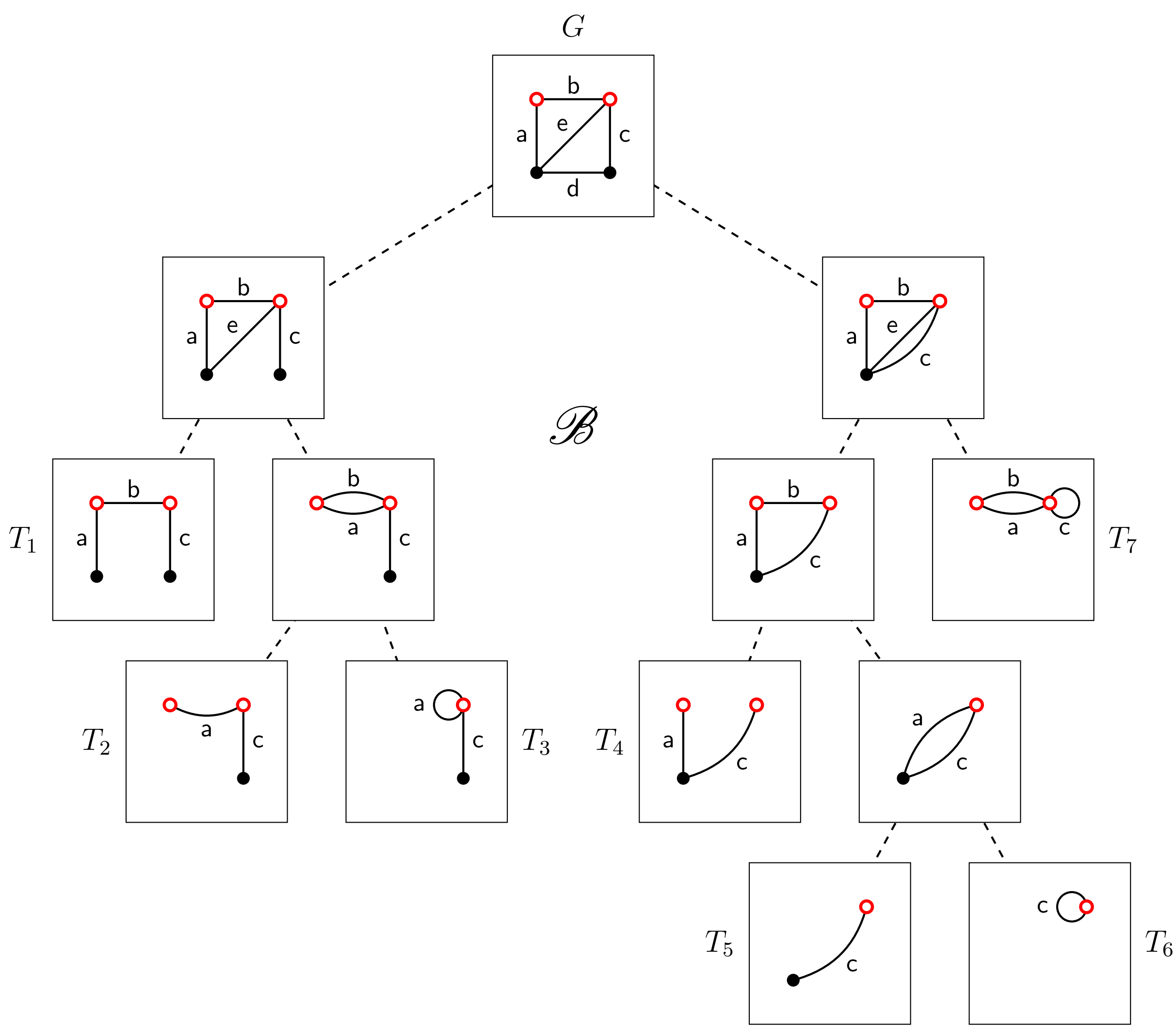


FIGURE 1. A restricted deletion/contraction tree (RDCT).

contracting a non-cut-edge. Similarly, every maximal sequence consisting of only contractions will eventually contain a loop, when the corresponding summand in (6) is 0.

Proposition 6.2. *In every expression of the form (6) arising from an RDCT for G, we have $\varepsilon_i = (-1)^{|E(G)|-|V(G)|+1}$ for all $i \in [s]$. That is,*

$$\tilde{\chi}(\Delta_U^G) = (-1)^{|E(G)|-|V(G)|+1} \sum_{i=1}^{s} \tilde{\chi}(\Delta_{U[T_i]}^{T_i}). \tag{7}$$

Proof. Fix $i \in [s]$ and let $T = T_i$. Obtaining T as a minor of G requires removing a total of $|E(G)| - |E(T)|$ edges via either deletion or contraction. The number of edges contracted must be $|V(G)| - |V(T)|$, because deletion preserves the number of vertices while contraction reduces it by 1. Therefore, the number of edges deleted is $(|E(G)| - |E(T)|) - (|V(G)| - |V(T)|)$. Since T is a tree this equals $|E(G)| - |V(G)| + 1$ and so $|V(T)| = |E(T)| - 1$. The recurrence of Theorem 5.2 implies that the sign $\varepsilon(T)$ is the number of edges deleted. ☐

Remark 6.3. Proposition 6.2 can also be proved topologically. Consider the homology group $H_1(G; \mathbb{R}) \cong \mathbb{R}^{|E|-|V|+1}$ (where G is regarded as a 1-dimensional cell complex). Each edge contraction is a homotopy equivalence, hence preserves H_1, while each edge deletion lowers the rank of H_1 by one. Since $H_1(T; \mathbb{R}) = 0$, the number of deletions must be $|E| - |V| + 1$.

Using these tools, we can now determine the sign of the Elser numbers $\mathsf{els}_k(G)$ for all G and k. The cases $k = 0$, $k = 1$, and $k \geqslant 2$ need to be treated separately.

Theorem 6.4. *For any graph G with two or more vertices and any $U \subseteq V(G)$, we have*

(a) *If $U = \varnothing$, then $(-1)^{|E(G)|+|V(G)|}\tilde{\chi}(\Delta_U^G) \leqslant 0$.*
(b) *If $|U| = 1$, then $(-1)^{|E(G)|+|V(G)|}\tilde{\chi}(\Delta_U^G) = 0$. (This is [Els84, Lemma 1].)*
(c) *If $|U| \geqslant 1$, then $(-1)^{|E(G)|+|V(G)|}\tilde{\chi}(\Delta_U^G) \geqslant 0$.*

Proof. Let $\mathscr{B}$ be an RDCT for G, with leaves labeled by tree minors $T_1, \ldots, T_s$. Then (7) may be rewritten as

$$(-1)^{|E(G)|-|V(G)|}\tilde{\chi}(\Delta_U^G) = \sum_{i=1}^{s} -\tilde{\chi}(\Delta_{U[T_i]}^{T_i}).$$

By Prop. 4.1, the summand $-\tilde{\chi}(\Delta_{U[T_i]}^{T_i})$ equals -1 only if $U[T_i] = \varnothing$ (so $U = \varnothing$), and it equals $+1$ only if $|U[T_i]| \geqslant 2$ (so $|U| \geqslant 2$ as well). ☐

Now we are ready to prove our main theorem:

Theorem 1.1. *Let G be a connected graph with at least two vertices, and let $k \geqslant 0$ be an integer. Then:*

(a) *If $k = 0$, then $\mathsf{els}_0(G) \leqslant 0$.*
(b) *If $k = 1$, then $\mathsf{els}_k(G) = 0$. (This is [Els84, Theorem 2].)*
(c) *If $k > 1$, then $\mathsf{els}_k(G) \geqslant 0$. That is, Elser's conjecture holds.*

Proof. By Theorem 3.4, we have

$$\mathsf{els}_k(G) = \sum_{U \subseteq V(G)} \operatorname{Sur}(k, |U|)\, (-1)^{|E(G)|+|V(G)|}\, \tilde{\chi}(\Delta_U^G) \tag{8}$$

and by Theorem 6.4 all summands are nonpositive, zero, or nonnegative according as $k = 0$, $k = 1$, or $k > 1$, implying the result. ☐

7. Proof of Theorem 1.2

We now consider the question of exactly when the inequalities in (a) and (c) of Theorem 1.1 are strict; equivalently, when $\mathsf{els}_k(G) \neq 0$. We will treat the cases $k = 0$ and $k = 2$ somewhat separately. Recall from Corollary 3.6 that if G contains a loop, then $\mathsf{els}_k(G) = 0$ for all k, and that $\mathsf{els}_k(G) = \mathsf{els}_k(\operatorname{Dep}(G))$. Therefore, we lose no generality by assuming throughout this section that G is simple. We begin with the combinatorial interpretation of Elser numbers that can be extracted from the work of the previous section.

Proposition 7.1. *Let G be a connected graph with at least two vertices, and let $\mathscr{B}$ be any RDCT for G, with leaves labeled $T_1, \ldots, T_s$. Then:*

(a) $\mathsf{els}_0(G) = -\#\{i : T_i \cong K_2\}$.

(b) For $k \geqslant 2$, the following are equivalent:
- $\mathsf{els}_k(G) > 0$*;*
- *there exists some $U \subseteq V(G)$ such that $|U| \leqslant k$ and $\tilde{\chi}(\Delta_U^G) \neq 0$;*
- *there exists a tree minor T_i of G, occurring as a leaf of $\mathscr{B}$, such that $|L(T_i)| \leqslant k$.*

Proof. Substituting (7) into (8) gives

$$\mathsf{els}_k(G) = -\sum_{U \subseteq V(G)} \mathrm{Sur}(k, |U|) \sum_{i=1}^{s} \tilde{\chi}(\Delta_{U[T_i]}^{T_i}). \tag{9}$$

When $k = 0$, equation (9) simplifies to

$$\mathsf{els}_0(G) = -\sum_{i=1}^{s} \tilde{\chi}(\Delta_{\varnothing}^{T_i})$$

which, together with Proposition 4.1, implies part (a).

When $k \geqslant 2$, all nonzero summands in (9) must have $1 \leqslant |U| \leqslant k$ (so that $\mathrm{Sur}(k, |U|) \neq 0$) and $U[T_i] \supseteq L(T_i)$ (by Proposition 4.1). In particular $|L(T_i)| \leqslant k$. On the other hand, if T_i is a tree minor with $\leqslant k$ leaves occurring as a leaf node of $\mathscr{B}$, then one can pull $L[T_i]$ back under the surjection $V(G) \to V(T_i)$ to obtain a set $U \subseteq V(G)$ with $U[T_i] = L(T_i)$ and $|U| = |U[T_i]|$, so (9) does indeed have a nonzero summand. $\square$

Proposition 7.1 is unsatisfactory in that it depends on the choice of a restricted deletion/contraction tree for G. We wish to remove this dependence and give a criterion for nonvanishing that depends only on G itself. Accordingly, the next goal is to show that *every* tree minor of G appears as a leaf of *some* RDCT.

We begin by recalling some of the theory of 2-connected graphs; see, e.g., [Wes96, chapter 4.2]. An **ear decomposition** of a graph G is a list of subgraphs $R_1, \ldots, R_m$ such that

(1) $E(G) = E(R_1) \cup \cdots \cup E(R_m)$;
(2) R_1 is a cycle; and
(3) for each $i > 1$, the graph R_i is a path that meets $R_1 \cup \cdots \cup R_{i-1}$ only at its endpoints.

It is known that G is 2-connected if and only if it has an ear decomposition ([Wes96, Thm. 4.2.8], attributed to Whitney). Most graphs have many ear decompositions; for instance, R_0 can be taken to be any cycle in G. It is easily seen that $m = |E(G)| - |V(G)| + 1$, the number of edges in the complement of a spanning tree T, suggesting that it ought to be possible to construct an ear decomposition by (essentially) adding a fundamental cycle of T in every iteration, where a fundamental cycle for T consists of an edge $xy \notin T$ together with the unique path in T from x to y. The following result, to our knowledge, has not appeared in the literature.

Theorem 7.2. *Let G be a 2-connected graph and let $T \subseteq G$ be a spanning tree. Then G has an ear decomposition $R_1 \cup \cdots \cup R_m$ such that $|E(R_i) \backslash T| = 1$ for every i.*

Proof. We construct the desired ear decomposition by an algorithm that we will first describe informally. For the cycle R_1, we can take any fundamental cycle with respect to T (that is, an edge outside T together with the unique path in T between its endpoints). At the i^{th} step of the algorithm, we will have constructed a 2-connected graph $G_i = R_1 \cup \cdots R_i$ such that $G_i \cap T$ is a spanning tree of G_i (these conditions are loop invariants of the algorithm). The algorithm then identifies an edge $e \notin T$ each of whose endpoints can be joined to G_i by (possibly trivial) paths in T; these two paths together with e form the ear R_{i+1}.

Here is the precise algorithm, including observations that justify its correctness.

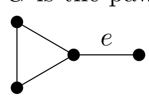
- **Initialization:** Let $i = 1$, let R_1 be any fundamental cycle of T, and let $G_1 = R_1$.
- **Loop** while $G_i \subsetneq G$:
  - If there exists an edge $e \in (G \backslash T) \backslash G_i$ with both endpoints in $V(G_i)$, then let $R_{i+1} = \{e\}$.
  - Otherwise:
    - Let x be a vertex in $V(G_i)$ with at least one neighbor outside $V(G_i)$.

    * Let T' be the subtree of T consisting of all paths that start at x and take their next step into $V(G_i)$.
    * Let T'' be the subtree of T consisting of all paths that start at x and take their next step outside $V(G_i)$.
    * Then $E(T)$ is the disjoint union of $E(T')$ and $E(T'')$, and $V(T') \cap V(T'') = \{x\}$.
    * There must be some edge $e = yz \in E(G)$ with one endpoint y in $V(T')\backslash\{x\}$ and one endpoint z in $V(T'')\backslash\{x\}$, otherwise x would be a cut-vertex of G.
    * In fact, $e \notin T$, since T' contains a path P_y from y to x and T'' contains a path P_z from x to z.
    * Let P' be the shortest subpath of P_y from y to a vertex in $V(G_i)$, and let $P'' = P_z$.
    * Set $R_{i+1} = P' \cup \{e\} \cup P''$.
- In either case, R_{i+1} is a path containing exactly one edge of T and that meets G_i only in its endpoints.
- Therefore, the graph $G_{i+1} = G_i \cup R_{i+1}$ is 2-connected. Moreover, it has $i+1$ ears, each of which contains exactly one edge outside T. Since T is acyclic, it follows that $E(G_{i+1}) \cap T$ is a spanning tree.
- Increment i and repeat.

□

Remark 7.3. The algorithm outlined in this proof is essentially equivalent to an algorithm sketched by Fedor Petrov on MathOverflow [Pet19] in response to a question by one of the authors.

A **tree minor** of G is a nontrivial tree of the form $G/C - D$, where C and D are subsets of $E(G)$. Note that C must be acyclic, and using $G/C - D$ is a tree it can be deduced that $|D| = |E(G)| - |V(G)| + 1$. For every RDCT $\mathscr{B}$ of G, every leaf of $\mathscr{B}$ is a tree minor. It is not true in general that every tree minor of G actually occurs in some binary tree $\mathscr{B}$, because the order of removing edges has to be arranged to avoid contracting a cut-edge or loop. For example, if G is the paw graph



then the tree minor consisting of the cut-edge e alone cannot appear as a leaf of $\mathscr{B}$. On the other hand, when G is 2-connected it is possible to achieve every tree minor.

Proposition 7.4. *Let G be 2-connected and let $G/C - D$ be any tree minor, where $C, D \subseteq E(G)$. Then it is possible to contract the edges of C and delete the edges of D in an order such that one never contracts a cut-edge or deletes a loop. Therefore, some RDCT of G contains $G/C - D$ as a leaf.*

Proof. Let $F = E(G)\backslash C\backslash D$. Then F is acyclic; in fact, $T = C \cup F$ must be a spanning tree of G since it is homotopy-equivalent to the tree $G/C - D = (C \cup F)/C$. By Theorem 7.2, there exists an ear decomposition $G = R_1 \cup \cdots \cup R_m$ such that $|R_i\backslash(C \cup F)| = |R_i \cap D| = 1$ for all i.

We show by induction on m that it is possible to order the contractions and deletions as desired. For the base case $m = 1$, then $G = R_1$ is an n-cycle and $|D| = |E(G)| - |V(G)| + 1 = 1$. First contract the edges in C, of which there can be at most $n-2$, to produce a smaller cycle G', then delete the edge in D which is not a cut edge.

If $m \geqslant 2$, first contract the edges of $C \cap R_m$. The result is a (possibly non-simple) graph consisting of $G_{m-1} = R_1 \cup \cdots \cup R_{m-1}$ with one additional edge (in D) joining the endpoints of R_m. That edge is not a cut-edge, so we can delete it, leaving the 2-connected graph G_{m-1}, and we are done by induction. □

This last observation yields an immediate answer to the question of when $\mathsf{els}_0(G) \neq 0$ (equivalently, by Theorem 1.1, when $\mathsf{els}_0(G) < 0$).

Theorem 7.5. *Let G be a connected simple graph. Then:*

- *If G is 2-connected, then $\mathsf{els}_0(G) < 0$.*
- *Otherwise, $\mathsf{els}_0(G) = 0$.*

Proof. The "otherwise" case is Theorem 1 of [Els84]. If G is 2-connected, then let H be any K_2 minor. By Prop 7.4, H appears as a leaf in some RDCT for G, so $\mathsf{els}_0(G) < 0$. □

At this point, we have proven Theorem 1.2 in the cases that $k = 0$ and $k = 1$. Accordingly, we assume throughout the rest of the section that $k \geqslant 2$ and $U \neq \varnothing$.

The **join** of two simplicial complexes Γ_1, Γ_2 on disjoint vertex sets is the simplicial complex $\Gamma_1 * \Gamma_2 = \{\sigma_1 \cup \sigma_2 : \ \sigma_1 \in \Gamma_1, \ \sigma_2 \in \Gamma_2\}$. A routine calculation shows that $\tilde{\chi}(\Gamma_1 * \Gamma_2) = -\tilde{\chi}(\Gamma_1)\tilde{\chi}(\Gamma_2)$. In particular, $\tilde{\chi}(\Gamma_1 * \Gamma_2) = 0$ if and only if $\tilde{\chi}(\Gamma_1)$ or $\tilde{\chi}(\Gamma_2)$ is zero.

Proposition 7.6. *Let G be a connected graph with a cut-vertex v. Let G_1, G_2 be connected subgraphs of G, each a union of cut-components of G with respect to v, such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{v\}$. Let $U \subseteq V(G)$ such that $v \in U$. Then*

$$\Delta_U^G = \Delta_{U \cap V(G_1)}^{G_1} * \Delta_{U \cap V(G_2)}^{G_2}$$

and consequently

$$\tilde{\chi}(\Delta_U^G) = -\tilde{\chi}\left(\Delta_{U \cap V(G_1)}^{G_1}\right)\tilde{\chi}\left(\Delta_{U \cap V(G_2)}^{G_2}\right).$$

Proof. Every nucleus of G must contain every cut-vertex by Proposition 2.2, so $\Delta_U^G = \Delta_{U \cup v}^G$ for all $U \subseteq V(G)$, so the assumption $v \in U$ is harmless. Let $N \in \mathcal{N}(G)$; then $N \cap G_1$ and $N \cap G_2$ are nuclei of G_1 and G_2, both containing v. Conversely, if N_1 and N_2 are nuclei of G_1 and G_2 that each contain v, then $N_1 \cup N_2$ is a nucleus of G (which of course contains v). Passing to U-nucleus complexes by complementing edge sets gives the desired result on joins. The equation for Euler characteristics follows from the remarks preceding the proposition. □

Proposition 7.7. *If G is 2-connected and $U \subseteq V(G)$ is nonempty, then $\tilde{\chi}\left(\Delta_U^G\right) = 0$ if and only if $|U| = 1$.*
Consequently, $\mathsf{els}_k(G) > 0$ for all $k \geqslant 2$.

Proof. The case $|U| = 1$ is Theorem 6.4(b). Thus, suppose $|U| \geqslant 2$. Let T be a spanning tree of G. Let F be the smallest subtree of T such that $V(F) \supseteq U$; in particular U contains all leaves of F. Then F is a tree minor of G, so by Proposition 7.4, the summand $\tilde{\chi}(\Delta_{U[F]}^F)$ (which equals -1 by Proposition 4.1) appears in some summation expression for $\tilde{\chi}(\Delta_U^G)$ arising from an RDCT. Equation (7) then implies that $\tilde{\chi}(\Delta_U^G) \neq 0$, and then Theorem 7.1(b) implies that $\mathsf{els}_k(G) > 0$ for all $k \geqslant 2$. □

Proposition 7.7 completes the proof of Theorem 1.2(a).

Proposition 7.8. *Let $U \subseteq V(G)$ be nonempty, let K be the collection of cut-vertices of G, let $U' = U \cup K$, and let $B_1, \ldots B_c$ be the 2-connected components of G.*
Then $\tilde{\chi}(\Delta_U^G) = 0$ if and only if $|V(B_i) \cap U'| \geqslant 2$ for every i.

Proof. Repeatedly applying Prop. 7.6 gives

$$\tilde{\chi}(\Delta_{U'}^G) = (-1)^{c-1}\prod_{i=1}^{c}\tilde{\chi}\left(\Delta_{U' \cap V(B_i)}^{B_i}\right)$$

which, by Proposition 7.7, is zero if and only if $|U' \cap V(B_i)| = 1$ for some i. □

Combining Proposition 7.1(b) with Proposition 7.8 implies the characterization of the positivity of $\mathsf{els}_k(G)$ in Theorem 1.2(b), completing the proof.

8. Monotonicity

In this section, we use the technical results of Sections 5 and 6, including the proof of Elser's conjecture itself, to prove a deletion-contraction type *inequality* for Elser numbers that is stronger than the original conjecture.

Theorem 8.1. *Let $e \in E(G)$ such that e is not a loop or cut-edge. Then*

$$\mathsf{els}_k(G) \geqslant \mathsf{els}_k(G/e) + \mathsf{els}_k(G \backslash e),$$

with equality for $k = 0$.

Proof. Let v_1 and v_2 be the endpoints of e. Note that $V(G) = V(G\backslash e)$, and that we can equate $V(G/e)$ with $V(G)\backslash\{v_1\}$. Abbreviating $n = |V(G)| = |V(G\backslash e)| = |V(G/e)| + 1$ and $m = |E(G)| = |E(G\backslash e)| + 1 = |E(G/e)| + 1$, we have

$$\mathsf{els}_k(G) - \mathsf{els}_k(G\backslash e) = (-1)^{m+n} \sum_{U \subseteq V(G)} \mathrm{Sur}(k, |U|)\, \tilde{\chi}(\Delta_U^G) - (-1)^{m+n-1} \sum_{U \subseteq V(G\backslash e)} \mathrm{Sur}(k, |U|)\, \tilde{\chi}(\Delta_U^{G\backslash e})$$

(by Theorem 3.4)

$$= (-1)^{m+n} \sum_{U \subseteq V(G)} \mathrm{Sur}(k, |U|) \left(\tilde{\chi}(\Delta_U^G) + \tilde{\chi}(\Delta_U^{G\backslash e})\right)$$

$$= (-1)^{m+n} \sum_{U \subseteq V(G)} \mathrm{Sur}(k, |U|)\, \tilde{\chi}(\Delta_{U/e}^{G/e})$$

(by Theorem 5.2)

$$= (-1)^{|E(G/e)|+|V(G/e)|} \left(\sum_{\substack{U \subseteq V(G) \\ v_1 \notin U}} \mathrm{Sur}(k, |U|)\, \tilde{\chi}(\Delta_{U/e}^{G/e}) + \sum_{\substack{U \subseteq V(G) \\ v_1 \in U}} \mathrm{Sur}(k, |U|)\, \tilde{\chi}(\Delta_{U/e}^{G/e}) \right)$$

$$= \mathsf{els}_k(G/e) + (-1)^{|E(G/e)|+|V(G/e)|} \sum_{\{v_1\} \subseteq U \subseteq V(G)} \mathrm{Sur}(k, |U|)\, \tilde{\chi}(\Delta_{U/e}^{G/e})$$

(regarding the first sum as over subsets of $V(G)\backslash\{v_1\} = V(G/e)$)

$$\geqslant \mathsf{els}_k(G/e)$$

(by Theorem 1.1). Note that when $k = 0$, the last sum vanishes and the last inequality is an equality. □

As in Section 6, we can iterate this recurrence until we obtain a tree. So for any graph G, we have that $\mathsf{els}_k(G)$ is bounded below by $\sum_i \mathsf{els}_k(T_i)$ for a collection of trees T_i when $k > 1$. We illustrate this in the following simple example.

Example 8.2. Let $n \geqslant 3$ and consider C_n, the cycle graph on n vertices. Let $e \in E(C_n)$ be any edge of C_n. Then $C_n\backslash e$ is P_n and C_n/e is C_{n-1}. By Theorem 8.1, we have

$$\mathsf{els}_k(C_n) \geqslant \mathsf{els}_k(C_{n-1}) + \mathsf{els}_k(P_n).$$

Iterating this gives

$$\mathsf{els}_k(C_n) \geqslant \sum_{i=1}^{n} \mathsf{els}_k(P_i) \geqslant \sum_{i=1}^{n} (i+1)^k - 2i^k + (i-1)^k$$

by Example 2.4.

9. Nucleus complexes: future directions

In this last section, we explore combinatorial and topological aspects of nucleus complexes, in many cases without giving proofs. We had initially intended to prove Elser's conjecture by computing their simplicial homology groups and thus their Euler characteristics. While this approach did not prove feasible, nucleus complexes nonetheless appear to be interesting objects in their own right, worthy of future study.

We begin with some easy observations. Let G be a connected graph and $U, U' \subseteq V(G)$. It follows easily from the definition of nucleus complexes that if $U \subseteq U'$, then $\Delta_{U'}^G \subseteq \Delta_U^G$. Moreover, in all cases, $\Delta_U^G \cap \Delta_{U'}^G = \Delta_{U \cup U'}^G$. On the other hand, $\Delta_U^G \cup \Delta_{U'}^G \subseteq \Delta_{U \cap U'}^G$, but equality need not hold.

A **matroid on ground set E** (more properly, a matroid independence complex) is a simplicial complex M on vertices E with the property that if $\sigma, \tau \in M$ and $|\sigma| > |\tau|$, then there is a vertex $v \in \sigma\backslash\tau$ such that $\tau \cup \{v\} \in M$. For a general reference on matroids, see, for example, [Oxl11]; for matroid complexes, see [Sta96]. Every connected graph G has an associated **graphic matroid** $M(G) = \{A \subseteq E(G) :\ A \text{ is acyclic}\}$ and **cographic matroid** $M^*(G) = \{A \subseteq E(G) :\ G\backslash A \text{ is connected}\}$ of dimensions $|V(G)|-2$ and $|E(G)|-|V(G)|$ respectively. These matroids are dual; that is, the facets of $M^*(G)$ are precisely the complements of facets of $M(G)$.

In fact, $\Delta^G_{V(G)}$ is precisely the cographic matroid $M^*(G)$. In particular, it is shellable, homotopy-equivalent to a wedge of spheres of dimension $|E(G)| - |V(G)|$, and has homology concentrated in that dimension. For arbitrary $U \subseteq V(G)$, the nucleus complex Δ^G_U is not in general a matroid complex. Nevertheless, experimental data gathered using Sage [Sage] supports the following conjecture.

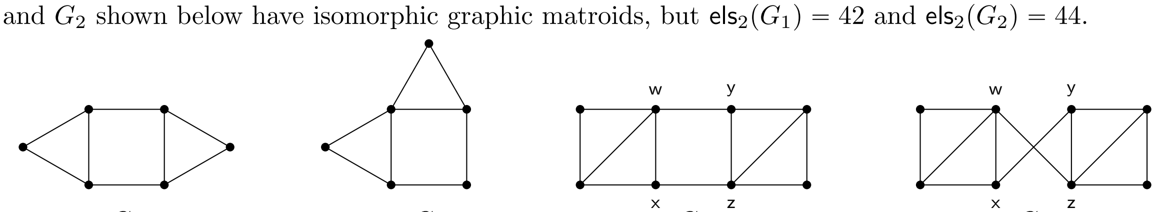
Conjecture 9.1. *Let G be a connected graph and $U \subseteq V(G)$. Then the reduced homology group $\tilde{H}_k(\Delta^G_U; \mathbb{R})$ is nonzero only if (i) $U = \varnothing$ and $k = |E(G)| - |V(G)| - 1$, or (ii) $|U| \geqslant 2$ and $k = |E(G)| - |V(G)|$.*

By Proposition 7.6, it is enough to prove the conjecture in the case that G is 2-connected. Using Sage, we have verified the conjecture computationally for all 2-connected graphs with 6 or fewer vertices

Problem 9.2. Compute the Betti numbers $\dim \tilde{H}_k(\Delta^G_U; \mathbb{R})$ combinatorially for arbitrary G, U, k.

A partial proof of Conjecture 9.1 can be obtained using Jonsson's theory of *pseudo-independence complexes*; we refer the reader to [Jon08, chapter 13] for the relevant definitions and theorems. In short, it can be shown that in all cases, the nucleus complex Δ^G_U is *pseudo-independent* (in the sense of [Jon08]) over its subcomplex $\Delta^G_{V(G)}$, and it is *strongly pseudo-independent* whenever U is a vertex cover. It follows that $\tilde{H}_k(\Delta^G_U; \mathbb{R}) = 0$ for all $k < |E(G)| - |V(G)|$ and all $U \subseteq V(G)$, and for all $k \neq |E(G)| - |V(G)|$ when U is a vertex cover. However, if U is not a vertex cover, then Δ^G_U sometimes fails to be SPI over $\Delta^G_{V(G)}$.

In another direction, one can ask how Elser numbers depend on the graphic matroid $M(G)$. Interestingly, while $\mathsf{els}_k(G)$ cannot be a matroid invariant for $k > 1$ (since it is not constant on trees with the same number of edges, all of which have the isomorphic graphic matroids), it turns out that $\mathsf{els}_0(G)$ is a matroid invariant for 2-connected graphs. This fact can be proven using Whitney's characterization of graphic matroid isomorphism in terms of 2-switches [Whi33]. Even in light of the $k = 0$ case of Theorem 8.1, it is not clear whether $\mathsf{els}_0(G)$ can be obtained from the Tutte polynomial: it is negative on 2-connected graphs but zero on graphs with a cut-vertex (cf. Theorem 1.2(b)), hence not multiplicative on direct sums. For $k \geqslant 2$, $\mathsf{els}_k(G)$ is not a matroid invariant even for 2-connected graphs; for example, the 2-connected graphs G_1 and G_2 shown below have isomorphic graphic matroids, but $\mathsf{els}_2(G_1) = 42$ and $\mathsf{els}_2(G_2) = 44$.



G_1 G_2 G_3 G_4

On the other hand, $M(G_3) \cong M(G_4)$, and $\mathsf{els}_k(G_3) = \mathsf{els}_k(G_4)$ for all $k \geqslant 2$. In general, if G' is obtained from G by replacing an edge cut $\{wy, xz\}$ with another edge cut $\{wz, xy\}$ (as for the pair G_3, G_4 above), then there is a bijection $\mathcal{N}(G) \to \mathcal{N}(G')$ that preserves vertex sets and edge set cardinalities, so $\tilde{\chi}(\Delta^G_U) = \tilde{\chi}(\Delta^{G'}_U)$ and $\mathsf{els}_k(G) = \mathsf{els}_k(G')$ for all U and k. It is possible that there are other special 2-switches with the same properties.

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