Optimal Energy Decay for the Damped Klein-Gordon Equation

Satbir Singh Malhi

B.A. Mathematics, GNDU (India), 2005

M.S. Mathematics, Georgia Southern University 2013

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Committee members

Milena Stanislavova, Chairperson

Mathew Johnson

Dionyssios Mantzavinos

Atanas Stefanov

Arvin Agah

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The Dissertation Committee for Satbir Singh Malhi certifies
that this is the approved version of the following dissertation :

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Milena Stanislavova, Chairperson

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Abstract

In this dissertation we study the long time dynamics of damped Klein-Gordon and damped fractional Klein-Gordon equations using C_0- Semigroup theory and its application. The C_0-semigroups are used to solve a large class of problems commonly known as evolution equations. Such models arise from delay differential equations and partial differential equations in many disciplines including physics, chemistry, biology, engineering, and economics. Water waves, sound waves and simple harmonic motion of strings are few important models of evolution equations. The Klein-Gordon equation is a relativistic version of the Schrödinger equation. It was named after Oskar Klein and Walter Gordon who proposed it to describe quantum particles in the framework of relativity. It describes the motion of spinless composite particles. Indeed, one of the most fundamental questions that should be asked when studying these equations is whether the solution (if it exist) goes to equilibrium (stable) state or behaves erratically as time evolves. Understanding these properties can help determine how robust a system is, as well as provides insight on the characteristics of the corresponding phenomena it is modeling.

In the first part we consider a one dimensional damped Klein-Gordon equation on the real line. It is well known fact that if there is no external force (i.e damping) acting in the system, the wave will oscillate forever in time since the energy is conserved in the system. An interesting question to ask is at what rate the energy starts leaving the system when we introduce damping force? This question was intensely studied in the last ten years. In this direction, Burq and Joly have proved that the energy decays at exponential rate if the damping force $\gamma(x)$ satisfies the geometric control condition (GCC) in a sense that there exist T, $\varepsilon > 0$, such that $\int_0^T \gamma(x(t))dt \geq \varepsilon$ along every straight line unit speed trajectory. However, GCC does not provide an optimal condition to ensure exponential

rate of energy decay. We address this problem in chapter 2 and provide optimal conditions on the damping coefficient γ under which the exponential decay holds in one-dimensional setting. In addition, we derive simple to verify necessary and sufficient conditions for such exponential rate of decay.

In the second part we relate the energy decay rate for the fractional damped wave equation to the order of its fractional derivative. In fact we prove that the energy decays at a polynomial rate if the order of derivative lies between $0 < s < 2$ and at an exponential rate when $s \geq 2$ provided the damping coefficient is non-zero and periodic. An important ingredient of the proof is the derivation of a new observability estimate for the fractional Laplacain. Such important estimate has potential applications in control theory.

Dedication

This dissertation is dedicated to my teachers who have supported me all the way since the beginning of my studies.

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Chapter 1

Introduction

In this chapter, we introduce the main concept of the theory of C_0-semigroups of bounded linear operators and its applications to partial differential equations. When we study the evolution of a system in the context of semigroups we break down the problem into transitional steps, that is the system evolves from one state to another state. When there exist a semigroup, instead of studying the initial value problem (IVP) directly, we can study it via the semigroup and its applicable theory. The theory of linear semigroup is very well developed in [23]. For example, linear semigroup theory provides necessary and sufficient conditions to determine the well-posedness of a problem. Furthermore the asymptotic behavior of the solution of these problems can be obtained with asymptotic theory of C_0-semigroup [28]. We will present the theory, along with several examples, which will motivate the development in later chapters. In this section we mainly focus on a special class of linear semigroups called C_0 semigroups or semigroups of strongly continuous bounded linear operators. The theory of these semigroups is presented with some examples which tend to arise in many areas of applications.

1.1 C_0-semigroups

We begin with some basic notions and the properties of the C_0-semigroup.

Definition 1. *A family $T(t)$, of bounded linear operators from a Banach space X into X is called a strongly continuous semigroup or in short a C_0-semigroup if*

1. $T(0) = I$ *(I is the identity operator on X).*

2. $T(t)T(s) = T(t+s)$ *(the semigroup property).*

3. $\lim_{t \longrightarrow 0} T(t)x = x, \ \forall x \in X$ *(Strongly continuous semigroup property).*

A semigroup of bounded linear operators, $(T(t))_{t>0}$ is uniformly continuous if

$$\lim_{t \downarrow 0} \|T(t) - I\| = 0.$$

Definition 2. *The generator A of a C_0- semigroup $T(t)$ is defined on the set*

$$D(A) = \left\{ x \in X : \lim_{t \longrightarrow 0} \frac{(T(t) - I)x}{t} \ \ exist \right\}$$

as the strong limit

$$Ax = \lim_{t \longrightarrow 0} \frac{(T(t) - I)x}{t}.$$

The set $D(A)$ is called the domain of the semigroup.

Next, we list some properties of C_0-semigroups and their generators, which will be used in the rest of the dissertation.

Theorem 3. *Let $T(t)$ be a C_0- semigroup. Then there exist constants $\omega \geq 0$ and $M \geq 1$, such that*

$$\|T(t)\| \leq Me^{\omega t} \quad for \ \ t \geq 0. \tag{1.1}$$

Theorem 4. *Let $T(t)$ be a C_0- semigroup and A be its generator. Then the following are true*

(a) For all $x \in X$, $t :\rightarrow T(t)x$ is a continuous function from $\mathscr{R}_0^+$ into X.

(b) For $x \in X$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x.$$

(c) For $x \in X$, $\int_0^t T(s)xds \in D(A)$ and

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

(d) For $x \in D(A)$, $T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

(e) For $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax\,d\tau = \int_s^t AT(\tau)x\,d\tau$$

(f) D(A), the domain of A, is dense in X and A is closed linear operator.

(g) A C_0-semigroup is uniquely determined by its generator.

To motivate linear semigroups result, we consider an abstract Cauchy problem

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t), \quad t \geq 0 \\ u(0) &= x \end{aligned} \tag{1.2}$$

where A is a linear operator with domain $D(A)$ in a Banach space X. A classical solution to the above initial value problem (IVP) is a continuous differentiable function $u : [0,\infty) \longrightarrow X$ taking its values in $D(A)$ and satisfying (1.2).

We say that the problem (1.2) is well posed if there exists a unique solution which depends continuously on initial data. A natural question here is the following: *What are the reasonable conditions we can impose on the Abstract Cauchy problem (1.2) or more specially on the linear operator A, so that the problem (1.2) is well-posed?* The C_0-semigroup theory approach provides an alternative to the existence and uniqueness of the evolution equation.

Let $T(t)$ be a C_0- semigroup and A be its generator, then by theorem 4, we have

$$\frac{d}{dt}T(t)x = AT(t)x, \;\; x \in D(A),$$

which implies that for each $x \in D(A)$, the problem (1.2) has a classical solution given by $u(t) = T(t)x$. In other words, we can say that the abstract Cauchy problem associated with the linear operator A is well-posed if A is the generator of a C_0- semigroup.

Theorem 5 (Well-Posedness Theorem)**.** *The IVP given by (1.2) is well posed iff A is the generator of a C_0-semigroup $T(t)$. In this case the unique solution of (1.2) is given by $u(t) = T(t)x$ for $x \in D(A)$.*

Next, we investigate the relationship between the linear operator A and its C_0- semigroup $T(t)$. For this we will try to answer the following two questions. First, for a given semigroup $T(t)$, how we can find its corresponding generator A. Second, for a given linear operator A, how can we ensure the existence of its corresponding C_0-semigroup. The complete answer to the first question is already presented in Theorem 4. Lets return to the second question. In most of the problems, we are given the operator A and one is interested to know for a given operator A, how to construct the corresponding C_0-semigroup $T(t)$. First, we consider the case when A is a bounded operator. This leads us to the following theorem.

Theorem 6. *A linear operator A is the generator of a uniformly continuous semigroup if and only if A is a bounded linear operator. In this case*

$$T(t) = e^{At} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \, : \, t \in \mathbb{R}^+.$$

In case when A is not a bounded linear operator, the convergence of the above such series is not well-defined. So, we no longer can construct the C_0-semigroup of unbounded linear operators through the above exponential series. The question is now, how can we construct the C_0-semigroup when the given linear operator A is unbounded? We further break this question into two parts. In the first part, we find the properties of A, which make the operator A a generator of a C_0-semigroup. Once this is done, we recover the C_0-semigroup $T(t)$ from its generator A. The answer to first of these questions is given by *Hille and Yosida*. Before we state the Hille-Yosida's theorem, we need the following definitions.

Definition 7. *A C_0- semigroup $T(t)$ is called a C_0- semigroup of contraction when $M = 1$ and $\omega = 0$ in (1.1). That is*

$$\|T(t)\| \leq 1 \qquad \forall\, t \geq 0.$$

Definition 8. *The resolvent set of A is denoted by $\rho(A)$ and is the set of all complex numbers λ for which $\lambda I - A$ is invertible. The resolvent of A is a family of bounded linear operators which is denoted by $R(\lambda, A)$ and is given by*

$$R(\lambda, A) = (\lambda - A)^{-1}, \ \ \text{where} \ \ \lambda \in \rho(A).$$

Theorem 9. *[Hille-Yosida Theorem for Contraction Semigroups] A linear (unbounded) operator A is the generator of a C_0-semigroup of contractions $T(t)$, $t \geq 0$ if any only if*

1. *A is closed and $\overline{D(A)} = X$.*

2. *The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every real $\lambda > 0$*

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}. \tag{1.3}$$

Proof. (Necessity)

Define

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x\, dt \quad for \ \ \lambda > 0 \ \ and \ \ x \in X \tag{1.4}$$

then $R(\lambda)$ is a bounded operator satisfying

$$\|R(\lambda)x\| = \int_0^\infty e^{-\lambda t}\|T(t)x\|\,dt \leq \frac{1}{\lambda}\|x\|. \tag{1.5}$$

and

$$R(\lambda)(\lambda I - A)x = x. \quad for\ \ x \in D(A). \tag{1.6}$$

Thus, $R(\lambda)$ is the inverse of $(\lambda I - A)$, it exist for all $\lambda > 0$ and satisfies (1.3).

(Sufficiency)

We define Yosida approximation of A by

$$A_\lambda = \lambda AR(\lambda : A) = \lambda^2 R(\lambda : A) - \lambda I. \texttt{ for every } \lambda > 0. \tag{1.7}$$

Then A_λ is a bounded linear operator. Therefore A_λ is a generator a uniformly continuous semigroup e^{tA_λ} satisfying

$$\|e^{tA_\lambda} - e^{tA_\mu}\| \leq t\|A_\lambda - A_\mu\| \quad \forall\, x \in X,\ \lambda, \mu > 0. \tag{1.8}$$

and

$$\lim_{\lambda\to\infty} A_\lambda x = Ax. \tag{1.9}$$

For $x \in D(A)$, we have

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leq t\|A_\lambda x - A_\mu x\| \leq t\|A_\lambda x - Ax\| + t\|Ax - A_\mu x\| \tag{1.10}$$

It follows that $e^{tA_\lambda}x$ converges uniformly on bounded intervals. Since $D(A)$ is dense in X

and $\|e^{tA_\lambda}\| \leq 1$, it follows that

$$\lim_{\lambda\to\infty} e^{tA_\lambda}x = T(t)x \ \ forevery \ \ x \in X. \tag{1.11}$$

Therefore the equation (1.11) implies that $T(t)$ satisfies the semigroup property with contraction. $\square$

Theorem 10. *[Hille-Yosida] A linear operator A is the infinitesimal generator of a C_0 group of bounded operators $T(t)$ satisfying $\|T(t)\| \leq Me^{\omega|t|}$ if and only if*

1. *A is closed and $\overline{D(A)} = X$.*

2. *Every real λ, $|\lambda| > \omega$ is in the resolvent set $\rho(A)$ of A and for such λ we have*

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(|\lambda| - \omega)^n}. \tag{1.12}$$

The Hille-Yosida Theorem is a powerful tool which gives us both necessary and sufficient conditions. From the proof, one should notice that the resolvent of A is represented in the form of Laplace Transformation of the C_0-semigroup. We should expect to obtain the semigroup by inverting the Laplace Transform.

Often the estimate (1.3) is hard to verify in examples, in particular since it involves the usually unknown resolvent. There are other results along the same lines such as the Lumer-Phillips Theorem[23, p. 14] and Stone Theorem [23, p. 41], which provide the answer to the question of the existence of C_0-semigroups in different settings.

Theorem 11. *[Stone Theorem] A is the generator of a C_0-group of unitary operator on a Hilbert spce H if and only if A is skew-adjoint.*

In the Lumer-Phillips theorem the assumption (1.3) will be replaced by conditions on A itself, namely its “dissipativity” and a range condition.

Theorem 12 (Lumer-Phillips)**.** *Let A be a densely-defined operator. If A generates a contraction semigroup, then it must be dissipative, and for each* $\lambda > 0$, *we must have* $(\lambda - A)[D(A)] = X$. *Conversely, if A is dissipative and there is a* $\lambda_0 > 0$ *such that* $(\lambda_0 - A)[D(A)] = X$, *then A must generate a contraction semigroup.*

1.1.1 Applications and Examples of C_0-semigroup

It is important that we recognize the problems to which C_0-semigroup theory can be applied. In this section we introduce some examples of C_0-semigroups. Many examples fall into the categories of: translations, fractional integration, harmonic functions, stochastic processes, diffusion equation and ergodic theory.

1.1.1.1 The Heat Equation

We consider the following heat equation on $X = C_B(\mathbb{R}^n)$, the space of bounded continuous functions on $\mathbb{R}^n$.

$$\begin{aligned} u_t &= \Delta u, \ \ 0 < t < \infty \qquad (1.13) \\ u(0) &= g(x), \quad x \in \mathbb{R}^n \end{aligned}$$

From [21], we know that for any $g \in X$, (1.13) admits a unique solution given by

$$u(x,t) = \int_{\mathbb{R}^n} K(x,y,t)g(y)dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)dy.$$

It can be easily verified that the above solution operator satisfies all the properties of C_0-semigroup. Hence the solution of (1.13) is a semigroup on X written as

$$(T(t)g)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)dy.$$

1.1.1.2 Klein-Gordon equations in $\mathbb{R}^n$

We consider the following Klein-Gordon equation in $\mathbb{R}^n$

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + u(x,t) = 0, & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \end{cases} \tag{1.14}$$

The equation (1.14) can be written as an Abstract Cauchy system:

$$\begin{aligned} u_t &= v \\ v_t &= \Delta u - u \end{aligned}$$

Then we have an evolution equation given by

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \mathscr{A} \begin{pmatrix} u \\ v \end{pmatrix}$$

where

$$\mathscr{A} = \begin{pmatrix} 0 & I \\ \Delta - I & 0 \end{pmatrix}$$

The operator $\mathscr{A}$ is defined on a Hilbert space

$$X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n),$$

which is equipped with the graph norm

$$\|(u,v)\|_X^2 = \int |u|^2 + |\nabla u|^2 + |v|^2 \, dx$$

Let $U = (u_1, u_2)$ and $V = (v_1, v_2)$, then we have

$$\begin{aligned}
\langle AU, V\rangle &= \langle (u_2, \Delta u_1 - u_1), (v_1, v_2)\rangle_{H^1 \times L^2} \\
&= \langle u_2, v_1\rangle_{H^1} + \langle \Delta u_1 - u_1, v_2\rangle_{L^2} \\
&= \langle \nabla u_2, \nabla v_1, \rangle_{L^2} + \langle u_2, v_1\rangle_{L^2} + \langle \Delta u_1, v_2\rangle_{L^2} - \langle u_1, v_2\rangle_{L^2} \\
&= -\langle u_2, \Delta v_1, \rangle_{L^2} + \langle u_2, v_1\rangle_{L^2} + \langle u_1, \Delta v_2\rangle_{L^2} - \langle u_1, v_2\rangle_{L^2} \\
&= \langle U, -AV\rangle
\end{aligned}$$

Therefore A is a skew adjoint operator. By Stone theorem

$S(t) = e^{tA}$ is a C_0-group of unitary operator.

1.2 The Spectral Theory of C_0-semigroups

The behavior of a dynamical system near some stationary solutions can be determined from a decomposition into invariant manifolds such as stable, unstable and center manifolds. These manifolds are invariant under the flow, and carry the solution near the stationary point characterized by their decay estimates. This approach has a long history of studying the local behavior of a dynamical system near stationary points. The fundamental idea of this approach is as follows: If the linearized system around a stationary solution has invariant manifolds with asymptotic decay rates that are disjoint, then one can acquire some versions of these manifolds for the nonlinear system. In infinite dimensions, the relation between linearlization and the non-linear equation is very subtle. Mostly the existence of invariant manifolds is derived from the group or semi group that arises from the linearlization around the stationary solution. The main idea is to relate the spectrum of the infinitesimal generator to the spectrum of its C_0- group. These types of spectral mapping problems are hard to prove for infinite dimensions.

It is well known that in finite dimensions, the behavior of a dynamical system can be determined from the spectrum of the operator. This result is known as a spectral mapping theorem, which states that the spectrum of the operator e^{tA} is given from the spectrum of A by exponentiation. In infinite dimensions, there are some examples where the spectral mapping theorem fails.

In this section, we will introduce the asymptotic behavior of the orbits $t \mapsto T(t)x$ of a C_0-semigroups and the conditions under which these orbits are stable (i.e converge to zero as $t \to \infty$), unstable or center manifold through spectral mapping theorems. The following three types of stability will be use in our study.

Definition 13. *Let $T(t)$ be a C_0-semigroup on a Banach space X, with generator A. Then $T(t)$ is said to be*

- *uniformly exponential stable, if there exist constants $M > 0$ and $\omega > 0$ such that*

$$\|T(t)\| \leq Me^{-\omega t} \quad \forall\, t \geq 0.$$

- *exponentially stable, if there exist constants $M > 0$ and $\omega > 0$ such that*

$$\|T(t)x\| \leq Me^{-\omega t}\|x\|_{D(A)} \;\; \forall\, t \geq 0 \;\; and \quad x \in D(A).$$

- *uniformly stable, if $\lim\limits_{t\to\infty} \|T(t)x\| = 0$ for all $x \in X$.*

The equation (1.1) implies that every C_0- semigroup is exponential bounded. Therefore, we can define the *uniformly growth bound* $\omega_0(T)$ of T as

$$\omega_0(T) := \inf\big\{\omega \in \mathbb{R} : \exists\, M > 0,\ \|T(t)\| \leq Me^{\omega t}, \forall\, t \geq 0\big\}.$$

Thus, the abstract Cauchy problem (1.2) is uniformly exponentially stable if and only if the growth bound $\omega_0(T)$ of its C_0-semigroup is negative.

The inequality (1.12) implies that the spectrum of the generator of a C_0- semigroup is always contained in some left half-plane. Therefore, we can define the spectral bound $s(A)$ of A by

$$s(A) := \sup\{Re(\lambda) : \lambda \in \sigma(A)\}.$$

If A is a bounded operator on a Banach space X, then

$$s(A) = \omega_0(T). \tag{1.15}$$

Equation (1.15) does not hold for unbounded operators. In general, it is true that ([28])

$$s(A) \leq \omega_0(T).$$

Next, we study the relation between the spectrum of the generator of a strongly continuous semigroup and the spectrum of the semigroup. Formally, one expect that $\sigma(T(t)) \setminus \{0\} = e^{\sigma(A)t}$. However this is not true in general for unbounded generators. We study the conditions which validate the spectral property and related principles of linear stability.

Definition 14. *A C_0- semigroup $(T(t))_{t>0}$ has the spectral mapping property if, for every $t > 0$,*

$$\sigma(T(t)) \setminus \{0\} = e^{\sigma(A)t}.$$

It is well know that the spectral identity $\sigma(T(t)) = e^{\sigma(A)t}$ holds for bounded operators . But if A generates a C_0-semigroup that can not be continued to a group, in this case the operator $T(t)$ is not invertible for each $t > 0$ i.e $0 \in \sigma(T(t))$. We have to subtract 0 from the spectrum of the semigroup operator $T(t)$ to get the spectral identity since zero does not belong to the range of the exponential function.

Theorem 15. *[Spectral Inclusion Theorem] Let $T(t)$ be a C_0- semigroup on a Banach space X, with generator A. Then we have the spectral inclusion relation*

$$\sigma(T(t)) \supset e^{\sigma(A)t}, \ \forall\, t \geq 0.$$

The inverse inclusion $\sigma(T(t)) \setminus \{0\} \subset e^{\sigma(A)t}$ generally fails. Since, the spectral mapping property always holds for the point and residual spectrum, its failure is completely determined by the continuous spectrum.

Theorem 16. *[Spectral Mapping Theorem for the point Spectrum] Let T be a C_0- semigroup on a Banach space X, with generator A. Then we have the spectral relation*

$$e^{t\sigma_p(A)} \subset \sigma_p(T(t)).$$

More precisely, if $\lambda \in \sigma_p(A)$, then $e^{\lambda t} \in \sigma_p(T(t))$; and, if $e^{\lambda t} \in \sigma_p(T(t))$, then there is some integer k such that $\lambda_k := \lambda + \frac{2\pi ik}{t} \in \sigma_p(A)$.

Theorem 17. *[Spectral Mapping Theorem for the Residual Spectrum] Let T be a C_0- semigroup on a Banach space X, with generator A. If $\lambda \in \sigma_r(A)$ and $\lambda_n : \lambda + \frac{2\pi in}{t} \notin \sigma_p$ for all $n \in Z$, then $e^{\lambda t} \in \sigma_r(T(t))$. If $e^{\lambda t} \in \sigma_r(T(t))$, then $\lambda_n : \lambda + \frac{2\pi in}{t} \notin \sigma_p$ for all $n \in Z$; and moreover, there is an integer k such that $\lambda_k \in \sigma_r(A)$.*

There are some important classes of C_0- semigroups for which the spectral mapping property holds. These include compact, eventually differential, positive and analytic semigroups.

If the spectral mapping property holds for a C_0-semigroup $T(t)_{t\geq 0}$ and its generator A, in such cases spectral bound is equal to the growth bound, $s(A) = \omega_0(T)$.

Next, we show that the spectral mapping property holds if we make an additional assumptions on the growth of the resolvent on the vertical lines. More precisely, for a semigroup on a Hilbert space the resolvent $R(A,.)$ of the generator A must be bounded along vertical lines to guarantee the spectral mapping property.

Theorem 18 (**Gearhart-Prüss).** *Let A generate a strongly continuous semigroup on a complex Hilbert space. The for each $t > 0$, we have*

$$\sigma(e^{tA}) \setminus \{0\} = \Big\{ e^{\lambda t} \mid \text{either } u_k = \lambda + \frac{2\pi k i}{t} \in \sigma(A) \text{ for some } k \in \mathbb{Z} \\ \text{or the sequence } \{\|(u_k - A)^{-1}\|_k \text{ is unbounded} \Big\}.$$

The following is an equivalent version of above theorem which is very useful in applications.

Theorem 19. *For $\lambda = a + it,\ a \in R^n \setminus \{0\}\ , t \in \mathbb{R}$, the function $t \longmapsto \|(\lambda - A)^{-1}\|$ remain bounded as $|t| \to \infty$. Then the spectral mapping theorem holds for the semigroup generated by the operator A.*

Proof of Theorem 19. We know that the spectral inclusion $e^{t\sigma(A)} \subset \sigma(e^{tA})$ always holds. Therefore, we just need to prove the reverse inclusion. We also know that the essential spectrum of A is given by $\sigma_{ess}(A) = \{i\lambda | \lambda \in \mathbb{R}, |\lambda| > m\}$ and the point spectrum consists of finitely many eigenvalues in $(-\infty, \infty)$. Consequently,

$$Z := \{z \in \mathbb{C} : \ |z| = 1\} \subset e^{t\sigma(A)}.$$

and for large t, $a + it \notin \sigma(A)$ for each $a \in \mathbb{R} \setminus \{0\}$.

To prove the reverse inclusion, we argue by contradiction. let us suppose if possible

$$\sigma(e^{tA}) \not\subset e^{t\sigma(A)}.$$

$\Rightarrow\ \exists\, \lambda \in \mathbb{C}$ such that

$$e^{\lambda t} \in \sigma(e^{tA}) \ \text{ where } \ \lambda \notin \sigma(A).$$

Notice $a := \mathrm{Re}\lambda \neq 0$, due to the fact $Z \subset e^{t\sigma(A)}$.

We define $\lambda_k := \lambda + \frac{2\pi i k}{t}$, then

$$e^{t\lambda_k} = e^{t\lambda} \notin e^{t\sigma(A)}.$$

Thus $\lambda_k \notin \sigma(A)$ for all $k \in \mathbb{Z}$. Since $e^{t\lambda} \in \sigma(e^{tA})$, Theorem 18 gives that $\{\|(\lambda_k - A)^{-1}\|\}_k$ is an unbounded sequence. But on the other hand if we take $t = \mathrm{Im}(\lambda) + 2\pi k/t$, we arrive at a contradiction to the fact that $\{\|(\lambda_k - A)^{-1}\|\}_k$ is unbounded.

$\square$

1.3 Optimal Energy Decay of Functions and Operator Semigrops

In the theory of Partial differential equations (PDE), one of the main questions to ask is whether the solution to these partial differential equations converge to their equilibrium. If the answer is yes, what is the rate of convergence? In the case of evolutionary PDE, one can address such problems by using operator theoretical methods involving C_0-semigroups.

The aim of this section is to give a simple and self-contained presentation on the asymptotic theory of C_0-semigroup and its applications to partial differential equations in more general setting. In this section, we introduce the results obtained by Alexander Borichev and Yuri Tomilov in their paper[9], in which they developed a technique of characterizing the rate of decay of orbits $t \mapsto T(t)x$ of a C_0-semigroups in resolvent terms of its generator. In the following chapters, we also provide some applications to this technique.

Many problems in mathematical physics can be formulated as an abstract Cauchy problem. We begin with the following abstract Cauchy problem

$$u'(t) = Au(t), \;\; t \geq 0 \tag{1.16}$$

$$u(0) = x$$

where A is a linear operator with domain $D(A)$ on a Banach space X. A classical solution to the above initial value problem is a continuous differential function $u : [0,\infty) \longrightarrow X$ taking its values in $D(A)$ and satisfies (1.16). A continuous function $u : [0,\infty) \longrightarrow X$ is a mild solution if there exist a sequence $(x_n) \subset D(A)$ such that for each n the above initial value problem with initial condition $u(0) = x_n$ has a classical solution $u(\cdot, x_n)$ with $\lim_{n\longrightarrow\infty} u(t,x_n) = u(t)$.

The study of the asymptotic behavior of the classical and mild solution of the abstract Cauchy problem is carry out with C_0-semigroup and its applicable theory.

Recall,

$$\frac{d}{dt}T(t)x = AT(t)x, \ \ x \in D(A),$$

which implies that for each $x \in D(A)$, the problem(1.16) has a classical solution given by $u(t) = T(t)x$. So we can say that the abstract Cauchy problem associated with a linear operator A is well-posed if A generates a C_0- semigroup.

Since our most of initial value problems only provide the operator A, so it is desirable to deduce asymptotic behavior of the solutions $u(t) = T(t)x$ of the initial value problem from information about A. It is well know that if A is bounded linear operator, then we have a very famous result, Spectral Mapping Theorem, which states that resolvent of the C_0-semigroup $T(t)$ can be obtained by exponentiation the resolvent of A i.e

$$\sigma(T(t)) = e^{t\sigma(A)}.$$

Which implies that the exponential growth of the solution of the initial value problem associated to a bounded operated A is determined by the location of the spectrum of A. In general, The Spectral Mapping Theorem is not true. There are some examples of unbounded operators where spectral mapping theorem fails. The failure of spectral mapping means that the spectrum of A no longer determine the asymptotic behavior of our above evolutionary system (1.16).

Many authors have tried to find additional conditions on the semigroup or on its generator under which the Spectral Mapping Theorem holds. Gearhart-Prüss theorem is one such result on the asymptotic behavior of linear autonomous evolution equations which state that a C_0 -semigroup $T(t)$ on a Hilbert space X has an exponential dichotomy if and only if the imaginary axis belongs to the resolvent set of its generator A and the resolvent of A, $R(is) = (is - A)^{-1}$ is uniformly bounded along the imaginary axis. Along this line, there are several results, and many of them can be seen in [23, 2, 7, 28]. In the sequel, we present the following results in the same direction.

1.3.1 Decay of Banach Space Semigroups

The following result was proved in [6, p. 803] and also see [7, p. 41-42].

Theorem 20. *Let $T(t)$ be a bounded C_0-semigroup on a Banach space X with generator A. Suppose the resolvent set $\rho(A)$ of A contains the imaginary axis. Then*

$$\|T(t)A^{-1}\| \longrightarrow 0, \quad t \longrightarrow \infty$$

That is, all the classical solutions of the abstract Cauchy problem (1.16) given by $u(t) = T(t)x, \;\; t \geq 0, \; x \in D(A)$, converge uniformly to zero if the operator A satisfies the conditions of Theorem 20.

A new approach along this line, which is initiated in [20] and later developed in [10, 11], in which authors associate the rate of decay of sufficiently smooth orbits for the semigroup $(T(t))_{t\geq 0}$ with the size of the resolvent $R(\lambda) = (\lambda - A)^{-1}$ of A on the imaginary axis. Batty and Duyckaerts, in the paper[8] gave a unified and simplified approach for estimating the decay rates for $\|T(t)A^{-1}\|$ in term of the growth of $R(is), \; s \in \mathbb{R}$. In particular the following theorem is proved there.

Theorem 21. *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Banach space X with generator A. Suppose the resolvent set $\rho(A)$ of A contains the imaginary axis. Then there exist*

C, $B > 0$ such that

$$\|T(t)A^{-1}\| \leq \frac{C}{M_{\log}^{-1}(t/C)},\ t \geq B$$

Where $M_{\log}^{-1}$ is the inverse of $M_{\log}$ defined on $[M_{\log}(0), \infty]$, which is through the following two equations.

$$M(\eta) = \max_{t \in [-\eta, \eta]} \|R(it, A)\|,\ \ \eta \geq 0$$

$$M_{\log}(\eta) := M(\eta)(\log(1 + M(\eta)) + \log(1 + \eta)),\ \eta \geq 0.$$

In a particular case, when $\alpha > 0$, $M(\eta) \leq C(1 + \eta^{\alpha}),\ \eta \geq 0$, above theorem gives

$$\|T(t)A^{-1}\| \leq C\left(\frac{\log t}{t}\right)^{\frac{1}{\alpha}},\ t \geq B. \tag{1.17}$$

It was conjectured in [8] that in the Hilbert space setting above rate of decay can be improved where one can remove the logarithmic factor in (1.17). Borichev and Tomilov proved the validity of this conjecture in their paper[9].

1.3.2 Decay of Hilbert Space Semigroup

To prove the main theorem of this section, we need the following two lemmas.

Lemma 1 (Gomilko [17]). *Let $(T(t))_{t \geq 0}$ be a C_0-semigroup on a Hilbert space H with generator A. Then $(T(t))_{t \geq 0}$ is bounded if and only if*

$$\mathbb{C}_+ \subset \rho(A),$$

$$\mathit{and}$$

$$\sup_{\xi > 0} \xi \int_{\mathbb{R}} \left(\|R(\xi + i\eta, A)x\|^2 + \|R(\xi + i\eta, A^*)\|^2 \right)\, d\eta < \infty\ \ \forall x \in H.$$

Proof. Assume that

$$\mathbb{C}_+ \subset \rho(A) \quad \& \quad \sup_{\xi>0} \xi \int_{\mathbb{R}} \left(\|R(\xi+i\eta,A)x\|^2 + \|R(\xi+i\eta,A^*)\|^2 \right) \, d\eta < \infty \;\; \forall x \in H.$$

We will show that $(T(t))_{t\geq 0}$ is bounded.

We have

$$R(\lambda,A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt.$$

By differentiating both sides with respect to λ, we get

$$-R(\lambda,A)^2 x = -\int_0^\infty t e^{-\lambda t} T(t)x \, dt = -\mathscr{L}(tT(t)x)$$

Taking inverse Laplace transformation on both sides, we get

$$tT(t)x = \frac{1}{2\pi i} \int_{\frac{1}{t}-i\infty}^{\frac{1}{t}+i\infty} e^{\lambda t} R(\lambda,A)^2 \, d\lambda$$

So, we have the following representation of $T(t)$.

$$\langle T(t)x, x^* \rangle = \frac{1}{2\pi i t} \int_{\frac{1}{t}-i\infty}^{\frac{1}{t}+i\infty} e^{\lambda t} \left\langle R^2(\lambda,A)x, x^* \right\rangle \, d\lambda, \quad t > 0$$

Using Hölder inequality together with the inequality $ab \leq \frac{a+b}{2}$, we get

$$\begin{aligned}
|\langle T(t)x, x^* \rangle| &\leq \frac{\xi}{2\pi} \int_{-\infty}^{\infty} |\langle R(\xi+i\tau,A)x, R(\xi-i\tau,A^*)x^* \rangle| \, d\tau \\
&\leq C\frac{\xi}{2\pi} \int_{-\infty}^{\infty} \|R(\xi+i\tau,A)x\| \|R(\xi-i\tau,A^*)x^*\| \, d\tau \\
&\leq C\frac{\xi}{2\pi} \int_{-\infty}^{\infty} \left(\|R(\xi+i\tau,A)x\|^2 + \|R(\xi-i\tau,A^*)x^*\|^2 \right) \, d\tau, \\
&< \infty
\end{aligned}$$

Thus the uniform bounded principal, implies that $(T(t))_{t\geq 0}$ is bounded.

For the reverse direction, assume $(T(t))_{t\geq 0}$ is bounded.

$$\begin{aligned}
R(\xi+i\tau,A)x &= \int_0^\infty e^{(-\xi-i\tau)t}T(t)x\,dt \\
&= \int_{-\infty}^{\infty} e^{-i\tau t}e^{-\xi t}T(t)\chi_{(0,\infty)}x\,dt \\
&= e^{-\xi t}\widehat{T(t)\chi_{(0,\infty)x}}
\end{aligned}$$

By Plancherel Theorem, we get

$$\|R(\xi+i\tau,A)x\|_{L^2_\tau} = \|e^{-\xi t}\widehat{T(t)\chi_{(0,\infty)x}}\|_{L^2_\tau} = \|e^{-\xi\cdot}\,T(.)\chi_{(0,\infty)}\|_{L^2_\tau} \leq \frac{C}{\xi}.$$

Similarly

$$\|R(\xi+i\tau,A^*)\|_{L^2_\tau} \leq \frac{C}{\xi}.$$

Therefore the above two inequalities implies

$$\sup_{\xi>0}\xi\int_{\mathbb{R}}(\|R(\xi+i\tau,A)\|^2+\|R(\xi+i\tau,A^*)\|^2)\,d\tau<\infty.$$

$\square$

Lemma 2. *Let $T(t)$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R}\subset\rho(A)$. Then for a fixed $\alpha>0$, we have*

$$\|R(\lambda,A)(-A)^{-\alpha}\|\leq C,\;\; Re\lambda>0,$$

if and only if

$$\|R(is,A)\| = O(|s|^{\alpha}),\ s \longrightarrow \infty.$$

Proof. It has been proved in [19, Lemma 3.2] that

$$\|R(\lambda,A)\| \leq C(1+|\lambda|^{\alpha}), \quad 0 < \text{Re } \lambda < 1,$$

is equivalent to

$$\|R(\lambda,A)(-A)^{-\alpha}\| \leq C,\ \ 0 < \text{Re } \lambda < 1.$$

So, we shall show that the condition

$$\|R(\lambda,A)\| \leq C(1+|\lambda|^{\alpha}), \quad 0 < \text{Re } \lambda < 1,$$

is equivalent to

$$\|R(is,A)\| = O(|s|^{\alpha}),\ s \longrightarrow \infty.$$

To prove this, we apply the maximum principle to the function

$$F(\lambda) = R(\lambda,A)\lambda^{-\alpha}\left(1 - \frac{\lambda^2}{B^2}\right)$$

on the set $D = \{\lambda \in \mathbb{C} : \text{Re } \lambda \geq 0,\ \varepsilon \leq |\lambda| \leq B,\ \varepsilon > 0\}$. The boundary of D consists of three parts, $B_1 = \{\lambda \in \mathbb{C} : \lambda = \varepsilon e^{i\theta}, 0 \leq \theta \leq \frac{\pi}{2}, \varepsilon > 0\}$, $B_2 = \{\lambda \in \mathbb{C} : \lambda = is, s \in \mathbb{R} \setminus 0\}$ and $B_3 = \{\lambda \in \mathbb{C} : \lambda = Be^{i\theta}, 0 \leq \theta \leq \frac{\pi}{2}, \varepsilon > 0\}$

Clearly for every fixed $\varepsilon > 0$, the set B_1 is compact. Therefore the continuity of $F(\lambda)$ implies that $F(\lambda)$ is bounded on B_1.

On B_2, we have the following estimate

$$|F(is)| = \|R(is,A)\|(|s|)^{-\alpha}\left(1+\frac{s^2}{B^2}\right) < C, \ \forall \ B$$

On B_3, we use the estimate

$$\|R(\lambda,R)\| \leq \frac{1}{\mathrm{Re}(\lambda)}$$

We get

$$\begin{aligned}|F(Be^{i\theta})| &= \|R(Be^{i\theta},A)\|B^{-\alpha}|\left(1-e^{2i\theta}\right)| \\ &\leq \frac{1}{B\cos\theta}\frac{1}{B^{\alpha}}2\sqrt{2}\cos\theta\sin\theta \\ &\leq C\frac{1}{B^{1+\alpha}}\end{aligned}$$

Thus by Maximal principle, $F(\lambda)$ is bounded on D. This gives

$$\|R(\lambda,A)\| \leq C(1+|\lambda|^{\alpha}), \ \mathrm{Re}\ \lambda > 0.$$

□

Theorem 22 (Borichev, Tomilov)**.** *Let $T(t)$ be a boubded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\alpha > 0$,*

$$\|R(is,A)\| = O(|s|^{\alpha}), \ s \longrightarrow \infty.$$

implies

$$\|T(t)A^{-1}x\| = O\left(\frac{1}{t^{1/\alpha}}\right), \ t \longrightarrow \infty, \ x \in H$$

Proof. Consider the operator $\mathscr{A}$ on $\mathscr{H} = H \oplus H$ given by the matrix operator

$$\mathscr{A} = \begin{pmatrix} A & (-A)^{-\alpha} \\ O & A \end{pmatrix},$$

with the domain $D(\mathscr{A}) = D(A) \oplus D(A)$.

$$R(\lambda, \mathscr{A}) = (\lambda I - \mathscr{A})^{-1} = \begin{pmatrix} \lambda I - A & -(-A)^{-\alpha} \\ O & \lambda I - A \end{pmatrix}^{-1},$$

therefore $R(\lambda, \mathscr{A})$ exist if and only if $\lambda \in \rho(A)$ and its given by

$$R(\lambda, \mathscr{A}) = \begin{pmatrix} \lambda I - A & -(-A)^{-\alpha} \\ O & \lambda I - A \end{pmatrix}^{-1} = \begin{pmatrix} (\lambda I - A)^{-1} & R^2(\lambda, A)(-A)^{-\alpha} \\ O & (\lambda I - A)^{-1} \end{pmatrix},$$

Define

$$\mathscr{T}(t) = \begin{pmatrix} T(t) & tT(t)(-A)^{-\alpha} \\ O & T(t) \end{pmatrix} \tag{1.18}$$

We claim that $\mathscr{T}(t)$ is a C_0-semigroup on $\mathscr{H}$ with generator $\mathscr{A}$.

1. $\mathscr{T}(0) = \begin{pmatrix} T(0) & O \\ O & T(0) \end{pmatrix} = I$

2. $\mathscr{T}(t+s) = \begin{pmatrix} T(t+s) & (t+s)T(t+s)(-A)^{-\alpha} \\ O & T(t+s) \end{pmatrix}$

$= \begin{pmatrix} T(t) & tT(t)(-A)^{-\alpha} \\ O & T(t) \end{pmatrix} \begin{pmatrix} T(s) & sT(s)(-A)^{-\alpha} \\ O & T(s) \end{pmatrix}$ for $t, s \geq 0$

Hence $\mathscr{T}(t)$ is a C_0-semigroup.

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D(\mathscr{A})$, we have

$$\begin{aligned}
\lim_{t\to 0} \frac{\mathscr{T}(t)x - x}{t} &= \lim_{t\to 0} \frac{\begin{pmatrix} T(t) & tT(t)(-A)^{-\alpha} \\ O & T(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}{t} \\
&= \begin{pmatrix} \lim_{t\to 0} \frac{T(t)x_1 - x_1}{t} + \lim_{t\to 0} T(t)(-A)^{-\alpha} x_2 \\ \lim_{t\to 0} \frac{T(t)x_2 - x_2}{t} \end{pmatrix} \\
&= \begin{pmatrix} Ax_1 + (-A)^{-\alpha} x_2 \\ Ax_2 \end{pmatrix} = \begin{pmatrix} A & (-A)^{-\alpha} \\ O & A \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{aligned}$$

Hence $(\mathscr{T}(t))_{t\geq 0}$ is a C_0-semigroup with generator $\mathscr{A}$ on $D(\mathscr{A}) = D(A) \oplus D(A)$.

By Lemma 2,

$$\|R(\lambda, A)(-A)^{-\alpha}\| \leq C, \ \ \mathrm{Re}\lambda > 0$$

For every $x = (x_1, x_2) \in \mathscr{H}$ and $\lambda \in C_+$,

$$\begin{aligned}
&\|R(\lambda, \mathscr{A})x\|^2 = \|R(\lambda,A)x_1 + R^2(\lambda,A)(-A)^{-\alpha}x_2\|^2 + \|R(\lambda,A)x_2\|^2 \\
&\leq (\|R(\lambda,A)x_1\| + \|R^2(\lambda,A)(-A)^{-\alpha}x_2\|^2)^2 + \|R(\lambda,A)x_2\|^2 \\
&\leq (\|R(\lambda,A)x_1\| + C\|R(\lambda,A)x_2\|)^2 + \|R(\lambda,A)x_2\|^2 \\
&\leq \|R(\lambda,A)x_1\|^2 + C^2\|R(\lambda,A)x_2\|^2 + 2C\|R(\lambda,A)x_1\|\|R(\lambda,A)x_2\| + \|R(\lambda,A)x_2\|^2 \\
&\leq \|R(\lambda,A)x_1\|^2 + C^2\|R(\lambda,A)x_2\|^2 + C\|R(\lambda,A)x_1\|^2 + C\|R(\lambda,A)x_2\|^2 + \|R(\lambda,A)x_2\|^2 \\
&\leq \|R(\lambda,A)x_1\|^2 + C^2\|R(\lambda,A)x_2\|^2 + 2C\|R(\lambda,A)x_1\|\|R(\lambda,A)x_2\| + \|R(\lambda,A)x_2\|^2 \\
&\leq (1+C)\|R(\lambda,A)x_1\|^2 + (1+C^2+C)\|R(\lambda,A)x_2\|^2 \\
&\leq \max(1+C, 1+C^2+C))(\|R(\lambda,A)x_1\|^2 + \|R(\lambda,A)x_2\|^2)
\end{aligned}$$

Hence

$$\|R(\lambda,\mathscr{A})x\|^2 \le C(\|R(\lambda,A)x_1\|^2+\|R(\lambda,A)x_2\|^2). \tag{1.19}$$

Similarly

$$\|R(\lambda,\mathscr{A}^*)x\|^2 \le C(\|R(\lambda,A^*)x_1\|^2+\|R(\lambda,A^*)x_2\|^2). \tag{1.20}$$

Since $(T(t))_{t\ge 0}$ is a family of bounded operators. By Lemma 1, we have

$$\sup_{\xi>0}\xi\int_{\mathbb{R}}(\|R(\xi+i\eta,A)\|^2+\|R(\xi+i\eta,A^*)\|^2)\,d\eta<\infty \tag{1.21}$$

for every $x\in\mathscr{H}$.

Combining the equations 1.19, 1.20 and 1.18, we get

$$\sup_{\xi>0}\xi\int_{\mathbb{R}}(\|R(\xi+i\eta,\mathscr{A})\|^2+\|R(\xi+i\eta,\mathscr{A}^*)\|^2)\,d\eta<\infty$$

for every $x\in\mathscr{H}$.

Now the reverse conclusion of Lemma 1 implies that $(\mathscr{T}(t))_{t\ge 0}$ is bounded on $\mathscr{H}$. By the definition (1.18) of $(\mathscr{T}(t))_{t\ge 0}$ and the fact $(T(t))_{t\ge 0}$ is bounded, we have

$$\sup_{t\ge 0}\|tT(t)(-A)^{-\alpha}\|<\infty.$$

Since $i\mathbb{R}=\rho(\mathscr{A})$ and $D(\mathscr{A})=\mathrm{Im}(\mathscr{A}^{-1})$ is dense in $\mathscr{H}$. Then by Theorem 20

$$\mathscr{T}(t)\to 0,\quad t\to\infty \tag{1.22}$$

for every $x\in\mathscr{H}$.

Furthermore, $i\mathbb{R} \subset \rho(A)$. Again by Theorem 20, we have

$$T(t) \to 0, \quad t \to \infty \tag{1.23}$$

for every $x \in H$. Equation (1.23) and (1.22) implies that

$$\|tT(t)(-A)^{-\alpha}x\| = o(1), \quad t \to \infty, \ x \in H.$$

Hence,

$$\|T(t)(-A)^{-\alpha}\| \leq \frac{C}{t}, \quad t \to \infty, \text{ on } H.$$

For $t > 0$ and $n \in \mathbb{N}$, we have

$$\|T(t)(-A)^{-n\alpha}\| = \|(T(t/n)(-A)^{-\alpha})^n\| \leq \frac{C^n}{t^n}. \tag{1.24}$$

Write $\frac{1}{\alpha} = n + \tau$, for some $n \in \mathbb{N}$ and $\tau \in [0,1)$. Using above estimate with moment inequality, see [14, ,CH II Theorem 5.34]

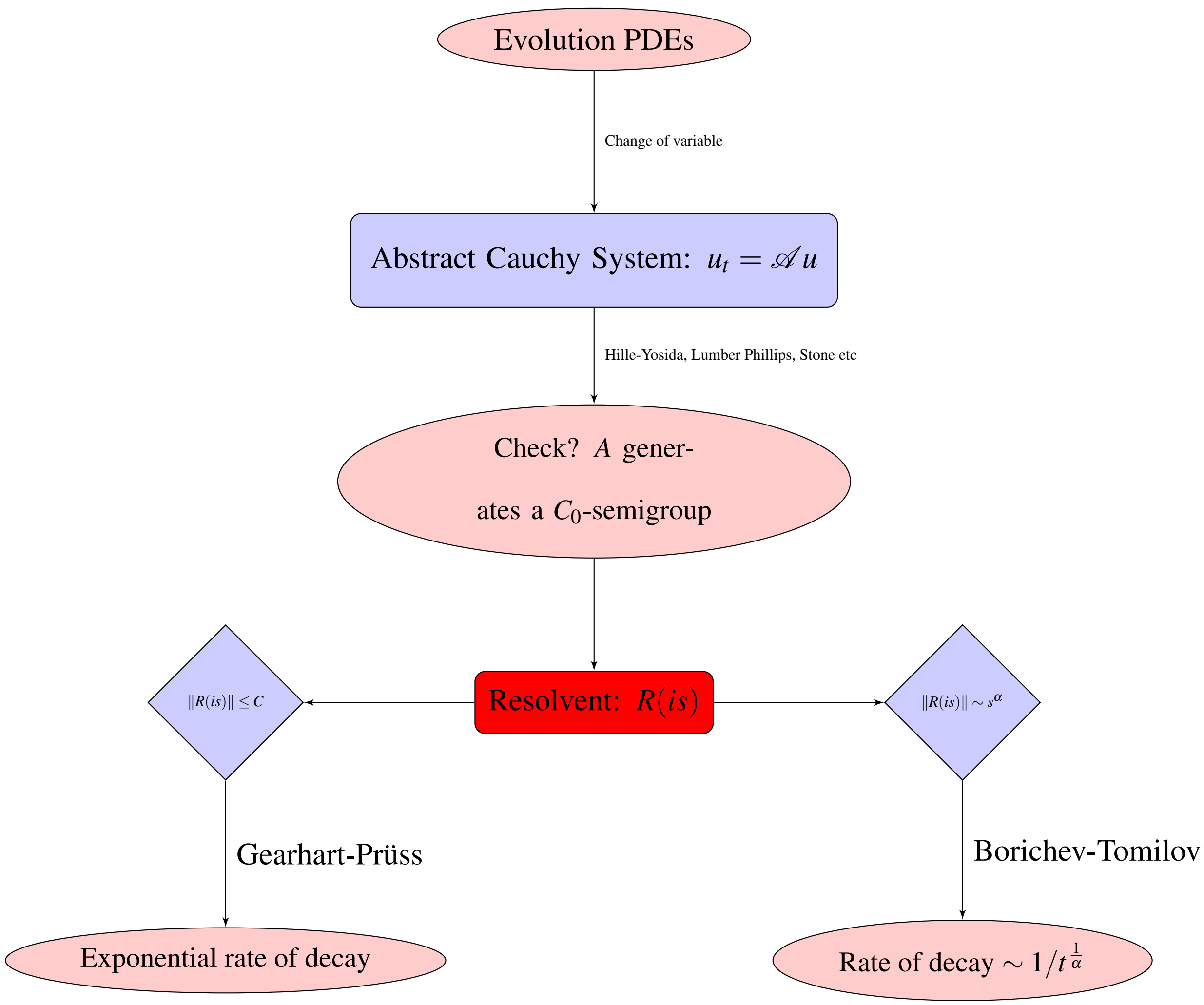
$$\begin{aligned}
\|T(t)A^{-1}\| = \|T(t)A^{-\frac{1}{\alpha}\alpha}\| &= \|T(t)A^{-(n+\tau)\alpha}\| \\
&= \|A^{(1-\tau)\alpha}T(t)A^{-(n+1)\alpha}\| \\
\leq C(\alpha)\|A^{\alpha}T(t)A^{-(n+1)\alpha}\|^{1-\tau}&\|T(t)A^{-(n+1)\alpha}\|^{\tau} \\
\leq C(\alpha)\|T(t)A^{-n\alpha}\|^{1-\tau}&\|T(t)A^{-(n+1)\alpha}\|^{\tau} \\
\leq C(\alpha)\left(\frac{C^n}{t^n}\right)^{1-\tau}\left(\frac{C^{n+1}}{t^{n+1}}\right)^{\tau} &= C(\alpha)\frac{C^{n+\tau}}{t^{n+\tau}}
\end{aligned}$$

So, we get

$$\|T(t)A^{-1}\| \leq \frac{C}{t^{\frac{1}{\alpha}}}.$$

□

The diagram below gives the overview of this chapter

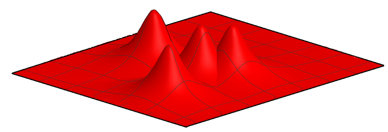


Chapter 2

On the long time behaviour of one dimensional damped Klein-Gordon equation

2.1 Introduction

In this chapter, the main object of study is the following damped Klein-Gordon equation



$$u_{tt} + \gamma(x)u_t - u_{xx} + u = 0. \quad (x,t) \in \mathbb{R} \times \mathbb{R} \qquad (2.1)$$

Where $\gamma(x)u_t$ represents a damping force proportional to the velocity u_t. This is a standard model in the theory. In the case $\gamma(x) = const.$, one can easily see that the energy function

$$E(u) = \frac{1}{2}\|u\|^2_{H^1(\mathbb{R})} + \frac{1}{2}\|u_t\|^2_{L^2(\mathbb{R})} = \frac{1}{2}\int_{-\infty}^{\infty} |u_x|^2 + |u|^2 + |u_t|^2 dx,$$

has an exponential decay as $t \to \infty$. Thus a natural question to ask is the following: under what conditions on $\gamma(x) \geq 0$, one can still guarantee such exponential (or slower algebraic) decay. This question was intensely researched in the last ten years. We present a brief (and definitely incomplete) overview of the recent results.

In this direction, Burq and Joly have proved in [12] exponential rate of decay of the semi-group under the geometric control condition (GCC) in a sense that there exist T, $\varepsilon > 0$, such that $\int_0^T \gamma(x(t))dt \geq \varepsilon$ along every straight line unit speed trajectory thus extending the previous work of Bardos, Lebeau, Rauch, and Taylor [3, 4, 26] of compact manifold to the whole space $\mathbb{R}^{\mathbb{N}}$. The region in fig 3 below is an example where GCC satisfied whereas the region in fig 2 GCC failed to satisfies. Notice that in [12] the authors also require additional uniform continuity requirement on the damping coefficient γ in order to use pseudo-differential calculus. The authors also provide counter examples [12](see fig 3 below) where exponential decay is expected but regularity hypothesis of GCC failed badly. However this is not in the case of compact manifold where this assumption is automatically true.

In the absence of geometric control condition, the same authors of [12] also provide a weaker hypothesis, namely network control condition (NCC) where the damping coefficient $\gamma(x)$ is strictly positive on a family of balls whose dilates cover $\mathbb{R}^N$ under which the solution of damped wave equation decays with logarithmic rate (still without loss of regularity). For a fixed periodic damping, Wunsch proved in [29] that without any geometric condition (see fig 4 below) there is at least a polynomial (certainly not optimal) decay (with loss of regularity).

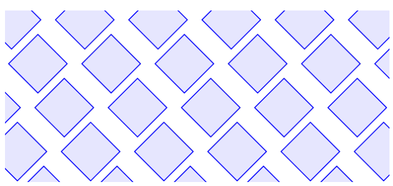


Fig 1

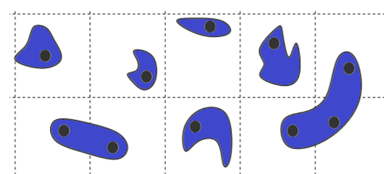


Fig 2

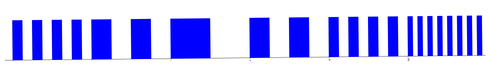


Fig 3

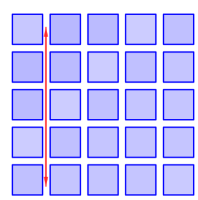


Fig 4

One can observe that in the case of compact manifold (see [1, 11, 27, 24] and references therein) the decay rate of the semigroup of damped wave equation highly depends on the way the damping coefficient γ vanishes. Several sharp result are obtained in different settings. One should expect same in the case of non compact setting. However, it is not clear in this case what is the optimal form of a damping coefficient which will ensure that one can expect exponential (or algebraic) energy decay to the solution of (2.1). The purpose of this paper is to find optimal conditions on the damping coefficient γ under which the exponential decay holds. In fact, we are able to provide a simple to verify necessary and sufficient condition for such an exponential decay in one spatial dimension.

2.1.1 Semigroup Representations and Main Result

In order to use C_0-semigroups theory, we recast the problem (2.1) as an abstract Cauchy problem. For this we define new variable $U = (u, u_t)^\top$, then equation (2.1) can be written as a dynamical system in the following form, where

$$U_t = \mathscr{A}U, \quad \mathscr{A} = \begin{pmatrix} 0 & I \\ \partial_x^2 - 1 & -\gamma(x) \end{pmatrix} \tag{2.2}$$

The operator $\mathscr{A}$ is defined on a Hilbert space $\mathscr{H} = H^1(\mathbb{R}) \times L^2(\mathbb{R})$, with domain $H^2(\mathbb{R}) \times H^1(\mathbb{R})$.

We can write $\mathscr{A}$ as

$$\mathscr{A} = A + B$$

where

$$A = \begin{pmatrix} 0 & I \\ \partial_x^2 - 1 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma(x) \end{pmatrix}$$

Clearly A is a self-adjoint operator and therefore generates a C_0-semigroup. Moreover, B is a bounded matrix operator since $\gamma(x)$ is bounded. Since every bounded perturbation of an operator also generates a C_0-semigroup, so $\mathscr{A}$ generates a C_0-semigroup, say $T(t)$. In fact, $T(t)$ is a semigroup of contractions (see Proposition 1 below).

The following is the main result of this chapter.

Theorem 23. *Assume $\gamma : \mathbf{R} \to \mathbf{R}$, with $\gamma \geq 0$ is a continuous and bounded function. The following statements are equivalent*

(i)

$$\liminf_{N\to\infty} \inf_{y\in\mathbf{R}} \frac{1}{N} \int_{y}^{y+N} \gamma(z)dz > 0. \tag{2.3}$$

(ii) $1 \in \rho(\mathscr{A})$ *and there exists* $\lambda_0 > 0$, *so that*

$$\|e^{t\mathscr{A}}(1-\mathscr{A})^{-1}\|_{H^1\times L^2 \to H^1\times L^2} \leq Ce^{-\lambda_0 t}.$$

Equivalently,

$$\|(u(t),u_t(t))\|_{H^1\times L^2} \leq Ce^{-\lambda_0 t}\|(u(0),u_t(0))\|_{H^2\times H^1}$$

whenever $(u(0),u_t(0)) \in H^2 \times H^1$.

(iii) $\lim_{t\to\infty} \|e^{t\mathscr{A}}\|_{H^2\times H^1 \to H^1\times L^2} = 0$.

(iv) For the semigroup generated by (2.1), $\sigma(\mathscr{A}) \cap i\mathbb{R} = \emptyset$.

The proof of the Theorem (23) is based on the semigroups techniques used in [29, 11, 16, 19], in which rather than estimating the norm of the solution directly, we use a result obtained by Gearhart-Prüss,[15, 25]. We use Theorem 24 which is a formulation given by Theorem 3 of [18] . More concretely, this result makes it possible to deduce exponential rate of decay of the energy of the solution by uniformly estimating the norm of the resolvent $(\mathscr{A} - \lambda I)^{-1}$ of the generator of the semigroup on the imaginary axis. Some additional remarks are in order.

Remarks:

1. The condition (2.3), in the context of γ bounded is equivalent to

$$\liminf_{N\to\infty} \inf_{y\in\mathbf{R}} \frac{1}{N}\int_y^{y+N} \gamma^p(z)dz > 0$$

for any $p > 1$. This is a consequence of the Hölder's inequality

$$\begin{aligned}\frac{1}{N}\int_y^{y+N} \gamma(z)dz &\leq \left(\frac{1}{N}\int_y^{y+N} \gamma^p(z)dz\right)^{\frac{1}{p}} \\ &\leq \|\gamma\|_{L^\infty}^{\frac{p-1}{p}} \left(\frac{1}{N}\int_y^{y+N} \gamma(z)dz\right)^{\frac{1}{p}}\end{aligned}$$

2. The implication $(ii) \Rightarrow (iii)$ above is of course trivial. The equivalence, namely the fact that $(iii) \Rightarrow (ii)$, means that as long as a solution starting with an initial data in $H^2 \times H^1$ goes to zero in the energy norm $H^1 \times L^2$, then this convergence must be exponential. In particular, this implies that algebraic convergence is impossible. However, exponential convergence is possible. This is of course in sharp contrast with the higher dimensional case, where algebraic convergence is possible [12, 29].

3. The equivalence $(iii) \Leftrightarrow (iv)$ is a particular case for the bounded semigroup (See Proposition 1) of the damped wave equation (2.1), of a more general theorem of Batty-Borichev-Tomilov([5], Theorem 1.4). See Theorem 25 below as well as the Corollary 2.

The following steps will be taken to complete the proof of Theorem 2.1. First, we show that our problem is well posed in the sense of C_0-semigroups and we describe the spectrum of the infinitesimal generator. Then we turn to compute the resolvent bound of the semigroup. The method we use here to find the resolvent bound is very functional analytical. However, this is the most technical part. At the end, we apply the Gearhart-Prüss Theorem 24 to deduce from the resolvent bound an estimate for the rate of energy decay of smooth solutions.

2.1.2 Preliminaries and Notations

In order to fix notations, the Fourier transform will henceforth take the form

$$\hat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-ix\xi}dx, \;\; f(x) = (2\pi)^{-1}\int_{\mathbf{R}} \hat{f}(\xi)e^{ix\xi}d\xi.$$

Henceforth, the constant C will change from line to line , but will always be independent of the spectral parameter. The constants C_δ and C_ε are different constant with dependence on δ and ε respectably. These constants also will change line to line throughout the presentation.

Proposition 1. *Let $\gamma \geq 0$ be a bounded function. Then, we have*

$$\|T(t)\|_{\mathscr{H}\to\mathscr{H}} \leq 1 \qquad \forall\;\; t \geq 0.$$

Proof. All we need for the proof is to take a sufficiently smooth and decaying initial data for (2.1), consider its solution at a later time and take a dot product with $u_t \in L^2(\mathbf{R})$. We obtain,

$$\partial_t(|u_t\|_{L^2}^2 + \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2) + \int \gamma |u_t|^2 dx = 0.$$

It follows that the energy function $E(t) = \|u_t(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \|u_x(t)\|_{L^2}^2$ is decaying with time, hence $E(t) \leq E(0)$, or equivalently $\|(u(t),u_t(t))\|_{\mathscr{H}} \leq \|(u(0),u_t(0))\|_{\mathscr{H}}$. □

Next, we have the following interesting corollary.

Corollary 1. *Let $\gamma \geq 0$ be a continuous function, so that* (2.3) *does not hold. That is*

$$\liminf_{N\to\infty} \inf_{y\in\mathbf{R}} \frac{1}{N}\int_y^{y+N} \gamma(z)dz = 0$$

Then, $\|e^{t\mathscr{A}}\|_{H^1\times L^2\to H^1\times L^2} = 1$, for all $t \geq 0$.

Proof. By Proposition 1, for $T(t) = e^{t\mathscr{A}}$, we have

$$\|T(t)\|_{H^1\times L^2\to H^1\times L^2} \le 1$$

and $T(0) = Id$. Clearly $\|T(0)\| = 1$. Assume for a contradiction, that for some $t_0 > 0$,

$$\|T(t_0)\|_{H^1\times L^2\to H^1\times L^2} = q < 1.$$

From the equivalent condition (iii) of Theorem 23 above, it follows that

$$\limsup_{t\to\infty} \|T(t)(1-\mathscr{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2} \ge c_0 > 0.$$

Say, $t_n \to \infty$, so that

$$\|T(t_n)(1-\mathscr{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2} \ge \frac{c_0}{2}.$$

Now,

$$\begin{aligned}
\frac{c_0}{2} &\le \|T(t_n)(1-\mathscr{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2} \\
&\le \|T(t_n)\|_{H^1\times L^2\to H^1\times L^2}\|(1-\mathscr{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2} \\
&\le q^{[\frac{t_n}{t_0}]}\|(1-\mathscr{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2}.
\end{aligned}$$

Since clearly, $\lim_n q^{[\frac{t_n}{t_0}]} = 0$, this is a contradiction.

□

The following result will be one of the main technical tools that allows us to deduce exponential decay from estimates on the resolvent.

Theorem 24 (Gearhart-Prüss)**.** *Let $e^{t\mathscr{A}}$ be a C_0-semigroup in a Hilbert space X and assume that there exists a positive constant $M > 0$ such that $\|e^{t\mathscr{A}}\| \le M$ for all $t \ge 0$. Let $\mu \in \rho(\mathscr{A})$, then the following are equivalent.*

(i) *There exists* $\lambda_0 > 0$ *and* C, *so that*

$$\|T(t)(\mu - \mathscr{A})^{-1}\|_{B(X)} \leq Ce^{-\lambda_0 t}$$

(ii) $i\mathbb{R} \subset \rho(\mathscr{A})$ *and*

$$\sup_{s\in\mathbb{R}} \|(\mathscr{A} - isI)^{-1}\|_{B(X)} < +\infty.$$

Another result, which will be useful for us is the following.

Theorem 25 (Batty-Borichev-Tomilov, [5], Theorem 1.4)**.** *Let* $e^{t\mathscr{A}}$ *be a bounded* C_0*-semigroup in a Banach space* X*. Then for* $\mu \in \rho(\mathscr{A})$*, the following are equivalent*

(i) $\sigma(\mathscr{A}) \cap i\mathbb{R} = \emptyset$

(ii) $\lim_{t\to\infty} \|T(t)(\mu - \mathscr{A})^{-1}\|_{B(X)} = 0.$

Note that in the case of the damped wave equation semigroup (2.2), say with $\mu = 1$, $(1-\mathscr{A})^{-1} : H^1 \times L^2 \to H^2 \times H^1$ and this map is onto. Thus, an application of Theorem 25 to this particular case yields the following

Corollary 2. *For the semigroup* $T(t)$ *of damped wave equation* (2.2)*, the following are equivalent*

(i) $\sigma(\mathscr{A}) \cap i\mathbb{R} = \emptyset$

(ii) $\lim_{t\to\infty} \|T(t)\|_{H^2\times H^1 \to H^1\times L^2} = 0$

2.1.3 Spectrum of $\mathscr{A}$

We begin by (formally) computing the resolvent of the operator $\mathscr{A}$ as follows:

Let $u = (u_1, u_2)^\top$ and $f = (f_1, f_2)^\top$ then

$$(isI - \mathscr{A})u = f$$

This is equivalent to

$$\begin{aligned} isu_1 - u_2 &= f_1 \\ (-\partial_x^2 + 1)u_1 + (is + \gamma(x))u_2 &= f_2 \end{aligned}$$

or

$$\begin{aligned} u_1 &= (-\partial_x^2 + 1 + is\gamma(x) - s^2)^{-1}\left((is + \gamma(x))f_1 + f_2\right) \\ u_2 &= (-\partial_x^2 + 1 + is\gamma(x) - s^2)^{-1}\left(isf_2 - (-\partial_x^2 + 1)f_1\right) \end{aligned}$$

Hence, if we introduce the resolvent operator $R(is) := (-\partial_x^2 + 1 + is\gamma(x) - s^2)^{-1}$, then resolvent operator of $\mathscr{A}$ is denoted by $\mathscr{R}(is, \mathscr{A})$ and is given by

$$\mathscr{R}(is, \mathscr{A}) = \begin{pmatrix} R(is)(is + \gamma(x)) & R(is) \\ \\ R(is)(is)(\gamma(x) + is) - I & R(is)(is) \end{pmatrix}. \tag{2.4}$$

From this, we see that in order to study $\mathscr{R}(is, \mathscr{A})$ it suffices to understand $R(is)$. In fact, by inspecting the form of the resolvent (2.4), we have the following.

Lemma 3. *The following are equivalent*

(i) $\mathrm{i}s \in \rho(\mathscr{A})$

(ii) $0 \in \rho(-\partial_x^2 + 1 + is\gamma(x) - s^2)$*, that is*

$$R(is) = (-\partial_x^2 + 1 + is\gamma(x) - s^2)^{-1} : L^2 \to L^2$$

and in addition, $R(is) : L^2 \to H^1$.

In fact, $\mathrm{i}s$ *is an eigenvalue of* $\mathscr{A}$ *if and only if* 0 *is an eigenvalue of* $-\partial_x^2 + 1 + \mathrm{i}s\gamma(x) - s^2$.

Henceforth, we denote $A_s := (-\partial_x^2 + 1 + is\gamma - s^2)$.

Note: In this lemma, we consider s fixed. In particular, we are not concerned with the behavior of the various norms as $|s| \to \infty$. This is a much more subtle issue, that we will deal with later.

According to Lemma 3, the set $\sigma(\mathscr{A}) \cap i\mathbb{R}$ can be characterized as those $is, s \in \mathbb{R}$, for which there exists $g_n \in H^2(\mathbf{R})$ with $\|g_n\|_{H^2} = 1$, so that

$$\lim_n \|A_s g_n\|_{L^2} = 0.$$

The purely imaginary spectrum $\sigma(\mathscr{A}) \cap i\mathbb{R}$ (if any!), naturally consists of two subsets - eigenvalues and the rest, which we call essential spectrum. Here, we depart from the usual definition, where eigenvalues of infinite multiplicities are considered as part of the essential spectrum. We will see though, that since eigenvalues do not appear in our setup, at least on the important set $\sigma(\mathscr{A}) \cap i\mathbb{R}$, this is not consequential. Namely, is is an eigenvalue, if there exists $g_s \neq 0$, $g_s \in H^2(\mathbf{R})$, so that $A_s g_s = 0$.

Proposition 2. *Let* $\gamma \geq 0, \gamma \neq 0$ *be a continuous function. Then,*

(i) $\mathscr{A}$ *has no purely imaginary eigenvalues.*

(ii) $i \in \sigma(\mathscr{A})$ *if and only if* $\sigma(\mathscr{A}) \supseteq \{i\lambda, \lambda \in \mathbb{R} : |\lambda| \geq 1\}$.

Finally, if there is $\delta > 0$*, so that* $\gamma(x) \geq \delta > 0$*, then* $\sigma(\mathscr{A}) \cap i\mathbb{R} = \emptyset$.

Proof. We show that there are no eigenvalues. First, we rule out the case that $s = 0$.

For $s = 0$, by Lemma 3, 0 will be an eigenvalue of $(-\partial_x^2 + 1)$. If so, there exist $g \neq 0$ such that $(-\partial_x^2 + 1)g = 0$, which is impossible -just take a dot product with g to conclude $\|g'\|_{L^2}^2 + \|g\|_{L^2}^2 = 0$, so $g = 0$.

So, take $s \neq 0$. Assume that there is an eigenvalue is , $s \neq 0$ of $\mathscr{A}$. Again by Lemma 3, 0 will be an eigenvalue of $(-\partial_x^2 + 1 + is\gamma - s^2)$. Let $f = f_1 + if_2, f \neq 0$ be the corresponding eigenfunction of eigenvalue 0. Then, taking real and imaginary part of the equation $(-\partial_x^2 + 1 + is\gamma - s^2)f = 0$, we obtain

$$\left| \begin{array}{l} (-\partial_x^2 + (1 - s^2))f_1 - s\gamma f_2 = 0, \\ (-\partial_x^2 + (1 - s^2))f_2 + s\gamma f_1 = 0. \end{array} \right.$$

Taking dot products with f_2 and f_1 respectively and subtracting, we obtain

$$\int_{\mathbf{R}} \gamma(x)(f_1^2 + f_2^2)dx = 0. \tag{2.5}$$

Recall $\gamma \geq 0$. Since $\gamma \neq 0$, let (a, b) be an interval on which $\gamma(x) > 0$. Then, (2.5) implies that $f_1(x) = f_2(x) = 0$ for $x \in (a, b)$. By the uniqueness theorem for second order ODE's, $f_1 = f_2 = 0$ for the intervals $(-\infty, a), (b, \infty)$, so $f_1 = f_2 = 0$, contradiction.

Clearly, if $\sigma(\mathscr{A}) \supseteq \{i\lambda, \lambda \in \mathbb{R} : |\lambda| \geq 1\}$, it follows that $i \in \sigma(\mathscr{A})$. Now, assume that $i \in \sigma(\mathscr{A})$. It follows that for a sequence g_n with $\|g_n\|_{H^2} = 1$, we have

$$(-\partial_x^2 + i\gamma)g_n = f_n,$$

where $\|f_n\|_{L^2} \to 0$. Taking dot product with g_n and then imaginary part yields

$$0 \le \int \gamma |g_n|^2 = \Im \langle f_n, g_n \rangle \le \|f_n\|_{L^2} \|g_n\|_{L^2} \to 0.$$

It follows that $\|\sqrt{\gamma} g_n\|_{L^2}^2 = \int \gamma |g_n|^2 \to 0$. Let $\tilde{f}_n := f_n - i\gamma g_n$. Clearly, $\|\tilde{f}_n\|_{L^2} \to 0$ and $-g_n'' = \tilde{f}_n$. Note that since $\|g_n''\|_{L^2} = \|\tilde{f}_n\|_{L^2} \to 0$, we have

$$1 = \|g_n\|_{H^2} \sim \|g_n''\|_{L^2} + \|g_n\|_{L^2},$$

whence $\liminf_n \|g_n\|_{L^2} > 0$.

Now, let $s \in \mathbb{R}$ such that $|s| > 1$. Consider $\mu := \sqrt{s^2 - 1} > 0$. Introduce $u_n := e^{i\mu x} g_n$, so $\liminf_n \|u_n\|_{L^2} = \liminf_n \|g_n\|_{L^2} > 0$. Compute

$$A_s u_n = (-\partial_x^2 + is\gamma - \mu^2)(g_n e^{i\mu x}) = e^{i\mu x}(-g_n'' - 2i\mu g_n' + is\gamma g_n)$$

We have

$$\|A_s u_n\|_{L^2} \le \|g_n''\|_{L^2} + 2\mu \|g_n'\|_{L^2} + |s| \|\gamma g_n\|_{L^2}$$

Since all of the quantities on the right were shown to converge to zero, it follows that $\lim_n \|A_s u_n\|_{L^2} = 0$, while $\liminf_n \|u_n\|_{L^2} > 0$. Thus, $is \in \sigma(\mathscr{A})$ for all $s \in \mathbb{R}$ such that $|s| > 1$.

For the last part, assume that $\gamma(x) \ge \delta$ and yet is is in $\sigma(\mathscr{A})$. We saw $s = 0$ is not an option. So, $s \ne 0$. That is

$$(-\partial_x^2 + 1 - s^2 + is\gamma) g_n = f_n. \tag{2.6}$$

Taking dot product with g_n and then imaginary parts yields

$$|s| \int \gamma |g_n|^2 dx \le |\langle f_n, g_n \rangle| \le \|f_n\| \|g_n\|.$$

It follows that

$$\delta|s|\int|g_n|^2dx \le \|f_n\|\|g_n\| \to 0,$$

so $\|g_n\| \to 0$. But from the equation (2.6),

$$\|g_n''\|_{L^2} \le C(|s^2-1|\|g_n\| + \|g_n\| + \|f_n\|) \to 0.$$

So, it follows that $\|g_n\|_{H^2} \to 0$, a contradiction.

$\square$

We now provide a sufficient condition for $\sigma(\mathscr{A}) \cap i\mathbb{R} \neq \emptyset$, which turns out, in a roundabout way, to be necessary as well.

Proposition 3. *Let $\gamma \ge 0$ be a bounded and continuous function, not identically zero. Assume that* (2.3) *does not hold, that is*

$$\liminf_{N\to\infty} \inf_{y\in\mathbf{R}} \frac{1}{N}\int_y^{y+N} \gamma(z)dz = 0. \tag{2.7}$$

Then, $\sigma(\mathscr{A}) \supseteq \{i\lambda, \lambda \in \mathbb{R} : |\lambda| \ge 1\}$.

Proof. By Proposition 2, it suffices to check that $i \in \sigma(\mathscr{A})$. It will be an element of the essential spectrum, since as we have shown there are no eigenvalues. By (2.7), we can find a sequences $y_j \in \mathbf{R}$, $N_j \to \infty$, so that

$$\lim_j \frac{1}{N_j}\int_{y_j}^{y_j+N_j} \gamma(z)dz = 0.$$

Consider $\Psi \neq 0 \in C_0^\infty(\mathbf{R})$ with $0 \le \Psi(z) \le 1$, so that $\Psi(z) = 0$ for $z < 0$ and $\Psi(z) = 0, z > 1$. Let $\varepsilon_j := N_j^{-1} \to 0$ and take u_j so that

$$u_j(x) := \sqrt{\varepsilon_j}\Psi(\varepsilon_j(x-y_j)).$$

Clearly, $\|u_j''\|_{L^2} \to 0$, while $\|u_j\|_{L^2} = \|\Psi\|_{L^2} = O(1)$.

Recall $A_s = (-\partial_x^2 + 1 + is\gamma - s^2)$. We compute the norm of A_s for $s = 1$ as follows

$$\|A_1(u_j)\|_{L^2} = \|(-\partial_x^2 + i\gamma)u_j\|_{L^2} \le C(\|u_j''\|_{L^2} + \|\gamma u_j\|_{L^2}).$$

We have already seen $\|u_j''\|_{L^2} \to 0$. For the other term,

$$\|\gamma u_j\|_{L^2}^2 \le \|\gamma\|_{L^\infty} \varepsilon_j \int \gamma(x) |\Psi(\varepsilon_j(x-y_j))|^2 dx \le \|\gamma\|_{L^\infty} \frac{1}{N_j} \int_{y_j}^{y_j+N_j} \gamma(z) dz.$$

It follows that $\lim_j \|\gamma u_j\|_{L^2} = 0$, whence Proposition 3 is established.

□

2.1.4 The Analysis of Control Hypothesis

Let us analyze (2.3) in a more quantitative way. It means that there exists κ_γ and N_γ, so that for all $N > N_\gamma$ and for all $y \in \mathbf{R}$, we have

$$\frac{1}{N} \int_y^{y+N} \gamma(z) dz \ge \kappa_\gamma. \tag{2.8}$$

We have the following technical lemma, which will be useful later on.

Lemma 4. *Let $\tilde{\gamma} \ge \gamma \ge 0$ are continuous functions, so that γ satisfies* (2.8). *Then, for every $x, y \in \mathbf{R}$*

$$exp\left(-\int_{\min(x,y)}^{\max(x,y)} \tilde{\gamma}(z) dz \right) \le e^{2N_\gamma \kappa_\gamma} e^{-\kappa_\gamma |x-y|}. \tag{2.9}$$

Proof. Consider the case $0 \le x < y$. Clearly, the case $x < y < 0$ follows by symmetry and then the case $x < 0 < y$ follows by applying the previous two cases to $x < 0 = y$ and $0 = x < y$.

We bound $\int_x^y \tilde{\gamma}(z)dz \geq 0$, if $y-x<N_\gamma$. When $y-x \geq N_\gamma$, we have by (2.8),

$$\int_x^y \tilde{\gamma}(z)dz \geq \kappa_\gamma(y-x).$$

Overall,

$$\begin{aligned} exp(-\int_{\min(x,y)}^{\max(x,y)} \tilde{\gamma}(z)dz) &\leq \begin{cases} 1 & y-x<N_\gamma \\ exp(-\kappa_\gamma(y-x)) & y-x \geq N_\gamma \end{cases} \\ &\leq e^{N_\gamma \kappa_\gamma} e^{-\kappa_\gamma(y-x)}. \end{aligned}$$

□

2.2 Proof of Theorem 23

We stat with a technical result that gives bounds for the resolvent, under the appropriate condition (2.3). For all practical purposes, this is essentially the implication $(i) \Rightarrow (ii)$ of Theorem 23. For technical reasons, however, we will need to assume (as a preliminary step) that the spectrum does not intersect the imaginary access, that is $\sigma(\mathscr{A}) \cap i\mathbb{R} = \emptyset$, so that the various quantities are well-defined. We remove this assumption later - in fact, we show, in a roundabout way, that indeed the property $\sigma(\mathscr{A}) \cap i\mathbb{R} = \emptyset$ follows from (2.3) alone.

2.2.1 The main technical step

Proposition 4. *Let $\gamma(x) \geq 0$ is a positive, continuous function, which satisfies* (2.3) *or equivalently* (2.9). *In addition, assume that $\sigma(\mathscr{A}) \cap i\mathbb{R} = \emptyset$. Accordingly, let $s \in \mathbf{R}$, $f \in L^2(\mathbf{R})$ and $u \in L^2(\mathbf{R})$ satisfy the resolvent equation*

$$(-\partial_x^2 + 1 + is\gamma(x) - s^2)u = f. \tag{2.10}$$

Then for every $\delta > 0$, there is a constant $C_{\delta,\kappa,N}$, so that for all $s \in \mathbb{R}$ such that $|s|^2 \in [0, 1-\delta) \cup (1+\delta, \infty)$, we have

$$\|u\|_{L^2(\mathbf{R})} \leq \frac{C_{\delta,\kappa,N}}{1+|s|} \|f\|_{L^2(\mathbf{R})}, \tag{2.11}$$

where κ, N are the quantitative bounds of γ from (2.8)*. The constants N and κ have subscript γ, however we will remove this in the rest of the presentation*

Proof. We begin by pairing the equation (2.10) with u and taking the real part, we obtain by using Cauchy-Schwartz, for $s^2 < 1 - \delta$

$$\|u'\|_{L^2}^2 + (1-s^2)\|u\|_{L^2}^2 = \Re\langle f, u\rangle \leq C_s \|f\|_{L^2(\mathbb{R})}^2 + \frac{1-s^2}{2}\|u\|_{L^2}^2.$$

It follows that

$$\|u\|_{H^1(\mathbb{R})}^2 \leq C_\delta \|f\|_{L^2(\mathbb{R})}^2,$$

Note that from this proof, the constant C_δ may blow up as $\delta \to 0$.

We now consider the case $|s|^2 \geq 1 + \delta$. We only consider the case when s is positive, however the case for negative s can be obtain by changing s to $-s$.

Let $0 < \varepsilon << 1$ be a small enough real, to be selected later. Introduce $\mu_s := \sqrt{s^2 - 1} \geq \sqrt{\delta} > 0$. Clearly, for $c_\delta |s| \leq \mu_s \leq C_\delta |s|$. Henceforth, all constants will implicitly depend on δ, but we will omit this dependence.

We introduce the operators $P_{\sim s}$, $P_{\sim -s}$ and $P_{\sim(s,-s)}$ through Fourier transformation as follows

$$
\begin{aligned}
\widehat{P_{\sim s}(f)}(\xi) &= \hat{f}(\xi)\ \psi\left(\frac{\xi-\mu_s}{\varepsilon}\right)\\
P_{\widehat{\sim(-s)}(f)}(\xi) &= \hat{f}(\xi)\ \psi\left(\frac{\xi+\mu_s}{\varepsilon}\right),\\
P_{\not\sim(s,-s)}(f)(\xi) &= (Id - P_{\sim s} - P_{\sim(-s)})f.
\end{aligned}
$$

where $\psi \in C_0^\infty(\mathbf{R})$ is an even function $\psi(z) = 1$ for $|z| < 1$ and $\psi(z) = 0, |z| > 2$.

Further, we use the simple notation for $P_{\sim s}u(x) := u_{\sim s}(x)$, $P_{\not\sim s}u(x) := u_{\not\sim s}(x)$ and $P_{\not\sim(s,-s)}(u(x)) := u_{\not\sim(s,-s)}(x)$.

Next, taking dot product of (2.10) with u and taking imaginary parts and Cauchy-Schwartz's inequality, yields the following estimates

$$
s\int_{\mathbb{R}} \gamma(x)|u|^2 dx \leq \|f\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}.
$$

Thus, we can conclude

$$
\|\sqrt{\gamma}u\|_{L^2} \leq \varepsilon\|u\|_{L^2} + C_\varepsilon \frac{\|f\|}{s} \tag{2.12}
$$

where C_ε is a constant which depends on ε.

Next, we apply $P_{\not\sim(s,-s)}$ on both side of the equation (2.10) to get

$$
(-\partial_x^2)u_{\not\sim(s,-s)}(x) - \mu_s^2 u_{\not\sim(s,-s)}(x) = -is(\gamma u)_{\not\sim(s,-s)}(x) + f_{\not\sim(s,-s)}(x).
$$

Applying Fourier Transformation on both sides and using that ξ is away from μ_s and $-\mu_s$, we get

$$
\widehat{u}_{\not\sim(s,-s)}(\xi) = \frac{-is}{(\xi^2-\mu_s^2)}\left(\widehat{(\gamma u)}_{\not\sim(s,-s)}(\xi)\right) + \frac{1}{\xi^2-\mu_s^2}\widehat{f}_{\not\sim(s,-s)}(\xi).
$$

On the support of $\widehat{u}_{\not\sim(s,-s)}(\xi)$, we clearly have $|\frac{-is}{(\xi^2-\mu_s^2)}| \leq C$, for some constant C. This gives the following estimate,

$$
\begin{aligned}
\|u_{\not\sim(s,-s)}\|_{L^2} &\leq C\left(\|(\gamma u)_{\not\sim(s,-s)}\|_{L^2}+\frac{\|f_{\not\sim(s,-s)}\|_{L^2}}{s}\right)\\
&\leq C\left(\|\gamma u\|_{L^2}+\frac{\|f\|_{L^2}}{s}\right)
\end{aligned}
$$

Then by (2.12), together with the fact that $\gamma \leq C\sqrt{\gamma}$ a.e, we get

$$
\|u_{\not\sim(s,-s)}\|_{L^2} \leq \varepsilon\|u\|_{L^2}+C_\varepsilon\frac{\|f\|_{L^2}}{s}. \tag{2.13}
$$

Next, we project $P_{\sim s}$ on both sides of the equation (2.10). Adding and subtracting $i\mu_s\gamma u_{\sim s}(x)$ we get

$$
-\partial_x^2 u_{\sim s}(x)+i\mu_s\gamma(x)u_{\sim s}(x)-\mu_s^2 u_{\sim s}(x)=f_{\sim s}(x)-is(\gamma u)_{\sim s}(x)+i\mu_s\gamma u_{\sim s}(x)
$$

Let $f=e^{i\mu_s x}F$ and $u=e^{i\mu_s x}U$ and observe that $P_{\sim s}(e^{i\mu_s x}g)=e^{i\mu_s x}P_{\sim 1}(g)$. We get

$$
-\partial_x^2 U_{\sim 1}(x)-2i\mu_s\frac{d}{dx}U_{\sim 1}(x)+i\mu_s\gamma U_{\sim 1}(x)=F_{\sim 1}(x)-is(\gamma U)_{\sim 1}+i\mu_s\gamma U_{\sim 1}(x).
$$

Hence, dividing by $-2i\mu_s$,

$$
\begin{aligned}
\frac{d}{dx}(U_{\sim 1}(x))-\frac{\gamma(x)}{2}U_{\sim 1}(x)=\frac{i}{2\mu_s}\partial_x^2 U_{\sim 1}(x)+\frac{i}{2\mu_s}F_{\sim 1}(x)+\frac{s}{2\mu_s}(\gamma U)_{\sim 1}(x)\\
-\frac{1}{2}\gamma(x)U_{\sim 1}(x).
\end{aligned}
$$

Using the integrating factor $e^{-\frac{1}{2}\int_0^x\gamma(y)dy}$, we solve in the form

$$
U_{\sim 1}(x)=-\int_x^\infty e^{\frac{1}{2}\int_y^x\gamma(z)dz}G(y)\,dy=-T(G),
$$

where $G = \frac{i}{2\mu_s}\partial_x^2 U_{\sim 1} + \frac{i}{2\mu_s}F_{\sim 1} + \frac{s}{2\mu_s}(\gamma U)_{\sim 1} - \frac{1}{2}\gamma U_{\sim 1}$ and T is an operator in the form

$$T(f)(x) = \int_x^\infty e^{\frac{1}{2}\int_y^x \gamma(z)dz} f(y)dy.$$

Note that by the bound (2.9), we have that

$$|T(f)(x)| \leq \int_x^\infty e^{2N\kappa} e^{-\kappa|x-y|}|f(y)|dy,$$

whence

$$\|Tf\|_{L^2} \leq \|e^{2N\kappa}e^{-\kappa|\cdot|}\|_{L^1}\|f\|_{L^2} = \frac{2e^{2N\kappa}}{\kappa}\|f\|_{L^2}.$$

In particular, the operator norm $\|T\|_{L^2\to L^2}$ depends only on N, κ.

Now, since $U_{\sim 1}(x) = e^{-i\mu_s x}u_{\sim s}(x)$, rewrite

$$\gamma U_{\sim 1}(x) = e^{-i\mu_s x}\gamma u_{\sim s}(x) = e^{-i\mu_s x}((\gamma u)(x) - \gamma(x)u_{\sim -s}(x) - \gamma(x)u_{\nsim(s,-s)}(x)).$$

Thus, introduce the effective right hand side

$$G_1 := \frac{i}{2\mu_s}\partial_x^2 U_{\sim 1} + \frac{i}{2\mu_s}F_{\sim 1} + \frac{s}{2\mu_s}(\gamma U)_{\sim 1} + e^{-i\mu_s x}(\gamma u - \gamma u_{\nsim(s,-s)}),$$

so that $u_{\sim s}(x)$ and $u_{\sim -s}(x)$ are now in the relation

$$u_{\sim s}(x) - \frac{1}{2}e^{i\mu_s x}T(e^{-i\mu_s x}\gamma(x)u_{\sim -s}(x)) = e^{i\mu_s x}T(G_1). \tag{2.14}$$

Multiplying the last equation by $\sqrt{\gamma}$ and by introducing a new linear operator $T_s f := \frac{1}{2}e^{i\mu_s x}\sqrt{\gamma}T(e^{-i\mu_s x}\sqrt{\gamma}f)$, we can record the last relation as follows

$$\sqrt{\gamma}u_{\sim s} - T_s(\sqrt{\gamma}u_{\sim -s}) = e^{i\mu_s x}\sqrt{\gamma}T(G_1). \tag{2.15}$$

Similar arguments apply to $u_{\sim -s}$. More concretely, projecting $P_{\sim(-s)}$ on both sides to the equation (2.10), and adding $i\mu_s\gamma u_{\sim(-s)}$, we get

$$-\partial_x^2 u_{\sim(-s)}(x) + i\mu_s\gamma(x)u_{\sim(-s)}(x) - \mu_s^2 u_{\sim(-s)}(x) = -is(\gamma u)_{\sim(-s)}(x) \qquad (2.16)$$
$$+ i\mu_s\gamma u_{\sim(-s)}(x) + f_{\sim(-s)}(x).$$

Letting now $f = e^{-i\mu_s x}\bar{F}$ and $u = e^{-i\mu_s x}\bar{U}$ and observing that

$$P_{\sim(-s)}(e^{-i\mu_s x}g) = e^{-i\mu_s x}P_{\sim 1}(g).$$

By (2.16), we obtain the equation

$$\frac{d}{dx}\bar{U}_{\sim 1}(x) + \frac{\gamma(x)}{2}\bar{U}_{\sim 1}(x) = -\frac{i}{2\mu_s}\partial_x^2\bar{U}_{\sim 1}(x) - \frac{s}{2\mu_s}(\gamma(x)\bar{U})_{\sim 1}(x)$$
$$+\frac{1}{2}\gamma(x)\bar{U}_{\sim 1}(x) - \frac{i}{2\mu_s}\bar{F}_{\sim 1}(x)$$

With the help of the integrating factor $e^{\frac{1}{2}\int_0^x \gamma(z)dz}$, we solve the equation (by integrating from $-\infty$ to x) as follows

$$\bar{U}_{\sim 1}(x) = \int_{-\infty}^{x} e^{\frac{1}{2}\int_x^y \gamma(z)dz} D(y)dy = T^*(D), \qquad (2.17)$$

where the right hand side is $D = -\frac{i}{2\mu_s}\partial_x^2\bar{U}_{\sim 1} - \frac{s}{2\mu_s}(\gamma(x)\bar{U})_{\sim 1} + \frac{1}{2}\gamma\bar{U}_{\sim 1} - \frac{i}{2\mu_s}\bar{F}_{\sim 1}$. Again,

$$\gamma\bar{U}_{\sim 1}(x) = e^{i\mu_s x}\gamma u_{\sim -s}(x) = e^{i\mu_s x}(\gamma(x)u(x) - \gamma(x)u_{\sim s}(x) - \gamma(x)u_{\not\sim(s,-s)}(x))$$

The effective right hand side becomes

$$D_1 := -\frac{i}{2\mu_s}\partial_x^2\bar{U}_{\sim 1} - \frac{s}{2\mu_s}(\gamma(x)\bar{U})_{\sim 1} - \frac{i}{2\mu_s}\bar{F}_{\sim 1} + \frac{1}{2}e^{i\mu_s x}\gamma u - \gamma u_{\not\sim(s,-s)})$$

and we obtain the following reformulation of (2.17),

$$u_{\sim -s} + \frac{1}{2} e^{-i\mu_s x} T^*(e^{i\mu_s x} \gamma u_{\sim s}) = e^{-i\mu_s x} T^*(D_1). \tag{2.18}$$

Again, a multiplication with $\sqrt{\gamma}$ resolves (2.18) to

$$\sqrt{\gamma} u_{\sim -s} + T_s^*(\sqrt{\gamma} u_{\sim s}) = e^{-i\mu_s x} \sqrt{\gamma} T^*(D_1). \tag{2.19}$$

Where $T_s^* f := \frac{1}{2} e^{-i\mu_s x} \sqrt{\gamma} T^*(e^{i\mu_s x} \sqrt{\gamma} f)$

Combining (2.15) and (2.19) allows us to control $\sqrt{\gamma} u_{\sim \pm s}$ and ultimately $u_{\sim \pm s}$. Indeed,

$$\begin{aligned} \sqrt{\gamma} u_{\sim s} &= T_s(\sqrt{\gamma} u_{\sim -s}) + e^{i\mu_s x} \sqrt{\gamma} T(G_1) \\ &= T_s(-T_s^*(\sqrt{\gamma} u_{\sim s}) + e^{-i\mu_s x} \sqrt{\gamma} T^*(D_1)) + e^{i\mu_s x} \sqrt{\gamma} T(G_1) \end{aligned}$$

whence we obtain the following operator equation for $\sqrt{\gamma} u_{\sim s}$

$$(Id + T_s T_s^*)(\sqrt{\gamma} u_{\sim s}) = T_s(e^{-isx} \sqrt{\gamma} T^*(D_1)) + e^{isx} \sqrt{\gamma} T(G_1).$$

Since $(Id + T_s T_s^*)$ is a symmetric operator, $(Id + T_s T_s^*) \geq Id$, we have that it is invertible (in fact, $\|(Id + T_s T_s^*)^{-1}\|_{L^2 \to L^2} \leq 1$),

$$\begin{aligned} \|\sqrt{\gamma} u_{\sim s}\|_{L^2} &\leq \|T_s(e^{-i\mu_s x} \sqrt{\gamma} T^*(D_1)) + e^{i\mu_s x} \sqrt{\gamma} T(G_1)\|_{L^2} \\ &\leq C(\|G_1\| + \|D_1\|), \end{aligned} \tag{2.20}$$

where in the last step, we have used that T, T_s, together with their adjoints are bounded on L^2, with bounds depending upon κ, N only.

So, it remains to find suitable bounds for $\|G_1\|_{L^2}, \|D_1\|_{L^2}$. We just provide the bounds for $\|G_1\|$, as the bounds for $\|D_1\|$ proceed in an identical way. Clearly,

$$\|\frac{i}{2\mu_s}F_{\sim 1}\|_{L^2} \leq \frac{C}{s}\|F\|_{L^2} = \frac{C}{s}\|f\|_{L^2}.$$

By Plancherel's

$$\begin{aligned}
\|\frac{i}{2\mu_s}\partial_x^2 U_{\sim 1}\|_{L^2} &\leq \frac{C}{s}\|\xi^2\widehat{U_{\sim 1}}\|_{L^2} \leq \frac{C}{s}\|\xi^2\widehat{U}(\xi)\psi\left(\frac{\xi}{\varepsilon}\right)\|_{L^2} \leq \frac{C\varepsilon^2}{s}\|U_{\sim 1}\|_{L^2} \\
&\leq \varepsilon\|u_{\sim s}\|_{L^2(\mathbb{R})},
\end{aligned}$$

provided $C\sqrt{2}\varepsilon \leq 1$. Next, by (2.12),

$$\|\frac{1}{2}(\gamma U)_{\sim 1} + e^{-i\mu_s x}\gamma u\|_{L^2} \leq \|\gamma U\|_{L^2} + \|\gamma u\| = 2\|\gamma u\|_{L^2} \leq \varepsilon\|u\|_{L^2} + C_\varepsilon\frac{\|f\|}{s}.$$

Finally, by (2.13),

$$\|\gamma u_{\not\sim(s,-s)}\|_{L^2} \leq C\|u_{\not\sim(s,-s)}\|_{L^2} \leq \varepsilon\|u\|_{L^2} + C_\varepsilon\frac{\|f\|}{s}.$$

Altogether, we obtain

$$\|G_1\| + \|D_1\| \leq C\varepsilon\|u\|_{L^2} + C_\varepsilon\frac{\|f\|}{s}. \tag{2.21}$$

Based on (2.20) and (2.21), we obtain the following estimate

$$\|\sqrt{\gamma}u_{\sim s}\|_{L^2} \leq C\varepsilon\|u\|_{L^2} + C_\varepsilon\frac{\|f\|}{s}.$$

Clearly, the same estimate holds for $\|\sqrt{\gamma}u_{\sim -s}\|_{L^2}$.

In order to get estimates for $\|u_{\sim s}\|_{L^2}, \|u_{\sim -s}\|_{L^2}$, one can now use the forms (2.14) and

(2.18), to deduce

$$\begin{aligned}\|u_{\sim s}\|+\|u_{\sim -s}\| &\leq C(\|\sqrt{\gamma}u_{\sim -s}\|+\|\sqrt{\gamma}u_{\sim s}\|+\|G_1\|+\|D_1\|)\\ &\leq C\varepsilon\|u\|_{L^2}+C_\varepsilon\frac{\|f\|}{s}.\end{aligned}$$

Finally, with some absolute constant C (and with some $C_\varepsilon \sim \varepsilon^{-1}$), we have

$$\|u\|_{L^2} \leq \|u_{\sim s}\|+\|u_{\sim -s}\|+\|u_{\not\sim(s,-s)}\| \leq C\varepsilon\|u\|_{L^2}+C_\varepsilon\frac{\|f\|}{s}.$$

Clearly, a choice of ε such that $C\varepsilon<\frac{1}{2}$, we obtain the desired bound (2.11).

$\square$

Next, we need an estimate for $L^2 \to H^1$ bounds of the resolvent $(-\partial_x^2+1+is\gamma(x)-s^2)^{-1}$.

Proposition 5. *Let $\gamma\geq 0, \gamma\neq 0$ be a continuous function, that satisfies* (2.9)*, with constants κ, N. In addition, assume $\sigma(\mathscr{A})\cap i\mathbb{R}=\emptyset$.*

Let $\delta>0$ and $|s|^2\in(0,1-\delta)\cup(1+\delta,\infty)$. Recalling $R(is)=(-\partial_x^2+1+is\gamma(x)-s^2)^{-1}$, we have the following estimates

$$\|R(is)\|_{L^2\to H^1}\leq C_{\delta,\kappa,N} \tag{2.22}$$

$$\|R(is)\|_{H^{-1}\to L^2}\leq C_{\delta,\kappa,N}$$

As a consequence,

$$\|(is-\mathscr{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2}\leq C_{\delta,\kappa,N}. \tag{2.23}$$

Proof. Let $u\in H^1(\mathbb{R})$ be the solution of (2.24)

$$(-\partial_x^2+1+is\gamma(x)-s^2)u=f \tag{2.24}$$

for $f \in L^2$.

Taking dot product of (2.24) with u yields,

$$\langle -\partial_x^2 u, u\rangle + (1-s^2)\langle u, u\rangle \quad \leq \quad \|f\|_{L^2}\|u\|_{L^2}$$

Hence,

$$\|u\|_{H^1}^2 \leq \|f\|_{L^2}\|u\|_{L^2} + (s^2-1)\|u\|_{L^2}^2$$

By Proposition 4, we get

$$\|u\|_{H^1}^2 \quad \leq \quad C_{\delta,\kappa,N}\|f\|_{L^2}\frac{\|f\|_{L^2(\mathbb{R}^2)}}{1+|s|} + C_{\delta,\kappa,N}\frac{(s^2-1)}{(1+|s|)^2}\|f\|_{L^2(\mathbb{R})}^2.$$

This proves

$$\|R(is)\|_{L^2\to H^1} \leq C_{\delta,\kappa,N}.$$

Hence by duality

$$\|R(is)\|_{H^{-1}\to L^2} \leq C_{\delta,\kappa,N}. \tag{2.25}$$

We now focus on (2.23), that is we show that the resolvent $R(is,\mathscr{A})$ of $\mathscr{A}$ is bounded in $H^1(\mathbb{R})\times L^2(\mathbb{R})$. We estimate the norm of $R(is,\mathscr{A})$ as follows,

$$\begin{aligned}\left\| R(is,\mathscr{A})\begin{pmatrix} f \\ g \end{pmatrix}\right\|_{H^1\times L^2} &= \|R(is)(is+\gamma(x))f\|_{H^1} + \|R(is)g\|_{H^1} \\ &+ \|(R(is)(is)(\gamma(x)+is)-I)f\|_{L^2} + \|R(is)(is)g\|_{L^2}\end{aligned}$$

This implies that to estimate the norm of the resolvent operator $R(is,\mathscr{A})$ as an operator on $H^1\times L^2$, we need to obtain the following bounds

$$\|R(is)\| = O(1) : L^2 \to H^1,$$

$$\|R(is)(is+\gamma(x))\| = O(1) : H^1 \to H^1,$$

$$\|sR(is)\| = O(1) : L^2 \to L^2,$$

$$\|R(is)(is)(\gamma(x)+is)-I)\| = O(1) : H^1 \to L^2.$$

The estimates for $sR(is)$ and $R(is)$ are in (2.11) and (2.22) respectively. In order to estimate

$$\|R(is)(is)[\gamma(x)+is)]-I\|_{H^1\to L^2},$$

we use that

$$R(is)(is)[\gamma(x)+is)]-I = R(is)(\partial_x^2-1),$$

and hence, combining (2.25) together with the fact that $\partial_x^2 : H^1 \to H^{-1}$ is continuous. For $f \in H^1(\mathbb{R})$, we have

$$\begin{aligned}
\|(R(is)(is)[\gamma(x)+is)]-I)f\|_{L^2} &= \|R(is)(\partial_x^2-1)f\|_{L^2} \leq C\|(1-\partial_x^2)f\|_{H^{-1}} \\
&= C\|f\|_{H^1}
\end{aligned}$$

This proves:

$$R(is)(is)(\gamma(x)+is)-I = O(1) : H^1 \to L^2 \tag{2.26}$$

It remains to estimate the norm of

$$R(is)(is+\gamma(x)) : H^1 \to H^1.$$

We rewrite the above operator as

$$R(is)(is+\gamma(x)) = \frac{1}{is}[1+R(is)(\partial_x^2-1)] \tag{2.27}$$

If $f \in H^1$ and $\tilde{u} = R(is)(\partial_x^2-1)f \in H^1$, then

$$(-\partial_x^2+1+is\gamma(x)-s^2)\tilde{u} = (\partial_x^2-1)f \in H^{-1} \tag{2.28}$$

Pair the equation (2.28) with $\tilde{u}$ and take the real part to get,

$$\|\partial_x\tilde{u}\|_{L^2}^2 - (s^2-1)\|\tilde{u}\|_{L^2}^2 \leq \|(-\partial_x^2+1)f\|_{H^{-1}}\|\tilde{u}\|_{H^1} \leq \|f\|_{H^1}\|\tilde{u}\|_{H^1}.$$

By Cauchy Schwartz inequality, we get

$$\|\tilde{u}\|_{H^1}^2 \leq 2(s^2-1)\|\tilde{u}\|_{L^2}^2 + \|f\|_{H^1}^2. \tag{2.29}$$

Next, when we estimate the L^2- norm of $\tilde{u} = R(is)(\partial_x^2-1)f$, we used (2.26) to get

$$\|R(is)(\partial_x^2-1)f\|_{L^2} \leq C\|f\|_{H^1} \tag{2.30}$$

Combining the estimates (2.29) and (2.30) proves that

$$R(is)(\partial_x^2-1) = O(|s|) : H^1(\mathbb{R}) \to H^1(\mathbb{R}).$$

Then by the equation (2.27), we have

$$\|R(is)(is+\gamma(x))f\|_{L^2} \leq C\|f\|_{H^1}$$

Hence, $(is-\mathscr{A})^{-1} = O(1) : H^1 \times L^2 \to H^1 \times L^2$. $\square$

2.2.2 Proof of Theorem 23: the implication $(i) \Rightarrow (ii)$

Take any $\gamma \geq 0$, a continuous, bounded and non-negative function, that satisfies (2.3). We would now like to prove exponential decay of the semigroup, as required in (ii) of Theorem 23. This is basically what Proposition 4 does, except that it in addition also assumes $\sigma(\mathscr{A}) \cap i\mathbb{R} = \emptyset$. This eventually turns out to be the case, but we have not proved that yet.

Instead, we proceed by an approximating argument. More specifically, fix $\varepsilon > 0$ and consider $\gamma_\varepsilon(x) := \gamma(x) + \varepsilon$ and the corresponding operator $\mathscr{A}_\varepsilon$. We immediately observe two things. First, since $\gamma_\varepsilon \geq \varepsilon > 0$, we have by Proposition 2, that $\sigma(\mathscr{A}_\varepsilon) \cap i\mathbb{R} = \emptyset$. Second, γ_ε satisfies (2.8) with the constants κ, N of γ. Hence, γ_ε satisfies (2.9). Thus, we are ready to apply Proposition 4 to γ_ε. For a fixed $\delta > 0$ and $|s|^2 \in (1-\delta, 1+\delta)$, we have the estimate

$$\|(-\partial_x^2 + 1 + is(\gamma+\varepsilon) - s^2)^{-1}\|_{L^2 \to L^2} \leq \frac{C_{\delta,\kappa,N}}{1+|s|}. \tag{2.31}$$

In particular, note that the above bound is *independent* upon the parameter $\varepsilon > 0$. One can now take $\varepsilon \to 0+$ in order to obtain the operator $(-\partial_x^2 + 1 + is\gamma - s^2)^{-1}$, together with the desired bounds on its $L^2 \to L^2$ operator norm. This could be justifies in at least two ways. One is to show that for a fixed s, the family $\{(-\partial_x^2 + 1 + is(\gamma+\varepsilon) - s^2)^{-1}\}_{\varepsilon>0}$ is Cauchy in $B(L^2)$, by using the resolvent identity. More or less equivalently, we can directly construct $(-\partial_x^2 + 1 + is\gamma - s^2)^{-1}$ by the resolvent identity and the Neumann theorem as follows

$$\begin{aligned}&(-\partial_x^2 + 1 + is\gamma - s^2)^{-1} := \\ &(-\partial_x^2 + 1 + is(\gamma+\varepsilon) - s^2)^{-1}(Id - is\varepsilon(-\partial_x^2 + 1 + is(\gamma+\varepsilon) - s^2)^{-1})^{-1}.\end{aligned}$$

Indeed, in the formula above, the first inverse exists by (2.31), while the second inverse exists by von Neumann for all small enough ε, since

$$\|is\varepsilon(-\partial_x^2 + 1 + is(\gamma+\varepsilon) - s^2)^{-1}\|_{L^2 \to L^2} \leq C|s|\varepsilon \frac{C_{\delta,\kappa,N}}{1+|s|} < \frac{1}{2}.$$

Now that we have constructed $(-\partial_x^2+1+is\gamma-s^2)^{-1}$ for all $s\in\mathbb{R}$ such that $|s|^2\in(0,1-\delta)\cup(1+\delta,\infty)$, we deduce the bound

$$\|(-\partial_x^2+1+is\gamma-s^2)^{-1}\|_{L^2\to L^2}\leq\frac{C_{\delta,\kappa,N}}{1+|s|},\tag{2.32}$$

by simply letting $\varepsilon\to 0+$ in (2.31). In addition, this shows that $\{i\lambda:|\lambda|\neq 1\}\subset\rho(\mathscr{A})$, that is the whole imaginary line, with the possible exception of $\pm i$ are in the resolvent set of $\mathscr{A}$.

Now, we show that $\pm i$ also belong to the resolvent set of $\mathscr{A}$. Indeed, otherwise, we will have by Proposition 2, that $\sigma(\mathscr{A})\supset\{i\lambda:|\lambda|>1\}$, which is a contradiction. Thus, we have established that $\pm i\in\rho(\mathscr{A})$ or

$$\|(-\partial_x^2\pm i\gamma)^{-1}\|_{L^2\to L^2}\leq C.$$

Next, we show that (2.32) holds in a neighborhood of $|s|=1$ as well. We have by the resolvent identity

$$\begin{aligned}&(-\partial_x^2+1+is\gamma-s^2)^{-1}-(-\partial_x^2+i\gamma)^{-1}\\=\ &(-\partial_x^2+1+is\gamma-s^2)^{-1}[s^2-1+i\gamma(1-s)](-\partial_x^2+i\gamma)^{-1},\end{aligned}$$

whence we can represent

$$\begin{aligned}&(-\partial_x^2+1+is\gamma-s^2)^{-1}=\\&(-\partial_x^2+i\gamma)^{-1}(Id-(s-1)(s+1-i\gamma)(-\partial_x^2+i\gamma)^{-1})^{-1}.\end{aligned}$$

Clearly, for $s\in\mathbb{R}$ with $|s-1|<<1$, say $(10+\|\gamma\|_{L^\infty})|s-1|\|(-\partial_x^2+i\gamma)^{-1}\|_{L^2\to L^2}\leq\frac{1}{2}$, the right-hand side is a well-defined operator and in addition

$$\|(-\partial_x^2+1+is\gamma-s^2)^{-1}\|_{L^2\to L^2}\leq 2\|(-\partial_x^2+i\gamma)^{-1}\|_{L^2\to L^2}.$$

Thus, $s\to\|(-\partial_x^2+1+is\gamma-s^2)^{-1}\|_{L^2\to L^2}$ is bounded in a neighborhood of $s=1$ and similarly, in a neighborhood of $s=-1$. In the same fashion as in Proposition 5, we conclude that

$$\sup_{s\in\mathbf{R}}\|(is-\mathscr{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2}\leq C<\infty.$$

By the Gearhart-Prüss theorem, $\|T(t)(1-\mathscr{A})^{-1}\|_{H^1\times L^2\to H^1\times L^2}\leq Ce^{-\lambda_0}t$, for some $\lambda_0>0$. Since, $(1-\mathscr{A})^{-1}:H^1\times L^2\to H^2\times H^1$ and it is onto, we conclude that

$$\|T(t)g\|_{H^1\times L^2}\leq Ce^{-\lambda_0 t}\|g\|_{H^2\times H^1},$$

as stated.

Next, the implication $(ii)\Rightarrow(iii)$ is of course trivial. The equivalence of (iii) and (iv) is the essence of Theorem 25, see also Corollary 2. Finally, the implication $(iv)\Rightarrow(i)$ is contained in Proposition 3. This finishes the proof of Theorem 23.

Chapter 3

On the energy decay rates for the 1D damped fractional Klein-Gordon equation

In this chapter, we consider the fractional Klein-Gordon equation in one spatial dimension, subjected to a damping coefficient, which is non-trivial and periodic, or more generally strictly positive on a periodic set. We show that the energy of the solution decays at the polynomial rate $O(t^{-\frac{s}{4-2s}})$ for $0 < s < 2$ and at some exponential rate when $s \geq 2$. Our approach is based on the asymptotic theory of C_0 semigroups in which one can relate the decay rate of the energy in terms of the resolvent growth of the semigroup generator. The main technical result is a new observability estimate for the fractional Laplacian, which may be of independent interest.

3.1 Introduction

In this chapter, we consider the energy decay of the following fractional damped Klein-Gordon equation

$$u_{tt} + \gamma(x)u_t + (-\partial_{xx})^{s/2}u + mu = 0, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}, \tag{3.1}$$

where $m > 0$ and $\gamma(x) \geq 0$ is bounded below by a positive constant on a 2π-periodic set. The parameter s refers to the fractional order of the spatial derivative and describes the fractional nature of the equation. Here and throughout, $u(x,t)$ is generally a complex-valued function, and the pseudo-differential operator $(-\partial_{xx})^{s/2}$ is defined through its Fourier multiplier

$$\widehat{(-\partial_{xx})^{s/2}} f(\xi) = |\xi|^s \hat{f}(\xi),\ \xi \in \mathbb{R}.$$

The function $\gamma(x)$ denotes the damping force, which travels with velocity u_t and causes the loss of energy decay in the system. This energy decay is the main object of study in this article.

For the case $s = 2$, the operator $-\partial_{xx}$ denotes the positive Laplacian. In this case, (3.1) reduces to the well know classical Damped Klein-Gordon equation. It has been studied extensively in the last decade by many authors.

We show that for low order fractional power $0 < s < 2$, the rate of decay is algebraic. This is in sharp contrast with the case $s \geq 2$, where the solution has exponential rate of decay. So, it appears that $s = 2$ is exactly a threshold value, which separates the algebraic from exponential rate of decay, but unfortunately our method does not address the optimality of this exponent. This remains an open question for future investigations.

The main result of this chapter is as follow.

Theorem 26. *Let $m > 0$ and $0 \leq \gamma(x) \in L^\infty$ and that there exist $\varepsilon > 0$ and a $2\pi\mathbb{Z}$- invariant open set $\Omega \subset \mathbb{R}$ such that $\gamma(x) \geq \varepsilon$ for a.e. $x \in \Omega$. Then there exists $C > 0$ so that*

- *for $0 < s < 2$, we have*

$$\|(u(t), u_t(t))\|_{H^{s/2} \times L^2} \leq \frac{C}{1 + t^{\frac{s}{4-2s}}} \|(u(0), u_t(0))\|_{H^s \times H^{s/2}}. \tag{3.2}$$

- *for $s \geq 2$, there exists $\lambda_0 > 0$, so that*

$$\|(u(t), u_t(t))\|_{H^{s/2} \times L^2} \leq Ce^{-\lambda_0 t} \|(u(0), u_t(0))\|_{H^{s/2} \times L^2}. \tag{3.3}$$

The proof of Theorem 26 is based on the semigroup technique used in [29, 11, 16, 19], in which rather than estimating norm of the solution directly, we used the following two classical results. Gearhart-Prüss Theorem [15, 25] and Borichev-Tomilov Theorem in [9] make it possible to deduce sharp rates of energy decay from appropriate growth bounds on the norm of the resolvent of the semigroup‘s generator.

3.2 Observability Estimates

We start with a few preliminary notations.

3.2.1 Function spaces, Fourier transforms, symbols

The spaces $L^p(\mathbf{R}), 1 \leq p \leq \infty$ are defined in a standard way. The Fourier transform for us will be given by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx, \ \ f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi.$$

The operator $-\partial_{xx}$ can be realized as $\widehat{-\partial_{xx} f}(\xi) = \xi^2 \hat{f}(\xi)$. For any $s > 0$, one can write $\widehat{(-\partial_{xx})^{s/2} f}(\xi) = |\xi|^s \hat{f}(\xi)$.

The fractional Sobolev spaces $H^s(\mathbf{R})$ can be identified as the set of all functions f, so that $[(-\partial_{xx})^{s/2} + 1] f \in L^2(\mathbf{R})$. Alternatively, the norm is defined as follows

$$\|f\|^2_{H^s(\mathbf{R})} = \int_{-\infty}^{\infty} (1 + \xi^2)^s |\hat{f}(\xi)|^2 d\xi < \infty.$$

For periodic functions defined on $[-1,1]$, which are sufficiently smooth, there is the usual Fourier series representation

$$f = \sum_k f_k e^{ik\pi x}, f_k = \frac{1}{\sqrt{2}} \int_{-1}^{1} f(x) e^{-ik\pi x} dx,$$

with $\|f\|^2_{L^2[-1,1]} = \sum_k |f_k|^2$. The fractional operator $(-\partial_{xx})^{s/2}$ using functional calculus is defined through

$$(-\partial_{xx})^{s/2} f = \sum_{k=-\infty}^{\infty} (\pi|k|)^s f_k e^{ik\pi x},$$

for sufficiently smooth functions $f \in L^2[-1,1]$.

3.2.2 Main observability lemma for the fractional Laplacian

The following estimate, which may be of interest in its own right, gives L^2 control of the resolvent of the free Laplacian on its spectra, modulo an error term.

Theorem 27. *Let $s > 0$, $\lambda \geq 1$ and $\Omega \subset \mathbb{R}$ be a non-empty, $2\pi\mathbb{Z}$ invariant open set. For all $\lambda \in \mathbb{R}$, let $((-\partial_{xx})^{s/2} - \lambda)u = f$. Then, there exists C, so that*

$$\|u\|_{L^2} \leq C(< \lambda >^{\frac{1}{s}-1} \|f\|_{L^2} + \|u\|_{L^2(\Omega)}). \tag{3.4}$$

The observability estimate for $s = 2$ has been proved by Burq and Zworski in [13] on a two-dimensional compact manifold. Recently, Wunsch [29] extended these estimates to $\mathbb{R}^n$ under a periodic setting. In this note we prove similar observability estimate for the fractional case. In fact, in the case of one-dimension our estimate contains an additional decay factor, which helps us to improve Wunsch's results in the one-dimensional setting.

Let us explain the idea behind such result. Clearly, the difficult case is when $\lambda > 0$ and large. Since the spectrum, $\sigma((-\partial_{xx})^{s/2}) = \sigma_{a.c.}((-\partial_{xx})^{s/2}) = [0,\infty)$, we cannot expect $[(-\partial_{xx})^{s/2} - \lambda]^{-1}$ to be bounded on L^2, and it is not. Instead, (3.4) asserts that such an L^2 resolvent bound almost holds (with an additional decay rate of $\lambda^{\frac{1}{s}-1}$, which is important for our purposes), modulo an extra "control" term.

The method of proof is to first establish the above estimate on the bounded interval $[-1,1]$. We then use the technique of Wunsch, [29] to extend the result to the real line $\mathbb{R}$.

3.2.2.1 Observability on intervals

We start with an elementary lemma.

Lemma 5. *Let $s > 0$. Then, there exists d_s, D_s, so that for every $0 < x < y$*

$$d_s \max(x,y)^{s-1}|x-y| \leq |x^s - y^s| \leq D_s \max(x,y)^{s-1}|x-y|. \tag{3.5}$$

Proof. Start with the function $f_s(z) = \frac{1-z^s}{1-z}$, defined for $z \in [0,1]$. Clearly this is a continuous function on $[0,1]$ (defined at $z = 1$ via $f(1) = s$), so it has a minimum and maximum, say d_s, D_s. That is,

$$d_s(1-z) \leq 1 - z^s \leq D_s(1-z).$$

Without loss of generality $x \leq y$ and apply the previous inequality to $z = \frac{x}{y}$. This shows (3.5). $\square$

Lemma 6. *Let $s > 0$. Consider the following damped fractional Laplace equation on $[-1,1]$*

$$((-\partial_{xx})^{s/2} - \lambda)u = f, x \in [-1,1]. \tag{3.6}$$

Then for every $\delta > 0$ there is C_δ so that

$$\|u\|_{L^2[-1,1]} \leq C_\delta [< \lambda >^{\frac{1}{s}-1} \|f\|_{L^2[-1,1]} + \|u\|_{L^2[-\delta,\delta]}] \tag{3.7}$$

for solutions u of (3.6)*, where $< \lambda > := (1 + |\lambda|^2)^{1/2}$.*

Proof. We can always assume that u, f are real, otherwise split in real and imaginary parts. We split the argument in the cases where f is an even function (in which case u is also even function) and then when f is an odd function (u odd respectively).

Case I: u, f are even functions: For u, f even, we can expend u and f in cosine series as follows

$$u = \sum_{k=0}^{\infty} u_k \cos(k\pi x),\ f = \sum_{k=0}^{\infty} f_k \cos(k\pi x)$$

In this case,

$$(-\partial_{xx})^{s/2} u(x) = \sum_{k=0}^{\infty} (\pi k)^s u_k \cos(k\pi x),$$

Assume first that $\lambda = -\pi^s \sigma^s, \sigma > \frac{1}{2}$. Then, taking a dot product with u in (3.6), we have

$$-\lambda \|u\|^2 < \|(-\partial_{xx})^{s/4} u\|^2 - \lambda \|u\|^2 = \langle f, u\rangle \leq -\frac{\lambda}{2}\|u\|^2 + \frac{C}{|\lambda|}\|f\|^2$$

Thus, we have better estimate in this case

$$\|u\|_{L^2} \leq \frac{C}{|\lambda|}\|f\|_{L^2} \tag{3.8}$$

Next, let us take $\lambda = \pi^s \sigma^s, \sigma > \frac{1}{2}$. Let $k_0 =]\sigma[$, that is, the closest integer to σ using the smaller integer when σ is a half number. Then for every $k \neq k_0$, we have

$$u_k = \frac{1}{\pi^s (k^s - \sigma^s)} f_k, k \neq k_0. \tag{3.9}$$

We wish to estimate the function

$$\tilde{u} = \sum_{k \neq k_0} u_k \cos(\pi k x) = u - u_{k_0} \cos(\pi k_0 x)$$

first. By Lemma 5, we have that $|k^s - \sigma^s| \sim |k - \sigma| \max(k, \sigma)^{s-1}, k \neq k_0$.

Case I: $s \geq 1$ In this case, we can further take $|k^s - \sigma^s| \geq C|k - \sigma|\sigma^{s-1}, k \neq k_0$. We have

$$\|\tilde{u}\|_{L^2}^2 = \sum_{k \neq k_0, k \geq 0} |u_k|^2 \leq \frac{1}{\pi^{2s}\sigma^{2s-2}} \sum_{k \neq k_0, k \geq 0} \frac{C}{|k - \sigma|^2} f_k^2 \leq \frac{C}{\pi^{2s}\sigma^{2s-2}}\|f\|^2 = \frac{C}{\lambda^{2-\frac{2}{s}}}\|f\|^2.$$

Thus,

$$\|\tilde{u}\|_{L^2} \leq C < \lambda >^{\frac{1}{s}-1} \|f\|_{L^2} \tag{3.10}$$

Case II: $0 < s < 1$ In this case, we have

$$\|\tilde{u}\|_{L^2}^2 = \sum_{k \neq k_0, k \geq 0} |u_k|^2 \leq \frac{C}{\pi^{2s}} \sum_{k \neq k_0, k \geq 0} \frac{\max(k,\sigma)^{2(1-s)}}{|k-\sigma|^2} f_k^2.$$

We split the sum in two pieces, $k \in (\sigma/2, 2\sigma)$ and the rest. We have

$$\begin{aligned}
&\sum_{k \neq k_0, k \geq 0 : k \in (\sigma/2, 2\sigma)} \frac{\max(k,\sigma)^{2(1-s)}}{|k-\sigma|^2} f_k^2 \\
&\leq C_s \sigma^{2(1-s)} \sum_{k \neq k_0, k \geq 0 : k \in (\sigma/2, 2\sigma)} \frac{1}{|k-\sigma|^2} f_k^2 \leq C_s \lambda^{\frac{2}{s}-2} \|f\|_{L^2}^2,
\end{aligned}$$

since in this case $\max(k,\sigma) \leq 2\sigma$ and $\sigma \sim \lambda^{\frac{1}{s}}$.

In the other case, that is $k \leq \sigma/2$ or $k \geq 2\sigma$, we have that $|k-\sigma| \sim \max(k,\sigma)$, so

$$\sum_{k \neq k_0, k \geq 0 : k \leq \sigma/2 \text{ or } k \geq 2\sigma} \frac{\max(k,\sigma)^{2(1-s)}}{|k-\sigma|^2} f_k^2 \leq \sup_{k \leq \sigma/2 \text{ or } k \geq 2\sigma} \frac{1}{\max(k,\sigma)^{2s}} \|f\|_{L^2}^2 \leq \frac{1}{\lambda^2} \|f\|_{L^2}^2.$$

The estimate in this case is exceptionally good, but this is just a small piece of the sum. In all cases, we conclude (3.10).

Next, we estimate

$$\begin{aligned}
\int_{-\delta}^{\delta} |u(x)|^2 dx &= \int_{-\delta}^{\delta} |u_{k_0} \cos(\pi k_0 x) + \tilde{u}(x)|^2 dx \\
&= 2|u_{k_0}|^2 \int_0^{\delta} \cos^2(\pi k_0 x) dx + 2\int_{-\delta}^{\delta} u_{k_0} \cos(\pi k x) \tilde{u}(x) dx + \int_{-\delta}^{\delta} |\tilde{u}(x)|^2 dx \\
&\geq |u_{k_0}|^2 \delta \left(1 + \frac{\sin(2\pi k_0 \delta)}{2\pi k_0 \delta}\right) - C|u_{k_0}| \|\tilde{u}\|_{L^2}.
\end{aligned}$$

Note $(1 + \frac{\sin(2\pi k_0 \delta)}{2\pi k_0 \delta}) > 1 - \frac{2}{\pi}$, so we can bound from below

$$\int_{-\delta}^{\delta} |u(x)|^2 dx \geq \frac{\delta(1-\frac{2}{\pi})}{2} u_{k_0}^2 - C\|\tilde{u}\|_{L^2}^2 \geq C_\delta u_{k_0}^2 - \frac{C}{\lambda^{2-\frac{2}{s}}}\|f\|^2.$$

Thus,

$$u_{k_0}^2 \leq C_\delta \left(<\lambda>^{\frac{2}{s}-2} \|f\|_{L^2}^2 + \|u\|_{L^2[-\delta,\delta]}^2 \right). \tag{3.11}$$

Hence by combining the estimates (3.10) and (3.11) , we get

$$\|u\|_{L^2[-1,1]} \leq C_\delta \left(<\lambda>^{\frac{1}{s}-1} \|f\| + \|u(x)\|_{L^2[-\delta,\delta]} \right).$$

Lastly, let $-\frac{\pi^s}{2^s} < \lambda < \frac{\pi^s}{2^s}$. In this case, we applied the same arguments as above on

$$u = u_0 + \sum_{k=1}^{\infty} u_k \cos(\pi k x)$$

to get $\|\tilde{u}\|_{L^2} \leq C\|f\|_{L^2}$, while $|u_0|^2 \leq C_\delta \left(\int_{-\delta}^{\delta} |u(x)|^2 dx + \|f\|^2 \right)$. Finally, we conclude that in all three cases,

$$\|u\|_{L^2[0,1]} \leq C_\delta \left(<\lambda>^{\frac{1}{s}-1} \|f\|_{L^2} + \|u\|_{L^2[-\delta,\delta]} \right).$$

Case II: u, f are odd functions For u, f odd functions, we can expand u and f in sine series as follows

$$u = \sum_{k=0}^{\infty} a_k \sin(k\pi x),\ f = \sum_{k=0}^{\infty} f_k \sin(k\pi x)$$

Again, for $\lambda < -\frac{\pi^s}{2^s}$, we have the estimate (same as above)

$$\|u\| \leq \frac{C}{|\lambda|}\|f\|.$$

For $\lambda = \pi^s \sigma^s s,\ \sigma > \frac{1}{2}$, we have (same as above in (3.10))

$$\|\tilde{u}\|_{L^2} \leq C < \lambda >^{\frac{1}{s}-1} \|f\|.$$

where in this case $\tilde{u} = \sum_{k \neq k_0} u_k \sin(\pi k x) = u - u_{k_0} \sin(\pi k_0 x)$. Next, we estimate

$$\begin{aligned} \int_{-\delta}^{\delta} |u(x)|^2 dx &= \int_{-\delta}^{\delta} |u_{k_0} \sin(\pi k_0 x) + \tilde{u}(x)|^2 dx \\ &= 2|u_{k_0}|^2 \int_0^{\delta} \sin^2(\pi k_0 x) dx + 2\int_{-\delta}^{\delta} u_{k_0} \sin(\pi k x) \tilde{u}(x) dx + \int_{-\delta}^{\delta} |\tilde{u}(x)|^2 dx \\ &\geq |u_{k_0}|^2 \delta (1 - \frac{\sin(2\pi k_0 \delta)}{2\pi k_0 \delta}) - C|u_{k_0}| \|\tilde{u}\|_{L^2}. \end{aligned}$$

Now, observe $z \to \frac{sin(z)}{z}$ can be close to 1, but in any case, we have

$$(1 - \frac{\sin(2\pi k_0 \delta)}{2\pi k_0 \delta}) \geq c \min(1, (k_0 \delta)^2) \geq c\delta^2.$$

Note that in this last estimate, we used $k_0 \geq 1$, so c is independent on k_0! Consequently,

$$\int_{-\delta}^{\delta} |u(x)|^2 dx \geq c\delta^3 |u_{k_0}|^2 - C|u_{k_0}| \|\tilde{u}\|_{L^2} \geq c\delta^3 |u_{k_0}|^2 - C_\delta \|\tilde{u}\|_{L^2}^2 \geq c\delta^3 |u_{k_0}|^2 - \frac{C_\delta}{\lambda^{2-\frac{2}{s}}} \|f\|^2.$$

Hence,

$$\|u\|_{L^2[-1,1]}^2 \leq 2(u_{k_0}^2 + \|\tilde{u}\|_{L^2}^2) \leq C_\delta \left(< \lambda >^{\frac{2}{s}-2} \|f\|^2 + \int_{-\delta}^{\delta} |u(x)|^2 dx \right).$$

Case III u, f **are arbitrary functions** In this case, we split u and f in even and odd parts and derive estimates for each of them. Putting it all together, we get

$$\begin{aligned}
\|u\|^2_{L^2[-1,1]} &= \|u_{even}\|^2_{L^2[-1,1]} + \|u_{odd}\|^2_{L^2[-1,1]} \\
&\leq C_\delta \left(\frac{\|f_{even}\|^2 + \|f_{odd}\|^2}{|\lambda|^{2-\frac{2}{s}}} + \int_{-\delta}^{\delta} (u^2_{even}(x) + u^2_{odd}(x))dx \right) \\
&= C_\delta \left(\frac{\|f\|^2}{\lambda^{2-\frac{2}{s}}} + \int_{-\delta}^{\delta} u^2(x)dx \right).
\end{aligned}$$

Hence,

$$\|u\|_{L^2[-1,1]} \leq C_\delta \left(\lambda^{\frac{1}{s}-1} \|f\|_{L^2[-1,1]} + \|u\|_{L^2[-\delta,\delta]} \right)$$

□

This finishes the proof of the observability estimate (3.7). Next, we extend Lemma 6 to the whole line $\mathbb{R}$ by using a technique similar to Wunsch, [29].

3.2.2.2 Observability on intervals implies observability for a H_α

Introduce the operators

$$H^s_\alpha := [(-i\partial_x - \alpha)^2]^{s/2} \text{ for } \alpha \in \mathbb{R}.$$

Equivalently, one may define H_α through the Fourier transform

$$\widehat{H^s_\alpha f}(k) = |k-\alpha|^s \hat{f}(k).$$

Observe the relation

$$(-i\partial_x - \alpha)^2 = e^{i\alpha\cdot}(-\partial_{xx})e^{-i\alpha\cdot}.$$

Since multiplication by $e^{\pm i\alpha x}$ is an unitary operator on $L^2[-1,1]$, the relation above is an unitary equivalence between $(-i\partial_x - \alpha)^2$ and $-\partial_{xx}$. Consequently, H^s_α is a self-adjoint operator, so by Stone theorem, iH^s_α generates a C_0-group of unitary operators on a Hilbert space, which we denote by $U_\alpha(t) = e^{itH_\alpha^s}$. In addition, and since one can define $g(-\partial_{xx})$ for very general functions g (for example $C[0,\infty)$), we have

$$g((-i\partial_x - \alpha)^2) = e^{i\alpha\cdot} g(-\partial_{xx}) e^{-i\alpha\cdot}. \tag{3.12}$$

In particular, applying (3.12) to the functions $t^{s/2}$ and $e^{it^{s/2}}$,

$$H_\alpha^s = e^{i\alpha\cdot}(-\partial_{xx})^{s/2} e^{-i\alpha\cdot}; \;\; e^{itH_\alpha^s} = e^{i\alpha\cdot} e^{itH_0^s} e^{-i\alpha\cdot}. \tag{3.13}$$

The observability estimate for H_α^s on flat torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is as follows.

Lemma 7. *Let $\Gamma \subset \mathbb{T}$ be open and non-empty. For all $\alpha \in [0,1)$, we have*

$$(H_\alpha^s - \lambda)u = f \Rightarrow \;\; \|u\|_{L^2(\mathbb{T})} \leq C\left(< \lambda >^{\frac{1}{s}-1} \|f\|_{L^2(\mathbb{T})} + \|u\|_{L^2(\Gamma)} \right) \tag{3.14}$$

with constants independent of $\alpha|$ and $|\lambda| \geq 1 \in \mathbb{R}$.

Proof. Note that for $\alpha = 0$, we have $H_0^s = (-\partial_{xx})^{s/2}$, and in this case the result is proved in Lemma 6. Next, assume $\alpha \neq 0$.

By the results in [22] and since H_α^s is a self-adjoint operator, the estimate (3.14) is equivalent to Schrödinger observability for H_α^s. That is, we need to establish that for every, non-empty $\omega \subset \mathbb{T}$ and every $T > 0$, there exist $C(T,\omega)$ such that

$$\|f\|_{L^2}^2 \leq C \int_0^T \|e^{itH_\alpha^s} f\|_{L^2(\omega)}^2 \, dt$$

Next, fix a non-empty open set ω. By H_0^s-observability, we have for every $T > 0$

$$\begin{aligned} \|f\|_{L^2}^2 &= \|e^{-i\alpha x} f\|_{L^2}^2 \leq C \int_0^T \|e^{itH_0^s}[e^{-i\alpha\cdot} f]\|_{L^2(\omega)}^2 dt = \\ &= C \int_0^T \|e^{i\alpha\cdot} e^{itH_0^s} e^{-i\alpha\cdot} f\|_{L^2(\omega)}^2 dt = C \int_0^T \|e^{itH_\alpha^s} f\|_{L^2(\omega)}^2 dt. \end{aligned}$$

This proves the Schrödinder observability, with the same constants as $\alpha = 0$. Hence by Theorem 5.1 of Miller [22] , the estimate (3.14) holds for all $s > 0$. $\square$

3.2.2.3 Observability for H_α implies observability

For $g \in \langle x\rangle^{-s} H^{-\infty}(\mathbb{R})$ with $s > 1$. We define the periodization of g as follows

$$\Pi g(x) = \sum_{n\in\mathbb{Z}} g(x + 2\pi n).$$

Also, for $\alpha \in \mathbb{R}$, we set

$$\Pi_\alpha g = \Pi(e^{i\alpha x} g)$$

Lemma 8. *For $g \in \langle x\rangle^{-s} H^{-\infty}(\mathbb{R})$ with $s > 1$, we have*

$$\|g\|^2_{L^2(\mathbb{R})} = \int_{[0,1)} \|\Pi_\alpha g\|^2_{L^2(\mathbb{T})}\, d\alpha. \tag{3.15}$$

Moreover, if $\Omega \subset \mathbb{R}$ is $2\pi\mathbb{Z}$-invariant and Ω_0 denotes its projection to $\mathbb{T}$, we have

$$\|g\|^2_{L^2(\Omega)} = \int_{[0,1)^2} \|\Pi_\alpha g\|^2_{L^2(\Omega_0)}\, d\alpha. \tag{3.16}$$

For the proof of the lemma, we refer to Lemma 5, [29].

Note that $((-\partial_{xx})^{s/2} - \lambda)u = f$ implies

$$e^{i\alpha x}((-\partial_{xx})^{s/2} - \lambda)e^{-i\alpha x}[e^{i\alpha x} u] = e^{i\alpha x} f$$

In terms of the operator Π, we get $(H_\alpha - \lambda)(\Pi_\alpha u) = \Pi_\alpha f$. By Lemma (7), we conclude

$$\|\Pi_\alpha u\|^2_{L^2(\mathbb{T})} \le C(<\lambda>^{\frac{2}{s}-2} \|\Pi_\alpha f\|^2_{L^2(\mathbb{T})} + \|\Pi_\alpha u\|^2_{L^2(\Omega_0)})$$

By Lemma 8, we may integrate both sides over the set $[0,1)$ to obtain

$$\|u\|^2_{L^2(\mathbb{R})} \leq C(<\lambda>^{\frac{2}{s}-2}\|f\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(\Omega)})$$

This is of course (3.4) and so the proof of Theorem 27 is complete.

3.2.3 Resolvent estimate

From the observability estimate above, we prove the following resolvent estimate for our damped problem.

Proposition 6. *Assume that $m > 0$, $\gamma(x) \geq 0$ and $\gamma \in L^\infty$ and there exist $\varepsilon > 0$ and a $2\pi\mathscr{Z}$-invariant set $\Omega \in \mathbb{R}$ such that $\gamma(x) \geq \varepsilon$ for a.e. $x \in \mathbb{R}$. For the equation*

$$((-\partial_{xx})^{s/2} + m + ik\gamma(x) - k^2)u = f \tag{3.17}$$

we have the following:

- *For $0 < s < 2$,*
$$\|u\|_{L^2(\mathbb{R})} \leq C <k>^{\frac{4}{s}-3}\|f\|_{L^2(\mathbb{R})}, \tag{3.18}$$
- *For $s \geq 2$,*
$$\|u\|_{L^2(\mathbb{R})} \leq C <k>^{\frac{2}{s}-2}\|f\|_{L^2(\mathbb{R})}. \tag{3.19}$$

Proof. We begin by pairing the equation (3.17) with u, taking the real part and using Cauchy inequality. For $|k| \leq k_0 = \sqrt{m}/2$, we get

$$\|u\|^2_{H^{s/2}(\mathbf{R})} + (m-k^2)\|u\|^2_{L^2(\mathbf{R})} \quad \leq \quad \|f\|_{L^2(\mathbf{R})}\|u\|_{L^2(\mathbf{R})} \leq \frac{\|f\|^2_{L^2(\mathbf{R})}}{4(m-k^2)} + (m-k^2)\|u\|^2_{L^2(\mathbf{R})}$$

This implies that

$$\|u\|_{H^{s/2}(\mathbf{R})} \leq C\|f\|_{L^2(\mathbf{R})}.$$

Next we assume that $|k| > k_0$. We apply Theorem 27 to equation (3.17) with the damping term on the right-hand side and $\lambda = k^2 - m$. Noting that $< \lambda > \sim < k^2 >$, we get

$$\|u\|_{L^2(\mathbf{R})} \leq C\left(< k >^{\frac{2}{s}-2} \|f\|_{L^2(\mathbf{R})} + < k >^{\frac{2}{s}-1} \|\gamma(x)u\|_{L^2(\mathbf{R})} + \|u\|_{L^2(\Omega)} \right). \quad (3.20)$$

Choose Ω to be contained in the set where $\gamma \geq \varepsilon$ a.e. for some $\varepsilon > 0$. We obtain

$$\|u\|_{L^2(\Omega)} \leq \varepsilon^{-1}\|\gamma(x)u\|_{L^2(\mathbf{R})},$$

so (3.20) becomes

$$\|u\|_{L^2(\mathbf{R})} \leq C\left(< k >^{\frac{2}{s}-2} \|f\|_{L^2(\mathbf{R})} + (< k >^{\frac{2}{s}-1} + \varepsilon^{-1})\|\gamma(x)u\|_{L^2(\mathbf{R})} \right). \quad (3.21)$$

Pairing the equation (3.17) with u and taking the imaginary part, we get for $k \geq k_0$,

$$\|\sqrt{\gamma(x)}u\|^2_{L^2(\mathbb{R})} \leq \frac{C}{< k >}\|f\|\|u\| \quad (3.22)$$

Combining these estimates and observing that $\gamma \leq C\sqrt{\gamma(x)}$ a.e. yields

$$\|u\|_{L^2(\mathbf{R})} \leq C\left(< k >^{\frac{2}{s}-2} \|f\|_{L^2(\mathbf{R})} + \frac{(< k >^{\frac{2}{s}-1} + \varepsilon^{-1})}{< k >^{1/2}}\|f\|^{1/2}_{L^2(\mathbf{R})}\|u\|^{1/2}_{L^2(\mathbf{R})} \right) \quad (3.23)$$

Applying Cauchy-Schwarz, we obtain

$$\|u\|_{L^2(\mathbf{R})} \leq C(< k >^{\frac{2}{s}-2} + < k >^{\frac{4}{s}-3} + < k >^{-1})\|f\| \quad (3.24)$$

By analyzing the cases $s \in (0,2)$ and $s \geq 2$ separately (here k is large), we finally conclude

$$
\begin{aligned}
\|u\|_{L^2(\mathbf{R})} &\leq C < k >^{\frac{4}{s}-3} \|f\|_{L^2(\mathbf{R})}, \quad s \in (0,2) \\
\|u\|_{L^2(\mathbf{R})} &\leq C < k >^{\frac{2}{s}-2} \|f\|_{L^2(\mathbf{R})}, \quad s \geq 2
\end{aligned}
$$

This completes the proof. $\square$

3.3 Resolvent estimates and proof of Theorem 26

We begin by recasting (3.1) as an abstract Cauchy problem. Define $U = (u, u_t)^T$, then equation (3.1) can be written as a dynamical system:

$$U_t = \mathscr{A} U$$

where

$$\mathscr{A} = \begin{pmatrix} 0 & I \\ -(-\partial_{xx})^{s/2} - m & -\gamma(x) \end{pmatrix},$$

where we take $D(\mathscr{A}) = H^s(\mathbf{R}) \times H^{s/2}(\mathbf{R})$. The basic Hilbert space is $\mathscr{H} = H^{s/2}(\mathbf{R}) \times L^2(\mathbf{R})$. The fact that $\mathscr{A}$ generates a semigroup, under this setup, is standard.

Next, we compute the resolvent of the operator $\mathscr{A}$. Let $u = (u_1, u_2)'$ and $f = (f_1, f_2)'$. Then

$$(ikI - \mathscr{A})u = f$$

is equivalent to

$$
\begin{aligned}
iku_1 - u_2 &= f_1 \\
((-\partial_{xx})^{s/2} + m)u_1 + (ik + \gamma(x))u_2 &= f_2
\end{aligned}
$$

or

$$
\begin{aligned}
u_1 &= ((-\partial_{xx})^{s/2}+m+ik\gamma(x)-k^2)^{-1}((ik+\gamma(x))f_1+f_2) \\
u_2 &= iku_1-f_1.
\end{aligned}
$$

Hence, the resolvent of $\mathscr{A}$ is

$$
R(ik,\mathscr{A})=\begin{pmatrix} R(ik)(ik+\gamma(x)) & R(ik) \\ \\ ikR(ik)(\gamma(x)+ik)-I & ikR(ik) \end{pmatrix},
$$

where $R(ik)=((-\partial_{xx})^{s/2}+m+ik\gamma(x)-k^2)^{-1}$. Note that

$$
R(ik)^*=R(-ik).
$$

Recall that our basic resolvent estimate, Proposition 6, provides bounds for the resolvent $R(ik)$, acting as operators on $L^2(\mathbf{R})$ into itself. On the other hand, $R(ik)$ are smoothing operators. The next result allows us to obtain bounds between different Sobolev spaces.

Proposition 7. *Let* $0<s<2$. *Then,*

$$
\|R(ik)\|_{L^2\to H^{s/2}}+\|R(ik)\|_{H^{-s/2}\to L^2}\leq C<k>^{\frac{4}{s}-2}. \tag{3.25}
$$

For $s\geq 2$*,*

$$
\|R(ik)\|_{L^2\to H^{s/2}}+\|R(ik)\|_{H^{-s/2}\to L^2}\leq C<k>^{\frac{2}{s}-1}. \tag{3.26}
$$

Proof. Let u be the solution of

$$
((-\partial_{xx})^{s/2}+m+ik\gamma(x)-k^2)u=f \tag{3.27}
$$

where $f \in L^2$. Taking dot product with u in (3.27) and taking the real part yields

$$\begin{aligned} \langle(-\partial_{xx})^{s/2}u,u\rangle + (m-k^2)\langle u,u\rangle &= \mathrm{Re}\langle f,u\rangle \\ \|u\|^2_{H^{s/2}} &\le \|f\|_{L^2}\|u\|_{L^2} + k^2\|u\|^2_{L^2} \end{aligned}$$

By Proposition 6 for $s \in (0,2)$, $\|u\|_{L^2} \le C < k >^{\frac{4}{s}-3} \|f\|_{L^2}$, so we obtain

$$\|u\|^2_{H^{s/2}} \le \|f\|_{L^2}\left(< k >^{\frac{4}{s}-3} \|f\|_{L^2(\mathbb{R})}\right) + k^2 < k >^{\frac{8}{s}-6} \|f\|^2_{L^2(\mathbb{R})}$$

This proves

$$\|R(ik)\|_{L^2 \to H^{s/2}} \le C < k >^{\frac{4}{s}-2},$$

and by duality $\|R(ik)\|_{H^{-s/2} \to L^2} \le C < k >^{\frac{4}{s}-2}$. For $s \ge 2$, we apply Proposition 6 and we similarly obtain

$$\|u\|^2_{H^{s/2}} \le \|f\|_{L^2}\left(< k >^{\frac{2}{s}-2} \|f\|_{L^2(\mathbb{R})}\right) + k^2 < k >^{\frac{4}{s}-4} \|f\|^2_{L^2(\mathbb{R})}$$

This proves (3.26). □

Next, we put together the results from Proposition 6, together with Proposition 7 to obtain the following result on the composite resolvent $R(ik,\mathscr{A})$.

Proposition 8. *For* $0 < s < 2$*, there is*

$$\|R(ik,\mathscr{A})\|_{H^{s/2} \times L^2} \le C < k >^{\frac{4}{s}-2}, \tag{3.28}$$

while for $s \ge 2$ *, we have*

$$\|R(ik,\mathscr{A})\|_{H^{s/2} \times L^2} \le C. \tag{3.29}$$

Proof of Proposition (8)**.** First we consider the case $0 < s < 2$. Write $R(ik, \mathscr{A})$ as follows

$$\left\| R(ik, \mathscr{A}) \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{H^{s/2} \times L^2} = \|R(ik)(ik + \gamma(x))f\|_{H^{s/2}} + \|R(ik)g\|_{H^{s/2}} + \|(ikR(ik)(\gamma(x) + ik) - I)f\|_{L^2} + \|ikR(ik)g\|_{L^2}$$

The estimates for the terms involving g follow easily from the established estimates. Indeed, from (3.25), we have

$$\|R(ik)g\|_{H^{s/2}} \leq C < k >^{\frac{4}{s}-2} \|g\|_{L^2},$$

while from (3.7), we have

$$\|ikR(ik)g\|_{L^2} \leq C|k| < k >^{\frac{4}{s}-3} \|g\|_{L^2} \leq C < k >^{\frac{4}{s}-2} \|g\|_{L^2}.$$

So, it remains to establish the bounds

$$\|R(is)(ik)(ik + \gamma(x))\| = O(|k|^{\frac{4}{s}-2}) : H^{s/2} \to H^{s/2} \tag{3.30}$$

$$\|R(ik)(ik)(\gamma(x) + ik) - I)\| = O(|k|^{\frac{4}{s}-2}) : H^{s/2} \to L^2. \tag{3.31}$$

Once, (3.30) and (3.31) are established, we conclude

$$\left\| R(ik, \mathscr{A}) \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{H^{s/2} \times L^2} \leq C \| \begin{pmatrix} f \\ g \end{pmatrix} \|_{H^{s/2} \times L^2},$$

and Proposition 8 will be proved.

Next, we estimate $R(ik)(ik)[\gamma(x) + ik)] - I : H^{s/2} \to L^2$. Elementary manipulations show that

$$R(ik)(ik)[\gamma(x) + ik)] - I = -R(ik)((-\partial_{xx})^{s/2} + m) \tag{3.32}$$

Combining (3.25), together with the fact that $(-\partial_{xx})^{s/2} : H^{s/2} \to H^{-s/2}$ is continuous, we obtain for $f \in H^{s/2}(\mathbb{R})$

$$\begin{aligned}
&\|(R(ik)(ik)[\gamma(x)+ik)] - I)f\|_{L^2} = \|R(ik)((-\partial_{xx})^{s/2}+m)f\|_{L^2} \le \\
&\le C|k|^{\frac{4}{s}-2}\|((-\partial_{xx})^{s/2}+m)f\|_{H^{-s/2}} \le C|k|^{\frac{4}{s}-2}\|f\|_{H^{s/2}}
\end{aligned}$$

This proves (3.31).

It remains to estimate $\|R(is)(ik+\gamma(x))\|_{H^{s/2}\to H^{s/2}}$. A variant of (3.32)reads

$$R(ik)(ik+\gamma(x)) = \frac{1}{ik}[I - R(ik)((-\partial_{xx})^{s/2}+m)],$$

Let $u = R(ik)((-\partial_{xx})^{s/2}+m)f$, then

$$((-\partial_{xx})^{s/2}+m+ik\gamma(x)-k^2)u = ((-\partial_{xx})^{s/2}+m)f$$

Pairing this equation with u and taking real parts and applying Cauchy-Schwarz, we get,

$$\begin{aligned}
\|(-\partial_{xx})^{s/4}u\|_{L^2}^2 - (k^2-m)\|u\|_{L^2}^2 &\le \|((-\partial_{xx})^{s/2}+m)f\|_{H^{-s/2}}\|u\|_{H^{s/2}} \\
&\le \|f\|_{H^{s/2}}\|u\|_{H^{s/2}}.
\end{aligned}$$

Therefore,

$$\|u\|_{H^{s/2}}^2 \le C(k^2\|u\|_{L^2}^2 + \|f\|_{H^{s/2}}^2). \tag{3.33}$$

Next, when we estimate $\|u\|_{L^2}$, we used (3.25) to get

$$\begin{aligned}
\|u\|_{L^2} &= \|R(ik)((-\partial_{xx})^{s/2}+m)f\|_{L^2} \le C|k|^{\frac{4}{s}-2}\|(-\partial_{xx})^{s/2}+m)f\|_{H^{-s/2}} \le \\
&\le C|k|^{\frac{4}{s}-2}\|f\|_{H^{s/2}}
\end{aligned}$$

Plugging this estimate back in (3.33), we obtain $\|u\|_{L^2} \leq C|k|^{\frac{4}{s}-1}\|f\|_{H^{s/2}}$. As a consequence,

$$R(ik)((-\partial_{xx})^{s/2}+m) = O(|k|^{\frac{4}{s}-1}) : H^{s/2}(\mathbb{R}) \to H^{s/2}(\mathbb{R}),$$

whence for large $|k|$,

$$\begin{aligned} \|R(ik)(ik+\gamma(x))\|_{H^{s/2}\to H^{s/2}} &= k^{-1}\|I - R(ik)((-\partial_{xx})^{s/2}+m)\|_{H^{s/2}\to H^{s/2}} \leq \\ &\leq Ck^{-1}(1+|k|^{\frac{4}{s}-1}) \leq C|k|^{\frac{4}{s}-2}, \end{aligned}$$

which is (3.30). Hence, for $0 < s < 2$, we get

$$R(ik, \mathscr{A}) = (ik - \mathscr{A})^{-1} = O(|k|^{\frac{4}{s}-2}) : H^{s/2} \times L^2 \to H^{s/2} \times L^2.$$

Similarly, for $s \geq 2$, we have

$$R(ik, \mathscr{A}) = (ik - \mathscr{A})^{-1} = O(|k|^{\frac{2}{s}-1}) : H^{s/2} \times L^2 \to H^{s/2} \times L^2$$

So, in fact, we have decay in k of the resolvent for $s > 2$. $\square$

Having proved Proposition 8, we are ready for the proof of our main result, Theorem 26. For the case $0 < s < 2$, we apply the Borichev-Tomilov Theorem 22 with $\alpha = \frac{4}{s} - 2 > 0$. Then, the semigroup satisfies the following bound

$$\|e^{t\mathscr{A}}(\mu - \mathscr{A})^{-1}\|_{H^{s/2}\times L^2 \to H^{s/2}\times L^2} \leq Ct^{-\frac{s}{4-2s}},$$

for any $\mu \in \rho(\mathscr{A})$, say $\mu = 1$. Equivalently,

$$\|e^{t\mathscr{A}}f\|_{H^{s/2}\times L^2} \leq Ct^{-\frac{s}{4-2s}}\|(1-\mathscr{A})f\|_{H^{s/2}\times L^2} \leq Ct^{-\frac{s}{4-2s}}\|f\|_{H^s\times H^{s/2}},$$

since $\mathscr{A} : H^s \times H^{s/2} \to H^{s/2} \times L^2$.

For $s \geq 2$, by Gearhart-Prüss Theorem 24 the energy of the damped fractional Klein-Gordon is decaying exponentially and more precisely, we have the bound (3.3).

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