Sharp time asymptotics for the quasi-geostrophic equation, the Boussinesq system and near plane waves of reaction-diffusion models

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Abstract

Through this dissertation we present the sharp time decay rates for three equations, namely quasi–geostrophic equation (SQG), Boussinesq system (BSQ) and plane wave of general reaction-diffusion models. In addition, in each case, we provide the dominant part of the solution which leads to the long term asymptotic profiles of each model.

The first two equations, arising in fluid dynamics, model some aspect of the shallow waters with horizontal and vertical structures. Indeed, quasi–geostrophis equation models the horizontal inertia forces of a flow. As a result of that, atmospheric and oceanographic flows which take place over horizontal length scales, which are very large compare to their vertical length scales, are studied by SQG equation. On the other hand BSQ system models some vertical aspect of the flow, namely the speed, pressure and the temperature of the flow. In coastal engineering, BSQ type equations have a vast application in computer modeling. Lastly, a plane wave is a constant-frequency wave whose wavefronts (surfaces of constant phase) are infinite parallel planes of constant peak-to-peak amplitude normal to the phase velocity vector.

In order to study these equations, we made some developments in the "scaling variable" methods, so that it fits over models. In particular, we now have a good understanding of this method when it is applied to the equations with fractional dissipations.

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Contents

List of Figures

Chapter 1

Introduction

1.1 Fourier Transform, function spaces and multipliers

In this Section , we introduce some basic Sobolev spaces, weighted L^2 spaces and some relevant estimates that will be useful in the sequel. We start with several notations. In mathematics the space of the rapidly decreasing functions on $\mathbb{R}^n$ is called the Schwartz space $\mathscr{S}$. It is defined to be

$$\mathscr{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \ \|f\|_{\alpha,\beta} < \infty, \forall \alpha, \beta \in \mathbb{N}^n \right\},$$

where α, β are multi–index, $C^\infty(\mathbb{R}^n)$ is the set of smooth functions on $\mathbb{R}^n$ to $\mathbb{C}$, and

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|.$$

On the Schwartz class, we can define the Fourier transform and its inverse via

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx, \ \ f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi$$

Consequently, since $\widehat{-\Delta f}(\xi) = |\xi|^2 \hat{f}(\xi)$, we define the operators $|\nabla|^a := (-\Delta)^{a/2}, a > 0$, via its action on the Fourier side $\widehat{|\nabla|^a f}(\xi) = |\xi|^a \hat{f}(p)$. More generally, the operators $f(|\nabla|)$, for reasonable functions f, are acting as multipliers by $f(|\xi|)$. We will also make use of the following notation - *we say that* $\mathbf{m}$ *is a symbol of order* $a, a \in \mathbb{R}$, if it is a smooth function on $\mathbb{R}^n \setminus \{0\}$, satisfying for all multi-indices $\alpha \in \mathbf{N}^n$,

$$|\partial^\alpha \mathbf{m}(\xi)| \leq C_\alpha |\xi|^{a-|\alpha|}.$$

It is actually enough to assume this inequality for a finite set of indices, say $|\alpha| \leq n$. The prototype will be something of the form $m(\xi) = |\xi|^a$, but note that a will be often negative in our applications. We will schematically denote a symbol of order a by $\mathbf{m}_a$.

The L^p spaces are defined by the norm $\|f\|_{L^p} = \left(\int |f(x)|^p \, dx \right)^{\frac{1}{p}}$, while the weak L^p spaces are

$$L^{p,\infty} = \left\{ f : \|f\|_{L^{p,\infty}} = \sup_{\lambda>0} \left\{ \lambda \, |\{x : |f(x)| > \lambda\}|^{\frac{1}{p}} \right\} < \infty \right\}.$$

In this context, recall the Hausdorff–Young inequality which reads as follows: For $p,q,r \in (1,\infty)$ and $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$

$$\|f * g\|_{L^p} \leq C_{p,q,r} \|f\|_{L^{q,\infty}} \|g\|_{L^r}.$$

For an integer n and $p \in (1,\infty)$, the Sobolev spaces are the closure of the Schwartz functions in the norm $\|f\|_{W^{k,p}} = \|f\|_{L^p} + \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^p}$, while for a non-integer s one takes

$$\|f\|_{W^{s,p}} = \|(1-\Delta)^{s/2} f\|_{L^p} \sim \|f\|_{L^p} + \||\nabla|^s f\|_{L^p}.$$

The Sobolev embedding theorem states $\|f\|_{L^q(\mathbb{R}^n)} \leq C \||\nabla|^s f\|_{L^p(\mathbb{R}^n)}$, where $1 < p < q < \infty$ and $n(\frac{1}{p} - \frac{1}{q}) = s$, with the usual modification for $q = \infty$, namely $\|f\|_{L^\infty(\mathbb{R}^n)} \leq C_s \|f\|_{W^{s,p}(\mathbb{R}^n)}$, $s > \frac{n}{p}$. In particular, an estimate that will be useful for us, is

$$\|(|\nabla|^\perp)^{-\beta} f\|_{L^q} \leq C \|f\|_{L^p}, \;\; 1 < p < q < \infty, \beta = n(\frac{1}{p} - \frac{1}{q}) \tag{1.1}$$

This follows from the Mikhlin's criteria for $L^p, 1 < p < \infty$ boundedness. Note that these estimates hold in a more general setting, when $(|\nabla|^\perp)^{-\beta}$ is replaced by an arbitrary symbol of order $-\beta$, that is

$$\|m_{-\beta}(\nabla)f\|_{L^\infty} \le C_\varepsilon(\|f\|_{L^{\frac{n}{\beta}+\varepsilon}} + \|f\|_{L^{\frac{n}{\beta}-\varepsilon}}). \tag{1.2}$$

We will give a proof of this in the proposition 1.1.3. Another useful ingredient will be the Gagliardo - Nirenberg interpolation inequality,

$$\||\nabla|^s f\|_{L^p} \le \||\nabla|^{s_1} f\|_{L^q}^{\theta} \||\nabla|^{s_2} f\|_{L^r}^{1-\theta},$$

where $s = \theta s_1 + (1-\theta)s_2$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

For the arguments related to the optimal decay rates in chapter (2), we will need to argue in the weighted spaces. For any $m \ge 0$ we define the Hilbert space $L^2(m)$ as follow

$$L^2(m) = \left\{ f \in L^2 : \ \|f\|_{L^2(m)} = \left(\int_{\mathbb{R}^2} (1+|x|^2)^m |f(x)|^2 dx \right)^{\frac{1}{2}} < \infty \right\} \tag{1.3}$$

One can show by means of Hölder's, $L^2(2) \hookrightarrow L^p(\mathbb{R}^2)$, whenever $1 \le p < 2$. Indeed, for any $f \in L^p(\mathbb{R}^2)$

$$\begin{aligned} \|f\|_{L^p}^p &= \int_{\mathbb{R}^2} |f|^2 dx = \int_{\mathbb{R}^2} \frac{(1+|x|^2)^{\frac{mp}{2}} |f|^2}{(1+|x|^2)^{\frac{mp}{2}}} dx \\ &\le C \left(\int_{\mathbb{R}^2} \frac{1}{(1+|x|^2)^{\frac{mp}{2-p}}} dx \right)^{\frac{2-p}{2}} \int_{\mathbb{R}^2} (1+|x|^2)^m |f|^2 dx. \end{aligned}$$

First integral is bounded for $1 \le p < 2$ and $m \ge 1$. Case $p = 2$ is clear.

1.1.1 The kernel representation of the fractional Laplacian

We recall the following kernel representation formula for negative powers of Laplacian. This is nothing, but a fractional integral - for $a \in (0,2)$,

$$|\nabla|^{-a} f(x) = c_a \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|^{2-a}} dy. \tag{1.4}$$

Next, for positive powers, we have similar formula. More specifically, for $a \in (0,2)$,

$$|\nabla|^a f(x) = C_a p.v. \int_{\mathbb{R}^2} \frac{f(x)-f(y)}{|x-y|^{2+a}} dy. \tag{1.5}$$

see Proposition 2.1, [13]). Next, we have the following result, due to Cordoba-Cordoba. This is a well known relation, and we ignore the proof.

Lemma 1.1.1. *(Lemma 2.4, 2.5, [13]) For $p : 1 \leq p < \infty$, $a \in [0,2]$ and $f \in W^{a,p}(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} |f(x)|^{p-2} f(x) [|\nabla|^a f](x) dx \geq 0. \tag{1.6}$$

If in addition, $p = 2^n, n = 1, 2, \ldots$, there is the stronger coercivity estimate

$$\int_{\mathbb{R}^2} |f(x)|^{p-2} f(x) [|\nabla|^a f](x) dx \geq \frac{1}{p} \| |\nabla|^{\frac{a}{2}} [f^{\frac{p}{2}}] \|^2_{L^2(\mathbb{R}^2)}. \tag{1.7}$$

1.1.2 Littlewood–Paley operators

We need to quickly introduce some elementary Littlewood-Paley theory. To introduce the Littlewood-Paley decomposition, we write for each $j \in \mathbb{Z}$,

$$A_j = \left\{ \xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1} \right\}.$$

The Littlewood-Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{i \in \mathbb{Z}} \in \mathscr{S}$ such that

$$supp\, \widehat{\Phi}_j \subset A_j, \;\; \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi), \quad or \;\; \Phi_j = 2^{jn}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi) = \begin{cases} 1, & \xi \in \mathbb{R}^n \backslash \{0\} \\ 0, & \xi = 0. \end{cases}$$

Therfore for a general function $\psi \in \mathscr{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi), \quad for \ \ \xi \in \mathbb{R}^n \backslash \{0\}.$$

In addition, if $\psi \in \mathscr{S}_0$, then the above equality holds for any $\xi \in \mathbb{R}^n$. That is, for $\psi \in \mathscr{S}_0$,

$$\sum_{j=-\infty}^{\infty} \Phi_j * \psi = \psi,$$

and hence

$$\sum_{j=-\infty}^{\infty} \Phi_j * f = f, \quad f \in \mathscr{S}.$$

For notational purposes, we define

$$\mathring{\Delta}_j f = \Phi_j * f.$$

The following Bernestein's inequalities are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernestein type inequalities for fractional derivatives.

Proposition 1.1.2. *Leq* $\alpha \geq 0$*, and* $1 \leq p \leq q \leq \infty$.

- *If* f *satisfies*

  $$supp \ \widehat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq K 2^j\}$$

  for some integer j *and a constant* $K > 0$*, then*

  $$\| |\nabla|^\alpha f \|_{L^q(\mathbb{R}^n)} \leq C_1 2^{\alpha j + jn(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)},$$

  where C_1 *is a constant depending on* K, α, p *and* q *only.*

- *If* f *satisfies*

  $$supp \ \widehat{f} \subset \{\xi \in \mathbb{R}^n : K_1 2^j \leq |\xi| \leq K_2 2^j\}$$

for some integer j and a constants $0 < K_1 \leq K_2$, then

$$C_1\, 2^{\alpha j}\|f\|_{L^q(\mathbb{R}^n)} \leq \||\nabla|^{\alpha} f\|_{L^q(\mathbb{R}^n)} \leq C_2 2^{\alpha j + jn(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^n)},$$

where C_2 is a constant depending on K_1, K_2, α, p and q only.

As an application of Littlewood–Paley theory we prove (1.2) , which is a replacement of (1.1).

Proposition 1.1.3. *Let $p = \infty$ and $\beta < n$, then*

$$\|(|\nabla|^{\perp})^{-\beta} f\|_{L^\infty} \leq C_\varepsilon(\|f\|_{L^{\frac{n}{\beta}+\varepsilon}} + \|f\|_{L^{\frac{n}{\beta}-\varepsilon}}). \tag{1.8}$$

Proof. Let $\widehat{\Delta_k f}(\xi) = \widehat{\Phi}_0(2^{-k}\xi)\hat{f}(\xi)$, where Φ_0 is as it is defined above, then

$$\|(\nabla^{\perp})^{-\beta} f\|_{L^\infty} \leq \sum_{k=0}^{\infty} \|\Delta_k((\nabla^{\perp})^{-\beta} f)\|_{L^\infty} + \sum_{k=0}^{\infty} \|\Delta_{-k}((\nabla^{\perp})^{-\beta} f)\|_{L^\infty}.$$

We make use of the above Bernstein inequality several times to control each of these terms. Indeed,

$$\begin{aligned}\sum_{k=0}^{\infty} \|\Delta_k((\nabla^{\perp})^{-\beta} f)\|_{L^\infty} &\leq \sum_{k=0}^{\infty} 2^{-k\beta}\|\Delta_k f\|_{L^\infty} \leq \sum_{k=0}^{\infty} 2^{-k\beta + nk(\frac{1}{\frac{n}{\beta}+\delta})}\|\Delta_k f\|_{L^{\frac{n}{\beta}+\delta}} \\ &\leq \|f\|_{L^{\frac{n}{\beta}+\delta}} \sum_{k=0}^{\infty} 2^{-k\beta(1-\frac{n}{n+\beta\delta})} \leq C\|f\|_{L^{\frac{n}{\beta}+\delta}}.\end{aligned}$$

In the same way,

$$\begin{aligned}\sum_{k=0}^{\infty} \|\Delta_{-k}((\nabla^{\perp})^{-\beta} f)\|_{L^\infty} &\leq \sum_{k=0}^{\infty} 2^{k\beta}\|\Delta_{-k} f\|_{L^\infty} \leq \sum_{k=0}^{\infty} 2^{k\beta - nk(\frac{1}{\frac{n}{\beta}-\delta})}\|\Delta_k f\|_{L^{\frac{n}{\beta}-\delta}} \\ &\leq \|f\|_{L^{\frac{n}{\beta}-\delta}} \sum_{k=0}^{\infty} 2^{k\beta(1-\frac{n}{n-\beta\delta})} \leq C\|f\|_{L^{\frac{n}{\beta}-\delta}}.\end{aligned}$$

□

1.1.3 Commutator bounds

For future discussions we state some commutator bounds. Some are standard estimates, and some are proven here. The classical by now product rule estimate, usually attributed to Kato-Ponce can be stated as follows.

Lemma 1.1.4. *Let $a \in (0,1)$ and $1 < p,q,r < \infty$, so that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then, there exists $C = C_{p,q,r,a}$*

$$\| |\nabla|^a [fg] \|_{L^p} \leq C_{p,q,r,a} (\| |\nabla|^a f \|_{L^q} \|g\|_{L^r} + \| |\nabla|^a g \|_{L^q} \|f\|_{L^r})$$

The following commutator lemma is proved in [26] in details.

Lemma 1.1.5. *Let s_1, s_2 be two reals so that $0 \leq s_1$ and $0 \leq s_2 - s_1 \leq 1$. Let p,q,r be related via the Hölder's $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, where $2 < q < \infty$, $1 < p,r < \infty$. Finally, let $\nabla \cdot V = 0$. Then for any $a \in [s_2 - s_1, 1]$*

$$\| |\nabla|^{-s_1} [|\nabla|^{s_2}, V \cdot \nabla] \varphi \|_{L^p} \leq C \| |\nabla|^a V \|_{L^q} \| |\nabla|^{s_2 - s_1 + 1 - a} \varphi \|_{L^r} \tag{1.9}$$

In addition, we have the following end-point estimate. For $s_1 > 0$, $s_2 > 0$, $s_3 > 0$ and $s_1 < 1$, $s_3 < 1$, $s_2 < s_1 + s_3$, there is[1]

$$\| |\nabla|^{-s_1} [|\nabla|^{s_2}, |\nabla|^{-s_3} V \cdot \nabla] \varphi \|_{L^2} \leq C \|V\|_{L^\infty} \| |\nabla|^{s_2 - s_1 + 1 - s_3} \varphi \|_{L^2}. \tag{1.10}$$

Lemma 1.1.6. *For any integer m and $\alpha \in (1,2)$, there is $C = C_\alpha$, so that*

$$\| [|\nabla|^{\alpha/2}, |\xi|^2] f \|_{L^2(\mathbb{R}^2)} \leq C \| |\xi|^{2 - \frac{\alpha}{2}} f \|_{L^2(\mathbb{R}^2)}. \tag{1.11}$$

[1] Note that in the statement of (1.10), one does not necessarily need precisely the form $|\nabla|^{-s_3} V$. In fact, the estimate applies for any Fourier multiplier Q, with the property that $\|QV_k\|_{L^\infty} \sim 2^{-ks_3} \|V_k\|_{L^\infty}$

Proof. Recall, that for $s \in (0,2)$

$$\begin{aligned}[|\nabla|^s, g]f(x) &= |\nabla|^s(gf) - g\,|\nabla|^s f = c_s \int \frac{f(x)g(x) - f(y)g(y)}{|x-y|^{2+s}} dy - \\ &- g(x) c_s \int \frac{f(x) - f(y)}{|x-y|^{2+s}} dy = c_s \int \frac{f(y)(g(x) - g(y))}{|x-y|^{2+s}} dy.\end{aligned}$$

Introduce a smooth partition of unity, that is a function $\psi \in C_0^\infty(\mathbb{R})$, $supp\ \psi \subset (\frac{1}{2}, 2)$, so that

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k}|\xi|) = 1, \xi \in \mathbb{R}^2, \xi \neq 0.$$

Introduce another C_0^∞ function $\Psi(z) = z^2 \psi(z)$, so that we can decompose

$$|\xi|^2 = \sum_{k=-\infty}^{\infty} |\xi|^2 \psi(2^{-k}|\xi|) = \sum_{k=-\infty}^{\infty} 2^{2k} \Psi(2^{-k}|\xi|).$$

We can then write

$$\begin{aligned}F(\xi) &:= [|\nabla|^{\frac{\alpha}{2}}, |\xi|^2]f = \sum_k 2^{2k} [|\nabla|^{\frac{\alpha}{2}}, \Psi(2^{-k}\cdot)]f(\xi) \\ &= \sum_k 2^{2k} \int \frac{f(y)(\Psi(2^{-k}\xi) - \Psi(2^{-k}y))}{|\xi - y|^{2+\frac{\alpha}{2}}} dy.\end{aligned}$$

Introducing

$$F_k := \int \frac{|f(y)||\Psi(2^{-k}\xi) - \Psi(2^{-k}y)|}{|\xi - y|^{2+\frac{\alpha}{2}}} dy,$$

we need to control

$$\begin{aligned}\|F\|_{L^2}^2 &= \sum_l \int_{|\xi|\sim 2^l} |F(\xi)|^2 d\xi = \sum_l \int_{|\xi|\sim 2^l} \left| \sum_k 2^{2k} F_k(\xi) \right|^2 d\xi = \\ &= \sum_l \int_{|\xi|\sim 2^l} \left| \sum_{k>l+10} 2^{2k} F_k(\xi) \right|^2 d\xi + \sum_l \int_{|\xi|\sim 2^l} \left| \sum_{k=l-10}^{l+10} 2^{2k} F_k(\xi) \right|^2 d\xi + \\ &+ \sum_l \int_{|\xi|\sim 2^l} \left| \sum_{k<l-10} 2^{2k} F_k(\xi) \right|^2 d\xi =: K_1 + K_2 + K_3\end{aligned}$$

We first consider the cases $k > l+10$. One can estimate easily F_k point-wise. More specifically, since in the denominator of the expression for F_k, we have $|\xi - y| \geq \frac{1}{2}|\xi| \geq 2^{k-3}$,

$$|F_k(\xi)| \leq 2^{-k(2+\frac{\alpha}{2})} \int |f(y)||\Psi(2^{-k}y)|dy \leq C2^{-k(1+\frac{\alpha}{2})}\|f\|_{L^2(|y|\sim 2^k)},$$

whence

$$\begin{aligned} K_1 &\leq \sum_l 2^{2l} \sum_{k_1 > l+10} \sum_{k_2 > l+10} 2^{k_1(1-\frac{\alpha}{2})}\|f\|_{L^2(|y|\sim 2^{k_1})} 2^{k_2(1-\frac{\alpha}{2})}\|f\|_{L^2(|y|\sim 2^{k_2})} \\ &\leq \sum_{k_1}\sum_{k_2} 2^{2\min(k_1,k_2)} 2^{k_1(1-\frac{\alpha}{2})}\|f\|_{L^2(|y|\sim 2^{k_1})} 2^{k_2(1-\frac{\alpha}{2})}\|f\|_{L^2(|y|\sim 2^{k_2})} \\ &\leq C\sum_k 2^{k(4-\alpha)}\|f\|^2_{L^2(|y|\sim 2^k)} \leq C\||\xi|^{2-\frac{\alpha}{2}} f\|^2. \end{aligned}$$

where we have used $\sum_{l:l<\min(k_1,k_2)-10} 2^{2l} \leq C2^{2\min(k_1,k_2)}$.

For the case $k < l-10$, we perform similar argument, since

$$|F_k(\xi)| \leq C2^{-l(2+\frac{\alpha}{2})} 2^k \|f\|_{L^2(|y|\sim 2^k)}.$$

So,

$$\begin{aligned} K_3 &\leq C\sum_l 2^{2l} 2^{-l(4+\alpha)} \sum_{k_1 < l-10} \sum_{k_2 < l-10} 2^{3k_1}\|f\|_{L^2(|y|\sim 2^{k_1})} 2^{3k_2}\|f\|_{L^2(|y|\sim 2^{k_2})} \\ &\leq C\sum_{k_1}\sum_{k_2} 2^{3k_1}\|f\|_{L^2(|y|\sim 2^{k_1})} 2^{3k_2}\|f\|_{L^2(|y|\sim 2^{k_2})} 2^{-(2+\alpha)\max(k_1,k_2)} \\ &\leq C\sum_k 2^{k(4-\alpha)}\|f\|^2_{L^2(|y|\sim 2^k)} \leq C\||\xi|^{2-\frac{\alpha}{2}} f\|^2. \end{aligned}$$

Finally, for the case $|l-k| \leq 10$, we use

$$|\Psi(2^{-k}\xi) - \Psi(2^{-k}y)| \leq 2^{-k}|\xi - y||\nabla\Psi(2^{-k}(\xi - y))| \leq C2^{-k}|\xi - y|,$$

so that

$$|F_k(\xi)| \le C2^{-k} \int_{|y|\sim 2^k} \frac{|f(y)|}{|\xi - y|^{1+\frac{\alpha}{2}}} dy = C2^{-k} |f| \chi_{|y|\sim 2^k} * \frac{1}{|\cdot|^{1+\frac{\alpha}{2}}}.$$

Thus,

$$\begin{aligned} K_2 &\le C \sum_k \int_{|\xi|\sim 2^k} 2^{2k} \left| |f| \chi_{|y|\sim 2^k} * \frac{1}{|\cdot|^{1+\frac{\alpha}{2}}} \right|^2 d\xi \\ &\le C \sum_k 2^{2k} \| |f| \chi_{|y|\sim 2^k} * \frac{1}{|\cdot|^{1+\frac{\alpha}{2}}} \|^2_{L^2(|\xi|\sim 2^k)} \\ &\le C \sum_k 2^{k(4-\alpha)} \| |f| \chi_{|y|\sim 2^k} * \frac{1}{|\cdot|^{1+\frac{\alpha}{2}}} \|^2_{L^{\frac{4}{\alpha}}(|\xi|\sim 2^k)} \\ &\le C \sum_k 2^{k(4-\alpha)} \|f\|^2_{L^2(|\xi|\sim 2^k)} \le C \| |\xi|^{2-\frac{\alpha}{2}} f \|^2. \end{aligned}$$

where we have used the Hausdorf-Young's inequality

$$\|f \chi_{|y|\sim 2^k} * \frac{1}{|\cdot|^{1+\frac{\alpha}{2}}} \|_{L^{\frac{4}{\alpha}}} \le C \| \frac{1}{|\cdot|^{1+\frac{\alpha}{2}}} \|_{L^{\frac{4}{2-\alpha},\infty}} \|f\|_{L^2(|\xi|\sim 2^k)} \le C \|f\|_{L^2(|\xi|\sim 2^k)}.$$

□

In the sequel we need to control the commutator $[\partial_1 |\nabla|^{-\frac{\alpha}{2}}, |\xi|^2]$. In fact, this commutator is morally like $[|\nabla|^{1-\frac{\alpha}{2}}, |\xi|^2]$, which was indeed considered in Lemma 1.1.6. However, there does not appear to be an easy way to transfer the estimate (1.11) to it, so we state the relevant estimate here.[2]

Lemma 1.1.7. *For any integer $a \in (0,1)$ there exists $C = C_a$ so that*

$$\| [\partial_1 |\nabla|^{-a}, |\xi|^2] f \|_{L^2} \le C \| |\xi|^{1+a} f \|_{L^2}. \tag{1.12}$$

Proof. For the proof of (1.12), recall the representation formula (1.4). We will reduce to the same

[2]In fact, it can be reduced to a similar expression as in the proof of (1.11), so we prove them simultaneously.

expressions as above. With the partition of unity displayed above, write

$$
\begin{aligned}
[\partial_1|\nabla|^{-a},|\xi|^2]f(\xi) = c_a \sum_{k=-\infty}^{\infty} 2^{2k}[\partial_1|\nabla|^{-a},\psi(2^{-k}\cdot)]f = \\
= \quad c_a \sum_{k=-\infty}^{\infty} 2^{2k}[\partial_{\xi_1}\int_{\mathbb{R}^2}\frac{\psi(2^{-k}y)f(y)}{|\xi-y|^{2-a}}dy - \psi(2^{-k}\xi)\partial_{\xi_1}\int_{\mathbb{R}^2}\frac{f(y)}{|\xi-y|^{2-a}}dy] = \\
= \quad c_a(a-2)\sum_{k=-\infty}^{\infty} 2^{2k}\int_{\mathbb{R}^2}\frac{\xi_1-y_1}{|\xi-y|}\frac{(\psi(2^{-k}y)-\psi(2^{-k}\xi))f(y)}{|\xi-y|^{2-a}}dy
\end{aligned}
$$

Taking absolute values and estimating yields the bound

$$
|[\partial_1|\nabla|^{-a},|\xi|^2]f(\xi)| \leq C_a \sum_{k=-\infty}^{\infty} 2^{2k}\int_{\mathbb{R}^2}\frac{|\psi(2^{-k}y)-\psi(2^{-k}\xi)||f(y)|}{|\xi-y|^{3-a}}dy
$$

This is of course exactly the same expression as before for the F_k, with $a := 1-\frac{\alpha}{2}$. Therefore, we can apply the same estimates to obtain

$$
\|[\partial_1|\nabla|^{-a},|\xi|^2]f\|_{L^2(\mathbb{R}^2)} \leq C\||\xi|^{1+a}f\|_{L^2}.
$$

This establishes (1.12). □

1.2 Gronwall's inequality

In the following we frequently use an important relation in PDE's concepts, called Gronwall's inequality. We shall use it in two different versions. First version, used in the regularity problem is stated as follow,

Lemma 1.2.1. *Let α, β and u be real–valued functions defined on the interval I. Assume that β and u are continuous and that the negative part of α is integrable on every closed and subinterval part of I. Then, If β is non-negative and if u satisfies the integral inequality*

$$
u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds, \quad for\ any\ t \in I,
$$

then,

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s) \cdot \exp\left(\int_s^t \beta(r)dr\right)ds, \quad for\ any\ t \in I.$$

The elementary proof of this lemma is as it follows.

Proof. define

$$v(s) = \exp\left(-\int_a^s \beta(r)dr\right)\int_a^s \beta(r)u(r)dr, \quad s \in I.$$

Then

$$v'(s) = \Bigg(\underbrace{u(s) - \int_a^s \beta(r)u(r)dr}_{\leq \alpha(s)}\Bigg)\beta(s)\ \exp\left(-\int_a^s \beta(r)dr\right),$$

where we used the assumed inequality for the upper estimate. Since β and the exponential are non–negative, this gives an upper estimate for the derivative of v. Since $v(a) = 0$, integration of this inequality from a to t gives

$$v(t) \leq \int_a^t \alpha(s)\beta(s)\left(-\int_s^t \beta(r)dr\right)ds.$$

Using the definition of $v(t)$ for the first step, and then this inequality and the function equation of the exponential function, we optain

$$\int_a^t \beta(s)u(s)ds = \exp\left(\int_a^t \beta(r)dr\right)v(t) \leq \int_a^t \alpha(s)\beta(s)\exp\Bigg(\underbrace{\int_a^t \beta(r)dr - \int_a^s \beta(r)dr}_{\int_s^t \beta(r)dr}\Bigg)ds.$$

Substituting this result into the assumed integral inequality gives the above Gronwall's inequality. □

For our argument on the time decay problems, we shall need another version of the Gronwall's inequality as follows.

Lemma 1.2.2. *Let $\sigma > \mu > 0, \kappa > 0$ and $a \in [0,1)$. Let A_1, A_2, A_3 be three positive constants so* *that a function $I:[0,\infty)\to\mathbb{R}_+$ satisfies $I(\tau)\le A_1e^{-\gamma\tau}$, for some real γ and*

$$I(\tau)\le A_2e^{-\mu\tau}+A_3\int_0^\tau \frac{e^{-\sigma(\tau-s)}}{(\min(1,|\tau-s|)^a}e^{-\kappa s}I(s)ds. \tag{1.13}$$

Then, there exists $C=C_{a,\sigma,\mu,\kappa,\gamma}$, so that

$$I(\tau)\le C_{a,\sigma,\mu,\kappa,\gamma}(1+|A_1|+|A_2|+|A_3|)e^{-\mu\tau}.$$

The proof of Lemma 1.2.2 is rather elementary, but we provide it for completeness.

Proof. The proof is straightforward, by a bootstrapping argument. We show that every Lyapunov exponent less than $-\mu$ can be bootstrapped higher. First, relabeling $I(\tau)\to(1+|A_1|+|A_2|+|A_3|)^{-1}I(\tau)$, we may assume without loss of generality that $A_1=A_2=A_3=1$. Next, assume that $\gamma<\mu$ is a Lyapunov exponent, that is $I(\tau)\le Ce^{-\gamma\tau}$. We know by the *a priori* assumed boundedness of $I(\tau)$ there is such an exponent. Applying this in (1.13), we obtain an improved estimate for $I(\tau)$. Indeed,

$$I(\tau)\le e^{-\mu\tau}+Ce^{-\sigma\tau}\int_0^\tau \frac{e^{s(\sigma-\kappa-\gamma)}}{(\min(1,|\tau-s|)^a}ds$$

If $\sigma-\kappa-\gamma\neq 0$, we have for $\tau>1$,

$$\begin{aligned}\int_0^\tau \frac{e^{s(\sigma-\kappa-\gamma)}}{|(\min(1,|\tau-s|)^a}ds &\le \int_0^{\tau-1} e^{s(\sigma-\kappa-\gamma)}ds+e^{\tau(\sigma-\kappa-\gamma)}e^{|\sigma-\kappa-\gamma|}\int_{\tau-1}^\tau \frac{1}{|\tau-s|^a}ds\\ &\le \frac{e^{(\tau-1)(\sigma-\kappa-\gamma)}-1}{\sigma-\kappa-\gamma}+C_{a,\sigma,\kappa,\gamma}e^{\tau(\sigma-\kappa-\gamma)}.\end{aligned}$$

whence the bound

$$I(\tau)\le e^{-\mu\tau}+C_{a,\sigma,\kappa,\gamma}\min\left(e^{-\tau(\kappa+\gamma)},e^{-\sigma\tau}\right).$$

It follows that $\min(\mu,\gamma+\kappa,\sigma)>\gamma$ is a new, better Lyapunov exponent than γ.

In general, we can keep $\sigma-\kappa-\gamma$ away from zero (and so the previous argument valid in all cases), if we readjust the γ if necessary.

In practice, starting with $\gamma = 0$, we jump immediately to κ by the previous argument, since $\sigma - \kappa > 0$, by assumption. Since $\kappa < \mu$, we can apply the same argument again with $\gamma = \kappa$. At this point, either $2\kappa > \mu$ and we finish off (by readjusting slightly γ by taking it smaller, like $\gamma = \frac{2\kappa}{3}$, if it happens that, say $|\sigma - 2\kappa| \leq \frac{\kappa}{2}$). If not, that is if $2\kappa < \mu$, take $\gamma = 2\kappa$ to be our new Lyapunov exponent and repeat. Eventually, for some n_0, $n_0\kappa < \mu \leq (n_0+1)\kappa$ and we will reach a Lyapunov exponent μ. □

At this point it also worth to recall the Young's inequality,

Lemma 1.2.3. *Let $p, q > 0$ be strictly positive real numbers, that satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then,*

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}. \tag{1.14}$$

Proof. The proof is quite elementary. Indeed, considering the fact that exponential function is convex,

$$AB = e^{\ln(AB)} = e^{\ln(A)+\ln(B)} = e^{\frac{1}{p}\ln(A^p)+\frac{1}{q}\ln(B^q)} \leq \frac{1}{p}e^{\ln(A^p)} + \frac{1}{q}e^{\ln(B^q)} = \frac{A^p}{p} + \frac{B^q}{q}.$$

□

1.3 Operator Theory

This section is devoted to a simple presentation of the operator theory. In fact, it is restricted to the materials needed in the sequel. We first state the Banach space version of the implicit function theorem

Theorem 1.3.1. *Let $\mathbb{X}, \mathbb{Y}$ and $\mathbb{Z}$ be Banach spaces. Let the mapping $f : \mathbb{X} \times \mathbb{Y} \to \mathbb{Z}$ be continuously Fréchet differentiable. If $(x_0, y_0) \in \mathbb{X} \times \mathbb{Y}, f(x_0, y_0) = 0$, and $y \mapsto Df(x_0, y_0)(0, y)$ is a Banach space isomorphism from $\mathbb{Y}$onto $\mathbb{Z}$, then there exist neighborhoods U of x_0 and V of y_0 and* *a Fréchet differentiable function* $g : U \mapsto V$ *such that* $f(x, g(x)) = 0$ *and* $f(x, y) = 0$ *if and only if* $y = g(x)$*, for all* $(x, y) \in U \times V$.

Next we define the closed linear operators. These operators are more general than bounded operators, and therefore not necessarily continuous, but they still retain nice enough properties that one can define the spectrum and (with certain assumptions) functional calculus for such operators. Many important linear operators which fail to be bounded turn out to be closed, such as the derivative and a large class of differential operators.

Definition 1.3.2. *[Closed Linear Operator] Let* $\mathbb{X}$, $\mathbb{Y}$ *be two Banach spaces. A linear operator* $A : D(A) \to \mathbb{Y}$ *is closed if for every sequence* $\{x_n\}$ *in* $D(A)$ *converging to* x *in* $\mathbb{X}$ *such that* $Ax_n \to y \in \mathbb{Y}$ *as* $n \to \infty$ *one has* $x \in D(A)$ *and* $Ax = y$.

Definition 1.3.3. *Let* L *be a linear operator on the Banach space* $\mathbb{X}$*, then the resolvent set of* L *is defined to be*

$$\rho(L) = \{\lambda \in \mathbb{C} : (\lambda I - L) \text{ is invertable}\} \tag{1.15}$$

and its spectrum

$$\sigma(L) = \mathbb{C} \backslash \rho(L) = \{\lambda \in \mathbb{C} : (\lambda I - L) \text{ is not invertable}\}. \tag{1.16}$$

Definition 1.3.4. *Let* $\mathbb{X}$ *be a Banach space. A one parameter family of operators* $T(\cdot)$, $0 \leq t < \infty$*, of bounded linear operators from* $\mathbb{X}$ *into* $\mathbb{X}$ *is a* semigroup *of bounded linear operators on* $\mathbb{X}$ *if*

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s)$ *for every* $t, s \geq 0$ *(the semigroup property).*

A semigroup of bounded linear operators, $T(t)$*, is uniformly continuous if*

$$\lim_{t \to 0} \|T(t) - I\|_{\mathbb{X}} = 0.$$

The linear operator A defined by

$$D(A) = \{x \in \mathbb{X} : \lim_{t\to 0} \frac{T(t)x - x}{t} \ \ exists\}$$

and

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}\bigg|_{t=0} \ \ for \ \ x \in D(A)$$

is the infinitesimal generator *of the semigroup $T(t)$, $D(A)$ is the domain of A.*

Definition 1.3.5. *A semigroup $T(t), 0 \le t < \infty$, of bounded linear operators on $\mathbb{X}$ is a* strongly continuous *semigroup of bounded linear operators if*

$$\lim_{t\to 0} T(t)x = x, \ \ for \ \ x \in \mathbb{X}.$$

A strongly continuous semmigroup of bounded linear operators on $\mathbb{X}$ is called a semigroup of class C_0 *or simply a C_0* semigroup.

Lemma 1.3.6. *Let $T(t)$ beb a C_0 semigroup. There exist constants $w \ge 0$ and $M \ge 1$ such that*

$$\|T\|_{\mathbb{X}} \le Me^{wt} \ \ for \ \ 0 \le t < \infty.$$

In the above lemma if $w = 0$, $T(t)$ is called *uniformly bounded* and if $M = 1$ it is called a C_0 *semigroup of contraction*.

The next theorem, which is widely used in operator theory as well as the study of PDE's, characterizes the infinitesimal generator of C_0 semigroup of contraction. Conditions on the behavior of the resolvent of an operator A, which are necessary and sufficient for A to be infinitesimal generator of a C_0 semigroup of contraction.

Theorem 1.3.7. *(Hille–Yosida Theorem) A linear (unbounded) operator A is the infinitesimal generator of a C_0 semigroup of contractions $T(t)$, $t \ge 0$ if and only if*

(i) A is closed and $\overline{D(A)} = \mathbb{X}$.

(ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$

$$\|R(\lambda : A)\|_{\mathbb{X}} \leq \frac{1}{\lambda}.$$

Now let $T(t)$ be a C_0 semigroup satisfying $\|T(t)\|_{\mathbb{X}} \leq e^{\omega t}$ (for some $\omega \geq 0$). Consider $S(t) = e^{-\omega t}T(t)$. $S(t)$ is obviously a C_0 semigroup of contractions. If A is the infinitesimal generator of $T(t)$, then $A - wI$ is the infinitesimal generator of $S(t)$. On the other hand if A is the infinitesimal generator of contractions $S(t)$, then $A + \omega I$ is the infinitesimal generator of the C_0 semigroup $T(t)$ satisfying $\|T(t)\|_{\mathbb{X}} \leq e^{\omega t}$. Indeed, $T(t) = e^{\omega t}S(t)$. These remarks lead us to the characterization of the infinitesimal generators of C_0 semigroups satisfying $\|T(t)\|_{\mathbb{X}} \leq e^{\omega t}$.

Lemma 1.3.8. *A linear operator A is the infinitesimal generator of a C_0 semigroup satisfying $\|T(t)\|_{\mathbb{X}} \leq e^{\omega t}$ if and only if*

(i) A is closed and $\overline{D(A)} = \mathbb{X}$.

(ii) The resolvent set $\rho(A)$ of A contains the ray $\{\lambda : \Im\lambda = 0, \lambda > \omega\}$and for every $\lambda > 0$

$$\|R(\lambda : A)\|_{\mathbb{X}} \leq \frac{1}{\lambda - \omega}.$$

An important aspect of the above lemma is range of the resolvent, say the ray $\{\lambda : \Im(\lambda) = 0, \lambda > \omega\}$. This is of a great use in chapter (2).

1.3.0.1 Gearheart-Prüss Theorem

Let A be the generator of a strongly continuous semigroup e^{tA}, $t \geq 0$ on a Hilbert space H. The position of the spectrum $\sigma(e^{tA})$ of the semigroup is responsible for its stability: if $\sigma(e^{tA}) \subset D := \{z \in C : |z| < 1\}, t \neq 0$, then the semigroup is uniformly asymptotically stable. However, in any actual problem the generator A (and hopefully, its spectrum $\sigma(A)$) is given, not the semigroup e^{tA}, $t \geq 0$. The classical Lyapunov Theorem takes care of this problem: for a wide range of semigroups if $\sigma(A) \subset \mathbb{C}_{-} = \{z \in \mathbb{C} : \Re Z < 0\}$ then $\sigma(e^{tA}) \subset D, t \neq 0$. This class of semigroups includes analytic semigroups, most frequently arising in applications due to their connections to parabolic problems for PDE's.

There are examples showing that the aforementioned Lyapunov Theorem, however, does not generally work, therefore one needs another tool to derive information about the linear stability of the solution from the spectral information about the generator given by the linearized equation. This is where the following Gearhart-Prüss Theorem is used.

Theorem 1.3.9. *[Gearheart-Prüss Theorem] For a strongly continuous semigroup on a Hilbert space,* $\omega(A) < 0$ *if and only if* $\{z : \Re z \geq 0\} \subset \rho(A)$ *and* $\sup\{\|(z-A)^{-1}\| : \Re z > 0\} < \infty$.

Chapter 2

On the sharp time decay rates for the 2D generalized quasi-geostrophic equation and the Boussinesq system

2.1 Introduction

The initial value problem for the 2D Navier-Stokes equation

$$\begin{cases} u_t + u\cdot\nabla u - \Delta u = \nabla p, \ \ x\in 2, t>0 \\ u(0,x) := u_0(x), \nabla\cdot u = 0 \end{cases} \tag{2.1}$$

where $u = (u_1, u_2)$ is the fluid velocity and p is the pressure, is ubiquitous and much studied model in the modern PDE theory. Basic issues like global well-posedness remain elusively unresolved in spatial dimensions $n \geq 3$. In the case of two spatial dimensions though, the problem is globally well-posed. This is mostly due to the *vorticity formulation*. We subtract two equations to get

$$\partial_t(\partial_1 u_2 - \partial_2 u_1) + u\cdot\nabla(\partial_1 u_2 - \partial_2 u_1) + (\partial_1 u_1 + \partial_2 u_2)(\partial_1 u_2 - \partial_2 u_1) + \Delta(\partial_1 u_2 - \partial_2 u_1) = 0.$$

Now if use the divergence free property $\partial_1 u_1 + \partial_2 u_2 = 0$ and define the vorticity $\omega = \partial_2 u_1 - \partial_1 u_2$ then we will get the vorticity equation

$$\begin{cases} \omega_t + u\cdot\nabla\omega - \Delta\omega = 0, \ \ x\in\mathbb{R}^2, t>0 \\ \omega(0,x) := \omega_0(x), \end{cases} \tag{2.2}$$

where the vorticity ω, a scalar quantity, is given by $\omega = \nabla^{\perp}\vec{u} = \partial_1 u_2 - \partial_2 u_1$, where $\nabla^{\perp} = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}$.

Many generalizations of this model have been considered, in particular to respond to modeling situations where the actual physical dissipation is different than the one provided by the Laplacian, in particular in large scale atmospheric models and large scale ocean modeling, see [1, 8, 31]. In particular, we consider the following "umbrella" model

$$\begin{cases} \partial_t z + u \cdot \nabla z + |\nabla|^{\alpha} z = 0, \qquad x \in \mathbb{R}^2,\, t > 0, \\ u = (|\nabla|^{\perp})^{-\beta} z, \nabla \cdot u = 0. \end{cases} \tag{2.3}$$

where $\alpha > 1$ and $\beta \geq 0$, $(|\nabla|^{\perp})^{-\beta} = \nabla^{\perp} m_{-\beta-1}(|\nabla|) = \mathbf{m}_{-\beta}(\xi)$, where $\mathbf{m}_a$ is a symbol of order a, see section 1.1 for precise definition[1]. These type of equations frequently arise in fluid dynamics and as such, they have been widely studied, especially so in the last twenty years. We refer the reader to the works [1, 3, 7, 8, 13, 21, 31, 44, 60, 72] and references therein.

A few examples, that we would like to emphasize as model cases, are as follows. The 2D Fractional Navier-Stokes equation arises, if we take $z = \omega$ and $\beta = 1$,

$$\omega_t + u \cdot \nabla \omega + |\nabla|^{\alpha} \omega = 0. \tag{2.4}$$

If we let $z = \theta$ be the temperature of a flow, $\alpha > 1$ and $\beta = 0$ the resulting equation is the so-called active scalar equation,

$$\theta_t + u \cdot \nabla \theta + |\nabla|^{\alpha} \theta = 0, \tag{2.5}$$

where $u_1 = -R_2\theta, u_2 = R_1\theta$, and $R_j, j = 1,2$ are the Riesz transforms, given by the symbols $m_j(\xi) = i\frac{\xi_j}{|\xi|}$.

[1]Note that it is a requirement that $m_{-\beta-1}(|\nabla|)$ is a *radial symbol* of order $-\beta - 1$.

The Boussinesq system, with general dissipations, reads

$$\begin{cases} \partial_t u + u \cdot \nabla u + |\nabla|^\alpha u = -\nabla P + \theta \vec{e}_2, & x \in \mathbb{R}^2,\, t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + |\nabla|^\beta \theta = 0, & x \in \mathbb{R}^2,\, t > 0, \\ \nabla \cdot u = 0. \end{cases} \tag{2.6}$$

where u is the velocity of the fluid, θ is its temperature, P is the pressure and $\alpha, \beta > 0$ are the dissipation rates for the velocity and the temperature respectively.

We consider the equivalent vorticity formulation, with the usual scalar vorticity variable is given by $\omega = \partial_1 u_2 - \partial_2 u_1$. For the purposes of this work, we will only consider the diagonal case $\alpha = \beta$, that is in vorticity formulation, consists of the following coupled equations

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + |\nabla|^\alpha \omega = \partial_1 \theta, & x \in \mathbb{R}^2,\, t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + |\nabla|^\alpha \theta = 0, & x \in \mathbb{R}^2,\, t > 0, \\ u = (\nabla^\perp)^{-1} \omega,\ \nabla \cdot u = 0. \end{cases} \tag{2.7}$$

2.1.1 Previous results

A lot of work has been done on the question of well-posedness, regularity of the solutions to these systems. We do not even attempt to overview the results, as this is only tangentially relevant for the current work, but the following references contain lots of information about these issues, [1, 8, 9, 10, 26, 28, 29, 30, 31, 32, 35, 36, 40, 42, 55, 59, 60, 62, 64, 66, 67, 68, 69, 70, 71, 73]. As the purpose of our work is to study the long time behavior of the said models, we discuss some recent works on the topic. Most of the research has been done in the important (and classical) Navier-Stokes case in two and three dimensional cases. As the global regularity for this model remains a challenging open problem in 3D, some authors restricted themselves to weak solutions[2] or they considered eventual[3] behavior of strong solutions. In this regard, we would like to reference

[2] which may be non-unique
[3] that is, past eventual singularity formation

the following works, [7, 17, 18, 21, 23, 43, 49, 50, 51, 52, 53].

In [50], the author has exhibited lower time-decay bounds for the solutions, which match the upper bounds and are therefore sharp. The approach in [17, 18], for the same question, uses the method of the so-called scaling variables. This was pioneered in [21, 7], although the idea really took of after the work [17]. It showed not only the optimal decay rates for the Navier-Stokes equation (this was actually previously established in [6]), but it provided an explicit asymptotic expansion of the solution, which explains the specific conditions on the initial data in [6], under which there are better decay rates.

Here, we follow this idea, to provide an explicit asymptotic expansion for the two models under consideration - the generalized quasi-geostrophic equation (2.3) and the Boussinesq system (in vorticity formulation), (2.7). Note that we work exclusively in two spatial dimensions. There are several reasons for this - 2D is the natural playground for (2.3), while the IVP for the Boussinesq system, the three (and higher) dimensional case, faces the same difficulties as the Navier-Stokes problem, namely absence of a global regularity theory. Moreover, we explore relatively low levels of dissipation, which in some sense, brings the global regularity theory to its limits, and we are still able to analyze the asymptotic behavior. Another interesting feature that we deal with is the fractional dissipation. These have been studied in the recent literature, but there are certain technical (and conceptual!) difficulties associated with them, that we deal with advanced Fourier analysis methods.

2.1.2 The scaled variables

We now introduce the scaling variables, for the models under consideration. Basically, the method consists of introducing a new exponential time variable $\tau : e^{\tau} \sim t$ and the corresponding variables in x are rescaled to accommodate this scaling, by keeping the linear part of the equation autonomous. In this way, an algebraic decay in t will manifest itself as an exponential decay in τ. As is well-known, algebraic decays in time (especially non-integrable ones) are notoriously hard to propagate along non-linear evolution equations, while any (however small) exponential decay, due to its integrability, is more amenable to this type of analysis.

Although what mentioned abbove is important for us, it is not yet the main purpose. In fact, as we will see, our scaling creates a gap between the discrete and continuous spectrum of the linear part of the scaled equation. This makes the analysis of the scalled equations more convenient. Here are the details.

2.1.2.1 The scaled variables: the SQG equation

Consider the equation (2.3), and use the scaling variables to rewrite the variables in terms of

$$\xi = \frac{x}{(1+t)^{\frac{1}{\alpha}}}, \qquad \tau = \ln(1+t). \tag{2.8}$$

We define new functions $Z(\xi,\tau)$ and $U(\xi,\tau)$ correspond to $z(x,t)$ and $u(x,t)$ as follows:

$$z(x,t) = \frac{1}{(1+t)^{1+\frac{\beta-1}{\alpha}}}\, Z\left(\frac{x}{(1+t)^{\frac{1}{\alpha}}}, \ln(1+t)\right), \tag{2.9}$$

$$u(x,t) = \frac{1}{(1+t)^{1-\frac{1}{\alpha}}}\, U\left(\frac{x}{(1+t)^{\frac{1}{\alpha}}}, \ln(1+t)\right). \tag{2.10}$$

The choices of the parameters is clearly dictated by the stricture of the corresponding equation - the goal is to ensure an autonomous PDE in the new variables. Indeed, a simple calculation shows

$$\begin{aligned} z_t &= \frac{Z_\tau}{(1+t)^{2+\frac{\beta-1}{\alpha}}} - \frac{1}{\alpha}\,\frac{1}{(1+t)^{2+\frac{\beta-1}{\alpha}}}\,\frac{x}{(1+t)^{\frac{1}{\alpha}}}\cdot\nabla_\xi Z - \frac{1+\frac{\beta-1}{\alpha}}{(1+t)^{2+\frac{\beta-1}{\alpha}}} Z, \\ u\cdot\nabla z &= \frac{1}{(1+t)^{2+\frac{\beta-1}{\alpha}}} U\cdot\nabla_\xi Z. \end{aligned}$$

We also have $|\nabla|^\alpha z = \frac{1}{(1+t)^{2+\frac{\beta-1}{\alpha}}}|\nabla|^\alpha Z$. The proof is just simply a use of relation (1.5). Indeed,

$$
\begin{aligned}
|\nabla|^{\alpha} z &= \int_{\mathbb{R}^2} \frac{z(x)-z(y)}{|x-y|^{2+\alpha}} dy = \frac{1}{(1+t)^{1+\frac{\beta-1}{\alpha}}} \int_{\mathbb{R}^2} \frac{Z(\frac{x}{(1+t)^{\frac{1}{\alpha}}}) - Z(\frac{y}{(1+t)^{\frac{1}{\alpha}}})}{|x-y|^{2+\alpha}} dy \\
&= \frac{(1+t)^{\frac{2}{\alpha}}}{(1+t)^{2+\frac{2}{\alpha}+\frac{\beta-1}{\alpha}}} \int_{\mathbb{R}^2} \frac{Z(\frac{x}{(1+t)^{\frac{1}{\alpha}}}) - Z(\frac{y}{(1+t)^{\frac{1}{\alpha}}})}{|\frac{x}{(1+t)^{\frac{1}{\alpha}}} - \frac{y}{(1+t)^{\frac{1}{\alpha}}}|^{2+\alpha}} \frac{dy}{(1+t)^{\frac{2}{\alpha}}} \\
&= \frac{1}{(1+t)^{2+\frac{\beta-1}{\alpha}}} \int_{\mathbb{R}^2} \frac{Z(\xi)-Z(\xi')}{|\xi-\xi'|^{2+\alpha}} \, d\xi' = \frac{1}{(1+t)^{2+\frac{\beta-1}{\alpha}}} |\nabla|^{\alpha} Z.
\end{aligned}
$$

Hence, $Z(\xi,\tau)$ satisfies the equation

$$
Z_{\tau} = \mathscr{L}Z - U \cdot \nabla_{\xi} Z \tag{2.11}
$$

where

$$
\mathscr{L}Z = -|\nabla|^{\alpha} Z + \frac{1}{\alpha} \xi \cdot \nabla_{\xi} Z + \left(1 + \frac{\beta-1}{\alpha}\right) Z. \tag{2.12}
$$

Note that the relation $u = (|\nabla|^{\perp})^{-\beta} z$ transforms into $U = (|\nabla|^{\perp})^{-\beta} Z$. In addition, the property $\nabla \cdot u = 0$ clearly transforms into $\nabla \cdot U = 0$.

Next, we introduce the scaled variables for the Boussinesq system.

2.1.2.2 The scaled variables: the Boussinesq system

Similar to the SQG case, we use the scaled variables

$$
\xi = \frac{x}{(1+t)^{\frac{1}{\alpha}}}, \qquad \tau = \ln(1+t).
$$

We define new functions $W(\xi,\tau)$, $U(\xi,\tau)$ and $\Theta(\xi,\tau)$, corresponding to $\omega(x,t)$, $u(x,t)$ and $\theta(x,t)$ as follows

$$
\begin{aligned}
\omega(x,t) &= \frac{1}{(1+t)}\, W\left(\frac{x}{(1+t)^{\frac{1}{\alpha}}}, \ln(1+t)\right)\\
u(x,t) &= \frac{1}{(1+t)^{1-\frac{1}{\alpha}}}\, U\left(\frac{x}{(1+t)^{\frac{1}{\alpha}}}, \ln(1+t)\right)\\
\theta(x,t) &= \frac{1}{(1+t)^{2-\frac{1}{\alpha}}}\, \Theta\left(\frac{x}{(1+t)^{\frac{1}{\alpha}}}, \ln(1+t)\right)
\end{aligned}
$$

Then, we calculate

$$
\begin{aligned}
\omega_t &= \frac{W_\tau}{(1+t)^2} - \frac{1}{\alpha}\,\frac{1}{(1+t)^2}\frac{x}{(1+t)^{\frac{1}{\alpha}}}\cdot \nabla_\xi W - \frac{1}{(1+t)^2}W,\\
|\nabla|^\alpha \omega &= \frac{1}{(1+t)^2}\cdot |\nabla|^\alpha W, u\cdot\nabla\omega = \frac{1}{(1+t)^2}U\cdot\nabla W,\ \partial_1\theta = \frac{1}{(1+t)^2}\partial_1\Theta.
\end{aligned}
$$

We also have $|\nabla|^\alpha\omega = \frac{1}{(1+t)^2}|\nabla|^\alpha W$. Indeed by (1.5)

$$
\begin{aligned}
|\nabla|^\alpha\omega &= \int_{\mathbb{R}^2}\frac{\omega(x)-\omega(y)}{|x-y|^{2+\alpha}}dy = \frac{1}{1+t}\int_{\mathbb{R}^2}\frac{W(\frac{x}{(1+t)^{\frac{1}{\alpha}}}) - W(\frac{y}{(1+t)^{\frac{1}{\alpha}}})}{|x-y|^{2+\alpha}}dy\\
&= \frac{(1+t)^{\frac{2}{\alpha}}}{(1+t)^{2+\frac{2}{\alpha}}}\int_{\mathbb{R}^2}\frac{W(\frac{x}{(1+t)^{\frac{1}{\alpha}}}) - W(\frac{y}{(1+t)^{\frac{1}{\alpha}}})}{|\frac{x}{(1+t)^{\frac{1}{\alpha}}} - \frac{y}{(1+t)^{\frac{1}{\alpha}}}|^{2+\alpha}}\,\frac{dy}{(1+t)^{\frac{2}{\alpha}}}\\
&= \frac{1}{(1+t)^2}\int_{\mathbb{R}^2}\frac{W(\xi)-W(\xi')}{|\xi-\xi'|^{2+\alpha}}\,d\xi' = \frac{1}{(1+t)^2}\cdot|\nabla|^\alpha W.
\end{aligned}
$$

For the θ equation similar computation shows that

$$
\begin{aligned}
\theta_t &= \frac{\Theta_\tau}{(1+t)^{3-\frac{1}{\alpha}}} - \frac{1}{\alpha}\,\frac{1}{(1+t)^{3-\frac{1}{\alpha}}}\frac{x}{(1+t)^{\frac{1}{\alpha}}}\cdot\nabla_\xi\Theta - \frac{2-\frac{1}{\alpha}}{(1+t)^{3-\frac{1}{\alpha}}}\Theta,\\
|\nabla|^\alpha\theta &= \frac{1}{(1+t)^{3-\frac{1}{\alpha}}}|\nabla|^\alpha\Theta,\ u\cdot\nabla\theta = \frac{1}{(1+t)^{3-\frac{1}{\alpha}}}U\cdot\nabla\Theta.
\end{aligned}
$$

Therefore $W(\xi,\tau)$ and $\Theta(\xi,\tau)$ satisfy (with the $\mathscr{L}$ defined above in (2.12), but with $\beta = 1$)

$$\begin{cases} W_\tau = \mathscr{L}W - U\cdot\nabla_\xi W + \partial_1\Theta \\ \Theta_\tau = (\mathscr{L} + 1 - \frac{1}{\alpha})\Theta - (U\cdot\nabla_\xi\Theta) \end{cases} \tag{2.13}$$

Clearly, the relations $\nabla\cdot u = 0$ and $u = (|\nabla|^{\perp})^{-1}\omega$ continue to hold for the capital letter variables as well, that is $\nabla\cdot U = 0$ and $U = (|\nabla|^{\perp})^{-1}W$. In addition to the above equations we can define $p(x,t) = \frac{1}{(1+t)^{2-\frac{2}{\alpha}}}P\left(\frac{x}{(1+t)^{\frac{1}{\alpha}}}, \log(1+t)\right)$ and find the following equation for $U(\xi,\tau)$,

$$U_\tau = (\mathscr{L} - \frac{1}{\alpha})U - (U\cdot\nabla_\xi U) - \nabla P + \Theta\cdot e_2 \tag{2.14}$$

2.1.3 Main results

The main goal of this work is to establish the sharp time decay rates of (various norms of) the solutions to (2.3) and (2.7). Our results actually provide explicit asymptotic profiles, of which the precise asymptotic rates are a mere corollary.

Since it is clear that the equation for θ in (2.7) is basically[4] (2.3), it is essential that we start with (2.3). This is the content of our first result, but in order to state it, we shall need to introduce a function $G : \hat{G}(p) = e^{-|p|^\alpha}$, see section 2.1.4 for proper definitions and properties. This is a variant of the function $e^{-\frac{|x|^2}{2}}$, or the Oseen vortex in the case $\alpha = 2$.

Theorem 2.1.1. *(Global decay estimates for SQG) Let* $1 < \alpha < 2$, *and* $\alpha + \beta \leq 3$. *Then, assuming that the initial data* z_0 *is in* $L^2(2) \cap L^\infty$, *the Cauchy problem* (2.3) *has a unique, global solution in* $L^2(2) \cap L^\infty$. *Moreover, for all* $\varepsilon > 0$, *there is a constant* $C = C_{\alpha,\beta,\varepsilon}$ *and for all* $p \in [1,2]$ *and* $t \geq 0$,

$$\|z(t,\cdot) - \frac{\int_{\mathbb{R}^2} z_0(x)dx}{(1+t)^{\frac{2}{\alpha}}} G\left(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}}\right)\|_{L^p} \leq \frac{C}{(1+t)^{\frac{3}{\alpha} - \frac{2}{\alpha p} - \varepsilon}}. \tag{2.15}$$

Moreover, if $\beta > 1$, *we have that* (2.15) *holds for the full range of indices* $1 \leq p < \infty$.

[4] albeit the relation of u with θ is not a direct one, but through the vorticity ω

For generic initial data, that is $\int_{\mathbb{R}^2} z_0(x)dx \neq 0$, we have

$$\|z(t,\cdot)\|_{L^p} \sim (1+t)^{-\frac{2(p-1)}{\alpha p}}, \ \ 1 \leq p \leq 2.$$

which extends to all $1 \leq p < \infty$, provided $\beta > 1$.

Remarks:

- The condition $\beta > 1$ is probably a technical one, but it is needed in our arguments.
- In [17, 18], the authors go one step further in deriving explicitly the next order asymptotic profiles. The analysis required for this step is performed in higher order weighted L^2 space. This cannot be done, since the function G does not belong to the next order weighted space, namely $L^2(3)$, see Proposition 2.1.9. This is in sharp contrast with the case $\alpha = 2$, considered in [17, 18], where the function is in Schwartz class.
- Related to the previous point, we need to address a problem, where the function G and the heat kernel of the semigroup $e^{\tau \mathscr{L}}$ have limited decay at infinity. Thus, any attempt to use the dynamical system approach in [17] to construct stable manifolds faces serious obstacles. We take a completely different approach to the problem in that we use *a priori* estimates and estimates on the evolution operator to establish the asymptotic decomposition.

Our next result concerns (2.7).

Theorem 2.1.2. *(Global decay estimates for Boussinesq) Let $\alpha \in (1, \frac{3}{2})$. Consider the Cauchy problem for* (2.7)*, with initial data $w_0, \theta_0 \in Y := L^2(2) \cap L^\infty \cap H^1(\mathbb{R}^2)$. Then, the Cauchy problem* (2.7) *is globally well-posed in Y - that is for every $t > 0$, the solution $(w(t), \theta(t)) \in Y \times Y$.*

Moreover, for every $\delta > 0$, there exists $C = C(\alpha, \delta, \|w_0\|_Y, \|\theta_0\|_Y)$, so that for all $p \in [1,2]$ and *for all $t > 0$,*

$$\|w(t,\cdot) - \frac{\gamma_2(0)\partial_1 G\left(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}}\right)}{(1+t)^{\frac{3}{\alpha}-1}} - \frac{\gamma_1(0) G\left(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}}\right)}{(1+t)^{\frac{2}{\alpha}}}\|_{L^p} \le \frac{C}{(1+t)^{\frac{6}{\alpha}-3-\frac{2}{\alpha p}-\delta}},$$

$$\|\theta(t,\cdot) - \frac{\gamma_2(0) G\left(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}}\right)}{(1+t)^{\frac{2}{\alpha}}}\|_{L^p} \le \frac{C}{(1+t)^{\frac{5}{\alpha}-2-\frac{2}{\alpha p}-\delta}}, \tag{2.16}$$

where $\gamma_1(0) = \int_{\mathbb{R}^2} w_0(x)dx, \gamma_2(0) = \int_{\mathbb{R}^2} \theta_0(x)dx$. In particular, if $\gamma_2(0) \neq 0$, we have

$$\|w(t,\cdot)\|_{L^p} \sim \frac{1}{(1+t)^{\frac{3}{\alpha}-1-\frac{2}{\alpha p}}}, \|\theta(t,\cdot)\|_{L^p} \sim \frac{1}{(1+t)^{\frac{2}{\alpha}-\frac{2}{\alpha p}}},$$

Remarks:

- As in Theorem 2.1.1, the results can be extended to provide asymptotic expansions for w, θ in the norms $L^p, p \in (2, \infty)$, with the exact same statement.
- Note that the decay rate $(1+t)^{1-\frac{3}{\alpha}}$ in the expression for w is dominant over $(1+t)^{-\frac{2}{\alpha}}$.
- For $\alpha \in (\frac{4}{3}, \frac{3}{2})$, the correction term $\frac{\gamma_1(0)}{(1+t)^{\frac{2}{\alpha}}} G\left(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}}\right)$ is faster decaying than the error term and we can state the result as follows

$$\|w(t,\cdot) - \frac{\gamma_2(0)}{(1+t)^{\frac{3}{\alpha}-1}} \partial_1 G\left(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}}\right)\|_{L^p} \le \frac{C_{\alpha,\delta}\|(w_0, \theta_0)\|_Y}{(1+t)^{\frac{6}{\alpha}-3-\frac{2}{\alpha p}-\delta}},$$

In this section we provide the essential tools for the proof of the main result. In section (2.1.4) some useful estimates regarding the function $G(\cdot)$ are given. As it is clear from the main result, this function plays an important rule through the chapter. In section (2.1.5), we study the operator $\mathscr{L}$ - we establish the basic structure of its spectrum, as well as an explicit form of the semigroup $e^{\tau\mathscr{L}}$. The semigroup is shown to act boundedly on certain weighted L^2 spaces. This is helpful for the study of the non-linear evolutions problem, but it also helps us identify the spectrum, through the Hille-Yosida theorem, see section 2.1.8

In section 2.2, we develop the local and global well-posedness theory for the generalized quasi-geostrophic equation, both in the original variables and then in the scaled variables. This is done via standard energy estimates methods. Even at this level, the optimal decay estimates start to emerge, in the scaled variables context[5]. Our asymptotic results for the quasi-geostrophic model are in section 2.2.3. In it, we use the *a priori* information from Section 2.2, together with new estimates for the Duhamel's operator to derive the precise asymptotic profiles for the solutions. For the Boussinesq system, we provide the necessary local and global well-posedness theory in Section 2.3. Some of these results are basic and could have been recovered from earlier publications. Others provide new *a piori* estimates for the scaled variables system, which are used in section 2.3.4. In section 2.3.4, we provide the proof of our main result about the precise asymptotic profiles for the Boussinesq evolution.

2.1.4 The function G

The function G defined by $\hat{G}(p) = e^{-|p|^\alpha}, p \in \mathbb{R}^2$ will be used frequently in the sequel. We list and prove some important properties.

Lemma 2.1.3. *For any $p \in [2,\infty]$ and $\alpha \in (1,2)$,*

$$(1+|\xi|^2)\; G(\xi), (1+|\xi|^2)\nabla G(\xi) \in L^p_\xi \tag{2.17}$$

In particular, $G, \nabla G \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

Note: For $\alpha \in (1,2)$, the function G does not belong to $L^2(3)$, due to the lack of smoothness of $\hat{G}$ at zero (or what is equivalent to the lack of decay of G at ∞).

Proof. For the L^2 estimate, $\|G\|_{L^2} = \|\hat{G}\|_{L^2} < \infty$. Since $\widehat{G}$ is a radial function

$$\||\xi|^2 G(\xi)\|_{L^2} = \|\Delta_p \widehat{G}(p)\|_{L^2} = \|\Delta_p e^{-|p|^\alpha}\|_{L^2} = \|(\partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho)(e^{-\rho^\alpha})\|_{L^2}.$$

[5] But at this point, we cannot yet conclude the optimality of these estimates, as we are missing an estimate from below.

But, $(\partial_{\rho\rho}+\frac{1}{\rho}\partial_\rho)(e^{-\rho^\alpha}) = -\alpha(\alpha-1)\rho^{\alpha-2}e^{-\rho^\alpha}+\alpha^2\rho^{2(\alpha-1)}e^{-\rho^\alpha}$. Therefore, $\||\xi|^2 G(\xi)\|_{L^2}^2 \leq I_1+I_2$, where $I_1 = \|\rho^{\alpha-2}e^{-\rho^\alpha}\|^2_{L^2(\rho d\rho)}$, $I_2 = \|\rho^{2(\alpha-1)}e^{-\rho^\alpha}\|^2_{L^2(\rho d\rho)}$. We have

$$I_1 \leq \int_0^1 \frac{1}{\rho^{2(2-\alpha)-1}}\, d\rho + \int_1^\infty \rho^{2(\alpha-2)+1}e^{-2\rho^\alpha}\rho\, d\rho.$$

Since $2(2-\alpha)-1<1$, the first term is bounded. The second term is also bounded by the exponential decay, whence I_1 is bounded. The second term, $I_2 = \|\rho^{2(\alpha-1)}e^{-\rho^\alpha}\|^2_{L^2(\rho d\rho)}$ is also bounded - no singularity at zero and exponential decay at ∞. This proves the L^2 estimate.

For the L^∞ estimate we can use the Hausdorf-Young's to bound $\|G\|_{L^\infty} \leq \|\hat{G}\|_{L^1} < \infty$. Similarly,

$$\begin{aligned} & \||\xi|^2 G(\xi)\|_{L^\infty} \leq \|\Delta_p \widehat{G}(p)\|_{L^1} \leq \\ \leq\ & \alpha(\alpha-1)\int_0^\infty \rho^{\alpha-2}e^{-\rho^\alpha}\rho d\rho + \alpha^2\int_0^\infty \rho^{2(\alpha-1)}e^{-\rho^\alpha}\rho d\rho \\ \leq\ & \alpha(\alpha-1)\int_0^\infty \rho^{\alpha-1}e^{-\rho^\alpha} d\rho + \alpha^2\int_0^\infty \rho^{2\alpha-1}e^{-\rho^\alpha} d\rho < \infty. \end{aligned}$$

Now the interpolation between L^2 and L^∞ yields $(1+|\xi|^2)\ G(\xi) \in L^p_\xi, 2\leq p\leq\infty$.

Regarding the claims about ∇G, it is easy to see that $\||\xi|^2\nabla G\|_{L^2} = \|\Delta_p[pe^{-|p|^\alpha}]\|_{L^2} < \infty$. Indeed, the last conclusion follows easily from an identical argument as the one above, as the central issue was the singularity at zero for $\|\Delta_p e^{-|p|^\alpha}\|_{L^2}$. Now the situation is better as we multiply by p, which actually alleviates the singularity at zero. Similar is the argument about $\||\xi|^2\nabla G\|_{L^\infty}$, we omit the details. □

The following lemma will be used frequently in the next sections - it is an easy consequence of the Hausdorff-Young's inequality.

Lemma 2.1.4. *Let $\alpha>0$, then for any $t>0$ and $1\leq p\leq\infty$,*

$$\|e^{-t|\nabla|^\alpha} f\|_{L^p} \leq C\|f\|_{L^p} \tag{2.18}$$

$$\|e^{-t|\nabla|^\alpha}\nabla f\|_{L^p} \leq Ct^{-\frac{1}{\alpha}}\|f\|_{L^p} \tag{2.19}$$

Proof. Clearly,

$$e^{-t|\nabla|^\alpha} f = \int G_t(x-y) f(y) dy$$

where $\widehat{G_t}(p) = \widehat{G}(t^{\frac{1}{\alpha}} p)$. Then $\|e^{-t|\nabla|^\alpha} f\|_{L^p} \le \|G_t\|_{L^1} \|f\|_{L^p} = C\|f\|_{L^p}$, where $C = \|G\|_{L^1(\mathbb{R}^2)}$.

$$\|e^{-t|\nabla|^\alpha} \nabla f\|_{L^p} = t^{-\frac{1}{\alpha}} \left\| \int \nabla G(t^{-\frac{1}{\alpha}}(\cdot - y)) f(y) dy \right\|_{L^p} \le C t^{-\frac{1}{\alpha}} \|f\|_{L^p},$$

where $C = \|\nabla G\|_{L^1(\mathbb{R}^2)}$. □

2.1.5 Spectral theory for $\mathscr{L}$

The following result discusses the spectrum of $\mathscr{L}$ acting on $L^2(2)$.

Proposition 2.1.5. *Let $\mathscr{L}$ be as defined in* (2.12)*, then*

1. The discrete spectrum: *Let $k \in \mathbb{N} \cup \{0\}$ be fixed and $\sigma = (\sigma_1, \sigma_2)$ be such that $|\sigma| = \sigma_1 + \sigma_2 = k$. Then the function $\phi_\sigma(\xi)$ defined by*

$$\phi_\sigma(\xi) = \partial_1^{\sigma_1} \partial_2^{\sigma_2} G, \tag{2.20}$$

is an eigenfunction of $\mathscr{L}$ related to the eigenvalue $\lambda_k = 1 - \frac{3-\beta+k}{\alpha}$. As a consequence, λ_k has multiplicity of at least $k+1$.

2. The essential spectrum: *Let $\mu \in \mathbb{C}$ be such that $\Re\mu \le -\frac{1}{\alpha}$ and define, $\psi_\mu \in L^2$ such that*

$$\widehat{\psi_\mu}(p) = |p|^{-\alpha\mu} e^{-|p|^\alpha}. \tag{2.21}$$

Then ψ_μ is an eigenfunction of the operator $\mathscr{L}$ with the corresponding eigenvalue[6] *$\lambda = 1 + \mu - \frac{3-\beta}{\alpha}$. As these eigenvalues are not isolated, they belong to the essential spectrum, so*

$$\sigma_{ess}(\mathscr{L}) \supseteq \left\{ \lambda \in \mathbb{C} : \Re\lambda \le 1 - \frac{4-\beta}{\alpha} \right\}.$$

[6] Note however that all this eigenvalues are not isolated, hence they are in the essential spectrum.

Remark 2.1.6. *We show later (see Lemma 1 below) that in fact, the operator $\mathscr{L}$ has exactly one simple eigenvalue $\lambda_0 = 1 - \frac{3-\beta}{\alpha}$ corresponding to the eigenfunction $G \in L^2(2)$, while the rest of the spectrum has the form of*

$$\sigma(\mathscr{L}) = \left\{\lambda \in \mathbb{C} : \Re\lambda \leq 1 - \frac{4-\beta}{\alpha}\right\} \cup \{1 - \frac{3-\beta}{\alpha}\}$$

Proof. Regarding discrete spectrum, we start with a calculation, which will allow us to identify some of the eigenvalues. Let $\phi_0(\xi)$ be a radial function, i.e. $\widehat{\phi_0}(p) = g(|p|)$. Then

$$\begin{aligned}
\widehat{\mathscr{L}\phi_0}(p) &= -\widehat{|\nabla|^\alpha \phi_0} + \frac{1}{\alpha}\widehat{\xi \cdot \nabla_\xi \phi_0}(p) + \left(1 + \frac{\beta-1}{\alpha}\right)\widehat{\phi_0}(p) = \\
&= -|p|^\alpha \widehat{\phi_0}(p) - \frac{2}{\alpha}\widehat{\phi_0}(p) - \frac{1}{\alpha}\sum_{j=1}^{2} p_j \partial_j \widehat{\phi_0}(p) + \left(1 + \frac{\beta-1}{\alpha}\right)\widehat{\phi_0}(p) = \\
&= -|p|^\alpha g(|p|) - \frac{2}{\alpha}\widehat{\phi_0}(p) - \frac{1}{\alpha}\sum_{j=1}^{2} p_j\, g'(|p|)\, \frac{p_j}{|p|} + \left(1 + \frac{\beta-1}{\alpha}\right)\widehat{\phi_0}(p) = \\
&= \left(1 + \frac{\beta-3}{\alpha}\right)\widehat{\phi_0}(p) + \left(-|p|^\alpha g(|p|) - \frac{1}{\alpha}|p|\, g'(|p|)\right) \qquad (2.22)
\end{aligned}$$

Now if g satisfies,

$$-|p|^\alpha g(|p|) - \frac{1}{\alpha}|p|\, g'(|p|) = 0 \qquad (2.23)$$

then clearly $\lambda = \left(1 - \frac{3-\beta}{\alpha}\right)$ is an eigenvalue for $\mathscr{L}$. The solution of (2.23), gives the eigenfunction, $\widehat{\phi_0}(p) = e^{-|p|^\alpha}$ or $\phi_0 = G$.

Now, let ϕ_k be an eigenfunction corresponding to the eigenvalue $\lambda_k = \left(1 - \frac{3-\beta+k}{\alpha}\right)$, that is

$$\mathscr{L}\phi_k(\xi) = \left(1 - \frac{3-\beta+k}{\alpha}\right)\phi_k \qquad (2.24)$$

Taking a derivative ∂_j in (2.24), we obtain

$$
\begin{aligned}
&\left(1-\frac{3-\beta+k}{\alpha}\right)\partial_j\phi_k = \partial_j\mathscr{L}\phi_k(\xi) = \\
- &\quad |\nabla|^{\alpha}\partial_j\phi_k + \frac{1}{\alpha}\partial_j(\xi\cdot\nabla\phi_k) + \left(1-\frac{3-\beta+k}{\alpha}\right)\partial_j\phi_k \\
= &\quad -|\nabla|^{\alpha}\partial_j\phi_k + \frac{1}{\alpha}\partial_j\phi_k + \frac{1}{\alpha}\xi\cdot\nabla(\partial_j\phi_k) + \left(1-\frac{3-\beta+k}{\alpha}\right)\partial_j\phi_k(\xi) \\
= &\quad \mathscr{L}[\partial_j\phi_k] + \frac{1}{\alpha}\partial_j\phi_k.
\end{aligned}
$$

It follows that

$$\mathscr{L}[\partial_j\phi_k] = \left(1-\frac{3-\beta+(k+1)}{\alpha}\right)\partial_j\phi_k$$

It follows that $\left(1-\frac{3-\beta+k+1}{\alpha}\right)$ is an eigenvalue, corresponding to an eigenfunction $\partial_j\phi_k$. Thus, we have identified a family of eigenvalues and eigenvectors as follows. Fix $k\in\mathbb{N}$, and let (σ_1,σ_2) be so that $\sigma_1+\sigma_2=k$. Then, by induction, for the function $\phi_k := \partial_1^{\sigma_1}\partial_2^{\sigma_2}\phi_0$, we have (2.24).

This finishes off the characterization of the discrete spectrum. Note that what we have proved so far does not guarantee that there is not any more discrete spectrum, but merely an inclusion, as stated.

Regarding essential spectrum, we compute $\widehat{\mathscr{L}\psi_\mu}$. From the calculation (2.22), we have

$$\widehat{\mathscr{L}\psi_\mu}(p) = \left(\mu+1+\frac{\beta-3}{\alpha}\right)\widehat{\psi}_\mu(p),$$

whence ψ_μ is an eigenfunction. Indeed, $\psi_\mu\in L^2(2)$, when $\Re\mu\le -\frac{1}{\alpha}$. This is easy to see with a computation similar to the ones performed in Lemma 2.1.3.

$$\||\xi|^2\psi_\mu\|_{L^2}^2 = \|\Delta_p\hat{\psi}_\mu\|_{L^2}^2 = \int_0^\infty |(\partial_{\rho\rho}+\frac{1}{\rho}\partial_\rho)\rho^{-\alpha\mu}e^{-\rho^\alpha}|^2\rho d\rho.$$

The worst term (when $\alpha>1$) is exactly $\int_0^1\rho^{-(3+2\alpha\mu)}d\rho$, which converges for $\Re\mu < -\frac{1}{\alpha}$. □

Figure (**??**) shows the spectrum of the operator $\mathscr{L}$ in the spaces of $L^2(2)$. As it is clear from

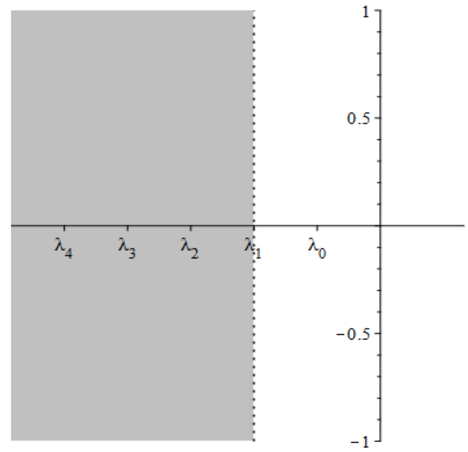

Figure 2.1: Spectrum of $\mathscr{L}$ in the space $L^2(2)$

the figure there is one isolated eigenvalue $\lambda_0 = 1 - \frac{3-\beta}{\alpha}$, and the rest of λ_i's lie in the essential spectrum.

Before move to the next section, we would like to emphasis that the eigenfunctions of $\mathscr{L}^*$ correspond to the discrete eigenvalues of $\mathscr{L}$ are given by the set $\{1, \xi, \cdots, \xi^k, \cdots\}$. Indeed, for any j, $\langle \mathscr{L}^* \xi^j, \partial^j G\rangle = \langle \xi^j, \mathscr{L}\partial^j G\rangle = \lambda_j \langle \xi^j, \partial^j G\rangle$. In other words $\mathscr{L}^*\xi^j = \lambda_j \xi^j$.

2.1.6 The semigroup $e^{\tau\mathscr{L}}$

The following proposition yields an explicit formula for the semigroup $e^{\tau\mathscr{L}}$. This is a variant of the formula displayed in [17], in the case $\alpha = 2, \beta = 1$.

Proposition 2.1.7. *The operator $\mathscr{L}$ defines a C_0 semigroup on $L^2(2)(\mathbb{R}^2)$, $e^{\tau\mathscr{L}}$. In fact, we have the following formula for its action*

$$\widehat{(e^{\tau\mathscr{L}} f)}(p) = e^{(1-\frac{3-\beta}{\alpha})\tau} e^{-a(\tau)|p|^\alpha} \widehat{f}(e^{-\frac{\tau}{\alpha}} p), \tag{2.25}$$

$$(e^{\tau\mathscr{L}} f)(\xi) = \frac{e^{(1-\frac{1-\beta}{\alpha})\tau}}{a(\tau)^{\frac{2}{\alpha}}} \int_{\mathbb{R}^2} G\left(\frac{\xi-\eta}{a(\tau)^{\frac{1}{\alpha}}}\right) f(e^{\frac{\tau}{\alpha}}\eta)\, d\eta, \tag{2.26}$$

where $a(\tau)=1-e^{-\tau}$. *In particular, for* $1\le p\le\infty$,

$$\|e^{\tau\mathscr{L}}f\|_{L^p} \leq Ce^{(1-\frac{1-\beta}{\alpha}-\frac{2}{\alpha p})\tau}\|f\|_{L^p} \tag{2.27}$$

$$\|e^{\tau\mathscr{L}}\nabla f\|_{L^p} \leq C\frac{e^{(1-\frac{2-\beta}{\alpha}-\frac{2}{\alpha p})\tau}}{a(\tau)^{\frac{1}{\alpha}}}\|f\|_{L^p}. \tag{2.28}$$

Remark: Note that $a(\tau)\sim\min(1,\tau)$. This will be used frequently in the sequel.

Proof. The generation of the semigroup would follow, once we prove that the function $g:\hat{g}(\tau,p):=e^{(1-\frac{3-\beta}{\alpha})\tau}e^{-a(\tau)|p|^\alpha}\widehat{f}(p\cdot e^{-\frac{\tau}{\alpha}})$ satisfies $\partial_\tau\hat{g}(\tau,p)=\widehat{\mathscr{L}g(\tau,\cdot)}$. Clearly, $\hat{g}(0,p)=\hat{f}(p)$, so $g(0,\xi)=f(\xi)$. Next, we compute $\partial_\tau\hat{g}(\tau,p)$. We have

$$\begin{aligned}
&\partial_\tau\widehat{g}(\tau,p)=\\
=\;&\left[(1-\frac{3-\beta}{\alpha}-a'(\tau)|p|^\alpha)\widehat{f}(p\cdot e^{-\frac{\tau}{\alpha}})-\frac{1}{\alpha}e^{-\frac{\tau}{\alpha}}p\cdot\nabla_p\widehat{f}(p\cdot e^{-\frac{\tau}{\alpha}})\right]e^{\tau(1-\frac{3-\beta}{\alpha})}e^{-a(\tau)|p|^\alpha}\\
=\;&\left(1+\frac{\beta-3}{\alpha}\right)\widehat{g}(p)+(a(\tau)-1)|p|^\alpha\widehat{g}(p)-\frac{1}{\alpha}e^{-\frac{\tau}{\alpha}}p\cdot\nabla_p\widehat{f}(p\cdot e^{-\frac{\tau}{\alpha}})e^{\tau(1-\frac{3-\beta}{\alpha})}e^{-a(\tau)|p|^\alpha}
\end{aligned}$$

where we have used the relation $a'(\tau)=1-a(\tau)$. Next, by (2.22), we have

$$\widehat{\mathscr{L}g(\tau,\cdot)} = -|p|^\alpha\widehat{g}(p)-\frac{1}{\alpha}\sum_{j=1}^2 p_j\partial_j\widehat{g}(p)+\left(1+\frac{\beta-3}{\alpha}\right)\widehat{g}(p).$$

But,

$$\begin{aligned}
&\frac{1}{\alpha}\sum_{j=1}^2 p_j\partial_j\widehat{g}(p)=\\
&=\frac{1}{\alpha}\sum_{j=1}^2 p_j\left(-\alpha a(\tau)p_j|p|^{\alpha-2}\widehat{f}(p\cdot e^{-\frac{\tau}{\alpha}})+e^{-\frac{\tau}{\alpha}}\partial_j\widehat{f}(p\cdot e^{-\frac{\tau}{\alpha}})\right)e^{\tau(1-\frac{3-\beta}{\alpha})}e^{-a(\tau)|p|^\alpha}\\
&=-a(\tau)|p|^\alpha\widehat{f}(p\cdot e^{-\frac{\tau}{\alpha}})e^{\tau(1-\frac{3-\beta}{\alpha})}e^{-a(\tau)|p|^\alpha}+\frac{1}{\alpha}e^{-\frac{\tau}{\alpha}}p\cdot\nabla_p\widehat{f}(p\cdot e^{-\frac{\tau}{\alpha}})e^{\tau(1-\frac{3-\beta}{\alpha})}e^{-a(\tau)|p|^\alpha}.
\end{aligned}$$

Altogether,

$$
\begin{aligned}
\widehat{\mathscr{L}g(\tau,\cdot)} &= -|p|^{\alpha}\widehat{g}(p) + \left(1+\frac{\beta-3}{\alpha}\right)\widehat{g}(p) + a(\tau)|p|^{\alpha}\widehat{g}(p) - \\
&- \frac{1}{\alpha}e^{-\frac{\tau}{\alpha}}p\cdot\nabla_p\widehat{f}(p\cdot e^{-\frac{\tau}{\alpha}})e^{\tau(1-\frac{3-\beta}{\alpha})}e^{-a(\tau)|p|^{\alpha}}.
\end{aligned}
$$

An immediate inspection reveals that $\partial_\tau\widehat{g}(\tau,p) = \widehat{\mathscr{L}g(\tau,\cdot)}(p)$ and so the semigroup formula (2.25) is established. The formula (2.26) is just a Fourier inversion of (2.25). Regarding the estimate (2.27), we proceed as follows

$$
\|e^{\tau\mathscr{L}}f\|_{L^p} \le e^{(1-\frac{1-\beta}{\alpha})\tau}\|G_{a(\tau)^{\frac{1}{\alpha}}}\|_{L^1}\|f(e^{\frac{\tau}{\alpha}}\cdot)\|_{L^p} = e^{(1-\frac{1-\beta}{\alpha}-\frac{2}{\alpha p})\tau}\|G\|_{L^1}\|f\|_{L^p}.
$$

For (2.28), note that integration by parts yields

$$
\begin{aligned}
(e^{\tau\mathscr{L}}\partial_j f)(\xi) &= \frac{e^{(1-\frac{1-\beta}{\alpha})\tau}}{a(\tau)^{\frac{2}{\alpha}}}\int_{\mathbb{R}^2} G\left(\frac{\xi-\eta}{a(\tau)^{\frac{1}{\alpha}}}\right)(\partial_j f)(e^{\frac{\tau}{\alpha}}\eta)d\eta = \\
&= \frac{e^{(1-\frac{2-\beta}{\alpha})\tau}}{a(\tau)^{\frac{3}{\alpha}}}\int_{\mathbb{R}^2}\partial_j G\left(\frac{\xi-\eta}{a(\tau)^{\frac{1}{\alpha}}}\right)f(e^{\frac{\tau}{\alpha}}\eta)d\eta,
\end{aligned}
$$

whence

$$
\|(e^{\tau\mathscr{L}}\nabla f)(\xi)\|_{L^p} \le \frac{e^{(1-\frac{2-\beta}{\alpha}-\frac{2}{\alpha p})\tau}}{a(\tau)^{\frac{1}{\alpha}}}\|\nabla G\|_{L^1}\|f\|_{L^p}.
$$

□

We need a variant of Proposition A.2 in [17], which discusses the commutation of the semigroup with differential operators.

Lemma 2.1.8. *We have the following commutation relation for $e^{\tau\mathscr{L}}$*

$$
\nabla e^{\tau\mathscr{L}} = e^{\frac{\tau}{\alpha}}e^{\tau\mathscr{L}}\nabla \tag{2.29}
$$

Proof. Let $u(x,\tau) = e^{\tau \mathscr{L}} f(x)$, then u satisfies the following equation

$$\begin{cases} u_\tau = \mathscr{L} u, \\ u(0,x) = f(x). \end{cases}$$

Clearly, taking a derivative ∂_j in (2.12) yields, for $j = 1,2$

$$\begin{cases} (\partial_j u)_\tau = \partial_j(\mathscr{L} u) = \mathscr{L} \partial_j u + \frac{1}{\alpha} \partial_j u, \\ \partial_j u(x,0) = \partial_j f(x), \end{cases}$$

which has the solution $\partial_j u(x,\tau) = e^{\tau[\mathscr{L} + \frac{1}{\alpha}]} \partial_j f(x)$. In other words

$$\nabla e^{\tau \mathscr{L}} f(x) = e^{\frac{\tau}{\alpha}} e^{\tau \mathscr{L}} \nabla f(x).$$

□

2.1.7 Semigroup estimates

We need to address an important question, namely the behavior of the bounded operators $e^{\tau \mathscr{L}}$ on $L^2(2)$. The next Proposition does that. More precisely, we are interested in the decay of the operator norms $\|e^{\tau \mathscr{L}}\|_{L^2(2) \to L^2(2)}$. Importantly, better decay estimates hold, when the functions have mean value zero. The long proof of this proposition is postponed to Appendix (A).

Proposition 2.1.9. *Let $f \in L^2(2)$, $\hat{f}(0) = 0$ and $\gamma = (\gamma_1, \gamma_2) \in \mathbf{N}^2, |\gamma| = 0,1$ and $0 < \varepsilon << 1$. Then there exists $C = C_\varepsilon > 0$, such that for any $\tau > 0$,*

$$\|\nabla^\gamma (e^{\tau \mathscr{L}} f)\|_{L^2(2)} \leq C \frac{e^{\left(1 - \frac{4-\beta}{\alpha} + \varepsilon\right)\tau}}{a(\tau)^{\frac{|\gamma|}{\alpha}}} \|f\|_{L^2(2)}, \tag{2.30}$$

or

$$\|\nabla^{\gamma}(e^{\tau\mathscr{L}}f)\|_{L^2(2)} \leq C\|f\|_{L^2(2)} \cdot \begin{cases} \frac{1}{\tau^{\frac{|\gamma|}{\alpha}}}, & \tau \leq 1 \\ e^{\left(1-\frac{4-\beta}{\alpha}+\varepsilon\right)\tau}, & \tau > 1 \end{cases} \tag{2.31}$$

2.1.8 The decay estimates for $e^{\tau\mathscr{L}}$ give a description of the spectrum of $\mathscr{L}$

In this section, we show that the spectral inclusions in Proposition 2.1.5 are actually equalities. We also compute explicitly the Riesz projection $\mathscr{P}_0$ onto the eigenvalue of $\mathscr{L}$ with the largest real part. In Proposition 2.1.5, we have already identified G as being an eigenfunction for $\mathscr{L}$ corresponding to an eigenvalue $\lambda_0 = 1 - \frac{3-\beta}{\alpha}$. On the other hand, applying Proposition 2.1.9, for functions with $\hat{f}(0) = 0$ and $\gamma = (0,0)$, implies

$$\|e^{\tau\mathscr{L}}f\|_{L^2(2)} \leq C_\varepsilon e^{\left(1-\frac{4-\beta}{\alpha}+\varepsilon\right)\tau}\|f\|_{L^2(2)}. \tag{2.32}$$

Denote the co-dimension one subspace $X_0 = \{f \in L^2(2) : \hat{f}(0) = 0\}$. Clearly, the operator $\mathscr{L}$ acts invariantly on X_0, since for every $f \in L^2(2) : \int f(\xi)d\xi = 0$, we have $\int_{\mathbb{R}^2} \xi \cdot \nabla f d\xi = 0$, whence $\int \mathscr{L}f(\xi)d\xi = 0$.

Introduce $\mathscr{L}_0 := \mathscr{L}|_{X_0}$, with domain $D(\mathscr{L}_0) = D(\mathscr{L}) \cap X_0 = H^\alpha \cap X_0$. By the Hille-Yosida theorem, this estimate (2.32) implies that the set $\{\lambda : \Re\lambda > \left(1 - \frac{4-\beta}{\alpha}\right)\}$ is in the resolvent set of $\mathscr{L}_0$, since the integral representing $(\lambda - \mathscr{L})^{-1}$, namely $\int_0^\infty e^{-\lambda\tau}e^{\tau\mathscr{L}}f d\tau$, converges by virtue of (2.32).

Combining this with the results from Proposition 2.1.5, we conclude that $\sigma(\mathscr{L}) \cap \{\lambda : \Re\lambda > \left(1 - \frac{4-\beta}{\alpha}\right)\}$ is a singleton - the eigenvalue $\lambda_0 = 1 - \frac{3-\beta}{\alpha}$, which is simple, with eigenfunction G. We conclude the following lemma.

Lemma 1. *For the operator $\mathscr{L}$ acting on $L^2(2)$, there is the following description of its spectrum*

$$\sigma(\mathscr{L}) = \{1 - \frac{3-\beta}{\alpha}\} \cup \sigma_{ess}(\mathscr{L}); \;\; \sigma_{ess}(\mathscr{L}) = \{\lambda : \Re\lambda \leq \left(1 - \frac{4-\beta}{\alpha}\right)\},$$

Its Riesz projection $\mathscr{P}_0$ corresponding to the largest (real-part) eigenvalue $\lambda_0 = 1 - \frac{3-\beta}{\alpha}$, is given *by*

$$\mathscr{P}_0 f(\xi) = \left(\int_{\mathbb{R}^2} f(\xi) d\xi \right) G(\xi)$$

We just need to show the part about the normalization of $\mathscr{P}_0$. Indeed, since $\mathscr{P}_0^2 f = \langle G, 1 \rangle \mathscr{P}_0 f = \hat{G}(0) \mathscr{P}_0 f = \mathscr{P}_0 f$, since $\hat{G}(0) = 1$.

Denote the projection $\mathscr{Q}_0 = Id - \mathscr{P}_0$ over the complementary part of the spectrum, so that $\mathscr{L}_0 = \mathscr{Q}_0 \mathscr{L} \mathscr{Q}_0$. Also, $\mathscr{Q}_0 : L^2(2) \to X_0$. Now, (2.32) can be reformulated as

$$\|\nabla^\gamma e^{\tau \mathscr{L}_0} f\|_{L^2(2)} \leq C_\varepsilon \frac{e^{\left(1 - \frac{4-\beta}{\alpha} + \varepsilon\right)\tau}}{a(\tau)^{\frac{|\gamma|}{\alpha}}} \|f\|_{L^2(2)}. \tag{2.33}$$

for any function f, since $e^{\tau \mathscr{L}_0} f = e^{\tau \mathscr{L}} \mathscr{Q}_0 f$ and the entry $\mathscr{Q}_0 f$ has mean value zero, so (2.32) is applicable.

In addition, we can derive estimates for the action of the semigroup $e^{\tau \mathscr{L}}$ on $L^2(2)$, without the crucial mean value zero property $\hat{f}(0) = 0$.

Proposition 2.1.10. *Let $f \in L^2(2)$. Then, there exists a constant C, so that*

$$\|\nabla^\gamma (e^{\tau \mathscr{L}} f)\|_{L^2(2)} \leq C \frac{e^{\left(1 - \frac{3-\beta}{\alpha}\right)\tau}}{a(\tau)^{\frac{|\gamma|}{\alpha}}} \|f\|_{L^2(2)}. \tag{2.34}$$

Proof. We use the decomposition

$$f = \mathscr{P}_0 f + \mathscr{Q}_0 f = \langle f, 1 \rangle G + [f - \langle f, 1 \rangle G].$$

Thus,

$$e^{\tau \mathscr{L}} f = \langle f, 1 \rangle e^{\tau(1 - \frac{3-\beta}{\alpha})} G + e^{\tau \mathscr{L}_0} [\mathscr{Q}_f]$$

It follows that

$$
\begin{aligned}
\|e^{\tau\mathscr{L}}f\|_{L^2(2)} &\leq C|\langle f,1\rangle|e^{\tau(1-\frac{3-\beta}{\alpha})}\|G\|_{L^2(2)}+C_\varepsilon e^{\left(1-\frac{4-\beta}{\alpha}+\varepsilon\right)\tau}\|f\|_{L^2(2)} \\
&\leq Ce^{\tau(1-\frac{3-\beta}{\alpha})}\|f\|_{L^2(2)},
\end{aligned}
$$

where we have used (2.33) and $|\langle f,1\rangle| \leq C\|f\|_{L^2(2)}$. Similar estimates can be derived, as before, for $\nabla^\gamma e^{\tau\mathscr{L}}$, we omit the details. □

2.2 Local and global well-posedness of the SQG and its long term behavior

The local and global theory of the Cauchy problem for SQG has been well-studied in the literature. Local and global well-posedness holds under very general conditions on initial data. Regardless, we will present a few results for our problem (2.3). This is necessary, since we assume a non-standard relation between u and z, but also because we need precise properties, beyond the scope of the well-posedness. Then, we will turn to properties of the rescaled equation, (2.11). We will do so, both in L^p spaces as well as in $L^2(2)$ spaces - the reason is that we will use some of our preliminary results as *a priori* estimates in the subsequent lemmas.

Our first results are about the well-posedness of the standard model (2.3) in L^p spaces.[7]

2.2.1 Global well-posedness and a priori estimates in L^p spaces

Lemma 2.2.1. *Suppose that $z_0 \in L^1 \cap L^\infty =: X$. Then,* (2.3) *is globally well-posed in the space X. Moreover, for every $p \in [1,\infty]$, $t \to \|z(\cdot,t)\|_{L^p}$ is non-increasing in time.*

Proof. We first prove the local existence of the strong solution in the space $C([0,T);X)$, that is, with T to be determined, we are looking for a fixed point of the integral equation

$$
z(\xi) = e^{-t|\nabla|^\alpha}z_0 - \int_0^t e^{-(t-s)|\nabla|^\alpha}\nabla(u\cdot z)\,ds. \tag{2.35}
$$

[7]The results can be made more precise, in individual L^p spaces, rather than in *all* L^p spaces. We will not do so here, because our goal is to extend to $L^2(2)$, which is yet smaller space.

According to Lemma (2.1.4) $\|e^{-t|\nabla|^{\alpha}}z_0\|_{L^1\cap L^\infty} \leq C_0\|z_0\|_{L^1\cap L^\infty}$. For any $T>0$ and $t\in(0,T)$, consider

$$Q(z_1,z_2) := \int_0^t e^{-(t-s)|\nabla|^{\alpha}}\nabla(u_1\cdot z_2)\, ds,$$

where u_1 is given by $u_1 = (\nabla^{\perp})^{-\beta}z_1$. For $t\in(0,T)$, using (2.19)

$$\begin{aligned}
\|Q(z_1(t),z_2(t))\|_{L^1} &= \|\int_0^t e^{-(t-s)|\nabla|^{\alpha}}\nabla(u_1\cdot z_2)\, ds\|_{L^1} \leq C\int_0^t \frac{1}{(t-s)^{\frac{1}{\alpha}}}\|(u_1\cdot z_2)\|_{L^1}\, ds \\
&\leq Ct^{1-\frac{1}{\alpha}}\sup_{0\leq s\leq T}\|u_1(s,\cdot)\|_{L^\infty}\sup_{0\leq s\leq T}\|z_2(s,\cdot)\|_{L^1} \leq \\
&\leq C_\varepsilon T^{1-\frac{1}{\alpha}}\sup_{0\leq s\leq T}(\|z_1(s,\cdot)\|_{L^{\frac{2}{\beta}+\varepsilon}} + \|z_1(s,\cdot)\|_{L^{\frac{2}{\beta}-\varepsilon}})\sup_{0\leq s\leq T}\|z_2(s,\cdot)\|_{L^1} \\
&\leq CT^{1-\frac{1}{\alpha}}\sup_{0\leq s\leq T}\|z_1\|_X\sup_{0\leq s\leq T}\|z_2\|_X.
\end{aligned}$$

where we have used the Sobolev embedding estimate (1.8). Similarly,

$$\|Q(z_1,z_2)\|_{L^\infty} \leq CT^{1-\frac{1}{\alpha}}\sup_{0\leq s\leq T}\|u_1\|_{L^\infty}\sup_{0\leq s\leq T}\|z_2\|_{L^\infty} \leq CT^{1-\frac{1}{\alpha}}\sup_{0\leq s\leq T}\|z_1\|_X\sup_{0\leq s\leq T}\|z_2\|_X.$$

Finally, following similar path, we also have

$$\|Q(z_1,z_1)-Q(z_2,z_2)\|_X \leq CT^{1-\frac{1}{\alpha}}(\|z_1\|_X+\|z_2\|_X)\|z_1-z_2\|_X.$$

Upon introducing $Y_T := \{z : \sup_{0\leq t\leq T}\|z(t,\cdot)\|_X \leq 2C_0\|z_0\|_X\}$ and taking into account the estimates above, we realize that the mapping (2.35)has a fixed point in the metric space $C([0,T],X)$, for small enough $T = T(\|z_0\|_X)$. In fact, the argument shows that $T \sim \|z_0\|_X^{-\frac{\alpha}{\alpha-1}}$.

For the global existence, we need to show that the $t\to\|z(t,\cdot)\|_{L^p}$ does not blow up in finite time. In fact, we show that the $t\to\|z(t,\cdot)\|_{L^p}$ is non-increasing, which will allow us to conclude global existence as well. To that end, we dot product the equation (2.3) with $|z|^{p-2}z, p\in(1,\infty)$ to get

$$\frac{1}{p}\partial_t\|z\|_{L^p}^p + \int_{\mathbb{R}^2} |\nabla|^\alpha z \cdot |z|^{p-2} z d\xi = 0.$$

By the positivity estimate (1.6), we have $\int_{\mathbb{R}^2} |\nabla|^\alpha z \cdot |z|^{p-2} z d\xi \geq 0$. Therefore, $\partial_t\|z\|_{L^p}^p \leq 0$, and $t \to \|z(t,\cdot)\|_{L^p}$ is non-increasing in time. For $p = 1, p = \infty$ the monotonicity follows from an approximation argument from the cases $1 < p < \infty$.

□

Our next result is about *a priori* estimates in L^p spaces, but this time in the rescaled variable formulation, (2.11). Note that the global existence of the rescaled equation is not in question anymore, due to Lemma 2.2.1. However, we show fairly precise decay estimates for the norm of the solution Z. This fairly elementary lemma already shows the advantage of the rescaled variables approach and its far reaching consequences.

Lemma 2.2.2. *Let $Z_0 \in L^1 \cap L^\infty(\mathbb{R}^2)$, $\alpha \in (1,2)$, $0 \leq \beta < 2$ and $p \in [1,\infty)$. Then the unique global strong solution Z of* (2.11) *satisfies*

$$\|Z(\tau)\|_{L^p} \leq \|Z_0\|_{L^p} e^{-\tau(\frac{2}{p\alpha}-1-\frac{\beta-1}{\alpha})}. \tag{2.36}$$

Proof. If we dot product (2.11) with $Z|Z|^{p-2}$, we have by the positivity estimate (1.6), $\int_{\mathbb{R}^2} |\nabla|^\alpha Z \cdot |Z|^{p-2} Z d\xi \geq 0$. Furthermore, using the divergent free property of $U(\xi)$

$$\begin{aligned} \frac{1}{p}\frac{d}{d\tau}\|Z\|_{L^p}^p &\leq \frac{1}{\alpha}\int (\xi \cdot \nabla_\xi Z) Z|Z|^{p-2}\, d\xi - \int (U \cdot \nabla_\xi Z) Z|Z|^{p-2} d\xi + \\ &+ \left(1 + \frac{\beta-1}{\alpha}\right)\|Z\|_{L^p}^p = \left(1 + \frac{\beta-1}{\alpha} - \frac{2}{\alpha p}\right)\|Z\|_{L^p}^p, \end{aligned} \tag{2.37}$$

therefore, we arrive at

$$\frac{1}{p}\frac{d}{d\tau}\|Z\|_{L^p}^p + (\frac{2}{\alpha p} - 1 - \frac{\beta-1}{\alpha})\|Z\|_{L^p}^p \leq 0.$$

Now we use the Gronwall's inequality to finish the proof. □

The above lemma shows *a priori* bound for $\|Z(\tau,\cdot)\|_{L^p}$, for any $p \in [1,\infty]$, and a decay rate for $p < \frac{2}{\alpha+\beta-1}$, but it is not giving any decay rate for $p \geq \frac{2}{\alpha+\beta-1}$. On the other hand, as we shall see later, the decay rate predicted by Lemma 2.2.2 is in fact optimal for $p = 1$, but certainly not so, for any other value of p. We can bootstrap the results of Lemma 2.2.2 in the next lemma to find, what *it will turn out to be, the optimal decay rate*[8] *for any* $p \geq 1$.

Lemma 2.2.3. *Let* $Z_0 \in L^1 \cap L^\infty(\mathbb{R}^2), 1 \leq p \leq \infty$ *and* $\alpha \in (1,2)$, $\alpha + \beta \leq 3$. *Then, there exists constant* $C = C_{p,\alpha,\beta}$, *so that the unique global strong solution* Z *of* (2.11) *satisfies*

$$\|Z(\tau,\cdot)\|_{L^p} \leq C_{p,\alpha,\beta}\|Z_0\|_{L^1\cap L^\infty} e^{-(\frac{3-\beta-\alpha}{\alpha})\tau}. \tag{2.38}$$

Proof. Note that it is enough to prove (2.38) for $p = 2^n, n = 1,2,\ldots$. Indeed, since we have already shown (2.38) (this is basically the statement of Lemma 2.2.2) for $p = 1$, the result for general $p < \infty$ will follow from the result for $p = 2^n$, by applying the Gagliardo-Nirenberg's inequality between $p = 1$ and $p = 2^n, n >> 1$.

So, assume $p = 2^n$, so that the estimate (1.7) is available to us. Taking again dot product $|Z|^{p-2}Z$ and taking into account (1.7) which implies $\int_{\mathbb{R}^2} |\nabla|^\alpha Z \cdot |Z|^{p-2} Z d\xi \geq p^{-1} \||\nabla|^{\frac{\alpha}{2}}[Z^{p/2}]\|_{L^2}^2$. We further add $C\|Z\|_{L^p}^p$, for some large C, to be determined. We have

$$\frac{1}{p}\frac{d}{d\tau}\|Z\|_{L^p}^p + C\|Z\|_{L^p}^p + \frac{1}{p}\||\nabla|^{\frac{\alpha}{2}}[Z^{p/2}]\|_{L^2}^2 \leq \left(C + 1 + \frac{\beta-1}{\alpha} - \frac{2}{\alpha p}\right)\|Z\|_{L^p}^p$$

By Sobolev embedding, we have $\frac{c_\alpha}{p}\|Z\|^p_{L^{\frac{2p}{2-\alpha}}} \leq \frac{1}{p}\||\nabla|^{\frac{\alpha}{2}}[Z^{p/2}]\|_{L^2}^2$. By Gagliardo- Nirenberg's, with $\gamma = \frac{2p-2}{2p-2+\alpha}$, $\|Z\|_{L^p} \leq \|Z\|^\gamma_{L^{\frac{2p}{2-\alpha}}}\|Z\|_{L^1}^{1-\gamma}$, whence by Young's inequality

[8] for generic data

$$
\begin{aligned}
\frac{1}{p}\frac{d}{d\tau}\|Z\|_{L^p}^p &+ C\|Z\|_{L^p}^p+\frac{c\alpha}{p}\|Z\|^p_{L^{\frac{2p}{2-\alpha}}} \leq \left(C+1+\frac{\beta-1}{\alpha}-\frac{2}{\alpha p}\right)\|Z\|^{p\gamma}_{L^{\frac{2p}{2-\alpha}}}\|Z\|_{L^1}^{p(1-\gamma)} \\
&\leq \varepsilon_0\|Z\|^p_{L^{\frac{2p}{2-\alpha}}}+\frac{\left(C+1+\frac{\beta-1}{\alpha}-\frac{2}{\alpha p}\right)^{\frac{1}{1-\gamma}}}{\varepsilon_0^{\frac{\gamma}{1-\gamma}}}\|Z\|_{L^1}^p
\end{aligned}
$$

and $\varepsilon_0>0$ is a fixed number, say we select it $\varepsilon_0=\frac{c\alpha}{p}$. Then

$$
\begin{aligned}
\frac{1}{p}\frac{d}{d\tau}\|Z\|_{L^p}^p+C\|Z\|_{L^p}^p &\leq \frac{\left(C+1+\frac{\beta-1}{\alpha}-\frac{2}{\alpha p}\right)^{\frac{1}{1-\gamma}}}{\varepsilon_0^{\frac{\gamma}{1-\gamma}}}\|Z\|_{L^1}^p \leq \\
&\leq \frac{\left(C+1+\frac{\beta-1}{\alpha}-\frac{2}{\alpha p}\right)^{\frac{1}{1-\gamma}}}{\varepsilon_0^{\frac{\gamma}{1-\gamma}}}\|Z_0\|_{L^1}^p e^{-p\tau(\frac{3-\beta-\alpha}{\alpha})},
\end{aligned}
$$

where we have used Lemma (2.2.2) to estimate $\|Z(\tau,\cdot)\|_{L^1}$. Denoting $\mu:=(\frac{3-\beta-\alpha}{\alpha})\geq 0$, select $C=\mu+1$. We have

$$
I'(\tau)+p(\mu+1)I(\tau)\leq D\|Z_0\|_{L^1}^p e^{-p\mu\tau},
$$

where $I(\tau)=\|Z(\tau)\|_{L^p}^p$, $D=p^{1+\frac{\gamma}{1-\gamma}}\frac{\left(\mu+2+\frac{\beta-1}{\alpha}-\frac{2}{\alpha p}\right)^{\frac{1}{1-\gamma}}}{c_\alpha^{\frac{\gamma}{1-\gamma}}}$. Now we use the Gronwall's inequality to derive the estimate

$$
I(\tau)\leq e^{-p(\mu+1)\tau}I(0)+\frac{D}{p}\|Z_0\|_{L^1}^p e^{-p\mu\tau}.
$$

Taking p^{th} root and simplifying yields the final estimate

$$
\|Z(\tau)\|_{L^p}\leq (\|Z_0\|_{L^p}+\left(\frac{D}{p}\right)^{\frac{1}{p}}\|Z_0\|_{L^1})e^{-\mu\tau}\leq (1+\left(\frac{D}{p}\right)^{\frac{1}{p}})\|Z_0\|_{L^1\cap L^\infty}e^{-\mu\tau}.
$$

For the case $p=\infty$, we take limits in the previous identity, for fixed $\tau>0$, as $p\to\infty$. Note that $\lim_{p\to\infty}\left(\frac{D}{p}\right)^{\frac{1}{p}}=1$, so (2.38)holds true in this case with $C=2$. □

2.2.2 Global solutions and a priori estimates in $L^2(2)$

From the previous section, we know that the SQG equation in its standard form, namely (2.3), has global solutions in L^p. Thus, the rescaled equation (2.11) also has unique global (strong) solutions in L^p. We now would like to understand the Cauchy problem in the smaller space $L^2(2)$. In particular, even if the initial data is well-localized, say $Z(0,\cdot) \in L^2(2)$, it is not *a priori* clear why the solution $Z(\tau)$ will stay in $L^2(2)$ for (any) later time $\tau > 0$. In other words, one needs to start with the local well-posedness for (2.11), and then we shall upgrade it to a global one, by means of *a priori* estimates on $\|Z(\tau)\|_{L^2(2)}$.

Theorem 2.2.4. *Suppose that* $Z_0 \in L^2(2)(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) =: X$. *Then* (2.11) *has a unique global strong solution* $Z \in C^0([0,\infty]; L^2(2)(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$, *with* $Z(0) = Z_0$. *In addition, there is the a priori estimate*

$$\|Z(\tau)\|_{L^2(2)\cap L^\infty} \leq Ce^{-\tau(\frac{3-\alpha-\beta}{\alpha})}\|Z_0\|_{L^2(2)\cap L^\infty}, \tag{2.39}$$

where C is an absolute constant.

Proof. We set up a local well-posedness scheme for the integral equation corresponding to (2.11), with initial data $Z(0) = f$, namely

$$Z(\tau) = e^{\tau\mathscr{L}}f - \int_0^\tau e^{(\tau-s)\mathscr{L}}\nabla\cdot(UZ)\,ds, \tag{2.40}$$

where $U = U_Z = (|\nabla|^\perp)^{-\beta}Z$. We have, according to (2.27) and (2.34),

$$\|e^{\tau\mathscr{L}}f\|_{L^2(2)} + \|e^{\tau\mathscr{L}}f\|_{L^\infty} \leq C(e^{(1-\frac{1-\beta}{\alpha})\tau} + e^{(1-\frac{3-\beta}{\alpha})\tau})\|f\|_{L^2(2)\cap L^\infty}$$

Thus, with $T \leq 1$ to be determined later, set

$$Y_T := \{Z(\tau,\cdot) \in X : \sup_{0\leq s\leq T}\|Z(s,\cdot)\|_X \leq 2C(e^{(1-\frac{1-\beta}{\alpha})} + e^{(1-\frac{3-\beta}{\alpha})})\|f\|_X\},$$

where the bound in Y is selected to be twice the value of the bound above, at $\tau = 1$. For the non-linear term, we have for each $\tau \in (0,T)$,

$$
\begin{aligned}
&\| \int_0^\tau e^{(\tau-s)\mathscr{L}} \nabla \cdot (U_{Z_1} Z_2)\, ds \|_{L^\infty} \leq C \int_0^\tau C \frac{e^{(1-\frac{2-\beta}{\alpha})(\tau-s)}}{a(\tau-s)^{\frac{1}{\alpha}}} \|U_{Z_1}(s) Z_2(s)\|_{L^\infty} ds \leq \\
\leq\ & C \sup_{0\leq s\leq T} \|U_{Z_1}\|_{L^\infty} \sup_{0\leq s\leq T} \|Z_2\|_{L^\infty} \int_0^\tau \frac{1}{(\tau-s)^{\frac{1}{\alpha}}} ds \leq \\
\leq\ & C T^{1-\frac{1}{\alpha}} \sup_{0\leq s\leq T} (\|Z_1\|_{L^{\frac{2}{\beta}+\varepsilon}} + \|Z_1\|_{L^{\frac{2}{\beta}-\varepsilon}}) \sup_{0\leq s\leq T} \|Z_2\|_{L^\infty} \\
\leq\ & C T^{1-\frac{1}{\alpha}} \sup_{0\leq s\leq T} \|Z_1\|_X \sup_{0\leq s\leq T} \|Z_2\|_X,
\end{aligned}
$$

where we have used (2.28), $e^{(1-\frac{2-\beta}{\alpha})(\tau-s)} \leq 3$, $a(\tau-s) = 1-e^{-(\tau-s)} \sim (\tau-s)$, for $0<s<\tau\leq 1$, the Sobolev embedding estimate (1.8) and finally the fact that $X = L^2(2)\cap L^\infty \hookrightarrow L^1 \cap L^\infty$. For the other norm in the definition of X, we have by Lemma 2.1.8,

$$
\begin{aligned}
&\| \int_0^\tau e^{(\tau-s)\mathscr{L}} \nabla \cdot (U_{Z_1}\cdot Z_2)\, ds \|_{L^2(2)} = \int_0^\tau e^{-\frac{(\tau-s)}{\alpha}} \|\nabla \cdot e^{(\tau-s)\mathscr{L}} (U_{Z_1}\cdot Z_2)\|_{L^2(2)}\, ds \\
\leq\ & C \int_0^\tau \frac{e^{-\frac{(\tau-s)}{\alpha}} e^{(1-\frac{3-\beta}{\alpha})(\tau-s)}}{a(\tau-s)^{\frac{1}{\alpha}}} \|U_{Z_1}(s)\cdot Z_2(s)\|_{L^2(2)}\, ds \leq \\
\leq\ & C \sup_{0\leq s\leq T} \|U_{Z_1}(s)\|_{L^\infty} \sup_{0\leq s\leq T} \|Z_2(s)\|_{L^2(2)} \int_0^\tau \frac{1}{(\tau-s)^{\frac{1}{\alpha}}} ds \leq \\
\leq\ & C T^{1-\frac{1}{\alpha}} \sup_{0\leq s\leq T} \|Z_1\|_X \sup_{0\leq s\leq T} \|Z_2\|_{L^2(2)}.
\end{aligned}
$$

Having these two bilinear estimates allows us to conclude that for sufficiently small T, of the form $T \sim \|f\|_X^{-\frac{\alpha}{\alpha-1}}$ (which should also be taken $T\leq 1$), we have local well-posedness in the space X.

Regarding global existence in $X = L^2(2)\cap L^\infty$, we obviously need *a priori* estimates for the solution to prevent potential blow up. We already have those in L^∞ and in L^2, by the results of Lemma 2.2.3. Thus, it remains to control the norm $J(\tau) := \int_{\mathbb{R}^2} |\xi|^4 |Z(\tau,\xi)|^2 d\xi$. To this end, take a dot product of the equation (2.11) with $|\xi|^4 Z$. We have

$$
\begin{aligned}
&\frac{1}{2}\partial_\tau \int |\xi|^4 Z^2 d\xi + \int |\xi|^4 |\nabla|^\alpha Z \cdot Z d\xi = \\
=\ & \frac{1}{\alpha}\int (\xi\cdot\nabla_\xi Z)|\xi|^4 Z\, d\xi - \int (U\cdot\nabla_\xi Z)|\xi|^4 Z\, d\xi + (1+\frac{\beta-1}{\alpha}) \int |\xi|^4 Z^2 d\xi.
\end{aligned}
$$

We first analyze the terms on the right hand-side. Integration by parts yields

$$\frac{1}{\alpha}\int(\xi\cdot\nabla_\xi Z)|\xi|^4 Z d\xi = -\frac{3}{\alpha}\int|\xi|^4 Z^2 d\xi$$
$$-\int(U\cdot\nabla_\xi Z)|\xi|^4 Z\, d\xi = 2\int|\xi|^2(\xi\cdot U)Z^2 d\xi.$$

Note that by Young's inequality, we have for all $\varepsilon > 0$

$$|\int|\xi|^2(\xi\cdot U)Z^2 d\xi| \leq C\int|\xi|^3\|U\|_{L^\infty}Z^2(\xi)d\xi \leq \varepsilon\int|\xi|^4 Z^2(\xi)d\xi + C\varepsilon^{-3}\|U\|_{L^\infty}^4\|Z\|_{L^2}^2.$$

By the Sobolev embedding (1.8) and Lemma 2.2.3, we have

$$\|U\|_{L^\infty} \leq C(\|Z\|_{L^{\frac{2}{\beta}+\varepsilon}} + \|Z\|_{L^{\frac{2}{\beta}-\varepsilon}}) \leq Ce^{-(\frac{3-\beta-\alpha}{\alpha})\tau},$$

so for every $\varepsilon > 0$, we have the estimate

$$|\int|\xi|^2(\xi\cdot U)Z^2 d\xi| \leq \varepsilon\int|\xi|^4 Z^2(\xi)d\xi + C\varepsilon^{-3}e^{-6\tau(\frac{3-\beta-\alpha}{\alpha})}.$$

The term $\int|\xi|^4|\nabla|^\alpha Z\cdot Z d\xi$ will give rise to some harder error terms (involving commutators between the $|\nabla|^{\alpha/2}$ and the weights), which we need to eventually control. It turns out that the most advantageous way to reign in the error terms is to split the weight $|\xi|^4$ between the two entries. More precisely,

$$\begin{aligned}
&\int|\xi|^4|\nabla|^\alpha Z\cdot Z d\xi = \int|\xi|^2|\nabla|^\alpha Z\cdot|\xi|^2 Z d\xi = \langle|\xi|^2|\nabla|^{\alpha/2}[|\nabla|^{\alpha/2}Z], |\xi|^2 Z\rangle = \\
= \ & \langle|\nabla|^{\alpha/2}|\xi|^2[|\nabla|^{\alpha/2}Z], |\xi|^2 Z\rangle - \langle[|\nabla|^{\alpha/2}, |\xi|^2][|\nabla|^{\alpha/2}Z], |\xi|^2 Z\rangle = \\
= \ & \langle|\xi|^2[|\nabla|^{\alpha/2}Z], |\nabla|^{\alpha/2}[|\xi|^2 Z]\rangle - \langle[|\nabla|^{\alpha/2}, |\xi|^2][|\nabla|^{\alpha/2}Z], |\xi|^2 Z\rangle = \\
= \ & \langle|\xi|^2|\nabla|^{\alpha/2}Z, |\xi|^2|\nabla|^{\alpha/2}Z\rangle + \langle|\xi|^2|\nabla|^{\alpha/2}Z, [|\nabla|^{\alpha/2}, |\xi|^2]Z\rangle \\
- \ & \langle[|\nabla|^{\alpha/2}, |\xi|^2][|\nabla|^{\alpha/2}Z], |\xi|^2 Z\rangle \\
= \ & \int|\xi|^4||\nabla|^{\frac{\alpha}{2}}Z|^2 d\xi + \langle|\xi|^2|\nabla|^{\alpha/2}Z, [|\nabla|^{\alpha/2}, |\xi|^2]Z\rangle - \langle[|\nabla|^{\alpha/2}, |\xi|^2][|\nabla|^{\alpha/2}Z], |\xi|^2 Z\rangle.
\end{aligned}$$

Denote the error terms

$$E := \langle |\xi|^2 |\nabla|^{\alpha/2} Z, [|\nabla|^{\alpha/2}, |\xi|^2] Z\rangle - \langle [|\nabla|^{\alpha/2}, |\xi|^2][|\nabla|^{\alpha/2} Z], |\xi|^2 Z\rangle.$$

Putting it all together implies

$$\frac{1}{2}J'(\tau) + (\frac{4-\alpha-\beta}{\alpha} - \varepsilon)J(\tau) + \int |\xi|^4 ||\nabla|^{\frac{\alpha}{2}} Z|^2 d\xi \leq |E| + C\varepsilon^{-3} e^{-6\tau(\frac{3-\beta-\alpha}{\alpha})} \quad (2.41)$$

$$\lesssim \||\xi|^2 |\nabla|^{\alpha/2} Z\|_{L^2} \|[|\nabla|^{\alpha/2}, |\xi|^2] Z\|_{L^2}$$
$$+ \|[|\nabla|^{\alpha/2}, |\xi|^2][|\nabla|^{\alpha/2} Z]\|_{L^2} \||\xi|^2 Z\|_{L^2} + \varepsilon^{-3} e^{-6\tau(\frac{3-\beta-\alpha}{\alpha})}. \quad (2.42)$$

By Gagliardo-Nirenberg's inequality

$$\||\xi|^{2-\frac{\alpha}{2}} g\|_{L^2} \leq C \||\xi|^2 g\|_{L^2}^{1-\frac{\alpha}{4}} \|g\|_{L^2}^{\frac{\alpha}{4}}.$$

Continuing with our arguments above (see (2.41)), we conclude from Lemma 1.1.6 that

$$\frac{1}{2}J'(\tau) + (\frac{4-\alpha-\beta}{\alpha} - \varepsilon)J(\tau) + \||\xi|^2 |\nabla|^{\alpha/2} Z\|_{L^2}^2 \leq$$
$$\leq \varepsilon \||\xi|^2 |\nabla|^{\alpha/2} Z\|_{L^2}^2 + \varepsilon \||\xi|^2 Z\|_{L^2}^2 + C_\varepsilon \|Z\|_{L^2}^2.$$

All in all, for all $\varepsilon < 1$, we have by Lemma 2.2.3,

$$\frac{1}{2}J'(\tau) + (\frac{4-\alpha-\beta}{\alpha} - 2\varepsilon)J(\tau) \leq C_\varepsilon \|Z\|_{L^2}^2 \leq C\|Z_0\|_{L^1 \cap L^\infty}^2 e^{-2\tau(\frac{3-\beta-\alpha}{\alpha})}.$$

By Gronwall's, we finally conclude that

$$J(\tau) \leq J(0) e^{-2\tau(\frac{4-\alpha-\beta}{\alpha} - 2\varepsilon)} + C\|Z_0\|_{L^1 \cap L^\infty}^2 e^{-2\tau(\frac{3-\alpha-\beta}{\alpha})}.$$

As a consequence

$$\||\xi|^2 Z(\tau)\|_{L^2} \leq C\|Z_0\|_{L^2(2) \cap L^\infty} e^{-\tau(\frac{3-\alpha-\beta}{\alpha})}.$$

This completes the proof of Theorem 2.2.4. □

2.2.3 Global dynamics of the solutions of the SQG model

Theorem 2.2.4 already provides pretty good estimate about the behavior of the solutions to the rescaled equation (2.11), in particular the solution Z disperses at ∞, with the rate $e^{-\tau(\frac{3-\alpha-\beta}{\alpha})}$. An important problem in this situations is whether or not this is optimal, that is whether there is a lower bound with the same exponential function, at least for generic data. It turns out that this is indeed the case. In fact, we have a more precise result, namely an asymptotic expansion.

Before we continue with the formal statement of the main result, we need a simple algebraic observation, which is important in the sequel. Recall the generalized Biot-Savart law that we imposed, $u = u_z = (|\nabla|^{\perp})^{-\beta}z$. This naturally transformed into the relation $U = U_Z = (|\nabla|^{\perp})^{-\beta}Z$ between the "scaled" velocity U and its vorticity Z. We claim that

$$U_G \cdot \nabla G = 0. \tag{2.43}$$

Indeed, since G is a radial function[9], say $G(\xi) = \zeta(|\xi|)$, we have that $\nabla G = (\xi_1, \xi_2)\frac{\zeta'(|\xi|)}{|\xi|}$. On the other hand, $U_G = (|\nabla|^{\perp})^{-\beta}G = |\nabla|^{\perp}m_{-\beta-1}(|\nabla|)G$, so $U_G = |\nabla|^{\perp}h(|\xi|)$, where h is a radial function representing $[m_{-\beta-1}(|\nabla|)G]$. That is, $h(|\xi|) = [m_{-\beta-1}(|\nabla|)G](\xi)$. It follows that $U_G = (-\xi_2, \xi_1)\frac{h'(|\xi|)}{|\xi|}$. Thus,

$$U_G \cdot \nabla G = (-\xi_2, \xi_1)\frac{h'(|\xi|)}{|\xi|} \cdot (\xi_1, \xi_2)\frac{\zeta'(|\xi|)}{|\xi|} = 0.$$

We are now ready to state the main theorem of this section.

Theorem 2.2.5. *Let $Z_0 \in L^2(2) \cap L^\infty(\mathbb{R}^2)$, $\varepsilon > 0$, $\alpha \in (1,2), \alpha+\beta \le 3$. Denote $\gamma(0) := \int_{\mathbb{R}^2} Z_0(\xi)d\xi$. Then there exists $C_\varepsilon > 0$ such that for any $\tau > 0$,*

$$\|Z(\tau,\cdot) - \gamma(0)e^{-\tau(\frac{3-\alpha-\beta}{\alpha})}G\|_{L^2(2)} \le C_\varepsilon e^{-\tau(\frac{4-\alpha-\beta}{\alpha}-\varepsilon)}. \tag{2.44}$$

[9] as the Fourier transform of a radial one

Assuming in addition that $\beta > 1$, we also have

$$\|\nabla[Z(\tau,\cdot) - \gamma(0)e^{-\tau(\frac{3-\alpha-\beta}{\alpha})}G]\|_{L^2(2)} \le C_\varepsilon e^{-\tau(\frac{4-\alpha-\beta}{\alpha}-\varepsilon)}. \tag{2.45}$$

In particular if $\int_{\mathbb{R}^2} Z_0(\xi)d\xi = 0$, then

$$\|Z\|_{L^2(2)} \le C_\varepsilon e^{-\tau(\frac{4-\alpha-\beta}{\alpha}-\varepsilon)}.$$

Remarks:

- We would like to point out that the existence of solution Z (and subsequently $\gamma(\tau)$ and $\tilde{Z}(\tau)$) is not in question anymore, due to the results obtained in Theorem 2.2.4. The purpose of this theorem is just to obtain better *a priori* estimates, in the form described in above.
- The requirement $\beta > 1$, imposed so that (2.45) holds is likely only a technical one, but we cannot remove it with our methods.

Proof. (Theorem 2.2.5)

According to the results in section 2.1.8, $\lambda_0 = -\frac{3-\alpha-\beta}{\alpha} \le 0$ is an isolated and simple eigenvalue for the operator $\mathscr{L}$ on $L^2(2)$, with eigenfunction G, while the rest of the spectrum is the essential spectrum, which we have identified before, $\sigma_{ess}(\mathscr{L}) = \{\lambda : \Re\lambda \le -\frac{4-\alpha-\beta}{\alpha}\}$. We have also found the spectral projection $\mathscr{P}_0 f = \langle f, 1\rangle G$ and $\mathscr{Q}_0 = Id - \mathscr{P}_0$. Thus, we can write

$$Z(\tau,\cdot) = \gamma(\tau)G(\xi) + \tilde{Z}(\tau,\cdot), \tag{2.46}$$

where $\gamma(\tau) = \langle Z(\tau,\cdot), 1\rangle = \int_{\mathbb{R}^2} Z(\tau,\xi)d\xi$, $\tilde{Z}(\tau) = \mathscr{Q}_0 Z(\tau,\cdot)$. Projecting the equation (2.11), with respect to the spectral decomposition provided by $\mathscr{P}_0$ and $\mathscr{Q}_0$, we obtain an ODE for γ and a PDE for $\tilde{Z}(\tau)$. More precisely,

$$\begin{aligned}\partial_\tau \gamma &= \langle \mathscr{L} Z, 1\rangle - \langle U\cdot\nabla Z, 1\rangle = \\ &= \langle -|\nabla|^\alpha Z + \frac{1}{\alpha}\xi\cdot\nabla_\xi Z + \left(1+\frac{\beta-1}{\alpha}\right) Z, 1\rangle - \langle \nabla(U\cdot Z), 1\rangle = \frac{\alpha+\beta-3}{\alpha}\gamma(\tau).\end{aligned}$$

Integrating this first order ODE yields the formula $\gamma(\tau) = \gamma(0)e^{-\tau\frac{3-\alpha-\beta}{\alpha}}$. For the PDE governing $\tilde{Z}(\tau)$, and recalling $\mathscr{L}_0 = \mathscr{L}\mathscr{Q}_0$, we obtain

$$\tilde{Z}_\tau = \mathscr{L}_0\tilde{Z} - \mathscr{Q}_0[U\cdot\nabla Z] = \mathscr{L}_0\tilde{Z} - \mathscr{Q}_0[U\cdot\nabla(\gamma(0)e^{-\tau\frac{3-\alpha-\beta}{\alpha}}G + \tilde{Z})].$$

In its equivalent integral formulation,

$$\tilde{Z}(\tau) = e^{\tau\mathscr{L}_0}\tilde{Z}(0) - \int_0^\tau e^{(\tau-s)\mathscr{L}_0}\mathscr{Q}_0[U\cdot\nabla(\gamma(0)\, e^{-\tau\frac{3-\alpha-\beta}{\alpha}}G + \tilde{Z}(s,\cdot)]\, ds. \tag{2.47}$$

Note that the commutation relation $\mathscr{Q}_0\nabla = \nabla$, whence one can remove $\mathscr{Q}_0$ in front of the nonlinearity. By (2.33), we can estimate

$$\begin{aligned}&\|\tilde{Z}(\tau)\|_{L^2(2)} \le \|e^{\tau\mathscr{L}_0}\tilde{Z}(0)\|_{L^2(2)} + \\ &+ \int_0^\tau \|e^{(\tau-s)\mathscr{L}_0}\left((U_G + U_{\tilde{Z}})\nabla\cdot(\gamma(0)\, e^{-s\frac{3-\alpha-\beta}{\alpha}}G + \tilde{Z}(s)\right)\|_{L^2(2)} ds \\ &\le \|e^{\tau\mathscr{L}_0}\tilde{Z}(0)\|_{L^2(2)} + |\gamma(0)|\int_0^\tau e^{-\frac{(\tau-s)}{\alpha}} e^{-s\frac{3-\alpha-\beta}{\alpha}}\|\nabla\cdot e^{(\tau-s)\mathscr{L}_0}(U_{\tilde{Z}}\cdot G)\|_{L^2(2)} ds \\ &+ \int_0^\tau e^{-\frac{(\tau-s)}{\alpha}}\|\nabla\cdot e^{(\tau-s)\mathscr{L}}(U\cdot\tilde{Z})\|_{L^2(2)} ds =: I_1 + I_2 + I_3,\end{aligned}$$

where we have used (2.43). Clearly by (2.30)

$$I_1 \le C e^{-\tau\left(\frac{4-\beta-\alpha}{\alpha}-\varepsilon\right)}\|\tilde{Z}(0)\|_{L^2(2)}.$$

Regarding I_2, we have

$$I_2 \;\leq\; |\gamma(0)|\int_0^\tau \frac{e^{-\frac{(\tau-s)}{\alpha}}e^{-s\frac{3-\alpha-\beta}{\alpha}}e^{-(\tau-s)\left(\frac{3-\beta-\alpha}{\alpha}\right)}\|U_{\tilde{Z}}\cdot G\|_{L^2(2)}}{a(\tau-s)^{\frac{1}{\alpha}}}ds$$

Now to bound $\|U_{\tilde{Z}}\cdot G\|_{L^2(2)}$ we look at two different cases, namely $0\leq\beta<1$ and $1\leq\beta<2$. If $0\leq\beta\leq1$, then we can use lemma (2.17) to get

$$\begin{aligned}\|U_{\tilde{Z}}\cdot G\|_{L^2(2)} &\leq \|U_{\tilde{Z}}\|_{L^{\frac{2}{1-\beta}}}\|(1+|\xi|^2)G\|_{L^{\frac{2}{\beta}}}\\ &\leq C\|U_{\tilde{Z}}\|_{L^{\frac{2}{1-\beta}}}\leq C\||\nabla|^\beta U_{\tilde{Z}}\|_{L^2}\leq C\|\tilde{Z}\|_{L^2}\leq\|\tilde{Z}\|_{L^2(2)}.\end{aligned}$$

If $1\leq\beta<2$, then for some $0<\varepsilon<<1$ we have

$$\begin{aligned}\|U_{\tilde{Z}}\cdot G\|_{L^2(2)} &\leq \|U_{\tilde{Z}}\|_{L^{\frac{2}{\varepsilon}}}\|(1+|\xi|^2)G\|_{L^{\frac{2}{1-\varepsilon}}}\\ &\leq C\|U_{\tilde{Z}}\|_{L^{\frac{2}{\varepsilon}}}\leq C\||\nabla|^\beta U_{\tilde{Z}}\|_{L^{\frac{2}{\beta+\varepsilon}}}\leq C\|\tilde{Z}\|_{L^{\frac{2}{\beta+\varepsilon}}}\leq C\|\tilde{Z}\|_{L^2(2)}.\end{aligned}$$

In the last inequality we used the fact that for $1<p<2$, $L^p\hookrightarrow L^2(2)$ and Lemma (2.17). Therefore

$$I_2\leq C\int_0^\tau\frac{e^{-(\tau-s)\left(\frac{4-\beta-\alpha}{\alpha}\right)}e^{-s\frac{3-\alpha-\beta}{\alpha}}}{(\min(1,|\tau-s|)^{\frac{1}{\alpha}}}\|\tilde{Z}(s)\|_{L^2(2)}ds.$$

Finally, we make use of (2.34) to get

$$\begin{aligned}I_3 &\leq \int_0^\tau\frac{e^{-\frac{(\tau-s)}{\alpha}}e^{-(\tau-s)\left(\frac{3-\beta-\alpha}{\alpha}\right)}\|U(s)\|_{L^\infty}\|\tilde{Z}(s)\|_{L^2(2)}}{a(\tau-s)^{\frac{1}{\alpha}}}\,ds\\ &\leq C\int_0^\tau\frac{e^{-(\tau-s)\left(\frac{4-\beta-\alpha}{\alpha}-\varepsilon\right)}e^{-s\left(\frac{3-\beta-\alpha}{\alpha}\right)}}{(\min(1,|\tau-s|)^{\frac{1}{\alpha}}}\|\tilde{Z}(s)\|_{L^2(2)}\,ds,\end{aligned}$$

where we have used that $a(\tau)\sim\min(1,\tau)$, the Sobolev inequality and Theorem 2.2.4 to conclude

$$\|U(s)\|_{L^\infty}\leq C(\|Z(s)\|_{L^{\frac{2}{\beta}+\varepsilon}}+\|Z(s)\|_{L^{\frac{2}{\beta}-\varepsilon}})\leq Ce^{-s\left(\frac{3-\beta-\alpha}{\alpha}\right)}. \tag{2.48}$$

We are now in a position to use the Gronwall's inequality, more precisely the version displayed in Lemma 1.2.2. We apply it with $I(\tau) = \|\tilde{Z}(\tau)\|_{L^2(2)}$, $\mu = \frac{4-\alpha-\beta}{\alpha} - \varepsilon, \sigma = \frac{4-\alpha-\beta}{\alpha}, \kappa = \frac{3-\alpha-\beta}{\alpha}$ and $a = \frac{1}{\alpha} < 1$, for $\varepsilon << 1$. Recall that by the *a priori* estimates in Theorem 2.2.4, we have

$$\|\tilde{Z}(\tau)\|_{L^2(2)} \leq \|Z(\tau)\|_{L^2(2)} + |\gamma(0)|e^{-\tau(\frac{3-\alpha-\beta}{\alpha})}\|G\|_{L^2(2)} \leq Ce^{-\tau(\frac{3-\alpha-\beta}{\alpha})} \leq C,$$

for all $\tau > 0$, since $3 \geq \alpha + \beta$. Thus, all the requirements of Lemma 1.2.2 are met and we obtain the bound

$$\|\tilde{Z}(\tau)\|_{L^2(2)} \leq C_\varepsilon e^{-\tau(\frac{4-\alpha-\beta}{\alpha}-\varepsilon)}. \tag{2.49}$$

Regarding the proof of (2.45), we proceed in a similar fashion. We need to control $\|\partial\tilde{Z}\|_{L^2(2)}$, for large τ, say $\tau \geq 1$. Applying $\partial = \partial_1, \partial_2$ to the integral equation (2.47) and taking $\|\cdot\|_{L^2(2)}$, we obtain

$$\begin{aligned}
&\|\partial\tilde{Z}(\tau)\|_{L^2(2)} \lesssim e^{-\tau(\frac{4-\alpha-\beta}{\alpha}-\varepsilon)}\|\tilde{Z}(0)\|_{L^2(2)} + \\
+ &\int_0^\tau \frac{e^{-\frac{(\tau-s)}{\alpha}}e^{-s\frac{3-\alpha-\beta}{\alpha}}}{\min(1,\tau-s)^{\frac{1}{\alpha}}}\|e^{(\tau-s)\mathscr{L}_0}\nabla(U_{\tilde{Z}}\cdot G)\|_{L^2(2)}ds + \\
+ &\int_0^\tau \frac{e^{-\frac{(\tau-s)}{\alpha}}}{\min(1,\tau-s)^{\frac{1}{\alpha}}}\|e^{(\tau-s)\mathscr{L}_0}\nabla(U\cdot\tilde{Z})\|_{L^2(2)}ds \lesssim e^{-\tau(\frac{4-\alpha-\beta}{\alpha}-\varepsilon)} + \\
+ &\int_0^\tau \frac{e^{-(\tau-s)(\frac{5-\alpha-\beta}{\alpha}-\varepsilon)}e^{-s\frac{3-\alpha-\beta}{\alpha}}}{\min(1,\tau-s)^{\frac{1}{\alpha}}}\|\nabla[U_{\tilde{Z}}(s)G]\|_{L^2(2)}ds + \\
+ &\int_0^\tau \frac{e^{-(\tau-s)(\frac{5-\alpha-\beta}{\alpha}-\varepsilon)}}{\min(1,\tau-s)^{\frac{1}{\alpha}}}\|\nabla[U(s)\tilde{Z}(s)]\|_{L^2(2)}ds
\end{aligned}$$

We estimate $\|\nabla[U_{\tilde{Z}}(s)G]\|_{L^2(2)} \leq \|\nabla U_{\tilde{Z}}(s)G\|_{L^2(2)} + \|U_{\tilde{Z}}(s)\nabla G\|_{L^2(2)}$. Following the strategy above, for $\beta \leq 1$ and then for $\beta > 1$, we arrive at

$$\|\nabla[U_{\tilde{Z}}(s)G]\|_{L^2(2)} \lesssim \|\tilde{Z}(s)\|_{L^2(2)} + \|\partial\tilde{Z}(s)\|_{L^2(2)} \lesssim e^{-s(\frac{4-\alpha-\beta}{\alpha}-\varepsilon)} + \|\partial\tilde{Z}(s)\|_{L^2(2)},$$

where we have used (2.49). For the other term, it is relatively easy to bound

$\|\nabla[U(s)\tilde{Z}(s)]\|_{L^2(2)}$, when $\beta > 1$,

$$\begin{aligned}\|\partial[U(s)\tilde{Z}(s)]\|_{L^2(2)} &\lesssim \|\partial U(s)\|_{L^\infty}\|\tilde{Z}(s)\|_{L^2(2)} + \|U(s)\|_{L^\infty}\|\partial\tilde{Z}(s)\|_{L^2(2)} \\ &\lesssim e^{-s(\frac{3-\alpha-\beta}{\alpha})}e^{-s(\frac{4-\alpha-\beta}{\alpha}-\varepsilon)} + e^{-s\left(\frac{3-\beta-\alpha}{\alpha}\right)}\|\partial\tilde{Z}(s)\|_{L^2(2)}.\end{aligned}$$

where we have used (recalling $U \sim |\nabla|^{-\beta}Z$), $\|\partial U(s)\|_{L^\infty} \le C(\|Z\|_{L^{\frac{2}{\beta-1}+\varepsilon}} + \|Z\|_{L^{\frac{2}{\beta-1}-\varepsilon}}) \le Ce^{-s(\frac{3-\alpha-\beta}{\alpha})}$, (2.49), (2.48). Plugging it together yields

$$\|\partial\tilde{Z}(\tau)\|_{L^2(2)} \lesssim e^{-\tau(\frac{4-\alpha-\beta}{\alpha}-\varepsilon)} + \int_0^\tau \frac{e^{-(\tau-s)(\frac{5-\alpha-\beta}{\alpha}-\varepsilon)}e^{-s(\frac{3-\alpha-\beta}{\alpha})}}{\min(1,\tau-s)^{\frac{1}{\alpha}}}\|\partial\tilde{Z}(s)\|_{L^2(2)}. \tag{2.50}$$

This puts us in a position to use the Gronwall's lemma 1.2.2. Note that in order to do that, we need any *a priori* exponential bound on $\|\partial Z(\tau)\|_{L^2(2)}$, similar to Theorem 2.2.4 for $\|Z(\tau)\|_{L^2(2)}$. This is actually easy to achieve, one just has to differentiate the equation and perform very coarse energy estimates[10]. As a result, Lemma 1.2.2 applies and we obtain

$$\|\partial\tilde{Z}(\tau)\|_{L^2(2)} \lesssim e^{-\tau(\frac{4-\alpha-\beta}{\alpha}-\varepsilon)},$$

as is the statement of (2.45). □

Note that for $\beta > 1$ and $2 < p < \infty$, we have

$$\|Z\|_{L^p} \le \|\partial Z\|_{L^2} \le Ce^{-\tau(\frac{3-\alpha-\beta}{\alpha})}. \tag{2.51}$$

It is now easy to conclude the main result, Theorem 2.1.1.

Proof of theorem (2.1.1). Realizing that $L^2(2) \hookrightarrow L^p, 1 \le p \le 2$, one just needs to translate the L^p

[10] which will give very inefficient exponential bounds on $\|\partial Z(\tau)\|_{L^2(2)}$, but that is all we need to jump start Lemma 1.2.2

estimates for Z, in the language of the original variable z.

$$\begin{aligned}
& \|z(t,\cdot)-\frac{\gamma(0)}{(1+t)^{\frac{2}{\alpha}}}G(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}})\|_{L^p}= \\
= \quad & \frac{1}{(1+t)^{1+\frac{\beta-1}{\alpha}-\frac{2}{\alpha p}}}\|Z(\tau,\cdot)-\gamma(0)e^{-\tau(\frac{3-\alpha-\beta}{\alpha})}G(\cdot)\|_{L^p} \\
\leq \quad & \frac{C_\varepsilon}{(1+t)^{1+\frac{\beta-1}{\alpha}-\frac{2}{\alpha p}}}e^{-\tau(\frac{4-\beta-\alpha}{\alpha}-\varepsilon)} \leq \frac{C_\varepsilon}{(1+t)^{\frac{3}{\alpha}-\frac{2}{\alpha p}-\varepsilon}}.
\end{aligned}$$

Moreover, in a similar manner, for $\beta > 1$ and $2 \leq p < \infty$ one has,

$$\begin{aligned}
& \|z(t,\cdot)-\frac{\gamma(0)}{(1+t)^{\frac{2}{\alpha}}}G(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}})\|_{L^p}= \\
&= \frac{1}{(1+t)^{1+\frac{\beta-1}{\alpha}-\frac{2}{\alpha p}}}\|Z(\tau,\cdot)-\gamma(0)e^{-\tau(\frac{3-\alpha-\beta}{\alpha})}G(\cdot)\|_{L^p} \\
&\leq \frac{C_{\alpha,\beta}}{(1+t)^{1+\frac{\beta-1}{\alpha}-\frac{2}{\alpha p}}}\|\partial\left[Z(\tau,\cdot)-\gamma(0)e^{-\tau(\frac{3-\alpha-\beta}{\alpha})}G(\cdot)\right]\|_{L^2} \\
&\leq \frac{C_{\alpha,\beta,\varepsilon}}{(1+t)^{1+\frac{\beta-1}{\alpha}-\frac{2}{\alpha p}}}e^{-\tau(\frac{4-\beta-\alpha}{\alpha}-\varepsilon)} = \frac{C_{\alpha,\beta,\varepsilon}}{(1+t)^{\frac{3}{\alpha}-\frac{2}{\alpha p}-\varepsilon}}.
\end{aligned}$$

□

2.3 Local and global existence of the solutions to the Boussinesq system and its long term behavior

The results of this section closely mirror Section 2.2. Consequently, we omit many of the arguments, when they are virtually the same. There are however a few important distinctions, which we will highlight herein.

2.3.1 Global regularity for the vorticity (ω, θ) Boussinesq system in $L^p(\mathbb{R}^2)$

Our first result is, non-surprisingly, is a local existence and uniqueness result in $L^p(\mathbb{R}^2)$. Most of the claims in this lemma are either well-known or follows a classical argument, but we provide a sketch of the proof for completeness.

Lemma 2.3.1. *Suppose that* $\omega_0, \theta_0 \in L^p$, $1 \le p \le \infty$. *Then there exists* $T = T(\|(\omega_0, \theta_0)\|_{L^1 \cap L^\infty})$, *such that unique strong solutions* $\omega, \theta \in C([0,T); L^1 \cap L^\infty)$ *exist.*

Moreover, the solutions $\omega(t), \theta(t)$ *exist globally. In addition, the function* $t \to \|\theta(t,\cdot)\|_{L^p}, 1 \le p \le \infty$ *is non-increasing,* $\|\theta(t,\cdot)\|_{L^p} \le \|\theta_0\|_{L^p}, 1 < p < \infty$, *while*

$$\|u(t,\cdot)\|_{L^2} \le \|u_0\|_{L^2} + t\|\theta_0\|_{L^2}.$$

Proof. For the local existence, we work in the space $X = L^1 \cap L^\infty = \cap L^p$. The strong solutions of the system of equations (2.7) are solutions of the integral equations

$$\begin{cases} \omega(\xi,t) = e^{-t|\nabla|^\alpha}\omega_0 + \int_0^t e^{-(t-s)|\nabla|^\alpha}\nabla(u\cdot\omega)\, ds - \int_0^t e^{-(t-s)|\nabla|^\alpha}\partial_1\theta\, ds, \\ \theta(\xi,t) = e^{-t|\nabla|^\alpha}\theta_0 + \int_0^t e^{-(t-s)|\nabla|^\alpha}\nabla(u\cdot\theta)\, ds. \end{cases} \tag{2.52}$$

By (2.18), we have that

$$\|e^{-t|\nabla|^\alpha}\omega_0\|_X + \|e^{-t|\nabla|^\alpha}\theta_0\|_X \le C(\|\omega_0\|_X + \|\theta_0\|_X)$$

One can now consider the space $Y := \{(\omega,\theta) : \sup_{0\le t\le T}[\|\omega\|_X + \|\theta\|_X] \le 2C(\|\omega_0\|_X + \|\theta_0\|_X)\}$. For the bilinear forms

$$Q_1(\omega_1,\omega_2) = \int_0^t e^{-(t-s)|\nabla|^\alpha}\nabla(u\cdot\omega)\, ds, Q_2(\omega_1,\theta) = \int_0^t e^{-(t-s)|\nabla|^\alpha}\nabla(u\cdot\theta)\, ds$$

where $u = (\nabla^\perp)^{-1}\omega_1$, we establish the estimates

$$\begin{aligned} \|Q_1(\omega_1,\omega_2) - Q_1(\tilde{\omega}_1,\tilde{\omega}_2)\|_X &\le CT^{1-\frac{1}{\alpha}}(\|(\omega_1,\omega_2)\|_X + \\ &+ \|(\tilde{\omega}_1,\tilde{\omega}_2)\|_X)(\|\omega_1-\tilde{\omega}_1\|_X + \|\omega_2-\tilde{\omega}_2\|_X) \end{aligned}$$

and

$$\begin{aligned}\|Q_2(\omega_1,\theta)-Q_2(\tilde{\omega}_1,\tilde{\theta})\|_X &\leq CT^{1-\frac{1}{\alpha}}(\|(\omega_1,\theta)\|_X+\\ &+ \|(\tilde{\omega}_1,\tilde{\theta})\|_X)(\|\omega_1-\tilde{\omega}_1\|_X+\|\theta-\tilde{\theta}\|_X),\end{aligned}$$

for $j=1,2$. This is done in an identical manner as in the proof of Lemma 2.2.1. It remains to deal with the integral term $\int_0^t e^{-(t-s)|\nabla|^\alpha}\partial_1\theta\, ds$. For it, we have

$$\begin{aligned}\|\int_0^t e^{-(t-s)|\nabla|^\alpha}\partial_1(\theta-\tilde{\theta})\, ds\|_{L^1\cap L^\infty} &\leq C\int_0^t \frac{1}{(\tau-s)^{\frac{1}{\alpha}}}\|\theta-\tilde{\theta}\|_{L^1\cap L^\infty}ds\\ &\leq CT^{1-\frac{1}{\alpha}}\sup_{0<s<T}\|\theta(s)-\tilde{\theta}(s)\|_{L^1\cap L^\infty},\end{aligned}$$

for $0<t<T$. All in all, we can guarantee that with an appropriate choice of T, the non-linear map given by (2.52)has a fixed point ω,θ in the space X.

Regarding the global well-posedness, we can continue the solution, as long as the norm $t\to\|\theta(t,\cdot)\|_{L^p}$ stay under control. First, for $1<p<\infty$, take dot product of the θ equation with $|\theta|^{p-2}\theta$, $p\in(1,\infty)$ and using the fact the positivity estimate (1.6), we obtain

$$\frac{1}{p}\partial_t\|\theta(t,\cdot))\|_{L^p}^p \leq \frac{1}{p}\partial_t\|\theta\|_{L^p}^p+\int_{\mathbb{R}^2}|\theta|^{p-2}\theta\cdot|\nabla|^\alpha\theta dx=0$$

It follows that $t\to\|\theta(t,\cdot)\|_{L^p}$ is non-increasing in any interval $(0,t)$, whence the solution is global and $\|\theta(t,\cdot)\|_{L^p}\leq\|\theta_0\|_{L^p}$. For $p=1, p=\infty$, we use approximation arguments to establish the same result.

Finally, we use this information to establish the global well-posedness of the u equation in (2.6). Taking dot product with u, we obtain

$$\begin{aligned}\frac{1}{2}\partial_t\|u(t,\cdot)\|_{L^2}^2 &\leq \frac{1}{2}\partial_t\|u(t,\cdot)\|_{L^2}^2+\||\nabla|^{\frac{\alpha}{2}}u\|_{L^2}^2=\langle u_2,\theta\rangle\leq\|u_2\|_{L^2}\|\theta(t)\|_{L^2}\\ &\leq \|u_2(t)\|_{L^2}\|\theta_0\|_{L^2}\end{aligned}$$

It follows that

$$\|u(t,\cdot)\|_{L^2} \leq \|u_0\|_{L^2} + t\|\theta_0\|_{L^2},$$

which provides the necessary bound to conclude global regularity, as stated. $\square$

The next lemma provides a global existence and uniqueness result for the (ω, θ) system.

Lemma 2.3.2. *Let* $\alpha > 1$. *Then, assuming* $\omega_0 \in L^2, \theta_0 \in H^{\frac{\alpha}{2}}$, *the Cauchy problem* (2.7) *has unique global solutions. In addition, for any* $T > 0$, *there exists* $C = C_{T, \|\omega_0\|_{L^2}, \|\theta_0\|_{H^{\frac{\alpha}{2}}}} > 0$, *so that the solutions satisfy*

$$\sup_{0 \leq t \leq T} \|\omega\|_{L^2} + \sup_{0 \leq t \leq T} \||\nabla|^{\frac{\alpha}{2}} \theta\|_{L^2} \leq C. \tag{2.53}$$

Remark: The constant C_T obtained in this argument is exponential in T, which is very non-efficient, as we shall see later on.

Proof. The global regularity for (2.7) is of course very similar to the global regularity established in Lemma 2.3.1. For the energy estimates, needed for (2.53), we can dot product the first equation in (2.7) with ω and the second one with $|\nabla|^\alpha \theta$ to get the following energy estimate

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \||\nabla|^{\frac{\alpha}{2}}\theta\|_{L^2}^2\right) &+ \||\nabla|^{\frac{\alpha}{2}}\omega\|_{L^2}^2 + \||\nabla|^{\alpha}\theta\|_{L^2}^2 \leq \left|\int \omega \cdot \partial_1 \theta d\xi\right| + \\ &+ \left|\langle [|\nabla|^{\frac{\alpha}{2}}, u\cdot\nabla]\theta, |\nabla|^{\frac{\alpha}{2}}\theta\rangle\right| := I_1 + I_2. \end{aligned}$$

Then for some $0 < \gamma < 1$,

$$\begin{aligned} I_1 = \left|\int \omega \cdot \partial_1 \theta d\xi\right| &\leq \||\nabla|^{\frac{\alpha}{2}}\omega\|_{L^2}\|\partial_1|\nabla|^{-\frac{\alpha}{2}}\theta\|_{L^2} \leq \varepsilon\||\nabla|^{\frac{\alpha}{2}}\omega\|_{L^2}^2 + C_\varepsilon\|\partial_1|\nabla|^{-\frac{\alpha}{2}}\theta\|_{L^2}^2 \\ &\leq \varepsilon\||\nabla|^{\frac{\alpha}{2}}\omega\|_{L^2}^2 + C_\varepsilon\||\nabla|^{\alpha}\theta\|_{L^2}^{2\gamma}\|\theta\|_{L^2}^{2(1-\gamma)} \leq \varepsilon\||\nabla|^{\frac{\alpha}{2}}\omega\|_{L^2}^2 + \varepsilon\||\nabla|^{\alpha}\theta\|_{L^2}^2 + C_\varepsilon\|\theta_0\|_{L^2}^2. \end{aligned}$$

We also have

$$I_2 = \left|\langle[|\nabla|^{\frac{\alpha}{2}}, u\cdot\nabla]\theta, |\nabla|^{\frac{\alpha}{2}}\theta\rangle\right| \quad \leq \quad \||\nabla|^{-\frac{\alpha}{2}}[|\nabla|^{\frac{\alpha}{2}}, u\cdot\nabla]\theta\|_{L^2}\||\nabla|^{\alpha}\theta\|_{L^2}$$

We can make use of the inequality (1.9) with $a=1, s_1=s_2=\frac{\alpha}{2}, p=2, q=\frac{8}{4-\alpha}$ and $r=\frac{8}{\alpha}$ to get

$$\begin{aligned}
\||\nabla|^{-\frac{\alpha}{2}}[|\nabla|^{\frac{\alpha}{2}}, u\cdot\nabla]\theta(t)\|_{L^2} &\leq C\|\theta\|_{L^{\frac{8}{\alpha}}}\|\nabla u\|_{L^{\frac{8}{4-\alpha}}} \leq C\|\theta_0\|_{L^{\frac{8}{\alpha}}}\|\omega\|_{L^{\frac{8}{4-\alpha}}} \\
&\leq C\|\theta_0\|_{L^{\frac{8}{\alpha}}}\||\nabla|^{\frac{\alpha}{4}}\omega\|_{L^2} \leq C\|\theta_0\|_{L^{\frac{8}{\alpha}}}\||\nabla|^{\frac{\alpha}{2}}\omega\|_{L^2}^{\frac{1}{2}}\|\omega\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

where we have used the Sobolev embedding and the Gagliardo-Nirenberg's inequality. Then,

$$I_2 \leq \varepsilon\||\nabla|^{\frac{\alpha}{2}}\omega\|_{L^2}^2 + \varepsilon\||\nabla|^{\alpha}\theta\|_{L^2}^2 + C_\varepsilon(\|\theta_0\|_{L^{\frac{8}{\alpha}}}\|\omega\|_{L^2}^{\frac{1}{2}})^4.$$

Therefore, for $\varepsilon < \frac{1}{2}$, we can hide the terms $\||\nabla|^{\frac{\alpha}{2}}\omega\|_{L^2}^2$ and $\||\nabla|^{\alpha}\theta\|_{L^2}^2$ and we obtain

$$\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \||\nabla|^{\frac{\alpha}{2}}\theta\|_{L^2}^2\right) \leq C\|\theta_0\|_{L^{\frac{8}{\alpha}}}^4\left(\|\omega\|_{L^2}^2 + \||\nabla|^{\frac{\alpha}{2}}\theta\|_{L^2}^2\right) + C\|\theta_0\|_{L^2}^2.$$

We use Gronwall's to conclude (2.53). □

2.3.2 Some a priori estimates for the scaled vorticity Boussinesq problem (W, Θ) in L^p

We now turn our attention to the scaled vorticity system. By the results of Lemma 2.3.2 and Lemma 2.3.3, such solutions exist globally, by virtue of the change of variables. Now that we have a global solution, together with the global estimate (2.57), we can actually obtain global *a priori* estimates for Θ in all L^p spaces.

Lemma 2.3.3. *Let $p \geq 1$, and $\Theta_0 \in L^1 \cap L^\infty(\mathbb{R}^2) \cap H^\alpha(\mathbb{R}^2)$, $W_0 \in L^2$. Then for any $\tau > 0$, $\Theta \in$*

$C^0([0,\tau];L^p)$, *there exists* $C = C_{\alpha,p}$ *such that*

$$\|\Theta(\tau,\cdot)\|_{L^p} \le C_{\alpha,p}\|\Theta_0\|_{L^p(\mathbb{R}^2)} e^{(2-\frac{1}{\alpha}-\frac{2}{\alpha p})\tau}. \tag{2.54}$$

Proof. We take a dot product of the Θ equation in (2.13)with $|\Theta|^{p-2}\Theta$, $p \ge 1$. We obtain

$$\frac{1}{p}\partial_\tau\|\Theta\|^p_{L^p} + \int_{\mathbb{R}^2} |\nabla|^\alpha\Theta|\Theta|^{p-2}\Theta d\xi = (2-\frac{1}{\alpha}-\frac{2}{\alpha p})\|\Theta\|^p_{L^p}.$$

Recall however that $\int_{\mathbb{R}^2} |\nabla|^\alpha\Theta|\Theta|^{p-2}\Theta d\xi \ge 0$, by Lemma 1.1.1. Thus, integrating this inequality yields (2.54). □

Lemma (2.3.3) provides us with a decay rate for $\Theta(\tau,\cdot)$ for $1 \le p < \frac{2}{2\alpha-1}$, but clearly an increasing exponential bound for $p \ge \frac{2}{2\alpha-1}$. However, we can use it in the next step to get a decay rate for any $p \ge 1$.

Lemma 2.3.4. *Let* $p \ge 1$, *and* $\Theta_0 \in L^1 \cap L^\infty(\mathbb{R}^2) \cap H^\alpha(\mathbb{R}^2)$, $W_0 \in L^2$. *Then for any* $\tau > 0$, $\Theta \in C^0([0,\tau];L^p)$, *there exists* $C = C_{\alpha,p}$ *such that*

$$\|\Theta(\tau,\cdot)\|_{L^p} \le C_{\alpha,p}\|\Theta_0\|_{L^p(\mathbb{R}^2)} e^{(2-\frac{3}{\alpha})\tau}. \tag{2.55}$$

Proof. Similar to the lemma (2.3.3) for any $p = 2^n$, $n \ge 1$ we have the following energy estimate

$$\frac{1}{p}\partial_\tau\||\Theta|\|^p_{L^p} + \frac{c_\alpha}{p}\|\Theta\|^p_{L^{\frac{2p}{2-\alpha}}} \le (2-\frac{1}{\alpha}-\frac{2}{\alpha p})\|\Theta\|^p_{L^p}.$$

Assuming C be a large number, we add $C\||\Theta|\|^p_{L^p}$ to both sides, we have

$$\begin{aligned}
&\frac{1}{p}\partial_\tau\||\Theta|\|^p_{L^p} + C\||\Theta|\|^p_{L^p} + \frac{c_\alpha}{p}\|\Theta\|^p_{L^{\frac{2p}{2-\alpha}}} \le (2-\frac{1}{\alpha}-\frac{2}{\alpha p}+C)\|\Theta\|^p_{L^p} \\
&\le (2-\frac{1}{\alpha}-\frac{2}{\alpha p}+C)\|\Theta\|^{\gamma p}_{L^{\frac{2p}{2-\alpha}}}\|\Theta\|^{(1-\gamma)p}_{L^1} \\
&\le \varepsilon_0\|\Theta\|^p_{L^{\frac{2p}{2-\alpha}}} + \frac{\left[(2-\frac{1}{\alpha}-\frac{2}{\alpha p}+C)\right]^{\frac{1}{1-\gamma}}}{\varepsilon_0^{\frac{\gamma}{1-\gamma}}}\||\Theta|\|^p_{L^1}.
\end{aligned}$$

where $\gamma = \frac{2(p-1)}{2p-2+\alpha}$. In other words

$$
\begin{aligned}
&\frac{1}{p}\partial_\tau \||\Theta|\|_{L^p}^p + C\||\Theta|\|_{L^p}^p \le \frac{\left[(2-\frac{1}{\alpha}-\frac{2}{\alpha p}+C)\right]^{\frac{1}{1-\gamma}}}{\varepsilon_0^{\frac{\gamma}{1-\gamma}}}\||\Theta|\|_{L^1}^p \\
&\le \frac{\left[(2-\frac{1}{\alpha}-\frac{2}{\alpha p}+C)\right]^{\frac{1}{1-\gamma}}}{\varepsilon_0^{\frac{\gamma}{1-\gamma}}} e^{p(2-\frac{3}{\alpha})\tau}
\end{aligned}
$$

Finally, we use Gronwall's inequality to finish the proof. □

We can use above lemma to find some decay rate for $U(\tau,\cdot)$. We need this to be able to get some bounds for W in higher L^p spaces.

Lemma 2.3.5. *Let $U_0 \in L^2(\mathbb{R}^2)$. Then for any $\tau > 0$, $U \in C^0([0,\tau];L^2)$, there exists $C = C_{\alpha,p}$ such that*

$$
\|U(\tau,\cdot)\|_{L^2} \le C_{\alpha,p}\|U_0\|_{L^2(\mathbb{R}^2)} e^{(2-\frac{3}{\alpha})\tau}. \tag{2.56}
$$

Proof. If we dot product the equation (2.14) with U we get the following relation

$$
\frac{1}{2}\partial_\tau \|U\|_{L^2}^2 + \||\nabla|^{\frac{\alpha}{2}} U\|_{L^2}^2 = \frac{1}{\alpha}\int (\xi\cdot\nabla U)U d\xi + (1-\frac{1}{\alpha})\|U\|_{L^2}^2 + \int \theta\cdot U d\xi.
$$

Then

$$
\begin{aligned}
&\partial_\tau \|U\|_{L^2}^2 + 2\||\nabla|^{\frac{\alpha}{2}} U\|_{L^2}^2 = 2(1-\frac{2}{\alpha})\|U\|_{L^2}^2 + \int \theta\cdot U d\xi \\
\le\ & 2(1-\frac{2}{\alpha})\|U\|_{L^2}^2 + \|\Theta\|_{L^2}\|U\|_{L^2} \le 2(1-\frac{2}{\alpha}+\varepsilon)\|U\|_{L^2}^2 + C_\varepsilon\|\Theta\|_{L^2}^2 \\
\le\ & 2(1-\frac{2}{\alpha}+\varepsilon)\|U\|_{L^2}^2 + C_\varepsilon e^{2(2-\frac{3}{\alpha})\tau}.
\end{aligned}
$$

Now we Use the Gronwall's inequality to complete the proof. □

The next lemma provides *a priori* estimates for W and Θ in L^2 spaces, which allows us to conclude global regularity.

Lemma 2.3.6. *Let $\alpha \in (1, \frac{3}{2})$, $W_0 \in L^2$. Then the solution W of* (2.13)*, satisfies*

$$\|W(\tau,\cdot)\|_{L^2} + \|\Theta(\tau,\cdot)\|_{L^2} \leq Ce^{(2-\frac{3}{\alpha})\tau}, \tag{2.57}$$

$$\sup_{0\leq\tau<\infty} \int_0^\tau \left(\||\nabla|^{\frac{\alpha}{2}} W(s)\|_{L^2}^2 + \||\nabla|^{\frac{\alpha}{2}} \Theta(s)\|_{L^2}^2 \right) ds \leq C \tag{2.58}$$

for some $C = C(\|W_0\|_{L^2}, \|\Theta_0\|_{L^2}, \alpha)$, independent on τ.

Proof. We dot product the first equation in (2.13) with W, and the second equation with Θ. We also use the trick used in lamma (2.2.3), i.e. we add the term $A(\|W\|_{L^2}^2 + \|\Theta\|_{L^2}^2)$, where A is a large constant to be determined. Then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|W\|_{L^2}^2 + \|\Theta\|_{L^2}^2 \right) &+ A(\|W\|_{L^2}^2 + \|\Theta\|_{L^2}^2) + \||\nabla|^{\frac{\alpha}{2}} W\|_{L^2}^2 + \||\nabla|^{\frac{\alpha}{2}} \Theta\|_{L^2}^2 \\ &\leq \left| \int \partial_1 \Theta W d\xi \right| + (A + 1 - \frac{1}{\alpha})\|W\|_{L^2}^2 + (A + 2 - \frac{2}{\alpha})\|\Theta\|_{L^2}^2. \end{aligned}$$

But by Gagliardo-Nirenberg (and taking into account that $1 - \frac{\alpha}{2} < \frac{\alpha}{2}$) and Young's inequalities,

$$\begin{aligned} &\left| \int \partial_1 \Theta W d\xi \right| \leq \||\nabla|^{1-\frac{\alpha}{2}} \Theta\|_{L^2} \||\nabla|^{\frac{\alpha}{2}} W\|_{L^2} \leq \varepsilon \||\nabla|^{\frac{\alpha}{2}} \Theta\|_{L^2}^2 + \varepsilon \||\nabla|^{\frac{\alpha}{2}} W\|_{L^2}^2 + C_\varepsilon \|\Theta\|_{L^2}^2 \\ \leq\ & \varepsilon \||\nabla|^{\frac{\alpha}{2}} \Theta\|_{L^2}^2 + \varepsilon \||\nabla|^{\frac{\alpha}{2}} W\|_{L^2}^2 + C_\varepsilon e^{2(2-\frac{3}{\alpha})\tau}. \end{aligned}$$

whence, using the estimate for $\|\Theta\|_{L^2}$ from (2.54)(with $p = 2$). We also have

$$\begin{aligned} &(A + 1 - \frac{1}{\alpha})\|W\|_{L^2}^2 \leq C(A + 1 - \frac{1}{\alpha})\|\nabla U\|_{L^2}^2 \leq \\ \leq\ & C(A + 1 - \frac{1}{\alpha})\|U\|_{L^2}^{2\gamma} \||\nabla|^{1+\frac{\alpha}{2}} U\|_{L^2}^{2(1-\gamma)} \leq C(A + 1 - \frac{1}{\alpha})\|U\|_{L^2}^{2\gamma} \||\nabla|^{\frac{\alpha}{2}} W\|_{L^2}^{2(1-\gamma)} \\ \leq\ & \varepsilon \||\nabla|^{\frac{\alpha}{2}} W\|_{L^2}^2 + \frac{[C(A + 1 - \frac{1}{\alpha})]^{\frac{1}{1-\gamma}}}{\varepsilon^{\frac{\gamma}{1-\gamma}}} \|U\|_{L^2}^2 \\ \leq\ & \varepsilon \||\nabla|^{\frac{\alpha}{2}} W\|_{L^2}^2 + \frac{[C(A + 1 - \frac{1}{\alpha})]^{\frac{1}{1-\gamma}}}{\varepsilon^{\frac{\gamma}{1-\gamma}}} e^{2(2-\frac{3}{\alpha})\tau}. \end{aligned}$$

Considering the estimate for $\|\Theta\|_{L^2}$ from (2.54)(with $p=2$)

$$\frac{d}{dt}\left(\|W\|_{L^2}^2+\|\Theta\|_{L^2}^2\right)+2A(\|W\|_{L^2}^2+\|\Theta\|_{L^2}^2)+2(1-2\varepsilon)\||\nabla|^{\frac{\alpha}{2}}W\|_{L^2}^2$$
$$+\quad 2(1-2\varepsilon)\||\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2\le\frac{2[C(A+1-\frac{1}{\alpha})]^{\frac{1}{1-\gamma}}}{\varepsilon^{\frac{\gamma}{1-\gamma}}}\,e^{2(2-\frac{3}{\alpha})\tau}.$$

At this point we choose $A=2(\frac{3}{\alpha}-2)$. Then the last relation has two consequences. First we can drop the term $2(1-2\varepsilon)\||\nabla|^{\frac{\alpha}{2}}W\|_{L^2}^2+2(1-2\varepsilon)\||\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2$, so

$$\frac{d}{dt}\left(\|W\|_{L^2}^2+\|\Theta\|_{L^2}^2\right)+4(\frac{3}{\alpha}-2)(\|W\|_{L^2}^2+\|\Theta\|_{L^2}^2)\le\frac{[C(\frac{5}{\alpha}-3)]^{\frac{1}{1-\gamma}}}{\varepsilon^{\frac{\gamma}{1-\gamma}}}\,e^{2(2-\frac{3}{\alpha})\tau}.$$

and then use the Gronwall's inequality for the following inequality and get the decay rate (2.57).

Second consequence to get

$$\int_0^\tau(\||\nabla|^{\frac{\alpha}{2}}W(t)\|_{L^2}^2+\||\nabla|^{\frac{\alpha}{2}}\Theta(t)\|_{L^2}^2)dt\le\left(\|W_0\|_{L^2}^2+\|\Theta_0\|_{L^2}^2\right)+\frac{C_\varepsilon}{2(\frac{3}{\alpha}-2)}.$$

This implies (2.58). □

We need some *a priori* estimates for $\|W\|_{L^p}$ for some $p>2$, as these will be necessary in our subsequent considerations. This turns out to be non-trivial. To this end, it turns out that it is easier to control $\|W\|_{H^1},\|\Theta\|_{H^1}$ and then use Sobolev embedding to control $\|W\|_{L^p},\|\Theta\|_{L^p},1<p<\infty$. In this way, we get the control needed, but we end up needing to require smoother H^1 initial data.

Proposition 2.3.7. $W_0,\Theta_0\in H^1$. *Then, the global solution satisfies* $W,\Theta\in C^0([0,\tau];$ $H^1(\mathbb{R}^2))$. *Moreover,*

$$\|W(\tau)\|_{H^1}+\|\Theta(\tau)\|_{H^1}\le Ce^{(2-\frac{3}{\alpha})\tau}. \tag{2.59}$$

$C=C(\|W_0\|_{H^1},\|\Theta_0\|_{H^1},\alpha)$, *independent on* τ.

Proof. Local well-posedness in the space H^1, for the original (unscaled) equations works as in

Lemma 2.3.2, so we omit it. Thus, we have local solutions for the scaled system as well. We now need to establish *a priori* estimates to show that these are global.

We differentiate each of the equations in (2.13) and then we proceed similar to the proof of Lemma 2.3.6. Namely, we dot product it with[11] ∂W and $\partial\Theta$ respectively. We add the two resulting equations to obtain the following energy inequality

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left(\|\partial W\|_{L^2}^2+\|\partial\Theta\|_{L^2}^2\right)+\||\nabla|^{\frac{\alpha}{2}+1}W\|_{L^2}^2+\||\nabla|^{\frac{\alpha}{2}+1}\Theta\|_{L^2}^2 \leq \\ \leq & \ \Big|\int \partial_1\partial\Theta\partial W d\xi\Big|+(1-\frac{1}{\alpha})\|\partial W\|_{L^2}^2+2(1-\frac{1}{\alpha})\|\partial\Theta\|_{L^2}^2+|\langle \partial U\nabla W,\partial W\rangle| \\ + & \ |\langle \partial U\nabla\Theta,\partial\Theta\rangle|. \end{aligned}$$

By Gagliardo-Nirenbergs' and Young's

$$\|\partial W\|_{L^2}^2+\|\partial\Theta\|_{L^2}^2 \leq \varepsilon(\|\nabla|^{\frac{\alpha}{2}+1}W\|_{L^2}^2+\|\nabla|^{\frac{\alpha}{2}+1}\Theta\|_{L^2}^2)+C_\varepsilon(\|W\|_{L^2}^2+\|\Theta\|_{L^2}^2)$$

Next,

$$\begin{aligned} \Big|\int \partial_1\partial\Theta\partial W d\xi\Big| & \leq C\||\nabla|^{\frac{\alpha}{2}+1}\Theta\|_{L^2}\||\nabla|^{2-\frac{\alpha}{2}}W\|_{L^2} \\ & \leq \varepsilon(\|\nabla|^{\frac{\alpha}{2}+1}W\|_{L^2}^2+\|\nabla|^{\frac{\alpha}{2}+1}\Theta\|_{L^2}^2)+C_\varepsilon\|W\|_{L^2}^2, \end{aligned}$$

where in the last estimate we have used that $2-\frac{\alpha}{2}<1+\frac{\alpha}{2}$. Finally,

$$\begin{aligned} |\langle \partial U\cdot\nabla W,\partial W\rangle| & = |\langle \nabla\cdot(\partial U W),\partial W\rangle| \leq C\|\nabla|^{\frac{\alpha}{2}+1}W\|_{L^2}\||\nabla|^{1-\frac{\alpha}{2}}(\partial U\, W)\|_{L^2} \\ & \leq \varepsilon\|\nabla|^{\frac{\alpha}{2}+1}W\|_{L^2}^2+C_\varepsilon\||\nabla|^{1-\frac{\alpha}{2}}(\partial U\, W)\|_{L^2}^2 \end{aligned}$$

[11]Here ∂ means either ∂_1 or ∂_2

By product estimates, (1.1.4) and Sobolev embedding

$$\begin{aligned} &\||\nabla|^{1-\frac{\alpha}{2}}(\partial U\, W)\|_{L^2} \le C(\||\nabla|^{1-\frac{\alpha}{2}}\partial U\|_{L^{\frac{8}{4-\alpha}}}\|W\|_{L^{\frac{8}{\alpha}}} + \||\nabla|^{1-\frac{\alpha}{2}}W\|_{L^{\frac{8}{4-\alpha}}}\|\partial U\|_{L^{\frac{8}{\alpha}}}) \\ \le\;& C\||\nabla|^{1-\frac{\alpha}{4}}\partial U\|_{L^2}\||\nabla|^{1-\frac{\alpha}{4}}W\|_{L^2} \le C\||\nabla|^{1-\frac{\alpha}{4}}W\|_{L^2}^2 \le \||\nabla|^{1+\frac{\alpha}{2}}W\|_{L^2}^{\frac{2-\frac{\alpha}{2}}{1+\frac{\alpha}{2}}}\|W\|_{L^2}^{\frac{\frac{3\alpha}{2}}{1+\frac{\alpha}{2}}}, \end{aligned}$$

where we have used $\partial U \sim W$ (in all Sobolev spaces) and Gagliardo-Nirenberg's. This allows us to estimate by Young's

$$|\langle \partial U \cdot \nabla W, \partial W\rangle| \le 2\varepsilon\||\nabla|^{\frac{\alpha}{2}+1}W\|_{L^2}^2 + C_\varepsilon\|W\|_{L^2}^{\frac{3\alpha}{\alpha-1}}.$$

Clearly, the appropriate estimate, obtained in the same way holds for

$$|\langle \partial U \nabla\Theta, \partial\Theta\rangle| \le 2\varepsilon\||\nabla|^{3\frac{\alpha}{2}+1}\Theta\|_{L^2}^2 + C_\varepsilon\|W\|_{L^2}^{\frac{\alpha}{\alpha-1}}.$$

All in all, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\partial W\|_{L^2}^2 + \|\partial\Theta\|_{L^2}^2\right) + (1-6\varepsilon)(\||\nabla|^{\frac{\alpha}{2}+1}W\|_{L^2}^2 + \||\nabla|^{\frac{\alpha}{2}+1}\Theta\|_{L^2}^2) \\ &\le C_\varepsilon(\|W\|_{L^2}^{\frac{3\alpha}{\alpha-1}} + \|W\|_{L^2}^2 + \|\Theta\|_{L^2}^2). \end{aligned}$$

Set $\varepsilon = \frac{1}{10}$. For every $A > 0$, there is $c_{\alpha,A}$, so that $\||\nabla|^{\frac{\alpha}{2}+1}W\|_{L^2}^2 \ge A\|\partial W\|_{L^2}^2 - c_{A,\alpha}\|W\|_{L^2}^2$ and similar for Θ, so we end up with

$$\frac{d}{dt}\left(\|\partial W\|_{L^2}^2 + \|\partial\Theta\|_{L^2}^2\right) + A\left(\|\partial W\|_{L^2}^2 + \|\partial\Theta\|_{L^2}^2\right) \le C_{A,\alpha}e^{2(2-\frac{3}{\alpha})\tau}.$$

where we have used the exponential bounds from (2.57). Setting sufficiently large A, namely $A > 2(\frac{3}{\alpha} - 2)$, and applying Gronwall's yields the result. □

As an immediate corollary, we have control of the L^p norms for W.

Corollary 2.3.8. *Let $W_0, \Theta_0 \in H^1$. Then, for all $p \in (2,\infty)$, there is the bound*

$$\|W(\tau,\cdot)\|_{L^p} \le C(\|W_0\|_{H^1}, \|\Theta_0\|_{H^1}, \alpha, p) e^{(2-\frac{3}{\alpha})\tau}. \tag{2.60}$$

2.3.3 Global regularity for the scaled vorticity Boussinesq problem (W,Θ) in $L^2(2)\cap L^\infty(\mathbb{R}^2)$

The next lemma is a local well-posedness result, which is a companion to Theorem 2.2.4.

Lemma 2.3.9. *Suppose that $W_0, \Theta_0 \in L^2(2)\cap L^\infty$. Then, there exists time $T = T(\|(W_0,\Theta_0)\|_{L^2(2)\cap L^\infty})$, so that the system of equation* (2.13) *has unique local solution $W, \Theta \in C^0([0,T]; L^2(2)\cap L^\infty)$ with $W(0) = W_0$ and $\Theta(0) = \Theta_0$.*

Proof. We are looking for strong solutions in the space $X = L^2(2)\cap L^\infty$, that is a solutions of the following system of integral equations

$$\begin{aligned} W(\tau) &= e^{\tau\mathscr{L}} W_0 - \int_0^\tau e^{(\tau-s)\mathscr{L}} \nabla(U\cdot W) ds + \int_0^\tau e^{(\tau-s)\mathscr{L}} (\partial_1\Theta) ds, \\ \Theta(\tau) &= e^{\tau(\mathscr{L}+1-\frac{1}{\alpha})}\Theta_0 - \int_0^\tau e^{(\tau-s)(\mathscr{L}+1-\frac{1}{\alpha})} \nabla(U\cdot\Theta) ds \end{aligned}$$

For the free solutions, according to (2.34) and (2.27),

$$\|e^{\tau\mathscr{L}} W_0\|_{L^2(2)\cap L^\infty} + \|e^{\tau(\mathscr{L}+1-\frac{1}{\alpha})}\Theta_0\|_{L^2(2)\cap L^\infty} \le Ce^\tau(\|W_0\|_{L^2(2)\cap L^\infty} + \|\Theta_0\|_{L^2(2)\cap L^\infty}).$$

For $0 < T < 1$, to be determined, introduce the space

$$Y_T := \{(W,\Theta) : \sup_{0\le\tau\le T} [\|W(\tau,\cdot)\|_X + \|\Theta(\tau,\cdot)\|_X] \le 2Ce(\|W_0\|_{L^2(2)\cap L^\infty} + \|\Theta_0\|_{L^2(2)\cap L^\infty}).\}.$$

According to (2.28), (2.29) and (2.34),

$$\begin{aligned}
&\| \int_0^\tau e^{(\tau-s)\mathscr{L}} \nabla(U\cdot W) ds\|_{L^2(2)\cap L^\infty} \le \\
\le\ & \int_0^\tau \frac{e^{-\frac{(\tau-s)}{\alpha}}(e^{(1-\frac{2}{\alpha})(\tau-s)} + e^{(1-\frac{1}{\alpha})(\tau-s)})}{a(\tau-s)^{\frac{1}{\alpha}}} \|U\cdot W\|_{L^2(2)\cap L^\infty}\, ds \\
\le\ & C \sup_{0\le\tau\le T} \|UW\|_{L^2(2)\cap L^\infty} \int_0^\tau \frac{1}{|\tau-s|^{\frac{1}{\alpha}}} ds \le CT^{1-\frac{1}{\alpha}} \sup_{0\le\tau\le T}\|U\|_{L^\infty} \sup_{0\le\tau\le T}\|W\|_{L^2(2)\cap L^\infty}.
\end{aligned}$$

and similarly

$$\| \int_0^\tau e^{(\tau-s)(\mathscr{L}+1-\frac{1}{\alpha})} \nabla(U\cdot\Theta) ds\|_{L^2(2)\cap L^\infty} \le CT^{1-\frac{1}{\alpha}} \sup_{0\le\tau\le T}\|U\|_{L^\infty} \sup_{0\le\tau\le T}\|\Theta\|_{L^2(2)\cap L^\infty}.$$

Recalling that $U = (\nabla^\perp)^{-1}W$, we further estimate by (1.8),

$$\|U\|_{L^\infty} \le C(\|W\|_{L^{2+\varepsilon}} + \|W\|_{L^{2-\varepsilon}}) \le C\|W\|_{L^2(2)\cap L^\infty},$$

since $L^2(2) \hookrightarrow L^{2-\varepsilon}$ and $L^2(2)\cap L^\infty \hookrightarrow L^1\cap L^\infty \hookrightarrow L^{2+\varepsilon}$. Finally,

$$\| \int_0^\tau e^{(\tau-s)\mathscr{L}} (\partial_1\Theta) ds\|_{L^2(2)\cap L^\infty} \le CT^{1-\frac{1}{\alpha}} \sup_{0\le\tau\le T}\|\Theta\|_{L^2(2)\cap L^\infty}.$$

Clearly, appropriate estimate hold for the differences, whence the integral equations provide a contraction mapping in the space Y_T, provided,

$$T^{1-\frac{1}{\alpha}} << \frac{1}{2Ce(\|W_0\|_{L^2(2)\cap L^\infty} + \|\Theta_0\|_{L^2(2)\cap L^\infty})}.$$

□

Our next result provides a global regularity for the W,Θ system in the space $L^2(2)$.

Lemma 2.3.10. *The system of equations* (2.7)*, with* $W_0,\Theta_0 \in X = L^2(2)\cap L^\infty$*, and also* $W_0,\Theta_0 \in$

$H^1(\mathbb{R}^2)$ *has an unique global solution, in space* X*. There exists* $C = C(\|W_0\|_X, \|\Theta\|_X)$ *such that*

$$\sup_{0\leq\tau<\infty} \|W(\tau,\cdot)\|_{L^2(2)} + \|\Theta(\tau,\cdot)\|_{L^2(2)} \leq C. \tag{2.61}$$

Remark: The estimate by a constant is very inefficient, as we shall see in section 2.3.4. One could improve the argument below, at a considerable technical price, to obtain better decay estimates. Since the results in section 2.3.4 will supersede these anyways, we choose to present the simpler arguments.

Proof. The existence of a local solutions are guaranteed by Lemma 2.3.9. So, it remains to establish energy estimates, which keep the relevant $L^2(2)$ norms under control. Note that the unweighted portion of the norm has an exponential decay, by (2.54)and (2.57). So, it remains to control the weighted norms.

We run a preliminary argument only on the Θ variable. As usual, this is easier, due to the lack of problematic term $\partial_1\Theta$, which appears in the equation for W. We dot product the Θ equation in (2.13) with $|\xi|^4\Theta$. We have

$$\begin{aligned}&\frac{1}{2}\frac{d}{d\tau}\int |\xi|^4\Theta^2 d\xi + \int |\xi|^4|\nabla|^\alpha\Theta\cdot\Theta d\xi + (\frac{4}{\alpha}-2)\int|\xi|^4\Theta^2 d\xi\\ &= -\int (U\cdot\nabla_\xi\Theta)|\xi|^4\Theta d\xi.\end{aligned}$$

Then

$$-\int (U\cdot\nabla_\xi\Theta)|\xi|^4\Theta d\xi \;=\; 2\int |\xi|^2(\xi\cdot U)\Theta^2 d\xi.$$

But

$$\left|\int |\xi|^2(\xi\cdot U)\Theta^2 d\xi\right| \leq C\int |\xi|^3\|U\|_{L^\infty}|\Theta|^2 d\xi \leq \varepsilon\int|\xi|^4|\Theta|^2 d\xi + C\varepsilon^{-3}\|U\|_{L^\infty}^4\|\Theta\|_{L^2}^2.$$

Now, according to (1.1), for every $\delta > 0$

$$\begin{aligned}\|U\|_{L^\infty} \le C_\delta(\|W\|_{L^{2+\delta}} + \|W\|_{L^{2-\delta}}) &\le C_\delta(e^{(2-\frac{3}{\alpha})\tau} + \|W\|_{L^2}^{\frac{2-2\delta}{2-\delta}}\|W\|_{L^1}^{\frac{\delta}{2-\delta}}) \\ &\le C_\delta + C_\delta\|W\|_{L^2(2)}^{\frac{\delta}{2-\delta}}.\end{aligned}$$

We also have

$$\begin{aligned}&\int |\xi|^4\Theta|\nabla|^\alpha\Theta d\xi = \langle|\xi|^2|\nabla|^{\frac{\alpha}{2}}|\nabla|^{\frac{\alpha}{2}}\Theta, |\xi|^2\Theta\rangle = \\ =\ & \langle|\nabla|^{\frac{\alpha}{2}}[|\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta], |\xi|^2\Theta\rangle - \langle[|\nabla|^{\frac{\alpha}{2}}, |\xi|^2]\ [|\nabla|^{\frac{\alpha}{2}}\Theta], |\xi|^2\Theta\rangle = \\ =\ & \langle|\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta, |\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\rangle + \langle|\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta, [|\nabla|^{\frac{\alpha}{2}}, |\xi|^2]\Theta\rangle - \\ -\ & \langle[|\nabla|^{\frac{\alpha}{2}}, |\xi|^2]\ [|\nabla|^{\frac{\alpha}{2}}\Theta], |\xi|^2\Theta\rangle \\ =\ & \int |\xi|^4||\nabla|^{\frac{\alpha}{2}}\Theta|^2 d\xi + \langle|\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta, [|\nabla|^{\frac{\alpha}{2}}, |\xi|^2]\Theta\rangle - \langle[|\nabla|^{\frac{\alpha}{2}}, |\xi|^2]\ [|\nabla|^{\frac{\alpha}{2}}\Theta], |\xi|^2\Theta\rangle\end{aligned}$$

Now if we define $I(\tau) = \int |\xi|^4\Theta^2 d\xi$, and put all above together we have the following relation

$$\begin{aligned}&\frac{1}{2}I'(\tau) + \left(\frac{4}{\alpha} - 2 - 10\varepsilon\right)I(\tau) + \int |\xi|^4||\nabla|^{\frac{\alpha}{2}}\Theta|^2 d\xi \\ &\le |\langle|\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta, [|\nabla|^{\frac{\alpha}{2}}, |\xi|^2]\Theta\rangle| + |\langle[|\nabla|^{\frac{\alpha}{2}}, |\xi|^2][|\nabla|^{\frac{\alpha}{2}}\Theta], |\xi|^2\Theta\rangle| + C_{\delta,\varepsilon}\|W(\tau,\cdot)\|_{L^2(2)}^{\frac{4\delta}{2-\delta}}.\end{aligned}$$

We can use Lemma 1.1.6 to get

$$\begin{aligned}&|\langle|\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta, [|\nabla|^{\frac{\alpha}{2}}, |\xi|^2]\Theta\rangle| \le \||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}\|[|\nabla|^{\frac{\alpha}{2}}, |\xi|^2]\Theta\|_{L^2} \\ \le\ & \||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}\||\xi|^{2-\frac{\alpha}{2}}\Theta\|_{L^2} \le \||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}\||\xi|^2\Theta\|_{L^2}^{1-\frac{\alpha}{4}}\|\Theta\|_{L^2}^{\frac{\alpha}{4}} \\ \le\ & \varepsilon(\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2 + \||\xi|^2\Theta\|_{L^2}^2) + C_\varepsilon.\end{aligned}$$

For the other term we have

$$
\begin{aligned}
& |\langle[|\nabla|^{\frac{\alpha}{2}},|\xi|^2][|\nabla|^{\frac{\alpha}{2}}\Theta],|\xi|^2\Theta\rangle| \leq \||\xi|^2\Theta\|_{L^2}\|[|\nabla|^{\frac{\alpha}{2}},|\xi|^2][|\nabla|^{\frac{\alpha}{2}}\Theta]\|_{L^2} \\
\leq \;& \||\xi|^2\Theta\|_{L^2}\||\xi|^{2-\frac{\alpha}{2}}[|\nabla|^{\frac{\alpha}{2}}\Theta]\|_{L^2} \leq \||\xi|^2\Theta\|_{L^2}\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^{1-\frac{\alpha}{4}}\||\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^{\frac{\alpha}{4}} \\
\leq \;& \varepsilon(\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2+\||\xi|^2\Theta\|_{L^2}^2)+C_\varepsilon\||\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \frac{1}{2}I'(\tau)+\left(\frac{4}{\alpha}-2-20\varepsilon\right)I(\tau)+(1-5\varepsilon)\int|\xi|^4||\nabla|^{\frac{\alpha}{2}}\Theta|^2 d\xi \\
& \leq C_\varepsilon+C_{\delta,\varepsilon}\|W(\tau,\cdot)\|_{L^2(2)}^{\frac{4\delta}{2-\delta}}+C_\varepsilon\||\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2.
\end{aligned}
$$

Choosing $\varepsilon=\frac{1}{200}$ and applying Gronwall's and then using of (2.58) implies that for every $\delta>0$, there is C_δ, so that

$$
\||\xi|^2\Theta(\tau,\cdot)\|_{L^2}\leq C_\varepsilon+C_\delta e^{-(\frac{4}{\alpha}-2-\delta)\tau}+C_\delta\sup_{0<s<\tau}\|W(s,\cdot)\|_{L^2(2)}^{\frac{2\delta}{2-\delta}}. \tag{2.62}
$$

for every $\delta>0$. In addition, we obtain the L^2_τ bound

$$
\int_0^\tau\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta(\tau,\cdot)\|_{L^2}^2 d\tau\leq C+C_\delta\sup_{0<s<\tau}\|W(s,\cdot)\|_{L^2(2)}^{\frac{4\delta}{2-\delta}}. \tag{2.63}
$$

We are now ready for the bounds for W, which are always harder. If we dot product in (2.13), the first equation with $|\xi|^4W$, we have the energy equalities

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{d\tau}\int|\xi|^4W^2 d\xi+\int|\xi|^4|\nabla|^\alpha W\cdot W d\xi+(\frac{3}{\alpha}-1)\int|\xi|^4W^2 d\xi \\
= \;& -\int(U\cdot\nabla_\xi W)|\xi|^4W d\xi+\int\partial_1\Theta\,|\xi|^4W d\xi
\end{aligned}
$$

Then $-\int (U\cdot\nabla_\xi W)|\xi|^4 W d\xi = 2\int |\xi|^2(\xi\cdot U)W^2 d\xi$. We can bound this term as follows

$$\left|\int |\xi|^2(\xi\cdot U)W^2 d\xi\right| \;\le\; C\int |\xi|^3\|U\|_{L^\infty}|W|^2 d\xi \le \varepsilon\int |\xi|^4|W|^2 d\xi + C\varepsilon^{-3}\|U\|_{L^\infty}^4\|W\|_{L^2}^2.$$

Again, according to (1.1), for every $\delta > 0$

$$\|U\|_{L^\infty} \le C_\delta(\|W\|_{L^{2+\delta}} + \|W\|_{L^{2-\delta}}) \le C(e^{(2-\frac{3}{\alpha})\tau} + \|W\|_{L^2}^{\frac{2-2\delta}{2-\delta}}\|W\|_{L^1}^{\frac{\delta}{2-\delta}}).$$

Taking into account (2.54), (2.60), $L^2(2)\hookrightarrow L^1$ and Young's inequality, allows us to estimate

$$\left|\int |\xi|^2(\xi\cdot U)W^2 d\xi\right| \le 2\varepsilon\int |\xi|^4|W|^2 d\xi + C_{\varepsilon,\delta}\|W(\tau,\cdot)\|_{L^2(2)}^{\frac{4\delta}{2-\delta}}.$$

We also have, similar to the Θ variable calculation,

$$\begin{aligned}\int |\xi|^4 W|\nabla|^\alpha W d\xi \;&=\; \||\xi|^2||\nabla|^{\frac{\alpha}{2}}W\|_{L^2}^2 + \langle |\xi|^2|\nabla|^{\frac{\alpha}{2}}W, [|\nabla|^{\frac{\alpha}{2}}, |\xi|^2]W\rangle \\ &-\; \langle [|\nabla|^{\frac{\alpha}{2}}, |\xi|^2][|\nabla|^{\frac{\alpha}{2}}W], |\xi|^2 W\rangle\end{aligned}$$

Now if we take $J(\tau) = \int |\xi|^4 W^2 d\xi$, and put all above together we have the following relation

$$\begin{aligned}&\frac{1}{2}J'(\tau) + \left(\frac{3}{\alpha} - 1 - 10\varepsilon\right)J(\tau) + \int |\xi|^4||\nabla|^{\frac{\alpha}{2}}W|^2 d\xi \\ &\le |\langle |\xi|^2|\nabla|^{\frac{\alpha}{2}}W, [|\nabla|^{\frac{\alpha}{2}}, |\xi|^2]W\rangle| + |\langle [|\nabla|^{\frac{\alpha}{2}}, |\xi|^2][|\nabla|^{\frac{\alpha}{2}}W], |\xi|^2 W\rangle| \\ &+ \left|\int |\xi|^4(\partial_1\Theta)W d\xi\right| + C_\varepsilon + C_{\varepsilon,\delta}\|W(\tau,\cdot)\|_{L^2(2)}^{\frac{4\delta}{2-\delta}} \\ &= I_1 + I_2 + I_3 + C_\varepsilon + C_{\varepsilon,\delta}\|W(\tau,\cdot)\|_{L^2(2)}^{\frac{4\delta}{2-\delta}}\end{aligned}$$

We can use Lemma 1.1.6 to get

$$
\begin{aligned}
I_1 &= |\langle|\xi|^2|\nabla|^{\frac{\alpha}{2}}W,[|\nabla|^{\frac{\alpha}{2}},|\xi|^2]W\rangle| \leq \||\xi|^2|\nabla|^{\frac{\alpha}{2}}W\|_{L^2}\|[|\nabla|^{\frac{\alpha}{2}},|\xi|^2]W\|_{L^2} \\
&\leq \||\xi|^2|\nabla|^{\frac{\alpha}{2}}W\|_{L^2}\||\xi|^{2-\frac{\alpha}{2}}W\|_{L^2} \leq \||\xi|^2|\nabla|^{\frac{\alpha}{2}}W\|_{L^2}\||\xi|^2W\|_{L^2}^{1-\frac{\alpha}{4}}\|W\|_{L^2}^{\frac{\alpha}{4}} \\
&\leq \varepsilon(\||\xi|^2|\nabla|^{\frac{\alpha}{2}}W\|_{L^2}^2+\||\xi|^2W\|_{L^2}^2)+C_\varepsilon,
\end{aligned}
$$

where we have used the bounds (2.57) for $\|W\|_{L^2}$. Next, regarding I_2, we have

$$
\begin{aligned}
I_2 &= |\langle[|\nabla|^{\frac{\alpha}{2}},|\xi|^2][|\nabla|^{\frac{\alpha}{2}}W],|\xi|^2W\rangle| \leq \||\xi|^2W\|_{L^2}\|[|\nabla|^{\frac{\alpha}{2}},|\xi|^2][|\nabla|^{\frac{\alpha}{2}}W]\|_{L^2} \\
&\leq \||\xi|^2W\|_{L^2}\||\xi|^{2-\frac{\alpha}{2}}|\nabla|^{\frac{\alpha}{2}}W\|_{L^2} \leq \||\xi|^2W\|_{L^2}\||\xi|^2|\nabla|^{\frac{\alpha}{2}}W\|_{L^2}^{1-\frac{\alpha}{4}}\||\nabla|^{\frac{\alpha}{2}}W\|_{L^2}^{\frac{\alpha}{4}} \\
&\leq \varepsilon(\||\xi|^2W\|_{L^2}^2+\||\xi|^2|\nabla|^{\frac{\alpha}{2}}W\|_{L^2}^2)+C_\varepsilon\||\nabla|^{\frac{\alpha}{2}}W\|_{L^2}^2.
\end{aligned}
$$

I_3 is normally a problematic term, but now we have the decay estimates for $\|\Theta\|_{L^2(2)}$, which we have proved in the first part of this Lemma. We have

$$
\begin{aligned}
I_3 &= \left|\langle\partial_1\Theta,|\xi|^4W\rangle\right| \leq \left|\langle|\xi|^2\partial_1\Theta,|\xi|^2W\rangle\right| \leq \left|\langle\partial_1|\nabla|^{-\frac{\alpha}{2}}|\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta,|\xi|^2W\rangle\right| \\
&+ \left|\langle[\partial_1|\nabla|^{-\frac{\alpha}{2}},|\xi|^2]\ [|\nabla|^{\frac{\alpha}{2}}\Theta],|\xi|^2W\rangle\right| := I_{3,1}+I_{3,2}.
\end{aligned}
$$

$I_{3,1}$ is estimated as follows

$$
\begin{aligned}
I_{3,1} &= \left|\langle\partial_1|\nabla|^{-\frac{\alpha}{2}}|\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta,|\xi|^2W\rangle\right| \leq C\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}\||\nabla|^{1-\frac{\alpha}{2}}[|\xi|^2W]\|_{L^2} \\
&\leq \||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}\||\xi|^2W\|_{L^2}^{\frac{2\alpha-2}{\alpha}}\||\nabla|^{\frac{\alpha}{2}}[|\xi|^2W]\|_{L^2}^{\frac{2-\alpha}{\alpha}} \leq C_\varepsilon\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2+ \\
&+ \varepsilon(\||\xi|^2W\|_{L^2}^2+\||\nabla|^{\frac{\alpha}{2}}[|\xi|^2W]\|_{L^2}^2)
\end{aligned}
$$

We bound the last term, by Lemma 1.1.6,

$$\||\nabla|^{\frac{\alpha}{2}}[|\xi|^2W]\|_{L^2} \le \||\xi|^2|\nabla|^{\frac{\alpha}{2}}W\|_{L^2} + \|[|\nabla|^{\frac{\alpha}{2}},|\xi|^2]W\|_{L^2}$$
$$\le \||\xi|^2|\nabla|^{\frac{\alpha}{2}}W\|_{L^2} + C\||\xi|^{2-\frac{\alpha}{2}}W\|_{L^2} \le \||\xi|^2|\nabla|^{\frac{\alpha}{2}}W\|_{L^2} + C(\|W\|_{L^2} + \||\xi|^2W\|_{L^2}).$$

Collecting terms together yields the following estimate for $I_{3,1}$ and using (2.59),

$$I_{3,1} \le 2\varepsilon(\||\xi|^2W\|_{L^2}^2 + \||\xi|^2|\nabla|^{\frac{\alpha}{2}}W\|_{L^2}^2) + C_\varepsilon\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2 + Ce^{2(2-\frac{3}{\alpha})\tau}.$$

Assuming the validity of (1.12), we proceed to bound $I_{3,2}$.

$$\begin{aligned} I_{3,2} &= \left|\langle[\partial_1|\nabla|^{-\frac{\alpha}{2}},|\xi|^2]\ [|\nabla|^{\frac{\alpha}{2}}\Theta],|\xi|^2W\rangle\right| \le \||\xi|^2W\|_{L^2}\|[\partial_1|\nabla|^{-\frac{\alpha}{2}},|\xi|^2]\ [|\nabla|^{\frac{\alpha}{2}}\Theta]\|_{L^2} \\ &\le \||\xi|^2W\|_{L^2}\||\xi|^{1+\frac{\alpha}{2}}|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2} \le \||\xi|^2W\|_{L^2}\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^{\frac{2+\alpha}{4}}\||\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^{\frac{2-\alpha}{4}} \\ &\le \varepsilon\||\xi|^2W\|_{L^2}^2 + \||\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2 + C_\varepsilon\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2 \\ &\le \varepsilon\||\xi|^2W\|_{L^2}^2 + C + C_\delta\|W\|_{L^2(2)}^{\frac{4\delta}{2-\delta}} + C_\varepsilon\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2, \end{aligned}$$

where we have made use of (2.63). Combining all the estimates, we obtain the following energy inequality

$$\begin{aligned} &\frac{1}{2}J'(\tau) + \left(\frac{3}{\alpha} - 1 - 20\varepsilon\right)J(\tau) + (1-5\varepsilon)\int |\xi|^4||\nabla|^{\frac{\alpha}{2}}W|^2 d\xi \\ \le\ & C_\varepsilon + C_\delta\|W\|_{L^2(2)}^{\frac{4\delta}{2-\delta}} + C_\varepsilon(\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta\|_{L^2}^2 + \||\nabla|^{\frac{\alpha}{2}}W\|_{L^2}^2) \end{aligned}$$

Applying Gronwall's and taking into account the L^2_τ integrability results (2.58) and (2.63), and $\|W\|_{L^2(2)}^2 \le J(\tau) + C$, we conclude for every $\delta > 0$

$$\begin{aligned} J(\tau) &\le J(0)e^{-2(\frac{3}{\alpha}-1-20\varepsilon)\tau} + C_\varepsilon\tau e^{-2(\frac{3}{\alpha}-1-20\varepsilon)\tau} + C_\delta \sup_{0<s<\tau} J(\tau)^{\frac{2\delta}{2-\delta}} + \\ &+ C_\varepsilon\int_0^\tau(\||\xi|^2|\nabla|^{\frac{\alpha}{2}}\Theta(s,\cdot)\|_{L^2}^2 + \||\nabla|^{\frac{\alpha}{2}}W(s,\cdot)\|_{L^2}^2)ds \le C_\varepsilon + C_\delta \sup_{0<s<\tau} J(\tau)^{\frac{2\delta}{2-\delta}} \end{aligned}$$

Selecting small ε and solving this inequality for $\sup_{0<s<\tau} J(\tau)$ implies the $\sup_{0<s<\tau} J(\tau) \leq C$, for all times τ. Inputting this last estimate in (2.62) implies the desired bound for $\|\Theta\|_{L^2(2)}$ as well. □

2.3.4 Global dynamics of the solutions of the Boussinesq model

It is the time to compute the optimal decay rate in $L^2(2)$ for the solution of the Boussinesq model (2.13). Recall that the relevant operator $\mathscr{L}$ has the form

$$\mathscr{L} = -|\nabla|^\alpha + \frac{1}{\alpha}\xi \cdot \nabla_\xi + 1,$$

with $\lambda_0(\mathscr{L}) = 1 - \frac{2}{\alpha}$ and $\sigma_{ess}(\mathscr{L}) \subset \{\lambda : \Re\lambda \leq 1 - \frac{3}{\alpha}\}$.

Theorem 2.3.11. *Suppose $\alpha \in (1, \frac{3}{2})$ and $W_0, \Theta_0 \in Y := L^2(2)(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$. Then for every $\delta > 0$, there exists $C = C_\delta(\|W_0\|_Y, \|\Theta_0\|_Y) > 0$, such that for any $\tau > 0$, the solutions W, Θ for the system of equations* (2.13) *obey*

$$\begin{aligned} &\|W - \gamma_2(0)e^{-(\frac{3}{\alpha}-2)\tau}\partial_1 G - \gamma_1(0)e^{-(\frac{2}{\alpha}-1)\tau}G\|_{L^2(2)} + \|\Theta - \gamma_2(0)e^{-(\frac{3}{\alpha}-2)\tau}G\|_{L^2(2)} \\ &\leq Ce^{-2(\frac{3}{\alpha}-2-\delta)\tau}. \end{aligned} \tag{2.64}$$

where $\gamma_1(0) := \int W_0(\xi)d\xi$, and $\gamma_2(0) := \int \Theta_0(\xi)d\xi$. In particular, if $\widehat{W}_0(0) = \widehat{\Theta}_0(0) = 0$ then

$$\|W\|_{L^2(2)} + \|\Theta\|_{L^2(2)} \leq C_\delta e^{-2(\frac{3}{\alpha}-2-\delta)\tau}. \tag{2.65}$$

Proof. Using the spectral decomposition for $\mathscr{L}$, described in section 2.1.8, write

$$W(\tau) = \gamma_1(\tau)G(\xi) + \widetilde{W}(\tau) \tag{2.66}$$

$$\Theta(\tau) = \gamma_2(\tau)G(\xi) + \widetilde{\Theta}(\tau) \tag{2.67}$$

where $\gamma_1(\tau) := \langle W(\tau), 1\rangle$, $\gamma_2(\tau) := \langle \Theta(\tau), 1\rangle$, $\widetilde{W} = \mathscr{Q}_0 W(\tau, \cdot)$ and $\widetilde{\Theta} = \mathscr{Q}_0\Theta(\tau, \cdot)$. Then, we derive the equations for γ_1, γ_2 as before - namely

$$\begin{aligned}\partial_\tau \gamma_1 &= \langle W_\tau, 1\rangle = \langle \mathscr{L} W, 1\rangle - \langle U\cdot\nabla W, 1\rangle + \langle \partial_1\Theta, 1\rangle \\ &= \langle \mathscr{L} W, 1\rangle = \langle W, \mathscr{L}^* 1\rangle = (1-\frac{2}{\alpha})\langle W, 1\rangle = (1-\frac{2}{\alpha})\ \gamma_1(\tau)\end{aligned}$$

Similarly, $\partial_\tau\gamma_2 = (2-\frac{3}{\alpha})\ \gamma_2(\tau)$. Solving the ODE's results in the formulas

$$\gamma_1(\tau) = \gamma_1(0)e^{(1-\frac{2}{\alpha})\tau}, \gamma_2(\tau) = \gamma_2(0)e^{(2-\frac{3}{\alpha})\tau}.$$

For the projections over the essential spectrum, we have the following PDE's

$$\begin{aligned}\widetilde{W}_\tau &= \mathscr{L}\widetilde{W} - \mathscr{Q}_0[U\cdot\nabla W - \partial_1\Theta] = \mathscr{L}\widetilde{W} - \mathscr{Q}_0[U\cdot\nabla(\gamma_1(0)\ e^{(1-\frac{2}{\alpha})\tau}G + \widetilde{W})] + \\ &+\ \mathscr{Q}_0[\partial_1(\gamma_2(0)\ e^{(1-\frac{2}{\alpha})\tau}G + \widetilde{\Theta})], \\ \widetilde{\Theta}_\tau &= (\mathscr{L}+1-\frac{1}{\alpha})\widetilde{\Theta} - \mathscr{Q}_0[U\cdot\nabla\Theta] = (\mathscr{L}+1-\frac{1}{\alpha})\widetilde{\Theta} - \mathscr{Q}_0[U\cdot\nabla(\gamma_2(0)\ e^{(2-\frac{3}{\alpha})\tau}G + \widetilde{\Theta})].\end{aligned}$$

We represent them via the Duhamel's formula

$$\begin{aligned}\widetilde{W}(\tau) &= e^{\tau\mathscr{L}}\widetilde{W_0} - \int_0^\tau e^{(\tau-s)\mathscr{L}}\mathscr{Q}_0[U\cdot\nabla(\gamma_1(0)\ e^{(1-\frac{2}{\alpha})s}G + \widetilde{W}(s))]\ ds + \\ &+\ \int_0^\tau e^{(\tau-s)\mathscr{L}}\mathscr{Q}_0[\partial_1\Theta(s)]\ ds, \\ \widetilde{\Theta}(\tau) &= e^{\tau(\mathscr{L}+1-\frac{1}{\alpha})}\widetilde{\Theta_0} - \int_0^\tau e^{(\tau-s)(\mathscr{L}+1-\frac{1}{\alpha})}\mathscr{Q}_0[U\cdot\nabla(\gamma_2(0)\ e^{(2-\frac{3}{\alpha})s}G + \widetilde{\Theta}(s))]\ ds.\end{aligned}$$

One term deserves a special attention, as it is explicit. Note that $\mathscr{Q}_0\partial_1 = \partial_1$, since $\mathscr{P}_0\partial_1 = 0$. Also for $\kappa > 0$, since G is an eigenfunction, with eigenvalue $1-\frac{2}{\alpha}$, we have $e^{\kappa\mathscr{L}}G = e^{(1-\frac{2}{\alpha})\kappa}G$. By

Lemma 2.1.8 that

$$
\begin{aligned}
&\int_0^\tau e^{(\tau-s)\mathscr{L}}\mathscr{Q}_0[\partial_1\Theta(s)]\,ds = \int_0^\tau e^{(\tau-s)\mathscr{L}}[\partial_1[\gamma_2(0)\,e^{(2-\frac{3}{\alpha})s}G+\widetilde{\Theta}(s))]]\,ds = \\
= \;& \gamma_2(0)\int_0^\tau e^{(2-\frac{3}{\alpha})s}e^{-\frac{\tau-s}{\alpha}}\partial_1 e^{(\tau-s)\mathscr{L}}[G]ds + \int_0^\tau e^{-\frac{\tau-s}{\alpha}}\partial_1 e^{(\tau-s)\mathscr{L}}\widetilde{\Theta}(s)ds = \\
= \;& \gamma_2(0)\partial_1 G\int_0^\tau e^{(2-\frac{3}{\alpha})s}e^{-\frac{\tau-s}{\alpha}}e^{(1-\frac{2}{\alpha})(\tau-s)}ds + \int_0^\tau e^{-\frac{\tau-s}{\alpha}}\partial_1 e^{(\tau-s)\mathscr{L}}\widetilde{\Theta}(s)ds = \\
= \;& \gamma_2(0)(e^{(2-\frac{3}{\alpha})\tau}-e^{(1-\frac{3}{\alpha})\tau})\partial_1 G + \int_0^\tau e^{-\frac{\tau-s}{\alpha}}\partial_1 e^{(\tau-s)\mathscr{L}}\widetilde{\Theta}(s)ds.
\end{aligned}
$$

Clearly, at this point, it makes more sense to introduce the new variable,

$$
W_1(\tau,\xi) := \tilde{W}(\tau,\xi) - \gamma_2(0)(e^{(2-\frac{3}{\alpha})\tau}-e^{(1-\frac{3}{\alpha})\tau})\partial_1 G =: \tilde{W} - e^{(2-\frac{3}{\alpha})\tau}G_1(\tau,\xi).
$$

Note that the decay rate $e^{(2-\frac{3}{\alpha})\tau}$ along the G_1 direction is slower than the decay rate $e^{(1-\frac{2}{\alpha})\tau}$ of the evolution along the G direction. Also, G_1 is basically $\partial_1 G$ multiplied by a bounded function of τ and hence an element of $L^2(2)\cap L^\infty$ etc. For future reference,

$$
\|W_1\|_X - Ce^{(2-\frac{3}{\alpha})\tau} \le \|\tilde{W}\|_X \le \|W_1\|_X + Ce^{(2-\frac{3}{\alpha})\tau}. \tag{2.68}
$$

for all Banach spaces in consideration herein.

We write the equations for W_1 and $\tilde{\Theta}$ as follows

$$
\begin{aligned}
W_1(\tau) \;&=\; e^{\tau\mathscr{L}}\widetilde{W_0} - \int_0^\tau e^{(\tau-s)\mathscr{L}}\mathscr{Q}_0[U\cdot\nabla(\gamma_1(0)\,e^{(1-\frac{2}{\alpha})s}G + e^{(2-\frac{3}{\alpha})s}G_1 + W_1(s))]\,ds + \\
&+\; \int_0^\tau e^{-\frac{\tau-s}{\alpha}}\partial_1 e^{(\tau-s)\mathscr{L}}\widetilde{\Theta}(s)ds. \\
\widetilde{\Theta}(\tau) \;&=\; e^{\tau(\mathscr{L}+1-\frac{1}{\alpha})}\widetilde{\Theta_0} - \int_0^\tau e^{(\tau-s)(\mathscr{L}+1-\frac{1}{\alpha})}\mathscr{Q}_0[U\cdot\nabla(\gamma_2(0)\,e^{(2-\frac{3}{\alpha})s}G + \widetilde{\Theta}(s))]\,ds.
\end{aligned}
$$

Note that $U = e^{(1-\frac{2}{\alpha})s}U_G + e^{(2-\frac{3}{\alpha})s}U_{G_1} + U_{W_1}$ and $U_G\cdot G = 0$.

We start the estimates for $\widetilde{\Theta}$. We have

$$
\begin{aligned}
\|\widetilde{\Theta}\|_{L^2(2)} &\leq Ce^{(2-\frac{4}{\alpha}+\delta)\tau}\|\widetilde{\Theta}(0)\|_{L^2(2)} \\
&+ |\gamma_2(0)|\int_0^\tau e^{(2-\frac{3}{\alpha})s}\|e^{(\tau-s)(\mathscr{L}+1-\frac{1}{\alpha})}\mathscr{Q}_0[U\cdot\nabla G]\|_{L^2(2)}\,ds+ \\
&+ \int_0^\tau \|e^{(\tau-s)(\mathscr{L}+1-\frac{1}{\alpha})}\mathscr{Q}_0[U\cdot\nabla\widetilde{\Theta}(s)]\|_{L^2(2)}\,ds =: Ce^{(2-\frac{4}{\alpha}+\delta)\tau}+J_1+J_2
\end{aligned}
$$

We have for all $\delta>0$ small enough, there is C_δ,

$$
\begin{aligned}
J_1 &= \int_0^\tau e^{(2-\frac{3}{\alpha})s}\|e^{(\tau-s)(\mathscr{L}+1-\frac{1}{\alpha})}\mathscr{Q}_0[U\cdot\nabla G]\|_{L^2(2)}\,ds \\
&\lesssim \|U_{G_1}G\|_{L^2(2)}\int_0^\tau \frac{e^{(2-\frac{5}{\alpha}+\delta)(\tau-s)}e^{2(2-\frac{3}{\alpha})s}}{(a(\tau-s))^{\frac{1}{\alpha}}}ds \\
&+ \int_0^\tau \frac{e^{(2-\frac{5}{\alpha}+\delta)(\tau-s)}e^{(2-\frac{3}{\alpha})s}}{(\min(1,|\tau-s|)^{\frac{1}{\alpha}}}\|U_{W_1}(s,\cdot)\cdot\nabla G\|_{L^2(2)}ds \lesssim e^{2(2-\frac{3}{\alpha})\tau}+ \\
&+ \int_0^\tau \frac{e^{(2-\frac{5}{\alpha}+\delta)(\tau-s)}e^{(2-\frac{3}{\alpha})s}}{(\min(1,|\tau-s|)^{\frac{1}{\alpha}}}(e^{(2-\frac{3}{\alpha})s})^{1-\varepsilon}ds \leq C_\delta e^{2(2-\frac{3}{\alpha}-\delta)\tau}.
\end{aligned}
$$

where we have used Lemma 2.1.3, Gagliardo-Nirenberg's, (2.57), $L^2(2)\hookrightarrow L^1$, (2.61), to estimate

$$
\begin{aligned}
\|U_{W_1}\nabla G\|_{L^2(2)} &\leq \|U_{W_1}\|_{L^{\frac{2}{\varepsilon}}}\|(1+|\xi|^2)|\nabla G|\|_{L^{\frac{2}{1-\varepsilon}}} \leq C\|U_{W_1}\|_{L^{\frac{2}{\varepsilon}}} \leq C\|W_1\|_{L^{\frac{2}{1+\varepsilon}}} \\
&\leq C\|W_1\|_{L^2}^{1-\varepsilon}\|W_1\|_{L^1}^{\varepsilon} \leq C(e^{(2-\frac{3}{\alpha})s})^{1-\varepsilon}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J_2 &= \int_0^\tau \|e^{(\tau-s)(\mathscr{L}+1-\frac{1}{\alpha})}\mathscr{Q}_0[U\cdot\nabla\widetilde{\Theta}(s)]\|_{L^2(2)}\,ds \\
&\leq C\int_0^\tau \frac{e^{(2-\frac{5}{\alpha}+\delta)(\tau-s)}}{(\min(1,|\tau-s|)^{\frac{1}{\alpha}}}\|U(s)\|_{L^\infty}\|\widetilde{\Theta}(s)\|_{L^2(2)}ds
\end{aligned}
$$

Thus, we need a good estimate of $\|U(s)\|_{L^\infty}$. We have by (1.8)

$$
\|U(s,\cdot))\|_{L^\infty} \leq C(\|W(s,\cdot)\|_{L^{2+\varepsilon}}+\|W(s,\cdot)\|_{L^{2-\varepsilon}}).
$$

By the *a priori* estimate (2.60), we have a good control of $\|W(s,\cdot)\|_{L^{2+\varepsilon}}$, namely $\|W(s,\cdot)\|_{L^{2+\varepsilon}} \leq Ce^{(2-\frac{3}{\alpha})s}$. For $\|W(s,\cdot)\|_{L^{2-\varepsilon}}$, we can control it by (2.61), but this is not efficient for our arguments - we need some, however small, decay in s, which we can then input in the Gronwall's, (1.13). To achieve that, we proceed by Gagliardo-Nirenberg's estimate. Taking account once again $L^2(2) \hookrightarrow L^1$, and the bounds (2.57),

$$\|W(s,\cdot)\|_{L^{2-\varepsilon}} \leq \|W(s,\cdot)\|_{L^2}^{\frac{2-2\varepsilon}{2-\varepsilon}} \|W(s,\cdot)\|_{L^1}^{\frac{\varepsilon}{2-\varepsilon}} \leq C(e^{(2-\frac{3}{\alpha})s})^{\frac{2-2\varepsilon}{2-\varepsilon}}.$$

All in all, for all $\delta > 0$,

$$\|U(s,\cdot)\|_{L^\infty} \leq C_\delta e^{-(\frac{3}{\alpha}-2-\delta)s}. \tag{2.69}$$

This results in the following estimates for J_2

$$J_2 \leq \int_0^\tau \frac{e^{(2-\frac{5}{\alpha}+\delta)(\tau-s)} e^{-(\frac{3}{\alpha}-2-\delta)s}}{(\min(1,|\tau-s|)^{\frac{1}{\alpha}}} \|\widetilde{\Theta}(s)\|_{L^2(2)} ds$$

Combining all the estimates obtained about[12] $\|\widetilde{\Theta}(s)\|_{L^2(2)}$, , we have

$$\|\widetilde{\Theta}(\tau)\|_{L^2(2)} \leq Ce^{-2(\frac{3}{\alpha}-2-\delta)\tau} + \int_0^\tau \frac{e^{(2-\frac{5}{\alpha}+\delta)(\tau-s)} e^{-(\frac{3}{\alpha}-2-\delta)s}}{(\min(1,|\tau-s|)^{\frac{1}{\alpha}}} \|\widetilde{\Theta}(s)\|_{L^2(2)} ds$$

Applying the Gronwall's, more precisely Lemma 1.2.2, we conclude

$$\|\widetilde{\Theta}(\tau)\|_{L^2(2)} \leq C_\delta e^{-(\frac{3}{\alpha}-2-\delta)\tau},$$

as stated.

[12]note that with our restrictions on α, $(\frac{3}{\alpha}-2) < \frac{4}{\alpha}-2$, so this is the slowest rate on the right hand sides of $\|\widetilde{\Theta}(\tau)\|_{L^2(2)}$.

For W_1, we get

$$
\begin{aligned}
& \|W_1\|_{L^2(2)} \leq Ce^{-(\frac{3}{\alpha}-1-\delta)\tau}\|\widetilde{W_0}\|_{L^2(2)} + \\
+ & \int_0^\tau e^{-\frac{(\tau-s)}{\alpha}}\|\nabla e^{(\tau-s)\mathscr{L}_0}[U\cdot(\gamma_1(0)e^{(1-\frac{2}{\alpha})s}G+e^{(2-\frac{3}{\alpha})s}G_1)+U\cdot W_1]\|_{L^2(2)}ds \\
+ & \int_0^\tau e^{-\frac{(\tau-s)}{\alpha}}\|\partial_1 e^{(\tau-s)\mathscr{L}_0}\widetilde{\Theta}(s)\|_{L^2(2)}ds \lesssim e^{(1-\frac{3}{\alpha}+\delta)\tau} + \\
+ & \int_0^\tau \frac{e^{(1-\frac{3}{\alpha}+\delta)(\tau-s)}e^{(2-\frac{3}{\alpha})s}\|U(|G|+|G_1|)\|_{L^2(2)}}{(a(\tau-s))^{\frac{1}{\alpha}}}\,ds + \\
+ & \int_0^\tau \frac{e^{(1-\frac{3}{\alpha}+\delta)(\tau-s)}\|U\|_{L^\infty}\|W_1\|_{L^2(2)}}{(a(\tau-s))^{\frac{1}{\alpha}}}\,ds \\
+ & \int_0^\tau \frac{e^{(1-\frac{4}{\alpha}+\delta)(\tau-s)}\|\widetilde{\Theta}(s)\|_{L^2(2)}}{(a(\tau-s))^{\frac{1}{\alpha}}}\,ds = e^{(1-\frac{3}{\alpha}+\delta)\tau}+I_1+I_2+I_3
\end{aligned}
$$

For I_1, we have

$$
\|U(|G|+|G_1|)\|_{L^2(2)} \leq \|(e^{(1-\frac{2}{\alpha})s}U_G+e^{(2-\frac{3}{\alpha})s}U_{G_1})(|G|+|G_1|)\|_{L^2(2)}+\|U_{W_1}(|G|+|G_1|)\|_{L^2(2)}.
$$

The first term is easily estimated, since $G, G_1 = \partial_1 G \in L^2(2)$ (whence $U_G, U_{G_1} \in L^\infty$ by Sobolev embedding and Lemma 2.1.3)

$$
\|(e^{(1-\frac{2}{\alpha})s}U_G+e^{(2-\frac{3}{\alpha})s}U_{G_1})(|G|+|G_1|)\|_{L^2(2)} \leq Ce^{(2-\frac{3}{\alpha})s},
$$

whence the contribution of these terms is no more than

$$
C\int_0^\tau \frac{e^{(1-\frac{3}{\alpha}+\delta)(\tau-s)}e^{2(2-\frac{3}{\alpha})s}}{\min(1,|\tau-s|)^{\frac{1}{\alpha}}}ds \leq Ce^{2\tau(2-\frac{3}{\alpha})}.
$$

For U_{W_1} terms, we can use Lemma 2.1.3, the Sobolev inequality and $L^2(2) \hookrightarrow L^{\frac{2}{1+\varepsilon}}$ to get

$$
\begin{aligned}
&\|U_{W_1}(s)(|G|+|G_1|)\|_{L^2(2)} = \|U_{W_1} \cdot (1+|\xi|^2)(|G|+|G_1|)\|_{L^2} \\
&\leq \|U_{W_1}\|_{L^{\frac{2}{\varepsilon}}} \|(1+|\xi|^2)(|G|+|G_1|)\|_{L^{\frac{2}{1-\varepsilon}}} \leq C\|U_{W_1}\|_{L^{\frac{2}{\varepsilon}}} \leq C\|\nabla U_{W_1}\|_{L^{\frac{2}{1+\varepsilon}}} \\
&\leq C\|W_1\|_{L^{\frac{2}{1+\varepsilon}}} \leq C\|W_1(s)\|_{L^2(2)}.
\end{aligned}
$$

All together, the contribution of I_1 is estimated by

$$
I_1 \leq Ce^{-2(\frac{3}{\alpha}-2)\tau} + \int_0^\tau \frac{e^{-(\frac{3}{\alpha}-1-\delta)(\tau-s)}e^{-(\frac{3}{\alpha}-2)s}}{\min(1,|\tau-s|)^{\frac{1}{\alpha}}} \|W_1(s)\|_{L^2(2)} ds
$$

Regarding I_2, we first need an appropriate estimate on $\|U\|_{L^\infty}$, which is fortunately already given by (2.69). This then gives the bound for I_2,

$$
I_2 \leq \int_0^\tau \frac{e^{-(\frac{3}{\alpha}-1-\delta)(\tau-s)}e^{-(\frac{3}{\alpha}-2-\delta)s}}{\min(1,|\tau-s|)^{\frac{1}{\alpha}}} \|W_1(s)\|_{L^2(2)} ds
$$

Combining all estimates for $\|W_1(\tau)\|_{L^2(2)}$ yields

$$
\|W_1(\tau,\cdot)\|_{L^2(2)} \leq Ce^{-2(\frac{3}{\alpha}-2)\tau} + \int_0^\tau \frac{e^{-(\frac{3}{\alpha}-1-\delta)(\tau-s)}e^{-(\frac{3}{\alpha}-2-\delta)s}}{\min(1,|\tau-s|)^{\frac{1}{\alpha}}} \|W_1(s)\|_{L^2(2)} ds.
$$

Applying Lemma 1.2.2, with $\mu = 2(\frac{3}{\alpha}-2), \sigma = (\frac{3}{\alpha}-1-\delta), \kappa = (\frac{3}{\alpha}-2-\delta)$ yields

$$
\|W_1(\tau,\cdot)\|_{L^2(2)} \leq Ce^{-2(\frac{3}{\alpha}-2)\tau}.
$$

This is the statement of (2.64) and Theorem 2.3.11 is proved in full. □

At this point considering the relation $L^2(2) \hookrightarrow L^p, 1 \leq p \leq 2$, the proof of theorem (2.1.2) is just a matter of translating the L^p estimates of W and Θ into the original functions ω and θ.

Proof of theorem (2.1.2). We just simply transfer the estimates in (2.3.11) into the original x and t,

$$
\begin{aligned}
&\|\omega(t,\cdot)-\frac{\gamma_2(0)}{(1+t)^{\frac{3}{\alpha}-1}}\partial_1 G(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}})-\frac{\gamma_1(0)}{(1+t)^{\frac{2}{\alpha}}}G(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}})\|_{L^p}\\
&=\left[\int_{\mathbb{R}^2}\left|\frac{1}{1+t}W(\frac{x}{(1+t)^{\frac{1}{\alpha}}})-\frac{\gamma_2(0)}{(1+t)^{\frac{3}{\alpha}-1}}\partial_1 G(\frac{x}{(1+t)^{\frac{1}{\alpha}}})-\frac{\gamma_1(0)}{(1+t)^{\frac{2}{\alpha}}}G(\frac{x}{(1+t)^{\frac{1}{\alpha}}}))\right|^p dx\right]^{\frac{1}{p}}\\
&=\frac{(1+t)^{\frac{2}{\alpha p}}}{(1+t)}\left[\int_{\mathbb{R}^2}\left|W(\frac{x}{(1+t)^{\frac{1}{\alpha}}})-\frac{\gamma_2(0)}{(1+t)^{\frac{3}{\alpha}-2}}\partial_1 G(\frac{x}{(1+t)^{\frac{1}{\alpha}}})\right.\right.\\
&\left.\left.-\frac{\gamma_1(0)}{(1+t)^{\frac{2}{\alpha}-1}}G(\frac{x}{(1+t)^{\frac{1}{\alpha}}}))\right|^p\frac{dx}{(1+t)^{\frac{2}{\alpha}}}\right]^{\frac{1}{p}}\\
&=\frac{1}{(1+t)^{1-\frac{2}{\alpha p}}}\|W(\cdot)-\frac{\gamma_2(0)}{(1+t)^{\frac{3}{\alpha}-2}}\partial_1 G(\cdot)-\frac{\gamma_1(0)}{(1+t)^{\frac{2}{\alpha}-1}}G(\cdot)\|_{L^p}\\
&\le\frac{1}{(1+t)^{1-\frac{2}{\alpha p}}}\|W(\cdot)-\frac{\gamma_2(0)}{(1+t)^{\frac{3}{\alpha}-2}}\partial_1 G(\cdot)-\frac{\gamma_1(0)}{(1+t)^{\frac{2}{\alpha}-1}}G(\cdot)\|_{L^2(2)}\\
&\le\frac{C_\varepsilon}{(1+t)^{1-\frac{2}{\alpha p}}}e^{-2\tau(\frac{3}{\alpha}-2-\varepsilon)}\le\frac{C_\varepsilon}{(1+t)^{\frac{6}{\alpha}-3-\frac{2}{\alpha p}-\varepsilon}}.
\end{aligned}
$$

The L^p estimate for θ requires similar computations,

$$
\begin{aligned}
&\|\theta(t,\cdot)-\frac{\gamma_2(0)}{(1+t)^{\frac{2}{\alpha}}}G(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}})\|_{L^p}\\
&=\left[\int_{\mathbb{R}^2}\left|\frac{1}{(1+t)^{2-\frac{1}{\alpha}}}\Theta(\frac{x}{(1+t)^{\frac{1}{\alpha}}})-\frac{\gamma_2(0)}{(1+t)^{\frac{2}{\alpha}}}G(\frac{x}{(1+t)^{\frac{1}{\alpha}}})\right|^p dx\right]^{\frac{1}{p}}\\
&=\frac{(1+t)^{\frac{2}{\alpha p}}}{(1+t)^{2-\frac{1}{\alpha}}}\left[\int_{\mathbb{R}^2}\left|\Theta(\frac{x}{(1+t)^{\frac{1}{\alpha}}})-\frac{\gamma_2(0)}{(1+t)^{\frac{2}{\alpha}-2+\frac{1}{\alpha}}}G(\frac{x}{(1+t)^{\frac{1}{\alpha}}})\right|^p\frac{dx}{(1+t)^{\frac{2}{\alpha}}}\right]^{\frac{1}{p}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|\theta(t,\cdot)-\frac{\gamma_2(0)}{(1+t)^{\frac{2}{\alpha}}}G(\frac{\cdot}{(1+t)^{\frac{1}{\alpha}}})\|_{L^p}=\\
&=\frac{(1+t)^{\frac{2}{\alpha p}}}{(1+t)^{2-\frac{1}{\alpha}}}\left[\int_{\mathbb{R}^2}\left|\Theta(\xi)-\gamma_2(0)e^{-\tau(\frac{3}{\alpha}-2)}G(\xi)\right|^p d\xi\right]^{\frac{1}{p}}\\
&=\frac{(1+t)^{\frac{2}{\alpha p}}}{(1+t)^{2-\frac{1}{\alpha}}}\|\Theta(\cdot)-\gamma_2(0)e^{-\tau(\frac{3}{\alpha}-2)}G(\cdot)\|_{L^p}\\
&\leq\frac{(1+t)^{\frac{2}{\alpha p}}}{(1+t)^{2-\frac{1}{\alpha}}}\|\Theta(\cdot)-\gamma_2(0)e^{-\tau(\frac{3}{\alpha}-2)}G(\cdot)\|_{L^2(2)}\\
&\leq\frac{C_\varepsilon}{(1+t)^{1-\frac{1}{\alpha}-\frac{2}{\alpha p}}}\;e^{-\tau(2\frac{3}{\alpha}-2-\varepsilon)}\leq\frac{C_\varepsilon}{(1+t)^{\frac{5}{\alpha}-\frac{2}{\alpha p}-2-\varepsilon}}.
\end{aligned}
$$

□

Chapter 3

Sharp relaxation rates for plane waves of reaction- diffusion systems

It is well-known and classical result that spectrally stable traveling waves of a general reaction-diffusion system in one spatial dimension are asymptotically stable with exponential relaxation rates. In a series of works in the 1990's, [24, 33, 37, 63], the authors have considered plane traveling waves for such systems and they have succeeded in showing asymptotic stability for such objects. Interestingly, the (estimates for the) relaxation rates that they have exhibited, are all algebraic and dimension dependent. It was heuristically argued that as the spectral gap closes in dimensions $n \geq 2$, algebraic rates are the best possible.

In this chapter, we revisit this issue. We rigorously calculate the sharp relaxation rates in L^∞ based spaces, both for the asymptotic phase and the radiation terms. These turn out to be are indeed algebraic, but about twice better than the best ones obtained in these early works, although this can be mostly attributed to the inefficiencies of using Sobolev embeddings to control L^∞ norms by high order L^2 based Sobolev space norms. Finally, we explicitly construct the leading order profiles, both for the phase and the radiation terms. Our approach relies on the method of scaling variables, as introduced in [17, 18] and also developed in the chapter 2, and in fact provides sharp relaxation rates in a class of weighted L^2 spaces as well.

3.1 Introduction

In this chapter, we study the following general reaction-diffusion models

$$\begin{cases} u_t = \Delta u + f(u), & x \in \mathbb{R}^n \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where, $n \geq 2$, $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbf{R}^m$, $m \geq 1$, and $f \in C^4(\mathbb{R}^n, \mathbf{R}^m)$. More precisely, we will be interested in the dynamics of the solutions with initial data close to plane waves, that is the dynamics near plane waves. Existence and stability of such waves in the case $n = 1$ is a classical subject, with a vast literature associated to it.

In order to introduce the problem and some notations, assume that there exist steady states $\phi_\pm \in \mathbf{R}^m$, so that $f(\phi_\pm) = 0$. Next, we assume that $n = 1$ and there exists solutions of (3.1), in the form $u(t,x) = \phi(x - ct)$. That is, ϕ satisfies the one-dimensional profile equation,

$$\phi''(z) + c\phi'(z) + f(\phi(z)) = 0, z \in \mathbb{R}. \tag{3.2}$$

We also assume that $\lim_{z \to \pm\infty} \phi(z) = \phi_\pm$, with exponential rates of convergence, although the exponential rate of convergence can be replaced with a weaker, but nevertheless strong enough algebraic rate. In any case, our standing assumption is that for some $\upsilon > 0$, there is

$$|\phi(z) - \phi_-| \leq Ce^{\upsilon z}, z < 0; \ \ |\phi(z) - \phi_+| \leq Ce^{-\upsilon z}, z > 0$$

Finally, we assume that the localized function $\phi' : \phi' \in H^2(\mathbb{R})$. Another relevant object for the stability theory is the (one-dimensional) linearized operator about the wave, namely

$$L_1 = \partial_{zz} + c\partial_z + Df(\phi), \ \ D(L_1) = H^2(\mathbb{R}).$$

Saying that ϕ is spectrally stable amounts to $\sigma(L_1) \subset \mathbb{C}_- = \{\lambda : \Re\lambda \leq 0\}$. Very often, waves like that enjoy the strong spectral stability property, namely that[1] $\sigma(L_1) \subset \{0\} \cup \{\lambda : \Re\lambda \leq -\delta\}$ for some $\delta > 0$. It is a classical result by now that for the $n = 1$ problem $u_t = u_{xx} + f(u)$ such solutions are asymptotically stable, [27, 48], and in fact they enjoy exponential relaxation rates.

The situation becomes more interesting for the case of plane waves. We now introduce the notion of plane wave solutions. These are in the form $u(t,x) = \phi(\kappa \cdot x - ct)$, where $\kappa \in \mathbb{S}^{n-1}$. It is clear that ϕ satisfies the same one-dimensional profile equation, (3.2). In fact, without loss of generality, we may assume that $\kappa = (1,0,\ldots,0)$ as the problem is rotationally invariant. These solutions ϕ, if they exist, are referred to as *plane waves*. Since all statements we make for traveling plane waves in the form $\phi(x_1 - ct, x_2, \ldots, x_n)$ will be easily translatable for general plane waves of the form $\phi(\kappa \cdot x - ct)$ for arbitrary $\kappa \in \mathbb{S}^{n-1}$, we henceforth concentrate on the case of waves in the form $\phi(z - ct, x_2, \ldots, x_n)$. Passing to the moving frame of reference $x_1 - ct \to z$ renders the equation (3.1) in the form

$$u_t = \Delta u + c\partial_z u + f(u), x \in \mathbb{R}^n. \tag{3.3}$$

To reiterate, going forward, we consider stationary solutions of (3.3), instead of traveling waves for (3.1). This is, as discussed above, an equivalent problem.

The study of the plane waves and their stability has attracted a lot of interest over the last thirty years. The following, very incomplete, list [4, 5, 19, 20, 33, 34, 37, 38, 39, 48, 56, 63], consists of mostly recent references as well as various applications to the sciences.

We have already mentioned about asymptotic stability for these waves, so it is time for some rigorous introductions. More specifically, asymptotic stability in this context means that for any initial data u_0, close to the plane wave ϕ in an appropriate norm, there is an asymptotic phase $\sigma(t,y), x = (z,y)$, so that the radiation term tends to zero, i.e.

$$\lim_{t \to \infty} \|u(t,z,y) - \phi(z - \sigma(t,y))\|_X = 0, \tag{3.4}$$

for some appropriate function space X in the variables $(z,y) \in \mathbb{R} \times \mathbb{R}^{n-1}$. It should be mentioned

[1]Here observe that 0 is automatically in the spectrum as corresponding to a translational invariance or just by virtue of taking ∂_z in the profile equation (3.2).

that the introduction of a (t,y) dependent asymptotic phase σ is absolutely necessary in order for an estimate like (3.4) to hold true. See for example Remark 1.3 in [33].

Regarding specific results about asymptotic stability of plane waves, let us begin by stating that the general question has been resolved, for the generality that we are interested in, in a very satisfactory fashion, in the works [24, 33, 37, 63]. Subsequently, and in a more general context in [19, 20, 38, 39, 56]. For some of these later results, the authors consider degenerate systems appearing in certain combustion and biological applications, where the spectral gap property fails even in one spatial dimension. These works necessitates the introduction of exponentially weighted spaces to effectively create such spectral gap, but this will be outside the scope of this dissertation. We shall instead concentrate on the easier and yet not very well-understood case, where we start with a spectral gap in one spatial dimension, i.e. the setup in [33, 37, 63].

In order to summarize the state of the art, the results in these works establish that as soon as $n \geq 2$, there is an *algebraic in time estimate* for the relaxation rates in various Sobolev norms. This is indeed in sharp contrast with the case of one spatial dimension, where under the same spectral assumptions (see the discussion below Assumption 3.1.1), one can show, see [27, 48], that both the radiation and the phase go to zero at an exponential rate.

3.1.1 Linearized operators

Let us introduce the full linearized operator for the plane wave that arises. Let $u = \phi + w$, then

$$\begin{aligned} w_t &= \Delta(\phi + w) + c\partial_z(\phi + w) + f(\phi + w) \\ &= (\Delta\phi + c\partial_z\phi + f(\phi)) + \Delta w + c\partial_z w + Df(\phi)w + N(\phi, w) \\ &= \Delta w + c\partial_z w + Df(\phi)w + N(\phi, w). \end{aligned}$$

Therefore considering the (3.2), the linearized operator is

$$L = \Delta + c\partial_z + Df(\phi) = L_1 + \Delta_y, \;\; D(L) = H^2(\mathbb{R}^n).$$

Above we defined $L_1 = \partial_{zz} + c\partial_z + Df(\phi)$. Clearly, L is a closed operator. Due to our assumptions, ϕ is a bounded function, whence L is a (non self-adjoint) Schrödinger operator with a drift term. It is a classical fact that for the related one dimensional operator, we have $L_1[\phi'] = 0$, which is obtained by differentiating the profile equation (3.2) in z. This is of course nothing but a manifestation of the fact that the problem is translationally invariant and hence zero is an eigenvalue. As we have alluded to above, the spectral stability of the wave ϕ, as a solution to the one dimensional model (3.3), consist in the fact that $\sigma(L_1) \subset \{z : \Re z \leq 0\}$. Moreover, we shall need to require that in fact its spectrum is a fixed distance $\delta > 0$ away from the marginal axes $\Re z = 0$, except for the translational eigenvalue at zero, which we assume to be simple. More specifically, we make the following standing assumption henceforth.

Assumption 3.1.1. *We assume that there exists $\delta > 0$, so that the spectrum of L_1 in $H^1(\mathbb{R})$ satisfies*

$$\sigma(L_1) \setminus \{0\} \subset \{\lambda : \Re\lambda \leq -\delta\} \tag{3.5}$$

Moreover, the eigenvalue at zero is simple, with an eigenfunction ϕ'.

Having the spectral gap condition (3.5), and under appropriate conditions on f, ϕ, allows one to show that the wave ϕ is asymptotically stable, with exponential decay of the radiation term, with an exponential rate of essentially $e^{-(\delta-\varepsilon)t}$. This goes back to at least the classical works [4, 27]. In the case of plane waves, one has L instead of L_1 as a linearized operator, which destroys the spectral gap property. In fact, since $L = L_1 + \Delta_y$, a direct computation shows that $L[\phi'(z)e^{ik\cdot y}] = -k^2\phi'(z)e^{ik\cdot y} + L_1[\phi']e^{ik\cdot y}$, which since $L_1[\phi'] = 0$, leads to,

$$L[\phi'(z)e^{ik\cdot y}] = -k^2\phi'(z)e^{ik\cdot y},$$

whence it becomes immediately clear that the continuous spectrum of L contains the whole negative real axes. In particular, it touches the imaginary axes at zero, so that the corresponding semigroup e^{tL} has at best polynomial rate of decay. Heuristically, one expects no better from the nonlinear problem, so polynomial in time bounds seem indeed the best possible in (3.4).

This is however an open problem, and one of the goals of this dissertation is to establish this rigorously. In fact, we aim at establishing the optimal decay rates in these asymptotic results. We achieve that by requiring slightly more localized initial perturbations $v_0 := u_0 - \phi$, namely that v_0 resides in an appropriate (power) weighted L^2 space, see Section 3.1.2 below. Before we state our concrete results, let us discuss the setup of the asymptotic stability result. This part follows the work of Kapitula, [33], but note that we introduce weighted spaces for the purposes of our analysis later on.

3.1.2 Setup of the asymptotic profile equations

We start with the Riesz projection for L_1, associated with the isolated and simple eigenvalue at zero. Namely, for a small ε, introduce

$$P_0 u = \frac{1}{2\pi i}\int_{|\lambda|=\varepsilon} (\lambda - L_1)^{-1} d\,\lambda \tag{3.6}$$

As zero is a simple eigenvalue, with an eigenfunction ϕ', it follows by the Riesz representation theorem[2] that for $u \in L^2(\mathbb{R})$, $P_0 u = \langle \psi, u\rangle \phi'$, where $\psi \in H^2(\mathbb{R})$ and in fact $L^*\psi = 0$, with the normalization, $\langle \psi, \phi'\rangle = 1$, see [34]. In addition, we define $Q_0 = Id - P_0$, and both operators commute with L_1. While the operators P_0, Q_0 act upon functions of the first variable only, we may also consider their action on functions, which depend on the remaining variables t, y as well.

Recall the definition (1.3) of weighted spaces $L^2(m)(\mathbb{R}^{n-1})$, or $L^2(m)$ for short, define

$$H^1(m) := \{f : \mathbb{R}^{n-1} \to \mathbb{R} : f, \nabla_y f \in L^2(m)\}.$$

Note that all the spaces in this section are based on functions on $\mathbb{R}^{n-1}$, due to the fact that $y \in \mathbb{R}^{n-1}$. In anticipation of our analysis later, we introduce the spaces $(H^1(m) \cap W^{1,\infty})_y H^1_z$ for functions

[2] In this work, we only use real-valued functions, so the dot product is symmetric $\langle \psi, u\rangle = \langle u, \psi\rangle$

$f(y,z)$, where the norm is taken as follows

$$\|f\|^2_{(H^1(m)\cap W^{1,\infty})_y H^1_z} = \sum_{a,b\in\{0,1\}} [\int_{\mathbb{R}^n} |\nabla_z^a \nabla_y^b f(y,z)|^2 (1+|y|^2)^m dz dy + \sup_{y\in\mathbb{R}^{n-1}} \|\nabla_z^a \nabla_y^b f(z,y)\|^2_{L^2_z}]$$

As is clear from the definition above, we shall adopt the notion that all norms in the z variable shall be always taken first. Introduce the complementary subspaces

$$\mathscr{N} = \{u \in (H^1(m)\cap W^{1,\infty})_y H^1_z : u = P_0 u\}$$
$$\mathscr{R} = \{u \in (H^1(m)\cap W^{1,\infty})_y H^1_z : u = Q_0 u\}.$$

Clearly $(H^1(m)\cap W^{1,\infty})_y H^1_z = \mathscr{N} + \mathscr{R}$, in the sense that every function in the base space[3] $(H^1(m)\cap W^{1,\infty})_y H^1_z$ is uniquely representable as a sum of two functions in $\mathscr{N}$ and $\mathscr{R}$ respectively. We need the following lemma[4]

Lemma 3.1.2. *There exists $\varepsilon_0 > 0$ and a constant C, so that for all $w : \|w\|_{(H^1(m)\cap W^{1,\infty})_y H^1_z} < \varepsilon_0$, one can find unique and small $(v(w), \sigma(w)) \in \mathscr{R} \times H^1(m)\cap W^{1,\infty}$, so that*

$$\|v(w)\|_{(H^1(m)\cap W^{1,\infty})_y H^1_z} + \|\sigma(w)\|_{H^1(m)\cap W^{1,\infty}} < C\varepsilon_0$$

and

$$\phi(z) + w(z,y) = \phi(z - \sigma(y)) + v(z,y). \tag{3.7}$$

The proof of the lemma involves a standard application of the implicit function theorem 1.3.1. Note that we can apply Lemma 3.1.2 and in particular decomposition (3.7) for time dependent perturbations, so long as the smallness condition is satisfied.

[3]Here, we would like to note that our base space is a bit different than the one used by the previous authors, who preferred to use high order Sobolev spaces, which control $L^\infty(\mathbb{R}^n)$.

[4]see Lemma 2.2 in [33] for a similar statement, in high order Sobolev spaces.

Proof. Set up a mapping

$$\mathbb{G}(w;v,\sigma)(z,y)=\phi(z-\sigma(y))+v(z,y)-\phi(z)-w(z,y)$$

We will show first that $\mathbb{G}:(H^1(m)\cap W^{1,\infty})_y H^1_z\times\mathscr{R}\times(H^1(m)\cap W^{1,\infty})\to(H^1(m)\cap W^{1,\infty})_y H^1_z$. This follows easily from the mean value theorem, since

$$\mathbb{G}(w;v,\sigma)(z,y)=-\sigma(y)\int_0^1\phi'(z-\tau\sigma(y))d\tau+v(z,y)-w(z,y),$$

and $\phi'\in H^1(\mathbb{R})$. Clearly $\mathbb{G}(0,0,0)=0$, so by the implicit function theorem, it remains to check that

$$d\mathbb{G}(0,0,0)(\tilde{\sigma},\tilde{v})=-\phi'(z)\tilde{\sigma}+\tilde{v}$$

is an isomorphism on $(H^1(m)\cap W^{1,\infty})_y H^1_z$. To this end, let $h\in(H^1(m)\cap W^{1,\infty})_y H^1_z$ be an arbitrary element and we have to resolve the equation

$$-\phi'(z)\tilde{\sigma}+\tilde{v}=h. \tag{3.8}$$

Clearly, by the properties of $\mathscr{R}$ and $\mathscr{N}$, (3.8) has an unique solution, namely $\tilde{\sigma}(y)=-\langle h(\cdot,y),\psi(\cdot)\rangle$, while $\tilde{v}=Q_0 h\in\mathscr{R}$. Moreover, these mappings are linear and

$$\|\tilde{\sigma}\|_{H^1(m)\cap W^{1,\infty}}\le\|\psi\|_{L^2_z}\|h\|_{(H^1(m)\cap W^{1,\infty})_y L^2_z},$$
$$\|\tilde{v}\|_{H^1(m)\cap W^{1,\infty}H^1_z}\le C\|h\|_{(H^1(m)\cap W^{1,\infty})_y H^1_z}.$$

Thus, the implicit function theorem applies and in a neighborhood of zero, there are unique and small $\sigma(w)\in H^1(m)\cap W^{1,\infty}, v(w)\in\mathscr{R}$, so that $\mathbb{G}(w;v(w),\sigma(w))=0$. Equivalently, (3.7) holds.

□

Using the ansatz provided by (3.7), and as long as $\|w(t,\cdot)\|_{(H^1(m)\cap W^{1,\infty})_y H^1_z}<<1$, the equation

(3.3) is transformed into the following system of equations

$$\begin{cases} v_t = Lv + Q_0 H(\phi_\sigma, v) + Q_0 N_1(\sigma, \nabla_y \cdot \sigma, v) \\ \sigma_t = \Delta_y \sigma + N_2(\sigma, \nabla_y \cdot \sigma, v), \\ v(0) = v_0, \ \ \sigma(0) = \sigma_0 \end{cases} \tag{3.9}$$

where $\phi_\sigma(z) := \phi(z - \sigma(t,y))$ and[5]

$$\begin{aligned} &H(\phi_\sigma, v) = f(v + \phi_\sigma) - f(\phi_\sigma) - Df(\phi_\sigma) =: \frac{1}{2} D^2 f(\phi_\sigma) v^2 + E(v) \\ &N_2(\sigma, \nabla_y \cdot \sigma, v) = K_1(\sigma)(\nabla_y \cdot \sigma)^2 + K_2(\sigma)\Big(\langle \psi, H(\phi_\sigma, v) \rangle + (Df(\phi_\sigma) - Df(\phi))v \rangle \Big) \\ &N_1(\sigma, \nabla_y \cdot \sigma, v) = N_2(\sigma, \nabla_y \cdot \sigma, v)\phi'_\sigma + \big(Df(\phi_\sigma) - Df(\phi)\big)v + (\nabla_y \cdot \sigma)^2 \phi''_\sigma \\ &K_1(\sigma) = -\frac{\langle \psi, \phi''_\sigma \rangle}{\langle \psi, \phi'_\sigma \rangle}, \ \ K_2(\sigma) = \frac{1}{\langle \psi, \phi'_\sigma \rangle}. \end{aligned}$$

The derivation of (3.3) is done in great details in [33], see equations (2.28), (2.29) on p. 261 there. One of the important points, [33], is that with $\|\sigma\|_{L^\infty} << 1$ guaranteed by Lemma 3.1.2, we have that $\langle \psi, \phi'_\sigma \rangle = \langle \psi, \phi' \rangle + \langle \psi, \phi'_\sigma - \phi' \rangle = 1 + O(\sigma)$, whence the denominators in the coefficients $K_j(\sigma), j = 1,2$ are away from zero.

The error term is of the form

$$E(v) = f(v + \phi_\sigma) - f(\phi_\sigma) - Df(\phi_\sigma)v - \frac{1}{2} D^2 f(\phi_\sigma) v^2 = O(v^3), \tag{3.10}$$

under the assumption $f \in C^3(\mathbb{R})$ and ϕ is a bounded function. We provide further concrete estimate on $E(v)$ later on, where we shall need to assume $f \in C^4$, since spatial derivatives on E need to be taken. See the proof of Lemma 3.4.3 below.

[5]Here $D^2 f(\phi_\sigma) v^2$ is a quadratic form and it denotes the action of the Hessian matrix $D^2 f(\phi_\sigma)$ on (v,v). We will use the same convention later on for trilinear forms

3.1.3 Main results

As we have already discussed, we provide the sharp time decay rate for σ and v in (3.9). The following theorems are our main results.

Theorem 3.1.3. *Let $n \geq 2$ and $m > \frac{n}{2}+1$. There exists small $\varepsilon_0 > 0$ and a constant C, so that the stationary solutions of* (3.3) *are asymptotically stable. More precisely, for all $\varepsilon : 0 < \varepsilon < \varepsilon_0$ and for all $u_0 : \|u_0(z,y) - \phi(z)\|_{(H^1(m)\cap W^{1,\infty})_y H^1_z} < \varepsilon$, the solution to* (3.3) *with initial data u_0 is global and there exists $\sigma \in L^\infty(\mathbb{R}, (H^1(m)\cap W^{1,\infty}))$, so that*

$$u(t,z,y) = \phi(z-\sigma(t,y)) + v(t,z,y), \;\; v = Q_0 v \in L^\infty(\mathbb{R}, (H^1(m)\cap W^{1,\infty})_y H^1_z)$$

with

$$\|\sigma(t,\cdot)\|_{L^\infty_y} \leq C\varepsilon(1+t)^{-\frac{n-1}{2}} \tag{3.11}$$

$$\|\nabla_y \sigma(t,\cdot)\|_{L^\infty_y} \leq C\varepsilon(1+t)^{-\frac{n}{2}} \tag{3.12}$$

$$\|v\|_{L^\infty_{y,z}} \leq C\varepsilon(1+t)^{-(n+\frac{1}{2})} \tag{3.13}$$

Remarks:

- The estimates for v can be stated in a more precise form as follows

$$\|v\|_{L^\infty_{y,z}} \leq C(\varepsilon^2(1+t)^{-(n+\frac{1}{2})} + \varepsilon e^{-\frac{\delta}{2}t}),$$

  of which (3.13) is a corollary. In other words, there are two terms in the formula for v - one linear in ε, but decaying exponentially in t (coming from free solutions), while the other decaying at the right power rate, but quadratic in ε, which comes from the Duhamel's term and the nonlinearity respectively.

- The decay estimates in L^∞_{yz} norms (3.11), (3.13) should be compared with the estimates in [63], [33]. As the arguments in these papers require the use of Sobolev embedding into H^k

spaces, it only provides the bound $\|\sigma\|_{L^\infty} \leq C\varepsilon(1+t)^{-\frac{n-1}{4}}$, whereas (3.11) is clearly much better. In fact, (3.11) is sharp, as shown in Theorem 3.1.4 below. The estimate (3.13) for v above is also clearly superior to the one provided in [33].

- We have more estimates for σ, v than the one stated in Theorem 3.1.3. In particular, v, σ belong to weighted L^2 spaces and in fact, one can write estimates as follows - for every $0 \leq \tilde{m} \leq m$,
$$\left(\int_{\mathbb{R}^{n-1}} |\sigma(t,y)|^2 |y|^{2\tilde{m}} dy\right)^{1/2} \leq C\varepsilon(1+t)^{-\frac{1}{2}(\frac{n-1}{2}-\tilde{m})},$$
This estimate gives an algebraic decay for $\tilde{m} < \frac{n-1}{2}$, but they are true even if $\tilde{m}$ is larger, that is the corresponding weighted L^2 norms may be growing in t. In the case $\tilde{m} = 0$, these become the usual L^2 spaces. One can in fact see that the result, in this case exactly matches the L^2 bounds in [33].

- One disadvantage of our method is that one cannot get estimates for $\nabla_y^2 \sigma$ nor $\nabla_y^2 v$ (and higher order derivatives), due to a technical issue that arises in the scaled variable analysis, see the remark after Proposition 3.2.3 below. Such estimates are clearly possible, as was demonstrated in [33]. On the other hand, we believe that this is really a technical issue, which we have not explored further.

The rates established in Theorem 3.1.3 are sharp. Specifically, we have the following result, which we formulate as a separate theorem.

Theorem 3.1.4. *Under the assumptions of Theorem 3.1.3, the estimates* (3.11)*,* (3.12) *and* (3.13) *are sharp. More precisely, let* $u_0 : \|u_0(y,z) - \phi(z)\|_{(H^1(m)\cap W^{1,\infty})_y H^1_z} < \varepsilon$ *and* $\sigma_0 \in H^1(m) \cap W^{1,\infty}$, $v_0 = Q_0 v_0 \in (H^1(m) \cap W^{1,\infty})_y H^1_z$ *be the unique pair guaranteed by Lemma 3.1.2, so that*
$$u_0(y,z) = \phi(z - \sigma_0(y)) + v_0(z,y).$$

Then, we have the following

$$\left\|\sigma(t,\cdot)-\frac{(\int_{\mathbb{R}^{n-1}}\sigma_0(y)dy)}{(1+t)^{\frac{n-1}{2}}}G\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^\infty_y}\le\frac{C\varepsilon^2}{(1+t)^{\frac{n}{2}}}, \tag{3.14}$$

$$\left\|\partial_j\sigma(t,\cdot)-\frac{(\int_{\mathbb{R}^{n-1}}\sigma_0(y)dy)}{(1+t)^{\frac{n}{2}}}(\partial_jG)\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^\infty_y}\le\frac{C\varepsilon^2}{(1+t)^{\frac{n+1}{2}}}, \tag{3.15}$$

where $j=1,\dots,n-1$, $G(y)=(4\pi)^{-\frac{n-1}{2}}e^{-\frac{|y|^2}{4}}$*. In particular, assuming that* $\int_{\mathbb{R}^{n-1}}\sigma_0(y)dy\neq 0$*, we have the asymptotics*

$$\|\sigma(t,\cdot)\|_{L^\infty_y}\simeq\varepsilon(1+t)^{-\frac{n-1}{2}},\ \ \|\nabla\sigma(t,\cdot)\|_{L^\infty_y}\simeq\varepsilon(1+t)^{-\frac{n}{2}}$$

Regarding v*, we have that for*[6] $n\ge 3$*,*

$$\|v(t,z,y)+\frac{(\int_{\mathbb{R}^{n-1}}\sigma_0(y)dy)^2}{(4\pi)^{n-1}}\frac{e^{-\frac{|y|^2}{2(t+1)}}}{(t+1)^{n+\frac{1}{2}}}L_1^{-1}Q_0[\phi''](z)\|_{L^\infty_{z,y}}\le C(\varepsilon^2(1+t)^{-n-1}+\varepsilon e^{-\frac{\delta}{2}t}). \tag{3.16}$$

whereas for $n=2$*,*

$$\begin{aligned}\|v(t,z,y) \quad &+ \quad \frac{(\int_{\mathbb{R}}\sigma_0(y)dy)^2}{4\pi}\frac{e^{-\frac{|y|^2}{2(t+1)}}}{(t+1)^{\frac{5}{2}}}L_1^{-1}Q_0[\phi''](z)\|_{L^\infty_{z,y}}\\ &\le C(\varepsilon^3(1+t)^{-\frac{5}{2}}+\varepsilon^2(1+t)^{-3}+\varepsilon e^{-\frac{\delta}{2}t}).\end{aligned} \tag{3.17}$$

In particular, if $\int_{\mathbb{R}^{n-1}}\sigma_0(y)dy\neq 0$*, we have the asymptotics*

$$\|v(t,\cdot)\|_{L^\infty_{y,z}}\simeq\varepsilon^2(1+t)^{-n-\frac{1}{2}}. \tag{3.18}$$

Remarks:

- The asymptotic expansion for σ improves both in the order of ε and the decay rate - the leading order term is order $\varepsilon(1+t)^{-\frac{n-1}{2}}$, while the error is $\varepsilon^2(1+t)^{-\frac{n}{2}}$. This is due to the

[6] note that L_1 is invertible on $Q_0[L^2_z]$ or $L_1^{-1}Q_0$ is well defined

fact that the leading order term entirely originates from the free solution.

- In contrast, the expansion for v has a main term, which is $\varepsilon^2(1+t)^{-n-\frac{1}{2}}$ and two to three types of error terms - an exponentially decaying in t, but linear in ε (originating from initial data) and faster decaying, but still quadratic in ε terms, originating from various other nonlinear terms. In the case $n=2$, we recover yet another term, which decays like the main term, but it is order of ε smaller. Most importantly, the structure of the error terms guarantees (3.18).

3.2 Preliminary steps

In this section, we transform the evolution equation (3.9) into an equivalent one, through the use of the so-called scaling variables.

3.2.1 The evolution system in scaling variables

Introduce the scaling variables

$$\tau = \ln(1+t), \quad \eta_j = \frac{y_j}{\sqrt{1+t}}, j = 2, \ldots, n.$$

In these independent variables, set the new dependent variables V, Γ as follows

$$v(z,y,t) = \frac{1}{1+t} V\left(z, \frac{y}{\sqrt{1+t}}, \ln(1+t)\right), \quad \sigma(y,t) = \frac{1}{\sqrt{1+t}} \Gamma\left(\frac{y}{\sqrt{1+t}}, \ln(1+t)\right).$$

Straightforward computations show

$$\begin{aligned} v_t &= -\frac{1}{(1+t)^2} V - \frac{1}{2} \frac{1}{(1+t)^2} \frac{y}{\sqrt{1+t}} \cdot \nabla_\eta V + \frac{1}{(1+t)^2} V_\tau, \\ \Delta_y v &= \frac{1}{(1+t)^2} \Delta_\eta V, \end{aligned}$$

$$
\begin{aligned}
L_1 v &= \frac{1}{1+t} L_1 V, \quad H(\phi_\sigma, v) = \frac{1}{2}\frac{1}{(1+t)^2} D^2 f(\phi_{\frac{1}{\sqrt{1+t}}\Gamma}) V^2 + E((1+t)^{-1} V), \\
(\nabla_y \cdot \sigma)^2 \phi''_\sigma &= \frac{1}{(1+t)^2} (\nabla_\eta \cdot \Gamma)^2 \phi''_{\frac{1}{\sqrt{1+t}}\Gamma} \\
N_2(\sigma, \nabla_y \cdot \sigma, v) &= \frac{1}{(1+t)^2} K_1((1+t)^{-1/2}\Gamma)(\nabla_\eta \cdot \Gamma)^2 + \\
&+ \frac{2}{(1+t)^2} K_2((1+t)^{-1/2}\Gamma) D^2 f(\phi_{\frac{1}{\sqrt{1+t}}\Gamma}) \langle \psi, V^2 \rangle \\
&+ K_2((1+t)^{-1/2}\Gamma) \langle \psi, E((1+t)^{-1} V) \rangle + \\
&+ \frac{1}{1+t} K_2((1+t)^{-1/2}\Gamma) \langle \psi, (Df(\phi_{\frac{1}{\sqrt{1+t}}\Gamma}) - Df(\phi)) V \rangle \\
&=: \frac{1}{(1+t)^2} N_2(\Gamma, \nabla_\eta \cdot \Gamma, V) \\
\sigma_t &= -\frac{1}{2}\frac{1}{(1+t)^{\frac{3}{2}}}\Gamma - \frac{1}{2}\frac{1}{(1+t)^{\frac{3}{2}}}\frac{y}{\sqrt{1+t}} \cdot \nabla_\eta \cdot \Gamma + \frac{1}{(1+t)^{\frac{3}{2}}}\Gamma_\tau, \\
\Delta_y \sigma &= \frac{1}{(1+t)^{\frac{3}{2}}} \Delta_\eta \Gamma \\
N_1(\sigma, \nabla_y \cdot \sigma, v) &= \frac{1}{(1+t)^2} N_2(\Gamma, \nabla_\eta \cdot \Gamma, V) \phi'_{\frac{1}{\sqrt{1+t}}\Gamma} + \frac{1}{1+t}\big(Df(\phi_{\frac{1}{(\sqrt{1+t})}\Gamma}) - Df(\phi)\big) V + \\
&+ \frac{1}{(1+t)^2} (\nabla_\eta \cdot \Gamma)^2 \phi''_{\frac{1}{\sqrt{1+t}}\Gamma} =: \frac{1}{(1+t)^2} N_1(\Gamma, \nabla_\eta \cdot \Gamma, V).
\end{aligned}
$$

So, we have introduced a new set of nonlinearities, which in the new variables (τ, η) take the form

$$
\begin{aligned}
H(\Gamma, V) &= \frac{1}{2} D^2 f(\phi_{e^{-\frac{\tau}{2}}\Gamma}) V^2 + e^{2\tau} E(e^{-\tau} V), \\
N_2(\Gamma, \nabla_\eta \cdot \Gamma, V) &= K_1(e^{-\frac{\tau}{2}}\Gamma)(\nabla_\eta \cdot \Gamma)^2 + \frac{1}{2} K_2(e^{-\frac{\tau}{2}}\Gamma)\Big(D^2 f(\phi_{e^{-\frac{\tau}{2}}\Gamma}) \langle V^2, \psi \rangle \\
&+ e^{2\tau} K_2(e^{-\frac{\tau}{2}}\Gamma) \langle \psi, E(e^{-\tau} V) \rangle + 2e^\tau \langle \psi, (Df(\phi_{e^{-\frac{\tau}{2}}\Gamma}) - Df(\phi)) V \rangle \Big), \\
N_1(\Gamma, \nabla_\eta \cdot \Gamma, V) &= N_2(\Gamma, \nabla_\eta \cdot \Gamma, V) \phi'_{e^{-\frac{\tau}{2}}\Gamma} + e^\tau \big(Df(\phi_{e^{-\frac{\tau}{2}}\Gamma}) - Df(\phi)\big) V \\
&+ e^{-\frac{\tau}{2}} (\nabla_\eta \cdot \Gamma)^2 \phi''_{e^{-\frac{\tau}{2}}\Gamma}.
\end{aligned}
$$

Therefore the system (3.9) is transfered into the system

$$\begin{cases} V_\tau = (\mathscr{L}_\eta + \frac{1}{2})V + e^\tau L_1 V + Q_0 H(\Gamma, V) + Q_0 N_1(\Gamma, \nabla_\eta \cdot \Gamma, V) \\ \Gamma_\tau = \mathscr{L}_\eta \Gamma + e^{-\frac{\tau}{2}} N_2(\Gamma, \nabla_\eta \cdot \Gamma, V) \end{cases} \tag{3.19}$$

where H, N_1, N_2 are defined above and the operator $\mathscr{L}_\eta$ is defined as

$$\mathscr{L}_\eta = \Delta_\eta + \frac{1}{2}\eta \cdot \nabla_\eta + \frac{1}{2}. \tag{3.20}$$

We finish this section by stating the variation of constant formula for (3.19). Note that this is slightly non-standard, due to the τ dependence of the linear operator, i.e. the term $e^\tau L_1$, in the equation for V. It should be noted that L_1 generates a C_0 semigroup on the Sobolev space $H^1(\mathbb{R})$ (see Lemma 3.4.1 below), while the operator L_η generates a semigroup, but on specific weighted L^2 based spaces, see Section 3.2.2 below. Thus, since the action in the variable z and the variable η are independent, we may in fact write the system for (V, Γ) as follows

$$\begin{aligned} V &= e^{\tau(\mathscr{L}_\eta + \frac{1}{2})} e^{(e^\tau - 1)L_1} V_0 + \\ &+ \int_0^\tau e^{(\tau - s)(\mathscr{L}_\eta + \frac{1}{2})} e^{(e^\tau - e^s - 1)L_1} [Q_0 H(\Gamma, V) + Q_0 N_1(\Gamma, \nabla_\eta \cdot \Gamma, V)(s)] ds \end{aligned} \tag{3.21}$$

$$\Gamma = e^{\tau \mathscr{L}_\eta} \Gamma_0 + \int_0^\tau e^{(\tau - s)\mathscr{L}_\eta} e^{-\frac{s}{2}} N_2(\Gamma, \nabla_\eta \cdot \Gamma, V)(s) ds, \tag{3.22}$$

where V_0, Γ_0 are the initial data of the variables V and Γ. Note that by the scaling variables assignments, $V_0(z, y) = v_0(z, y), \Gamma_0(y) = \sigma_0(y)$.

It becomes clear by this last formulas that in order to study the long time properties of the system (3.21), (3.22), it will be helpful to know about spectral properties of L_η and estimates of the associated semigroup.

3.2.2 The operator L_η - spectral information and the associated semigroup

For this section, note that the spaces that we introduce are based on $\mathbb{R}^{n-1}$, instead of the usual $\mathbb{R}^n$. This is due to the fact that the scaling variables transformation is performed only in the variables $y \in \mathbb{R}^{n-1}$.

The following results are due to Gallay-Wayne, see Theorem A.1 in [17]. Note however that the operator $\mathscr{L}$ appearing in [17], satisfies $\mathscr{L}_\eta = \mathscr{L} - \frac{N-1}{2}$ and $N = n-1$. Moreover, proposition 2.1.5, proved in chapter 2, presents this proposition in 2 dimension for the operator $\mathscr{L}$ containing fractional derivative, instead of a full Laplacian.

Proposition 3.2.1. *Let $m \geq 0$ and $\mathscr{L}_\eta$ be the linear operator* (3.20) *acting on $L^2(m)$, and $G(\eta) = (4\pi)^{-\frac{n-1}{2}} e^{-\frac{|\eta|^2}{4}}$. Then, its spectrum consists of*[7] $\sigma(\mathscr{L}_\eta) = \sigma_d(\mathscr{L}_\eta) \cup \sigma_c(\mathscr{L}_\eta)$*, where*

1. *The discrete spectrum is*

$$\sigma_d(\mathscr{L}_\eta) = \left\{ \lambda_k \in \mathbb{C} : \lambda_k = -\frac{n+k-2}{2}; k = 0, 1, 2, \cdots \right\}.$$

2. *The essential spectrum is*

$$\sigma_{ess}(\mathscr{L}_\eta) = \left\{ \lambda \in \mathbb{C} : \Re\lambda \leq -\frac{n+5}{4} - \frac{m}{2} \right\}.$$

Moreover, for $m > \frac{n-1}{2}$, the largest element of $\Re\sigma(\mathscr{L}_\eta)$, i.e. the eigenvalue $\lambda_0 = -\frac{n-2}{2}$, is simple, with an eigenfunction G, which satisfies

$$\mathscr{L}_\eta G = \lambda_0 G, \ \ \sigma(\mathscr{L}_\eta) \setminus \{-\frac{n-2}{2}\} \subset \{\lambda : \Re\lambda \leq -\frac{n-1}{2}\}$$

In our next proposition, we discuss the semigroup generation properties.

Proposition 3.2.2. *The operator $\mathscr{L}_\eta$ defines a C_0 semigroup on $L^2(m)(\mathbb{R}^{n-1})$. We have the following formula for its action*

[7] this is a not necessarily disjoint partition, as some eigenvalues are embedded into the continuous spectrum

$$\widehat{(e^{\tau \mathscr{L}_\eta} f)}(\xi) = e^{-\frac{n-2}{2}\tau} e^{-a(\tau)|\xi|^2} \widehat{f}(e^{-\frac{\tau}{2}}\xi), \tag{3.23}$$

$$(e^{\tau \mathscr{L}_\eta} f)(\eta) = \frac{e^{\frac{\tau}{2}}}{\left(4\pi a(\tau)\right)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} G\left(\frac{\eta - \eta'}{2a(\tau)^{\frac{1}{2}}}\right) f(e^{\frac{\tau}{2}}\eta') d\eta', \tag{3.24}$$

where $a(\tau) = 1 - e^{-\tau}$.

The semigroup formulas (3.23) and (3.24) are also taken from [17] (see statement 4, Theorem A.1), with the readjustments due to the different constant and the fact that L_η acts on $n-1$ variables.

Finally, we state some estimates about the action of the semigroup $e^{\tau \mathscr{L}_\eta}$ on $L^2(m)(\mathbb{R}^{n-1})$. A version of these are in fact needed for the determination of the spectrum $\sigma(\mathscr{L}_\eta)$, but they have already been proved in Proposition A.2, [17]. Even though these are well-known, we state them explicitly and provide some calculations for them, as our normalizations are slightly different than [17], which may create an element of confusion.

3.2.3 Spectral projections and estimates for $e^{\tau \mathscr{L}_\eta}$ on $L^2(m)$

Fix $m > \frac{n}{2} + 1$. The spectral projections corresponding to the eigenspaces of $\mathscr{L}_\eta$ can be constructed explicitly, [17], but we will not do so here. Instead, we just construct the one corresponding to the first eigenvalue $\lambda_0(\mathscr{L}_\eta) = -\frac{n-2}{2}$. Recall that its eigenspace is one dimensional, spanned by G. Accordingly, we shall need an eigenvector e_* for the adjoint operator, so that $\mathscr{L}^* e_* = -\frac{n-2}{2} e_*$. But since

$$\mathscr{L}_\eta^* = \Delta_\eta - \frac{1}{2}\eta \cdot \nabla_\eta - \frac{n-2}{2}.$$

So, it is easy to see that $e_* = 1$ is an eigenfunction[8] for $\mathscr{L}_\eta^*$ and since our normalization for G is chosen so that $\langle 1, G\rangle = (4\pi)^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|\eta|^2}{4}} d\eta = 1$, it holds that $e_* = 1$. Thus, we have the

[8] belonging to the dual space $L^2(-m)(\mathbb{R}^{n-1})$

convenient formula

$$\mathscr{P}_0 f(\eta) = \left(\int_{\mathbb{R}^{n-1}} f(\eta')d\eta'\right) G(\eta) = \langle f, 1\rangle_\eta G(\eta)$$

and $\mathscr{Q}_0 = Id - \mathscr{P}_0$.

Proposition 3.2.3. *Let $m > \frac{n+1}{2}$. Then, for all $\alpha \in \mathbb{N}^{n-1}$, there exists $C_\alpha > 0$ such that*

$$\|\nabla^\alpha (e^{\tau \mathscr{L}_\eta} \mathscr{Q}_0 f)\|_{L^2(m)(\mathbb{R}^{n-1})} \leq C_\alpha \frac{e^{-\frac{n-1}{2}\tau}}{a(\tau)^{\frac{|\alpha|}{2}}} \|f\|_{L^2(m)(\mathbb{R}^{n-1})}, \tag{3.25}$$

for all $f \in L^2(m)$ and all $\tau > 0$.

Remark: The appearance of the factors $a(\tau)^{\frac{|\alpha|}{2}}$ in the denominator makes the control of second and higher order derivatives, such as $\nabla^2_\eta \Gamma, \nabla^2_\eta V$, problematic. The reason is that for $0 < \tau < 1$, $a(\tau) \sim \tau$ and we need an integrable in τ functions sitting on the right-hand side of (3.25).

Proof. This proposition is proved in [17], see Proposition A.2, we have just made the adjustments for the constants and the dimension of the space. Note that the exponent $\frac{n-1}{2}$ on the right hand side of the estimate is consistent with the assertion that $\sigma(\mathscr{L}_\eta \mathscr{Q}_0) \subset \{\Re\lambda \leq -\frac{n-1}{2}\}$.

We just copy estimate (92) from Proposition A.2 in [17], and we take into account that $\mathscr{L}_\eta = L - \frac{n-2}{2}$, where the operator L is the semigroup generator in [17]. Thus, we obtain (3.25). □

Finally, we need an estimate of the following type.

Proposition 3.2.4. *Let $m > \frac{n}{2}$ and $a \in \mathbb{N}$. Then,*

$$\|\nabla^a e^{\tau \mathscr{L}_\eta} f\|_{L^\infty(\mathbb{R}^{n-1})} \leq C \frac{e^{-\frac{n-2}{2}\tau}}{a(\tau)^{\frac{a}{2}}} \left(\|f\|_{L^\infty(\mathbb{R}^{n-1})} + \|f\|_{L^2(m)((\mathbb{R}^{n-1}))}\right). \tag{3.26}$$

We get the following improvement, when the semigroup is acting on the co-dimension one subspace $\mathscr{Q}_0[L^2(m)]$ and $m > \frac{n}{2} + 1$,

$$\|\nabla^a e^{\tau \mathscr{L}_\eta} \mathscr{Q}_0 f\|_{L^\infty(\mathbb{R}^{n-1})} \leq C \frac{e^{-\frac{n-1}{2}\tau}}{a(\tau)^{\frac{a}{2}}} \left(\|f\|_{L^\infty(\mathbb{R}^{n-1})} + \|f\|_{L^2(m)((\mathbb{R}^{n-1}))}\right). \tag{3.27}$$

Proof. We divide the proof into the cases of $\tau < 1$ and $\tau \geq 1$. For $\tau < 1$ we use the definition (3.24) in our calculations. Indeed,

$$\begin{aligned}
&\|\nabla^a e^{\tau \mathscr{L}_\eta} f\|_{L^\infty} \leq C \frac{e^{\frac{\tau}{2}}}{(a(\tau))^{\frac{n+a-1}{2}}} \| \int_{\mathbb{R}^{n-1}} \nabla^a G(\frac{\eta - \eta'}{(a(\tau))^{\frac{1}{2}}}) f(e^{\frac{\tau}{2}} \eta') d\eta' \|_{L^\infty} \\
\leq \;& C \frac{\|\nabla^a G(\frac{\cdot}{(a(\tau))^{\frac{1}{2}}})\|_{L^1(\mathbb{R}^{n-1})} \|f(e^{\frac{\tau}{2}} \cdot)\|_{L^\infty(\mathbb{R}^{n-1})}}{(a(\tau))^{\frac{n+a-1}{2}}} \leq \frac{C\|\nabla^a G\|_{L^1(\mathbb{R}^{n-1})} \|f\|_{L^\infty(\mathbb{R}^{n-1})}}{(a(\tau))^{\frac{a}{2}}} \\
\leq \;& \frac{C\|f\|_{L^\infty(\mathbb{R}^{n-1})}}{(a(\tau))^{\frac{a}{2}}}.
\end{aligned}$$

Since for $\tau < 1$, $e^{\frac{n-2}{2}\tau}$ is bounded, we have

$$\|\nabla^a e^{\tau \mathscr{L}_\eta} f\|_{L^\infty} \leq \frac{C e^{-\frac{n-2}{2}\tau}}{(a(\tau))^{\frac{a}{2}}} \|f\|_{L^\infty(\mathbb{R}^{n-1})}. \tag{3.28}$$

We now turn our attention to the case $\tau \geq 1$. We have,

$$\begin{aligned}
&\|\nabla^a e^{\tau \mathscr{L}_\eta} f\|_{L^\infty} \leq C e^{-\frac{n-2}{2}\tau} \|e^{-a(\tau)|\cdot|^2} |\cdot|^a \widehat{f}(e^{-\frac{\tau}{2}} \cdot)\|_{L^1} \\
&= C e^{-\frac{n-2}{2}\tau} \int_{\mathbb{R}^{n-1}} e^{-a(\tau)|\xi|^2} |\xi|^a |\widehat{f}(e^{-\frac{\tau}{2}} \xi)| d\xi \\
&= e^{-\frac{n-2}{2}\tau} e^{\frac{(n+a-1)\tau}{2}} \int_{\mathbb{R}^{n-1}} e^{-a(\tau)|e^{\frac{\tau}{2}} q|^2} |q|^a |\widehat{f}(q)| dq \\
&\leq C e^{\frac{a+1}{2}\tau} \left[\int_{a(\tau)|e^{\frac{\tau}{2}} q|^2 \leq 1} + \sum_{i=1}^{\infty} \int_{i \leq a(\tau)|e^{\frac{\tau}{2}} q|^2 \leq i+1} \right] \left(e^{-a(\tau)|e^{\frac{\tau}{2}} q|^2} |q|^a |\widehat{f}(q)| \right) dq \\
&:= J_1 + J_2.
\end{aligned}$$

Since $|\hat{f}(q)| \leq \|f\|_{L^1} \leq C\|f\|_{L^2(m)}$, because $m > \frac{n}{2}$, we have

$$\begin{aligned}
&e^{-\frac{a+1}{2}\tau} J_1 \leq \int_{a(\tau)|e^{\frac{\tau}{2}} q|^2 \leq 1} e^{-a(\tau)|e^{\frac{\tau}{2}} q|^2} |q|^j |\widehat{f}(q)| dq \leq \|f\|_{L^2(m)} \int_{a(\tau)|e^{\frac{\tau}{2}} q|^2 \leq 1} |q|^a dq \\
&\leq C\|f\|_{L^2(m)} \int_0^{\frac{e^{-\frac{\tau}{2}}}{a(\tau)^{\frac{1}{2}}}} r^{a+n-2} dr \leq C \frac{e^{-\frac{(a+n-1)\tau}{2}}}{a(\tau)^{\frac{a+n-1}{2}}} \|f\|_{L^2(m)} \leq C e^{-\frac{(a+n-1)\tau}{2}} \|f\|_{L^2(m)},
\end{aligned}$$

since for $\tau > 1$, $a(\tau) > \frac{1}{2}$. In other words,

$$J_1 \le Ce^{-\frac{(n-2)}{2}\tau}\|f\|_{L^2(m)}.$$

For J_2 in a similar way, we have

$$\begin{aligned}
e^{-\frac{a+1}{2}\tau}J_2 &\le \|f\|_{L^2(m)}\sum_{i=1}^{\infty}\int_{i\le a(\tau)|e^{\frac{\tau}{2}}q|^2\le i+1} e^{-a(\tau)|e^{\frac{\tau}{2}}q|^2}|q|^a dq \\
&\le C\|f\|_{L^2(m)}\sum_{i=1}^{\infty}e^{-i}\int_{i\frac{e^{-\frac{\tau}{2}}}{a(\tau)^{\frac{1}{2}}}}^{(i+1)\frac{e^{-\frac{\tau}{2}}}{a(\tau)^{\frac{1}{2}}}} r^{a+n-2}dr \\
&\le C\|f\|_{L^2(m)}e^{-\frac{a+n-1}{2}\tau}\sum_{i=1}^{\infty}e^{-i}\left((i+1)^{a+n-1}-i^{a+n-1}\right) \le C\|f\|_{L^2(m)}e^{-\frac{a+n-1}{2}\tau}.
\end{aligned}$$

In other words,

$$J_2 \le Ce^{-\frac{(n-2)}{2}\tau}\|f\|_{L^2(m)}.$$

Therefore for $\tau > 1$ if we put both estimates for J_1 and J_2 together we get

$$\|\nabla^a e^{\tau\mathscr{L}_\eta}f\|_{L^\infty} \le Ce^{-\frac{n-2}{2}\tau}\|f\|_{L^2(m)}. \tag{3.29}$$

The proof of (3.26) is now is complete by putting the estimates (3.28) and (3.29) together. For the estimate (3.27), we use that $\mathscr{Q}_0 f = f - \langle f,1\rangle_\eta G$, so that $\langle \mathscr{Q}_0 f,1\rangle_\eta = \langle f,1\rangle_\eta - \langle f,1\rangle_\eta\langle G,1\rangle_\eta = 0$. So, $\widehat{\mathscr{Q}_0 f}(0) = 0$. Thus, in the estimates above, we have

$$\begin{aligned}
|\widehat{\mathscr{Q}_0 f}(q)| &= |\widehat{\mathscr{Q}_0 f}(q) - \widehat{\mathscr{Q}_0 f}(0)| \le |q|\|\nabla\widehat{\mathscr{Q}_0 f}\|_{L^\infty} \le C|q|\int_{\mathbb{R}^{n-1}}|\eta||\mathscr{Q}_0 f(\eta)|d\eta \\
&\le C|q|\|\mathscr{Q}_0 f\|_{L^2(m)},
\end{aligned} \tag{3.30}$$

where in the last inequality, we needed $m > \frac{n}{2}+1$. In addition,

$$\|\mathscr{Q}_0 f\|_{L^2(m)} \le \|f\|_{L^2(m)} + |\langle f,1\rangle_\eta|\|G\|_{L^2(m)} \le C\|f\|_{L^2(m)}.$$

Plugging these estimates in the argument above, we gain a power of $|q|$, which gains an extra power of $e^{-\frac{\tau}{2}}$ over the estimate (3.26), which is reflected on the right-hand side of (3.27). $\square$

3.3 Long time asymptotics - setup and further reductions

In this section, we study the precise asympotics of the radiation term V and the phase Γ.

3.3.1 Decomposing the evolution along the spectrum of $\mathscr{L}_\eta$

Due to the fairly explicit spectral information available about $\mathscr{L}_\eta$, see Proposition 3.2.1, and the semigroup estimates in Propositions 3.2.3 and 3.2.4, it is beneficial to consider the system (3.21), (3.22) in $L^2(m)$ based spaces. For the estimates to work, we need to take m to be large enough, say $m > \frac{n+1}{2}$. In this space, the operator $\mathscr{L}_\eta$ has at least one isolated eigenvalue $\lambda_0 = -\frac{n-2}{2}$ corresponding to the eigenfunction $G(\eta) = (4\pi)^{-\frac{n-1}{2}} e^{-\frac{|\eta|^2}{4}}$, recall $\eta \in \mathbb{R}^{n-1}$.

For conciseness, we set $\widetilde{f} = \mathscr{Q}_0 f$, that is all functions with a tilde hereafter will denote functions in $\mathscr{Q}_0(L^2(m))$. With this set up, we decompose the solutions of the system of equations (3.19) in the following way,

$$\begin{cases} V(z,\eta,\tau) = \alpha(z,\tau)G(\eta) + \widetilde{V}(z,\eta,\tau), \\ \Gamma(\eta,\tau) = \gamma(\tau)G(\eta) + \widetilde{\Gamma}(\eta,\tau), \end{cases} \tag{3.31}$$

where $\alpha(z,\tau) = \langle V,1\rangle_\eta = \int_{\mathbb{R}^{n-1}} V(z,\eta,\tau)d\eta$ and $\gamma(\tau) = \langle \Gamma,1\rangle_\eta = \int_{\mathbb{R}^{n-1}} \Gamma(\eta,\tau)d\eta$. In order to find the representations of α and γ we make $\langle\cdot,1\rangle$ in (3.19),

$$\begin{cases} \alpha_\tau = \langle V_\tau,1\rangle_\eta = \langle(\mathscr{L}_\eta + \frac{1}{2})V,1\rangle_\eta + e^\tau\langle L_1 V,1\rangle_\eta + \langle Q_0 H(\Gamma,V),1\rangle_\eta \\ \qquad + \langle Q_0 N_1(\Gamma,\nabla_\eta\Gamma,V),1\rangle_\eta \\ \gamma_\tau = \langle\Gamma_\tau,1\rangle_\eta = \langle\mathscr{L}_\eta\Gamma,1\rangle_\eta + e^{-\frac{\tau}{2}}\langle N_2(\Gamma,\nabla_\eta\Gamma,V),1\rangle_\eta. \end{cases}$$

Some of the terms in this system can be simplified. Clearly $\langle L_1 V, 1\rangle_\eta = L_1\alpha(z,\tau)$. Moreover,

$$\begin{aligned}&\langle(\mathscr{L}_\eta + \frac{1}{2})V, 1\rangle_\eta = \langle \Delta V, 1\rangle_\eta + \frac{1}{2}\langle \eta\cdot\nabla_\eta V, 1\rangle_\eta + \langle V, 1\rangle_\eta\\ &= \frac{1}{2}\int \eta\cdot\nabla V\, d\eta + \langle V,1\rangle_\eta = -\frac{n-3}{2}\alpha(z,\tau).\end{aligned}$$

Therefore, we obtain the ODE/PDE system

$$\begin{cases}\alpha_\tau(z,\tau) = -\frac{n-3}{2}\ \alpha(z,\tau) + e^\tau L_1\alpha(z,\tau) + \langle Q_0 H(\Gamma,V),1\rangle_\eta + \langle Q_0 N_1(\Gamma,\nabla_\eta\cdot\Gamma,V),1\rangle_\eta\\ \gamma_\tau = -\frac{n-2}{2}\ \gamma(\tau) + e^{-\frac{\tau}{2}}\langle N_2(\Gamma,\nabla_\eta\cdot\Gamma,V),1\rangle_\eta.\end{cases} \tag{3.32}$$

Recall now that by our construction in (3.9), we had $v = Q_0 v$ or equivalently $P_0 v = 0$. Clearly, such a property transfers to the scaling variables[9], that is $Q_0 V = V, P_0 V = 0$. Consequently,

$$P_0\alpha(\cdot,\tau) = P_0\langle V(\cdot,\eta,\tau),1\rangle_\eta = \langle P_0 V(\cdot,\eta,\tau),1\rangle_\eta = 0$$

or equivalently $\alpha(z,\tau) = Q_0\alpha(\cdot,\tau)$. Thus, the system (3.32), which consists of an ODE and a PDE, has the following integral representation,

$$\begin{aligned}\alpha(z,\tau) &= e^{-\frac{n-3}{2}\tau}e^{(e^\tau-1)L_1}Q_0\alpha(z,0) + \qquad (3.33)\\ &+ \int_0^\tau e^{-\frac{n-3}{2}(\tau-s)}e^{(e^\tau-e^s)L_1}Q_0\Big[\langle H(\Gamma,V),1\rangle_\eta(s) + \langle N_1(\Gamma,\nabla_\eta\cdot\Gamma,V),1\rangle_\eta(s)\Big]ds,\\ \gamma(\tau) &= e^{-\frac{n-2}{2}\tau}\gamma(0) + \int_0^\tau e^{-\frac{n-2}{2}(\tau-s)}e^{-\frac{s}{2}}\langle N_2(\Gamma,\nabla_\eta\cdot\Gamma,V),1\rangle_\eta(s)ds. \qquad (3.34)\end{aligned}$$

We also can find the representation of $\widetilde{V}$ and $\widetilde{\Gamma}$. For that, we project the system of equations (3.19) away from the eigenvector G. That is, we apply $\mathscr{Q}_0$ in (3.19). Note that all operations in the z variable commute with the operations in the η variables, such as $L_1\mathscr{Q}_0 = \mathscr{Q}_0 L_1, \mathscr{Q}_0 Q_0 = Q_0\mathscr{Q}_0$

[9] the operators P_0, Q_0 are acting in the variable z, which is independent on the action in the scaled variable η

and so on. We obtain

$$\begin{aligned}
\widetilde{V}_\tau &= (\mathscr{L}_\eta + \frac{1}{2})\widetilde{V} + e^\tau L_1 \tilde{V} + Q_0[\mathscr{Q}_0 H(\Gamma, V) + \mathscr{Q}_0 N_1(\Gamma, \nabla_\eta \cdot \Gamma, V)], \\
\widetilde{\Gamma}_\tau &= \mathscr{L}_\eta \widetilde{\Gamma} + e^{-\frac{\tau}{2}} \mathscr{Q}_0 N_2(\Gamma, \nabla_\eta \cdot \Gamma, V).
\end{aligned}$$

Note that once again $\widetilde{V}(z,\eta,\tau) = Q_0 \widetilde{V}(z,\eta,\tau)$. The system has the following integral representation,

$$\begin{aligned}
\widetilde{V}(z,\eta,\tau) &= e^{(\mathscr{L}_\eta+\frac{1}{2})\tau} e^{(e^\tau - 1)L_1} Q_0 \widetilde{V}_0 + \qquad (3.35) \\
&+ \int_0^\tau e^{(\mathscr{L}_\eta+\frac{1}{2})(\tau-s)} \mathscr{Q}_0 e^{(e^\tau - e^s)L_1} Q_0 \Big[H(\Gamma,V)(s) + N_1(\Gamma, \nabla_\eta \cdot \Gamma, V)(s)\Big] ds \\
\widetilde{\Gamma}(\eta,\tau) &= e^{\tau \mathscr{L}_\eta} \widetilde{\Gamma}_0 + \int_0^\tau e^{\mathscr{L}_\eta(\tau-s)} \mathscr{Q}_0 e^{-\frac{s}{2}} N_2(\Gamma, \nabla_\eta \cdot \Gamma, V)(s) ds. \qquad (3.36)
\end{aligned}$$

Thus, we have reduced matters to the system (3.33), (3.34), (3.35), (3.36). Our next goal is to show a small data, global regularity result for this system.

3.3.2 The function space

We now introduce a function space X. Of course, the time decay exponents are chosen appropriately so that the argument eventually closes. More specifically,

$$\begin{aligned}
&\|(\alpha,\beta,\widetilde{V},\widetilde{\Gamma})\|_X := \sup_{\tau>0} \Big\{ e^{(n-\frac{1}{2})\tau} \|\alpha(\cdot,\tau)\|_{H^1_z} + e^{\frac{n-2}{2}\tau} |\gamma(\tau)| \Big\} + \\
&+ \sup_{\tau>0} \Big\{ e^{(n-\frac{1}{2})\tau} \|\widetilde{V}\|_{L^2(m)H^1_z} + e^{(n-\frac{1}{2})\tau} \|\widetilde{V}\|_{L^\infty_\eta H^1_z} \Big\} + \\
&+ \sup_{\tau>0} \Big\{ e^{\frac{n-1}{2}\tau} \|\widetilde{\Gamma}\|_{H^1(m)} + e^{\frac{n-1}{2}\tau} \|\widetilde{\Gamma}\|_{L^\infty_\eta} + e^{\frac{n-1}{2}\tau} \|\nabla_\eta \widetilde{\Gamma}\|_{L^\infty_\eta} \Big\}.
\end{aligned}$$

Here, recall the convention $\|f\|_{L^\infty_\eta H^1_z} = \sup_\eta \|f(\cdot,\eta)\|_{H^1_z}$.

3.3.3 Asymptotics in the scaling variables system

The following is the main result, describing the asymptotics of the evolution in the scaling variables. We just note that by the setup in the scaling variables, the initial data in the scaling variables coincides with the initial data in the original variables.

Theorem 3.3.1. *There exists* $\varepsilon_0 > 0$ *and a constant* C_0*, so that for every* $\varepsilon : 0 < \varepsilon < \varepsilon_0$ *and initial data* $(\alpha_0, \gamma_0, \widetilde{V}_0, \widetilde{\Gamma}_0) = (\alpha, \gamma, \widetilde{V}, \widetilde{\Gamma})|_{\tau=0}$ *satisfying*

$$\|\alpha(\cdot,0)\|_{H^1_z} + |\gamma(0)| + \|\widetilde{V}_0\|_{H^1_z H^1(m)} + \|\widetilde{V}_0\|_{L^\infty_\eta H^1_z} + \|\widetilde{\Gamma}_0\|_{H^1(m)} + \|\widetilde{\Gamma}_0\|_{L^\infty_\eta} + \|\nabla_\eta \widetilde{\Gamma}_0\|_{L^\infty_\eta} < \varepsilon, \quad (3.37)$$

the system (3.33)*,* (3.34)*,* (3.35)*,* (3.36) *has an unique solution in the ball* $B_X(0, C_0\varepsilon)$*, with the given initial data. That is, it satisfies*

$$\|\alpha(\cdot,\tau)\|_{H^1_z} \le C_0 \varepsilon e^{-(n-\frac{1}{2})\tau}, |\gamma(\tau)| \le C_0 \varepsilon e^{-\frac{n-2}{2}\tau} \quad (3.38)$$

$$\|\widetilde{V}(\tau,\cdot)\|_{L^2(m) H^1_z \cap L^\infty_\eta H^1_z} \le C_0 \varepsilon e^{-(n-\frac{1}{2})\tau} \quad (3.39)$$

$$\|\widetilde{\Gamma}(\tau,\cdot)\|_{H^1(m) \cap L^\infty_\eta} + \|\nabla_\eta \widetilde{\Gamma}(\tau,\cdot)\|_{L^\infty_\eta} \le C_0 \varepsilon e^{-\frac{n-1}{2}\tau}. \quad (3.40)$$

In particular, taking into account (3.31)*,*

$$\|V(\tau,\cdot)\|_{L^2(m) H^1_z \cap L^\infty_\eta H^1_z} \le C_0 \varepsilon e^{-(n-\frac{1}{2})\tau} \quad (3.41)$$

$$\|\Gamma(\tau,\cdot)\|_{H^1(m) \cap L^\infty_\eta} + \|\nabla_\eta \Gamma(\tau,\cdot)\|_{L^\infty_\eta} \le C_0 \varepsilon e^{-\frac{n-2}{2}\tau}. \quad (3.42)$$

The proof of Theorem 3.3.1 occupies Section 3.4 below. We only mention that as a consequence of it and the relations (3.41), (3.42), we derive the asymptotics of the solutions (v, σ) of the system (3.9). More precisely, taking into account the scaling variables definition, we obtain

$$\|\sigma(t,\cdot)\|_{L^\infty_y} = \frac{1}{\sqrt{1+t}}\|\Gamma(t,\cdot)\|_{L^\infty_y} \le C\varepsilon_0(1+t)^{-\frac{n-1}{2}},$$
$$\|\nabla_y\sigma(t,\cdot)\|_{L^\infty_y} = \frac{1}{1+t}\|\Gamma(t,\cdot)\|_{L^\infty_y} \le C\varepsilon_0(1+t)^{-\frac{n}{2}}$$
$$\|v\|_{L^\infty_{\eta,z}} \le \|v\|_{L^\infty_\eta H^1_z} = \frac{1}{1+t}\|V(\tau,\cdot)\|_{L^\infty_\eta H^1_z} \le C\varepsilon_0(1+t)^{-(n+\frac{1}{2})}$$

These are precisely the claims in (3.11), (3.12) and (3.13).

3.4 Long time asymptotics - Proof of Theorem 3.3.1

We perform a fixed point argument in a sufficiently small ball of X. To that end, we view the question for solvability as a fixed point problem in the schematic form

$$(\alpha,\gamma,\widetilde{V},\widetilde{\Gamma}) = \text{free solutions} + \Phi(\alpha,\gamma,\widetilde{V},\widetilde{\Gamma}),$$

where Φ is defined as the Duhamel terms in the right-hand sides of (3.33), (3.34), (3.35), (3.36). The existence and uniqueness of the fixed point will be established, once we can show that there exists a sufficiently small $\varepsilon > 0$ and a C (depending on parameters, but not on ε), so that whenever initial data satisfies (3.37), we have

- $$\|\text{free solutions}\|_X \le C\varepsilon, \tag{3.43}$$
- For all $(\alpha,\gamma,\widetilde{V},\widetilde{\Gamma}) \in X : \|(\alpha,\gamma,\widetilde{V},\widetilde{\Gamma})\|_X \le \varepsilon$, there is
$$\|\Phi(\alpha,\gamma,\widetilde{V},\widetilde{\Gamma})\|_X \le C\varepsilon^2. \tag{3.44}$$
- For all $(\alpha_j,\gamma_j,\widetilde{V}_j,\widetilde{\Gamma}_j) : \|(\alpha_j,\gamma_j,\widetilde{V}_j,\widetilde{\Gamma}_j)\|_X \le \varepsilon, j = 1,2$, there is

$$\|\Phi(\alpha_1,\gamma_1,\widetilde{V}_1,\widetilde{\Gamma}_1)-\Phi(\alpha_2,\gamma_2,\widetilde{V}_2,\widetilde{\Gamma}_2)\|_X \leq C\varepsilon\|(\alpha_1,\gamma_1,\widetilde{V}_1,\widetilde{\Gamma}_1)-(\alpha_2,\gamma_2,\widetilde{V}_2,\widetilde{\Gamma}_2)\|_X. \quad (3.45)$$

Due to the multilinear structure of the functional Φ, we can concentrate on (3.44), identical approach will yield (3.45). We start with the free solutions, as these only involve the mapping properties of the semigroups $e^{\tau\mathscr{L}_\eta}$ and e^{sL_1}.

Lemma 3.4.1. *The operator L_1 generates a semigroup on $H^1(\mathbb{R})$. In fact, under the Assumption 3.1.1, for all $\delta_1 < \delta$, there is a constant $C = C_{\delta_1}$,*

$$\|e^{sL_1}Q_0 f\|_{H^1(\mathbb{R})} \leq C_{\delta_1}e^{-\delta_1 s}\|f\|_{H^1(\mathbb{R})}. \quad (3.46)$$

In the applications, we will use $\delta_1 := \frac{\delta}{2}$.

The proof of Lemma 3.4.1 involves the spectral gap property assumption. It is done by combining appropriate resolvent estimates and the Gearheart-Prüss theorem.

Proof. The proof of the bound (3.46) follows from the Gearheart-Prüss theorem in the following way. Since, by our assumption (3.5) the spectrum is to the left of any vertical line in the complex plane $\{z : \Re z = -\delta_1\}$, $0 < \delta_1 < \delta$, it will suffice to show that for a fixed such δ_1,

$$\sup_{\mu\in\mathbb{R}} \|(L_1+\delta_1+i\mu)^{-1}\|_{H^1(\mathbb{R})\to H^1(\mathbb{R})} = C_{\delta_1} < \infty. \quad (3.47)$$

Indeed, the Gearheart-Prüss theorem guarantees that if $\sigma(L_1) \subset \{z : \Re z < -\delta_1\}$ and (3.47) holds, then the operator $L_1+\delta_1$ generates a semigroup with strictly negative growth bound, that is - there exists $\varepsilon > 0$, so that $\|e^{s(L_1+\delta_1)}\|_{H^1(\mathbb{R})\to H^1(\mathbb{R})} \leq C_{\delta_1}e^{-\varepsilon s}$ or, equivalently

$$\|e^{sL_1}\|_{H^1(\mathbb{R})\to H^1(\mathbb{R})} \leq C_{\delta_1}e^{-s(\varepsilon+\delta_1)} \leq C_{\delta_1}e^{-s\delta_1},$$

which is (3.46).

Thus, it suffices to establish (3.47). To this end, fix δ_1 and observe that since the resolvent $(L_1+z)^{-1}$ is analytic $B(H^1(\mathbb{R}))$ valued function on $\{z : \Re z > -\delta\}$, it is continuous in the same region and in particular, for each N, there is C_N,

$$\sup_{\mu\in\mathbb{R}:|\mu|<N} \|(L_1+\delta_1+i\mu)^{-1}\|_{H^1(\mathbb{R})\to H^1(\mathbb{R})} = C_{\delta_1,N} < \infty$$

Thus, the real issue is to establish the bounds in (3.47) for all large enough μ. So, we setup $g \in H^1(\mathbb{R})$ and $f = (L_1+\delta_1+i\mu)^{-1}g$ or equivalently

$$f'' + cf' + Wf + \delta_1 f + i\mu f = g, \tag{3.48}$$

where $W = Df(\phi)$ is a bounded, real-valued potential.

The existence of such an $f \in H^1(\mathbb{R})$ is not in any doubt, by the spectral assumptions, we just need *a posteriori* uniform in μ estimates for it, for all large enough μ. We take a dot product of (3.48) with f. Taking imaginary parts of the said dot product leads to the identity

$$\mu\|f\|^2 + c\Im\langle f', f\rangle = \Im\langle g, f\rangle.$$

Applying the Cauchy-Schwartz inequality and after some algebraic manipulations, we obtain that for every $\varepsilon > 0$, there is C_ε, so that

$$\mu\|f\|^2 \le \frac{\mu}{2}\|f\|^2 + \frac{C}{\mu}(\|f'\|^2 + \|g\|^2).$$

So, we get the *a posteriori* estimate

$$\|f\|^2 \le \frac{C}{\mu^2}(\|f'\|^2 + \|g\|^2). \tag{3.49}$$

We now take the real-part of the dot produc of (3.48) with f. We similarly obtain for every $\varepsilon > 0$,

$$\|f'\|^2 \leq \varepsilon \|f'\|^2 + D_\varepsilon[\|f\|^2 + \|g\|^2].$$

Plugging in (3.49) into this last inequality yields

$$\|f'\|^2 \leq \varepsilon \|f'\|^2 + \frac{M_\varepsilon}{\mu^2}(\|f'\|^2 + \|g\|^2) + D_\varepsilon \|g\|^2.$$

Selecting $\varepsilon = \frac{1}{4}$ and then μ so large so that $\frac{M_\varepsilon}{\mu^2} < \frac{1}{4}$, we arrive at

$$\|f'\|^2 \leq D\|g\|^2.$$

Combining the last estimate with (3.49) yields the desired, uniform in μ estimate (3.47). □

Using the positivity properties of the function G, we have the following

Lemma 3.4.2. *Let $1 \leq p \leq \infty$, then there is the pointwise inequality*

$$\|e^{\tau \mathscr{L}_\eta} f(\cdot, \eta)\|_{L^p_z(\mathbb{R})} \leq e^{\tau \mathscr{L}_\eta} \|f(\cdot, \eta)\|_{L^p_z(\mathbb{R})} \tag{3.50}$$

Proof. Based on the semigroup definition of (3.24), and considering the fact that $G(\cdot)$ is a positive function of the variable η,

$$\begin{aligned} \|e^{\tau \mathscr{L}_\eta} f(\cdot, \eta)\|_{L^p_z(\mathbb{R})} &= \frac{e^{\frac{\tau}{2}}}{(4\pi a(\tau))^{\frac{n-1}{2}}} \left\| \int_{\mathbb{R}^{n-1}} G\Big(\frac{\eta - \eta'}{2(a(\tau))^{\frac{1}{2}}}\Big) f(\cdot, e^{\frac{\tau}{2}}\eta') d\eta' \right\|_{L^p_z(\mathbb{R})} \leq \\ \leq \ \frac{e^{\frac{\tau}{2}}}{(4\pi a(\tau))^{\frac{n-1}{2}}} &\int_{\mathbb{R}^{n-1}} G\Big(\frac{\eta - \eta'}{2(a(\tau))^{\frac{1}{2}}}\Big) \|f(\cdot, e^{\frac{\tau}{2}}\eta')\|_{L^p_z(\mathbb{R})} d\eta' = e^{\tau \mathscr{L}_\eta} \|f(\cdot, \eta)\|_{L^p_z(\mathbb{R})} \end{aligned}$$

□

3.4.1 Control of the free solutions

For the free solution term of α, we have by (3.46), with

$$e^{-\frac{n-3}{2}\tau}\|e^{(e^\tau-1)L_1}Q_0\alpha(z,0)\|_{H^1_z} \le Ce^{-\frac{n-3}{2}\tau}e^{-\frac{\delta}{2}e^\tau}\|\alpha(z,0)\|_{H^1_z} \le C\varepsilon e^{-(n-\frac{1}{2})\tau},$$

where we gave up an exponential decay in e^τ. For the free solution term of γ, we clearly have $e^{-\frac{n-2}{2}\tau}|\gamma(0)| \le \varepsilon e^{-\frac{n-2}{2}\tau}$.

For the free solution of $\tilde{V}$, we need to control two terms. We have by (3.25) and (3.46)

$$\|e^{(\mathscr{L}_\eta+\frac{1}{2})\tau}e^{(e^\tau-1)L_1}Q_0\widetilde{V}_0\|_{L^2(m)H^1_z} \le Ce^{-\frac{\delta}{2}e^\tau}e^{-\frac{n-2}{2}\tau}\|\widetilde{V}_0\|_{L^2(m)H^1_z} \le C\varepsilon e^{-(n-\frac{1}{2})\tau},$$

where we gave up an exponential decay in e^τ as well. For the other free solution term of $\tilde{V}$, we have by (3.50), (3.46) and (3.25)

$$\begin{aligned}
\|e^{(\mathscr{L}_\eta+\frac{1}{2})\tau}e^{(e^\tau-1)L_1}Q_0\widetilde{V}_0\|_{L^\infty_\eta H^1_z} &\le C\|e^{(\mathscr{L}_\eta+\frac{1}{2})\tau}\|e^{e^\tau L_1}Q_0\widetilde{V}_0\|_{H^1_z}\|_{L^\infty_\eta} \le \\
&\le Ce^{-\frac{(n-2)}{2}\tau}e^{-\frac{\delta}{2}e^\tau}(\|\widetilde{V}_0\|_{L^\infty_\eta H^1_z}+\|\widetilde{V}_0\|_{L^2(m)H^1_z}) \le C\varepsilon e^{-(n-\frac{1}{2})\tau}.
\end{aligned}$$

For the free solution of the $\tilde{\Gamma}$, we have by (3.25) and (3.27),

$$\|e^{\tau\mathscr{L}_\eta}\tilde{\Gamma}_0\|_{L^\infty_\eta\cap L^2(m)} \le Ce^{-\frac{n-1}{2}\tau}\|\tilde{\Gamma}_0\|_{L^\infty_\eta\cap L^2(m)}.$$

For the terms $\|\nabla_\eta e^{\tau\mathscr{L}_\eta}\tilde{\Gamma}_0\|_{L^\infty_\eta\cap L^2(m)}$, we split our considerations in two cases, $\tau<1, \tau\ge 1$. We consider the case $\tau<1$ first. By a formula equivalent to (3.24)

$$\begin{aligned}
\|\nabla_\eta e^{\tau\mathscr{L}_\eta}\tilde{\Gamma}_0\|_{L^\infty_\eta\cap L^2(m)} &\le \frac{C}{\left(a(\tau)\right)^{\frac{n-1}{2}}}\|\int_{\mathbb{R}^{n-1}}G\left(\frac{\eta'}{2a(\tau)^{\frac{1}{2}}}\right)\nabla_\eta\tilde{\Gamma}_0(e^{\frac{\tau}{2}}(\eta-\eta'))d\eta'\|_{L^\infty_\eta\cap L^2(m)} \\
&\le C\|\nabla_\eta\tilde{\Gamma}_0\|_{L^\infty_\eta\cap L^2(m)} \le C\varepsilon e^{-\frac{n-1}{2}\tau}.
\end{aligned}$$

since $e^{\frac{n-1}{2}\tau}$ is bounded for $0 < \tau \leq 1$. Finally for $\tau > 1$, we have that $a(\tau) \geq \frac{1}{2}$, so we conclude from (3.27)

$$\|\nabla_\eta e^{\tau \mathscr{L}_\eta} \tilde{\Gamma}_0\|_{L^\infty_\eta} \leq C e^{-\frac{n-1}{2}\tau} \|\tilde{\Gamma}_0\|_{L^2(m)} \leq C\varepsilon e^{-\frac{n-1}{2}\tau}$$

This completes the cases of the free solutions.

Below, we shall use the semigroup estimates on the Duhamel terms in the same way we have used them on the free solutions. This will bring about certain norms on the nonlinear terms, so we need to prepare these estimates.

3.4.2 Estimates on the nonlinear terms $H(\Gamma, V)$, $N_1(\Gamma, \nabla_\eta \Gamma, V)$ and $N_2(\Gamma, \nabla_\eta \Gamma, V)$

We first note that due to (3.31), we have the following estimates

$$\begin{aligned} \|V\|_{L^2(m)H^1_z} + \|V\|_{L^\infty H^1_z} &\leq \|\alpha(s,\cdot)\|_{H^1_z}(\|G\|_{L^\infty_\eta} + \|G\|_{L^2(m)}) \\ &+ \|\tilde{V}(s,\cdot)\|_{L^2(m)H^1_z} + \|\tilde{V}(s,\cdot)\|_{L^\infty_\eta H^1_z}, \\ \|\Gamma\|_{H^1(m)} + \|\Gamma\|_{W^{1,\infty}_\eta} &\leq |\gamma(s)|(\|G\|_{H^1(m)} + \|G\|_{W^{1,\infty}_\eta}) + \|\tilde{\Gamma}(s,\cdot)\|_{H^1(m)} + \|\tilde{\Gamma}(s,\cdot)\|_{W^{1,\infty}_\eta}. \end{aligned}$$

Thus, if $(\alpha, \gamma, \widetilde{V}, \widetilde{\Gamma}) \in X : \|(\alpha, \gamma, \widetilde{V}, \widetilde{\Gamma})\|_X < \varepsilon$, we conclude that the corresponding (V, Γ), given by (3.31) satisfy

$$\|V(s,\cdot)\|_{L^2(m)H^1_z} + \|V(s,\cdot)\|_{L^\infty_\eta H^1_z} \leq C\varepsilon e^{-(n-\frac{1}{2})s}, \tag{3.51}$$

$$\|\Gamma(s,\cdot)\|_{H^1(m)} + \|\Gamma(s,\cdot)\|_{W^{1,\infty}_\eta} \leq C\varepsilon e^{-\frac{n-2}{2}s}, \tag{3.52}$$

With that in mind, we present the following lemma.

Lemma 3.4.3. *Let* (V, Γ) *be as in* (3.31) *and* $(\alpha, \gamma, \widetilde{V}, \widetilde{\Gamma}) \in X : \|(\alpha, \gamma, \widetilde{V}, \widetilde{\Gamma})\|_X < \varepsilon$. *Then, the* *nonlinearities $H(\Gamma,V)$, $N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)$ and $N_2(\Gamma,\nabla_\eta\cdot\Gamma,V)$ obey the following bounds*

$$\|H(\Gamma,V)(s)\|_{L^2_\eta(m)H^1_z} \le C\varepsilon^2 e^{-(2n-1)s}. \tag{3.53}$$

$$\|N_2(\Gamma,\nabla_\eta\cdot\Gamma,V)\|_{L^2(m)} + \|N_2(\Gamma,\nabla_\eta\cdot\Gamma,V)\|_{L^\infty_\eta} \le C\varepsilon^2 e^{-(n-2)s}, \tag{3.54}$$

$$\|Q_0N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)\|_{L^2(m)H^1_z} + \|Q_0N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)\|_{L^\infty_\eta H^1_z} \le C\varepsilon^2 e^{-(n-\frac{3}{2})s} \tag{3.55}$$

Remark: Note that the spectral projections $Q_0, \mathscr{Q}_0$ appear in front of all nonlinearities displayed above. In almost all cases, that is for (3.53) and (3.54), this does not make a difference in the bounds (i.e. the exponents on the right-hand side). The appearance of Q_0 in (3.55) though makes a difference (and even then, for only one term). Nevertheless, the estimate (3.55) without Q_0 holds with the weaker exponent $e^{-(n-2)s}$ on the right-hand side.

Proof. Note that by Sobolev embedding, we have the *a priori* bound on $\|V\|_{L^\infty}$ as follows

$$\|V(s)\|_{L^\infty_{z,\eta}} \le C\|V(s,\cdot)\|_{L^\infty_\eta H^1_z} \le C\varepsilon e^{-(n-\frac{1}{2})s}. \tag{3.56}$$

We start with the estimate for $H(\Gamma,V) = \frac{1}{2}D^2f(\phi_{e^{-\frac{s}{2}}\Gamma})V^2 + e^{2s}E(e^{-s}V)$. We have the pointwise bound

$$|\partial_z[D^2f(\phi_{e^{-\frac{s}{2}}\Gamma})V^2]| \le C[|D^3f(\phi_{e^{-\frac{s}{2}}\Gamma})||\phi'||V|^2 + |D^2f(\phi_{e^{-\frac{s}{2}}\Gamma})||V||\partial_zV|].$$

Due to the Taylor's remainder formula, we can represent the error term as follows

$$e^{2s}E(e^{-s}V) = \frac{e^{-s}}{6}\int_0^1 D^3f(\phi_{e^{-\frac{s}{2}}\Gamma} + pe^{-s}V)V^3(1-p)^3dp,$$

whence by taking into account that $f \in C^4$ and ϕ,ϕ',V are bounded functions, we have the pointwise bound

$$|\partial_z e^{2s}E(e^{-s}V)| \le Ce^{-s}[|\partial_zV||V|^2 + |V|^3|\phi'_{e^{-\frac{s}{2}}\Gamma}| + |\partial_zV||V|^3e^{-s}]. \tag{3.57}$$

Altogether, we get the pointwise bounds

$$|H[\Gamma,V]|+|\partial_z[H[\Gamma,V]|\leq C[|V|^2+|V||\partial_z V|].$$

So, by (3.56) and (3.51), we conclude

$$\|H(\Gamma,V)(s)\|_{L^2_\eta(m)H^1_z}\leq C\|V\|_{L^\infty_{z,\eta}}[\|V\|_{L^2(m)L^2_z}+\|\partial_z V\|_{L^2(m)L^2_z}]\leq C\varepsilon^2 e^{-(2n-1)s}.$$

Next, we deal with $N_2(\Gamma,\nabla_\eta\Gamma,V)$. Recall

$$\begin{aligned}&N_2(\Gamma,\nabla_\eta\cdot\Gamma,V)=K_1(e^{-\frac{s}{2}}\Gamma)(\nabla_\eta\cdot\Gamma)^2+\frac{1}{2}K_2(e^{-\frac{s}{2}}\Gamma)D^2f(\phi_{e^{-\frac{s}{2}}\Gamma})\langle V^2,\psi\rangle\\&+\frac{1}{2}K_2(e^{-\frac{s}{2}}\Gamma)\left(e^{2s}\langle\psi,E(e^{-s}V)\rangle+2e^s\langle\psi,(Df(\phi_{e^{-\frac{s}{2}}\Gamma})-Df(\phi))V\rangle\right).\end{aligned}$$

Before we get on with N_2, recall that $|K_1(\sigma)|=O(1),|K_2(\sigma)|=O(1)$. Thus, $|K_1(e^{-\frac{s}{2}}\Gamma)(\nabla_\eta\cdot\Gamma)^2|\leq C|\nabla_\eta\Gamma|^2$. We have by (3.52),

$$\|K_1(e^{-\frac{s}{2}}\Gamma)(\nabla_\eta\cdot\Gamma)^2\|_{L^2(m)}\leq C\|\nabla_\eta\cdot\Gamma\|_{L^2(m)}\|\nabla_\eta\cdot\Gamma\|_{L^\infty_\eta}\leq C\varepsilon^2e^{-(n-2)s}$$

Regarding the other terms, we estimate away the term $K_2(e^{-\frac{s}{2}})$ by a constant and

$$\begin{aligned}&\|D^2f(\phi_{e^{-\frac{s}{2}}\Gamma})\langle V^2,\psi\rangle\|_{L^2(m)}+2e^s\|\langle\psi,(Df(\phi_{e^{-\frac{s}{2}}\Gamma})-Df(\phi))V\rangle\|_{L^2(m)}+\\+\quad&e^{2s}\|\langle\psi,E(e^{-s}V)\rangle\|_{L^2(m)}\leq C\|V\|_{L^2_zL^2(m)}\|V\|_{L^\infty_{\eta,z}}+Ce^{\frac{s}{2}}\|V\|_{L^\infty_\eta L^2_z}\|\Gamma\|_{L^2(m)}+\\+\quad&Ce^{-s}\|V\|^2_{L^\infty_{\eta,z}}\|V\|_{L^2_zL^2(m)}\leq C\varepsilon^2e^{-\frac{3n-4}{2}s}\leq C\varepsilon^2e^{-(n-2)s}.\end{aligned}$$

For the estimate of $\|N_2(\Gamma,\nabla_\eta\Gamma,V)\|_{L^\infty_\eta}$, we have

$$\|K_1(e^{-\frac{s}{2}}\Gamma)(\nabla_\eta\cdot\Gamma)^2\|_{L^\infty_\eta}\leq C\|\nabla_\eta\Gamma\|^2_{L^\infty_\eta}\leq C\varepsilon^2e^{-(n-2)s}.$$

For the other terms

$$\begin{aligned}
&\|D^2 f(\phi_{e^{-\frac{s}{2}}\Gamma})\langle V^2,\psi\rangle\|_{L^\infty_\eta} + 2e^s\|\langle\psi,(Df(\phi_{e^{-\frac{s}{2}}\Gamma}) - Df(\phi))V\rangle\|_{L^\infty_\eta} + \\
+ \quad & e^{2s}\|\langle\psi,E(e^{-s}V)\rangle\|_{L^\infty_\eta} \leq C\|V\|_{L^\infty_{\eta,z}}\|V\|_{L^\infty_\eta L^2_z} + Ce^{\frac{s}{2}}\|V\|_{L^\infty_\eta L^2_z}\|\Gamma\|_{L^\infty_\eta} \\
+ \quad & Ce^{-s}\|V\|^2_{L^\infty_{\eta,z}}\|V\|_{L^\infty_\eta L^2_z} \leq C\varepsilon^2 e^{-\frac{3n-4}{2}s} \leq C\varepsilon^2 e^{-(n-2)s}.
\end{aligned}$$

This completes the analysis of $N_2(\Gamma,\nabla_\eta\Gamma,V)$ and (3.54) is established.

Finally, we discuss the proof of (3.55), that is the control of the N_1 term in the relevant norms. Recall

$$\begin{aligned}
Q_0 N_1(\Gamma,\nabla_\eta\cdot\Gamma,V) \quad &= \quad N_2(\Gamma,\nabla_\eta\cdot\Gamma,V)Q_0[\phi'_{e^{-\frac{s}{2}}\Gamma}] \\
&+ \quad Q_0[e^s\big(Df(\phi_{e^{-\frac{s}{2}}\Gamma}) - Df(\phi)\big)V + e^{-\frac{s}{2}}(\nabla_\eta\cdot\Gamma)^2\phi''_{e^{-\frac{s}{2}}\Gamma}].
\end{aligned}$$

For the first term, note that since $Q_0[\phi'] = 0$ and (3.52),

$$\|Q_0[\phi'_{e^{-\frac{s}{2}}\Gamma}]\|_{H^1_z} = \|Q_0[\phi'_{e^{-\frac{s}{2}}\Gamma} - \phi']\|_{H^1_z} \leq Ce^{-\frac{s}{2}}\|\Gamma\|_{L^\infty} \leq C\varepsilon e^{-\frac{n-1}{2}s}.$$

We thus easily have by (3.54),

$$\begin{aligned}
&\|N_2(\Gamma,\nabla_\eta\cdot\Gamma,V)Q_0[\phi'_{e^{-\frac{s}{2}}\Gamma}]\|_{L^2(m)H^1_z\cap L^\infty_\eta H^1_z} \leq \\
&\leq C\|N_2(\Gamma,\nabla_\eta\cdot\Gamma,V)\|_{L^2(m)\cap L^\infty_\eta}\|Q_0[\phi'_{e^{-\frac{s}{2}}\Gamma}]\|_{H^1_z} \leq C\varepsilon^3 e^{-\frac{3n-5}{2}s}.
\end{aligned}$$

For the next term, we use the boundedness of Q_0 in the function spaces that we use, to conclude

$$\begin{aligned}
&\|e^s\big(Df(\phi_{e^{-\frac{s}{2}}\Gamma}) - Df(\phi)\big)V\|_{L^2(m)H^1_z\cap L^\infty_\eta H^1_z} \leq \\
&\leq Ce^{\frac{s}{2}}[\|\Gamma\|_{L^2(m)} + \|\Gamma\|_{L^\infty_\eta}](\|V\|_{L^\infty_\eta H^1_z} + \|V\|_{L^\infty_{\eta,z}}) \\
&\leq C\varepsilon^2 e^{-\frac{3n-4}{2}s} \leq C\varepsilon^2 e^{-(n-\frac{3}{2})s}.
\end{aligned}$$

For the last term, we have

$$\|e^{-\frac{s}{2}}(\nabla_\eta\cdot\Gamma)^2\phi''_{e^{-\frac{s}{2}}\Gamma}\|_{L^2(m)H^1_z}\le Ce^{-\frac{s}{2}}\|\nabla_\eta\cdot\Gamma\|_{L^\infty_\eta}\|\nabla_\eta\cdot\Gamma\|_{L^2(m)}\le C\varepsilon^2e^{-(n-\frac{3}{2})s},$$
$$\|e^{-\frac{s}{2}}(\nabla_\eta\cdot\Gamma)^2\phi''_{e^{-\frac{s}{2}}\Gamma}\|_{L^\infty_\eta H^1_z}\le Ce^{-\frac{s}{2}}\|\nabla_\eta\Gamma\|^2_{L^\infty_\eta}\le C\varepsilon^2e^{-(n-\frac{3}{2})s}.$$

Putting everything together, we arrive at (3.55). Note that for $n\ge 3$, the dominant decay term for $e^{-(n-\frac{3}{2})s}$ came only from the contribution of the term $Q_0[e^{-\frac{s}{2}}(\nabla_\eta\cdot\Gamma)^2\phi''_{e^{-\frac{s}{2}}\Gamma}]=e^{-\frac{s}{2}}(\nabla_\eta\Gamma)^2Q_0[\phi''_{e^{-\frac{s}{2}}\Gamma}]$, since[10] $Q_0[\phi'']\neq 0$. For $n=2$, the decay terms $e^{-\frac{3n-5}{2}}=e^{-(n-\frac{3}{2})s}=e^{-\frac{s}{2}}$, so two terms contribute at the same rate. Even in this case though, the contribution of $N_2(\Gamma,\nabla_\eta\cdot\Gamma,V)Q_0[\phi'_{e^{-\frac{s}{2}}\Gamma}]$ is of order $\varepsilon^3e^{-s/2}$ versus $\varepsilon^2e^{-s/2}$ for $Q_0[e^{-\frac{s}{2}}(\nabla_\eta\cdot\Gamma)^2\phi''_{e^{-\frac{s}{2}}\Gamma}]$. □

3.4.3 Estimates on the Duhamel's terms

The following elementary lemmas will be useful as well.

Lemma 3.4.4. *If $c,d>0: c\neq d$, then*

$$\int_0^\tau e^{-d(\tau-s)}\left(\frac{1}{\sqrt{\tau-s}}+1\right)e^{-cs}ds\le C_{c,d}e^{-\min(c,d)\tau}. \tag{3.58}$$

Let $b\in\mathbb{R}$, $\delta>0$ and $c\ge 0$ then

$$\int_0^\tau e^{b(\tau-s)}e^{-\delta(e^\tau-e^s)}e^{-cs}ds\le C_{b,\delta}e^{-(c+1)\tau}. \tag{3.59}$$

Proof. The estimate (3.58) is standard. We estimate the integrals $\int_0^{\tau-1}\tau...ds$ and $\int_{\tau-1}^\tau..ds$ separately. We have that

$$\int_0^{\tau-1}e^{-d(\tau-s)}\left(\frac{1}{\sqrt{\tau-s}}+1\right)e^{-cs}ds\le e^{-d\tau}\left(\frac{e^{(d-c)(\tau-1)}-1}{d-c}\right)\le\frac{e^{-\min(d,c)\tau}}{|d-c|}.$$

[10] Since ϕ' is the eigenvector for the simple eigenvalue at zero for L_1, we have that $Q_0[g]\neq 0$ for all $g\neq\phi'$

For the other term,

$$\int_{\tau-1}^{\tau} \frac{e^{-d(\tau-s)}}{\sqrt{\tau-s}} e^{-cs} ds \leq e^c e^{-c\tau} \int_{\tau-1}^{\tau} \frac{1}{\sqrt{\tau-s}} ds \leq e^c e^{-c\tau} \leq e^c e^{-\min(d,c)\tau}.$$

Proof of (3.59)

Since $\lim_{h\to 0+} \frac{e^h-1}{h} = 1$, fix $h_0 > 0$, so that for all $0 < h < h_0$, we have $e^h - 1 \geq \frac{1}{2}h$. We can, without loss of generality take $h_0 \leq 1$.

We split the integration in (3.59) in two intervals $s \in (\tau - h_0, \tau)$ and $s \in (0, \tau - h_0)$. For the latter, we have that $e^\tau - e^s \geq e^\tau - e^{\tau-h_0} = e^\tau(1-e^{-h_0})$. So,

$$\begin{aligned} \int_0^{\tau-h_0} e^{b(\tau-s)} e^{-\delta(e^\tau-e^s)} e^{-cs} ds &\leq e^{-\delta(1-e^{-h_0})e^\tau} \int_0^{\tau-h_0} e^{b(\tau-s)} ds \leq e^{-\delta(1-e^{-h_0})e^\tau} e^{|b|\tau} \tau \\ &\leq C_{b,\delta} e^{-(c+1)\tau}, \end{aligned}$$

where we obtain a much better, exponential in e^τ, decay rate. For the case $s \in (\tau - h_0, \tau)$, observe first that by the choice of h_0, we have

$$e^\tau - e^s = e^s(e^{\tau-s} - 1) \geq \frac{1}{2} e^s (\tau - s) \geq \frac{1}{8} e^\tau (\tau - s).$$

We need to control $e^{-c\tau} \int_{\tau-h_0}^{\tau} e^{-\frac{\delta}{8} e^\tau (\tau-s)} ds$, as follows

$$e^{-c\tau} \int_{\tau-h_0}^{\tau} e^{-\frac{\delta}{8} e^\tau (\tau-s)} ds \leq e^{-c\tau} \int_0^1 e^{-\frac{\delta}{8} e^\tau s} ds \leq 8 e^{-(c+1)\tau} \int_0^\infty e^{-\delta z} dz = \frac{8}{\delta} e^{-(c+1)\tau}.$$

□

We are now ready to deal with the Duhamel's term contributions, that is estimates (3.44).

3.4.3.1 The Duhamel's portion of $\alpha(z,\tau)$ in (3.33)

We have by (3.46)

$$\begin{aligned}
& \| \int_0^\tau e^{-\frac{n-3}{2}(\tau-s)} e^{(e^\tau - e^s)L_1} Q_0 \left[\langle H(\Gamma,V),1\rangle_\eta(s) + \langle N_1(\Gamma,\nabla_\eta\cdot\Gamma,V),1\rangle_\eta(s) \right] ds \|_{H^1_z} \\
\leq\ & C\int_0^\tau e^{-\frac{n-3}{2}(\tau-s)} e^{-\frac{\delta}{2}(e^\tau-e^s)} [\|\langle H(\Gamma,V),1\rangle_\eta(s)\|_{H^1_z} + \|\langle N_1(\Gamma,\nabla_\eta\cdot\Gamma,V),1\rangle_\eta(s)\|_{H^1_z}] ds \\
\leq\ & C\int_0^\tau e^{-\frac{n-3}{2}(\tau-s)} e^{-\frac{\delta}{2}(e^\tau-e^s)} [\|H(\Gamma,V)(s)\|_{H^1_z L^2_\eta(m)} + \|N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)(s)\|_{H^1_z L^2_\eta(m)}] ds
\end{aligned}$$

According to (3.53) and (3.55), the last expression is controlled by

$$C\varepsilon^2 \int_0^\tau e^{-\frac{n-3}{2}(\tau-s)} e^{-\frac{\delta}{2}(e^\tau-e^s)} e^{-(n-\frac{3}{2})s} ds \leq C\varepsilon^2 e^{-(n-\frac{1}{2})\tau},$$

where in the last step, we have used (3.59).

3.4.3.2 The Duhamel's portion of $\gamma(\tau)$ in (3.34)

$$\begin{aligned}
& \int_0^\tau e^{-\frac{n-2}{2}(\tau-s)} e^{-\frac{s}{2}} |\langle N_2(\Gamma,\nabla_\eta\cdot\Gamma,V),1\rangle_\eta(s)| ds \leq \\
& \leq C\int_0^\tau e^{-\frac{n-2}{2}(\tau-s)} e^{-\frac{s}{2}} \|N_2(\Gamma,\nabla_\eta\cdot\Gamma,V),1\rangle_\eta(s)\|_{L^2(m)} ds
\end{aligned}$$

The last expression is controlled, in view of (3.54), by

$$C\varepsilon^2 \int_0^\tau e^{-\frac{n-2}{2}(\tau-s)} e^{-\frac{s}{2}} e^{-(n-2)s} ds \leq C\varepsilon^2 e^{-\frac{n-2}{2}\tau}.$$

3.4.3.3 The Duhamel's portion of $\tilde{V}$ in (3.35)

We first take the norm $\|\cdot\|_{L^2(m)H^1_z}$. Let $l \in \{0,1\}$. We obtain from (3.50), (3.25) and (3.46) and Fubini's

$$\begin{aligned}
&\|\int_0^\tau e^{(\tau-s)(\mathscr{L}_\eta+\frac{1}{2})}\mathscr{Q}_0 e^{(e^\tau-e^s)L_1}Q_0\Big[H(\Gamma,V)(s)+N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)(s)\Big]ds\|_{L^2(m)H^1_z} = \\
= \;& \|\int_0^\tau e^{(\tau-s)(\mathscr{L}_\eta+\frac{1}{2})}\mathscr{Q}_0\nabla^l_z e^{(e^\tau-e^s)L_1}Q_0\Big[H(\Gamma,V)(s)+N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)(s)\Big]\|_{L^2_zL^2_\eta(m)}ds \\
\leq \;& \int_0^\tau e^{-\frac{n-2}{2}(\tau-s)}\|\nabla^l_z e^{(e^\tau-e^s)L_1}Q_0\Big[H(\Gamma,V)(s)+N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)(s)\Big]\|_{L^2_zL^2_\eta(m)}ds \leq \\
\leq \;& C\int_0^\tau e^{-\frac{n-2}{2}(\tau-s)}e^{-\frac{\delta}{2}(e^\tau-e^s)}[\|H(\Gamma,V)(s)\|_{H^1_zL^2_\eta(m)}+\|N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)(s)\|_{H^1_zL^2_\eta(m)}]ds
\end{aligned}$$

Next, we deal with $\|\cdot\|_{L^\infty_\eta H^1_z}$. We have from (3.27)

$$\begin{aligned}
&\|\int_0^\tau e^{(\tau-s)(\mathscr{L}_\eta+\frac{1}{2})}\mathscr{Q}_0 e^{(e^\tau-e^s)L_1}Q_0\Big[H(\Gamma,V)(s)+N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)(s)\Big]ds\|_{L^\infty_\eta H^1_z} = \\
= \;& \|\int_0^\tau e^{(\tau-s)(\mathscr{L}_\eta+\frac{1}{2})}\mathscr{Q}_0\nabla^l_z e^{(e^\tau-e^s)L_1}Q_0\Big[H(\Gamma,V)(s)+N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)(s)\Big]\|_{L^\infty_\eta L^2_z}ds \\
\leq \;& \int_0^\tau e^{-\frac{n-2}{2}(\tau-s)}\|\nabla^l_z e^{(e^\tau-e^s)L_1}Q_0\Big[H(\Gamma,V)(s)+N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)(s)\Big]\|_{L^\infty_\eta L^2_z\cap L^2(m)L^2_z}ds \leq \\
\leq \;& \int_0^\tau e^{-\frac{n-2}{2}(\tau-s)}e^{-\frac{\delta}{2}(e^\tau-e^s)}\|H(\Gamma,V)(s)\|_{H^1_zL^2_\eta(m)\cap L^\infty_\eta L^2_z}ds \\
+ \;& \int_0^\tau e^{-\frac{n-2}{2}(\tau-s)}e^{-\frac{\delta}{2}(e^\tau-e^s)}\|N_1(\Gamma,\nabla_\eta\cdot\Gamma,V)(s)\|_{H^1_zL^2_\eta(m)\cap L^\infty_\eta L^2_z}ds
\end{aligned}$$

In view of (3.53) and (3.55), we control both contributions by

$$C\varepsilon^2\int_0^\tau e^{-\frac{n-2}{2}(\tau-s)}e^{-\frac{\delta}{2}(e^\tau-e^s)}e^{-(n-\frac{3}{2})s}ds \leq C\varepsilon^2 e^{-(n-\frac{1}{2})\tau},$$

where again in the last step, we have used (3.59).

3.4.3.4 The Duhamel's portion of $\tilde{\Gamma}$ in (3.36)

For $l \in \{0,1\}$, we obtain from (3.25)

$$\begin{aligned} & \| \int_0^\tau e^{(\tau-s)\mathscr{L}_\eta} \mathscr{Q}_0 e^{-\frac{s}{2}} N_2(\Gamma, \nabla_\eta \cdot \Gamma, V)(s) ds \|_{H^1(m)} \leq \\ \leq \quad & C \int_0^\tau e^{-\frac{n-1}{2}(\tau-s)} e^{-\frac{s}{2}} \|N_2(\Gamma, \nabla_\eta \cdot \Gamma, V)(s)\|_{L^2(m)} ds \end{aligned}$$

Next, for the norm $\|\cdot\|_{L^\infty_\eta}$, we obtain from (3.27)

$$\begin{aligned} & \| \int_0^\tau e^{(\tau-s)\mathscr{L}_\eta} \mathscr{Q}_0 e^{-\frac{s}{2}} N_2(\Gamma, \nabla_\eta \cdot \Gamma, V)(s) ds \|_{L^\infty_\eta} \leq \\ \leq \quad & C \int_0^\tau e^{-\frac{n-1}{2}(\tau-s)} e^{-\frac{s}{2}} \|N_2(\Gamma, \nabla_\eta \cdot \Gamma, V)(s)\|_{L^2(m)\cap L^\infty} ds \end{aligned}$$

Finally, for $\|\nabla[\cdot]\|_{L^\infty_\eta}$, we obtain from (3.27)

$$\begin{aligned} & \| \int_0^\tau \nabla_\eta e^{(\tau-s)\mathscr{L}_\eta} \mathscr{Q}_0 e^{-\frac{s}{2}} N_2(\Gamma, \nabla_\eta \cdot \Gamma, V)(s) ds \|_{L^\infty_\eta} \leq \\ \leq \quad & C \int_0^\tau \frac{e^{-\frac{n-1}{2}(\tau-s)}}{\sqrt{a(\tau-s)}} e^{-\frac{s}{2}} \|N_2(\Gamma, \nabla_\eta \cdot \Gamma, V)(s)\|_{L^2(m)\cap L^\infty} \end{aligned}$$

By (3.54), we control the last three integrals by

$$C\varepsilon^2 [\int_0^\tau \frac{e^{-\frac{n-1}{2}(\tau-s)}}{\sqrt{\tau-s}} e^{-\frac{s}{2}} e^{-(n-2)s} ds + \int_0^\tau e^{-\frac{n-1}{2}(\tau-s)} e^{-\frac{s}{2}} e^{-(n-2)s} ds] \leq C\varepsilon^2 e^{-\frac{n-1}{2}\tau},$$

where in the last stage, we have used (3.58).

3.5 Sharpness of the decay rates and asymptotic profiles

In this section, we discuss the sharpness of these rates as well as the asymptotic profiles.

3.5.1 The asymptotic profiles for σ

The statements for Γ are straightforward as the decay rate for $\gamma(\tau)$ (see (3.38)), $e^{-\frac{n-2}{2}\tau}$ is strictly slower than the decay rate for $\tilde{\Gamma}$, which is $e^{-\frac{n-1}{2}\tau}$. In addition, by examining the evolution equation for $\gamma(\tau)$, (3.34) and the subsequent estimates in Section 3.4, we see that

$$\gamma(\tau) = \gamma(0)e^{-\frac{n-2}{2}\tau} + O(e^{-\frac{n-1}{2}\tau}) = \langle\Gamma(0,\cdot),1\rangle_\eta e^{-\frac{n-2}{2}\tau} + O(e^{-\frac{n-1}{2}\tau}) \quad (3.60)$$

$$= (\int_{\mathbb{R}^{n-1}} \sigma_0(y)dy)e^{-\frac{n-2}{2}\tau} + O(e^{-\frac{n-1}{2}\tau}). \quad (3.61)$$

It follows that

$$\|\Gamma(\tau,\cdot) - (\int_{\mathbb{R}^{n-1}} \sigma_0(y)dy)e^{-\frac{n-2}{2}\tau}G(\cdot)\|_{L^\infty_\eta} \leq C\varepsilon^2 e^{-\frac{n-1}{2}\tau}.$$

By the estimates for $\nabla_\eta\tilde{\Gamma}$ in L^∞_η, it follows that

$$\|\nabla[\Gamma(\tau,\cdot) - (\int_{\mathbb{R}^{n-1}} \sigma_0(y)dy)e^{-\frac{n-2}{2}\tau}G(\cdot)]\|_{L^\infty_\eta} \leq C\varepsilon^2 e^{-\frac{n-1}{2}\tau}.$$

Translating back to the original variables,

$$\left\|\sigma(t,\cdot) - \frac{(\int_{\mathbb{R}^{n-1}} \sigma_0(y)dy)}{(1+t)^{\frac{n-1}{2}}} G\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^\infty_y} \leq \frac{C\varepsilon^2}{(1+t)^{\frac{n}{2}}},$$

$$\left\|\nabla_y\sigma(t,\cdot) - \frac{(\int_{\mathbb{R}^{n-1}} \sigma_0(y)dy)}{(1+t)^{\frac{n}{2}}} (\nabla_y G)\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{L^\infty_y} \leq \frac{C\varepsilon^2}{(1+t)^{\frac{n+1}{2}}},$$

These are precisely the estimates (3.14), (3.15).

3.5.2 Asymptotic profiles for the radiation term v

The goal in this section is to isolate a leading order term, $\bar{V}$ for V, which decays at the leading order rate $e^{-(n-\frac{1}{2})\tau}$. A quick look at the estimates for the free solutions in Section 3.4.1 confirms that they decay exponentially in e^τ.

Next, going to the Duhamel terms, assume for the moment $n \geq 3$. We have seen that the leading order nonlinearity is exactly $Q_0[e^{-\frac{s}{2}}(\nabla_\eta\cdot\Gamma)^2\phi''_{e^{-\frac{s}{2}}\Gamma}]$, which decays of the order $e^{-(n-\frac{3}{2})s}$ (and thus produces through the Duhamel's operator an object with a decay of about $e^{-(n-\frac{1}{2})\tau}$), while all the others are of rates of at least $e^{-\frac{3n-5}{2}s}$ (and thus produce, through the Duhamels operator terms of decay of at least $e^{-\frac{3n-3}{2}\tau}$). *Note that in this argument, we certainly need to establish lower bound for the Duhamel's operator, which is acting on what we believe is the main term, $Q_0[e^{-\frac{s}{2}}(\nabla_\eta\cdot\Gamma)^2\phi''_{e^{-\frac{s}{2}}\Gamma}]$. So far, we have only established upper bounds and it is not clear a priori whether some hidden cancellation does not occur within the Duhamel's operator formalism.*

In order to establish the said lower bounds, we start by further reducing the leading order terms, by peeling off lower order (i.e. faster decaying) terms. Taking into account $\tilde{\Gamma}=O(e^{-\frac{n-1}{2}s})$ and $e^{-\frac{s}{2}}\Gamma=O(e^{-\frac{n-1}{2}s})$,

$$\begin{aligned}
Q_0[e^{-\frac{s}{2}}(\nabla_\eta\cdot\Gamma)^2\phi''_{e^{-\frac{s}{2}}\Gamma}] &= e^{-\frac{s}{2}}(\nabla_\eta\cdot\Gamma)^2 Q_0[\phi''_{e^{-\frac{s}{2}}\Gamma}] \\
&= e^{-\frac{s}{2}}(\nabla_\eta\cdot(\gamma(\tau)G+\tilde{\Gamma}))^2 Q_0[\phi''+(\phi''_{e^{-\frac{s}{2}}\Gamma}-\phi'')] \\
&= e^{-\frac{s}{2}}(\nabla_\eta\cdot(\gamma(s)G))^2 Q_0[\phi'']+O(e^{-(n-1)s}) \\
&= \gamma_0^2 e^{-(n-\frac{3}{2})s}(\nabla_\eta\cdot G)^2 Q_0[\phi'']+O(e^{-(n-1)s})
\end{aligned}$$

where in the last equality, we used $\gamma(s)=\gamma_0 e^{-\frac{n-2}{2}s}+O(e^{-\frac{n-1}{2}s})$. In view of the equations (3.19), we see that if the term $\bar{V}$ satisfies *the linear inhomogeneous equation*

$$\bar{V}_\tau=(\mathscr{L}_\eta+\frac{1}{2})\bar{V}+e^\tau L_1\bar{V}+\gamma_0^2 e^{-(n-\frac{3}{2})\tau}(\nabla_\eta\cdot G)^2 Q_0[\phi''],\bar{V}(0)=0. \tag{3.62}$$

where we recall that $\gamma_0=\langle\Gamma,1\rangle_\eta=\int_{\mathbb{R}^{n-1}}\sigma_0(y)dy$. Denote $H:=(\nabla_y\cdot e^{-\frac{|y|^2}{4}})^2=\frac{|y|^2}{4}e^{-\frac{|y|^2}{2}}$. Then, (3.62) reads

$$\bar{V}_\tau=(\mathscr{L}_\eta+\frac{1}{2})\bar{V}+e^\tau L_1\bar{V}+\gamma_0^2 e^{-(n-\frac{3}{2})\tau}Q_0[\phi''](z)H(\eta),\bar{V}(0,z,\eta)=0. \tag{3.63}$$

Due to the estimates that we had for the remaining nonlinearities (and more precisely (3.59), which upgrades the Duhamel's term by $e^{-\tau}$ over the non-linearity) , we will have the asymptotic estimate

$$\|V(\tau,\cdot)-\bar{V}(\tau,\cdot)\|_{(H^1(m)\cap W^{1,\infty})_\eta H^1_z} \leq C\varepsilon^2 e^{-n\tau}. \tag{3.64}$$

At this point, it is more advantageous to translating back to the original variables. In doing so, via the assignment $\bar{v}(z,y,t)=\frac{1}{1+t}\bar{V}(z,\frac{y}{\sqrt{1+t}},\ln(1+t))$, we obtain the following equation for $\bar{v}$

$$\bar{v}_t = L\bar{v} + \frac{(\int_{\mathbb{R}^{n-1}}\sigma_0(y)dy)^2}{(1+t)^{n+\frac{1}{2}}} H\left(\frac{y}{\sqrt{1+t}}\right) Q_0[\phi''], \bar{v}(0)=0, \tag{3.65}$$

where recall $L=L_1+\Delta_y$. Similarly, (3.64) translates into the following estimate for $v-\bar{v}$,

$$\|v(t,\cdot)-\bar{v}(t,\cdot)\|_{L^\infty_{yz}} \leq C\varepsilon^2(1+t)^{-(n+1)}. \tag{3.66}$$

We will now compute $\bar{v}$ to a leading order. As a solution to (3.65), we have the formula

$$\bar{v}(t) = c_0 \int_0^t e^{(t-s)L_1}[Q_0\phi''] \frac{e^{(t-s)\Delta_y}[H\left(\frac{\cdot}{\sqrt{1+s}}\right)]}{(1+s)^{n+\frac{1}{2}}} ds, c_0 := \frac{(\int_{\mathbb{R}^{n-1}}\sigma_0(y)dy)^2}{(4\pi)^{n-1}}.$$

Next, we need to compute $e^{(t-s)\Delta_y}[H\left(\frac{\cdot}{\sqrt{1+s}}\right)]$. Before we go any further, we take a moment to introduce another version of the Fourier transform, its inverse and some explicit formulas that will be useful.

$$\hat{f}(\xi) = \int_{\mathbb{R}^{n-1}} f(x)e^{-2\pi i x\cdot\xi}dx, \ \ f(x) = \int_{\mathbb{R}^{n-1}} \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

With this definition, $\widehat{e^{-a|x|^2}}(\eta) = \left(\frac{\pi}{a}\right)^{\frac{n-1}{2}} e^{-\frac{\pi^2|\eta|^2}{a}}$, so

$$\hat{H}(\eta) = -\frac{1}{16\pi^2}\Delta_\eta[\widehat{e^{-\frac{|\cdot|^2}{2}}}] = \frac{(2\pi)^{\frac{n-1}{2}}}{4} e^{-2\pi^2|\eta|^2}(1+c_1|\eta|^2).$$

for some constant c_1. Furthermore,

$$\begin{aligned} e^{(t-s)\Delta_y}\widehat{[H\left(\frac{y}{\sqrt{1+s}}\right)]}(\eta) &= e^{-4\pi^2(t-s)|\eta|^2}(1+s)^{\frac{n-1}{2}}\hat{H}(\eta\sqrt{1+s}) = \\ &= \frac{(2\pi)^{\frac{n-1}{2}}}{4}(1+s)^{\frac{n-1}{2}}e^{-2\pi^2(2t+1-s)|\eta|^2}(1+c_1(1+s)|\eta|^2). \end{aligned}$$

Eventually, in the term $(1+s)^{\frac{n+1}{2}}|\eta|^2e^{-2\pi^2(2t+1-s)|\eta|^2}$ produces lower order terms, so it can be dropped. Note that $2t+1-s>0$, when $s\in(0,t)$. Inverting the Fourier transform above yields

$$e^{(t-s)\Delta_y}[H\left(\frac{\cdot}{\sqrt{1+s}}\right)](y) = \left(\frac{1+s}{2t+1-s}\right)^{\frac{n-1}{2}} e^{-\frac{|y|^2}{2(2t+1-s)}} + l.o.t.$$

This allows us to write

$$\bar{v}(t) = c_0\int_0^t e^{(t-s)L_1}[Q_0\phi'']\frac{e^{-\frac{|y|^2}{2(2t+1-s)}}}{(2t+1-s)^{\frac{n-1}{2}}(1+s)^{\frac{n}{2}+1}}ds + l.o.t.$$

Introduce $M(t,s,y) := \frac{e^{-\frac{|y|^2}{2(2t+1-s)}}}{(2t+1-s)^{\frac{n-1}{2}}(1+s)^{\frac{n}{2}+1}}$ and note that the operator L_1 is invertible on $Q_0[L_z^2]$. Thus, performing an integration by parts,

$$\begin{aligned} I(t,y,z) &= \int_0^t M(t,s,y)e^{(t-s)L_1}[Q_0\phi'']ds = -M(t,s,y)e^{(t-s)L_1}L_1^{-1}Q_0[\phi'']|_0^t + \\ &+ \int_0^t e^{(t-s)L_1}[L_1^{-1}Q_0\phi'']\frac{\partial M}{\partial s}(t,s,y)ds = -L_1^{-1}Q_0[\phi'']M(t,t,y) + \\ &+ M(t,0,y)e^{tL_1}[L_1^{-1}Q_0[\phi'']] + \int_0^t e^{(t-s)L_1}[L_1^{-1}Q_0\phi'']\frac{\partial M}{\partial s}(t,s,y)ds. \end{aligned}$$

We argue that the leading order term is

$$-c_0L_1^{-1}Q_0[\phi'']M(t,t,y) = -c_0\frac{e^{-\frac{|y|^2}{2(t+1)}}}{(t+1)^{n+\frac{1}{2}}}L_1^{-1}Q_0[\phi''], \tag{3.67}$$

which clearly has a decay rate in $L^\infty_{y,z}$ of order $(1+t)^{-(n+\frac{1}{2})}$ as stated. We now need to show that the remaining two terms have faster decay rates. For the term $e^{tL_1}[L_1^{-1}Q_0\phi'']$, we have by Sobolev embedding and (3.46)

$$\|e^{tL_1}[L_1^{-1}Q_0\phi'']\|_{L_z^\infty} \leq C\|e^{tL_1}[L_1^{-1}Q_0\phi'']\|_{H_z^1} \leq C_\delta e^{-\frac{\delta}{2}t}\|L_1^{-1}Q_0\phi'']\|_{H_z^1}, \tag{3.68}$$

so it has an exponential decay in time. Similarly, splitting the integral

$$\int_0^t e^{(t-s)L_1}[L_1^{-1}Q_0\phi'']\frac{\partial M}{\partial s}(t,s,y)ds = \int_0^{t-\sqrt{t}} \dots ds + \int_{t-\sqrt{t}}^t \dots ds$$

allows us to estimate the former integral as follows,

$$\begin{aligned}
&\|\int_0^{t-\sqrt{t}} e^{(t-s)L_1}[L_1^{-1}Q_0\phi'']\frac{\partial M}{\partial s}(t,s,y)ds\|_{L_z^\infty} \leq \\
&\leq \int_0^{t-\sqrt{t}} \|e^{(t-s)L_1}[L_1^{-1}Q_0\phi'']\|_{L_z^\infty}|\frac{\partial M}{\partial s}(t,s,y)|ds \leq \\
&\leq C_\delta e^{-\frac{\delta}{2}\sqrt{t}}\|L_1^{-1}Q_0\phi'']\|_{H_z^1} \leq C(1+t)^{-(n+1)}.
\end{aligned}$$

since on the region of integration $t-s \geq \sqrt{t}$, and we can apply (3.68). For the latter integral, one can see that for $s \in (t-\sqrt{t},t)$, we have by (3.68), $\|e^{(t-s)L_1}[L_1^{-1}Q_0\phi'']\|_{L_z^\infty} \leq C_\delta$, so that

$$\begin{aligned}
&\|\int_{t-\sqrt{t}}^t e^{(t-s)L_1}[L_1^{-1}Q_0\phi'']\frac{\partial M}{\partial s}(t,s,y)ds\|_{L_{z,y}^\infty} \leq \\
&\leq \int_{t-\sqrt{t}}^t \|e^{(t-s)L_1}[L_1^{-1}Q_0\phi'']\|_{L_z^\infty}\|\frac{\partial M}{\partial s}(t,s,y)\|_{L_y^\infty}ds \\
&\leq C_\delta \int_{t-\sqrt{t}}^t \|\frac{\partial M}{\partial s}(t,s,y)\|_{L_y^\infty}ds \leq \frac{C}{(1+t)^{n+1}},
\end{aligned}$$

where in the last step, we have used that if $s \sim t$, then $\|\frac{\partial M}{\partial s}(t,s,y)\|_{L_y^\infty} \leq C(1+t)^{-n-\frac{3}{2}}$. All in all, summarizing the results from this section, we have established that

$$\|\bar{v} + c_0\frac{e^{-\frac{|y|^2}{2(t+1)}}}{(t+1)^{n+\frac{1}{2}}}L_1^{-1}Q_0[\phi'']\|_{L_{z,y}^\infty} \leq C(1+t)^{-n-1},$$

which combined with (3.66) leads us to (3.16).

For the case of $n = 2$, we saw that there are two terms in the nonlinearity (for the equation in the scaled variables) with dominant decay rate, namely $N_2(\Gamma, \nabla_\eta \cdot \Gamma, V)Q_0[\phi'_{e^{-\frac{s}{2}}\Gamma}]$ and $Q_0[e^{-\frac{s}{2}}(\nabla_\eta \cdot \Gamma)^2\phi''_{e^{-\frac{s}{2}}\Gamma}]$. We have just analyzed the second one, which produces (on a solution level and in the standard variables) the term found in (3.67), which is of order $\varepsilon^2(1+t)^{-\frac{5}{2}}$, for $n = 2$. On the other hand, the term $N_2(\Gamma, \nabla_\eta \cdot \Gamma, V)Q_0[\phi'_{e^{-\frac{s}{2}}\Gamma}]$ produces a solution less than $C\varepsilon^3(1+t)^{-\frac{5}{2}}$, and as such is lower order in ε, but of the same order in terms of power decay in t. These exact results are summarized in (3.16) and (3.17).

Appendix A

Proof of Proposition (2.1.9)

Proof. For simplicity in calculations we divide both sides of (2.25) by $e^{(1-\frac{3-\beta}{\alpha})\tau}$, then

$$\begin{aligned}
&\|\partial^\gamma(e^{\tau\mathscr{L}}f)\|^2_{L^2(2)} \le \int_{\mathbb{R}^2}|\partial^\gamma(e^{\tau\mathscr{L}}f)|^2 d\xi + \int_{\mathbb{R}^2}||\xi|^2\partial^\gamma(e^{\tau\mathscr{L}}f)|^2 d\xi \\
&= e^{2(1-\frac{3-\beta}{\alpha})\tau}\left(\int_{\mathbb{R}^2}|p^\gamma[e^{-a(\tau)|p|^\alpha}\widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp + \int_{\mathbb{R}^2}|\Delta_p[p^\gamma e^{-a(\tau)|p|^\alpha}\widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp\right) \\
&= e^{2(1-\frac{3-\beta}{\alpha})\tau}\left(\int_{\mathbb{R}^2}|p^\gamma[e^{-a(\tau)|p|^\alpha}\widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp\right. \\
&\left.+\gamma^2\int_{\mathbb{R}^2}|p^{|\gamma|-1}\nabla_p[e^{-a(\tau)|p|^\alpha}\widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp + \int_{\mathbb{R}^2}|p^\gamma\Delta_p[e^{-a(\tau)|p|^\alpha}\widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp\right).
\end{aligned}$$

At this point it is clear that it is better, for simplicity, to divide both sides by $e^{2(1-\frac{3-\beta}{\alpha})\tau}$. Then we want to control the right hand side of the following relation

$$\begin{aligned}
&\frac{\|\partial^\gamma(e^{\tau\mathscr{L}}f)\|^2_{L^2(2)}}{e^{2(1-\frac{3-\beta}{\alpha})\tau}} \le \int_{\mathbb{R}^2}|p^\gamma[e^{-a(\tau)|p|^\alpha}\widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp \qquad \text{(A.1)} \\
+\quad &\gamma^2\int_{\mathbb{R}^2}||p|^{|\gamma|-1}\nabla_p[e^{-a(\tau)|p|^\alpha}\widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp + \int_{\mathbb{R}^2}|p^\gamma\Delta_p[e^{-a(\tau)|p|^\alpha}\widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp \\
:=\quad &J_1 + J_2 + J_3.
\end{aligned}$$

Estimate for J_1

To control J_1 we divide the argument into two different cases, $\tau \le 1$ and $\tau > 1$. In the case of

$\tau \leq 1$, we have

$$\begin{aligned}
J_1 &= \int_{\mathbb{R}^2} |p^\gamma [e^{-a(\tau)|p|^\alpha} \widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp = e^{\frac{2\tau}{\alpha}(|\gamma|+1)} \int_{\mathbb{R}^2} |q|^{2|\gamma|} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |\widehat{f}(q)|^2 dq \\
&\leq e^{\frac{2\tau}{\alpha}(|\gamma|+1)} \int\limits_{\{q:0\leq 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \leq 1\}} |q|^{2|\gamma|} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |\widehat{f}(q)|^2 dq \\
&+ e^{\frac{2\tau}{\alpha}(|\gamma|+1)} \sum_{j=1}^{\infty} \int\limits_{\{q:\ j\leq 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \leq j+1\}} |q|^{2|\gamma|} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |\widehat{f}(q)|^2 dq \\
&= J_1^1 + J_1^2.
\end{aligned}$$

We can estimate

$$\begin{aligned}
J_1^1 &\leq e^{\frac{2\tau}{\alpha}(|\gamma|+1)} \int_{0\leq |q| \leq \frac{e^{-\frac{\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}} |q|^{2|\gamma|} |\hat{f}(q)|^2 dq \leq \frac{e^{\frac{2\tau}{\alpha}}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}} \|f\|_{L^2}^2 \leq \frac{C}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}} \|f\|_{L^2(2)} \\
&\leq C\, \frac{e^{\frac{-2\tau}{\alpha}(1-\varepsilon)}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}} \|f\|_{L^2(2)}^2.
\end{aligned}$$

We treat J_1^2 in a similar manner. Indeed,

$$\begin{aligned}
J_1^2 \quad &\leq e^{\frac{2\tau}{\alpha}(|\gamma|+1)} \sum_{j=1}^{\infty} e^{-j} \int\limits_{j\leq 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^\alpha \leq (j+1)} |q|^{2|\gamma|} |\hat{f}(q)|^2 dq \\
&\leq \frac{e^{\frac{2\tau}{\alpha}}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}} \sum_{j=1}^{\infty} e^{-(j+1)} (j+1)^{\frac{2|\gamma|}{\alpha}} \int\limits_{j\leq 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^\alpha \leq (j+1)} |\hat{f}|^2 dq \\
&\leq \frac{e^{\frac{2\tau}{\alpha}}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}} \|f\|_{L^2}^2 \sum_{j=1}^{\infty} e^{-(j+1)} (j+1)^{\frac{2|\gamma|}{\alpha}} \\
&\leq C \frac{e^{\frac{2\tau}{\alpha}}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}} \|f\|_{L^2(2)}^2 \leq \frac{C}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}} \|f\|_{L^2(2)}^2 \leq C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}} \|f\|_{L^2(2)}^2
\end{aligned}$$

After putting together the estimates for J_1^1 and J_1^2 we get

$$J_1 \leq C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)} \|f\|_{L^2(2)}^2}{a(\tau)^{\frac{2|\gamma|}{\alpha}}}.$$

Regarding the case $\tau > 1$, first note that in this range of τ, $0 < C < a(\tau) \leq 1$. Moreover,

$$|\hat{f}(q) - \hat{f}(0)| \leq 2\|\hat{f}\|_{L^\infty}, \quad |\hat{f}(q) - \hat{f}(0)| \leq |q|\|\nabla \hat{f}\|_{L^\infty},$$

then by interpolation, we conclude that for every $\varepsilon > 0$, we have

$$|\hat{f}(q) - \hat{f}(0)| \leq C_\varepsilon |q|^{1-\varepsilon} \||\nabla|^{1-\varepsilon} \hat{f}\|_{L^\infty} \leq C_\varepsilon |q|^{1-\varepsilon} \|f\|_{L^2(2)}, \tag{A.2}$$

where in the last inequality we have used that by Hausdorf-Young's
$\||\nabla|^{1-\varepsilon}\hat{f}\|_{L^\infty} \leq \int_{\mathbb{R}^2} |\xi|^{1-\varepsilon}|f(\xi)|d\xi \leq C\|f\|_{L^2(2)}$. Therefore,

$$\begin{aligned}
J_1 &= \int_{\mathbb{R}^2} |p^\gamma[e^{-a(\tau)|p|^\alpha} \widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp = e^{\frac{2\tau}{\alpha}(|\gamma|+1)} \int_{\mathbb{R}^2} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |q|^{2|\gamma|} |\widehat{f}(q)|^2 dq \\
&\leq e^{\frac{2\tau}{\alpha}(|\gamma|+1)} \|f\|^2_{L^2(2)} \int_{\mathbb{R}^2} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |q|^{2(|\gamma|+1-\varepsilon)} dq \\
&\leq e^{\frac{2\tau}{\alpha}(|\gamma|+1)} \|f\|^2_{L^2(2)} \int\limits_{\{q: 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \leq 1\}} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |q|^{2(|\gamma|+1-\varepsilon)} dq \\
&+ e^{\frac{2\tau}{\alpha}(|\gamma|+1)} \|f\|^2_{L^2(2)} \sum_{j=1}^\infty \int\limits_{\{q:\ j \leq 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \leq j+1\}} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |q|^{2(|\gamma|+1-\varepsilon)} dq = J_1^1 + J_1^2.
\end{aligned}$$

Now

$$\begin{aligned}
J_1^1 &= e^{\frac{2\tau}{\alpha}(|\gamma|+1)} \|f\|^2_{L^2(2)} \int\limits_{\{q: 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \leq 1\}} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |q|^{2(|\gamma|+1-\varepsilon)} dq \\
&\leq e^{\frac{2\tau}{\alpha}(|\gamma|+1)} \|f\|^2_{L^2(2)} \int\limits_{\{q: 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \leq 1\}} |q|^{2(|\gamma|+1-\varepsilon)} dq \\
&\leq Ce^{\frac{2\tau}{\alpha}(|\gamma|+1)} \|f\|^2_{L^2(2)} \int_0^{\frac{e^{-\frac{\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}} r^{2(|\gamma|+1-\varepsilon)+1} dq \\
&\leq C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)} \|f\|^2_{L^2(2)}}{a(\tau)^{\frac{2}{\alpha}(|\gamma|+2-\varepsilon)}} \leq C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)} \|f\|^2_{L^2(2)}}{a(\tau)^{\frac{2|\gamma|}{\alpha}}}.
\end{aligned}$$

In a similar way,

$$
\begin{aligned}
J_1^2 &= e^{\frac{2\tau}{\alpha}(|\gamma|+1)}\|f\|^2_{L^2(2)} \sum_{j=1}^{\infty} \int_{\{q:\ j\le 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \le j+1\}} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |q|^{2(|\gamma|+1-\varepsilon)} dq \\
&\le e^{\frac{2\tau}{\alpha}(|\gamma|+1)}\|f\|^2_{L^2(2)} \sum_{j=1}^{\infty} e^{-j} \int_{\{q:\ j\le 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \le j+1\}} |q|^{2(|\gamma|+1-\varepsilon)} dq \\
&\le e^{\frac{2\tau}{\alpha}(|\gamma|+1)}\|f\|^2_{L^2(2)} \sum_{j=1}^{\infty} e^{-j} \int_{(\frac{j}{a(\tau)})^{\frac{1}{\alpha}} e^{-\frac{\tau}{\alpha}}}^{(\frac{j+1}{a(\tau)})^{\frac{1}{\alpha}} e^{-\frac{\tau}{\alpha}}} r^{2(|\gamma|+1-\varepsilon)+1} dr \\
&\le C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}}{a(\tau)^{\frac{2}{\alpha}(|\gamma|+2-\varepsilon)}} \|f\|^2_{L^2(2)} \sum_{j=1}^{\infty} e^{-j}(j+1)^{2(|\gamma|+2-\varepsilon)} \le C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}\|f\|^2_{L^2(2)}}{a(\tau)^{\frac{2}{\alpha}(|\gamma|+2-\varepsilon)}} \\
&\le C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}\|f\|^2_{L^2(2)}}{a(\tau)^{\frac{2|\gamma|}{\alpha}}}.
\end{aligned}
$$

Therefore for $\tau > 1$ we have

$$
J_1 \le C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}\|f\|^2_{L^2(2)}}{a(\tau)^{\frac{2|\gamma|}{\alpha}}}.
$$

Estimate for J_2

To control J_2 first note that

$$
\nabla e^{-a(\tau)|p|^\alpha} = -\alpha\, a(\tau)\, p|p|^{\alpha-2} e^{-a(\tau)|p|^\alpha}. \tag{A.3}
$$

Therefore,

$$
\begin{aligned}
J_2 &= |\gamma|^2 \int_{\mathbb{R}^2} ||p|^{|\gamma|-1} \nabla_p [e^{-a(\tau)|p|^\alpha} \widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp \\
&\le \alpha^2 |\gamma|^2 a(\tau)^2 \int_{\mathbb{R}^2} ||p|^{|\gamma|-1} |p|^{\alpha-1}\, e^{-a(\tau)|p|^\alpha}\, \widehat{f}(pe^{\frac{-\tau}{\alpha}})|^2 dp \\
&+ \quad |\gamma|^2 e^{\frac{-2\tau}{\alpha}} \int_{\mathbb{R}^2} |p^{|\gamma|}\, e^{-a(\tau)|p|^\alpha} \cdot (\nabla \widehat{f})(pe^{\frac{-\tau}{\alpha}})|^2 dp := I_1 + I_2.
\end{aligned}
$$

A.0.0.1 Estimate for I_1

To control the first term I_1 we proceed as follows

$$
\begin{aligned}
\frac{I_1}{a(\tau)^2} &\leq \int_{\mathbb{R}^2} e^{-2a(\tau)|p|^\alpha}|p|^{2(\alpha+|\gamma|-2)}|\widehat{f}(p\cdot e^{-\frac{\tau}{\alpha}})|^2 dp = \\
&= e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\int_{\mathbb{R}^2} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha}|q|^{2(\alpha+|\gamma|-2)}|\widehat{f}(q)|^2 dq \\
&\leq e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)} \int\limits_{\{q:2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha\leq 1\}} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha}|q|^{2(\alpha+|\gamma|-2)}|\widehat{f}(q)|^2 dq \\
&+ e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\sum_{j=1}^{\infty} \int\limits_{\{q:\ j\leq 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha\leq j+1\}} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha}|q|^{2(\alpha+|\gamma|-2)}|\widehat{f}(q)|^2 dq \\
&= I_1^1 + I_1^2.
\end{aligned}
$$

We can estimate

$$
\begin{aligned}
I_1^1 &\leq e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\int_{|q|\leq \frac{e^{-\frac{\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}} |q|^{2(\alpha+|\gamma|-2)}|\hat{f}(q)|^2 dq = \\
&= e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\int_{|q|\leq \frac{e^{-\frac{\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}} |q|^{2(\alpha+|\gamma|-2)}|\hat{f}(q)-\hat{f}(0)|^2 dq
\end{aligned}
$$

Using the relation (A.2), we obtain

$$
\begin{aligned}
&I_1^1 \leq e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\|f\|^2_{L^2(2)}\int_{|q|\leq \frac{e^{-\frac{\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}} |q|^{2(\alpha+\gamma-2)}|q|^{2(1-\varepsilon)} dq = \\
&= Ce^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\|f\|^2_{L^2(2)}\int_0^{\frac{e^{-\frac{\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}} r^{2(\alpha+|\gamma|-\varepsilon)-1}\, dr = C\frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}\|f\|^2_{L^2(2)}}{a(\tau)^{2(1+\frac{|\gamma|-\varepsilon}{\alpha})}}.
\end{aligned}
$$

therefore, recalling that $a(\tau)\leq 1$,

$$
I_1^1 \leq \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}\|f\|^2_{L^2(2)}}{a(\tau)^{2(1+\frac{|\gamma|}{\alpha})}}.
$$

We treat I_1^2 in a similar manner. Again, using (A.2),

$$
\begin{aligned}
I_1^2 &\le e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)} \sum_{j=1}^{\infty} e^{-j} \int\limits_{j\le 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^{\alpha}\le (j+1)} |q|^{2(\alpha+|\gamma|-2)} |\hat{f}(q)-\hat{f}(0)|^2 dq \\
&\le e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)} \sum_{j=1}^{\infty} e^{-j} \int\limits_{j\le 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^{\alpha}\le (j+1)} |q|^{2(\alpha+|\gamma|-2)} |q|^{2(1-\varepsilon)} \|f\|^2_{L^2(2)} dq \\
&\le e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)} \|f\|^2_{L^2(2)} \sum_{j=1}^{\infty} e^{-j} \int\limits_{j\le 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^{\alpha}\le (j+1)} |q|^{2(\alpha+|\gamma|-1-\varepsilon)} dq \\
&\le e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)} \|f\|^2_{L^2(2)} \sum_{j=1}^{\infty} e^{-j} \int\limits_{e^{-\frac{\tau}{\alpha}} \frac{j^{1/\alpha}}{a(t)^{1/\alpha}} \le r \le e^{-\frac{\tau}{\alpha}} \frac{(j+1)^{1/\alpha}}{a(t)^{1/\alpha}}} r^{(2\alpha+2|\gamma|-1-2\varepsilon)} dr \\
&\le C \frac{e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)} \|f\|^2_{L^2(2)}}{a(\tau)^{2(1+\frac{|\gamma|}{\alpha})}} \sum_{j=1}^{\infty} e^{-j} j^{\frac{2(\alpha+|\gamma|-\varepsilon)}{\alpha}} e^{-\frac{2\tau}{\alpha}(\alpha+|\gamma|-\varepsilon)} \le C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)} \|f\|^2_{L^2(2)}}{a(\tau)^{2(1+\frac{|\gamma|}{\alpha})}}
\end{aligned}
$$

After putting together the estimates for I_1^1 and I_1^2 we get

$$
I_1 \le C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)} \|f\|^2_{L^2(2)}}{a(\tau)^{\frac{2|\gamma|}{\alpha}}}.
$$

Estimate for I_2

$$
\begin{aligned}
I_2 &\le C e^{-\frac{2\tau}{\alpha}} \int_{\mathbb{R}^2} |\, |p|^{|\gamma|-1} e^{-a(\tau)|p|^{\alpha}} (\nabla \widehat{f})(pe^{\frac{-\tau}{\alpha}})|^2 dp = \\
&= e^{\frac{2\tau}{\alpha}(|\gamma|-1)} \int_{\mathbb{R}^2} |q|^{2(|\gamma|-1)} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}} |\nabla \widehat{f}(q)|^2 dq \\
&\le e^{\frac{2\tau}{\alpha}(|\gamma|-1)} \sum_{j=0}^{\infty} \int\limits_{\{q:\ j\le 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}\le j+1\}} \left(|q|^{2(|\gamma|-1)} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}} |\nabla \widehat{f}(q)|^2 dq \right) = \\
&= I_2^1 + I_2^2.
\end{aligned}
$$

For I_2^1, we have by Hölder's

$$\begin{aligned} I_2^1 &\leq e^{\frac{2\tau}{\alpha}(|\gamma|-1)} \int_{0\leq|q|\leq\frac{e^{\frac{-\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}} |q|^{2(|\gamma|-1)}|\nabla\hat{f}(q)|^2 dq \leq \\ &\leq Ce^{\frac{2\tau}{\alpha}(|\gamma|-1)}\|\nabla\hat{f}\|^2_{L^{\frac{2}{\varepsilon}}}\left(\int_{0\leq|r|\leq\frac{e^{\frac{-\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}} r^{\frac{2(|\gamma|-1)}{1-\varepsilon}+1}dr\right)^{1-\varepsilon} \\ &= Ce^{\frac{2\tau}{\alpha}(|\gamma|-1)}\|\nabla\hat{f}\|^2_{L^{\frac{2}{\varepsilon}}}\frac{e^{-\frac{2\tau}{\alpha}(|\gamma|-1)-\frac{2\tau}{\alpha}(1-\varepsilon)}}{(2a(\tau))^{\frac{2|\gamma|}{\alpha}-\frac{2\varepsilon}{\alpha}}} \leq C\frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}}\|\nabla\hat{f}\|^2_{L^{\frac{2}{\varepsilon}}}. \end{aligned}$$

By Sobolev embedding, we have $\|\nabla\hat{f}\|^2_{L^{\frac{2}{\varepsilon}}} \leq C\|\nabla\hat{f}\|^2_{H^{1-\varepsilon}(\mathbb{R}^2)} \leq C\|(1-\Delta)\hat{f}\|^2_{L^2} =$ $= C\|f\|^2_{L^2(2)}$. Therefore

$$I_2^1 \leq C\frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}}\|f\|^2_{L^2(2)}.$$

For I_2^2, we estimate

$$\begin{aligned} I_2^2 &\leq e^{\frac{2\tau}{\alpha}(|\gamma|-1)}\sum_{j=1}^{\infty} e^{-j}\int_{j\leq 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^{\alpha}\leq(j+1)} |q|^{2(|\gamma|-1)}|\nabla\hat{f}|^2 dq \\ &\leq e^{\frac{2\tau}{\alpha}(|\gamma|-1)}\|\nabla\hat{f}\|^2_{L^{\frac{2}{\varepsilon}}}\sum_{j=1}^{\infty} e^{-j}\left[\int_{j\leq 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^{\alpha}\leq(j+1)} |q|^{\frac{2(|\gamma|-1)}{1-\varepsilon}}dq\right]^{1-\varepsilon}. \end{aligned}$$

But,

$$\int_{j\leq 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^{\alpha}\leq(j+1)} |q|^{\frac{2(|\gamma|-1)}{1-\varepsilon}}dq \leq C\left(\frac{j}{a(\tau)}\right)^{\frac{2(|\gamma|-1)}{1-\varepsilon}+2} e^{\frac{-\frac{2\tau}{\alpha}(|\gamma|-1)}{1-\varepsilon}-\frac{2\tau}{\alpha}},$$

so using again the bound $\|\nabla\hat{f}\|_{L^{\frac{2}{\varepsilon}}} \leq C\|f\|_{L^2(2)}$,

$$\begin{aligned} I_2^2 &\leq Ce^{\frac{2\tau}{\alpha}(|\gamma|-1)}\|f\|^2_{L^2(2)}\sum_{j=1}^{\infty} e^{-j}\left[\left(\frac{j}{a(\tau)}\right)^{\frac{2(|\gamma|-1)}{1-\varepsilon}+2} e^{\frac{-\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}{1-\varepsilon}-\frac{2\tau}{\alpha}}\right]^{1-\varepsilon} \\ &\leq C\frac{e^{-\frac{2\tau(1-\varepsilon)}{\alpha}}\|f\|^2_{L^2(2)}}{a(\tau)^{2\frac{|\gamma|}{\alpha}-2\varepsilon}}\sum_{j=1}^{\infty} e^{-j}j^{2\alpha+2|\gamma|-2\varepsilon} \leq C\frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}\|f\|^2_{L^2(2)}}{a(\tau)^{\frac{2|\gamma|}{\alpha}}}. \end{aligned}$$

Hence after putting together the estimates for I_2^1 and I_2^2 we have

$$I_2 \leq C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)} \|f\|^2_{L^2(2)}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}}.$$

Estimate for J_3

$$\begin{aligned} J_3 &= \int_{\mathbb{R}^2} |p^{|\gamma|} \Delta_p [e^{-a(\tau)|p|^\alpha} \widehat{f}(pe^{-\frac{\tau}{\alpha}})]|^2 dp \leq \int_{\mathbb{R}^2} |p^{|\gamma|} \, \Delta_p [e^{-a(\tau)|p|^\alpha}] \, \widehat{f}(pe^{\frac{-\tau}{\alpha}})|^2 dp \\ &+ \ 2 \int_{\mathbb{R}^2} |p^{|\gamma|} \, \nabla e^{-a(\tau)|p|^\alpha} \cdot \nabla(\widehat{f}(pe^{\frac{-\tau}{\alpha}}))|^2 dp + \int_{\mathbb{R}^2} |p^{|\gamma|} \, e^{-a(\tau)|p|^\alpha} \Delta_p(\widehat{f}(pe^{\frac{-\tau}{\alpha}}))|^2 dp. \end{aligned}$$

By (A.3) we have,

$$\begin{aligned} &\Delta_p [e^{-a(\tau)|p|^\alpha}] = \sum_{j=1}^{2} \partial_j \Big(-\alpha \, a(\tau) \, p_j |p|^{\alpha-2} e^{-a(\tau)|p|^\alpha} \Big) = \\ &= \ -\alpha \, a(\tau) \sum_{j=1}^{2} \Big(|p|^{\alpha-2} + (\alpha-2) \, \frac{p_j^2}{|p|} |p|^{\alpha-3} + \ p_j |p|^{\alpha-2} (-\alpha \, a(\tau)) \frac{p_j}{|p|} |p|^{\alpha-1} \Big) e^{-a(\tau)|p|^\alpha} \\ &= \ \Big(-\alpha^2 \, a(\tau) |p|^{\alpha-2} + \ \alpha^2 \, a(\tau)^2 |p|^{2(\alpha-1)} \Big) e^{-a(\tau)|p|^\alpha}. \end{aligned}$$

Hence, by allowing for a slight abuse of notations by using γ, which is a multi-index instead of $|\gamma|$, its length,

$$\begin{aligned} J_3 \ &\lesssim \ a(\tau)^2 \int_{\mathbb{R}^2} | \, |p|^{\alpha+|\gamma|-2} e^{-a(\tau)|p|^\alpha} \widehat{f}(pe^{-\frac{\tau}{\alpha}})|^2 dp + \\ &+ \ a(\tau)^4 \int_{\mathbb{R}^2} | \, |p|^{2(\alpha-1)+|\gamma|} e^{-a(\tau)|p|^\alpha} \widehat{f}(pe^{-\frac{\tau}{\alpha}})|^2 dp \\ &+ \ a(\tau)^2 e^{-\frac{2\tau}{\alpha}} \int_{\mathbb{R}^2} | \, |p|^{\alpha+|\gamma|-1} e^{-a(\tau)|p|^\alpha} (\nabla \widehat{f})(pe^{-\frac{\tau}{\alpha}})|^2 dp + \\ &+ \ e^{-\frac{4\tau}{\alpha}} \int_{\mathbb{R}^2} ||p|^\gamma e^{-a(\tau)|p|^\alpha} (\Delta \widehat{f})(pe^{-\frac{\tau}{\alpha}})|^2 dp \\ &:= \ I_3 + I_4 + I_5 + I_6, \end{aligned}$$

Estimate for I_3 and I_4

$$\begin{aligned}
\frac{I_3}{a(\tau)^2} &\leq \int_{\mathbb{R}^2} e^{-2a(\tau)|p|^\alpha} |p|^{2(\alpha+|\gamma|-2)} |\widehat{f}(p\cdot e^{-\frac{\tau}{\alpha}})|^2 dp = \\
&= e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)} \int_{\mathbb{R}^2} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |q|^{2(\alpha+|\gamma|-2)} |\widehat{f}(q)|^2 dq \\
&\leq e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)} \int\limits_{\{q: 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \leq 1\}} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |q|^{2(\alpha+|\gamma|-2)} |\widehat{f}(q)|^2 dq \\
&+ e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)} \sum_{j=1}^{\infty} \int\limits_{\{q:\ j\leq 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \leq j+1\}} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |q|^{2(\alpha+|\gamma|-2)} |\widehat{f}(q)|^2 dq \\
&= I_3^1 + I_3^2.
\end{aligned}$$

By comparing I_3 with I_1 it is clear that $I_3^1 = I_1^1$ and $I_3^2 = I_1^2$, and we treat them in the same way. Hence

$$I_3 \leq C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)} \|f\|^2_{L^2(2)}}{a(\tau)^{\frac{2|\gamma|}{\alpha}}}.$$

The estimate for I_4 proceeds in an identical manner, but we have a slightly different power of p, so we present it here briefly.

$$\begin{aligned}
&\frac{I_4}{a(\tau)^4} \leq \int_{\mathbb{R}^2} |\ |p|^{2(\alpha-1)+|\gamma|} e^{-a(\tau)|p|^\alpha} \widehat{f}(pe^{\frac{-\tau}{\alpha}})|^2 dp = \\
= \ & e^{\frac{\tau}{\alpha}(4\alpha+2|\gamma|-2)} \int_{\mathbb{R}^2} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{2}}|^\alpha} |q|^{4(\alpha-1)+2|\gamma|} |\widehat{f}(q)|^2 dq \\
= \ & e^{\frac{\tau}{\alpha}(4\alpha+2|\gamma|-2)} \sum_{j=0}^{\infty} \int\limits_{\{q:\ j\leq 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \leq j+1\}} \left(e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha} |q|^{4(\alpha-1)+2|\gamma|} |\widehat{f}(q)|^2 dq \right) \\
:= \ & I_4^2 + I_4^2.
\end{aligned}$$

Denoting by I_4^1 the integral corresponding to $2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha \leq 1$ and the rest with I_2^2, we have by

(A.2), $|\hat{f}(q)| = |\hat{f}(q) - \hat{f}(0)| \le C|q|^{1-\varepsilon}\|f\|_{L^2(2)}$,

$$\begin{aligned} I_4^1 &\le e^{\frac{\tau}{\alpha}(4\alpha+2|\gamma|-2)} \int\limits_{\{q: 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}\le 1\}} |q|^{4(\alpha-1)+2|\gamma|}|\widehat{f}(q)|^2 dq \le \\ &\le e^{\frac{\tau}{\alpha}(4\alpha+2|\gamma|-2)}\|f\|^2_{L^2(2)} \int\limits_{\{q: 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}\le 1\}} |q|^{4(\alpha-1)+2|\gamma|+2(1-\varepsilon)} dq \\ &= Ce^{\frac{\tau}{\alpha}(4\alpha+2|\gamma|-2)}\|f\|^2_{L^2(2)} \int\limits_{0\le r\le \frac{e^{-\frac{\tau}{\alpha}}}{(a(\tau))^{\frac{1}{\alpha}}}} r^{4(\alpha-1)+2|\gamma|+2(1-\varepsilon)+1} dr \le C\frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}\|f\|^2_{L^2(2)}}{(a(\tau))^{4+\frac{2|\gamma|}{\alpha}}}. \end{aligned}$$

For I_4^2, we have

$$\begin{aligned} I_4^2 &\le e^{\frac{\tau}{\alpha}(4\alpha+2|\gamma|-2)} \sum_{j=1}^{\infty} \int_{\{q:\ j\le 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}\le j+1\}} |q|^{4(\alpha-1)+2|\gamma|} e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}} |\widehat{f}(q)|^2 dq \\ &\le e^{\frac{\tau}{\alpha}(4\alpha+2|\gamma|-2)} \sum_{j=1}^{\infty} e^{-j} \int_{\{q:\ j\le 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}\le j+1\}} |q|^{4(\alpha-1)+2|\gamma|} |\widehat{f}(q)-\widehat{f}(0)|^2 dq \\ &\le e^{\frac{\tau}{\alpha}(4\alpha+2|\gamma|-2)}\|f\|^2_{L^2(2)} \sum_{j=1}^{\infty} e^{-j} \int_{\{q:\ j\le 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}\le j+1\}} |q|^{4(\alpha-1)+2|\gamma|} |q|^{2(1-\varepsilon)} dq \\ &\le e^{\frac{\tau}{\alpha}(4\alpha+2|\gamma|-2)}\|f\|^2_{L^2(2)} \sum_{j=1}^{\infty} e^{-j} \int_{(\frac{j}{a(\tau)})^{\frac{1}{\alpha}} e^{\frac{-\tau}{\alpha}} \le r \le (\frac{j+1}{a(\tau)})^{\frac{1}{\alpha}} e^{\frac{-\tau}{\alpha}}} r^{4(\alpha-1)+2|\gamma|+2(1-\varepsilon)+1} dr \\ &\le e^{\frac{\tau}{\alpha}(4\alpha+2|\gamma|-2)}\|f\|^2_{L^2(2)} \frac{e^{-\frac{\tau}{\alpha}(4\alpha+2|\gamma|-2\varepsilon)}}{(a(\tau))^{4+\frac{2|\gamma|}{\alpha}-\frac{2\varepsilon}{\alpha}}} \sum_{j=1}^{\infty} e^{-j} j^{2(2\alpha+|\gamma|-\varepsilon)} \le C\ \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}}{(a(\tau))^{4+\frac{2|\gamma|}{\alpha}}}\|f\|^2_{L^2(2)}. \end{aligned}$$

Therefore

$$I_4 \le C\ \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}\|f\|^2_{L^2(2)}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}}$$

Estimate for I_5

$$
\begin{aligned}
\frac{I_5}{a(\tau)^2} &\le Ce^{-\frac{2\tau}{\alpha}}\int_{\mathbb{R}^2}|\,|p|^{\alpha+|\gamma|-1}e^{-a(\tau)|p|^\alpha}(\nabla\widehat{f})(pe^{\frac{-\tau}{\alpha}})|^2dp=\\
&= e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\int_{\mathbb{R}^2}|q|^{2(\alpha+|\gamma|-1)}e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha}|\nabla\widehat{f}(q)|^2dq\\
&\le e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\sum_{j=0}^{\infty}\int_{\{q:\ j\le 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha\le j+1\}}\left(|q|^{2(\alpha+|\gamma|-1)}e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^\alpha}|\nabla\widehat{f}(q)|^2dq\right)\\
&= I_5^1+I_5^2.
\end{aligned}
$$

For I_5^1, we have by Hölder's

$$
\begin{aligned}
I_5^1 &\le e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\int_{|q|\le\frac{e^{\frac{-\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}}|q|^{2(\alpha+|\gamma|-1)}|\nabla\hat{f}(q)|^2dq\le\\
&\le Ce^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\|\nabla\hat{f}\|^2_{L^{\frac{2}{\varepsilon}}}\left(\int_{0\le|r|\le\frac{e^{\frac{-\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}}r^{\frac{2(\alpha+\delta-1)}{1-\varepsilon}+1}dr\right)^{1-\varepsilon}\\
&= Ce^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\|\nabla\hat{f}\|^2_{L^{\frac{2}{\varepsilon}}}\frac{e^{-\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)-\frac{2\tau}{\alpha}(1-\varepsilon)}}{(2a(\tau))^{2+\frac{2|\gamma|}{\alpha}-\frac{2\varepsilon}{\alpha}}}\le C\frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}}{(a(\tau))^{2+\frac{2|\gamma|}{\alpha}}}\|\nabla\hat{f}\|^2_{L^{\frac{2}{\varepsilon}}}.
\end{aligned}
$$

However, by Sobolev embedding, we have $\|\nabla\hat{f}\|^2_{L^{\frac{2}{\varepsilon}}}\le C\|\nabla\hat{f}\|^2_{H^{1-\varepsilon}(\mathbb{R}^2)}\le C\|(1-\Delta)\hat{f}\|^2_{L^2}=C\|f\|^2_{L^2(2)}$.

For I_5^2, we estimate

$$
\begin{aligned}
I_5^2 &\le e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\sum_{j=1}^{\infty}e^{-j}\int_{j\le 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^\alpha\le(j+1)}|q|^{2(\alpha+|\gamma|-1)}|\nabla\hat{f}|^2dq\\
&\le e^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\|\nabla\hat{f}\|^2_{L^{\frac{2}{\varepsilon}}}\sum_{j=1}^{\infty}e^{-j}\left[\int_{j\le 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^\alpha\le(j+1)}|q|^{\frac{2(\alpha+|\gamma|-1)}{1-\varepsilon}}dq\right]^{1-\varepsilon}.
\end{aligned}
$$

But,

$$
\int_{j\le 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^\alpha\le(j+1)}|q|^{\frac{2(\alpha+|\gamma|-1)}{1-\varepsilon}}dq\le C\left(\frac{j}{a(\tau)}\right)^{\frac{2(\alpha+|\gamma|-1)}{1-\varepsilon}+2}\ e^{\frac{-\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}{1-\varepsilon}-\frac{2\tau}{\alpha}},
$$

so using again the bound $\|\nabla \hat{f}\|_{L^{\frac{2}{\varepsilon}}} \leq C\|f\|_{L^2(2)}$,

$$
\begin{aligned}
I_5^2 &\leq Ce^{\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}\|f\|^2_{L^2(2)}\sum_{j=1}^{\infty} e^{-j}\left[\left(\frac{j}{a(\tau)}\right)^{\frac{2(\alpha+|\gamma|-1)}{1-\varepsilon}+2} e^{\frac{-\frac{2\tau}{\alpha}(\alpha+|\gamma|-1)}{1-\varepsilon}-\frac{2\tau}{\alpha}}\right]^{1-\varepsilon} \\
&\leq C\frac{e^{-\frac{2\tau(1-\varepsilon)}{\alpha}}\|f\|^2_{L^2(2)}}{a(\tau)^{2+2\frac{|\gamma|}{\alpha}-2\varepsilon}}\sum_{j=1}^{\infty} e^{-j} j^{2\alpha+2|\gamma|-2\varepsilon} \leq C\frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}\|f\|^2_{L^2(2)}}{a(\tau)^{2+\frac{2|\gamma|}{\alpha}}}.
\end{aligned}
$$

Hence after putting together the estimates for I_5^1 and I_5^2 we have

$$
I_5 \leq C\frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}\|f\|^2_{L^2(2)}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}}.
$$

Estimate for I_6

In the same way we can get the estimate for I_6. Indeed,

$$
\begin{aligned}
& I_6 \leq e^{-\frac{4\tau}{\alpha}}\int_{\mathbb{R}^2}\left| |p|^{|\gamma|}e^{-a(\tau)|p|^{\alpha}}(\widehat{\Delta f})(pe^{\frac{-\tau}{\alpha}})\right|^2 dp \\
=\ & e^{\frac{2\tau}{\alpha}(|\gamma|-1)}\int_{\mathbb{R}^2}|q|^{2|\gamma|}e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}}|\Delta\widehat{f}(q)|^2 dq \\
\leq\ & e^{\frac{2\tau}{\alpha}(|\gamma|-1)}\int_{\{q: 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|\leq 1\}}\left(|q|^{2|\gamma|}e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}}|\Delta\widehat{f}(q)|^2 dq\right) \\
+\ & e^{\frac{2\tau}{\alpha}(|\gamma|-1)}\sum_{j=1}^{\infty}\int_{\{q:\ j\leq 2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|\leq j+1\}}\left(|q|^{2|\gamma|}e^{-2a(\tau)|q\cdot e^{\frac{\tau}{\alpha}}|^{\alpha}}|\Delta\widehat{f}(q)|^2 dq\right) \\
=\ & I_6^1 + I_6^2.
\end{aligned}
$$

For I_6^1,

$$
\begin{aligned}
I_6^1 &\leq e^{\frac{2\tau}{\alpha}(|\gamma|-1)}\int_{|q|\leq\frac{e^{-\frac{\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}}|q|^{2|\gamma|}|\Delta\hat{f}(q)|^2 dq = \\
&= \frac{e^{\frac{2\tau}{\alpha}(|\gamma|-1)}e^{-\frac{2\tau|\gamma|}{\alpha}}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}}\int_{|q|\leq\frac{e^{-\frac{\tau}{\alpha}}}{(2a(\tau))^{\frac{1}{\alpha}}}}|\Delta\hat{f}(q)|^2 dq \leq\leq C\frac{e^{-\frac{2\tau}{\alpha}}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}}\|f\|^2_{L^2(2)}.
\end{aligned}
$$

For I_6^2, we have

$$\begin{aligned} I_6^2 &\leq e^{\frac{2\tau}{\alpha}(|\gamma|-1)} \sum_{j=1}^{\infty} e^{-j} \int_{j \leq 2a(\tau)|qe^{\frac{\tau}{\alpha}}|^{\alpha} \leq (j+1)} |q|^{2|\gamma|} |\Delta \hat{f}(q)|^2 dq \\ &\leq \frac{e^{\frac{2\tau}{\alpha}(|\gamma|-1)} e^{-\frac{2\tau|\gamma|}{\alpha}}}{a(\tau)^{\frac{2|\gamma|}{\alpha}}} \sum_{j=1}^{\infty} e^{-j}(j+1)^{\frac{2|\gamma|}{\alpha}} \int |\Delta \hat{f}(q)|^2 dq \leq C \frac{e^{-\frac{2\tau}{\alpha}}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}} \|f\|_{L^2(2)}^2. \end{aligned}$$

Therefore,

$$I_6 \leq C \frac{e^{-\frac{2\tau}{\alpha}}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}} \|f\|_{L^2(2)}^2 \leq C \frac{e^{-\frac{2\tau}{\alpha}(1-\varepsilon)}}{(a(\tau))^{\frac{2|\gamma|}{\alpha}}} \|f\|_{L^2(2)}^2.$$

Putting it all together finishes off the proof. □

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