A CANONICAL FORM FOR THE DIFFERENTIAL EQUATIONS

OF CURVES IN n-DIMENSIONAL SPACE

by

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From 1900 to 1925 Professor Wilczynski published many articles on the projective differential geometry of curves, surfaces, and curves on surfaces. These papers appeared for the most part in the Transactions of the American Mathematical Society, and the American Journal of Mathematics. His work on space curves is to be found either in an article[3] of 1905 or a text book[2] of 1906. This book incorporated the work of several articles on curves and ruled surfaces. The method of approach was based on Lie's theory of continuous groups. However, the forms obtained were not truly canonical.

In 1928 Professor Stouffer[1] published some canonical forms for the differential equations of curves in a plane or in ordinary space. The method of approach was simple and did not employ Lie's theory. However, the canonical form for space curves was not a generalization of that for plane curves.

At the suggestion of Professor Stouffer the problem of determining a canonical form or forms for curves in n-dimensional space was undertaken.

The canonical form here obtained is an exact generalization of the canonical form for curves in ordinary space as mentioned above. The results are valid for any space of three or more dimensions regardless of whether thenumber of dimensions is odd or even.

The writer wishes to express his gratitude to Professor Stouffer for the inspiration, helpful suggestions, and kindly criticism given in the solution and presentation of this problem.

1, 2, 3 See bibliography at the end.

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§ 1. INTRODUCTION

A study of the projective differential properties of a curve in n-dimensional space may be based on the linear homogeneous differential equation

(1.00)

$$y^{(n+1)} + \binom{n+1}{1} p_1 y^{(n)} + \binom{n+1}{2} p_2 y^{(n-1)} + \cdots + p_{n+1} y = 0,$$

where p_i are functions of the independent variable x.

If the curve be represented in homogeneous coordinates by n+1 functions $y_i(x)$ then the coefficients p_i may be so determined as to make the y_i a fundamental set of solutions of 1.00, provided the curve does not lie in a space of less than n dimensions, in which case the Wronskian $|y_1^{(n)}, \cdots, y_n', y_{n+1}|$ is different from zero.

Conversely, since any fundamental set of solutions of 1.00 consists of n+1 linearly independent functions, these may be taken as the homogeneous coordinates of a curve in n-dimensional space. However, any set of linear combinations

(1.01) $$\bar{y}_i = \sum c_{ij} y_j, \quad i, j = 1, 2, \cdots, n+1,$$

of a first fundamental set of solutions y_j is itself a fundamental set provided that the determinant $|c_{ij}|$ is different from zero. We see then that with each differential equation there is associated not one but an infinite number of curves all related by projective transformations.

In general the solutions of 1.00 are power series in $(x-x_0)$, valid in the neighborhood of x_0, where x_0 is a regular point for the differential equation. Hence, we may study by this method only those differential properties of the curve which are invariant under the projective transformation 1.01.

Moreover, neither the parametric representation of the curve nor the associated differential equation is unique. A transformation

$$\xi = \xi(x) \tag{1.02}$$

of the independent variable or a transformation

$$\eta = \lambda Y \tag{1.03}$$

of the dependent variable, where ξ and λ are arbitrary functions of x, will leave the curve unaltered. These are the most general transformations converting the differential equation 1.00 into another of the same form and order.*

* See Wilczynski, Projective Differential Geometry of Curves and Ruled Surfaces, Leipzig, Teubner, Chapter 2.

§ 2. TRANSFORMATIONS OF THE VARIABLES

Let the differential equation 1.00 be transformed by 1.02 into a new equation

(2.00)

$$\eta^{(n+1)} + \binom{n+1}{1}\pi_1\eta^{(n)} + \binom{n+1}{2}\pi_2\eta^{(n-1)} + \cdots + \pi_{n+1}\eta = 0,$$

where

(2.01) $$\eta^{(m)} = d^m\eta \,/\, d\xi^m.$$

Below are given the first five terms of the m th derivative of y used in the above transformation.

(2.02)

$$\begin{aligned}
y^{(m)} = {} & \eta^{(m)}(\xi')^m + \eta^{(m-1)}\binom{m}{2}(\xi')^{m-2}\xi'' \\
& + \eta^{(m-2)}\left[\binom{m}{3}(\xi')^{m-3}\xi''' + 3\binom{m}{4}(\xi')^{m-4}(\xi'')^2\right] \\
& + \eta^{(m-3)}\left[\binom{m}{4}(\xi')^{m-4}\xi^{iv} + 10\binom{m}{5}(\xi')^{m-5}\xi'''\xi'' \right. \\
& \qquad \left. + 15\binom{m}{6}(\xi')^{m-6}(\xi'')^3\right] \\
& + \eta^{(m-4)}\left[\binom{m}{5}(\xi')^{m-5}\xi^{v} + 15\binom{m}{6}(\xi')^{m-6}\xi^{iv}\xi'' \right. \\
& \qquad + 10\binom{m}{6}(\xi')^{m-6}(\xi''')^2 \\
& \qquad + 105\binom{m}{7}(\xi')^{m-7}(\xi'')^2\xi''' \\
& \qquad \left. + 105\binom{m}{9}(\xi')^{m-9}(\xi'')^4\right] \\
& + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

Wilczynski published (Ibid., p. 20-21.) a general method for obtaining in order the terms of 2.02. The work is increasingly difficult after the first two terms.

Substitution of the expressions $y^{(n)}$ in 1.00 gives for the first four coefficients π_i

(2.03)

$$\xi'\pi_1 = p_1 + \frac{n}{2}\psi ,$$

$$(\xi')^2\pi_2 = p_2 + (n-1)\psi p_1 + \frac{n-1}{3}\left[\psi' + \frac{3n-2}{4}\psi^2\right] ,$$

$$(\xi')^3\pi_3 = p_3 + \frac{3(n-2)}{2}\psi p_2 + (n-2)\left[\psi' + \frac{3n-5}{4}\psi^2\right] p_1$$

$$+ \frac{n-2}{4}\left[\psi'' + (2n-3)\psi'\psi + \frac{(n-1)(n-2)}{2}\psi^3\right] ,$$

$$(\xi')^4\pi_4 = p_4 + 2(n-1)\psi p_3 + 2(n-3)\left[\psi' + \frac{3n-8}{4}\psi^2\right] p_2$$

$$+ (n-3)\left[\psi'' + (2n-5)\psi'\psi + \frac{(n-2)(n-3)}{2}\psi^3\right] p_1$$

$$+ \frac{n-3}{5}\left[\psi''' + \frac{5n-12}{2}\psi''\psi + \frac{5n-11}{3}\psi'\psi'\right.$$

$$+ \frac{15n^2-70n+76}{6}\psi'\psi^2$$

$$\left. + \frac{15n^3-105n^2+230n-152}{48}\psi^4\right] ,$$

where $\qquad \psi = \xi''/\xi'$

Equation 2.00 on being transformed by 1.03 takes the form

(2.04)

$$Y^{(n+1)} + \binom{n+1}{1} P_1 Y^{(n)} + \binom{n+1}{2} P_2 Y^{(n-1)} + \cdots + P_{n+1} Y = 0,$$

where

$$\lambda P_1 = \lambda \pi_1 + \lambda' ,$$

(2.05)

$$\lambda P_m = \lambda \pi_m + \binom{m}{1} \lambda' \pi_{m-1} + \cdots + \binom{m}{m-1} \lambda^{(m-1)} \pi_1 + \lambda^{(m)} .$$

The derivatives of η used in the above transformation were obtained from the relation

(2.06)

$$\eta^{(m)} = \lambda Y^{(m)} + \binom{m}{1} \lambda' Y^{(m-1)} + \binom{m}{2} \lambda'' Y^{(m-2)} + \cdots + \lambda^{(m)} Y .$$

§ 3. CANONICAL EXPANSIONS

Let it be assumed that there exists among the coefficients P_i of equation 2.04 and their derivatives some relation invariant under further transformations of the form 1.02 and 1.03 only if the functions ξ and λ are such that ξ'' and λ' are identically zero. The differential equation 2.04 is then said to be in a canonical form.

It will be our purpose to obtain such a canonical form and to determine the transformations 1.02 and 1.03 which derive said form from the original equation 1.00. Further, we shall seek some geometrical relation, between the curve and its related system of reference, characterizing this canonical form.

We may without loss of generality consider solutions of 2.04 in the neighborhood of $\xi=0$, since any value ξ_0 of the parameter could be transformed into $\bar{\xi}=0$ by means of the simple relation $\xi = \bar{\xi} + \xi_0$. It is evident from the form of 2.03 that this transformation would not disturb the canonical form.

Because of the above, and the discussion of 1.01, we are justified in restricting our study to the projective differential properties, near $\xi=0$, of any one of the entire family of curves associated with the canonical equation 2.04. In so doing we study the projective differential properties common to all curves of the family.

Let us consider then a particular fundamental set of solutions obtained by means of the well known theory of ordinary linear differential equations. We assume a solution of the form

$$(3.00) \qquad Y(\xi) = Y(0) + Y'(0)\,\xi + Y''(0)\,\frac{\xi^2}{2!} + \cdots ,$$

where the derivatives of order greater than n are determined by the differential equation 2.04 and its derivatives. The series 3.00 may be written in the form

(3.01) $$Y(\xi) = Y_0[X_1] + Y_0'[X_2] + \cdots + Y_0^{(n)}[X_{n+1}] \; ,$$

where the coefficients X_i are themselves power series in ξ. These n+1 coefficients X_i form a fundamental set, valid in the neighborhood of $\xi=0$, with Wronskian equal to unity for $\xi=0$.

On computing these coefficients we notice that they would be greatly simplified were $\bar{P}_1 = 0$. From the form of 2.03 and 2.05 it follows that $\bar{P}_1$ would be zero if $\lambda' + \lambda\pi_1 = 0$. Further, this form would remain invariant under transformations similar to 1.02 and 1.03 only if the functions corresponding to ξ'' and λ' were identically zero. Hence the function λ of the transformation 1.03 is assumed to satisfy the differential equation $\lambda'/\lambda = -\pi_1$ so that $\bar{P}_1 = 0$.

The curve under consideration is then expressed in the parametric form by

(3.02)
$$X_i = \frac{\xi^{i-1}}{(i-1)!} + A_{i\,n+1}\frac{\xi^{n+1}}{(n+1)!} + A_{i\,n+2}\frac{\xi^{n+2}}{(n+2)!} + \cdots ,$$

where $i = 1, 2, \ldots, n+1$,

(3.03)

$$\begin{array}{l|l}
A_{1\,n+1} = -P_{n+1} , & A_{1\,n+2} = -P'_{n+1} , \\
A_{i\,n+1} = -\binom{n+1}{i-1}P_{n+2-i} , & A_{2\,n+2} = -P_{n+1} - \binom{n+1}{1}P'_{n} , \\
A_{n+1\,n+1} = -(n+1)P_1 = 0 , & A_{i\,n+2} = -\binom{n+1}{i-2}P_{n+3-i} - \binom{n+1}{i-1}P'_{n+2-i} , \\
 & A_{n+1\,n+2} = -\binom{n+1}{n-1}P_2 .
\end{array}$$

The non-homogeneous coordinates $Z_i = X_{i+1} / X_1$ of a point on the curve are given by

(3.04)

$$Z_i = \frac{\varepsilon^i}{i!} + B_{i\,n+1}\frac{\varepsilon^{n+1}}{(n+1)!} + B_{i\,n+2}\frac{\varepsilon^{n+2}}{(n+2)!} + \cdot\ \cdot\ \cdot\ ,$$

where $i = 1, 2, \cdot\cdot\cdot, n$,

(3.05)

$$\begin{array}{ll|ll}
B_{1\,n+1} = A_{2\,n+1} & , & B_{1\,n+2} = A_{2\,n+2} - (n+2)A_{1\,n+1} & , \\
B_{i\,n+1} = A_{i+1\,n+1} & , & B_{i\,n+2} = A_{i+1\,n+2} & , \\
B_{n\,n+1} = P_1 = 0 & , & B_{n\,n+2} = A_{n+1\,n+2} & .
\end{array}$$

Finally, on eliminating the parameter ε of the expansions 3.04 we have the curve represented by the n-1 canonical expansions

(3.06)

$$Z_i = \frac{Z_1^i}{i!} + C_{i\,n+1}\frac{Z_1^{n+1}}{(n+1)!} + C_{i\,n+2}\frac{Z_1^{n+2}}{(n+2)!} + \cdot\ \cdot\ \cdot\ ,$$

where $i = 2, 3, \cdot\cdot\cdot, n$,

(3.07)

$$\begin{array}{ll|ll}
C_{2\,n+1} = -\binom{n+1}{2}P_{n-1} & , & C_{2\,n+2} = (n+1)^2 P_n \quad -\binom{n+1}{2}P'_{n-1} & , \\
C_{i\,n+1} = -\binom{n+1}{i}P_{n+1-i} & , & C_{i\,n+2} = -\binom{n+1}{i-1}P_{n+2-i} \quad -\binom{n+1}{i}P'_{n+1-i} & , \\
C_{n\,n+1} = 0 & , & C_{n\,n+2} = -\binom{n+1}{n-1}P_2 & *
\end{array}$$

It should be noted that the canonical form and therefore the coefficients P_i are still unknown. Expansions 3.06 represent the associated expansions of the canonical form. The coefficients C_{ij} are functions of the P_i whose proper values are yet to be determined.

§ 4. OSCULATING CURVE OF DEGREE n

It is common knowledge that five points in a plane determine a conic while six points in ordinary space determine a cubic. It is our purpose to show that $n+3$ points in an n-dimensional space determine a curve of degree n.

Let N be the number of points required to determine the $(n+1)^2$ constants a_{ij} of the parametric equations
(4.00)

$$\rho x_i = a_{i0} + a_{i1}t + a_{i2}t^2 + \cdots + a_{in}t^n ,$$
$$i = 1,2, \cdots ,n+1 .$$

It is assumed that the n+1 homogeneous coordinates are known for each of the N points. The parameter t takes on N different values $t_1, t_2, \quad, t_N$ for the assigned points. However, in fixing the parametric system we are at liberty to assign values to t at three of the points. For instance we might set $t_1 = 0$, $t_2 = 1$, and $t_3 = \infty$. The other N-3 values of t must be fixed by the homogeneous coordinates of the N points. Again, the proportionality factor ρ takes on N different values $\rho_1, \rho_2, \cdots, \rho_N$ for these points. However, the equations are homogeneous in ρ_i and a_{ij} hence only N-1 of the constants ρ_i are essential. For instance we might set $\rho_1 = 1$. Hence
(4.01)

$$N(n+1) = (n+1)^2 + (N-3) + (N-1) ,$$

from which it follows that $N = n+3$.

Let us consider now the limiting form of a curve of degree n determined by n+3 points, on the original curve 3.06, as these points approach coincidence. Generalizing* the concepts of contact between two plane curves, or two skew curves, we say that these two curves have contact of order n+2. Further, we call this particular curve of degree n an osculating curve.

* See note on next page.

We shall also use the corresponding# analytical definition. If the expansions for the curve of degree n agree with the expansions 3.06 up to and including the terms of order n+2 we say that the two curves have contact of order n+2. The former is said to osculate the latter. We could have used this definition entirely. However the geometrical concepts aid in simplifying the work.

From 3.02 it is evident that the homogeneous coordinates X_i of the point $\xi=0$ are $1,0,\ldots,0$. So long as there is no danger of ambiguity we shall for the sake of brevity call this the point $X(0)$, or just X. Similarly, the coordinates of X' are $(0,1,0,\ldots,0)$, and so on until finally the coordinates of $X^{(n)}$ are $(0,\ldots,0,1)$.

The tangent to the curve is determined by X and X', the osculating plane by X, X', and X'', and the osculating linear k-space by $X, X', X'', \ldots, X^{(k)}$ where $k \leq n-1$.

According to the geometrical concept of contact two curves with contact of the second order have a common tangent and a common osculating plane. Two curves with contact of order equal to or greater than n-1 have common osculating linear spaces up to and including the osculating linear (n-1)-space.

If the curve represented by 4.00 is to have osculating linear spaces in common with the original curve 3.02 several of the coefficients a_{ij} must be identically zero. Thus, $x_2, x_3, \ldots, x_{n+1}$ will be zero at the contact point t=0 only if $a_{20} = a_{30} = \cdots = 0$. The point x' will be on the tangent line* $X_3 = X_4 = \cdots = 0$ determined by X and X' only if $a_{31} = a_{41} = \cdots = 0$. Likewise, the point $x^{(k)}$ will be in the osculating linear k-space $X_{k+2} = \cdots = X_{n+1} = 0$ only if $a_{ik} = 0$, $i \geq k+2$.

The correspondence used here is an exact generalization of the relation connecting, "order of contact", "number of coincident points", and "agreement of expansions" as discussed in § 212, of Goursat-Hedrick, Mathematical Analysis, vol.I.

* Here x' is a point on the line tangent to the osculating curve 4.00 at the point t = 0. Here X' is the similar point on the tangent to the original curve. The capital letters X_i also represent the running coordinates of any point in space referred to the holyhedron determined by $X, X', \ldots, X^{(n)}$.

The equations 4.00 are now greatly simplified in that all terms below the main diagonal have been caused to vanish. This curve then cuts the reference n-1 space $X_n = 0$ in two points one of which is the point $t = 0$ counted n-1 times. Let us set $a_{nn} = 0$. In so doing we merely assign the coordinate $t = \infty$ to this other point in the n-1 space. Further, it will be most convenient to have $a_{10} = a_{21} = 1$. The first is made possible by a proper choice of ρ and the second by a proper choice of the unit point $t = 1$. Equations 4.00 now assume the form

(4.02)

$$\begin{array}{llllllll}
\rho x_1 = & 1 + a_{11} t + a_{12} t^2 + & \cdot & \cdot & \cdot & & + a_{1n} t^n , \\
\rho x_2 = & t + a_{22} t^2 + & \cdot & \cdot & \cdot & & + a_{2n} t^n , \\
\rho x_3 = & a_{32} t^2 + & \cdot & \cdot & \cdot & & + a_{3n} t^n , \\
\cdot & & \cdot & \cdot & \cdot & & \cdot \\
\rho x_n = & & & & & a_{n\,n-1} t^{n-1} & + \ast , \\
\rho x_{n+1} = & & & & & & a_{n+1\,n} t^n .
\end{array}$$

In order to determine the coefficients of the above expansions we first change to non-homogeneous coordinates by dividing ρx_{i+1} by ρx_1. Then

(4.03)
$$z_i = \frac{\rho x_{i+1}}{\rho x_1} = b_{ii} t^i + b_{i\,i+1} t^{i+1} + \cdot \cdot \cdot \cdot ,$$

where $b_{ii} = a_{i+1\,i}$, $b_{i\,i+1} = a_{i+1\,i+1} - a_{i+1\,i}\, a_{11}$,

$a_{ij} = 0$ for $j > n$.

Next we eliminate t between the expansions for z_i and z_1 thereby obtaining

$$(4.04) \qquad z_i = c_{ii} z_1^i + c_{i\,i+1} z_1^{i+1} + \cdot \cdot \cdot \cdot \cdot ,$$

where $i = 2, 3, \cdots, n$,

$$(4.05) \qquad c_{ii} = b_{ii} = a_{i+1\,i} ,$$

$$(4.06) \qquad c_{i\,i+1} = b_{i\,i+1} - i\, b_{ii} b_{12}$$
$$= a_{i+1\,i+1} - a_{i+1\,i}\, a_{11} - i\, b_{ii}(a_{22} - a_{11}) .$$

Since the coefficients of 3.06 and 4.04 must coincide up to and including terms of order n+2 we set $c_{ii} = 1 / i!$ and $c_{i+1\,i} = 0$ resulting at once in

(4.07) $\quad a_{i+1\,i} = 1 / i!\ , \quad i = 0,1,2,\cdots,n\ .$

Since $a_{nn} = a_{n+1\,n+1} = 0$ two simple equations are obtained from 4.06 for $i = n-1$ and $i = n$ resulting in $a_{11} = a_{22} = 0$, and consequently

(4.08) $\quad a_{i+1\,i+1} = 0, \quad i = 0,1,2,\cdots,n-1\ .$

In a similar fashion we find the coefficients of the third, fourth, and fifth diagonals of 4.02 to be

(4.09)

$$a_{i+1\,i+2} = \frac{(3n-2i)P_2}{i!\ 2(n+2)}\ , \quad i = 0,1,\ldots,n-2\ ,$$

(4.10)

$$a_{i+1\,i+3} = \frac{(4n-3i-4)P_3 - (3)(n-i-2)P_2'}{i!\ 3!\ (n+2)}\ ,$$

$$i = 0,1,\ldots,n-3\ ,$$

(4.11) $\quad a_{i+1\,i+4} =$

$$\frac{(n+2)(5n-4i-10)P_4 - 4(n+2)(n-i-3)P_3' + 12(n-3-i)(n-2-i)P_2^2}{i!\ 4!\ (n+2)^2}\ ,$$

$$i = 0,1,\cdots,n-4\ .$$

The results of 4.09, 4.10 and 4.11 were obtained from the relations

$$b_{i\,i+2} = a_{i+1\,i+2} - a_{12} / i! \,, \tag{4.12}$$

$$b_{i\,i+3} = a_{i+1\,i+3} - a_{13} / i! \,, \tag{4.13}$$

$$b_{i\,i+4} = a_{i+1\,i+4} - a_{14} / i! - a_{12} b_{i\,i+2} \,, \tag{4.14}$$

$$c_{i\,i+2} = b_{i\,i+2} - b_{13} / (i-1)! \,, \tag{4.15}$$

$$c_{i\,i+3} = b_{i\,i+3} - b_{14} / (i-1)! \,, \tag{4.16}$$

$$c_{i\,i+4} = b_{i\,i+4} - b_{15} / (i-1)! - (i+2)\, b_{13} c_{i\,i+2} - b_{13}^2 / 2(i-2)! \,, \tag{4.17}$$

where

$$a_{ij} = 0 \quad \text{for} \quad j > n \,, \tag{4.18}$$

$$c_{ij} = 0 \quad \text{for} \quad j < n+1 \,, \tag{4.19}$$

$$c_{i\,n+1} = C_{i\,n+1} / (n+1)! \,, \tag{4.20}$$

$$c_{i\,n+2} = C_{i\,n+2} / (n+2)! \,. \tag{4.21}$$

§ 5. PRINCIPAL TANGENT PLANE

Theorem.* *If two curves in n-dimensional space have contact of order m at a point, there exists a unique plane from any point of which one can project the two curves by means of two hyper-cones with contact of order m+1.*#

Let the two curves be represented by the expansions

(5.01) $$z_i = \sum \gamma_{ij} z_1^{\,j} ,$$

and

(5.02) $$z_i = \sum \delta_{ij} z_1^{\,j} ,$$

where

(5.03) $$\gamma_{ik} = \delta_{ik} ,$$

$$i = 2, 3, \cdots, n ,$$

$$j = 2, 3, \cdots, \infty ,$$

$$k = 2, 3, \cdots, m .$$

Furthermore

(5.04) $$\gamma_{i\,m+1} \neq \delta_{i\,m+1}$$

for at least one value of i. Here again the coordinates z_i are non-homogeneous and related to the homogeneous coordinates x_i by $z_i = x_{i+1} / x_1$.

* This theorem is a generalisation of one stated for ordinary space by Halphen. See Journal de l'Ecole Polytechnique, 1880, vol. 47, page 25. However, his proof is rather difficult and can not be readily extended to hyper-space. The present method of approach was first employed by Stouffer, for curves in ordinary space, in his lectures of 1927-28 at the University of Kansas.

\# It is assumed that $m \geq 2$, and $n \geq 3$.

Consider now a new system of reference differing from the original only in the (n+1)th vertex. With respect to the new system of reference the two curves in contact may be represented by

(5.05) $$Z_i = \sum \Gamma_{ij} Z_1^{\,j},$$

and

(5.06) $$Z_i = \sum \Delta_{ij} Z_1^{\,i},$$

where

(5.07) $$\Gamma_{ik} = \Delta_{ik}\,,$$
$$i = 2,3,\cdots,n\,,$$
$$j = 2,3,\cdots,\infty\,,$$
$$k = 2,3,\cdots,m\,.$$

It should be noted that the new system of reference is not yet determined, hence the coefficients Γ_{ij} and Δ_{ij} are unknown. However, since the property of contact is independent of the system of reference, we are justified in writing the equations 5.07 .

Our proof consists in showing that the (n+1)th vertex of the new system of reference can be chosen in a double infinity of ways so as to make

(5.08) $$\Gamma_{i\,m+1} = \Delta_{i\,m+1}\,,$$

for $$i = 2,3,\cdots,n-1.$$

The most general transformation relating any two sets of homogeneous coordinates in one to one correspondence is

(5.09)

$$\rho X_i = \sum \alpha_{ij} x_j\,,$$
$$i,j = 1,2,\cdots,n+1.$$

However, since in this instance the two systems of reference have their first n vertices in common, the transformation 5.09 becomes simply

(5.10)

$$\rho X_i = x_i + \alpha_{i\,n+1} x_{n+1}\,,$$

$$\rho X_{n+1} = x_{n+1}\,.$$

The two systems of non-homogeneous coordinates are then related by

$$(5.11) \qquad Z_i = \frac{\rho X_{i+1}}{\rho X_1} = \frac{z_i + \beta_i z_n}{1 - \sigma z_n},$$

$$Z_n = \frac{z_n}{1 - \sigma z_n},$$

where for the convenience in writing

$$(5.12) \qquad -\sigma = \alpha_{1\,n+1}, \quad \beta_i = \alpha_{i+1\,n+1}.$$

The first relation of 5.11 may be represented in the form of an expansion by
(5.13)

$$Z_i = (z_i + \beta_i z_n)(1 + \sigma z_n + \sigma^2 z_n^2 + \cdots) .$$

On substituting in the above the expansions 5.01 for the first curve, and again the expansions 5.02 for the second curve, we obtain two sets of expansions

$$(5.14) \qquad Z_1 = z_1 + \sum \mu_{1j} z_1^j ,$$

$$(5.15) \qquad Z_i = \sum \mu_{ij} z_1^j ,$$

and

$$(5.16) \qquad Z_1 = z_1 + \sum \nu_{1j} z_1^j ,$$

$$(5.17) \qquad Z_i = \sum \nu_{ij} z_1^j ,$$

$$i = 2,3,\cdots,n-1 ,$$
$$j = 2,3,\cdots,\infty .$$

Because of 5.03 we have

$$(5.18) \qquad \mu_{ik} - \nu_{ik} = 0 ,$$

$$i = 1,2,\cdots,n-1 ,$$
$$k = 2,3,\cdots, m ,$$

and

(5.19)

$$\mu_{i\,m+1} - \nu_{i\,m+1} = \gamma_{i\,m+1} + \beta_i \gamma_{m\,m+1} - \delta_{i\,m+1} - \beta_i \delta_{m\,m+1} ,$$

$$i = 2,3, \cdots, n-1.$$

On eliminating the parameter z_1 between 5.14 and 5.15, and again between 5.16 and 5.17, we obtain the expansions 5.05 and 5.06 for $i = 2,3, \cdots, n-1.$

The demonstration of the method employed in eliminating z_1 will be greatly condensed by the use of a function Q_{rs} defined by the following examples. Let

(5.20)

$$Q_{23} = \mu_{11}\mu_{12} + \mu_{12}\mu_{11} = 2\mu_{12}$$

be the sum of the products of the permutations of the elements μ_{ij} taken 2 at a time, such that the sum of the weights of the second subscripts is 3, where $\mu_{11} = 1.$ Similarly,

5.21)

$$Q_{24} = \mu_{11}\mu_{13} + \mu_{12}\mu_{12} + \mu_{13}\mu_{11} = 2\mu_{13} + \mu_{12}^2 ,$$

$$Q_{34} = \mu_{11}\mu_{11}\mu_{12} + \mu_{11}\mu_{12}\mu_{11} + \mu_{12}\mu_{11}\mu_{11} = 3\mu_{12} .$$

In order to eliminate the z_1^2 terms of 5.15 we square both members of 5.14, multiply by $\Gamma_{i2} = \mu_{i2}$, and subtract from 5.15. Then

(5.22)

$$Z_i - \Gamma_{i2} Z_1^2 = \Gamma_{i3} z_1^3 + \left[\mu_{i4} - \Gamma_{i2} Q_{24}\right] z_1^4 + \cdots ,$$

where $\Gamma_{i3} = \mu_{i3} - \Gamma_{i2} Q_{23}$.

In order to eliminate the z_1^3 term of 5.22 we cube both members of 5.14, multiply by Γ_{i3}, and subtract from 5.22. Then

(5.23)

$$Z_i - \Gamma_{i2} Z_1^2 - \Gamma_{i3} Z_1^3 = \Gamma_{i4} z_1^4 + \cdots \cdots ,$$

where $\Gamma_{i4} = \mu_{i4} - \Gamma_{i2} Q_{24} - \Gamma_{i3} Q_{34}$.

In a similar fashion we eliminate terms of higher degree in z_1. The coefficient $\Gamma_{i\,m+1}$ will be identically

(5.24)

$$\mu_{i\,m+1} - \Gamma_{i2} Q_{2\,m+1} - \Gamma_{i3} Q_{3\,m+1} - \cdots - \Gamma_{im} Q_{m\,m+1} \; .$$

No element μ_{ij} of $Q_{2\,m+1}$ will have a second subscript greater than m. Again, no element of $Q_{3\,m+1}$ will have a second subscript greater than m-1. Hence the first term $\mu_{i\,m+1}$ is the only element μ_{ij} of 5.24 with a second subscript of m+1.

From the symmetry of our notation it is evident that $\Delta_{i\,m+1}$ may be obtained from 5.24 by the substitution of ν_{ij} for μ_{ij}. It follows then from 5.18 that

(5.25)

$$\Gamma_{i\,m+1} - \Delta_{i\,m+1} = \mu_{i\,m+1} - \nu_{i\,m+1} \; .$$

Now it is a simple matter to choose β_i so as to make the difference 5.19, and consequently the difference 5.25, zero if $\gamma_{n\,m+1} \neq \delta_{n\,m+1}$. If however $\gamma_{n\,m+1} = \delta_{n\,m+1}$ we have merely to repeat our proof after interchanging the (k+1)th and (n+1)th vertices of reference where k represents a value of i for which $\gamma_{i\,m+1} \neq \delta_{i\,m+1}$.

Let us assume then that $\gamma_{n\,m+1} \neq \delta_{n\,m+1}$, and insist that

(5.26)

$$\beta_i = \frac{\delta_{i\,m+1} - \gamma_{i\,m+1}}{\gamma_{n\,m+1} - \delta_{n\,m+1}}, \quad i = 2,3,\cdots,n-1 \; .$$

The (n+1)th vertex of the new system of reference is of course represented by $X_1 = X_2 = \cdots = X_n = 0$, $X_{n+1} = 1$. This same vertex according to the relations 5.10 and 5.12 is represented in the original system of coordinates by

(5.27)

$$\begin{aligned} x_1 &= \sigma \, , & x_3 &= -\beta_2 \, , \\ & & \cdot \quad & \quad \cdot \quad \cdot \\ x_2 &= -\beta_1 \, , & x_n &= -\beta_{n-1} \, , \\ & & x_{n+1} &= 1 \; . \end{aligned}$$

where σ and $-\beta_1$ are arbitrary. Hence this vertex of projection may be any one of a double infinity of points lying in a plane. The two hyper-cones with contact of order m+1 are represented by the expansions 5.05 and 5.06 where

(5.28)
$$\Gamma_{ik} = \Lambda_{ik} ,$$
$$i = 2,3, \cdots ,n-1 ,$$
$$k = 2,3, \cdots ,m+1 .$$

Since σ and $-\beta_1$ are arbitrary while the constants $\beta_2, \beta_3, \cdots, \beta_{m-1}$ are definite it follows that the principal tangent plane 5.27 contains the vertices $(1,0,\cdots,0)$ and $(0,1,0,\cdots,0)$.

In this section we have made but a brief investigation of the principal plane. However, it is evident that there are many extensions. One interesting fact might be noted in passing. The equations 5.28 represent not only two hyper-cones in n-dimensional space but also two curves in the (n-1)-dimensional space $Z_n = 0$. These two curves may be projected by means of two hyper-cones with contact of order m+2. Furthermore, the vertex of projection may be any point of a unique plane in the (n-1)-dimensional space. This reasoning may be repeated until we obtain finally two ordinary cones with contact of order m+n-2 in three dimensional space.

§ 6. ASSOCIATED GEOMETRY OF THE CANONICAL FORM

We have assumed a canonical form though its nature is as yet unknown. It is our purpose now to determine the geometrical and analytical characteristics of said form.

According to 3.01 the point Y is given by the X coordinates $(1,0,\cdots,0)$, Y' by $(0,1,0,\ldots,0)$, Y'' by $(0,0,1,0,\ldots,0)$, $\cdots$ and $Y^{(n)}$ by $(0,\ldots,0,1)$. These $(n+1)$ points are the vertices of our canonical polyhedron of reference.

Suppose now that the $(n+1)$th vertex $Y^{(n)}$ of the system of reference is transformed into some other point $\bar{Y}^{(n)}$ by a change $\bar{\xi} = \bar{\xi}(\xi)$ in the independent variable. Corresponding to 2.02 we have

(6.00)

$$\bar{Y}^{(n)} = Y^{(n)}(\xi')^{n} + Y^{(n-1)}\binom{n}{2}(\xi')^{n-2}\xi'' + Y^{(n-2)}\left[\binom{n}{3}(\xi')^{n-3}\xi''' + 3\binom{n}{4}(\xi')^{n-4}(\xi'')^{2}\right] + \cdots,$$

where in this instance $\xi^{(k)} \equiv d^k\xi/d\bar{\xi}^k$. From the form of the above it is evident that $\bar{Y}^{(n)}$ will not contain $Y^{(n-1)}$ in combination with Y and the other first n derivatives of Y, only if

(6.01)

$$\binom{n}{2}(\xi')^{n-2}\xi'' = 0 .$$

Since a transformation with $\xi' = 0$ would be trivial it follows that ξ'' must be identically zero. That is, $\xi'' = 0$, $\xi = A\bar{\xi} + B$ where A and B are arbitrary constants of integration, and $\bar{\xi}'' \equiv d^2\bar{\xi}/d\xi^2 = 0$.

Again, suppose that $\ddot{Y}^{(n)}$ is transformed into some other point $\bar{Y}^{(n)}$ by a change $\dot{Y} = \bar{\lambda}\bar{Y}$ in the dependent variable. Corresponding to 2.06 we have

(6.02)

$$\bar{Y}^{(n)} = \lambda Y^{(n)} + \binom{n}{1}\lambda' Y^{(n-1)} + \binom{n}{2}\lambda'' Y^{(n-2)} + \cdots\cdots ,$$

where in this instance $\lambda = 1/\bar{\lambda}$, and both λ and $\bar{\lambda}$ are functions of ξ. From the form of the above it is evident that $\bar{Y}^{(n)}$ will not contain $Y^{(n-1)}$, in combination with Y and the other first n derivatives of Y, only if $\binom{n}{1}\lambda'$ is identically zero. That is $-\bar{\lambda}'/\bar{\lambda}^2 = 0$. Since a transformation with $\bar{\lambda} = \infty$ would be trivial it follows that $\bar{\lambda}'$ must be zero.

Since the transformed point $\bar{Y}^{(n)}$ will remain in the (n-1)-space determined by $Y, Y', Y'', \ldots, Y^{(n-2)}, Y^{(n)}$ only if $\xi'' = \bar{\lambda}' = 0$ it follows that just this property will characterize a canonical form.

Now the osculating (n-2)-space $Z_n = Z_{n-1} = 0$ is determined by the n-1 points $Y, Y', Y'', \ldots, Y^{(n-2)}$. Also the principal tangent plane, to the curve 3.06 and its osculating curve of degree n, contains the first two of these points. It is evident then that the osculating (n-2)-space and the principal tangent plane determine a unique space of (n-1) dimensions. Let us assume, for our canonical form, that the point $Y^{(n)}$ is in this unique (n-1)-space.

In order to determine the analytical characteristics of our canonical form we consider the general discussion of § 5 as applied to our two particular curves with contact of order m = n+2. For the first curve

(6.03)

$$\dot{Y}_{ij} \equiv \dot{C}_{ij} / j! ,$$

while for the second curve

(6.04) $$\dot{\delta}_{ij} \equiv C_{ij} .$$

Again we consider a second polyhedron of reference differing from our canonical polyhedron only in the (n+1)th vertex.

Corresponding to 5.10 the two systems of homogeneous coordinates are related by

(6.05)
$$\rho\bar{X}_1 = X_1 - \sigma X_{n+1},$$
$$\rho\bar{X}_i = X_i + \beta_{i-1} X_{n+1},$$
$$\rho\bar{X}_{n+1} = X_{n+1}.$$

The (n+1)th vertex (0, ···, 0, 1) of the new system is represented in the homogeneous coordinates of the canonical system by

(6.06)
$$X_1 = \sigma,$$
$$X_2 = -\beta_1,$$
$$\vdots$$
$$X_n = -\beta_{n-1},$$
$$X_{n+1} = 1.$$

Let us assume that

(6.07)
$$\gamma_{n\,n+1} - \delta_{n\,n+1} = \frac{C_{n\,n+3}}{(n+3)!} - c_{n\,n+3} \neq 0.$$

If

(6.08)
$$\delta_{n-1\,n+1} - \gamma_{n-1\,n+1} = c_{n-1\,n+3} - \frac{C_{n-1\,n+3}}{(n+3)!} = 0,$$

then according to 5.26 and 6.06, β_{n-1} is identically zero and the (n+1)th vertex of the new system of reference remains in the canonical (n-1)-space $X_n = 0$, or $Z_{n-1} = 0$.

It is evident that the converse of the last statement is true. Hence the relations 6.08 and $P_1 = 0$ characterize analytically our canonical form.

In order to determine geometrically some of the vertices of the canonical polyhedron of reference we next study the linear (n-1)-space

(6.09)
$$\begin{vmatrix} X_1 & X_2 & \cdot & \cdot & X_{n+1} \\ x'_1 & x'_2 & \cdot & \cdot & x'_{n+1} \\ x''_1 & x''_2 & \cdot & \cdot & x''_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{(n-1)} & x_2^{(n-1)} & \cdot & \cdot & x_{n+1}^{(n-1)} \end{vmatrix} = 0 ,$$

osculating the osculating curve 4.02. Here X_i represent the coordinates of a point in the linear space while x_i represent the expansions of 4.02.

The determinant of 6.09 is greatly simplified by the proper combination of rows. We eliminate the term with coefficient a_{in} in all elements of the determinant with the exception of the last row by subtracting t times the elements of row 3 from n times the elements of row 2, subtracting t times the elements of row 4 from n-1 times the elements of row 3, etc. Again, we eliminate the term with coefficient $a_{i\,n-1}$ in all elements of the determinant with the exception of the last two rows by subtracting t times the elements of row 3 from n-1 times the elements of 2, subtracting t times the elements of row 4 from n-2 times the elements of row 3, etc. In a similar manner we eliminate as many terms as possible. Finally, the elements of the i th column become

(6.10)
$$\begin{array}{rl} & X_i \quad , \\ n & a_{i0} + a_{i1}t \ , \\ (n-1) & a_{i1} + 2!\, a_{i2}t \ , \\ 2!\,(n-2) & a_{i2} + 3!\, a_{i3}t \ , \\ 3!\,(n-3) & a_{i3} + 4!\, a_{i4}t \ , \\ \cdot & \cdot \quad \cdot \quad \cdot \\ (n-1)! & a_{i\,n-1} + n!\, a_{in}t \ . \end{array}$$

According to 4.02, 4.07 and 4.09 we have $a_{ij} = 0$ for $j < i-1$, $a_{i\,i-1} = 1 / (i-1)!$ and $a_{ii} = 0$. Equation 6.09 then assumes the form

(6.11)

$$
\begin{array}{ccccccccc}
X_1 & X_2 & X_3 & X_4 & X_5 & \cdot & \cdot & \cdot & X_{n+1} \\
\end{array}
$$

$$
\left|
\begin{array}{ccccccccc}
n + * & * + t & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
* + 2\,a_{12}t & n-1 + * & * + t & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
2!(n-2)\,a_{12} + 3!\,a_{13}t & * + 3!\,a_{23}t & n-2 + * & t & 0 & \cdot & \cdot & \cdot & 0 \\
3!(n-3)\,a_{13} + 4!\,a_{14}t & 3!(n-3)\,a_{23} + 4!\,a_{24}t & * + 4!\,a_{34}t & n-3 & t & \cdot & \cdot & \cdot & 0 \\
\cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\right| = 0
$$

where other values of a_{ij} are given by 4.09, 4.10, 4.11.

Now the linear (n-1)-space 6.11, osculating the osculating curve of degree n at the point t, cuts the tangent line $X_3 = X_4 = \cdots = 0$ in the point

(6.12)
$$\begin{vmatrix} X_1 & X_2 \\ n & t \end{vmatrix} = 0 .$$

We noted that the osculating curve 4.02 cuts the (n-1)-space $X_n = 0$ in one point, other than t=0, to which the coordinate $t = \infty$ was assigned. At $t = \infty$ the point 6.12 has the homogeneous coordinates

(6.13)
$$\begin{vmatrix} X_1 & X_2 \\ 0 & 1 \end{vmatrix} = 0, \quad X_3 = X_4 = \cdots = 0 ,$$

or just $(0,1,0,\cdots,0)$.

The osculating linear (n-1)-spaces 6.11 intersect the plane $X_4 = X_5 = \cdots = 0$ in the one parameter family of lines

(6.14)
$$\begin{vmatrix} X_1 & X_2 & X_3 \\ n & t & 0 \\ 2\,a_{12}t & n-1 & t \end{vmatrix} = 0.$$

The above equation is a quadratic in t of the form

(6.15)
$$a\,t^2 + b\,t + c = 0 ,$$

where $a = X_1 - 2a_{12}X_3$, $b = -nX_2$, $c = n(n-1)X_3$.

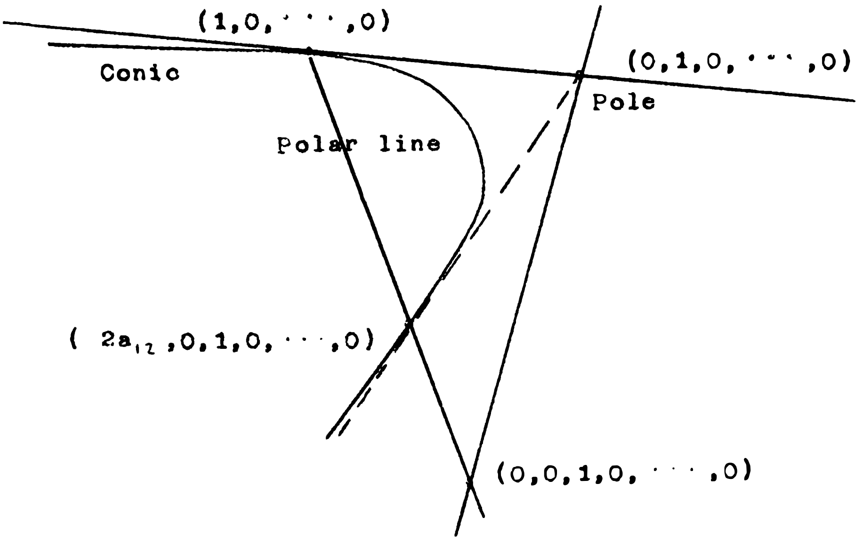
These straight lines envelope a curve given by the equation 6.15 and its first derivative with respect to t. In other words equation 6.15 has a double root and its discriminant is identically zero. The envelope of 6.14 is then the conic

(6.16)
$$n\,X_2^2 - 4(n-1)\,X_1X_3 + 8(n-1)\,a_{12}X_3^2 = 0 ,$$

where $$a_{12} = \frac{3n\, P_2}{2(n+2)} .$$

The polar line of the point $(0,1,0,\cdots,0)$ with regard to this conic is the line $X_2 = 0$ in the osculating plane.

The tangent, to the curve obtained by taking the first derivative of the expansions of the original curve 3.02, intersects the above polar line in the point $(0,0,1,0,\cdots,0)$.



§ 7. CANONICAL FORM OF THE DIFFERENTIAL EQUATION

The coefficients of the terms of degree n+3 in the expansions 3.02, 3.04 and 3.06 are easily computed from the relations

$$(7.00)\quad A_{1\,n+3} = * \qquad A'_{1\,n+2} + A_{n\,n+1}\,A_{1\,n+1},$$

$$(7.01)\quad A_{i\,n+3} = A_{i-1\,n+2} + A'_{i\,n+2} + A_{n\,n+1}\,A_{i\,n+1},$$

$$(7.02)\quad A_{n+1\,n+3} = A_{n\,n+2} + A'_{n+1\,n+2} + \qquad * \qquad ,$$

$$(7.03)\quad B_{2\,n+3} = A_{3\,n+3} - \frac{(n+2)(n+3)}{2}\,A_{1\,n+1}\;,$$

$$(7.04)\quad B_{i\,n+3} = A_{i+1\,n+3}\;,$$

$$(7.05)\quad B_{n\,n+3} = A_{n+1\,n+3}\;,$$

$$(7.06)\quad C_{2\,n+3} = B_{2\,n+3} - (n+3)\,B_{1\,n+2}\;,$$

$$(7.07)\quad C_{3\,n+3} = B_{3\,n+3} - \frac{(n+3)(n+2)}{2}\,B_{1\,n+1}\;,$$

$$(7.08)\quad C_{i\,n+3} = B_{i\,n+3}\;,$$

$$(7.09)\quad C_{n\,n+3} = B_{n\,n+3}\;.$$

Since 7.08 is entirely different from 7.06 or 7.07 it is evident that the values of the coefficient

$$(7.10)\qquad \frac{C_{n-1\,n+3}}{(n+3)!}$$

are divided into the three cases of n=3, n=4 and n> 4.

By means of 7.06, 7.03, 7.01, 3.05, 3.03 we obtain

$$(7.11)\quad \frac{C_{2\,6}}{6!} = -\,\frac{10P_4 - 16P_3' + 6P_2'' - 36P_2^{\,2}}{6!}\;.$$

By means of 7.07, 7.04, 7.01, 3.05, 3.03 we obtain

$$(7.12)\qquad \frac{c_{77}}{7!} = -\frac{-100P_4 + 20P_3' + 10P_2'' - 100P_2^2}{7!}.$$

By means of 7.08, 7.04, 7.01, 3.03 we obtain for $n > 4$

$$(7.13)\qquad \frac{c_{n-1\ n+3}}{(n+3)!} =$$

$$= \frac{\binom{n+1}{4}P_4 + 2\binom{n+1}{3}P_3' + \binom{n+1}{2}P_2'' - \binom{n+1}{2}P_2^2}{(n+3)!}.$$

The corresponding coefficient for the osculating curve of degree n is readily obtained from 4.17 and 4.14 for $i = n-1$. We have

(7.14)

$$c_{n-1\ n+3} = \frac{(n-2)a_{14} - (n-1)a_{25}}{(n-1)} + \frac{a_{12}^2}{(n-1)} + \frac{a_{12}b_{13}}{(n-2)}$$

$$- \frac{b_{13}^2}{2(n-3)!} - (n+1)b_{13}\,c_{n-1\ n+1},$$

where

$$(7.15)\qquad a_{12} = \frac{3nP_2}{2(n+2)}, \quad a_{23} = \frac{(3n-2)P_2}{2(n+2)} \quad \text{by 4.09},$$

$$(7.16)\qquad b_{13} = \frac{-P_2}{n+2} \qquad \text{by 7.15 and 4.12},$$

$$(7.17)\qquad c_{n-1\ n+1} = \frac{-(n+1)nP_2}{2(n+1)!} \qquad \text{by 4.20 and 3.07}.$$

However, $a_{14} = a_{25} = 0$ for $n=3$, $a_{25} = 0$ for $n=4$,

(7.18)

$$a_{14} = \frac{5(n+2)(n-2)P_4 - 4(n-3)(n+2)P_3' + 12(n-3)(n-2)P_2^2}{4!\,(n+2)^2}$$

for $n > 4$,

(7.19)

$$a_{25} = \frac{(n+2)(5n-14)P_4 - 4(n-4)(n+2)P_3{}' + 12(n-4)(n-3)P_2{}^2}{4!\,(n+2)^2}$$

for $n > 4$.

Hence the values of 7.14 are divided into three cases similar to those of 7.10. They are

(7.20) $$c_{21} = \frac{6\,P_2{}^2}{4!\,5^2\,2!} ,$$

(7.21) $$c_{37} = \frac{120P_4 - 48P_3{}' + 48P_2{}^2}{4!\,6^2\,3!} ,$$

(7.22) $$c_{n-1\,n+3} = \frac{-(n-6)(n+2)P_4 - 8(n+2)P_3{}' + 6(n^2-8)P_2{}^2}{4!\,(n+2)^2\,(n-1)!} ,$$

$n > 4$.

Fortunately the difference between 7.14 and 7.10 is in the same form for all three cases. We find that

(7.23)

$$c_{n-1\,n+3} - \frac{C_{n-1\,n+3}}{(n+3)!}$$

$$= \frac{(n+2)(5P_4 - 8P_3{}' + 3P_2{}'') - 3(5n+12)P_2{}^2}{3!\,(n+2)^2\,(n+3)\,(n-1)!} .$$

We shall denote the numerator of the right hand member of 7.23 by $\overline{M}$. It is our purpose now to express $\overline{M}$ in terms of ξ , the original coefficients p_i , and their derivatives; and so to choose the transformation 1.02 as to make $\overline{M} = 0$.

We have already insisted that the function λ of the transformation 1.03 satisfy the differential equation

(7.24) $$\lambda' + \lambda\pi_1 = 0 .$$

This equation 7.24 and its first k-1 derivatives enable us to solve for the k th derivative of λ in terms of λ and the first k-1 derivatives of π_1 . We find

(7.25)

$$\lambda^{(k)} = (-1)^k \lambda \begin{vmatrix} \pi_1 & 1 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ \pi_1' & \pi_1 & 1 & 0 & 0 & \cdot & \cdot & 0 \\ \pi_1'' & 2\pi_1' & \pi_1 & 1 & 0 & \cdot & & 0 \\ \pi_1''' & 3\pi_1'' & 3\pi_1' & \pi_1 & 1 & \cdot & \cdot & 0 \\ \pi_1^{iv} & 4\pi_1''' & 6\pi_1'' & 4\pi_1' & \pi_1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \pi_1^{(k-1)} & \binom{k-1}{1}\pi_1^{(k-2)} & \binom{k-1}{2}\pi_1^{(k-3)} & \cdot & \cdot & \cdot & & \pi_1 \end{vmatrix}$$

Equation 2.05 with the above valuesfor the derivatives of λ then expresses P_m in terms of $\pi_1, \pi_2, \cdots, \pi_m$ and the first m-1 derivatives of π_1 . The resulting expressions for P_2 , P_3 , and P_4 are

(7 2 6)

$$P_2 = \begin{vmatrix} \pi_1 & 1 \\ \pi_1' & \pi_1 \end{vmatrix} - 2\pi_1\pi_1 + \pi_2 ,$$

$$P_3 = -\begin{vmatrix} \pi_1 & 1 & 0 \\ \pi_1' & \pi_1 & 1 \\ \pi_1'' & 2\pi_1' & \pi_1 \end{vmatrix} + 3\pi_1 \begin{vmatrix} \pi_1 & 1 \\ \pi_1' & \pi_1 \end{vmatrix} - 3\pi_2\pi_1 + \pi_3 ,$$

$$P_4 = \begin{vmatrix} \pi_1 & 1 & 0 & 0 \\ \pi_1' & \pi_1 & 1 & 0 \\ \pi_1'' & 2\pi_1' & \pi_1 & 1 \\ \pi_1''' & 3\pi_1'' & 3\pi_1' & \pi_1 \end{vmatrix} - 4\pi_1 \begin{vmatrix} \pi_1 & 1 & 0 \\ \pi_1' & \pi_1 & 1 \\ \pi_1'' & 2\pi_1' & \pi_1 \end{vmatrix}$$

$$+ 6\pi_2 \begin{vmatrix} \pi_1 & 1 \\ \pi_1' & \pi_1 \end{vmatrix} - 4\pi_3\pi_1 + \pi_4 .$$

With these as a pattern it is a simple matter to write down the corresponding expressions for P_5, P_6, . . . , P_{n+1}.

The values for $\pi_1, \pi_2, \pi_3, \pi_4$ were given by the relations of 2.03. From the first one of these we obtain

(7.27)

$$\xi'\pi_1 = f_1 + \frac{n}{2}\psi ,$$

$$(\xi')^2\pi_1' = f_1' - \psi f_1 + \frac{n}{2}\left[\psi' - \psi^2\right] ,$$

$$(\xi')^3\pi_1'' = f_1'' - 3\psi f_1' - \left[\psi' - 2\psi^2\right] f_1$$

$$+ \frac{n}{2}\left[\psi'' - 4\psi'\psi + 2\psi^3\right] ,$$

$$(\xi')^4\pi_1''' = f_1''' - 6\psi f_1'' - \left[4\psi' - 11\psi^2\right] f_1'$$

$$- \left[\psi'' - 7\psi\psi' + 6\psi^3\right] f_1$$

$$+ \frac{n}{2}\left[\psi''' - 7\psi''\psi - 4\psi'\psi' + 15\psi'\psi^2 - 6\psi^4\right] .$$

Now let $\mathfrak{P}_i$ be the same function of the coefficients p_i that P_i is of the coefficients π_i. (See 7.26)

By direct substitution we obtain from 7.26, 7.27 and 2.03 the expressions for P_i in terms of $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_i$. Below are given the results for P_2, P_3, and P_4 also certain derivatives of P_2 and P_3.

(7.28)

$$(\xi')^2 P_2 = \mathfrak{P}_2 - \frac{n+2}{12}\left[2\psi' - \psi^2\right] .$$

(7.29)

$$(\xi')^3 P_2' = \mathfrak{P}_2' - 2\psi\mathfrak{P}_2 - \frac{n+2}{6}\left[\psi'' - 3\psi'\psi + \psi^3\right],$$

$$(\xi')^3 P_3 = \mathfrak{P}_3 - 3\psi\mathfrak{P}_2 - \frac{n+2}{4}\left[\psi'' - 3\psi'\psi + \psi^3\right].$$

(7.30)

$$(\xi')^4 P_2'' = \mathfrak{P}_2'' - 5\psi\mathfrak{P}_2' - 2\left[\psi' - 3\psi^2\right]\mathfrak{P}_2$$

$$- \frac{n+2}{6}\left[\psi''' - 6\psi''\psi + 12\psi'\psi^2 - 3\psi'\psi' - 3\psi^4\right].$$

(7.31)

$$(\xi')^4 P_3' = \mathfrak{P}_3' - 3\psi\mathfrak{P}_3 - 3\psi\mathfrak{P}_2' - 3\left[\psi' - 3\psi^2\right]\mathfrak{P}_2$$

$$- \frac{n+2}{4}\left[\psi''' - 6\psi''\psi + 12\psi'\psi^2 - 3\psi'\psi' - 3\psi^4\right].$$

(7.32)

$$(\xi')^4 P_4 = \mathfrak{P}_4 - 6\psi\mathfrak{P}_3 - \left[(n+6)\psi' - \frac{n+24}{2}\psi^2\right]\mathfrak{P}_2$$

$$- \frac{3(n+2)}{10}\left[\psi''' - 6\psi''\psi + 12\psi'\psi^2 - 3\psi'\psi' - 3\psi^4\right]$$

(7.33)

$$+ \frac{(n+2)(5n+12)}{246}\left[4\psi'\psi' - 4\psi'\psi^2 + \psi^4\right].$$

(7.33)

$$(\xi')^4 P_2^2 = \mathfrak{P}_2^2 - \frac{(n+2)}{6}\left[2\psi' - \psi^2\right]\mathfrak{P}_2 + \left(\frac{n+2}{12}\right)^2\left[2\psi' - \psi^2\right]^2.$$

On substituting 7.32, 7.31, 7.30, 7.33 into the expression for $\overline{M}$ we obtain

(7.34)
$$\overline{M} = \frac{1}{(\xi')^4}\left(M + 15\psi\theta_3\right),$$

where

(7.35)
$$\theta_3 = 3\mathfrak{P}_2' - 2\mathfrak{P}_3,$$

(7.36)
$$M = (n+2)\left[5\mathfrak{P}_4 - 5\mathfrak{P}_3' + 3\mathfrak{P}_2''\right] - 3(5n+12)\mathfrak{P}_2^2.$$

From 7.29 it is evident that $\overline{\theta}_3$ is an invariant* of ranke 3 represented by

(7.37)
$$\overline{\theta}_3 = 3P_2' - 2P_3 = \frac{1}{(\xi')^3}\theta_3$$

The difference

(7.38)
$$\frac{c_{n\,n+3}}{(n+3)!} - c_{n\,n+3}$$

is readily computed from the relations 7.07, 7.08 and 4.16. The values of these coefficients are divided into the two cases of $n=3$ and $n>3$. However, in each case the above difference is equal to $\overline{\theta}_3$ except for a constant factor. Hence the assumption of 6.07 is equivalent to assuming that $\overline{\theta}_3 \neq 0$.

If θ_3 is different from zero it is always possible to make $\overline{M} = 0$ by choosing ψ such that

(7.39)
$$\psi = -M/15\theta_3.$$

From the form of 7.34 it is evident that the form $\overline{M} = 0$ is preserved under further transformations of the independent variable only if the function ψ of the transformation is zero.

Suppose that the transformation 1.02 is followed by the transformation 1.03. Relations 3.05 express the new coefficients in terms of π_i, λ and the derivatives of λ.

*See Stouffer, Bulletin of the American Mathematical Society, vol. 34, page 292.

By direct substitution we find that the same function $\overline{\overline{M}}$ of these new coefficients that $\overline{M}$ is of the coefficients π_i is such that $\overline{\overline{M}} = \overline{M}$. Hence the function $\overline{M}$ is invariant under the dependent variable transformation and the form $\overline{M} = 0$ is therefore not disturbed.

From the first equation of 2.03 it is evident that the form $P_1 = 0$ is not disturbed by the transformation of the independent variable if $\psi = 0$. From the first equation of 2.05 it is evident that the form $P_1 = 0$ is preserved under further transformations of the dependent variable only if the function λ' of the transformation is zero.

FUNDAMENTAL THEOREM I. *Any linear homogeneous differential equation 1.00 may be transformed into a canonical form 2.00 by means of the transformation 1.02 with* $\xi''/\xi' = -M/15\theta_3$, *followed by the transformation 1.03 with* $\lambda'/\lambda = -\pi_1$, *if* $\theta_3 \neq 0$. *This canonical form is characterized by* $\overline{M} = \bar{P}_1 = 0$ *and is preserved under further transformations only if the functions* λ *and* ξ *of the transformations are such that* $\lambda' = \xi'' = 0$.

FUNDAMENTAL THEOREM II. *Geometrically the canonical form is characterized by the fact that the (n+1)th vertex* $Y^{(n)}$ *of the canonical polyhedron of reference is in the unique linear (n–1)–space determined by the osculating linear (n–2)–space and the principal tangent plane to the curve and its osculating curve of degree n. The osculating curve of degree n cuts the above mentioned unique linear (n–1)–space in the contact point t=0 counted (n–1) times, and in another point designated by* $t = x$. *The contact point t=0 is the first vertex point (1,0, . . . ,0) of the canonical polyhedron of reference. The linear (n–1)–space osculating the osculating curve of degree n at the point* $t = x$ *intersects the tangent line common to the original curve and its osculating curve of degree n* in the second point $(0,1,0,\ldots,0)$ of the canonical polyhedron of reference. The linear $(n-1)$-spaces osculating the osculating ing curve of degree n intersect the osculating plane common to the original curve and its osculating curve of degree n in a one parameter family of lines enveloping a conic. The polar line of the point $(0,1,0,\ldots,0)$ with respect to this conic is the line joining the first and third vertices of the canonical polyhedron. The third vertex $(0,0,1,0,\ldots,0)$ is at the intersection of the above mentioned polar line with the tangent line to the curve generated by Y'.

§ 8. INVARIANTS AND COVARIANTS

For the sake of brevity let us represent the linear equation 2.04 by $L(P_i)Y = 0$. Further, let this equation be transformed into another equation $L(\bar{P}_i)\bar{Y} = 0$, in the same canonical form as 2.04. The most general transformation relating these two equations is of the form $\bar{\xi} = \bar{\xi}(\xi)$, and $Y = \bar{\lambda}\,\bar{Y}$, where $\bar{\xi}'' = \bar{\lambda}' = 0$. From the form of 2.03 and 2.05 it is at once evident that each $\bar{P}_i$ differs from the corresponding coefficient P_i only by a constant factor. That is, the coefficients of the canonical form are definitely fixed except for constant factors. Again, from the form of 2.02 and 2.06 it is at once evident that $\bar{Y}^{(m)}$ differs from $Y^{(m)}$ only by a constant factor. Since $Y_i^{(m)}$ represent the homogeneous coordinates of the (m+1)th vertex of the canonical polyhedron of reference it follows that the point $\bar{Y}^{(m)}$ coincides with $Y^{(m)}$ for each value of m. That is, the vertices of the canonical polyhedron are absolutely fixed.

Furthermore, the functions P_i of the original coefficients $p_1, p_2, \ldots, p_i$ and their derivatives are in the form of relative invariants. In order to show this, let us consider two linear homogeneous differential equations $L(p_i)y = 0$ and $L(\bar{p}_i)\bar{y} = 0$ related by permissable transformations of the form of 1.02 and 1.03. Two such equations are said to be equivalent. According to the first fundamental theorem these two equations may be transformed into two canonical forms $L(P_i)Y = 0$ and $L(\bar{P}_i)\bar{Y} = 0$. Now each P_i is expressable in terms of $p_1, p_2, \quad, p_i$ as in 7.28, 7.29 and 7.32, and the corresponding coefficient $\bar{P}_i$ is the same function of $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_i$. Now the p_i are related to $\bar{p}_i$ and these in turn are related to $\bar{P}_i$ by definite transformations, hence P_i may be expressed in terms of $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_{n+1}$. But P_i and $\bar{P}_i$ are coefficients of two canonical forms hence, as we have just seen, can differ only by a constant factor. It follows then that each P_i function of $p_1, p_2, \ldots, p_i$ differs from the corresponding $\bar{P}_i$ function of $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_i$, and consequently, from the P_i function of $\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_i$ by at most a constant factor. Hence each P_i is in the form of a relative invariant.

In exactly the same way we show that the functions $Y^{(m)}$ of the original coefficients $p_1, p_2, \ldots, p_{n+1}$ and derivatives of y are in the form of relative covariants.

The invariants P_2, P_3, and P_4 are at once obtained from 7.28, 7.29 and 7.32 on the substitution of $-M/16\theta_3$ for ψ. The ~~other~~ invariants $P_2, P_3, \ldots, P_{n+1}$ form our complete system of invariants. All other invariants are combinations of these or their derivatives.

One simple invariant $\bar{\theta}_4$ might be noted in passing. The derivative of $\bar{\theta}_3$ is simply

(8.01)
$$\bar{\theta}_3' = \frac{1}{(\xi')^4}\left[\theta_3' - 3\psi\,\theta_3\right].$$

Hence from the form of 7.34 it is evident that $5\bar{\theta}_3' + \bar{M}$ is an invariant of rank 4. In order to conform to the notation of Wilczynski we shall denote this last invariant by $5(n+2)\bar{\theta}_4$. Then

(8.02)
$$\bar{\theta}_4 \equiv P_4 - 2P_3' + \frac{6}{5}P_2'' - \frac{3(5n+12)}{5(n+2)}P_2^2.$$

The covariants Y, Y', Y'' may be computed from 2.06, 2.02, 7.25, 7.27, and 7.39. On reversing the relations obtained from 2.06 for m = 0,1,2, we have

(8.03)
$$\lambda Y = \eta,$$
$$\lambda Y' = \eta' - \frac{\lambda'}{\lambda}\eta,$$
$$\lambda Y'' = \eta'' - 2\frac{\lambda'}{\lambda}\eta' - \left[\frac{\lambda''}{\lambda} - 2\left(\frac{\lambda'}{\lambda}\right)^2\right]\eta.$$

Again, on reversing the relations obtained from 2.02 for m = 0,1,2, we have

(8.04)
$$\eta = y ,$$
$$\xi'\eta' = y' ,$$
$$(\xi')^2\eta'' = y'' - \psi y' .$$

From 8.03, 8.04, 7.25, 7.27 and 7.39 we obtain

(8.05)
$$\lambda Y = y ,$$
$$\xi'\lambda Y' = y' + \left(p_1 + \frac{n}{2}\psi\right) y ,$$
$$(\xi')^2\lambda Y'' = y'' + \left[2p_1 + (n-1)\psi\right] y'$$
$$+ \left[p_1^2 + (n-1)p_1\psi + p_1' + \frac{n}{2}\psi' + \frac{n^2-2n}{4}\psi^2\right] y$$

where $\psi = -M/15\theta_3$.

The other covariants Y''', . . ., $Y^{(n)}$ of our complete system of covariants may be computed in a similar manner. The derivatives of order equal to or greater than n+1 are expressable in terms of these covariants by means of the canonical equation 2.04.

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