MATHEMATICAL ECONOMICS BEFORE COURNOT

by

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R. M. R.

September 8, 1948

CHAPTER I

THE PLAN OF THE DISSERTATION

This thesis is prompted by the strong feeling that doctrinal developments in the science of economics, taken in their entirety, have not been adequately interpreted. This is a feeling which is shared, I believe, by some well-informed people, though comments to this effect appear only sporadically in the literature. Whatever may be one's judgment, however, regarding the most significant lines of development in our field, there can be no doubt that a great mass of material lies hidden and unread simply because it is not accessible to most readers.

It is my present purpose to make readily available to an interested few that part of the material listed in the Jevons-Fisher bibliography of mathematico-economic literature which appeared before Cournot's *Recherches*. I am encouraged in the thought that after a careful examination of the mathematical sections of these books, we may draw some inferences which will enable us to take a more accurate view of the development of economics. This latter objective, however, is a purely secondary one.

The Jevons bibliography appeared in 1879 in the second edition to Jevon's Theory of Political Economy.[1] The Fisher bibliography is contained in the English translation of A. A. Cournot's Mathematical Principles of the Theory of Wealth.[2] These two bibliographies are reproduced in Appendices A and B, and discrepancies between them are noted there. As is well known, the Jevons-Fisher Bibliography is divided into four periods:

1. Ceva to Cournot (1711-1837);
2. Cournot to Jevons (1838-1870);
3. Jevons to Marshall (1871-1889);
4. Marshall to the time of Fisher's compilation (1890-1897).

If there be some doubt in the reader's mind as to the usefulness of a close examination of these works, I can only cite, in support of my own feeling of their significance, Jevon's own words in this regard. In his introductory lecture at the opening of the Session, 1876-1877, at University College, London, Faculty of Arts and Laws, Jevons made these remarks.

> "Now, too, that attention is at last being given to the mathematical character of the science, it is becoming apparent that a series of writers in France, Germany, Italy, and

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1. W. S. Jevons, Theory of Political Economy, 4th ed., London: Macmillan and Co., 1931, pp. 322-326.
2. A. A. Cournot, Researches into the Mathematical Principles of the Theory of Wealth, London: Macmillan and Co., 1897, pp. 173-176.

> England have made attempts towards a mathematical theory. Their works have been almost unnoticed, or, at any rate forgotten, mainly on the account of the prejudices against the line of inquiry they adopted. It is much to be desired that some competent mathematician and economist should seek these works out and prepare a compendious abstract of their contents, in the manner of Mr. Todhunter's valuable histories of mathematical science."[1]

I do not presume to meet the qualifications which Mr. Jevons required of the individual who should seek these works out, nor have I attempted to work over the entire bibliography. The present paper is confined to the first of the four periods indicated above--i.e., Ceva to Cournot--for the reason that there is an almost universal lack of familiarity with these early writers. There is the conviction, I am sure, even among those well schooled in the history of economic thought, that mathematical methodology in the field began with A. A. Cournot and that nothing of significance in this regard appeared before him. That this opinion should be changed is the thesis which I propose to defend.

This dissertation has been given the title of Mathematical Economics Before Cournot with some misgivings. It will be apparent that most of the material included has been suggested

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1. W. S. Jevons, "The Future of Political Economy," The Fortnightly Review, vol. 30 (New Series), p. 626.

by the basic bibliography of Jevons and Fisher. Other materials have been found, however, which should serve as valuable additions. Books which are known to everyone familiar with the basic literature of economic thought have been included because they have a mathematical significance which has been overlooked. In this category I should place some of the works of the Physiocrats.

A very careful survey of economic literature before 1838 has been made in an effort to uncover examples of economic writing in which there is a real attempt to use a mathematical method. I hope that I have touched upon all materials of significance. I have the feeling that not very much of importance has eluded my attention, though there is reason to believe that a detailed investigation of some of the early Italian writers and of some eighteenth and early nineteenth century Germans might prove especially fruitful.

One further comment should be made. In a work of this sort, it is always difficult to decide upon a plan of approach. I hesitated to present the material in strict chronological order for fear that authors of especial significance might not stand out clearly. I thought for a time of classifying the writings according to the fields into which we nowadays arbitrarily, and, I think, somewhat foolishly divide our studies. That is, I originally had the feeling that material important in Price and Income Theory should be taken up in one chapter, that anything of significance in Monetary Theory should be put in another chapter, and that the works of interest to the specialist in Public Finance should be arranged in another chapter. Unfortunately, early writers, interested for the most part in treating the study of economics as an entity, frequently took up so many matters in a fairly short work that some parts of a very brief article would have appeared in all chapters. Indeed, an important characteristic of the mathematical writer's approach to economics is his preoccupation with the essential unity of the economic system.

As the writing of the material from original sources progressed, I became impressed with a similarity of approach by writers of the same nationality. Then, too, since I am not a linguist, some personal difficulties were encountered in the frequent transition from one language to another. It was decided then, both for the convenience of the reader and of myself, to take up the material chronologically by nationality of writer. Any other way would have done as well perhaps, but this approach has the advantage of leaving a very definite feeling for the national contribution.

Briefly, the plan of the work is as follows. In Chapter II, we shall see what traces of mathematical methodology may be found in the literature with which everyone is familiar. Chapters III, IV, V, and VI contain the early contributions of the French, Italian, German, and English economists respectively. In Chapter VII will be found some comment on the analyses which seem important from a modern point of view. Finally, in Chapter VIII, we shall reflect at some length on the way in which our present body of theory has developed.

CHAPTER II

MATHEMATICAL METHODOLOGY IN THE FAMILIAR LITERATURE

Whenever anyone compiles, as did Jevons and Fisher, a special bibliography in a restricted field, there is a strong implication that the writings which are not included contain little of interest. A considerable effort has been made to discover some useful mathematical method in the works of important figures in the history of economic thought before 1838 who were omitted from the Jevons-Fisher bibliography.

Despite the unlikelihood of discovering anything new in the writings of the Classical economists, the giants of that school were examined first. In Smith, Ricardo, T. Malthus, Bentham, James Mill, and Senior, I failed to find more than fleeting passages which might be construed as mathematical, with the proviso, of course, that simple arithmetic examples not be so considered.

There is, to be sure, the statement of Malthus, quoted by Jevons, which led to an eager search of Malthus' writings. In a brief pamphlet, _Observations on the Effects of the Corn Laws, and of a Rise or Fall in the Price of Corn on the_

Agriculture and General Wealth of a Country, Malthus discusses the merits of a large manufacturing population. Manufactures and commerce, he says, may be beneficial to a country in a number of ways, but beyond a certain point the evils outweigh the advantages. It is not a question of whether a country should be entirely a manufacturing state or a purely agricultural one, but at what point a growth in manufacturing should be halted. Then he adds, "Many of the questions both in morals and politics seem to be of the nature of problems de maximus and minimus in Fluxions; in which there is always a point where a certain effect is the greatest, while on either side of this point it gradually diminishes."[1] At no other place, apparently, does he refer to the possibility of applying the calculus to economic problems. Despite the interest evidenced in his later writings in an inductive verification of his basic thesis regarding the relative increase of population and the means of subsistence, there is nothing which could be called mathematical in our sense of the term.

There can be no doubt but what both Ricardo and Senior had some grasp of the ideas which mathematical economists were later to explain so clearly. Nor can there be any question that their failure to use even the simplest geometric

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1. Thomas Malthus, Observations on the Effects of the Corn Laws, a reprint, Baltimore, The Johns Hopkins Press, 1932, p. 25.

aids left obscure much that was not properly cleared up for over half a century. Especially is this true of their concept of demand, though Knight contends, and I suppose rightly, that Senior did understand what was later to be called the law of diminishing utility.[1] Both Ricardo and Senior had some notion of varying inputs of the factors of production. Ricardo never did get away from the concept of "doses" of capital and labor, whereas Senior did see the necessity for varying only one factor in productivity analysis.[2] In no case, however, is there an explicit use of mathematics in analysis.

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1. See Nassau W. Senior, An Outline of the Science of Political Economy, a reprint, London: George Allen and Unwin, Ltd., 1938, pp. 11-12. "Not only are there limits to the pleasure which commodities of any given class can afford, but the pleasure diminishes in a rapidly increasing ratio long before those limits are reached. Two articles of the same kind will seldom afford twice the pleasure of one, and still less will ten give five times the pleasure of two. In proportion, therefore, as any article is abundant, the number of those who are provided with it, and do not wish, or wish but little, to increase their provision, is likely to be great; and, so far as they are concerned, the additional supply loses all, or nearly all, its utility. And in proportion to its scarcity the number of those who are in want of it, and the degree in which they want it, are likely to be increased, and its utility, or, in other words, the pleasure which the possession of a given quantity of it will afford, increases proportionally."
2. See David Ricardo, The Works of David Ricardo, McCulloch ed., London: John Murray, 1888, pp. 42-43 and p. 376. The latter page contains the "Table, Showing the Progress of Rent and Profit Under an Assumed Augmentation of Capital" to be found in Ricardo's Essay on the Influence of a Low Price of Corn on the Profits of Stock. Compare Senior, op. cit, pp. 84-85.

Among contemporary continental writers two stand out. Heinrich Friedrich von Storch (Henri Storch), that curious German who lived and taught in Russia and wrote for the most part in French, was frequently at great pains to put statistical data in functional relation in his many tables. Particularly is this true with regard to monetary relations.[1] He attempts in one passage to draw up an empirical table relating price and quantity taken, but he makes an altogether erroneous application of his data.[2] On the whole, one feels that Storch frequently is verging on some generalizations in mathematical terms, but that a disposition to stay away from specific laws prevents him.

The French follower of Smith, Jean-Baptiste Say, is frequently quoted as one of the early opponents of mathematical method. In the introduction to his Treatise on Political Economy is a comment which later will give us insight into the failure of the founders of the study of political economy to apply mathematics.

> "It would, however, be idle to imagine that greater precision, or a more steady direction could be given to this study, by the application of mathematics to the solution of its problems. The values with which political economy is concerned, admitting of the application to them of the terms plus and minus, are indeed within the range of mathematical

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1. Henri Storch, Cours d'Économie Politique, St. Petersburg: A. Pluchart & Comp., 1815, VI, 315.
2. Ibid., I, 191.

> inquiry; but being at the same time subject to the influence of the faculties, the wants and the desires of mankind, they are not susceptible of any rigorous appreciation, and cannot, therefore, furnish any data for absolute calculations. In political as well as in physical science, all that is essential is a knowledge of the connection between causes and their consequences. Neither the phenomena of the moral or material world is subject to strict arithmetrical computation."[1]

Already we see the difficulty in Say's mind, but it is made even more clear in a lengthy footnote which indicates strongly his extreme uneasiness about the question of method. For he goes on to point out that while we know that in any given year

> ". . . the price of wine will infallibly depend upon the quantity to be sold, compared with the extent of the demand . . . , if we are desirous of submitting these two data to mathematical calculation, their ultimate element must be decomposed before we can become thoroughly acquainted with them, or can, with any degree of precision, distinguish the separate influence of each."[2]

The difficulty is that we must have in hand so much information ". . . to determine the quantity (of the wine) to be put into circulation; itself but one of the elements of price."[3] On the demand side there are even more difficulties to be encountered, including the problem of determining the subjective evaluations of all the consumers. In short, all these many variables cannot be considered in a mathematical analysis and those who have

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1. J. B. Say, A Treatise on Political Economy, Translated from the French by C. R. Prinsep, Philadelphia: Grigg, Elliott, and Co., 1849, p. xxvi.
2. Ibid, p. xxvii.
3. Idem.

attempted to do it

> ". . . have not been able to enunciate these questions into analytical language, without divesting them of their natural complication, by means of simplifications, and arbitrary suppressions, of which the consequences, not properly estimated, always essentially change the condition of the problem, and pervert all its results; so that no other inference can be deduced from such calculations than from formula arbitrarily assumed."[1]

In short, since it is obviously impossible to write explicit functions involved, we cannot use mathematics at all. We shall have occasion to comment on this matter later.

It may be noted in passing that several Britons who were contemporaries of the Classical writers expressed clearly, in words, certain ideas which were later to gain universal acceptance when formulated in mathematical language. Among these "neglected British economists" are John Rooke and Mountifort Longfield. The former had an accurate conception of the use of marginal costs in analysis, and the latter clearly showed an understanding of the importance of marginal productivity in the theory of distribution.[2] Thomas Tooke's _Thoughts and Details on the High and Low Prices of Thirty Years, from 1793 to 1822_ contains ample

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1. _Idem._
2. E. R. A. Seligman, "Some Neglected British Economists," _Economic Journal_, vol. XIII, pp. 511-514 and pp. 525-533. See also John Rooke, _An Inquiry Into the Principles of National Wealth_, Edinburgh: A. Balfour & Co., 1834, and Mountifort Longfield, _Lectures on Political Economy._ Reprinted by the London School of Economics and Political Science, 1931, published originally at Dublin in 1834.

evidence that this author understood very well the nature of demand and the idea of elasticity of demand.[1] Refreshingly original as these thinkers were, however, their methodological approach was essentially the same as that of their better-known contemporaries.

If in the writings of the Classical period we find no use of mathematical method, to what reasonably well-known literatures may we turn? That group of Englishmen, whose gropings in the field of Political Arithmetic form perhaps the earliest important body of work in the literature of economics, come first to mind. Sir William Petty, John Graunt, Gregory King, and Charles Davenant were probably the most important of this group. Close attention to Petty reveals certain passages of apparent interest. Concerning his argument, Petty says,

> "The Method I take to do this, is not yet very usual; for instead of using only comparative and superlative Words, and intellectual Arguments, I have taken the course (as a specimen of the Political Arithmetick I have long aimed at) to express myself in Terms of Number, Weight, or Measure."[2]

1. Thomas Tooke, Thoughts and Details on the High and Low Prices of Thirty Years, from 1793 to 1822, 2nd ed., London: John Murray, 1824.
2. Sir William Petty, The Economic Writings of Sir William Petty, London: C. J. Clay and Sons, 1899, I, 244.

Certain isolated bits are indicative of attempts at fruitful mathematical generalization. Working, however, from admittedly inadequate statistical material, Petty was forced to make guesses which today seem preposterous; however much insight his works may indicate after careful interpretation, we can scarcely say that his writing is significantly mathematical.

King's law, on the other hand, comes close to being true mathematical economics. One statement of this law is as follows:

> "We estimate that a deficit in the harvest of wheat will cause the price of wheat to rise above the normal price in the proportions indicated: when the harvest of wheat is deficient by 1/10, 2/10, 3/10, 4/10, 5/10, the price of wheat will rise respectively by 3/10, 8/10, 16/10, 28/10, and 45/10."[1]

Whether this law was in fact enunciated by Gregory King or by Charles Davenant is a matter which at present does not concern us.[2] The point is that there is indicated a realization of a definite relationship between one varying quantity (presumably causal) and its effect varying in an inverse way.

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1. Henri Guitton, Essai Sur La Loi De King, Paris: Librairie Du Recueil Sirey, 1938, p. 10.
2. There is an interesting observation of King's which indicates his thorough understanding of the principles of demand. See his comments regarding the variance of quantities of ale consumed in England between 1688 and 1695 as a result of price changes consequent upon excises levied. Gregory King, Two Tracts, a reprint, Baltimore: The Johns Hopkins Press, 1936, p. 10.

It is not much of a step to lay down, in mathematically general language, a law of demand, but we cannot wish for what might have been.[1]

Passing to the vast literature of those writers who took an essentially mercantilist approach or who, if not adhering to basic mercantilist economic tenets, were concerned with approximately the same subject matter, we find little of interest. Thomas Mun, Thomas Wilson, and Thomas Culpepper yield nothing to the point. Josiah Child is conscious of the relationship between varying quantities, as is apparent in his treatment of the increase of capital stock in a country as primarily a function of the rate of interest,[2] and it is perfectly clear that he has a modern awareness of the functional relationship between the rate of investment in a country and the rate of interest.[3] A recognition of the importance of functional relationships in economic life is emphasized by Jean Bodin in his early exposition of a mechanistic quantity theory of money, but conscious mathematical method is lacking.[4]

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1. For a discussion of the significance of political arithmetic in the history of economic thought see Jacob H. Hollander, "The Dawn of a Science," Adam Smith, 1776-1926, Chicago: The University of Chicago Press, 1928, pp. 3-9. I should take issue with Hollander's statement that political arithmetic represented the "main current" of thought prior to 1750, for the reason that there was no discernible "current" of economic thought at this early date.
2. Sir Josiah Child, A New Discourse of Trade, 5th ed., Glasgow: Robert and Andrew Foulis, 1751, p.11.
3. Ibid., p. 38.
4. Jean Bodin, Response of Jean Bodin to the Paradoxes of Malestroit and the Paradoxes, Translated from the French Second Edition, Paris, 1578, by George Albert Moore, Washington: The Country Dollar Press, 1946.

James Steuart probably comes as close to a mathematical notation as anyone mentioned in this section. In a consideration of money and of the balance of payments of a country, he lets A, B, C, and D stand for variable quantities in his discussion.

> "Let this quantity of coin, necessary for circulating the paper money, be called (B) and let the paper be called (C); consequently (A) will be equal to the sum of (B) plus (C). Again we have said, that all balances owing by nation to nation, are paid either in coin, in the metals, or in bills; and that bank paper can be of no use in such payments. Let the quantity of the metals, coin, or bills, going out or coming into the country for payment ofsuch balance, be called (D). These short designations premised, we may reason with more precision."[1]

The reasoning is so easy to follow, and the relationships disclosed are so very simple that I have not included this particular example with the English writings in Chapter VI. It is perhaps as significant as some material to be discussed later at considerable length.

3

We come now to that illustrious group of men who called themselves les économistes and who comprise the first cohesive "school" of economic thought--the Physiocrats. Of the Physiocrats, the only writer in either the Jevons or the Fisher bibliographies is Condorcet, one of the lesser figures on the periphery of the

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1. Sir James Steuart, An Inquiry Into the Principles of Political Economy, London: Miller and Cadell, 1767, II, 170.

main group whose *Vie De Turgot* throws some light on the life of the statesman whom he admired so much. The brief but complicated and obscure passage in Condorcet's *Life of Turgot* will be reported in some detail in Chapter III. It is sufficient at this point to note that it purports to show Turgot's ideas concerning the most equitable way of changing from a system of predominantly indirect taxes to one which rests solely (or predominantly) on direct taxes on the *produit net*. One might expect to find an analysis in Turgot's works similar to that of Condorcet, but it is disappointing to discover that there is little in Turgot of an explicitly mathematical nature. In his letters the French statesman refers to a problem in physics or in mathematics, and in one instance he uses an algebraic notation in discussing an economic matter. In considering the possibility of supporting the government by the arbitrary issue of paper money, he indicates the ". . . progression following which the paper money ought to be augmented each year."[1] He supposes there to be in circulation 1,200,000,000 livres, and that the expenses of the government per annum are one-fourth this amount. If the King issues 300,000 livres in paper to pay for state expenses, such a depreciation will take place that

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1. Gustave Schelle, editor, *Oeuvres de Turgot*, Paris, Libraire Felix Alcan, 1913, I., 149.

in the next year a greater absolute sum must be issued. In the first year the total amount in circulation will be

$$a + \frac{a}{4} = 1.200 + 300 = 1.500$$

and in the second year the amount will be

$$a + \frac{a}{4} + \frac{a + \frac{a}{4}}{4} = 1.500 + \frac{1.500}{4} = 1.975$$

Unfortunately, at this precise point the letter in which this analysis is contained comes to an abrupt end.

It should be noted in passing that Turgot, as well as Quesnay and Dupont de Nemours, had an adequate working concept of the principles involved in demand analysis. Their empirical studies of the elasticity of demand for grains, as reflected in total selling value, never bring them to the point, however, of defining the concept about which they write so much.[1]

In the light of present-day preoccupation with economic aggregates, the Physiocratic approach to economic analysis is of especial significance. Upon a first consideration, the famous device invented by Quesnay for use in that analysis, the Tableau Économique, seems a truly mathematical one. As is well known, the Tableau shows, by arithmetical example, how

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1. See, for example, Dupont de Nemours, De l'exportation et de l'importation des Grains, from the Collections des Economistes, Paris, 1911.

investment in agriculture gives rise to a true economic surplus called net product, which passes first as rent to the proprietary class of society and subsequently is expended on the goods of both the farmer and the industrialist.[1] That Quesnay and the chief Physiocrats were on the threshold of a true mathematical method cannot be denied. As has been pointed out in a clever article which appeared recently,

> "They tried first to show how a given national income, if it is to be maintained, must be distributed among the various types of expenditures and, secondly, to indicate the factors which might alter the size of that income. Thus the analogy with the dependent variable which Keynes wants to explain is very close. . . . They conceived total production as a function of aggregate expenditure. . . . The rate of investment in farming directly determines agricultural production and the other variables, according to a system of functional relationships which are well described by Turgot."[2]

Although this sounds interesting, we must not be deceived by the modern terminology. While the Physiocrats may well have been ". . . the first to build a macro-static theory in terms of production, income, and expenditures,"[3] the only mathematical

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1. For those interested in reviewing the matter, the two best short presentations are to be found in Auguste Oncken, Oeuvres de Quesnay, Paris: Jules Peelman, 1888, Analyse du Tableau Economique, pp. 305-328 and in the facsimile reproduction for the British Economic Association entitled Tableau Oeconomique, London: 1894.
2. Maurice Lamontagne, "Some French Contributions to Economic Theory," The Canadian Journal of Economics and Political Science, 13:514-515, November, 1947.
3. Ibid, p. 516.

device used is arithmetical. The *Tableau Économique* could well be translated into algebraic functions. The difficulty is that, in order to discern the proper relationships, we should have to work through a good deal of unexciting, eighteenth-century French prose, for certainly the mathematical ideas do not come ready-made from the *Tableau.* And of course we are not interested in the present essay with what we might do to the original essentially non-mathematical presentation.

4

Generally speaking we shall find that attempts to use a mathematical method are made by those who adhere to no "school," who on the whole take up economic investigation with few prejudgments, and who achieve frequently an amazing quality of originality. These writers defy classification; they will not be put into categories. It is even very difficult to make generalizations as to their background and their work, though a fairly safe one seems to be that nearly all were either university professors or engineers.

There are only a few indications that within the four countries represented the writers influenced each other. Nevertheless the materials which follow will be handier as sources of reference if some kind of classification is made. We pass, then, to a consideration, by countries, of these individualists in our science. The first body of literature presented is that of the French.

CHAPTER III

THE EARLY FRENCH MATHEMATICAL ECONOMISTS

1

Edme Mariotte (1620-1684)

Mariotte is included only because he is listed in both the Jevons and Fisher bibliographies. The section cited is brief and contains only arithmetic notation, but the idea expressed is nearly the same as that of Daniel Bernoulli. The essays of Mariotte are almost entirely in the field of physics as is indicated by the several titles of his essays: De la Percussion ou Choc des Corps, La Nature de l'Air, Le Froid et le Chaud, Le Traite des Couleurs, etc. The passages cited by both Jevons and Fisher are found in the last article in the Works entitled Essai de Logique.[1] In the "Avis au Lecteur" it is stated that the Essai de Logique was first published in Paris in 1678, and it should be noted that this material appeared considerably earlier than did that of Ceva, who is quite generally credited with doing the first writing of this sort.

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1. Edme Mariotte, Oeuvres de M. Mariotte, Leide: 1717. Another edition, published in 1740 at La Haye, came to hand first. The text is the same in both editions; in fact, the printed pages are identical except for title page and preface.

Principle 97, which is cited in the bibliographies, is here quoted verbatim.

> "It is not the magnitude nor the number of things which must be considered in selecting as between good things and evil; but the magnitude of the pleasures and pains which they cause."[1]

Article III of Part II is entitled *Des Principes des Propositions Morales.* The author concludes some opening remarks with the statement that if it is difficult to explain a natural effect in physics on account of the many possible causes of that effect, it is more difficult to do so in the field of morals because of the even greater complexity of causal relationships to be considered. He then proceeds to give warning that, following Principle 97, one ought not to be deceived by the magnitude of things instead of considering the advantages and inconveniences which might result from them.

Suppose that a man who has 20,000 crowns is thinking of wagering them against 100,000 crowns, the result to turn on, say, a single throw of the dice. One might think that this would be a good bet at odds of five to one. But in this case one must not consider the actual physical quantities of the sums involved; one must consider further the advantages and the inconveniences which would result. Twenty thousand crowns suffice to keep a man in ease, and one hundred thousand crowns

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1. *Ibid.*, p. 619.

more would increase his good fortune in a ratio of perhaps three to two or three to one. But if the man lose his 20,000 crowns, he falls into complete poverty, and the ratio as between having enough to live on comfortably and having nothing at all is almost infinite, or as 100,000 to one. Thus one may judge well that 20,000 crowns should not be played against 100,000 crowns at a single stroke, though perhaps it would be all right to risk twenty crowns against one hundred. And one may make use of the principles involved here in all similar cases.[1]

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1. Ibid., p. 667. I have followed the text rather more closely than usual in order to achieve precision in conveying the central idea.

2

Daniel Bernoulli (1700-1782)

Since Daniel Bernoulli was a member of an illustrious family of Swiss mathematicians, it may seem strange to include him among the French writers. He is taken up in this chapter simply for the reason that his main idea is the same as that of Mariotte, although it is more highly developed.

Bernoulli was not interested primarily in economic problems. His treatment of utility theory results from his consideration of the mathematics of probability, with reference particularly to games of chance.[1] His basic tenet is well known, largely because of Marshall's reference to "Bernoulli's suggestion."[2] The central idea is that in considering additions to total satisfaction resulting from incremental additions to one's wealth (fortune) it is important to include as a criterion of measurement the size of one's fortune as well as the size of the increment. Let us consider in detail the "theorem" of Bernoulli.

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1. Harold T. Davis, The Theory of Econometrics, Bloomington, Indiana: The Principia Press, 1941, pp. 55-56. This reference contains an explanation of the problem of the "St. Petersburg paradox," which Bernoulli wished to solve.
2. Alfred Marshall, Principles of Economics, 1st. ed., London: The Macmillan Co., 1890, p. 180.

In Figure I let the line AB represent the amount of wealth (Vermögen) which an individual possesses before engaging in a fair bet. Then over the extension BR from AB, let the curve BGS be constructed. The ordinates of this curve CG, DH, EL, FM, etc., represent the "advantages" which result from gains in wealth BC, BD, BE, BF, etc. Furthermore let m', n', p', q', etc., be quantities which indicate how often the gains BC, BD, BE, BF, etc., can occur. Then the average "advantage" can be represented by

$$PO = \frac{m' \cdot CG + n' \cdot DH + p' \cdot EL + q' \cdot FM + \ldots}{m' + n' + p' + q' + \ldots}$$

If now we erect AQ perpendicular to AR and mark off AN = PO, then NO - AB, i.e., BP, we may suppose to be the amount of gain to be had if the individual in question wins his bet. The question now is, what <u>ought</u> to be the sum which he should gamble against this amount BP? To discover this sum, we must continue the curve BGS in the opposite direction in such a way that the abscissa Bp represents the possible loss resulting from the bet and the ordinate po represents the "disadvantage" (Nachteil) resulting from such a possible loss. Now in a fair bet the "disadvantage" of the possible loss has to be equal to the "advantage" of the possible gain; consequently, we must take An = AN or po = PO. Therefore Bp shows the initial amount which no one

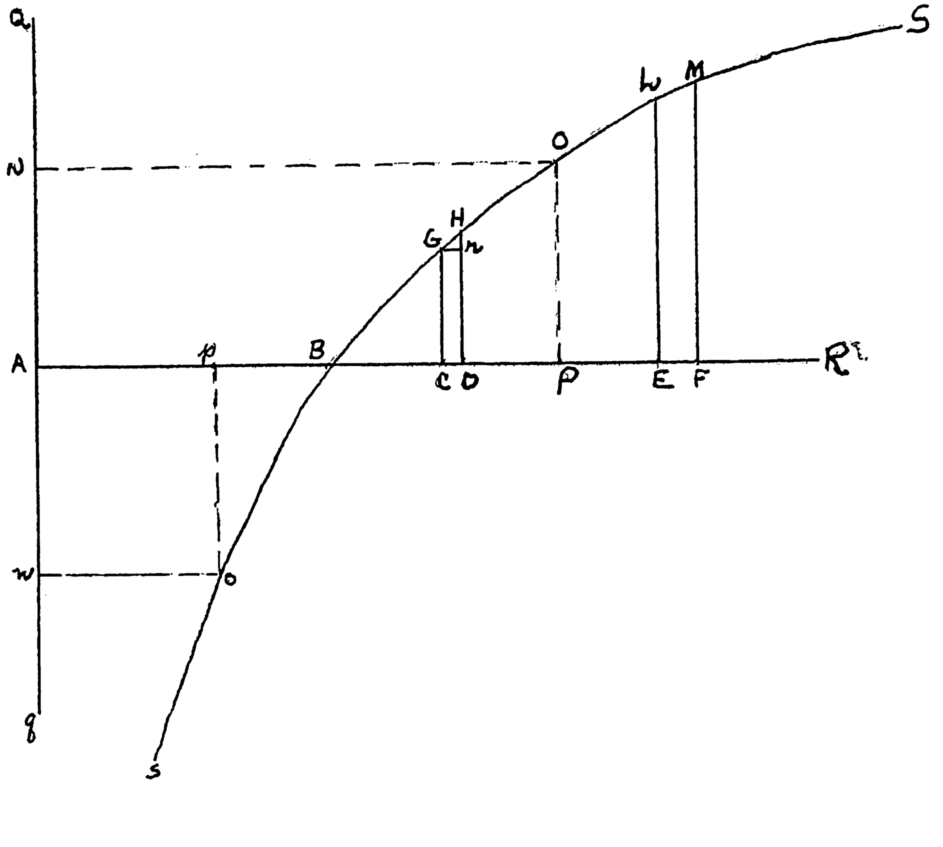

Figure 1.

who considers properly his economic situation should exceed in making a wager which is likely, if successful, to net an amount BP. Thus the scholars are in error who say that a prospective wager may be considered simply on its own merits and that it is all right for a person to risk a sum precisely equal to the sum to be gained. This would be true only if the curve sBS were a straight line or if the sum wagered is infinitely small relative to one's resources (Vermögen).[1]

We must now pass from this general consideration to a matter which above all others, according to Bernoulli, ought to be investigated. Let us refer again to Figure I. From this point on the analysis is to be made in terms of infinitely small quantities, so we shall consider BC and BD as almost equal so that their difference CD becomes infinitely small. If we then draw Gr parallel to BR, rH becomes the infinitely small "advantage" which accrues to anyone with a "fortune" AC upon the acquisition of an infinitely small gain CD. The "advantage" then, while directly proportional to the gain CD is inversely proportional to the "fortune" in question AC.

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1. The word "Vermögen" is apparently difficult to translate. In most cases it seems that the English word "fortune" approximates it, but in this instance "fortune" is hardly accurate. According to Bernoulli, "Vermögen is anything that can bring satisfaction to an individual; for most people the main part of their "Vermögen" is their "work power"(Arbeitskraft)."

If we let

$$AC = x$$
$$CD = dx$$
$$CG = y$$
$$rH = dy$$
and $$AB = \alpha$$

and if we take b to be a constant value, we have then

$$dy = \frac{b.dx}{x}$$

and

$$y = b.\log \frac{x}{\alpha}^{1.}$$

The curve sBS is therefore a logarithmic curve whose subtangent is always equal to b and whose asymptote is the straight line Qq.

---

1. If $dy = \frac{b.dx}{x}$

Then $y = b \log x + c$
When $y = o$, $o = b \log \alpha + c$
or $c = -b \log \alpha$
$y = b \log x - b \log \alpha$
$= b (\log x - \log \alpha)$
$= b \log \frac{x}{\alpha}$

This analysis follows closely the German text found in Die Grundlage der modernen Wertlehre: Daniel Bernoulli, Versuch einer neuen Theorie der Wertbestimmung von Glücksfällen (Specimen Theoriae novae de Mensura Sortis), Translated from the Latin by Dr. Alfred Pringsheim, Leipzig, Duncker & Humblot, 1896, pp. 31-37.

3

François Véron de Forbonnais (1722-1800)

At least two of the works of Forbonnais contain nothing which could be called mathematico-economic writing.[1] The *Élémens du commerce* appears in both the Jevons and the Fisher bibliographies, with especial attention directed to Chapter IX, *De la circulation de l'argent*.[2]

Forbonnais discusses the implications of a bimetallic monetary standard. He supposes in Europe a common ratio which indicates the relative value of gold and silver. If this ratio be one to fifteen, and we let a equal a pound of gold and b a pound of silver then

$$a = 15b.$$

If a government change this ratio to make gold relatively more valuable, the equation would become, say,

$$a = 16b.$$

Under such conditions, traders in neighboring countries would bring gold to the country where it was overvalued, an amount of gold a exchanging for an amount of silver 16b,

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1. His *Recherches et considérations sur les finances de France depuis l'année 1595 jusqu'à l'année 1721* and his translation of a work by Charles King entitled *Le négotiant anglois* will attract only the economic historian and the student of statistics.
2. Fisher calls attention to Chapter VIII also.

the profit of the traders being b. The traders' profit is a loss to the people of the country in which gold is overvalued.

There follow these sentences.

> "Il ne suffit pas encore que le législateur observe la proportion du poids que suivent les états voisins. Comme le degré de finesse ou le titre de ses monnoies dépend de sa volonté, il faut qu'il se conforme à la proportion unanimement établie entre les parties de la plus grande finesse, dont chaque métal est susceptible.
>
> "S'il ne donne pas àses monnoies le plus grand degré de finesse, il faut que les termes diminués soient continuellement proportionnels aux plus grands termes."[1]

Let gold of the highest degree of fineness equal 16c and silver of the highest degree of fineness equal 6d. If then one wishes to coin gold only half as fine as it is possible to coin it, we can represent this degree of fineness by 8c. Maintaining the same weight proportion between gold and silver, the new silver coins must be equivalent to 3d in fineness for 8c:3d::16c:6d. If this proportion be not observed so that silver is equal to 4d in fineness, gold will flow into the country, and the difference d is the profit of those exchanging gold for silver. The state which overvalues either gold or silver is impoverished both

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1. François Véron de Forbonnais, Élémens du commerce, 2nd. ed., Leyde: 1754, II, 125.

"actually and relatively." Thus it follows that it is to the interest of the state to maintain the precise ratio between gold and silver, both as to weight and fineness, which other states have.

This brief section then closes with the following:

> "Dans les suppositions que nous avons établies,
>
> $$a + 16c = 15b + 6d$$
>
> $$a + 8c = 15b + 3d$$
>
> Et ainsi du reste. Ou bien si l'une de ces proportions est rompue, il faut la rétablir par l'autre:
>
> $$a + 16c = 30b + 3d :: a + 16c = 15b + 6d$$
>
> $$a + 8c = 7\tfrac{1}{2}b + 6d :: a + 8c = 15b + 3d$$" [1]

In Chapter VIII of the Élémens there is another short mathematical passage similar in its simplicity. This chapter, entitled Du change, reads like the treatment of international exchange rates which one would expect to find in a modern elementary textbook in economics. After explaining the mint par of exchange (le pair du prix du change) and after showing how actual exchange rates may vary from the mint par (cette altération est appellée le cours du prix du change), the author proceeds to explain how arbitrage operations affect international exchange rates.

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1. Ibid., p. 127.

Those who have international remittances to make are always careful to make them in the most advantageous way possible.

> "La science de ce commerce consiste donc à saisir toutes les inégalités favorables que présentent les prix du change entre deux villes, et entre ces deux villes et les autres. Car si cinq places de commerce s'éloignent entr'elles du pair du prix du change dans la même proportion, il n'y aura aucune opération lucrative à faire entr'elles: l'intéret de l'argent et les frais de commissions tourneroient en pure perte. Cette égalité reciproque entre le cours du prix du change de plusieurs place s'appelle le pair politique."[1]

Let us suppose an equilibrium condition such that

$$a = b$$

$$b = c$$

$$c = a$$

Since these are all equal quantities, the currencies represented by the letters must be at mint par.

Suppose now that

$$a = b$$

$$b = c$$

$$c = a + d$$

Parity is disturbed. By exchanging b for c, one can obtain a + d. But a only is required to obtain b, so

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1. Ibid., p. 110.

there is a profit of d made in the exchange. Parity ". . . will be reestablished if these quantities increase among themselves equally:

$$a + d = b + d$$
$$b + d = c + d$$
$$c + d = a + d.$$

This parity corresponds to the "political par of exchange," or to the equality of the "cours" between several places."[1]

If we assume a new alteration in parity so that

$$a + d = b + d$$
$$b + d = c + d$$
$$c + d = a + d + f,$$

we shall have a case in which the exchange will be made as in the preceding example and a profit f can be made by the proper trading. Or to put the matter another way, we may assume

$$a + d = b + d$$
$$b + d = c + d$$
$$c + d = a + d - f$$

The owner of c + d, if he were to exchange directly for a + d - f would obviously lose an amount f. To avoid this, he will exchange for b + d and then for a + d,

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1. Ibid., p. 111.

4

Achille Nicolas Isnard (-1802 or 1803)

"Nous osons prédire que l'instruction réparera un jour les maux du genre du main."[1] So begins Isnard's preliminary discourse on the necessity of instruction in the subject with which this work is concerned. We may feel that he is another early economist who will apologize for his preconceived notions regarding public policy. Very quickly, however, he pleads for the use of pure reason in economic matters and condemns the doctrines of the Physiocrats, the English, and the Mercantilist followers of Colbert. Especially will he show that M. Quesnay and "the economists" erred in laying down the following propositions:

(1) Agriculture is the only source of wealth.
(2) Industry does not increase wealth.
(3) Taxes must be levied only on the net product of land.
(4) The Sovereign is a co-proprietor of the net product.[2]

Pierre Boven's remarks concerning Isnard's introduction are particularly interesting in view of what will be said in Chapter VIII concerning the pure theorist.

> "On voit du premier coup le mélange de préoccupations scientifiques et pratiques. D'ailleurs, nous ne sommes pas encore au temps où l'on est en droit de compter sur une démarcation

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1. A. N. Isnard, Traité des Richesses, London and Lausanne, François Grasset et Comp., 1781, p.4.
2. Ibid, p. xiii.

> nette entre les deux domaines. Mais les découvertes se font généralement quand on cherche autre chose."[1]

But we must pass to the mathematical sections which appear for the most part at the beginning of the first volume. Isnard points out the fact that one can compare homogeneous things with reference either to their quantities or their magnitudes, but among heterogeneous things it is necessary to find some homogeneous link. In the process of being exchanged, there come to be established among goods certain ratios of exchange, and these Isnard proposes to investigate. Of course exchange ratios are ordinarily expressed in terms of money, but if money does not exist, they can still be indicated. For example, given a certain quantity of one good equal in exchange to a certain quantity of another good, a representing the number of units of the first good, and b the number of units of the second, one may say that the value of the first good in terms of the second is equal to $\frac{b}{a}$. And if goods taken two by two have ratios between them, they also have such ratios taken all together. For example, if we have three goods of which the values are M, M', and M'' such that M:M'::1:2 and M':M''::3:5, we can write M:M':M''::3:6:10. These ratios give the value

1. Pierre Boven, Les Applications Mathématique à l'économie politique, Paris, 1912, p. 51.

of the goods.

> "The word value expresses then the ratio between two things when they are compared in exchange. In speaking of economic goods, one rarely uses the word value in an absolute sense. The word which properly expresses the absolute meaning which one would like to give to it is utility."[1]

Note now the questions which Isnard proposes.

> "Comment les choses acquierent-elles une valeur dans les échanges? Comment cette valeur dépend-elle de la quantité des choses et du besoin et des valeurs? Comment les besoins sont-ils subordonnés eux-mêmes aux quantités et aux valeurs? C'est ce qu'il faut découvrir successivement."[2]

At this date (1781) he was asking questions which were not seriously asked again for over half a century..

He considers now how exchange ratios are determined. First suppose a case in which there are owners of two goods, each owner wishing to dispose of a part of the good which he owns. If one owner wishes to get rid of a units of good M and the other owner wishes to dispose of b units of M', and since there is a possibility of exchanging only these two goods, then a units of M are given in exchange for

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1. Ibid, p. 17.
2. Loc. cit.; I have left these questions untranslated in order not to risk coloring their meaning.

b units of M'; hence we have aM equal bM', and consequently M:M'::$\frac{1}{a}$:$\frac{1}{b}$. The value of each unit of merchandise is then in inverse ratio to the quantity offered in exchange.

> "If instead of two goods we suppose three to be exchanged, or a greater number, it will be the same for the general value of the goods. Each unit of a good will be equal to the sum of the offers made by the owners of the other goods divided by thenumber of units of the good being considered, or, what is the same thing, the values of goods will be directly proportionate to the sum of the offers and inversely proportional to the quantity of the particular good whose value is being considered. But the offers being composed of several heterogeneous goods, it is not possible to deduce from the equality, or from the equation of which we have just spoken, the relation between two particular goods; in order to find the ratio between goods taken two by two, it is necessary to form as many equations as there are goods."[1]

We may pass now to a somewhat more complicated case.

> "Let there be three goods to be exchanged one against the other; let there be a units of M, b units of M' and c units of M''. Let the amount a of good M be divided into two parts, am and an, each of these quantities being the amount of M offered by the owner of M in order to get some M' and M''; let the quantity b of good M' be divided into two parts bp and bq, each of these quantities being the amount of M' offered by the owner of M' to get some M and M''; let the

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1. Ibid., pp. 18-19.

quantity c be divided into two parts cr and cs, each of these quantities being the amount of M'' offered by the owner of M'' in order to get some M and M'. These suppositions give three equations:

$$\begin{aligned} aM &= pbM' + rcM'' \\ bM' &= maM + scM'' \\ cM'' &= qbM' + naM \end{aligned}$$

One can deduce from these three equations the ratios of the goods taken two by two, and we have then

$$M:M':M'' :: \frac{r + p - pr}{a} : \frac{s + m - sm}{b} : \frac{n + q - nq}{c}$$

One can deduce also the value of each good relative to each other and the quantities which each owner will attract in exchange for his own offers . . . .

"It is easy to account for variations in values by variations in offers; one can see how, proceding from the supposition that m equals n, p equals q, and r equals s, to the supposition that the quantities n, q, and s equal zero, how, I say, between the extreme suppositions, progressive changes in the offers have an influence on all the values; to do this it is only necessary to make different suppositions in numbers in place of the algebraic quantities we have just used, supposing always, as we have done up to now, that the quantities of merchandise remain constant."

We have covered enough of Isnard's chapter on value to be impressed with his clear idea of the interdependence of economic phenomena. There is much which remains to be reported in a full treatment, but since only typical material can be included, we shall conclude this section with a word regarding the introduction of money into his system.

> "If there are several goods M, M', M'', M''', M'''', etc., whose values are known and which are in ratio of a to b to c to d to e, etc., we can compare all the goods to one of them; thus we shall have
>
> M : M' :: a : b
> M : M'' :: a : c
> M : M''' :: a : d
> M : M'''' :: a : e, etc.
>
> The values of M', M'', M''', M'''' will be then $\frac{bM}{a}$, $\frac{cM}{a}$, $\frac{dM}{a}$, $\frac{eM}{a}$, etc."[1]

Thus one good in an economy can serve as a common measure of all the others, and we can always relate to this one good the values of all. As a matter of fact, this good, which serves as money, ought to have considerable value relative to its size and weight. The point Isnard wishes

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1. Ibid., p. 21.

especially to make, however, is that money has value as an economic good and that it does not have value simply as a matter of convention or because government authorities have said that it must have.[1]

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1. There are several other passages in the first fifty pages of Book One where mathematical notation is used, but they add nothing to the significance of Isnard. Frequently, in non-mathematical passages, he makes remarks of considerable interest. For example, he treats the wages of labor as a part of the general pricing problem, as simply another dependent variable in his system.

5

Marie Caritat Condorcet (1743-1794)

This section appears in a biography of the Physiocrat, Turgot, whom Seligman calls, ". . . the most cautious as well as the greatest of the Physiocrats . . . ."[1] Without question Condorcet, in writing on Turgot's ideas regarding tax reform, has erred in some respects, and the section containing the mathematical notation is attended with considerable obscurity. This material is taken from the French edition of 1774 and from an English translation which appeared in 1787.[2] There are minor discrepencies between the translation and the original work that I have tried to reconcile.

To understand this section, one must recall a fundamental doctrine of the Physiocrats. The Physiocrats laid down the rule that in whatever form a tax is levied it will be so shifted that it falls eventually on the produit net, that is, on the annual produce of the land after all expenses incurred in obtaining that produce (including taxes) have been deducted. For this reason the Physiocrats repeatedly state that the only just tax is one levied directly against the net product.

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1. E. R. A. Seligman, The Shifting and Incidence of Taxation, 5th ed., New York: Columbia University Press, 1932, p. 137.
2. M. C. Condorcet, Vie de M. Turgot, London: 1786, pp. 162 et.seq. and M. C. Condorcet, The Life of M. Turgot, Translated from the French, London: 1787, pp. 404 et.seq.

Condorcet tries to recapitulate Turgot's reasoning regarding the proposition that all indirect taxes should be suppressed and a direct tax on the produit net levied instead. There is apparently some lack of certainty regarding what constitutes an indirect tax, but Turgot has given a formal definition of these terms in a work of his own.

> "The tax which the land owner pays immediately out of his income is called a direct tax; the tax which is not directly assessed on the income is called an indirect tax. Indirect taxes may be reduced to three chief taxes: the tax on the tenant farmer, the tax on the profits of capital or industry, and the tax on commodities sold or consumed. The land owner, however, bears the burden of the indirect tax in two ways: through an increase of his expenses, and through a decrease of his income. The term, 'indirect tax', therefore, covers every tax except a direct tax on the net revenue of land."[1]

In Condorcet's report of Turgot's suggestion for changing indirect taxes to direct ones, he calculates all the indirect taxes at present being levied to learn what is the real net product of each property. He then arrives at the requisite rate of taxation, computes the new tax on each piece of land, and finally discovers how much more

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1. Seligman, op.cit., pp. 138-139. Seligman is paraphrazing a passage in Turgot's work, Explications sur le Sujet du Prix offert par la Société Royale de l'Agriculture de Limoges au Mémoire dans lequel on aurait le mieux demontré l'Effet de l'Impot Indirect sur le Revenu des Propriétaires de Bien-Fonds, reprinted in Daire, Oeuvres de Turgot, Vol. I, pp. 416-417.

or less the proprietor of land pays under the new system of direct taxes only. The mathematical analysis is as follows.

> Let a', a'', a'''........a'''^{n} represent the amount of the net product after taxes of units of land of different qualities.[1]
>
> Let b', b'', b'''........b'''^{n} express the amount of a direct tax previously placed on the several net products, a', a'', a'''........a'''^{n}.
>
> Let i', i'', i''',........i'''^{n} express the amount of the indirect tax levied on the net products of successive grades of land.
>
> Now $I' = \Sigma b$, $I'' = \Sigma i$, and we have then $I = I' + I''$.
>
> Let $\int \bar{a}$ equal the sum of all the net products a', a'', a'''........a'''^{n} [2] subject to the direct tax b and the indirect tax i.[3]

We may now proceed to consider the method of changing the indirect taxes into a direct tax. Suppose all indirect taxes changed into a direct tax by a single operation. If all imposts are removed, then the net product will become $a' + b' + i'$ and similarly for all other a's in the

---

1. Sometimes in the reading of this passage it appeared that a referred not to the net product of, say, a hectare of land, but to the land itself as a physical property. But the subsequent analysis can be meaningful only if a always refers to the net product of land.
2. "Nous désignerons par cette expression $\int \bar{a}$ la somme de toutes les valeurs a prises chacune autant de fois qu'il existe de terres de cette nature. . . ." Condorcet, <u>op.cit</u>., p. 163.
3. The sign $\int\overline{}$ is used whenever a summation is made by the author.

series.[1] Now I equals the sum of all taxes, and the amount of the total net product of all properties, with taxes removed, is $\int a + b + i$. Then $\frac{I}{\int a + b + i} \times (a' + b' + i')$

represents the amount of tax which the land that yielded the net product a' pays on the assumption that only a single direct tax is levied. In other words, rate of taxation times annual product of the property equals the annual tax.

On the supposition that the property which yields the product a' be farmed, the former tax i', which no longer exists, is now paid in rent to the landlord by the farmer. The owner of the land which yielded the net product a' then has to pay $\frac{I}{\int a + b + i} \times (a' + b' + i') - i'$.

Consequently, that which the proprietor must pay in addition to what he formerly paid will be $\frac{I}{\int a + b + i} \times (a' + b' + i') - b' - i'$.

The analysis outlined above undoubtedly presents some difficulty. An arithmetic example may serve to convey the essential ideas contained in this passage. Let us suppose a net product from three grades of land, a', a'', a'''. We

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1. This follows from the postulate of the Physiocrats that all taxes fall on the net product.

will assume that the net product from a' equals ten bushels, that the net product from a'' equals twenty bushels, and that the net product from a''' equals thirty bushels. Let us further assume that there are five units of land yielding net product a', four units yielding net product a'', and three units of land yielding net product a'''. In tabular form this appears as follows:

	Bushels	Number of units of land of given grade.
a'	10	5
a''	20	4
a'''	30	3

Now let us assume the direct tax levied at the beginning of the analysis to be a proportional tax of ten per cent. Thus $b' = 1$, $b'' = 2$, and $b''' = 3$. Finally, let us assume an indirect tax of two bushels to be levied against each unit of land of whatever grade so that $i' = 2$, $i'' = 2$, and $i''' = 2$.

Now $I = i' + i'' = \Sigma b + \Sigma i = 22 + 24 = 46$, and $\int \overline{a + b + i} = 220 + 22 + 24 = 267$. Thus the rate of taxation, assuming all former imposts removed and a single direct impost levied, is $\frac{46}{267}$ or .17228.

Now $a' + b' + i' = 10 + 1 + 2 = 13$ equals the sum of a' and the direct and indirect taxes.

$a'' + b'' + i'' = 20 + 2 + 2 = 24$ equal the sum of a'', and the direct and indirect taxes.

$a''' + b''' + i''' = 30 + 3 + 2 = 35$ equal the sum of a''' and all taxes.

Now $\frac{46}{267} \times 13 = 2.2397 =$ the direct tax paid by the owner of any property which yields net product a'.

Finally, $\frac{46}{267} \times 24 = 4.13483 =$ the direct tax paid by an owner of a unit of land which yields net product a'' and

$\frac{46}{267} \times 35 = 6.02996 =$ the direct tax paid by any proprietor of a unit of land of a grade which yields net product a'''.

It will be noted that if the land is farmed the landlord will be able to obtain from the farmer the preexisting amount of the indirect tax. Thus, the owner of a unit of land yielding net product a' would, on this ground, need to pay only 2.23970 - 2.00000 = .23970 bushels. We may sum up the matter by pointing out that before the change in taxes the landlord formerly paid a total of three bushels per annum. Now he pays a total of 2.23970 bushels. Under the new system of taxing then, in this particular numerical example, the proprietor of a grade of land which yields net product a' has an annual savings in taxes of .7603 bushels per annum. But the taxes will be higher than before for the other proprietors; this is what Condorcet means when he gives the formula $\frac{I}{\int \overline{a + b + i}} \times (a' + b' + i') - b' - i'$ equals the difference between the tax formerly paid and what is now paid.

Condorcet goes on to point out what is by now obvious. The farmer gains nothing by the removal of the indirect tax. The i that he no longer pays to the Government he now pays in rent to the landlord. The proprietors taken as a group will pay the same amount under the new system as under the old. The reasoning is as follows: They formerly had remaining to them, after taxes, $\int \overline{a + b} - I'$. After the tax change, they have the sum of $\int \overline{a + b + i} - \frac{I}{\int \overline{a + b + i}}$

$$\times \int \overline{a + b + i} = \int \overline{a + b + i} - I.$$

$$\text{The } \int \overline{i} = I''$$

$$\text{and } I = I' + I'',$$

$$\text{hence, } \int \overline{a + b + i} - I = \overline{a + b} - I'.$$

The case of the proprietor who finds himself actually worse off than he was before can result only from the fact that formerly he did not pay taxes in proportion to the net product of his land. Hence, anyone who pays more taxes than he did formerly does so rightly and in conformity with justice.

6

Nicolas-François Canard (1750-1833)

Perhaps the best known of the early mathematical writers is Nicolas-François Canard. Certainly Canard's work is among the first which can be termed extensively mathematical. As one reads the sections in his Principes d'Économie Politique concerning price theory and the shifting and incidence of taxes, there is an impression of being completely submerged in mathematical thinking and mathematical notation.

Almost without exception this work has been adversely criticized. Most writers who have commented on the book have been at least as faint in their praise as was Francis Horner in his comment in the Edinburgh Review in 1803. Among other things, he remarks that M. Canard ". . . appears to us to display very little sagacity and that . . . he has, without any necessity, affected to change the established form of expression."[1] Zawadzki feels that, generally speaking, the mathematical sections of the work are much less important than the non-mathematical parts. Specifically, he objects that Canard did not try to develop conditions of equilibrium in a system of equations which could be simultaneously solved

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1. [Francis Horner] "A Review of N. F. Canard's Principes d'Économie Politique," Edinburgh Review, vol. I, p. 431.

and that, instead of expressing his functional relations in a modern way, he transforms these relations into strict proportionalities.[1] The Italian Murray dismisses Canard, along with Ceva and Whewell, by saying that economic science can draw nothing of benefit from him.[2]

In the evaluation of Canard's work which will be found in Chapter VII there is stated my own highly favorable opinion of Canard's contribution. Let us pass then without further comment to a consideration of those parts of his work containing the mathematical analysis.

Canard wishes to examine the forces which determine prices. Now a price is simply the ratio of the value of any good to the value of a specified quantity of one of the precious metals. What forces determine these different ratios? Since everything which has a price is created by labor, the price of any object must be to that of any other object in proportion to the work which has gone into the creation of both. Canard realizes that this statement is only partially true, for he proceeds immediately to a further examination.

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1. Wladyslaw Zawadzki, Les mathématiques appliquées àl'économie politique, Paris: Marcel Rivière et Cie., 1914, pp. 34-35.
2. R. A. Murray, Leçons d'économie politique, Paris: Payot et Cie., 1920, p. 49.

In order to determine the general causes which determine price, it is necessary to look more closely at the principles governing men's conduct in their business relations. He takes it as axiomatic that every individual tends to procure for himself the greatest amount of pleasure (jouissance) possible and that each seller tries to get for the price of his labor the greatest amount that he can. Any buyer then will always try to procure merchandise at the lowest possible price; the seller, on the other hand, will always attempt to obtain the highest price possible. "C'est donc entre le besoin de l'acheteur et le besoin opposé du vendeur, que commence à se déterminer la valeur des choses."[1]

Consider now the buyers and sellers in a market.

> "There will necessarily be a difference between the price demanded by the sellers and the price offered by the buyers. This difference from the highest to the lowest price will form an interval (latitude) over which will take place the struggle of sellers and buyers. The sellers will try to take advantage of the need and competition of the buyers to make them pay the greater part of this interval; the buyers on their side will take advantage of the need and competition of the sellers in order to pay the smallest part possible of this interval."[2]

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1. N. F. Canard, Principes d'économie politique, Paris: F. Buisson, 1801, p. 28.
2. Idem.

Let the difference between the highest price demanded by the sellers and the lowest price offered by the buyers be L.

Let x equal that part of the difference which the sellers are content to take in addition to the lowest price.

Let L-x equal that portion of the difference which the buyers finally succeed in cutting off the highest price.

Finally, let

B equal the desire of the buyers to purchase,

N equal their competition,

b equal the desire of the sellers to dispose of their goods,

and n equal their competition.

The portion x of the interval L which is paid by the buyers will increase as BN increases, and for the same reason L-x will increase as "bn" increases. Thus,

$$x:BN::L-x:bn$$

$$bnx = BN(L-x)$$

$$\text{and} \quad x = \frac{BN}{BN + bn} L$$

This is Canard's "equation of determination" which expresses the equality of the moments of two opposing forces which are always tending toward equilibrium. "C'est au principe de l'équilibre de ces forces que se rapporte toute la théorie de l'économique politique, comme c'est au principe de l'équilibre du lévier, que se rapporte toute la statique."[1]

From the basic "equation of determination," a number of principles can be deduced. In the first place, if bn = 0, then x = L. This is the case in which there is no competition among the sellers, and the buyers pay to the monopolist the entire L. If, on the other hand, BN = 0, the case of a monopsonist, x = 0 and the sellers pay the entire difference between the highest and the lowest price at which the good might be sold.

Finally, one must determine the limits of L. The price of any good can never fall below the necessary wage of the labor (salaire nécessaire du travail) required in the production of the good. The necessary wage is one which assures to any who receive it a reward for his manual labor sufficient to keep him without any "superfluous pleasures." Apparently this

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1. Ibid., pp. 30-31. At this point F. Horner, in the article in the Edinburgh Review cited above makes the following comment: "He proceeds through 20 pages with this calculation, into which a great many more terms as well as new symbols are introduced; but our readers, we perceive, have already had enough of it."

must be sufficient to enable the laborer to maintain his family sufficiently well that his children may grow to become competent and vigorous workers.

What is the highest price which a monopolist might charge? At this point Canard shows that he has grasped the concept of elasticity of demand though he presents no mathematical measurements. If the good to be sold is not an absolute necessity, the number of buyers will diminish as the price rises. While the monopolist gains by the rise in price, he loses by the diminution of his sales, and he will reach a point where any further price rise will cause him actually to lose. If the object to be sold is an absolute necessity for consumers, it appears that the seller can raise his price to any point. However, in this case the price is limited by the natural wage of the buyer, for workers cannot pay for bread an amount greater than their own wage, and they cannot be expected to starve to death.

If P equal price and S equal the natural wage, the "equation of prices" between a single <u>branch</u> of sellers and a single <u>branch</u> of buyers will be

$$P = S + \frac{BN}{BN + bn} L.$$

It should be noted here that while S expresses that part of the price which will go to pay for the unskilled labor, the quantity $\frac{BN}{BN + bn} L$ will go to pay (1) the rent of land, (2) the return to capital, and (3) the return to the skilled labor or management.

Goods nearly always pass through several hands before being finally offered to the consumer. If a good is sold in the earliest stage of production for a price equal to $S + \frac{BN}{BN + bn} L$, and if the original buyer after some processing sells the good again, the expression for the price of the good at this level is as follows:

$$S' + \frac{B'N'}{B'N' + b'n'} L'$$

The price of a good from the earliest to the latest stages of production is summed by writing down what each <u>branch</u> of sellers in the series gets for his own production. Thus,

$$P = S + \frac{BN}{BN + bn} L + S' + \frac{B'N'}{B'N' + b'n'} L' + S'' + \frac{B''N''}{B''N'' + b''n''} L'' + \ldots$$

The consumer of any final product then pays the sum of all the labor involved plus a return to all the other factors used in the production of the goods.

There remains the question of the limits between which the price of a good which has passed through several stages of production may vary. Suppose first that a final consumer is the sole buyer of a good. He then pays to the final seller only the "salaire nécessaire" at the final stage and keeps the whole of the "latitude." The final seller then pays to the next to the last seller only the "salaire nécessaire," and so on. Then for this good, at the final stage,

$$P = S + S' + S'' + \dots$$

Thus a monopsonist placed at the final stage appropriates for himself the sum of all the successive "latitudes." A monopsonist at <u>any</u> stage in production takes for himself all the preceding intervals.

On the other hand, a monopolist at the first stage of production forces the first buyer to pay the whole of the latitude L. The buyer, in his role as seller, does the same with his buyer, and so on. But the first seller consents to sell only at the highest price possible and demands the gains of all the successive sellers, so that the highest possible price which can be paid for a good is

$$P = S + L + L' + L'' + \dots$$

In fact, however, there tends always to be competition. If at any stage of production unusual gains are made, competitors are attracted, and competition becomes such that an "absolute equilibrium" is established. "Absolute equilibrium" is defined as a state in which the forces BN, bn, B'N', b'n', etc., are all equal and the formula

$$P = S + \frac{BN}{BN + bn} L + S' + \frac{B'N'}{B'N' + b'n'} L' + S'' + \frac{B''N''}{B''N'' + b''n''} L'' + \dots$$

becomes

$$P = S + L + S' + L' + S'' + L'' + \dots$$

With some algebraic embellishments, this constitutes Canard's theory of price. It should be noted, however, that Canard is quite aware of the interaction of competitive forces as between "stages" of production within the same industry and as between industries in the economy, although he never uses the words "firm" or "industry." He is at some pains, though, to show that "abnormal" rents, wages, or profits attract resources and that the effects of abnormal gains ramify throughout the entire economy until the équilibre absolu is restored.

7

Barnabé Brisson (1777-1828)

Brisson, like some others in Jevons' bibliography, could well be passed over. Apparently Fisher thought him of little importance for the name does not appear in the later list. For the sake of completeness a brief consideration is given here.

In the Jevons bibliography appears this statement.

> "Brisson was a profound mathematician as well as an eminent engineer. He appears to have presented to the Institut in this year (1802), in conjunction with his friend Depuis de Torcy, an Essai, part of which has been printed in the 14e Cahier du Journal de L'École Polytechnique. Other portions of the memoir, however, are stated never to have been published, including the first part, which contained 'des considérations d'économie politique appliquée aux projets de communications nouvelles.' See 'Essai sur le système général de navigation intérieure de la France' by B. Brisson, Paris, 1829, p. ii. Some indications of the nature of Brisson's financial theory are found at pp. xiv, 160-61."[1]

The essay cited by Jevons as appearing in 1829 was published by the administrator of Brisson's estate shortly after his death in 1828. As stated in the preceding paragraph,

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1. Jevons, op.cit., Appendix V, p. 324.

the essay is primarily of technical interest. Apparently the section of the 1802 memoir having to do with the financial aspects of internal navigation did not seem closely allied to the principal subject matter and was for that reason not included in this publication. Nevertheless, as Jevons says, the rudiments of the theory are given and it would seem that these are sufficient.

Brisson is proposing a comprehensive system of canals of two classes for the whole of France. After the purely technical discussion, he estimates the probable expense of the project, his figures for the nine regions being 329,557,000 francs for canals of the first class and 745,926,000 francs for canals of the second class, a total of 1,075,483,000 francs for the entire project. This estimate of the actual cost of construction is probably liberal, but two elements of cost have been omitted. Supervision of construction amounts to perhaps five per cent of total cost and in addition there is the charge for interest on capital used while each separate canal is being built. The total expense becomes then nearly a fifth greater than the original estimate, or 1,284,000,000 francs.

The projects are to be undertaken and run by private firms which will have "concessions à perpétuité." Except, however, for a small number of canals which are immediately profitable because of their especially good location, the projects may not for some time yield a net revenue sufficient to attract private capital. It is therefore necessary to subsidize the ventures from the public treasury, and this action is justified on the grounds that there is a permanent accession to the public wealth of France and that there is likewise a saving in the reduced number of public ways which the state will have to provide in the future.

Brisson thinks it unwise to pay the subsidy in a lump sum at the beginning of construction, and he thinks it ought not to be paid over a short period of time as work proceeds. Instead, the payment from the public treasury ought to commence when a particular canal is open to traffic, the payment continuing over a number of years in order to supplement the toll receipts. State monies are then never lost on enterprises not completed, and the help to those undertaking the venture comes at precisely the time when it is needed, namely, before the areas which the canals cross are enriched, directly and indirectly, by the new lines of communication.

The author has some difficulty in deciding what portion of the whole the state should pay. Adequate statistics are lacking and, besides, calculations made today are likely to be inaccurate tomorrow. He thinks, though, that the government ought to pay one-sixth of the total expense of 1,284,000,000 francs. He believes that for any particular canal the Treasury payments might well continue over a twenty-five year period, equal payments to be made annually. Thus, supposing the construction of the entire project to be completed in sixty years and that the first canal does not come into operation for six years, the Treasury would, at the end of a six-year period, pay a sum of 276,074 francs. This sum is doubled the following year, trebled the third year, and so on. Thus the annual sum paid increases each year for twenty-five years, at which time it amounts to 6,901,850 francs. It remains at this rate for twenty-nine years and then decreases at a uniform rate for twenty-five years in order to reduce to zero the amount owed by the government at the end of sixty years of construction.[1] The period of sixty years is selected by Brisson on the ground that it seems ". . . assez proportionné à la grandeur de l'objet qu'on se propose."[2] It is a period not so long but that

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1. Barnabé Brisson, Essai sur le système général de navigation intérieure de la France, Paris: Carilian-Goeury Libraire, 1829, p. 131.
2. Idem.

a wise government could foresee clearly the need for this sort of construction. Yet it is a period in which the resources of the country are sufficient to build the canals without great sacrifice—the Treasury is not burdened and the price of day labor is not bid up to where agriculture and industry suffer.

The strictly mathematical analysis is found in a note at the end of the two essays. On the assumptions (given above) that the <u>average</u> length of time required to build a canal is six years and that the entire project requires sixty years, different portions of the system are completed at the end of the sixth year and an approximately equal completion takes place annually for fifty-four years after that. It is assumed that 1/55 of the total project is completed each year from the sixth through the sixtieth year. The construction companies have to meet their costs from the very first, and they advance 3,890,909 fr. 09c. francs the first year, double that the second year, and so on as shown in the second column of the table which follows.

M. Brisson wishes the government to pay one-sixth the total cost over a period of seventy-nine years, beginning at the end of the sixth year. At the end of the sixth year canals costing 23,345,454 fr. 55c. are opened to service.

"Il s'agit de chercher la somme qui, payée vingt-cinq fois au bout des septième, huitième. . . trente-unième, équivaudrait à 3,890, 909 fr. 09c. soldés à la fin de la sixième.

"Nommons A cette dernière somme, et x celle qu'on doit payer tous les ans: A peut être considéré comme divisé en vingt-cinq parties a', a'', ...a^{25}, respectivement équivalentes (comme étant données immédiatement) aux paiements à faire de x au bout de un, deux. . . vingt-cinq ans. Ou, on a, comme on sait, les équations suivantes, en nommant t l'unité, plus l'interet annuel que nous supposerons de 5 pour $\frac{0}{0}$.

$$a't = x$$

$$a''t^2 = x$$

$$a'''t^3 = x$$

$$\cdots\cdots\cdots\cdots$$

$$a^{25}t^{25} = x$$

"Multipliant la première par t^{24}, la deuxième par t^{23}, et ainsi de suite, on obtient

$$At^{25} = x(t^{24} + t^{23} + \ldots\ldots + t + 1)$$

$$= x\left[\frac{t^{25}-1}{t-1}\right];$$

ou

$$A[t^{26} - t^{25}] = x[t^{25} - 1];$$

ce qui donne

x = 276,074 francs."[1]

1. Ibid., pp. 160-161.

Beginning with the seventh year the state will pay this sum for twenty-five years (as indicated in the table which follows) for the canals finished at the end of the sixth year; beginning with the eighth year it will pay a like sum for twenty-five years for canals completed at the end of the seventh year. And so on. The total expense of the canals is 1,284,000,000 francs of which the government's share is to be 214,000,000. But the total government payments to the canal interests will be about twice that amount since the payments are not made in a lump sum but are spread over twenty-five year periods. An inspection of the third column of the table should make this clear.

NUMÉROS des années.	SOMMES À AVANCER par les compagnies exécutantes.	PART contributive de l'État.	INDICATION DES CANAUX auxquels ces parts se rapportent.	OBSERVATIONS.
1	3,890,909 f. 09 c.			
2	7,781,818 18			
3	11,672,727 27			
4	15,563,636 36			
5	19,454,545 45			
6	23,345,454 55			
7	*Idem.*	276,074 f.	pour les canaux terminés la 6ᵉ année.	
8	*Idem.*	552,148	— les 6ᵉ, 7ᵉ	
9	*Idem.*	828,222	— les 6ᵉ, 7ᵉ, 8ᵉ	Vingt-quatre années, pendant lesquelles la part de l'État va en croissant.
. . .	. . . .	. . . . . .	. . . .	
29	*Idem.*	6,349,702	— les 6ᵉ, 7ᵉ, 28ᵉ	
30	*Idem.*	6,625,776	— les 6ᵉ, 7ᵉ, 29ᵉ	
31	*Idem.*	6,901,850	— les 6ᵉ, 7ᵉ, 30ᵉ	
32	*Idem.*	*Idem.*	— les 7ᵉ, 8ᵉ, 31ᵉ	
. . .	. . . .	. . . . . .	. . . .	
37	*Idem.*	*Idem.*	— les 12ᵉ, 13ᵉ, 36ᵉ	
. . .	. . . .	. . . . . .	. . . .	
54	*Idem.*	*Idem.*	— les 29ᵉ, 30ᵉ, 53ᵉ	
55	*Idem.*	*Idem.*	— les 30ᵉ, 31ᵉ, 54ᵉ	Trente-une années, pendant lesquelles elle est au maximum et constante.
56	19,454,545 45	*Idem.*	— les 31ᵉ, 32ᵉ, 55ᵉ	
57	15,563,636 36	*Idem.*	— les 32ᵉ, 33ᵉ, 56ᵉ	
58	11,672,727 27	*Idem.*	— les 33ᵉ, 34ᵉ, 57ᵉ	
59	7,781,818 18	*Idem.*	— les 34ᵉ, 35ᵉ, 58ᵉ	
60	3,890,909 09	*Idem.*	— les 35ᵉ, 36ᵉ, 59ᵉ	
61	0	*Idem.*	— les 36ᵉ, 37ᵉ, 60ᵉ	
62	0	6,625,776	— les 37ᵉ, 60ᵉ	
63	0	6,349,702	— les 38ᵉ, 60ᵉ	Vingt-quatre années, pendant lesquelles elle diminue.
. . .	. . . .	. . . . . .	. . . .	
85	0	276,074	— la 60ᵉ	

(*Note de l'éditeur.*)

8

Jean Charles Léonard Simonde de Sismondi (1773-1842)

Sismondi's economic writings should be of especial interest to the student of economic aggregates or of cyclical fluctuations. Our concern is with Chapter IV of De la richesse commerciale, his second work in economics published long before his more famous histories. This chapter is entitled, "Income and Expenditure of Society; Their Balance."[1] The author proposes an inquiry as to what constitutes the income and the expenditure of a nation as a whole and how a nation may grow richer or poorer.

Briefly, the national income (le revenu) is the annual physical product of its labor less that part required to maintain the productive workers during the period of time considered. This latter amount is called the necessary wage (salaire nécessaire). The necessary wage, set aside from the previous year's production, is equal to the goods and services strictly necessary for the consumption of the productive workers if the national product is to be turned out; varying money prices of this physical quantity of goods and services have nothing to do with the total of the salaire nécessaire.

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1. J. C. L. Simonde de Sismondi, De la richesse commerciale, À Genève, J. J. Paschoud, Libraire, 1803, I., 81.

Following a discussion of the distribution of the national income among six classes (three of which are productive and three unproductive) Sismondi proceeds to treat the question of the increase or decrease of the wealth of a nation. This treatment is lengthy, and since it is not essential to an understanding of the mathematical section, it will only be indicated. Briefly, there are three main situations in which a nation may find itself. First, it may have no foreign trade. Even without foreign commerce it may from year to year grow richer, grow poorer, or remain stationary--according as it increases, decreases, or allows to remain constant the portion of the national income of one year set aside for <u>salaire nécessaire</u> to be expended in the following year. Second, a country may engage in foreign trade and have a persistent excess of imports over exports. The difference is assumed to be made up by borrowing from the nations with which the trade is carried on. Finally, a country may engage in foreign trade and have a persistent excess of exports over imports, the imbalance being made up by loans to those nations receiving the goods. In the latter two cases there may be increasing, decreasing, or constant wealth in the nation, though the country which makes loans abroad is likely to be in a better position than the one which borrows. The really important consideration has to do with the relative changes in the amounts set aside from year to year for the "necessary wage."

There follow examples in which figures are assumed for each of these main types as to income, expenditure, necessary wages, and the amounts of foreign trade. After these are given, Sismondi remarks in a footnote:

> "Those who are not familiar with algebraic language pay no attention to calculations which are presented to them in this form; on the contrary, those who have acquired the habit of considering ideas and numbers abstractly find it repugnant to make numerical suppositions which always seem to them unlikely or inexact: to keep both groups content, I shall make general in this note what is expounded in the text, and I shall adopt this time only the language of the exact sciences; but I repeat, it will be only this time, for to apply this language to a science which is not exact is to expose oneself to continual errors. Political economy is not founded solely on calculations; a great number of moral observations which cannot be submitted to calculation incessantly alter the facts; wishing constantly to abstract, the mathematician suppresses by chance essential figures in each of his equations."[1]

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1. Ibid., pp. 104-105.

There follows in the same footnote the mathematical analysis.

Let P equal the production of the national labor during the year,

Let N equal the necessary wage (<u>salaire nécessaire</u>) of the immediately preceding year,

so that P - N equals the national income.

Let D equal the national expenditure (as defined above).

Let X equal the difference between the necessary wage of the immediately preceding year and that advanced in the current year,

so that N + X equals the necessary wage for the current year.

Let C equal the debts owed <u>to</u> or <u>by</u> foreigners.

When a nation has no foreign trade, its consumption equals its production. Consumption is

$$D + \overline{N + X}$$

But $$D + \overline{N + X} = P$$

or $$D = P - \overline{N + X}$$

If a nation borrows from foreigners, its situation is thus:

$$D + \overline{N + X} = P + C$$

so that $$D = P + C - \overline{N + X}$$

If a nation lends each year to the foreigners, then

$$D + \overline{N + X} = P - C$$

and $$D = P - C - \overline{N + X}$$

Sismondi concludes that the progressive or regressive state of a nation depends on the value of X, that is, on the difference between the <u>salaire nécessaire</u> in one year and that of the year following.

Suppose first that $C = X = \frac{N}{10}$. Then on the "balance sheet" of the first nation

$$D = P - \frac{11N}{10}.$$

Without any foreign trade, this nation becomes richer each year by an amount $\frac{N}{10}$, which is the difference between P - N, its national income, and $P - \frac{11N}{10}$, its national expense.

For the second nation

$$D = P - N + \frac{N}{10} - \frac{N}{10}$$

or $$D = P - N$$

Although it imports each year goods from abroad beyond the value of its exports, and although it always becomes indebted with respect to the other nations, it is nevertheless in a stationary state, neither poorer nor richer.

For the "balance sheet" of the third nation,

$$D = P - N - \frac{2N}{10}$$

or $$D = P - \frac{12N}{10}$$

Hence this nation becomes richer each year by $\frac{N}{5}$, lending the amount $\frac{N}{10}$ to foreigners and using an equal amount each year to increase its internal production (by increasing the <u>salaire nécessaire</u>).

If instead of having $C = X$, $C = \frac{N}{20}$, and $X = \frac{N}{10}$, the equations given above differ and

$$1.\quad D = P - \frac{11N}{10}$$

$$2.\quad D = P + \frac{N}{20} - N - \frac{N}{10}$$

or

$$D = P - \frac{21N}{20}$$

$$3.\quad D = P - N - \frac{N}{10} - \frac{N}{20}$$

or

$$D = P - \frac{23N}{20}$$

If in each of these three cases D is compared with P - N, the national income, in all three cases it is seen that the nations become richer, but unequally so.

Next, if $C = \frac{N}{20}$ and $X = 0$, the three equations are

$$1.\quad D = P - N$$

$$2.\quad D = P - \frac{19N}{20}$$

$$3.\quad D = P - \frac{21N}{20}$$

Thus the first country is neither richer nor poorer, the second is impoverished, and the third is enriched.

Finally, if $C = \frac{N}{20}$ and $X = -\frac{N}{10}$, so that each nation diminishes annually by one-tenth the amount which it sets aside for the salaire nécessaire, there results:

$$1.\quad D = P - \frac{9N}{10}$$

$$2.\quad D = P - \frac{17N}{20}$$

$$3.\quad D = P - \frac{19N}{20}$$

Thus all three nations are impoverished but not with equal rapidity.[1]

In the rest of this work and in his Études sur l'Économie politique which followed twenty-five years later, Simonde de Sismondi, with one exception, keeps his promise not to use mathematical symbols.[2] This exception, however, is simply a practical application of the analysis given and constitutes nothing new.[3]

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1. Sismondi, op.cit., p. 122. This section containing the symbolic notation is not placed in quotation marks because of very free translation, but it follows the French text closely.
2. See not only Sismondi's Études but more especially his Nouveaux Principes D'Économie politique (1819) for additions to his doctrine. I was most impressed, however, with De la Richesse Commerciale.
3. The interested reader may find this passage on pp. 215-216 of De la Richesse Commerciale.

9

L. F. G. de Cazaux (   )

Cazaux was a prolific writer of the early nineteenth century who was apparently dissatisfied with current doctrines, especially those of J-B Say. We find in Cazaux, as in so many of his contemporaries, occasional rare insight and some passages which are startlingly accurate from a present-day point of view. For the most part, though, he does not make very good sense, and a summary of his opening remarks on value, in which appears his mathematical notation, is good for illustrative purposes.

The fundamental question in political economy is, "What is the measure of the value of things?" It is strange, according to Cazaux, that although men universally consider money as the measure of value, the scholars since Smith disagree with the common view. Say, for example, contends that when one exchanges a hundred bushels of wheat against ten francs, the hundred bushels of wheat are worth ten francs and the ten francs are worth one hundred bushels of wheat; but if at some other place a hundred bushels of wheat are worth eleven francs, this is perhaps as much because the francs are worth less as that the wheat is worth more. Cazaux does not agree, but just as we are on the point of admiring this man who neatly puts his finger on one of the misconceptions of Smith and his followers, we are dismayed by reasoning which leads to the following statement.

> "One sees clearly . . . that the value of money is directly proportional to the interest rate at which one may currently place it; and that, consequently, the value of things is directly proportional to this interest and to the price of things."[1]

Let the respective values of the same good at two different times or in two different places be designated by v and V.[2] Let p and P be the respective prices of the good at two different times or in two different places, and let i and I be the respective rates of interest on money at two different times or in two different places. Since values increase in direct ratio with the price of a good and with the interest on money,

$$v : V :: ip : IP$$

from which $$v = V\frac{ip}{IP}.$$

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1. L. F. G. de Cazaux, Élémens d'économie privée et publique, à Paris et à Toulouse, 1825, p. 20.
2. At this point Cazaux remarks in a footnote that contrary to the opinion of Say he thinks mathematical analysis ". . . est très susceptible d'être appliquée à l'Économie politique, science, à notre avis, tout de calcul." Idem.

Thus if the rates of interest are the same in two different places, i equals I and

$$v = V\frac{p}{P}$$

If the thing whose value is to be compared at two different times or in two different places is money, the p equals P and

$$v = V\frac{i}{I}.$$ [1]

Now come numerous applications of the equations to practical cases. One example should suffice. The interest rate in ancient Rome was generally twelve per cent. If the current rate in France be four per cent, what is the value of money in France relative to the value which it had in ancient Rome? Since i equals 4 and I equals 12, we have then

$$v = \frac{V}{3}$$

Money in modern France then has but 1/3 the value which it had in ancient Rome.

There is some use of algebra in a strange passage in which Cazaux tries to define <u>la richesse</u>. The definition is

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1. Cazaux did not trust the mathematical abilities of his readers. Each of the equations is given in words in footnotes.

as follows:

> "The ratio of the sum in silver which represents the annual income from one's property to the sum in silver which represents all the things which one wishes to acquire annually, is constantly, in all places and times, the exact expression of wealth, that is to say, of the power of enjoying annually things desired through the exchange of those things which one has annually to offer in return."[1]

If S represents the annual income in money terms from the things one possesses, S' the money value of the things one would like to have in the same year, and if R is the symbol for wealth, then

$$R = \frac{S}{S'}$$

Since men's desires are, generally speaking, always greater than their power to satisfy them and since in the minds of nearly everyone S' is always greater than S, no one ever becomes perfectly wealthy, no matter how big S may be. On such a basis R should never equal one. Morality and religion have, incidentally, a wealth-increasing influence in that they effect moderation in men's desires.

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1. Ibid., p. 36.

If, however, ease (<u>aisance</u>) is spoken of as a state attained when our basic needs, without luxury, are met and of poverty as a state below which these basic needs are not met, the formula is more useful. A state of ease may be taken as prevailing when R equals 1. If one has 150 units of income per annum and can buy the necessaries of life with 150 units of money, then R equals 1. Now the formula can be expanded. If T equals the annual taxes required to be paid, D equals interest on debts previously contracted, and F equals foolish expenditures, then

$$R = \frac{S - (T + D + F)}{S}$$

If $T + D + F = S$, $R = 0$. The only thing worse than this is to be in a situation where R is negative.

This is perhaps sufficient for most readers. It need only be added that the equations evolved apply to states as well as to individuals.

10

Auguste Walras (1801-1866)

The name of Auguste Walras appears in this chapter chiefly because it is found in the Jevons bibliography. As in the case of Condillac, Brisson, and Hermann, Walras was removed from the list by Fisher, presumably on the ground that his work is not mathematical. Jevons says of him,

> "If we are to trace out the 'filiation of ideas' by which M. (Léon) Walras was led to his theory, we should naturally look back to the work of his father, Auguste Walras, published at Paris in 1831, and entitled _De la nature de la richesse, et de l'origine de la valeur_. In this work we find, it is true, no distinct recognition of the mathematical method, but the analysis of value is often acute and philosophic. The principal point of the work moreover is true, that value depends upon _rarity_ . . . ."[1]

With this opinion I am in substantial agreement, but I must confess a certain uneasiness in summarily dismissing the elder Walras. Anyone who reads Chapter XVIII of his chief work must be impressed with his awareness of the need for mathematical method. In his discussion of the concept

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1. Jevons, _op.cit._, p. xxxix.

of *rareté*, Walras obviously needs help from mathematics.

> "Évidement, la rareté n'est et ne peut être autrechose que la rapport qui existe entre la somme des biens limités et la somme des besoins qui en reclament la jouissance. L'utilité est un rapport de *qualité* ou de nature. La rareté est un rapport de nombre ou de quantité."[1]

In the explanation of rareté which follows there is much concerning quantities which are in functional relation, and momentarily the reader expects a formal presentation of the ideas. There is only disappointment, however. *Rareté* is a mathematical relationship; it is a ratio of numbers or quantities.

> "Il partage la condition et la nature de tous les rapports, qui sont sujets à varier avec les termes qui les constituent et qui augmentent ou diminuent, suivant que leurs antécédens et leurs conséquens augmentent ou diminuent les uns par rapport aux autres."[2]

Then in a long passage which follows, there is an explicit recognition of the usefulness of mathematical method for

> ". . . la valeur est une chose susceptible de plus et de moins, et . . . la richesse proprement dite est

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1. Auguste Walras, *De la nature de la richesse et de l'origine de la valeur*, Paris: Librairie Félix Alcan, 1938 (published originally in 1831), p. 267.
2. *Ibid.*, p. 268.

> une grandeur, et, ce qui est encore plus important, une grandeur <u>appréciable</u>. Aussi bien, personne n'ignore que la richesse se compte et se mesure, et que l'économie politique relève de l'arithmetique. C'est par là qu'elle satisfait aux espérances des bons esprits qui se flattent, avec raison, de la voir un jour se placer au rang des sciences mathematiques, et arriver a la certitude qui distingue d'une manière si avantageuse cette importante branche de nos connaissances."[1]

I do not wish to enter into idle speculation, but the question does arise as to why Walras, who we know had a considerable training in mathematics, did not proceed to apply a method for which he so clearly felt a need. We may hazard two remarks in this connection. In the first place, he had some of the misgivings which Say had.[2] Shortly after the passage quoted in the preceding paragraph we find him remarking that the application of mathematics to economics must be based on observed facts. A more important deterrent, however, was Walras' recognition of the impossibility of measuring utility. In a most interesting comment found in a letter written by Walras is this comment.

> "L'obstacle fondamental qui s'oppose à ce que les mathématiques s'emparent de l'économie politique, comme elles l'ont fait de la mécanique, de la physique, de l'acoustique et de l'optique, réside dans l'impossibilité de déterminer une unité de mesure de l'utilité; une <u>unité</u> <u>besogneuse</u>."[3]

1. <u>Ibid</u>., p. 270.
2. <u>Vide</u> <u>supra</u>, pp. 10-12.
3. <u>Ibid</u>., p. 312.

CHAPTER IV

THE ITALIAN MATHEMATICAL WRITERS

1

Giovanni Ceva (1647 or 1648-1734)

We shall treat Giovanni Ceva very briefly. However interesting his doctrines may be to the student of monetary theory, his work in mathematical economics cannot be considered important.

So far as I have been able to determine, the generally held opinion is correct that Ceva was the first writer on economics to use symbolic notation.[1] It is employed, however, only for the purpose of keeping the variables more easily in mind and for the sake of convenience in writing down proportions; there is no algebraic manipulation of even the simplest sort.

Ceva speaks of both the "external" and the "internal" value of money. By "external" value he means the purchasing power of the circulating medium as a whole, though he apparently considers that medium to consist only of gold, silver, and copper coins. By "internal" value he means the price of specie.

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1. It has been mentioned that Mariotte, who used only arithmetical examples, actually preceded Ceva. *Vide supra*, p. 21.

His demonstration of the determination of the external value of money is as follows.

> At time I let the population a possess a quantity of money b, and let the external value of this money be c. Then in a time K let the population of the country be d, the quantity of money e, and the external value of the money f.

Then Ceva states his "theorem."

> "I say that the value c is to the value f as the ratio of population a to population d and reciprocally as the quantity of money e is to the quantity b."[1]

This seems strange and wonderful doctrine, but so far we have not been concerned with the truth or falsity of arguments judged according to the criteria of today. It is interesting to note at this point, however, that Ceva apparently connected increases in population with an increase in the value of money upon observing the sharp rises in the prices of commodities when soldiers moved into a town.[2]

Ceva's "proofs" of several propositions are similar to the one given. He lets certain letters stand for

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1. Ceva Giovanni, De Re Numaria: quoad fieri potuit geometrice tractata reprinted in Un Precursore della Econometria by Eugenio Masé-Dari, Modena: 1935, p. 40.
2. Ibid., p. 27. This is Masé—Dari's comment in an article which precedes the reprint of the Ceva work.

quantities and then, without giving his reasoning, produces his statement of a law. He is, of course, conscious of the relationships between quantities and states them in simple formulas. In fact, Boven comments on Ceva's explicit recognition of the interdependence of <u>all</u> economic quantities. Apparently he has reference to the following brief remark.

> "Non me latet, hanc reciprocam
> mutuamque dependentiam esse
> veluti inertissimam trutinam,
> utpote quae ab opinione pendent,
> quae nequit rem exacte aestimare."[1]

Boven translates this passage as follows.

> "Je me rends bien compte que cette
> dépendance réciproque et mutuelle
> est comme une balance très délicate
> qui dépend de l'appréciation, laquelle
> ne peut estimer exactement une chose."[2]

Perhaps the most important aspect of Ceva's work is summed up nicely in the following comment quoted by Masé-Dari from a brief biographical notice concerning Ceva in the Grande Enciclopedia Italiana.

> "Ebbe il merito di definire con chiarezza
> il concetto della ipotesi economica
> e di insistere sul valore della economia
> pura, considerandola come la sola scienza
> esatta possibile in questo campo."[3]

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1. <u>Ibid</u>., p. 39.
2. <u>Boven</u>, <u>op.cit</u>., p. 41. Someone who is competent to do so ought to make a translation of Ceva's work into a modern language. I had to have <u>De Re Numaria</u> read to me, and I must confess that the significance of this sentence escaped me.
3. Masé-Dari, <u>op.cit</u>., p. 5.

One further comment may be of interest. Masè-Dari reports that it is not unlikely that Beccaria had read this work of Ceva's and that it suggested his mathematical treatment of a problem in taxation.

2

Cesare Bonesana Beccaria (1738-1794)

Beccaria's brief mathematical work is his Economic Analysis of Smuggling taken from the periodical, Il Caffe.[1] Here is a passage of especial interest because it is among the first attempts to analyse a specific problem in practical affairs with the tools of pure theory. I have found mention of this article in several places, but to my knowledge it has never before been presented in detail.

Beccaria begins by defending the qualified use of algebra in economic analysis. Algebra, he says, can be used in any discussion where we talk about increasing or decreasing quantities, wherever, in fact, two or more quantities are in functional relation. Since the "political sciences" treat of the debits and credits of nations, of taxes, of money, and the like—all things which have to do with quantity and which require calculation—the mathematical method is useful, but only up to a point. Political principles depend in greater part upon the decisions of too many wills and upon emotions which permit of no precise measurements.

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1. Cesare Beccaria, Opere di Cesare Beccaria, Milano: 1822, I., 427.

As an example of the way in which an economic problem may be solved analytically, Beccaria investigates the matter of the profitability of smuggling. When the state imposes either an import or an export duty on goods, it ordinarily makes an evasion of the duty punishable by the loss of the merchandise involved. The risk of the ruling authority is thus proportional to the tax it might lose, the risk of the merchant to that of the value of the merchandise which might be confiscated. If the tax and the value of the merchandise are equal, the risk of the ruler and of the smuggler are equal. If the tax is greater than the value of the merchandise, the risk of the ruler is greater than that of the merchant. It should be added that if the risks of the merchant increases in proportion to the number of guards used by the ruling authority, they diminish in proportion to the volume of goods which the merchant tries to smuggle. These principles "being so clear as to need no further explanation," the author proceeds to the solution of what he considers an important problem in the "economic balance of the state," namely, "What value of a certain good should a merchant attempt to run by contraband so that, even though he lost part of it by confiscation, he would be at least as well off as he was when he started because of the gain of the smuggling?"[1] This is an important question because of the light which it throws on tariff-making.

Let u equal the value of the merchandise to be imported.

Let t equal the total amount of tax if all the goods came through customs.

Let x equal the value of the merchandise required to be smuggled if the merchant is to break even.

Finally, let d equal the difference between the tax and the value of the merchandise.

The value of the merchandise is to the tax as the proportion required to be smuggled to its corresponding tax, or

$$u : t :: x : \frac{tx}{u}$$

Now "we shall have for the condition of the problem the equation $x + \frac{tx}{u} = u$."[2]

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1. Ibid., I., 428. "Si cerca per quanto valore di una data mercé i mercanti dovrebbero defraudare la regalia, cosicché anche perdendo il resto si trovassero per il guadagno del contrabbando collo stesso capitale di prima?"
2. Ibid., I., 429. The condition to be fulfilled is that the merchant shall break even. The amount of his loss is the value of the goods to be imported minus the value of those he must successfully smuggle in order to lose nothing, or in symbols, $u - x$. His gain equals the fraction of the goods that are not seized by the customs officials multiplied by the total tax he would have paid did all goods enter legally, or, in symbols, $t\frac{x}{u}$. When his loss equals his gain, $u - x = t\frac{x}{u}$. Hence the expression $u = x + t\frac{x}{u}$ is the condition that the merchant make neither profit nor loss on his venture.

Multiplying by u, we have

$$ux + tx = uu$$

$$x\,(u + t) = uu$$

$$x = \frac{uu}{u + t}$$

Now the tax could be equal to the value of the good, that is, $t = u$. But the tax might be higher than the value of the good by the quantity d, i.e., $t = u + d$, or it might be lower than the value of the good, i.e., $t = u - d$.

Substituting in the equation

$$x = \frac{uu}{u + t}$$

When $t = u$, then $x = \frac{uu}{u + u} = \frac{uu}{2u} = \frac{u}{2}$.

When $t = u + d$, then $x = \frac{uu}{u + u + d} = \frac{uu}{2u + d}$ or less than $\frac{u}{2}$.

When $t = u - d$, then $x = \frac{uu}{u + u - d} = \frac{uu}{u - d}$ or more than $\frac{u}{2}$.

Those who wish to, says Beccaria, may plot the equation ux + tx = uu, [1] and an inspection of Figure 2 will help the reader to get in mind the results obtained by algebra. For example, when the tax is equal to the value of the merchandise, the value of the goods which must be smuggled is equal to one half the intrinsic value of the merchandise. Thus if the intrinsic value of the goods were $1000 and the tax $1000, $500 in taxes would be saved on $500 worth of goods smuggled in. The runner of contraband could therefore lose the other $500 worth of goods and be no worse off than when he started. This information should be of interest to the tariff maker, according to Beccaria, because he would have the advantage of knowing how much he should have to fear from the merchants of contraband given various tariff rates.

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1. Idem. I have used Becarria's notation of uu rather than u^2. The instructions for plotting the curve are so obscure that two well-educated Italians of my acquaintance could not make an adequate translation. "Supponendo nell'equazione ux + tx = uu indeterminata là t e la x e costante la u; il luogo dell'equazione sarà ad una iperbola fra gli assintoti, di cui le abscisse t prese sull'assintoto da una distanza u dall'angolo assintato, più la medesima distanza, saranno alle ordinate x parallele all'altro assintoto in ragione costante, cioè come il quadrato della potenza u. L'ispezione della figura, in chi la voglia costruire, rischiarirá tutti i differenti casi dell'equazione." The equation $ux + tx = u^2$ is a hyperbola as shown.

It perhaps should be added here that some mathematical notation appears elsewhere in Beccaria's works, although it is not of consequence. There is a passage in which he makes a very simple analysis of international gold and silver flows under bimetallism, not unlike the one of Forbonnais.[1] In another section he gives the rule for relating the value of one good to another in algebraic terms. If s equal the quantity of a good and p the number of possessors, and if m equal the number of buyers, t the taxes, c the labor expended to make the good, and i the transportation required to bring it to market, then

$$v : V :: \frac{mtci}{sp} : \frac{MTCI}{SP} .$$

There is nothing else in Beccaria's works at all comparable in scope to his analysis of smuggling.

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1. Ibid., p. 403.

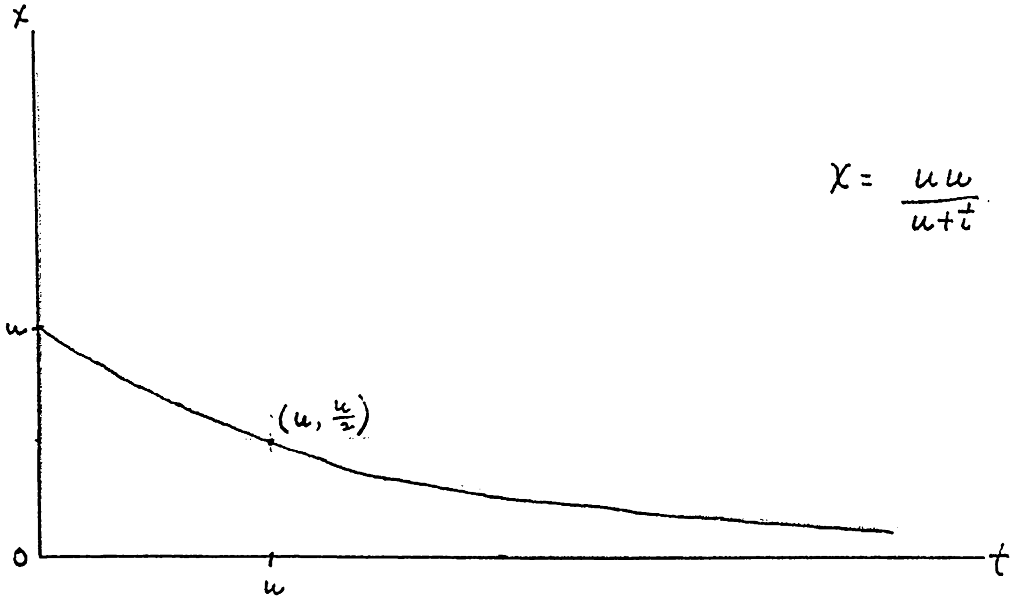

Figure 2.

3

Guglielmo Silio (

The only work in the Jevons-Fisher bibliography which was not obtained is Silio's Saggio sull'influenza dell' analisi nelle scienze politiche ed economiche applicata ai contrabbandi which is to be found in volume V of the collection Nuova Raccolta d'opuscoli di autori siciliani. The secondary sources do not, for the most part, mention Silio. In his short summary of mathematico-economic writers of the eighteenth century, Moret says that Silio doubtless imitated Beccaria in his treatment of matters concerning smuggling. The Sicilian posed and solved five problems concerning the running of contraband, but Moret gives none of his solutions. Taking into consideration all the circumstances favorable to the merchant and all those favorable to the customs officers, and assuming a certain value of the merchandise, Silio calculated the proper duties to be levied on the goods and the penalties to be assessed against those caught evading the law.

In Palgrave's Dictionary there is the statement that Silio advocated the application of the calculus to the solution of his series of problems ". . . to determine the best means for suppressing smuggling."[1] By means of mathematical analysis he demonstrated ". . . the error of supposing that a constantly increasing revenue would be obtained by continually raising the customs duties, and that moderate duties would yield a better financial result and would diminish smuggling, increase consumption, and stimulate foreign trade."[2]

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1. Palgrave's Dictionary of Political Economy, vol. III, p. 394.
2. Ibid., p. 395. An effort is now being made to secure a copy of Silio's work from Italy. As soon as a summary has been prepared, it will be inserted as Appendix C at the end of this volume.

4

Luigi Valeriani Molinari (

The works of Molinari should not be included in any bibliography of mathematical economics unless the appearance of a single simple formula and the use of a few A's, B's, C's and D's in a discussion of exchange rates so qualifies them. Written in the most obscure, pedantic Italian, Molinari's works are difficult to read and likely are not worth the effort.

The specific value of any good, according to our author, may be expressed by the simple formula $p = \frac{i}{o}$, where i stands for inchiesta (demand) and o for offerta (offer or supply). What p stands for is never clear. The specific value of any good is equal to the need for it as expressed by the demand divided by the quantity of the good available to satisfy the need as expressed by the supply--i.e., specific value is equal to demand divided by supply. Molinari makes it clear that i does not stand for the number of demanders nor does o stand for the number of suppliers. The individual who is trying to obtain one unit of a good does not have the same power to increase its value as does the individual who is looking for ten units, and similarly, a supplier who offers one unit of a good does not diminish value as does the supplier who offers ten units. The writer also makes the point that the supply of a good consists not only of the quantity of it on the market at a given time but also of those quantities which _may_ be brought forth (presumably by price rises).

The specific value of one good is to that of another as the demand divided by the supply of one good is to the demand divided by the supply of the other, or,

$$p : P :: \frac{i}{o} : \frac{I}{O} \quad [1]$$

One might think, as did a certain unnamed "celebrated author" who took Molinari to task in the matter, that the ratio $\frac{i}{o}$ would give us the price of one good and $\frac{I}{O}$ would give us the price of another good. But this is not so. If two items are being traded, the specific value of one is the price of the other, and vice versa. That is why we say that the price of a good has changed when we receive

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1. Luigi V. Molinari, _Operette concernenti quella parte del gius della genti, e pubblico, che dicesi pubblica economia_, Bologna: Ulisse Ramponi, 1815, pp. 24-26.

more or less of another commodity for it. Changes in price come about because of bilateral or unilateral changes in the specific values of goods traded.[1] There is no further mathematical treatment in Molinari's works.

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1. Luigi Molinari, Apologia della formola $p = \frac{i}{o}$, trattandosi del come si determini il prezzo delle cose tutte mercatabili, contro ciò che ne dice il celebre autore del nuovo prospetto delle scienze. Bologna: Marsigli, 1816, p. 6.

5

Francesco Fuoco

In a paper read at the meeting of the Econometric Society, Lausanne, September, 1931, Gustavo Del Vecchio calls attention to the fourth of Fuoco's Economic Essays entitled "The Use of Algebra in Political Economy." He lists the chapter headings of the essay which include "Is algebra adapted to apply to the objects of Political Economy?", "Examples of some of the principal applications of algebra to the aims of Political Economy," and "How far one can usefully extend the application of algebra to Political Economy?" Del Vecchio then comments as follows.

> "Fuoco is mentioned by Cossa, Jevons, and in Palgrave's Dictionary of Political Economy, but if he really treated all these subjects, eleven years before Cournot, in a conscious and constructive way, he could rightly be called an unknown economist.
>
> "Truly, his actual treatment of the problem is rather careless, and it exhibits several mistakes which mathematical economists ought to avoid. But still his exact general argument cannot be overlooked, and we should not forget his service in opposing the view adopted too hurriedly (on the occasion of the unhappy attempt of Canard) by J. B. Say, who espoused the position that mathematics cannot

> find a place in economic research. The confutation of such a great authority as Say in a question of this importance is sufficient to insure a place for the Italian writer in the history of economic thought."[1]

The statement of Del Vecchio is fairly accurate, though his judgment is tentative. I think we may say with assurance that Fuoco had a proper concept of the application of algebraic methods to economics, but he warns against the use of other than simple expressions and seems himself to have contributed nothing of consequence in the way of applying his method.

Fuoco contends that whenever we reason about quantity, whenever we deal with functions and with quantitative relationships, the use of algebraic language is not only easy and natural but very useful. Its usefulness is not confined to the establishment among quantities of "determinate relationships." Frequently it is of great help in expressing relationships among "indeterminate quantities," in which instances it is necessary to know only that as one quantity varies, other quantities vary functionally with it. Thus even though

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1. Gustavo Del Vecchio, "Francesco Fuoco, Opponent of J. B. Say on the Use of Algebra in Political Economy," Econometrica, 1:220, April, 1933.

certain laws have not been precisely verified by statistical study general tendencies may be expressed by means of algebraic expressions. Say, he avers, has misconstrued the purpose of the mathematical (algebraic) method, because the eminent French writer insists that problems can be solved only when the quantities expressed by letters are "real and possible," that "impossible" quantities cannot be used to solve a problem in a determinate manner.[1] Say, in short, insists on the impossibility of inserting the vast number of economic variables into equations. He seeks specific answers and feels that such calculations cannot be made.[2] But, contends Fuoco, Say forgets that mathematicians consider quantities which are negative, imaginary, indeterminate, infinitely small, infinitely large, and so on, and they are interested in concise statements of relationships as well as in solving problems.

With this kind of an introduction to the matter one expects great things. In my opinion, however, Fuoco goes no farther than, say, Canard, in furnishing analytical tools.

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1. Francesco Fuoco, Saggi Economici, Pisa: S. Nistri, 1827, Tomo II, Saggio IV, p. 79. "Il Say pensa con molto fondamento che i problemi non divengano solubili se non quando le quantità espresse colle lettere sono reali, e possibili, perchè le quantità impossibili non possono servire a risolvere un problema in una maniera determinata."
2. Vide Supra, pp. 10-12.

He uses only the simplest algebraic expressions, writes only explicit functions, and, with the exception of one or two clever ideas, offers nothing of particular consequence.

He begins with a strange device. Work (lavoro) is a word, which expresses the entirety of industrial skill or ability. It is the "single and unique" cause of production. We may substitute for it the idea of "force," and if we introduce the element of time (which economists do not take sufficiently into consideration) and the "velocity" of a given work force, we may set down in algebraic terms an instructive relationship.

Let the productive force be f, the time t, and q the quantity of product.

Then $q = ft$

or $f = \frac{q}{t}$

or $t = \frac{q}{f}$

Call velocity v, and since it is in direct proportion to force and in inverse proportion to time,

$$v = \frac{f}{t}, \text{ and therefore } f = vt$$

Substituting in the first equation

$$q = vt^2$$

Hence,

$$\frac{q}{v} = t^2 \quad \text{or} \quad t = \frac{q}{v}$$

$$\frac{q^2}{t} = v$$

Thus if two of the four quantities f, t, v, and q are given, the others can be determined.

The quantity of product can be increased or decreased or the period of work shortened without altering output according to the laws of proportion thus established among the quantities which contribute to production. Thus an increment of force or of velocity, a greater or smaller saving of time, would not only help to determine ahead of time the resultant change in quantity of product but also changes in cost of production. Therefore it is easy to resolve questions regarding the use of bigger or smaller machines and of better methods of production, and it is possible to determine the limits to the expansion of industry as a whole, at a certain time and in a certain state of society.[1]

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1. Ibid., pp. 92-95. Fuoco certainly claims too much for his simple formula $q = vt^2$. Apparently he tacitly assumes that more detailed relationships would be available in practice.

There follows a long discussion regarding value and price which we must omit. Fuoco is quite aware of the difficulties involved in equating the value of a good with the quantity of labor which goes into it. "If we let u equal value, and l the quantity of work required to produce that value, we have $u = l$. But what useful knowledge could ever be gained from this equation? And what precise idea should we have about l?"[1] There is implied the necessity of analyzing l, without which the idea of value remains "confused and indeterminate."

Let capital consumed in production be c, interest i, the wage of labor employed to make the good s, the profit resulting from managerial ability (talento direttore) p, and the profit resulting from the skillfulness of the worker (destrezza dell'operajo) d. Then

$$v = c + i + s + p + d$$

Now price consists of that quantity of value which is obtained for another quantity of value, both being calculated in money.[2]

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1. Ibid., p. 96.
2. Idem. "Il prezzo consiste in quella quantità di valore, che si ottiene per un altra quantità di valore calcolate entrambe in numerario."

Price then is determined on the basis of the quantities of value that are exchanged when goods are exchanged. But from this we must not assume that p = v, because p could be equal to, greater than, or less than v. If x equal the quantity by which p may be greater, and y the quantity by which it may be less, then we have three possible cases

$$p = v$$

$$p = v + x$$

$$p = v - y$$

These variations can result from more or less competition among sellers on the one hand andbuyers on the other hand. When the quantity offered equals the quantity demanded, the first equation holds true. When the quantity demanded is greater than that offered, the second equation is applicable. Finally, if the quantity offered exceeds the quantity demanded, the third equation describes the situation.

There is another way of expressing a relationship between price and other quantities, however, and Fuoco finds this more useful. Price increases or decreases in inverse ratio of the quantity of a good offered, but it varies directly with the need that someone has to buy <u>or</u>

sell. If b equals this need and q equals the quantity of the good, then

$$p = \frac{b}{q}$$ [1]

There follows some manipulation of this formula. For example, the striking conclusion may be reached that $b = pq$, i.e., that ". . . il bisogno è nella ragion composta diretta del prezzo, e della quantita del prodotto."[2] Or if the price of a good at one time is expressed by $P = \frac{B}{Q}$ and at another time by $p = \frac{b}{q}$, then clearly

$$P : p :: \frac{B}{Q} : \frac{b}{q}.$$

If in comparing these two quantities at different times the needs remain the same, the immediately preceding equation becomes

$$P : p :: \frac{1}{Q} : \frac{1}{q}$$

$$\text{or} \quad P : p :: q : Q$$

In order to do justice to Fuoco's method, I shall prolong the discussion in order to present what is his cleverest analysis. Let j be the value of the day's work

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1. Ibid., p. 98.
2. Ibid., p. 100.

of a man who in his lifetime can work a time K and who requires a period of instruction t; and let j' be the value of the day's work of a man whose trade is such that anyone could work at it without any instruction and without shortening his life by engaging in it. Let K' be the portion of his life to be spent at the trade. Now what the first worker earns during his lifetime, $(K - t)\,j$, must at least be equal to what he would earn in the second trade plus the value of the capital used in his instruction. This latter amount may be supposed equal to what he would have earned in the second trade during his period of instruction, or tj'. Thus

$$(K - t)\,j = K'j' + tj'$$

or

$$\frac{j}{j'} = \frac{K' - t}{K - t}$$

Thus far all men have been supposed equal in ability. In order to take into account differences in ability, let n represent the ratio of the value of a day's work as between two laborers in the first trade, one with ordinary talent, the other with superior talent. This quantity n is a ". . . function of the number which would express the probability that in a given number of men there would be found one of the superior ability possessed by the worker whom we are considering."[1] If j be the value of a day's labor of the ordinary worker, and y that of a superior man, we shall have $j = \frac{y}{n}$. Substituting this value in the preceding equation, we have

$$\frac{y}{j'} = n\frac{(K' - t)}{K - t}$$

And if j' be considered as a unit of value of a day's labor

$$y = n\frac{(K' - t)}{K - t}$$

Now if V be the natural value of any product; C, the value of the raw materials of which it is composed plus the interest on the capital used to make it, and T the number of days used in producing the good, then

$$V = T \times y + C = T \times n\frac{(K' + t)}{K - t} + C$$

All of this seems hardly important enough to merit further consideration, and Cournot's place in the history of thought is apparently not in jeopardy. One further

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1. *Ibid.*, p. 103.

comment should be made, however. Fuoco was familiar with the work of General Lloyd, which has been described, and he comments favorably on his equations. He also mentions the mathematical expressions to be found in Chapter IV and Chapter XIV of Pietro Verri's Meditazioni. A careful examination, however, of two editions of this work failed to reveal anything in the way of mathematical notation.[1]

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1. Compare Pietro Verri, Meditazioni sulla Economia Politica, Venezia: 1771, pp. 22-31 and 97-101 with the same work published as vol. XV of the "Scrittori Classici Italiani di Economia Politica," Milano: G. G. Destefanis, 1804, pp. 32-52 and 131-139. The actual symbols may appear in other editions or Fuoco himself may have made the easy translation of the following: "Il prezzo delle cose è in ragione diretta del numero de' compratori, e inversa del numero de' venditori." See p. 31 of the Venice edition of the Meditazioni.

CHAPTER V

THE GERMAN MATHEMATICAL WRITERS

1

Claus Krönoke (1771-1843)

It is disappointing to learn that Krönke's _Ausfürliche Unleitung zur Regulirung der Steuern_ contains nothing of a mathematical nature, for this is not true of an earlier work, _Das Steuerwesen, nach seiner Natur und seinen Wirkungen._ The latter work is divided into two approximately equal parts, the first treating generally of the relations between the value of the various factors of production and their products, the second having to do with principles of taxation. Algebraic symbols are used throughout the book, but their most important use is in the discussion of capital and interest. Because it is typical of Kröncke's method and because the principles involved recur throughout the work, this analysis is given in some detail.

Let m equal the natural rate of interest, that is, the rate at which a lender would part with an amount of capital if there were no risk involved. If there were an element of risk, the rate would be somewhat higher, say, $m + x$, where x indicates the risk factor.

If an owner of capital were to lend 100 units, a being one unit of capital, he would have at the end of a year the sum

$$100a + \frac{100am}{100} = (100 + m)a$$

If there were a possibility that of the 100 units of capital lent, one unit would be lost together with the interest on that unit, the lender could expect to have at the end of a year

$$99a + \frac{99a(m + x)}{100} = \left[99 + \frac{99}{100}(m + x) \right] a$$

Any lender must receive an equal amount by either method so that

$$\left[99 + \frac{99}{100}(m + x) \right] a = (100 + m)a$$

$$99 + \frac{99}{100}m + \frac{99}{100}x = 100 + m$$

$$\frac{99}{100}x = 1 + \frac{1}{100}m$$

$$99x = 100 + m$$

or

$$x = 1\frac{1}{99} + \frac{1}{99}m$$

Or, more generally, if from a number of units of capital, p, one unit is lost, together with the interest on it, we have

$$a\left(p + \frac{pm}{100}\right) = a\left[p-1 + \frac{(p-1)m}{100} + \frac{(p-1)x}{100}\right]$$

$$\frac{(p-1)x}{100} = 1 + \frac{m}{100}$$

$$x = \frac{100 + m}{p - 1}$$

The quantity x can never be much greater than this because competition among the lenders keeps it down; on the other hand, the amount x is in the nature of insurance against loss and must be received by the capitalist. This is not idle speculation; it is a practical matter, and the actual amount of x is being determined every day in the market.[1]

Capital is valuable because it brings forth yearly products for the owner. Frequently it is necessary to compare the value of this annual flow with the value of the

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1. Claus Krönoke, Das Steuerwesen nach seiner Natur und seinen Wirkungen, Darmstadt und Giessen: Heyer, 1804, pp. 54-59.

capital which gives rise to it. In the case of capital and interest, this comparison can be made easily.

Let the value of capital be a.

Let the annual "interests" or products (Interessen oder Produkte) be b.

From the preceding argument concerning the rate of interest

$$a : b = 100 : m + x$$

or $$a : b = \frac{100}{m+x} : 1$$

The interest or the annually received product, taken $\frac{100}{m+x}$ times, gives the value of the capital. And conversely, the value of the capital divided by $\frac{100}{m+x}$ or multiplied by $\frac{m+x}{100}$ gives the value of the annual interest (Zinsen).

There arise problems involving a factor of production and its value flow. One of the most common is: What is the present value, x, of a capital good which lasts n years and produces annually a product a?

Assume that $\frac{m+x}{100} = p$, so that the present values of the products forthcoming are:

At the end of the first year $= a \cdot \frac{1}{(1+p)}$

At the end of the second year $= a \cdot \frac{1}{(1+p)^2}$

At the end of the third year $= a \cdot \frac{1}{(1+p)^3}$

$\vdots$

At the end of the n-1 year $= a \cdot \frac{1}{(1+p)^{n-1}}$

At the end of the n year $= a \cdot \frac{1}{(1+p)^n}$

The sum of these values in the right-hand column is X.

$$\text{Let } \frac{1}{1+p} = u$$

$$\text{Then } X = a(u + u^2 + u^3 \ldots\ldots\ldots u^n)$$

$$\text{and if we write } X = aS$$

$$S = u + u^2 + u^3 \ldots\ldots\ldots u^n$$

$$\text{and } \frac{1}{u}S = 1 + u + u^2 \ldots\ldots\ldots u^{n-1}$$

Subtracting the second line from the first,

$$S(1-\frac{1}{u}) = u^n - 1$$

Thus

$$S = \frac{u^n-1}{1-\frac{1}{u}} = \frac{u^{n+1}-u}{u-1}$$

and $$X = a\left[\frac{u^{n+1}-u}{u-1}\right]$$

But $$U = \frac{1}{1+p}$$

So that

$$S = \frac{\frac{1}{(1+p)^{n+1}} - \frac{1}{1+p}}{\frac{1}{1+p} - 1} = \frac{\frac{1-(1+p)^n}{(1+p)^{n+1}}}{\frac{-p}{1+p}}$$

$$= \frac{1+p}{-p(1+p)^{n+1}} - \frac{(1+p)^{n+1}}{-p(1+p)^{n+1}} = \frac{1}{p} - \frac{1}{(1+p)^n p}$$

and finally

$$X = a(\frac{1}{p} - \frac{1}{(1+p)^n p})$$

For example, if the annual interest be 10% (i.e., $m+x = 10$), then $p = \frac{m+x}{100} = .10$. If the capital is to last twenty years, $n = 20$, and

$$S = \frac{1}{.10} - \frac{1}{(1.1)^{20} \times .10}$$

$$= 10 - \frac{1}{(1.1)^{20} \times .10}$$

$$\log.\ 1.1 = 0.0413927$$

$$\times 20$$

$$\log.\ (1.1)^{20} = 0.8278540$$

$$\log.\ 0.1 = 0.0000000-1$$

$$\log.\ (1.1)^{20} \times .10 = 0.8278540-1$$

$$\log.\ \frac{1}{(1.1)^{20} \times .10} = 0.1721460$$

which is the logarithm of 1.48644.

Therefore

$$S = 10-1.48644$$

$$= 8.51356$$

Thus if someone expects to receive at the end of every year for twenty years the product of a capital good, and if the rate of interest is ten per cent, the present worth of all the products will be equal to 8.51356 times the value of the product for a year.

This is the basic analysis, although Kröncke poses many different problems which may be solved given the essential formula.

2

Joseph Lang (

Lang begins his <u>Grundlinien der Politischen Arithmetick</u> with the contention that certain parts of the subject matter of Political Economy (National-Oekonomie) are susceptible of quantitative analysis. To the extent that economic quantities can be observed as increasing or decreasing with respect to other quantities which themselves grow larger or smaller, the principles of mathematics can be applied to them, and this study is called Political Arithmetic.[1]

It is most difficult to offer a compendium of Lang's work. His analysis proceeds step by step, and a full understanding of the material at any particular point requires a careful reading of everything which has gone before. However, a summary of the introduction and first two chapters gives an indication of the method used, which is, after all, what is of especial interest in this treatment.

The first quantity with which Political Arithmetic is concerned is the population of a country (die Bevölkerung).

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1. Joseph Lang, <u>Grundlinien der Politischen Arithmetick</u>, Kursk: C. Langner, 1811, p. 1.

Changes in population as a whole are of the utmost importance and exert a real influence on the national well-being. Perhaps even more important are the changes relative to each other of the classes into which a population is divided. Variations in the numbers of people in each class exert a powerful influence on the national wealth, and such variations are themselves conditioned by changing quantities in the economic system.

Every nation's population may be divided into three main classes. First, there is the producer class (Die Klasse der Produzenten), which contains all the people who participate in agriculture and in the finding and gathering of raw materials. Farmers, hunters, fishermen, miners, and the like, are included in this group. Second, there is the industrial class (Die industriöse Klasse), which includes everyone who busies himself with processing and fabricating raw materials and with distributing these products to the consumer. Included here are craftsmen, factory workers, merchants, even artists--in short, all who are in any way connected with business. Third, there is the official or service class (Die Klasse der Dienstthuer). This group is comprised of those who do not produce material things and who do nothing for the pleasure and enjoyment of others. Belonging to this category are state officials, the military, property owners who live from their rents, and owners of capital who live on their interest.

Each class brings forth annually a total product which is the result of the "mutual working together" of its members. The amount of all the food products and raw materials which the first group produces in a year and sells to the other two classes is the Landbaubalanz, which I shall call the "Agricultural balance" since the greater part of it consists of the products of the farmer. The mass of all manufactured and "artistic" products which in the course of a year pass from the second class to the other two is the industrial balance (Manufakturbalanz). Finally, the sum of all the services rendered by the members of the third category to the other two groups is the service balance (Dienstbalanz). These three balances comprise the Realbalanz, the money value of which is called the Geldbalanz. (At this point it is asserted that the exchange value of a good in terms of money is called its price.)

The author now proposes to relate each individual balance to the other two balances. The producer class sells each year to the industrial class a mass of food products and other natural products, and Lang lets the total amount of money which the producer class receives on this account equal a. Similarly the producer class receives a sum of money for goods sold to the group which renders services, this sum to be represented by the letter b. The money worth of the <u>Landbaubalanz</u> is then expressed

$$X = a + b.$$

Similarly Lang writes an expression for the money value of the <u>Manufakturbalanz</u>. Let the money value of the products which the industrial class sells to the producer class be c, and the amount of goods sold to the service class be equal in value to d. Then

$$Y = c + d.$$

Finally, in considering the <u>Dienstbalanz</u> the money value of the services sold by this group to the producer class is e, and the amount sold to the industrial class is f so that

$$Z = e + f.$$

Up to this point Lang has considered the money values <u>received</u> by each class, but he wishes to know also what amounts are paid out. The equations given above show that the money value which the producers pay to the industrialists is c and the amount paid to the "official" group is e. Thus the money amounts paid out for goods and services received by the producer class is

$$X' = c + e.$$

Similarly the industrial class pays an amount a to the producers for goods purchased and to the "official" class an amount f for services rendered so that

$$Y' = a + f.$$

Finally the service (official) class pays an amount b to the producers and an amount d to the manufacturers. Thus

$$Z' = b + d.$$

Since no class can pay a money amount greater than it receives,

$$X = X',$$

$$\text{or} \quad a + b = c + e.$$

$$Y = Y',$$

$$\text{or} \quad c + d = a + f.$$

$$Z = Z',$$

$$\text{or} \quad e + f = b + d.$$

So much for the introductory material. Lang argues for the necessity of having accurate statistical data regarding the total population, the numbers in each class, and so on. But general formulas can be arrived at concerning the most important relationships, even though their practical application must await the assembling of the requisite statistical data. To this task he now proceeds.

Let the whole population be A.

Let the number in the producer class be V.

Let the number in the industrial class be X.

Let the number in the service class be Y.

$$\text{Then } A = V + X + Y.$$

He is now ready to derive a more general formula for the money balances received by each class as expressed in "value units." In order to do this it is necessary to define the value unit (Wertheinheit). This he takes to be ". . . that quantity of foodstuffs and other raw materials which, on an average, an individual of the industrial class needs during the course of a year for his immediate and personal needs."[1] The value of such a quantity of goods is taken as equal to unity.

Defining the value unit as he does, Lang says that the value of all the materials furnished by the producers to

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1. *Ibid.*, p. 55.

the industrial class, to be used for the personal needs of the industrial workers, is equal to as many units as there are members of that class, or X. In addition, large amounts of raw materials are fabricated by the manufacturers, and the value of these materials stands in some relationship to the amount of materials required for personal needs. This relationship may be expressed by m. Thus the goods taken by the industrial group on this latter account equal mX. Finally Lang considers the quantities of food and raw materials used by the service class. A civil servant, an army colonel, or a clipper of coupons uses larger quantities of these, on an average, than an industrial worker. The relative value of the quantity of natural products which a member of this class uses is f, where $f>1$. Thus the value of the products taken by the service class as a whole equals fY. Adding these three divisions of the Landbaubalanz, the received money balance in terms of value units is

$$B = X + mX + fY$$

The amount received is equal to the amount paid out and the amount paid out may be divided into two parts, one going to the service class, the other to the industrial class. These two parts vary considerably relative to each other in different countries at different times. Let the relative portion which goes to the service class be g, where $g<1$, so that the part of the producers' income which goes to the service class equals

$$gB = g\,(X + mX + fY)$$

The part of the producers' income which passes to the industrialists is the portion of total income which remains when the part which goes to the official class has been subtracted. The relation of this part to the whole is therefore 1-g and the part which is spent on industrial goods is therefore

$$(1 - g)\,B = (1 - g)\,(X + mX + fY)$$

If the amount paid out be denoted by B' then

$$B' = gB + (1 - g)\,B = B$$

$$\text{therefore} \qquad B' = X + mX + fY$$

Similarly Lang arrives at expressions for the received money balance of the service group. It receives a part of its income from the producers and a part from the industrial class. From the former it has, from the preceding paragraph,

$$gB = g\,(X + mX + fY)$$

From the latter, if h be the "relationship" of this part to the value of the producers' balance,

$$hB = h\,(X + mX + fY).$$

Adding these parts the value of the Dienstbalanz or the received money balance of the service class is

$$D = gB + hB = (g + h) B$$

$$\text{or} \quad D = (g + h) (X + mX + fY)$$

And so on. In similar fashion expressions for the "expended" money balance of the service class, for the "received" money balance of the industrial class, and for the "expended" money balance of the industrial class may be written.

We may now pass to a brief examination of the chapter entitled "The Total Product and the Net Income of the Three Classes." Let U be the value of the natural products which the producer class uses per annum for its indirect needs, i.e., for the actual operation of, say, the farm or mine. Seed grain or animal feed would be an example of such a product for the farmer. Let V equal the value of the natural products which are consumed for the direct, i.e., personal, needs of this group. It will be recalled that the value of the Landbaubalanz, that is, the amounts received from the other two classes, is

$$B = X + mX + fY$$

These three items constitute the worth of the product which the producer has in a year, and this is equal to

$$U + V + B$$

or $$U + V + X + mX + fY$$

But note this. Those natural products which the producer uses for his "indirect" needs, i.e., for the "continuance" of his enterprise, may be considered as a "gift of nature" and are not to be viewed as a result of the activity of the producer himself. Thus ". . . das jährliche Totalproduct der Produzenten

$$= V + B$$

$$= V + X + mX + fY$$"[1]

To find the net income of the producer class (das reine Einkommen der Produzenten) certain subtractions must be made from the total product. First is the amount paid to the service class in fees, taxes, rent, interest, etc. This is

$$gB = g(X + mX + fY)$$

Second is the value of its instrumental outlay (Instrumental-Auslagen), that is, the value of the manufactured products used in conducting the enterprises. Let this value be

$$L = \ell B = \ell(X + mX + fY)$$

These instrumental outlays are only a part of the manufactured products which the producers receive from the industrial class. Therefore,

$$L < (1 - g) B$$

and $$l < (1 - g)$$

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1. ~~Ibid.~~, p. 66.

The value, then, which is to be subtracted from the total product of the producer class is

$$gB + L = (g + \ell)\,B = (g + \ell)\,(X + mX + fY)$$

and since the total product is

$$V + B = V + X + mX + fY$$

the net income of the producer class is

$$\begin{aligned} V + B - gB - L &= V + B - gB - \ell B \\ &= V + (1 - g - \ell)\,B \\ &= V + (1 - g - \ell)\,(X + mX + fY) \end{aligned}$$

And so the formulas for the net income of the other two classes might be deduced. In Lang's next chapter is considered the total product of the nation and its net income, in the next the average net income of the individuals of the nation, and finally money and the monetary circulation. The sixth chapter, on the application of the algebraic formulas of political arithmetic, is significant; especially so are the hypothetical economic tables of nations, wherein are considered the important economic aggregates of a national economy.

One further matter of interest may be noted. In the chapter on money and monetary circulation there is a statement of what we should call an equation of exchange. In Lang's notation it is

$$yZ = Px$$

where Z is the quantity of money in circulation, y the rapidity or velocity of circulation (*Die Zirkulationsgeschwindigkeit*), P the amount of goods and services exchanged in a year expressed in value units, and x is the "money price of a value unit" (*Geldpreis der Werteinheit*).

If the quantity of money in circulation be increased by an amount ΔZ there must also be an increase Δx in the money price of the value unit and we have

$$Y(Z + \Delta Z) = P(x + \Delta x)$$

Since $YZ = Px$ we have

$$Y\Delta Z = P\Delta x$$

and $$\Delta x = \frac{Y}{P}\Delta Z$$

This use of the sign "Δ" has been encountered only in the work of Lang.

3

Georg Von Buquoy (1781-1851)

Large sections of von Buquoy's books have no interest for the modern economist. Five-sixths of the pages of Die Theorie der Nationalwirthschaft are devoted to Book I which is entitled "The Direct Sources of National Wealth or the Technical Part of Political Economy." The word "technical" is an apt one. In this part are discussed the laws of physics affecting the movement of water wheels used in connection with the milling of flour, the laws of physics applicable to the proper loading and towing of wagons, the rules for pressing oil and wine, for running saw mills, for reducing iron ore, and so on. Even the part on commerce deals largely with such things as measures, weights, tariff rates, and the like. An even greater portion of the supplement, Das Nationalwirthschaftliche Prinzip, is devoted to an explanation of formulas taken for the most part from the field of physics.

Because of the importance of two particular passages, I shall pass to them quickly and treat them in some detail. The one which is of especial interest contains his suggestions for the uce of implicit functions in economic theory.

Buquoy says that one must be careful in using mathematics in the study of political economy. It is no wonder that Say became angered at the misuse of algebra, for he had not seen it properly employed in the statement of the great general laws having to do with economics. The mathematician should lead us not to dry and meaningless results but should reveal the laws according to which the quantitative influences of changable and interdependent quantities are felt. We may say that a quantity b increases or decreases as certain other quantities a, c, or d vary, but we never need to express the determined values of a, c, or d in order to find our effect b expressed in figures.

It is often sufficient in the study of political economy to use ". . . algebraic formulas as sentences expressed in symbols."[1] For this purpose the following kind of notation can be used. "Let $f(\overset{<}{X})$ signify a quantity dependent upon X in such a way that as X increases the quantity also increases. For example, we can understand by $f(\overset{<}{X})$ the market price of a commodity and by X the quantity of an actual demand. On the other hand, $f(\overset{>}{X})$ indicates a quantity which depends upon X in such a way that as X increases the quantity decreases. For instance,

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1. Georg von Buquoy, *Das Nationalwirthschaftliche Prinzip*, Leipsig, 1816, p. 335.

the market price of a commodity decreases if the competition, X, of sellers increases. The symbol $f\,\overset{<a>}{(X)}$ means that a certain quantity depends on X in such a way that as X increases it increases to a certain limit 'a'; beyond this limit, however, the quantity decreases as X increases. The symbol $f\,\overset{>b<}{(X)}$, on the other hand, indicates a quantity which decreases to a certain limit 'b' as X increases; beyond this limit it increases as X increases. Thus as division of labor increases natural price is lowered, but a limit will be reached beyond which natural price again increases.

"The application of such symbolic expressions, which conveniently express a whole chain of ideas in a single expression, becomes a great aid in the presentation of abstract combinations. . . . Thus, for instance, $F\,(\overset{<}{f}\,(\overset{<}{X}))$ indicates a quantity which increases as $f\,(X)$ increases, where X itself is an increasing quantity. . . . Now this can be put a little more clearly. Instead of saying $Z = F\left(\overset{<}{f}\,(\overset{<}{x})\right)$, have $Z = F\,(\overset{<}{y})$ and $y = f\,(\overset{<}{X})$. Similarly, instead of writing $U = F\left(\overset{<}{f}\,(\overset{>}{X})\right)$, one could write $U = F\,(\overset{<}{y})$ and $Y = f\,(\overset{>}{X})$; from which it would follow that with an increasing value of X the value of $f\,(X)$ decreases and therefore U decreases. Here, for instance, one could express by X the competition of sellers, by f (X) the market price of a good, and by U the business profit of the sellers. If one writes $Y = \varphi(\overset{<}{y}, \overset{>}{x})$ he expresses the fact that as y increases Y increases and as x increases Y decreases. Thus one might express the relationship between Y, the market price, and the competition among buyers and among sellers, where y stands for the former and x for the latter. Finally, should these quantities be substituted in the expression U = F (Y) there would result $U = \overset{<}{F}\left(\varphi(\overset{<}{y}, \overset{>}{x})\right)$; that is, with competition y among buyers and x among sellers, market price at the same time grows and diminishes, and so consequently does the business gain of the individual merchant."[1]

Thus we have for the first time, beautifully set down, instructions for the use of implicit functions in economic theory. It is astonishing, therefore, to discover that Buquoy does not make use of this device in his principal section on "natural price." He does express his theory in terms of equations, but he uses only explicit functions. Somewhat in the manner of Canard, he states algebraically the "costs" at different stages of production, where costs include the rent of land, profits, interest

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1. Ibid., pp. 335-336. This is a fairly free translation, but it follows the text closely enough to be considered a direct quotation.

on capital, and current outlay on labor and raw materials. Natural price is simply equal to a summation of all the costs encountered at various stages of production.[1]

However, in the section on agricultural techniques, at the very beginning of the first volume, there is a use of implicit functions in the treatment of a problem in which a gain is to be maximized. This analysis is so strikingly modern in all its implications that it deserves a detailed reporting.

The author wants to put depth of plowing on a "rational" basis so as to determine what depth may be of greatest advantage to the farmer. In order to do this, one must know according to what "law" yield increases as depth of plowing becomes greater and also according to what "law" costs increase with increasing depth. These two relationships can be expressed by two curves; along the X-axis are measured depths of plowing, and along the Y-axis are measured value of product and costs. With regard to total revenue, Buquoy postulates that as depth of plowing increases, total revenue increases but in ever decreasing proportion

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1. For this discussion see Georg von Buquoy, Die Theorie der Nationalwirthschaft, Leipzig: Breit, 1815, pp. 242-253. A reporting of this material would make this section unduly long and would add nothing of significance. Interested readers will find it simple enough and perhaps useful for some purposes.

and that at a certain point the curve becomes a straight line parallel to the X-axis. With regard to the curve of total cost, as depth of plowing increases, total costs increase in an increasing proportion so that at a relatively great value of the abscissa a great value of the ordinate appears. Then Buquoy reasons as follows.

> "Let the law according to which the value of the crop depends upon the depth of plowing be given by the equation $y = f(x)$ and let the law according to which costs depend upon depth of plowing be expressed by the equation $Y = F(x)$. Then the net revenue can be generally expressed at any depth by $f(x) - F(x)$, from which that value can be found at which the net revenue is a maximum. And this maximum will be at that value of x at which the first derived function of $f(x) - F(x)$ disappears and at which simultaneously the second derived function becomes negative."[1]

Except for the fact that he is not considering total revenue and total cost as a function of output of product, but rather as a function of depth of plowing (i.e., of a technical operation), this is the modern approach to the problem of maximization. The idea upon which partial equilibrium analysis was to be based had appeared in print.

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1. *Ibid.*, p. 54.

4

Johann Heinrich von Thünen (1783-1850)

That portion of von Thünen's work which is of greatest interest to the modern reader appeared some twelve years after the close of the period under present consideration. Der isolirte Staat was published originally in 1826 and revised slightly in 1842, but the second volume which contains his theory of marginal productivity and his development of the formula for the "natural wage" was not published until 1850. On a number of grounds the German writer must be considered in the first rank of economists. However, excellent articles concerning him have made available his most important doctrines, and the purpose of this section is to present a very brief analysis of the early work in which he developed his methodological technique.[1]

Von Thünen begins by describing his hypothetical economy. He imagines, for the sake of simplicity, a state

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1. See H. L. Moore, "Von Thünen's Theory of Natural Wages," Quarterly Journal of Economics, 9:291-304, April, 1895, and 9:388-408, July, 1895, and Eric Schneider, "Johann Heinrich von Thünen, Econometrica, 2:1-12, January, 1934. See also the excellent article of Arthur H. Leigh, "Von Thünen's Theory of Distribution and The Advent of Marginal Analysis," The Journal of Political Economy, 54:481-502, December, 1946.

in which there is one large city located precisely in the center of a broad plain consisting of land of equal fertility. The plain is traversed neither by canal nor by navigable river, and all transportation is by means of horse-drawn wagon. There is no city or town other than the one at the center, and the state is completely separated from the rest of the world by a great desert which surrounds the plain.

Having stated his conditions, von Thünen proceeds to an examination of the elements which go to make up the price of grain. He draws on his practical experience as owner of a great estate in Mecklenburg for the facts and figures necessary to his discussion. He begins with an extensive consideration of the effects of transportation costs on the prices which may be received for grain at varying distances from the central market, proceeds to a consideration of the relationship between grain prices and land rent, and finally analyzes production costs in detail. Two fundamental principles emerge from his detailed arithmetical treatment.

(1) The value of grain on an estate diminishes as the distance which separates the estate from the market increases.

(2) A part of the costs of production of grain is proportional to the area of land cultivated and another part is proportional to the size of the crops.

Von Thünen admits that his arithmetical analysis, based on figures taken from the Mecklenburg estate, does not necessarily fit all possible situations. A more general analysis can be made, however, in algebraic terms, and the results obtained in this way can be used in any particular case simply by substituting in the equations the applicable figures. He then proceeds to derive a formula which will represent generally the price of a grain.

"Let the product in grain be x, the gross revenue be ax thalers, the cost of seed be b thalers and the costs of preparing the ground for sowing be c thalers. Let there be between the gross revenue and the expenses which are proportional to the size of the crop a ratio represented by 1 : q, where q is always a fraction because the proportional expenses can never absorb the whole crop. Since 1 : q = ax : aqx, it follows that the proportional expenses are equal to aqx thalers. Certain costs will have to be met by money payments and that part of the expense which must be paid in money is represented by p. The part of the expense to be paid in grain is represented by 1 - p, where p is a fraction. Finally, the value of rye on the estate is equal to h thalers. The following relationships then hold.

The gross product $= \frac{ax}{h}$ scheffels of rye

The expenses of sowing $= \frac{b}{h}$ scheffels of rye

The expenses of preparation $= \frac{(1-p)c}{h}$ scheffels of rye + pc thalers

The expenses proportional to the size of the crop are

$$\frac{(1-p)aqx}{h} \text{ scheffels } + \text{ apqx thalers}$$

The land rent is then equal to

$$\left(\frac{ax}{h} - \frac{b + (1-p)c + (1-p)aqx}{h}\right) \text{ schef.} = p(aqx+c) \text{ thlr.}$$

If the land rent be assumed equal to 0, then

$$\left(\frac{ax}{h} - \frac{b + (1-p)c + (1-p)aqx}{h}\right) = p(aqx+c) \text{ thlr.}$$

Consequently,

$$(ax - b - (1-p)\ (aqx+c) \text{ schef.} = hp(aqx+c) \text{ thlr.}$$

and 1 scheffel $= \dfrac{hp(aqx + c)}{ax - b - (1-p)(aqx+c)}$ thlr.

"The aim of this algebraic operation has been to ascertain what may be the influence of greater or lesser production on the price of grain, the land rent being assumed equal to zero. In the formula just given, however, x appears in both the numerator and the denominator. For the purpose of observing the effect on price of increases or decreases in the quantity of grain it will be more convenient to write

$$\text{the price of 1 scheffel} = \frac{hp}{\frac{ax-b}{aqx+c} - (1-p)}$$

"Suppose now aqx + c = z; when z increases x likewise increases. Then $x = \frac{z-c}{aq}$. Putting this new expression for x in the immediately preceding equation we have

$$\frac{hp}{\frac{az - ac - baq}{aqz} - (1-p)} = \frac{hp}{\frac{a - \left(\frac{ac+baq}{z}\right)}{aq} - (1-p)}$$

Now $\frac{ac + baq}{z}$ will always become smaller as z increases; as the negative part of the denominator decreases the whole denominator increases. But as x and z increase together, when x increases the denominator becomes larger while the numerator does not change. It follows that the fraction which expresses the value or the price of rye becomes smaller and smaller as x increases and larger and larger as x decreases; the smaller x becomes, the greater the price of rye tends to be.

"This reasoning confirms the law that the cost of production of grains is higher as the degree of fertility of the soil is lower."[1]

Von Thünen remarks that it would hardly be worth the trouble of proving so obvious a principle by means of such a detailed calculation were it not that he wants to fix "once and for all" the point of view from which he is to undertake later some other researches. Von Thünen must have had his second volume in mind when he wrote this, for the remainder of the first volume consists of a lengthy inductive analysis of the costs of farming under varying conditions of culture at varying distances from the central market.

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1. J. H. vön Thunen, Der isolirte Staat, Hamburg: Perthes, 1826, pp. 34-37.

5

Friedrich Benedikt Wilhelm Hermann (1795-1868)

This is another of the authors on the Jevons list whom Fisher did not think worth retaining. There are two brief passages in which a mathematical notation appears, and these are reported here for the sake of completeness.

The slightly mathematical sections of the work appear in Chapter IV. In considering the determination of the price of a good the author examines the forces which operate on the buyers' side and those which operate on the sellers' side. It is cost which influences supply and costs must be considered solely from the point of view of the enterpriser. To start any business, one must have capital, and this may be subdivided into fixed capital (invested in plant) and money capital, which is used to pay for materials, labor, storage, and the like. It is the function of the enterpriser to combine all capital in the best possible way to produce a product. Into the product goes circulating or fluid capital which includes that part of fixed capital used up in production. In other words, depreciation on fixed capital is a part of the capital which "presents itself" in the final product. The entire value of the fluid capital contained in the product must be included in its price.

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1. Friedrich B. W. Herman, Staatswirthschaftliche Untersuchungen, München, A. Weber'schen, 1832, pp. 80-105.

One further expense enters into price. No matter what particular form the entrepreneur's capital takes during any period of production the owner of the capital must do without its use during that period. Capital which is used "for the good of the buyer" is made available only through some sacrifice on the part of the owner of the capital. Thus the "use of capital" becomes an element of cost. In brief the total cost of a product is equal to the sum of the capital which "has gone over into the product" plus the going rate for the "use" of all capital in the productive combination.

Let A equal the circulating or fluid capital (including depreciation of fixed equipment) which is contained in the product.

Let B equal the fixed capital used during production.

Let $\frac{p}{100}$ equal the rate paid on an average for the use of capital.

The costs are

$$A + (A + B) \cdot \frac{p}{100} .$$

Sometimes savings in costs can be effected by substituting one kind of capital for another in production, and it is possible to show algebraically how this may come about. Costs are equal to $A + (A + B) \cdot \frac{p}{100}$ as indicated above.

Let a be a portion of the circulating capital (e.g., labor cost), an outlay on which will not in the future recur because new fixed capital B' (perhaps a machine) is to be substituted for it. The annual depreciation, b, of the machine must be added to the outlay of the circulating capital, and the new costs are

$$A - a + b + (A-a+b+B+B') \cdot \frac{p}{100} .$$

And these must be smaller than the former costs

$$\text{or} \qquad b + (B'+b) \cdot \frac{p}{100} < a + a \cdot \frac{p}{100}$$

The change, therefore, is advantageous if

$$p < \frac{100\ (a-b)}{B'+b-a}$$

$$\text{or} \qquad b < \frac{a\ (100+p)\ -B'p}{100+p}$$

$$\text{or} \quad a > \frac{b(100+p) + B'p}{100 + p}$$

$$\text{or} \quad B' < \frac{(a-b)\,(100+p)}{p}$$ [1]

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1. *Ibid.*, pp. 87-88.

CHAPTER VI

THE ENGLISH MATHEMATICAL WRITERS

1

Francis Hutcheson (1691-1746)

Francis Hutcheson is mentioned in the Jevons bibliography but was removed from the list by Fisher. In the preface to the second edition of The Theory of Political Economy, Jevons puts Hutcheson in his third category, with those

> "... who, without any parade of mathematical language or method, have nevertheless carefully attempted to reach precision in their treatment of quantitative ideas, and have thus been led to a more or less complete comprehension of the true theory of utility and wealth."[1]

He speaks of his use of mathematical symbols as being "crude and premature" but not absurd and he comments that ". . . Hutcheson's views have not received the attention they deserve."[2]

In An Essay on the Nature and Conduct of the Passions and Affections there is no explicit use of mathematical method and Jevons notes this. There is, however, a good

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1. Jevons, op. cit., p. xxv.
2. Ibid., p. 322.

deal of language expressive of quantity in this work. For example, ". . . the Moment of Good in any Object, is in a Compound Proportion of the Duration and Intenseness" and "The Ratio of the Hazard of acquiring or retaining any Good must be multiplied into the Moment of the Good; so also the Hazard of avoiding any Evil is to be multiplied into the Moment of it, to find its comparative value."[1]

In an earlier work there is an explicit employment of symbols.[2] In the edition which was obtained first (5th) there was nothing of mathematics in the section cited by Jevons. A careful perusal revealed this statement in the preface which may be of some interest.

> "In this . . . edition there are Additions interspersed. . . and some Mathematical Expressions are left out, which, upon Second Thoughts, appear'd useless and were disagreeable to some readers."[3]

In the editions of 1729, however, we find one of the first known attempts at symbolic demonstration.

Certain propositions or axioms are laid down, among which are the following.

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1. Francis Hutcheson, An Essay on the Nature and Conduct of the Passions and Affections, London: 1728, pp. 39-40.
2. Francis Hutcheson, An Inquiry into the Original of Our Ideas of Beauty and Virtue, 3rd ed., corrected, London: 1729, pp. 186-197, pp. 286-289.
3. Francis Hutcheson, An Inquiry Into the Original of Our Ideas of Beauty and Virtue, 5th ed., London: 1753, p. xxii.

"The moral Importance of any Agent, or the Quantity of publick Good produc'd by him, is in a compound Ratio, of his Benevolence and Abilitys; or (by substituting the initial letters for the Words, as M = Moment of Good, and μ = Moment of Evil) M = BA. . . .

"But as the natural Consequences of our Actions are various, some good to ourselves and evil to The Publick; and others evil to ourselves, and good to the Publick; or either useful both to ourselves and others, or pernicious to both; the entire Spring of Good Actions is not always Benevolence alone; or of Evil, Malice Alone; (nay, sedate Malice is rarely found) but in most actions we must look upon Self-love as another Force, sometimes conspiring with Benevolence, and assisting it, when we are excited by our views of private Interest, as well as publick Good, and sometimes opposing Benevolence, when the good Action is anyway difficult or painful in the Performance, or detrimental in its consequences tothe Agent. In the former Case, M = (B+S) x A = BA + SA; and therefore BA = M - SA = M - I,[1] and

$$B = \frac{M - I}{A}.$$

In the latter Case, M = (B-S) x A = BA - SA; therefore, BA = M + SA = M + I, and $B = \frac{M + I}{A}$.[2]

---

1. I = S x A from a preceding paragraph.
2. Ibid., p. 186 and p. 187.

This is scarcely more mathematics than a college sophomore encounters in a course in introductory logic and is certainly far removed from the symbolic logic of today. Such a passage does serve to remind us, however, that the philosopher (e.g., Spinoza) has long made use of algebraic and geometric tools whenever he comes to deal with quantities in some way related.

2

Major General Henry Lloyd (1720-1783)

An Essay on the Theory of Money by Major General Henry Lloyd is the earliest work in English listed in the Fisher bibliography and the earliest, save one, in the Jevons list.[1] Seligman refers to Lloyd's work as the first attempt in English to apply mathematics to economics.[2]

The author proposes to make clear to his readers the different effects which money has upon the "industry, manners, and the different species of government established among mankind." Money or "Universal Merchandise" is a source of great benefit to all peoples of the world. The advantages which societies derive from the use of money cause them ". . . to augment its course; for which reason they introduced the use of banks, public notes, etc."[3] The first eight chapters of the essay (81 pages) are concerned with the salutary effects on an economy of a plentiful monetary medium in general and of a public bank

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1. The earlier works listed by Jevons are the two books by Francis Hutcheson just examined.
2. Seligman, op.cit., Economic Journal, p. 352.
3. Henry Lloyd, An Essay on the Theory of Money, London: 1771, p. ix.

in particular. Lloyd argues that the population and industry of a nation are in "proportion" to the quantity of money in circulation, and that liberty in any country is in proportion to the equality of the circulation, despotism being proportionate to the inequality of the circulation. Luxury, corruption of manners, and national poverty are again in proportion to the inequality of the circulation, whereas the state of the arts and sciences are in "compound ratio of the quantity of circulation and liberty."

Our interest in this essay begins with Chapter IX. "The price of any merchandise whatever, is in an inverse ratio of its quantity, that is, the less there is of any commodity brought to market, the dearer it will be and vice versa.

> "Let us suppose that the whole mass of national productions, whether natural or artificial, be divided into a given number of parts, and likewise that the money and paper currency be also divided into a given number of equal parts, so that one or more of these correspond to one or more of those. The number of the parts of money or paper, which are to be given in exchange for any commodity is what we call the <u>price</u> of it; fromthis definition it follows, that by increasing, or diminishing the quantity of the money and paper currency (which I shall hereafter denominate by the general term <u>circulation</u>) or that of the commodities,

> the price or ratio between them will vary, in proportion to that increase or diminution; consequently, it is impossible to fix it, without prejudicing the buyer or seller.
>
> "Though the price of any commodity is in fact in a compound ratio, direct as the quantity of circulation, and inverse as that of merchandise, yet it may be simplified. . . ."[1]

For example, let c equal the quantity of the circulation and m the quantity of merchandise. If p equals price, or the "proportion between them,"

$$\frac{c}{m} = p.$$

If $\frac{c = 10}{m = 1}$, p = 10, which is the same thing as saying that ten portions of circulation will correspond to each portion of M. If C, which is equal to 10, be multiplied by 10, then C = 100, M = 1, and p = 100, or ten times what it was before. Contrariwise, if C be divided by 10 we have C = 1, M = 1, and p = 1. Let M, on the other hand be multiplied by 10 and C = 10, M = 10, and p = 1, that is, "one part only of c corresponds to M, whose value is diminished in proportion as

1. Ibid., pp. 82-84.

its quantity is increased; consequently, the price of M, will be in an inverse ratio of its quantity, compared with that of circulation; we shall therefore include all the variations in the price of M, in the two following formularies."[1]

Let the proportion between M and C be p, and let the "augmentation or diminution" of M be Y. Then the following equations hold:

$$\frac{C}{M} = p \qquad \frac{C}{M \times Y} = \frac{p}{y} \qquad \frac{C}{\frac{M}{Y}} = p \times Y.$$

Some "learned authors" have said that it is an error to reason that the price of merchandise increases in proportion to the increase in circulation, and in proof of their point they cite the failure of prices in Europe to rise twenty-fold after a twenty-fold increase in circulation following the importation of treasure from America. But this is no argument at all, because these authors have failed to consider the great increase in trade which took place at the same time.

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1. Ibid., p. 86.

The equations developed above are basic to a discussion of international metal flows, of equilibrium under bimetallism, and of the determination of the rate of interest.

Interest is ". . . the price which is given by the borrower for the use of a certain sum for a certain time."[1] It follows from this definition, that the ". . . interest will be in a compound ratio, direct as the number of borrowers; and inverse as the number of lenders; that is, the greater the number of borrowers, compared with that of the lenders, the higher will be the rate of interest, and vice versa. It is plain that the more money is to be lent, and the fewer the borrowers, the lower will be the interest: and the less money to be lent and the more the borrowers, the higher will be the interest: as the quantity of money to be lent will be in proportion to that of the circulation, we say that the rate of interest will be in inverse ratio to the quantity of circulation."[2]

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1. The unsophisticated and accurate treatment of interest as being the payment for the use of borrowed funds is typical of seventeenth and eighteenth century writers. They were perfectly aware, too, of the reasons for business borrowing.
2. Ibid, pp. 127-128.

Let C = the lenders

Let B = the borrowers

Let I = the common interest

Let Y = the variation in the quantity of money to be lent, or circulation.

Then there are three basic equations

$$(1)\ \frac{B}{C} = I$$

$$(2)\ \frac{B}{C \times Y} = \frac{I}{Y}$$

$$(3)\ \frac{B}{\frac{C}{Y}} = I \times Y.$$

General Lloyd goes on to say that "as y is variable" a number of corollaries may be deduced. These are of interest to the monetary theorist, but for the purpose of investigating the author's methodology we have proceeded far enough.

The difficulties which Lloyd encountered in his analysis are, of course, apparent. I have quoted him directly at some length in order to convey the precise terminology which he uses. When he says that the rate of interest is equal to the number of borrowers divided by the number of lenders, he obviously does not mean that at all. He is talking about a relationship between quantities of money, as is evident when he says that the rate of interest "will be in inverse ratio to the quantity of circulation." Even though we read into his symbols his presumed meaning, we must take his rules of strict proportionality literally, for it is quite apparent from context that, say, equation (3) above means exactly what it says. If we make allowance for the fact that C and B refer to quantities of money (not numbers of people), then we must accept the statement that if the quantity of money in circulation be halved, the interest rate will be doubled. Lloyd might have had certain mental reservations regarding the matter of strict proportionality, but they never are made explicit.

3

Samuel Gale (   )

Samuel Gale is mentioned in Seligman's article "On Some Neglected British Economists."[1] After calling attention to the work of Major General Henry Lloyd, Seligman makes the following comments.

> "A more elaborate effort to apply mathematics to economics, which has eluded the attention of all writers on the subject, is the interesting work by the American, Samuel Gale, entitled An Essay on the Nature and Principles of Public Credit. The original work, a substantial volume of two hundred and thirty-four pages, was published anonymously in London in 1784, although the preface dated at Charles-Town, South Carolina, in 1782, was signed by Gale. A second part, entitled Essay II on the Nature and Principles of Public Credit, appeared in the same year (1784) with a preface dated St. Augustine, East Florida, 1783, and a third part, with a long title of about ten lines, appeared in 1786, not published, but 'printed, to save the labour of copying the manuscript.' The books by Gale are by all means the most comprehensive and detailed examples of early mathematico-economic literature, and will repay careful examination."[2]

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1. E. R. A. Seligman, "On Some Neglected British Economists," Economic Journal, 13:352.
2. Idem. Gale may not have been an American. He mentions a long separation from his family in England, and I received the impression that he might well have intended to be in this country only temporarily.

Gale is certainly one of the most distinguished economists of his time. The three essays are confined almost exclusively to matters of public finance, but his treatment of monetary theory is indeed worthwhile. His mathematical analysis, being restricted almost entirely to what we should today call the "mathematics of investment," is not especially important from the point of view of the economic theorist, but it is by all odds the most involved treatment made prior to the beginning of the nineteenth century.

The mathematical sections of Gale's work are found exclusively in the first essay. Since "symbolic" demonstrations recur throughout some 234 pages, I can give only examples of his treatment, but they will be more than sufficient to convey the idea of the author's method.

He begins by laying down the

> ". . . primary or simple properties of those kinds of debts, the consequent properties or effects of which are intended to be investigated; together, also, with the primary principles or axioms, whereby those consequent properties or effects are intended to be deduced.
>
> "The debts, then, intended for the present subject of inquiry, are such (by way of definition) that the CAPITAL or PRINCIPAL be not demanded by the creditor; but, that his demand

> (the which he shall also be at liberty to assign, or transfer, at any time, to any other person) shall consist of a certain quantity annuity or periodical payment; either,
>
> 1st, To continue in perpetuity; or, until it shall afterwards cease, in consequence of a valuable consideration to be in future agreed on: --or,
>
> 2ndly, To continue a certain definite term of time, and then to cease: --or,
>
> 3rdly, To continue (indefinitely as to time) until the capital or principal be repaid."[1]

The three types of annuities may be termed, in order, perpetual annuities, determinate annuities, and redeemable annuities or annuity stocks. The first two may be redeemed (or repurchased) by a future agreement with the annuitants; but in such an eventuality the annuitant is not restricted as to the price which he may receive, whereas the owner of an annuity stock cannot receive more than the nominal or face value of the stock.

Now "in all matters of a mathematical nature, it is indispensably necessary, that certain self-evident FIRST PRINCIPLES should be admitted, by way of axioms (or foundations as it were) from whence to proceed . . . ."[2] Gale therefore postulates the following:

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1. Samuel Gale, An Essay on the Nature and Principles of Public Credit, London: 1784, pp. 12-13.
2. Ibid., p. 14.

(1) A time of peace will always be followed by a time of war and a time of war by a time of peace.

(2) The expenses of a nation will be greater in a time of war than in a time of peace.

(3) There shall be a standard money in the economy in terms of which all values shall be expressed.

(4) The relative value of all commodities in the economy shall be determined by ". . . the DEMAND that there shall be for such commodities respectively, and the ease or difficulty with which such demand may be supplied."

(5) There is a quality in money which makes it productive of a rent, or interest. Since the demand for money is greater during a war time than during a peace, it is postulated that the interest rate will be higher in time of war than in time of peace.

"These are the only principles that are required as data, and it is presumed that they are all so perfectly obvious as to be admitted without the remotest hesitation."[1]

Gale is now ready to prove certain propositions. It is the first one in which we shall be primarily interested. Two alternatives are open to a people confronted with the extraordinary expenses of prosecuting a war. They can raise the necessary funds by taxation "or other contributions" while the war is going on, or they can borrow by "establishing annuity stocks redeemable at pleasure." These stocks would bear the rate of interest current during

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1. *Ibid.*, pp. 14-15.

the war, but when the war is over the debt will be refunded at the peacetime rate of interest which (it is postulated) is always lower than the rate prevailing during war-time. If the latter method of financing the war be chosen, the people will be required to pay into the treasury a semi-annual "war-revenue" which would be received over a period including both the time of war and the time of peace. Gale will show that under either method of financing the "damages or burthens" to the people are equal. And if the monetary load be the same in either case, then it can be shown that a public debt is advantageous in a number of ways. Two of these advantages are described at the end of the passage to be quoted presently.

We may now pass to a detailed consideration of Gale's proof of the above proposition. I originally contemplated paraphrasing this section, but because of its complexity I have had it photographed for the convenience of the reader. Little or no difficulty will be encountered in reading the material in the text proper, which closely parallels the "symbolic" demonstration.

The algebra is not at all hard if two formulas are recalled from elementary algebra. The sum of a geometric progression containing "n" terms, the first term of which is "a" and the ratio of which is "r", may be written

$$S = \frac{a - ar^n}{1 - r}$$

The present value "A" of an annuity payment "R" made "n" years hence is

$$A = \frac{R}{(1 + i)^n}$$

SECT. III.

Of the damages or burthens ſuſtained by the members of a ſtate in ſupporting the expences of a war:—That incurring a Public Debt is not only the moſt convenient method of raiſing the money, but (if properly conducted) muſt alſo be productive of an actual ſaving.

THE firſt reflection that preſents itſelf to the mind from a conſideration of the beforementioned poſtulata (in page 14 and 15) is this,—that although the monies for the expences of a war, ſhould be *actually* raiſed by taxes or contributions during the time of its continuance; yet nevertheleſs, the actual *damages* or *burthens* ſuſtained by the members of the ſtate, on account of their contributions to thoſe expences, will be, *not only* the monies by them ſo paid or contributed, but alſo the intereſt thereon; becauſe, they would (at any given period of time) have been by ſo much the RICHER, if ſuch payments or contributions had not been required of them, and they had purſued the ſame induſtry and œconomy.

* For example,—Let it be ſuppoſed that out of a term of twenty years, a nation ſhall have a war during

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SYMBOLIC DEMONSTRATIONS.

* 1ſt.—If during any time, or number of half years T, in which the rate of the increaſe of money by intereſt be R, a nation ſhall carry on a war at any half-yearly expence S, which ſaid ſum S ſhall be paid into the treaſury at the beginning

during the first eight years, at an expence of *ten millions* per year, or rather *five millions* per half year; which sum (let it again be supposed) the service shall require to be paid into the treasury at the beginning of each half year:—Let it also be supposed, that the rate of interest for money in such nation (taking it on the average) during the time of war, be *five per cent. per annum*; and, (seeing from the 5th postulatum that the rate of interest will naturally be higher in time of war than in time of peace) let it be supposed also, for example sake, that in time of peace it be *three and an half per cent. per annum*; and that the interest be payable *half-yearly*.

Now the sum to be expended in this case will be £.80,000,000; but the amount thereof, together with the damages or interest thereon, will, at the end of the said *eight* years of war be £.99,324,050; which, together with the interest thereon during the remaining *twelve* years of the said twenty, will amount to £.150,619,000; which

ginning of each half year,—Then, it is evident that the expences, with the interest or damages thereon, will, at the end of each respective half year, be as follows, viz.

Half years - - 1, 2, 3, 4, &c. to T
Expences or damages RS, R^2S, R^3S, R^4S, &c. to R^TS
the sum of which is $RS+R^2S+R^3S+R^4S+$&c. to $+R^TS$
Which (per progressionis naturam) is equal to - - - - - - - - - - - $\frac{R^{T+1}-R}{R-1}\times S$

2d.—And the said expences, with the interest or damages thereon, at the end of any farther time t (of peace), in which the rate of increase by interest be r, will of course be - - - - - - - - - - $\frac{R^{T+1}-R}{R-1}\times r^{t}S$

which of course will be the quantity of the damages or burthens sustained at the end of the twenty years, in case the whole of the expences should be so paid by taxes or contributions during the continuance of the war: because (as said before) the members of the state will (at the end of the said term) be so much *poorer* than they would have been, if they had pursued the same industry and œconomy, and those taxes or contributions had not been required of them.

I must request that the reader would not be discomposed at the *apparent* extravagance of the abovementioned amount; because, on reflection, he will find, that in case a state shall consist of *ten millions* of people, this immense burthen will be less than *ten shillings and three pence a year* for each, both for the payment of the interest and the principal.

It could not indeed be pretended, that if the war did not happen, or if the expences thereof should be otherwise provided, the whole of this £.5,000,000 per half year would be actually lent on interest by its proprietors, or otherwise accumulated into a CAPITAL:—It is however perfectly evident, from that natural desire which is implanted in the breasts of men for bettering their condition, that a considerable part thereof would actually be accumulated into a capital:—A part of it would, it is perfectly clear, be *actually* lent on interest:—Another part would be employed by its proprietor in the still more profitable road of commerce and industry:—so far the capitals of the members of the state would evidently be increased:—And, although a remaining part would be expended or *consumed* by its owners, in the enjoyment of a more comfortable way of living; yet,

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yet, ſuch is the natural and unalterable connection between *cauſes* and *effects*, that even this part would not be unworthy of the conſideration of its rent or intereſt.—It is evident, on the ſlighteſt reflection, that this laſt-mentioned part will pertain chiefly to the lower orders of the ſtate; and theſe, by being deprived of it, will naturally cauſe an increaſe in the number of impotent poor, whereby the *private* taxes for their ſupport muſt neceſſarily be increaſed; ſo that the burthens of the other members will be increaſed, and the accumulation of the capitals of the members of the ſtate at large will conſequently be checked or diminiſhed, by ſo much the more, as the lower orders of the ſtate ſhall be unable to ſupport their proportionate parts.

The ſtate, conſidered as a body politic, muſt evidently be ſo much the poorer alſo, on account of the war; becauſe, the capital, the wealth, or the reſources, of a ſtate at large, can certainly be no other than the capital, the wealth, or the reſources, of the ſeveral members *conjointly* that compoſe ſuch ſtate.—But the view in which the loſſes ſuſtained by the ſtate, conſidered as a body politic, will perhaps appear moſt ſtriking, is, the *loſs* of the induſtry of thoſe members who are carried forth to war; the entire loſs of all thoſe members who fall by the ſword; and the *conſequent loſs* of that progeny which in the natural courſe and order of things would have proceeded from them; by whom the AGRICULTURE, the MANUFACTURES, and the COMMERCE of the ſtate, would be extended, if the direful ſcourge of war did not unhappily intervene.

Each of theſe reflections equally ſupport the principle of extending the conſideration, not only

to

to the actual *expences* of a war, but alſo, to the rent or intereſt thereof: and the more theſe reflections ſhall be contemplated, the more will that principle be ſupported by them.—It is not neceſſary however to dwell farther on the reaſonableneſs of it; becauſe, it is preſumed that no one (*at leaſt* no one in a commercial country) will heſitate a moment at the admiſſion of the *poſtulata* in page 14 and 15; and theſe being admitted, the conſideration of the *intereſt* as well as the actual *expences* muſt neceſſarily follow.

Hence then, if inſtead of raiſing the whole of the money by taxes during the continuance of the war, the ſame money ſhould be raiſed by incurring a public debt for any part thereof, ſuch method of raiſing the money will be attended with *advantage* or *diſadvantage* compared with the former method, (excluſive of matters of conveniency) according as the amount of *ſuch expences*, with the *intereſt* thereon, at the end of the ſame term of time, computed at the ſame rates of intereſt, ſhall be *leſſer* or *greater* than the amount beforementioned.

In order to determine whether the raiſing of the money by incurring a public debt for a part thereof, would be *advantageous* or *diſadvantageous*, compared with the former method of raiſing the whole by taxes during the continuance of the war, if we aſcertain the periodical quantity of a ſtanding revenue to be raiſed during the whole of the aforeſaid term of twenty years, which, in its damages or burthens to be ſuſtained by the members of the ſtate computed according to the abovementioned rates of intereſt, ſhall be equal to the burthens or damages beforementioned; then, ſuch method of raiſing the money will of courſe be

advantageous

advantageous or *disadvantageous*, compared with the former method, according as ſuch ſtanding revenue ſhall be more or leſs than ſufficient *actually* to defray the beforementioned expences and the intereſt thereon.

† Now, if a ſtanding *war-revenue* ſhould be eſtabliſhed (payable half-yearly) during the whole of the aforeſaid term of twenty years, which, together with the intereſt thereon during the ſaid term, computed at the ſame rates as before, ſhall be equal to the beforementioned damages or burthens (viz. £.150,619,000—ſee page 24) ſuch ſtanding

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SYMBOLIC DEMONSTRATIONS *continued from p. 24.*

† 3d.—Now, if inſtead of raiſing the whole expences S per half year by taxes during the aforeſaid time T, a certain revenue or half-yearly ſum s ſhould be raiſed during the whole time $T+t$; then, the damages or burthens ſuſtained by the members of the ſtate during the part T (of the time $T+t$) in which the rate of increaſe by intereſt be R, will be as follows, half-yearly, viz.

Half years - - 1, 2, 3, 4, &c. to T

Damages or burthens s, Rs, R^2s, R^3s, &c. to $R^{T-1}\times s$

the ſum of which is $s+Rs+R^2s+R^3s+$&c. to $+R^{T-1}\times s$

Which (per progreſſionis naturam) is equal to $\frac{R^T-1}{R-1}\times s$

4th.—Which ſaid damages or burthens, at the end of the farther part t of the ſaid time $T+t$, in which the rate of increaſe by intereſt be r, will become - $\frac{R^T-1}{R-1}\times r^t s$

5th.—And the damages or burthens ſuſtained by the members of the ſtate from the continuance of the ſaid revenue s during the ſaid farther part t of the time $T+t$ will (by the ſame progreſſional ſeries as N° 3) be $\frac{r^t-1}{r-1}\times s$

6th.

ſtanding war-revenue will be the half-yearly ſum of £.2,557,200.

In this caſe it is evident that a debt muſt be incurred during the war:—And ſeeing that this revenue comes into the treaſury at the *end* of each half year, whereas it is neceſſary that the money for the ſervice of the war ſhould (according to the former ſuppoſition) be lodged in the treaſury at the *beginning* of each half year, the debt muſt conſequently grow from the beginning, by an half-yearly quantity equal to the difference between the

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6th.—Wherefore the damages or burthens ſuſtained from the ſaid revenue s during the whole time $T+t$ will be - - - - - - - - $\overline{\frac{R^T-1}{R-1}\times r^t s+\frac{r^t-1}{r-1}\times s}$

OR (which is all the ſame) - - $\overline{\frac{R^T-1}{R-1}\times r^t+\frac{r^t-1}{r-1}}\times s$

7th.—Hence then, it being required that the revenue s ſhall be ſuch, as that the damages or burthens thereof during the whole time $T+t$, ſhall be equal to the damages or burthens ſuſtained during the ſame time $T+t$ by raiſing the half-yearly ſum S during the time T, we ſhall have (per N° 2. p. 24) - - - - $\overline{\frac{R^T-1}{R-1}\times r^t+\frac{r^t-1}{r-1}}\times s=\frac{R^{T+1}-R}{R-1}\times r^t S$

8th.—And hence - - - - $s=\dfrac{\frac{R^{T+1}-R}{R-1}\times r^t S}{\frac{R^T-1}{R-1}\times r^t+\frac{r^t-1}{r-1}}$

9th.—And, ſeeing that the revenue s does not come into the treaſury till the end of the half year, whereas it is neceſſary (according to N° 1, page 23) that S ſhould be brought into the treaſury at the beginning of the half year, the half-yearly expences of the war, and the present worth of the half-yearly war-revenue.

If therefore, from the half-yearly expences of the war - - - -	£. 5,000,000
we subtract the present worth of (£.2,557,200) the revenue payable at the end of the half-year, viz. -	2,494,830
The remainder will be the half-yearly quantity by which (including the interest thereon) the debt will grow during the war - - - -	2,505,170

The amount of which at the end of the eight years of war, will be the quantity of debt incurred, viz. £. 49,764,700.

OR

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year, a debt (which let be called D) must consequently be incurred, growing from the beginning, by the half yearly quantity $S-\frac{s}{R}=\frac{\overline{SR-s}}{R}$; wherefore, the debt incurred at the end of each respective half year, will be as follows; viz.

Half years - 1, 2, 3, &c. to T

Increase of the debt $\overline{SR-s}$, $\overline{SR-s}\times R$, $\overline{SR-s}\times R^2$, &c. to $\overline{SR-s}\times R^{T-1}$

The sum of which at the end of the time T, (by the same series as N° 3, in p. 28) will be $D=\frac{R^T-1}{R-1}\times\overline{SR-s}$

10th.—OR (by reduction) - $D=\frac{R^{T+1}-R}{R-1}\times\overline{S-\frac{s}{R}}$

11th.—OR (by reduction again, which also agrees with the difference between N° 1, in page 24, and N° 3, in page 28, and thereby forms a proof or check on the work, viz.) - - - - $D=\frac{R^{T+1}-R}{R-1}\times S-\frac{R^T-1}{R-1}\times s$

5 12th.

OR otherwise—Seeing that the whole amount of the expences, with the interest thereon computed to the end of the war (in page 24) is - -	£.99,324,050
Whereas, the standing *war-revenue* (viz. £.2,557,200 per half year) with the interest thereon, computed to the end of the war, will amount only to - - - - - - - - -	49,559,350
Their difference will of course be the quantity of debt outstanding at the end of the war, viz. - - -	£.49,764,700

* The war being now ended, and the rate of interest for money falling (according to the foregoing admission) to $3\frac{1}{2}$ per cent. per annum; this debt may be converted into redeemable annuity-stocks bearing the said rate of $3\frac{1}{2}$ per cent. because, if any of the former lenders should refuse to comply with the decrease that shall naturally take place in the rate of interest when the extraordinary demand for money ceases, other lenders will

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12th.—* The war being ended, and the rate of increase by interest falling, according to the natural course and order of things, to any lower rate r, and the said debt D being accordingly converted into a redeemable stock of annuities bearing the said rate r, the half-yearly interest thereon will become - - - - - - $D\times\overline{r-1}$

13th.—And the remaining part of the half-yearly revenue s will then become an half-yearly *sinking fund*; whereby the said Debt D (and of course the interest thereon) will be periodically reduced during the time t of peace; viz. - - - - - - - - - - $\overline{s-D\times\overline{r-1}}$

14th.—Which (for contraction sake) let be put equal to - - - - - - - - - - - - - - - - s

15th.

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will naturally be found, whereby the former ones may be paid off.

Hence then, the half-yearly interest of the debt will become - -	£. 870,880
Which, being subtracted from the half-yearly standing war-revenue -	2,557,200
The remainder will become an half-yearly *sinking fund*, by which the debt (and of course the interest thereon) will be reduced; viz. -	£. 1,686,320

* Now this sinking fund will, at the end of the aforesaid

15th.—* Now the periodical quantity by which this sinking fund s will redeem the debt D during the time *t* of peace, will be half-yearly, as follows; viz.

Half years - 1, 2, 3, 4, &c. to *t*

Redemptions - s, *r*s, r^2s, r^3s, &c. to $r^{t-1}\times s$

The sum of which (by the same progressional series as N° 3, in page 28) will be - - - - - - - $\frac{r^t-1}{r-1}\times s$

16th.—Hence, the debt at the end of the said time *t* of peace will be - - - - - - - - $D-\frac{r^t-1}{r-1}\times s$

Which (per N° 13 and 14, in page 31) is - - - - $=D-\frac{r^t-1}{r-1}\times s-D\times\overline{r-1}$

Which (per N° 9, in page 30) is

$$=\frac{R^T-1}{R-1}\times\overline{SR-s}-\frac{r^t-1}{r-1}\times s-\frac{R^T-1}{R-1}\times\overline{SR-s}\times\overline{r-1}$$

Which (by reduction) is - - - $=\frac{R^{T+1}-R}{R-1}\times r^tS-\frac{R^T-1}{R-1}\times r^t+\frac{r^t-1}{r-1}\times s$

Which (per N° 7, in page 29) is $=$ O.

Wherefore, in this case, there can be neither profit nor loss on account of the debt.

aforesaid twelve years (or rather twenty-four half years) of peace, amount to £. 49,764,700 the same as the debt.—Wherefore, if this sinking fund be actually applied to the redemption of the debt, the debt will be entirely paid off; and consequently, the members of the state will be neither richer nor poorer, than they would have been, had the whole expences of the war been raised by taxes during the time of its continuance; admitting the same industry and œconomy to prevail in the *one* case as in the *other*.

It will hereafter be shewn (in the sixth section) that the burthens or damages sustained by the members of the state, in consequence of taxes, (admitting them always to be judiciously laid and applied) will be much mitigated by the effects of the additional vigour, which the proper application of them will give to the circulation of money; by which (as will hereafter appear) the rate of interest will be in a great measure governed. But, as the rate of interest will (according to the fifth postulatum, page 15) naturally be higher in time of war than in time of peace; and as we are at present *only* comparing the damages or burthens that result from the *one* and from the *other* method of raising the money, on a supposition that the rate of interest be pre-given; the effects of the CIRCULATION do not immediately appertain to our present comparison; and therefore need not at present be brought under consideration.—It must however be perfectly obvious, on the least serious reflection, that the more equal the taxes shall be in *war* and in *peace*, the more regular, uniform, and steady, will be the circulation of money; and from thence, the value of other property will be the more regular, uniform, and steady also; which is certainly a very

D

very great argument in favour of a public debt, even if there were no other arguments in its favour: but there are ſtill many other arguments in favour of a public debt.

*It muſt be obſerved, in the foregoing caſe, that the whole of the beforementioned ſtanding *war-revenue*, viz. £.2,557,200 will be on hand at the end of the twenty-fourth half year of peace; a part of which, viz. £.2,513,220 would clear off the debt; and the other part, viz. £.43,980 pays the

17th.—*But, it muſt be obſerved, that (per definition, page 13) the capital not being demandable, the laſt half-year's produce of the ſinking-fund, viz. $r^{t-1}\times s$ may be applied to the ſervice of the new war, and the intereſt to be paid thereon, will, in ſuch caſe, be only according to the rate r, as in time of peace; viz. the half-yearly ſum of - - - - - - - $r^{t-1}\times s\times\overline{r-1}$

18th.—Whereas, if there had been no debt, this ſum muſt have been attended with the increaſed rate R, occaſioned by the extraordinary demand for money, which would be the half-yearly ſum of - - $r^{t-1}\times s\times\overline{R-1}$

19th.—Wherefore, the debt will be productive of an actual half-yearly ſaving of - - - $r^{t-1}\times s\times\overline{R-r}$

20th.—The preſent worth of which at the beginning of the half year, (at which time the money S for the ſervice of the war, is pre-ſuppoſed to be required in the treaſury, per N° 1, page 24) is - - $\frac{r^{t-1}\times s\times\overline{R-r}}{R}$

21ſt.—Hence then, the ſelf-ſame revenues which will be equal *only* to the burthens ſuſtained by ſupporting a war for any time T out of $T+t$, at any expence S per half year, where there is no public debt; will be ſufficient, where there is a public debt, to ſupport a war for the ſelf-ſame time, at an half-yearly expence of - - - - - - - $S+\frac{r^{t-1}\times s\times\overline{R-r}}{R}$

the half-year's intereſt thereof:—It muſt alſo be obſerved, that as the CAPITAL of *annuity-ſtocks* (per definitions in page 13) is not demandable by the creditors, government can retain the aforeſaid ſum of £.2,513,220, and apply it to the ſervice of the new war, now (by the ſuppoſition) commencing; in which caſe, the intereſt to be paid thereon will be only at the rate of $3\frac{1}{2}$ per cent. per annum, viz. the beforementioned half-yearly ſum of - - - - - - - £.43,980

Whereas, if there had been no debt, it muſt have been attended with the increaſed rate of intereſt occaſioned by the extraordinary demand for money, which (admitting it as before to be 5 per cent. per annum) would be the half-yearly ſum of - - - - - 62,830

Wherefore, the debt will be productive of an actual half-yearly ſaving of - £.18,850

The preſent worth of which at the beginning of the half year (at which time the money is preſuppoſed to be required to be lodged in the treaſury), is £.18,396.

Hence then, the ſelf-ſame revenue (or burthen) that muſt be ſuſtained by the members of the ſtate, in ſupporting a war of the beforementioned time, at an expence of £.5,000,000 per half year, or £.10,000,000 per year, where there is no public debt; will ſupport a war for the ſame time, at an expence of £.5,018,390 per half year, or £.10,036,780 per year, where there is a public debt.

A further comment may be made regarding Gale's work. His monetary theory, though developed without the use of algebraic symbols, indicates a clear idea of the functional relationships existing between the various quantities which affect the value of money. He distinguishes between that part of the circulation which is available to be lent and that part which is not, and he shows how the two parts vary relative to each other as the interest rate varies. He explains the concept of monetary velocity, and is the first writer whom I have encountered to use that precise term.

Oddly, though, he never extends the use of his algebraic analysis. "The comparative value of money, will not be *inversely*, in a *simple ratio*, as the quantity of circulating money in any nation; but, as the PRODUCT of the multiplication of the *quantity*, by the *force* or *velocity* of the *circulation*, inversely."[1] Momentarily we expect such a statement to be "translated," but Gale's conception of the proper use of mathematical methodology does not include this extension.

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1. *Ibid.*, p. 144.

4

E. R.

In his article "On Some Neglected British Economists," E. R. A. Seligman cites this anonymous writer as the man who first took exception to the Ricardian labor theory of value. He also says of him that "It is worthy of note that he . . . gives a mathematical treatment of the incidence of taxation on land--with one exception the earliest attempt made in England to apply mathematics to economics."[1] A serious attempt has been made, with no success, to discover who E. R. may have been.

This Essay on some General Principles of Political Economy, on Taxes upon Raw Produce, and on Commutation of Tithes should be of interest to the general student of the history of economic thought. The views expressed are so enlightened relative to the ones prevalent in 1822 that there is a temptation to set down his thoughts too fully. Since the mathematical sections are extensive, only those remarks absolutely essential to the argument are reported.

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1. Seligman, op.cit., p. 352. The earlier attempt to which Seligman refers is that of Major General Henry Lloyd. As Seligman furthur points out, neither Jevons nor Fisher has included this writer in his bibliography.

In his preface, the author says that he has attempted ". . . to discover how far that style of reasoning, which has hitherto been confined to the stricter sciences, may be safely applied to some of the principles and some of the conclusions of Political Economy."[1] This "style of reasoning" does not appear during the first half of the essay, which is devoted largely to a refutation of Ricardian value theory. Notwithstanding the high authority of Ricardo, E. R. thinks

> ". . . that the real and permanent exchangeable value of a commodity is not dependent immediately either on the quantity of labour which is condensed in it, or on the quantity of labour which is condensed in the object given for it. Exchangeable value of a commodity depends upon the relation which the demand for it bears to the quantity in which it can be supplied. Commodities are not sold because they are produced, but because they are demanded. . . . The quantity of labour only affects the value as it affects the supply."[2]

The term demand

> ". . . may be employed either as a measure of the quantity of produce which would be purchased, if the price remained the same, or of the whole sum which persons are willing to expend in the purchase of that produce rather than diminish their consumption, if

1. E. R., An Essay on Some General Principles of Political Economy, on Taxes upon Raw Produce, and on Commutation of Tithes, London: 1822, p. v.
2. Ibid., pp. 7-8.

> the price rises, or with an actual increase of consumption, if the price falls. Mr. Say lays it down as a rule, that 'the value of every commodity rises always in a direct ratio to the demand, and in an inverse ratio to the supply.' This, however, is not correct, if the word demand be employed in the former of its two senses; for though the price may depend upon the demand, it will not be proportional to the demand . . . ."[1]

But if the word demand is used in the latter of the two senses indicated above, the Mr. Say's principle is correct. If, that is, ". . . we employ the term demand to signify the sum actually expended in the purchase of any commodity, or the quantity of exchangeable commodities really given for it, we may say correctly that when the supply is constant, the price varies as the demand. . . when the demand and supply both vary, the price will vary as the demand directly, and the supply inversely."[2]

Passing over the interesting argument on wages and on the incidence of taxes upon raw produce, we proceed to the section which contains the mathematical analysis of the effects of a commutation of tithes.

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1. Ibid., p. 12. The italics are mine.
2. Ibid., pp. 18-19. The italics are mine. I have felt it necessary to quote the above passages at some length because a familiarity with the author's concept of demand is necessary to an understanding of the argument which follows.

The Church of England clergy at that time was supported by a levy of one-tenth on the produce of land.

> "The payment of tithe is grating and vexatious to the farmer; it is a cause of frequent quarrels between the parishioners and their rector, who generally is obliged to sacrifice his income to his desire of being respected, by consenting to receive far less than he might justly claim."[1]

And the farmers, too, are likely to be prejudiced against the doctrine of one who ". . . is the consumer of one-tenth of the hard earned reward of their labour."[2] Further arguments against tithes may be advanced on moral and political grounds, and the author feels that if the expediency of their commutation be proved, the question then arises as to who ought to bear the expense of commutation. Should the proprietor of land pay it, or should the consumer, or should both? He disagrees with the judgment of Ricardo that the consumer pays the tithe and therefore ought to bear the charges of commutation. Using an abstract method, he proposes to show in what proportion the landowner might benefit by commutation.

"Suppose, then, the whole titheable soil of Great Britain to consist of classes of such degrees of fertility that each, with the same labour, would produce one twenty-fifth

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1. <u>Ibid.</u>, p. 43.
2. <u>Idem.</u>

less than the class immediately above it. Suppose that exactly the same quantity of each class is appropriated to the growth of corn, and that the state of agriculture is such that six of these classes are so cultivated. Also let there be the same number (say N) acres of each class, and let p be the produce per acre of the richest of these classes. Then the whole quantity produced (Q) will be represented by the following equation.

$$Q = Np\left\{1 + \left(1 - \frac{1}{25}\right) + \left(1 - \frac{2}{25}\right) + \left(1 - \frac{3}{25}\right) + \left(1 - \frac{4}{25}\right) + \left(1 - \frac{5}{25}\right)\right\}$$

$$= Np\left(6 - \frac{1}{25}(1 + 2 + 3 + 4 + 5)\right) = Np \cdot \frac{135}{25}$$ "[1]

The tithe being removed, poorer land is cultivated until the poorest land in the next year will bear nine-tenths $Np(1 - \frac{5}{25})$.[2]

This equals $\frac{18}{25}$ Np, and therefore the poorest grade of land in the year after commutation will produce $Np\left\{1 - \frac{7}{25}\right\}$;

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1. *Ibid.*, p. 57.
2. Upon the removal of the tithe, assuming that costs remain the same, the producers get the full amount of their produce. They will then proceed to cultivate poorer land until the yield at the margin is the same as before, viz., $\frac{9}{10} Np(1 - \frac{5}{25})$.

Consequently two additional grades of land will be added. If Q_1 be the quantity produced in the next year,

$$Q_1 = Q + Np\left\{\left(1-\frac{6}{25}\right) + \left(1-\frac{7}{25}\right)\right\} = Np\,\frac{172}{25}$$

As a result of the increase of supply, the demand (that is, the total sum expended) is affected. Eaters consume more food, but they do not spend as much for it.[1] In this sense demand diminishes. For the sake of making a calculation, the author assumes ". . . what will not be considered as an improbable occurrence, that the diminution is such that the new demand is to the former demand as 344 to 405."[2] Such price varies directly with demand and inversely with supply

$$\frac{\text{the price after commutation}}{\text{the price before commutation}} = \frac{\frac{344}{172}}{\frac{405}{135}} = \frac{2}{3}$$

Since the price is now two-thirds what it was ". . . the proprietors of the first, second, third, fourth, fifth, and sixth classes of land will gain two-thirds of the former value

1. The author, like many of his contemporaries, had a good practical idea of elasticity of demand, especially as it may be reflected in total receipts.
2. *Ibid.*, p. 58.

of the tithe of these lands respectively."[1] He then concludes that

> ". . . these proprietors, the first year, at least, after the abolition, ought to contribute towards a commutation two-thirds of the value of their respective tithes, leaving the consumers to pay the remainder. By this arrangement, the consumers will be considerable gainers; for they save one-third the price of the whole titheable produce, and they are required to contribut only one-third of the price of one-tenth of that produce, that is, one one-thirtieth of the whole. Their gain, therefore, is to their contribution in the ratio of ten to one. The proprietors also, inasmuch as they are likewise consumers, will participate in this benefit."[2]

Admitting that the above results may not be accurate if applied to actual conditions in England, it has nevertheless been shown that it is possible to arrive at an "accurate and arith metical" result if proper data are given. Now the author proceeds to a general formula by which such calculations can always be made.

Let C = the quantity of titheable produce in general, not subject to tithe.

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1. Ibid., p. 59.
2. If the consumer saves one-third the price of the titheable produce and yet contributes only one-third the price of one-tenth of the produce, would not the landlord, while retaining two-thirds of the remitted tithe, lose a great deal in total revenue by a reduction in price of 1/3, if we assume (with E. R.) elasticity of demand to be less than one? The argument is apparently faulty at this point, but we are interested primarily in the methodology.

Let Q = the amount of titheable produce grown in the year of commutation. $Q_1, Q_2, Q_3, \ldots Q_r$ are amounts grown in years subsequent to commutation.

Let $D, D_1, D_2, \ldots D_r$ and $P, P_1, P_2, \ldots P_r$ be respectively the demands and the prices corresponding to $Q, Q_1, Q_2, \ldots Q_r$.

Let the difference in the fertility of the several classes of soil be such that each class, with the same labour, produces 1/nth less than the class immediately above it.

Let $N_1, N_2, \ldots N_m$ be the number of acres of each class of titheable land used to grow titheable produce.

Let p equal the produce of the best grade of land and let mth grade of land be the poorest in cultivation in the year of commutation.

Since price $\propto \frac{\text{demand}}{\text{supply}}$,

$$\frac{P_r}{P} = \frac{\frac{D_r}{Q_r}}{\frac{D}{Q}}$$

or

$$P_r = P \cdot \frac{D_r Q}{D Q_r}$$

$$\text{if } (P=1),\ P_r = \frac{D_r Q}{DQ_r}$$

Also

$$Q = C + p \left\{ N_1 + N_2 \left(1 - \frac{1}{n}\right) + N_3 \left(1 - \frac{2}{n}\right) + \ldots + N_m \left(1 - \frac{m-1}{n}\right) \right\}$$

After tithes are commuted

$$Q_1 = Q + P \left\{ N_{m+1} \left(1 - \frac{m}{n}\right) + N_{m+2} \left(1 - \frac{m+1}{n}\right) + \ldots + N_{m+v} \left(1 - \frac{m+v-1}{n}\right) \right\}$$

where v will be determined by the equation

$$1 - \frac{m+v-1}{n} = \frac{9}{10}\left(1 - \frac{m-1}{n}\right)$$

or

$$v = \frac{1}{10}(n-m+1)$$ [1]

Let G_1 be the gain of the landlord (of, say, the mth class) whose contribution to commutation we wish to ascertain, and let P_1 be the price at which the grain will sell. Then the additional money rent of this proprietor is

$$P_1 \cdot G_1 = \frac{D_1 Q}{DQ_1} \cdot G_1$$

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1. See footnote 2, p. 173 of this section.

If the gain of this proprietor were to continue the same in all succeeding years, it would be easy indeed to compute the amount of his contribution; he would simply give the present value of an annuity equal to this gain. But D_1 and P_1, the new demand and price, are insufficient to make profitable the continued cultivation of land which produces only 1/10 the crop of the mth class, land will be thrown out of cultivation, and ". . . the poorest soil now cultivated will be such the produce of which, at the price P_1, will sell for the same sum as nine-tenths the produce of an equal quantity of the mth class sold for at the price P. Now the quantity to produce a certain sum $\propto \frac{1}{\text{price}}$,

$$\therefore \frac{\frac{9}{10} \cdot P \cdot \left(1 - \frac{m-1}{n}\right)}{\text{equivalent quantity per acre}} = \frac{P_1}{P}$$

$$\therefore \text{the equivalent quantity} = \frac{9}{10} \cdot \frac{P}{P_1} \cdot P\left(1 - \frac{m-1}{n}\right)$$

Hence, the landlord's gain in produce will be

$$N \cdot P \cdot \left(1 - \frac{m-1}{n}\right) \cdot \left\{1 - \frac{9}{10} \cdot \frac{P}{P_1}\right\}, \text{ say } G_2.$$

The price at which G_2 will be sold, $P_2 = \frac{D_2Q}{DQ_2}$

The value of this gain will be $P_2 \cdot G_2 = \frac{D_2Q}{DQ_2} \cdot G_2$

And, generally, in the rth year the value of the landlord's gain will be

$$\frac{D_rQ}{DQ_r} \cdot G_r$$

Therefore, the landlord's whole gain, estimated in money, will be the present value of the series,

$$\frac{D_1Q}{DQ_1} \cdot G_1 + \frac{D_2Q}{DQ_2} \cdot G_2 + \frac{D_3Q}{DQ_3} \cdot G_3 \ldots + \frac{D_rQ}{DQ_r} \cdot G_r$$

Now in order to compute the additional rent which the landlord would receive upon the abolition of tithes, we have simply to know what the N's, n's, m's, and D's may be. The N's, n's, and m's ". . . may be found from a survey of the different soils of which Great Britain consists, and of the quantity of each kind which is or may be cultivated. The connexion of demand with produce, or the value of the quantities D_1, D_2, etc., may be ascertained by examining tables of the produce or consumption of many past years, and comparing these with the corresponding population and prices, allowing for the change in the value of money; for we may presume that the value of D will, in future, be proportional to its average value for the past. Thus, I think, it has been proved, that, in all possible cases, the landlord's contribution is a function of quantities whose value may be ascertained before the system of commutation is carried into effect."[1]

E. R. goes on to say that he has avoided suggesting practical plans for commutation, although he is led to believe that his speculation might serve as a guide when it comes to practical legislation. The results of work of this sort may seldom have too definite a form, but he feels that perhaps "an artificial science might be formed, which should promote the practical discovery of political truth. We might reason strictly from definitions, and upon hypotheses framed, as nearly as our knowledge will permit, like what we have reason to believe is the actual state of things. We might substitute proportions for dependencies, and might obtain

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1. Ibid., p. 66. The italics in the last sentence, except for the word "before," are my own.

results, not indeed entirely and literally true in practice, but results to which practical truths yet undiscovered are probably analogous. This analogy, by directing our experimental inquiries, might be expected to lead to the discovery of . . . practical truths. . . ."[1]

Such a thinker has at least gone a part of the way toward an economic method which one day would enable political economy to have some pretensions toward achieving the status of a science.

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1. *Ibid.*, pp. 70-71.

5

T. Perronet Thompson (1783-1869)

Possibly the most important writer in the English group is T. Perronet Thompson. An article *On the Instrument of Exchange*, which appears in Volume 1 of the *Westminister Review*, is the contribution of importance to the present study. It is nominally on monetary theory, and as such it would hold only a modicum of interest for the modern reader. Oddly enough its importance lies in the fact that Thompson enunciates very clearly in these few pages a principle which is basic to modern price theory. Specifically laid down is the idea that if total gain is to be maximized, marginal revenue and marginal cost must be equated. Thompson even uses the calculus to make this point, thus becoming the third writer to employ the calculus in economic analysis and the second writer to have a clear conception of the idea of maximizing a difference between total gain and total cost.

The article was written ostensibly as comment on two articles which had appeared not long previously. One of these, *On the Means of Arresting the Progress of National Calamity*, by Sir John Sinclair, had advanced the view that the quantity of money in the English economy must be increased or the English people would suffer dire economic consequences. The second, The Question Concerning the Depreciation of our Currency Stated and Examined, by William Huskisson, had taken the opposing view that the currency of a country could not be depreciated without harmful results to all classes within the state. Thompson's objection to both papers is that neither author states in a systematic way the effects which alterations in the quantity of money may have on the economy.

Thompson begins his analysis with a tiresome and uninteresting speculation regarding the origin of money. He then passes to a section on monetary theory which contains no material pertinent to our subject. In this section he states carefully a concept of the supply of and demand for money, and he shows how, upon the introduction of gold coins, such coins would at first circulate at a premium because of their convenience, how eventually the supply of such coins would become "complete," and finally how further injections of such money into the economy would bring about gradual inflation. Such an inflation he contends may bring about ". . . an increase of production, employment and wealth, . . . in some particular branches of trade, in consequence of the direction given by the government to the additional coins; but it would be balanced by an equal diminution in some other branches."[1]

1. T. Perronet Thompson, "On the Instrument of Exchange," Westminster Review, 1:181, April, 1824.

The section which is of interest in the present inquiry begins as follows:

> "In a State where the receipts and disbursements of the people had been made only in commodities, much trouble would be saved if the government was to fabricate paper billets having a certain value specified in each, as for instance, a bushel of wheat, and deliver them in its payments in lieu of the commodities specified; engaging to receive them again for the same value in discharge of taxes, and at all times to return the specified commodities upon demand. And in consequence of the convenience attending the employment of billets as the instrument of exchange, a number of them would be neither returned in discharge of taxes nor in demand of payment. And for every billet so retained in circulation, it is clear that the commodity which had been received when it was issued would be in the hands of the government, over and above the receipts of the taxes or just revenue; and that a corresponding quantity of some commodities which had been previously employed as the instrument of exchange would be restored to their ordinary usefulness."[1]

Now it might be expected that such billets would circulate for a time at a premium because of their considerable convenience, but as the number in circulation increased, they would tend toward circulation at par. So long as the

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1. Ibid., p. 183.

government were willing to redeem the billets for one bushel of wheat, their value in terms of wheat could never rise much above nor fall much below one bushel.

What would happen if the government were to cease to pay wheat upon demand? It does not follow that the value of the billets would drop to zero; on the contrary, if the government made the billets legal tender, ". . . men would make an attempt to continue the circulation, and what they attempted would succeed."[1]

The people might not like this, but after some short period of uneasiness and agitation the billets would continue to serve as the instrument of exchange and would circulate at par so long as no greater number of billets were issued than were received in discharge of taxes in any given period of time. Thompson feels that it is as well established as any economic fact that an irredeemable paper currency will circulate without depreciation so long as it is made legal tender and so long as the quantity of it be not increased appreciably.

However, if the government were to issue in any given period of time a greater number of billets than were received in that period in discharge of taxes, an increase of paper

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1. *Ibid.*, p. 184.

prices would occur.

> "For example, if exclusively of the number occupied in the discharge of the taxes, four thousand billets were in circulation at any particular instant where three thousand circulating at par would be sufficient, --then the billets would be depreciated by one fourth, or a billet purporting to be for a bushel of wheat would in fact exchange only for the value of three quarters; or, which is the same thing, the paper prices of commodities would rise by one third."[1]

The government not having kept its promise to redeem billets for a bushel of wheat, other people are bound to give for a billet not what the government had said it would give but what the state of the market allows.

We come to the heart of the matter when Thompson examines in detail the consequences of a systematic increase of an irredeemable paper currency. Suppose that the entire monetary medium consists of billets circulating at par. Suppose further that the government continues to issue, say, daily, a number of billets somewhat in excess of the number returned in discharge of taxes in this small period of time. The number authorized by the return of taxes is called the _legitimate_ issue. Any number over and above the legitimate issue is called the _superfluous_ issue, and these two taken together are the _actual_ issue. The _nominal_ value of any quantity of the paper equals the number of bushels of wheat

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1. _Ibid._, p. 185.

actually expressed upon the paper, while the substantial value equals the number of bushels of wheat which at any moment may be actually had in exchange for a billet.

Assume successive daily issues of billets constant in amount but containing a superfluous issue. During the first day no depreciation is felt, but on successive days depreciation takes place. On any given day,

> ". . . the sum of the augmentations to the number in circulation would be equal to the superfluous issue of one day multiplied by the number of days during which the superfluous issues had been carried on; --from which the depreciation may be found. And what the government would substantially receive on the same day in exchange for the superfluous issue, would be expressed by the product of itself and of the fraction which expresses the depreciation. And what it would substantially lose on the same day by the diminution of value of the legitimate part of the issue, or the paper received for the taxes and re-issued, would be expressed by the nominal amount of the legitimate daily issue multiplied by the same fraction."[1]

The gain of the government on any day is equal to the first of the quantities mentioned above minus the second. In other words, the government on any day gains by the nominal amount of superfluous issue minus the actual daily issue

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1. Ibid., p. 186.

times the fraction which expresses the depreciation. Whenever the actual daily issue times the fraction which expresses the depreciation becomes equal to the daily superfluous issue, marginal cost to the government equals marginal revenue. At this point further superfluous issues avail the government nothing for marginal losses exceed marginal revenue. As Thompson puts it, ". . . if the superfluous issues were continued after this period, the government would begin to lose; for it would suffer more by the diminution of what it obtained for the paper received for the taxes, than it would obtain for the superfluous issue."[1] What a familiar ring this last statement has to the instructor in economic theory who has attempted to explain the principle of maximization of gain by a firm without the proper mathematical tool. Thompson goes on to point out that, if the superfluous issue were continued, the government would eventually begin to incur losses.

Up to this point the reasoning may have been difficult to follow. Note, however, how the mathematical notation which follows immediately clears up the matter.

> Let A equal the number of billets, each nominally equal to one bushel of wheat, which are, ". . . sufficient when at par for the whole circulation including the payment of taxes."[2]

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1. Ibid., p. 187. The italics are my own.
2. *Ibid.*, p. 187.

Let b equal the number of billets used each day to pay taxes.

Let s equal the daily superfluous issue.

Let p equal the legitimate issue.

Let t equal the number of days in which there has been a superfluous issue.

Let z equal the fraction which expresses the depreciation.

Now, supposing (1) the daily issues, and (2) the nominal tax receipts to be uniform daily:

$$z = 1 - \frac{A-b}{A-b+st}$$

Simplifying,

$$z = \frac{st}{A-b+st}$$

Now the bushels of wheat received for the superfluous issue during any small period of time, dt, is equal to sdt-(sz)dt. (In words, this is the amount gained in purchasing power by the government on account of a daily superfluous issue). The bushels of wheat lost on the remainder of the issue is equal to (pz)dt. The gain (dG) during any small period of time, dt, is

$$dG = s \cdot dt - (s+p)\, dt \cdot z$$

$$\text{or} \qquad dG = s \cdot dt - (s+p)\, dt \cdot \frac{st}{A-b+st}$$

Simplifying,

$$dG = s \cdot \frac{A-b-pt}{A-b+st}\, dt$$

Integrating this expression and correcting it so as to be zero when t is zero,

$$G = \frac{stp}{s}(A-b)\ \mathrm{Log}\ \frac{A-b+st}{A-b} - pt.$$

This latter expression gives the total gain of the government in any time t expressed in bushels of wheat.[1]

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1. Some minor changes have been made in the mathematical notation in order to make this passage easier to read. For example, instead of the expression dt Thompson uses t. This was still the common method of indicating a derivative when Thompson was at Cambridge. Thompson also uses the word "fluent" to indicate an integral. The integration indicated in the text is as follows:

$$dy = sdt - (s+p)zdt$$

$$= sdt - (s+p)s\frac{t}{A-b+st}\,dt$$

$$y = s\int dt - s(s+p)\int\frac{tdt}{A-b+st}$$

Integrating

$$y = st - s(s+p)\left[\frac{t}{s} - \frac{A-b}{s^2}\log(A-b+st)\right] + C$$

but y = 0, when t = 0

$$0 = 0 - s(s+p) \bullet \left[\frac{A-b}{s^2}\log(A-b)\right] + C$$

$$C = -\frac{s+p}{s}(A-b)\log(A-b)$$

$$Y = st - s(s+p)\left[\frac{t}{s} - \frac{A-b}{s^2}\log(A-b+st)\right]$$

$$- \frac{s+p}{s}(A-b)\log(A-b)$$

$$= \frac{s+p}{s}(A-b)\log\frac{A-b+st}{A-b} - pt.$$

Note also Thompson's statement that "When the depreciation is such that $(s+p)zdt = sdt$ the daily gain will be at an end." This is the condition that $dG = 0$. Here we have of course a different situation than that of the firm which seeks, in the short period, an output at which it maximizes net revenue, for in this instance it is the government which seeks to maximize a gain by fiscal manipulation through time. The principle involved, however, is precisely the same; it is the necessary condition for a maximum G.

Thompson goes on to say that when

$$(s+p)zdt = sdt$$

$$z = \frac{s}{s+p}$$

or when

$$\frac{st}{A-b+st} = \frac{s}{s+p}$$

$$t = \frac{A-b}{p}$$

When t is any greater than this there is a loss during any period dt. If the depreciation is carried further the government incurs losses. Those interested may set the integral given above equal to zero in order to see that net losses begin when

$$\frac{\frac{z}{1-z}}{\log\frac{1}{1-z}} = \frac{s+p}{p}.$$

The remainder of the article consists of additions to the basic analysis and of suggestions for the practical management of the British monetary system. It closes with a rather interesting comment regarding an equitable way of tax/ing income. In brief, Thompson feels that taxation must start above the class which performs manual labor and even above the class which contains the lowest order of mental workers. In no case should an income tax be a percentage which would approach to the whole of income. He suggests that there be an exemption of a certain amount and that there should be levied a proportional tax upon incomes over that amount. Graphically expressed "The scale. . . should be nothing at a certain income and approach to some reasonable percentage as to an asymptote."[1]

The mathematical expression of his idea is as follows:

> "The equation to the curve is $y = M \cdot \frac{m}{n} \cdot \frac{x-a}{x}$ where x is the abscissa measured from a point without the curve, a the distance from this point to the vertex of the curve, y the ordinate, M a given line, and m/n a given fraction. If x represents the income, a the income at which taxation is to commence, m/n the uniform rate levied on the excess of x above a, the ordinate y will vary as the percentage which the proposed

1. *Ibid.*, p. 204.

> scale assigns to the income represented by x. Or the proportion of the ordinate to the given line M, will always be that of the numerator to the denominator of the fraction which expresses the percentage on the income.--If a line is drawn parallel to the abscissa at a distance equal to M times m/n and on the same side of it as the curve, it will be an asymptote; for when x is indefinitely increased, y approaches to being equal to M times m/n."[1]

In the Jevons bibliography is noted a postscript to the article "On the Instrument of Exchange" which appeared in an 1830 issue of the Westminster Review. This postscript contains some additions and corrections, only one of which is of interest in this paper. It concerns the remarks on a just scale of taxation just quoted, and because it is one of only three geometric treatments in the whole of the material covered I am citing it in full.

"The equation to the curve is $y = M \cdot \frac{x-a}{x}$, where x is the abscissa measured from a point A outside the curve, a the distance from this point to the vertex, y the ordinate, and M a given straight line BC drawn from the vertex at right angles to the axis. If x represents the whole or actual income, a the income after which taxation is to commence,

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1. Idem.

and M the percentage uniformly levied on the excess of x above a, y will represent the percentage on the whole income represented by x. If through C a straight line is drawn parallel to the axis, it will be an asymptote to the curve; for when x is indefinitely increased, y approaches to being equal to BC.

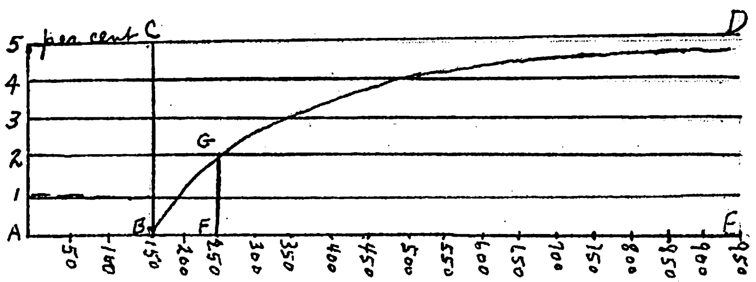

Figure 3

"If AE is divided into equal parts of which AB contains 150, and if BC is divided into five equal parts each of which represents one percent, and straight lines parallel to AE are drawn through the points of division, the perpendicular FG drawn from any point F in the line BE will show the percentage levied on the whole income represented by AE. Thus on an income of 250 l. the percentage is two per cent, as represented in the figure; on one of 500 l. it would be three and one-half per-cent; and in like manner in other cases. As the income becomes large, the percentage on it approaches to five percent.

A similar mode of representation may be applied to any other proposed scale or system of variation; the narrowing or falling away of the space included between the curved line and BE, representing the relief given to the smaller incomes."[1]

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1. T. Perronet Thompson, "Postscript to the Article on the Instrument of Exchange," Westminster Review, 12:527, January-April, 1830.

6

William Whewell (1794-1866)

William Whewell, a Cambridge professor with a considerable reputation in his day as a mathematician and philosopher, is remembered in economics as a result of three papers which appeared in 1829, 1831, and 1850.[1] Whewell, like Canard, is frequently mentioned in the literature dealing with this period, and has been almost as much maligned as the French author. That some of the adverse criticism of Whewell's work has not been justified is apparent. It is quite true that in the earlier article there is pretty much of an uncritical attempt to "translate" Ricardo into mathematical language. The second article, while purporting to be an exposition of Ricardian doctrine, is full of well-taken comment regarding the accuracy of the reasoning of the master.

Those writers who have touched on the early mathematical economists have been prone to comment at length without

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1. William Whewell, "Mathematical Exposition of Some Doctrines of Political Economy," Cambridge Philosophical Transactions, Vol. III, 1829, pp. 191-230. In Vol. IV, 1831, pp. 155-198 appeared his "Mathematical Exposition of Some of the Leading Doctrines in Mr. Ricardo's 'Principles of Political Economy and Taxation.'" A third article, which was published in Vol. IX, 1850, is not included here because it falls outside the period under consideration.

giving their readers any idea of the method used by the men whom they criticize. We shall, therefore, consider Whewell's work at sufficient length to exhibit clearly his method.

Whewell does not contend that any different results can be obtained by the use of mathematical symbols and reasoning than by ordinary processes. But he does hope ". . . that some parts of this science of Political Economy may be presented in a more systematic and connected form. . . and more simply and clearly, by the use of mathematical language than without such help."[1] Books on political economy, he contends, contain more complex numerical calculation and deal continually with quantities, yet they are frequently hard to follow because of failure to make use of the tools of the mathematician. Any systematic algebraic interpretation must, however, ". . . borrow the elements and axioms which are its materials from that higher department of the science of Political Economy, which is concerned with the moral and social principles of man's actions and relations."[2] The principles are "few and general" (though not necessarily "true and applicable"), and it would seem that the cases to which these principles are applied in economics are certainly not less complicated nor less general

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1. Whewell, Cambridge Philosophical Transactions, Vol. III, 1829, p. 192.
2. Ibid., p. 193.

than those of mechanics, in which field mathematics has already proved such a help. In fact,

> ". . . if men had, without the aid of consistent mathematical calculation, attempted to make a system of mechanical philosophy, there would have been three errors difficult to avoid. They might have assumed their principles wrongly; they might have reasoned falsely from them in consequence of the complexity of the problem; or they might have neglected the disturbing causes which interfered with the effect of the principal forces. And the making mechanics into a mathematical science supplied a remedy for all these defects. It made it necessary to state distinctly the assumptions, and these thus were open to a thorough examination; it made the reasonings almost infallible; and it gave results which could be compared with practice so as to show whether the problem was approximately solved or not. It appears, I think, that the sciences of Mechanics and Political Economy are so far analagous, that something of the same advantage may be looked for from the application of mathematics in the case of Political Economy. And this must be remarked, that in this we are so far from claiming for it the rank of a science mathematically demonstrated, that we do not thus assert it to be a near approximation to the business of the world, any more than the doctrines of Mechanics are to actual practice, if we neglect friction and resistance, and the imperfection of materials, and suppose moreover the laws of motion to be questionable. But we hold this method of investigation to be the best way of separating the theories which have been advanced, into the different kinds of truth, or of falsehood, of which they may happen to consist. And we conceive, that by doing this we shall better enable men to form their opinions of the value of these several parts."[1]

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1. _Ibid._, pp. 194-195. To compare these remarks with a modern view, see E. H. P. Brown, _The Framework of the Pricing System_, London: Chapman and Hall, 1936, Chapter I.

I have quoted these remarks partly for their inherent interest and partly to clarify attitudes toward Whewell's work. He goes on in the next paragraph to say that the subject which he wishes to discuss belongs only to the <u>second</u> of the three processes "liable to error"--namely, the "deduction of conclusions from fundamental propositions." For example, there are all sorts of divergent opinions on matters having to do with rent and taxes held by men who reason from approximately the same general principles. Whewell feels that these differences come about because of assumptions introduced <u>during the course of the investigation</u>, and he contends that mathematical analysis will indicate the point at which <u>someone</u> has erred.

Before beginning his analysis the author lists six axioms which he accepts as principles. It would be interesting to discuss his comments on each of them, but I shall have to be content with a mere listing.[1]

> Axiom 1. Rent is the excess of the produce of the land above the usual profits of the capital which is employed upon it.

> Axiom 2. When the produce of any land would equal or exceed the usual profits of the capital requisite to cultivate it, it will be cultivated; and not otherwise; and when new capital can be applied to land so that the additional produce will equal or exceed the usual profits of the capital, the capital will be so applied; and not otherwise.

> Axiom 3. If from any cause the value of the produce of a given quantity of land increases, new

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1. The fourth axiom, the one regarding the relationship between price and supply is discussed later.

land will be cultivated, or new capital will be employed on land already cultivated; and this will be done till there is some land cultivated, or some capital agriculturally employed, which returns no more than the usual profits of capital without any surplus.

Axiom 4. The increase of price is proportional to the deficiency of the supply.

Axiom 5. The rate of price of any article will be such that the whole price of any portion is equal to the capital employed to produce it together with the usual profits. (Later this axiom is stated more succinctly and more clearly as "Price equals cost of production plus profits.")

Axiom 6. If any tax be imposed on one employment of capital, (for instance, on agriculture), the profits of the capitalist will not be affected by it.[1]

Given these six principles let us proceed to Whewell's analysis in order that we may see precisely what he does and how he carries on his investigation.[2]

Let it be supposed that there are various qualities of soil which are 1st, 2nd, 3rd, . . . mth, nth, and that the quantities of each of these soils are respectively a_1,

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1. These axioms are quoted verbatim. Ibid., pp. 196, 198, 199, 201, 202, and 203.
2. The following analysis follows the text very closely. I have changed wording here and there and have tried to put the material in a form somewhat easier to follow than the original, but this passage should be considered equivalent to a direct quotation.

a_2, a_3, . . . a_m, . . . a_n acres or units of land; that the capital employed on one acre in the different cases is c_1, c_2, c_3, . . . c_m, . . . c_n shillings or units of money: that the produce of one acre of each quality is respectively, r_1, r_2, r_3, . . . r_m, . . . r_n quarters or units of produce. Let it be supposed also that the price of a quarter (or other unit) of corn is p shillings (units of money), and that the annual return requisite to replace a capital, c, with the usual profit is qc (q-1 being a fraction which expresses the rate of profit).

In general we shall consider only the average produce and rate of all the soils, except the last quality, the last quality being that which is brought into cultivation, or thrown out, by the changes considered.

Let a = whole number of acres in cultivation,

c = average capital employed on an acre,

r = average produce per acre.

Then

ar = whole produce,

arp = its price (i.e., the total selling value of all the produce),

ac = whole capital,

acq = sum requisite to replace the whole capital with profit.

Hence by Axiom 1 (i.e., Rent is = produce - profits)

arp-acq = the whole rent, it being assumed there are no taxes.

If taxes are to paid on each acre, depending in any manner on the quality; viz. on the first quality t_1 per A, on the second t_2, on the mth, t_m, on the nth, t_n, the whole tax is $a_1t_1 + a_2t_2 + a_3t_3 + \ldots a_mt_m + \ldots a_nt_n$, and is equal to at, where t is the average tax per A; (t_1, etc., are expressed in units of money).

The rent is now the excess of the price of the produce above the deduction, which are the tax and the profits on the capital; for the capitalist cannot pay more or less, as appears by the same reasonings as those which were used when there was no tax. That is, the whole rent is now

$$arp - at - acq$$

In this case a, r, p, c may be different from what they were before, in consequence of the introduction of t, and we have to examine what the alterations are which will thus take place; q is supposed to be unaltered by Axiom 6 (i.e., Taxes do not affect the rate of profits).

In consequence of the tax, let it be supposed thatthe price p becomes p'; and that the last quality of soil a_n is thrown out of cultivation. Hence the whole number of acres in cultivation now is $a - a_n$, and the whole produce $ar - a_nr_n$. Let the produce of the last quality be of the whole produce a fraction u, so that $ar - a_nr_n = ar(1-u)$, the produce during the tax. Also the capital ac is diminished by $a_n c_n$, which we suppose to be a portion v of the whole capital, and hence the capital employed after the tax is ac (1 -v). Hence,

	Without the Tax	With the Tax
Rent	$arp - acq$	$ar(1-u)p' - (a-a_n)t - ac(1-v)q$
Return and profits	acq	$ac(1-v)q$
Tax	0	$(a - a_n)\ t$
Whole price of the produce	arp	$ar\ (1 - u)\ p'$

The rent and return with profits in the first case, and the rent, return with profits and tax in the second, are equal to the whole price of the produce.

Hence by the imposition of the tax,

Diminution of rent $=$ $arp - ar\ (1 - u)\ p' + (a-a_n)\ t - acqv$.

If the whole tax be of the whole price of the produce a fraction K, constant or variable with the variations of price, etc.,

Diminution of rent $=$ $arp - ar\ (1 - u)\ (1 - K)\ p' - acqv$,

Diminution of profits, etc., $=$ $acqv$,

Increase of price $=$ $ar\ (1 - u)\ p' - arp$

Tax $=$ $Kar\ (1 - u)\ p'$.

The tax therefore is the sum of the diminution of rent, the diminution of return to capital, and the increase of price;

that is, of what is taken from the landlord, what is taken from the capitalist, and what is taken from the consumer, as it manifestly must be. How each of these portions is determined is now considered.

The diminution of the capitalist's share arises from the diminution of the capital employed, by the throwing of the land a_n out of cultivation; the rate of profits q is supposed to remain the same at first. If this vary, however, the consequences of its variation may be traced afterwards. Various suppositions may be made, of which our formulae will give us the consequences.

We may suppose that no land is thrown out of cultivation in consequence of the tax: that the demand increases, and with it the price, so as to keep a_n in cultivation. That this must be so, we must have

$pr_n - c_n q$ not less than 0 by Axiom 2 (Land will be cultivated if produce = or profits),

and $p'r_n - t_n - c_n q$ not less than 0, for the same reason.

Suppose the soil a_n to be exactly of the limiting quality, so that by the 5th and 6th Axioms (Price = cost of production plus profits, and taxes do not affect the rate of profits).

$pr_n - c_nq = 0$, $p'r_n - t_n - c_nq = 0$, and let

$t_n = K_np'r_n$. Therefore $p'r_n(1 - K_n) = c_nq$.

Hence, $pr_n = c_nq$. Therefore $p'(1 - K_n) = p$.

Here also $u = 0$ and $V = 0$, because no land is thrown out; therefore, putting for p' its value,

$$\text{Increase of price} = \frac{arp}{1 - K_n} - arp = \frac{K_n arp}{1 - K_n}.$$

$$\text{Diminution of rent} = arp - ar\,\frac{1 - K}{1 - K_n}\,p = \frac{K - K_n}{1 - K_n}.$$

$$\text{Diminution of profits} = 0.$$

$$\text{Tax} = \frac{Karp}{1 - K_n}$$

If the tax bear a given ratio to the produce for all soils $K_n = K$, the diminution of rent is 0, the increase of price and the amount of the tax are equal, and the whole tax falls upon the consumers.

This is the case considered by the writers who follow Mr. Ricardo, and the conclusion depends entirely, as appears from the investigation, on the supposition that the supply is perfectly unaffected by the tax.

It may be observed also that if poorer soils be taxed at a lower proportional rate than average, the tax falls on rent, and is taken from price. Here K_n is less than K. If the poorest soil pays no tax, the whole of the tax levied on other soils falls on rent, even on the supposition above mentioned. In this case $K_n = 0$.[1]

Enough of Whewell's analysis has been followed to give an idea of his method. There remains, however, to show one specific development of his which is presented explicitly for the first time in the paper of 1829. This is the origination of a means of measuring the responsiveness of quantity changes to corresponding changes in price.

One writer says that Whewell was the first to reach ". . . the stage of strict formulation of the idea of elasticity of demand."[2] It is pointed out that Whewell ". . . inverts Marshall's formula, which is read from price to quantity in the case of demand."[3] It has been called to my attention that, strictly speaking, the measurement which Whewell derives is that of "flexibility of prices."[4] Whewell defines

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1. Ibid., pp. 205-209.
2. D. H. MacGregor, "Marshall and His Book," Economica, Vol. IX (New Series), No. 36, November, 1942, p. 316.
3. Idem.
4. Professor Howey has made me aware of this distinction, which is mentioned in Frank Knight's article on "Demand" in the Encyclopaedia of the Social Sciences. In that article is this statement, "The direction of the causal relation assumed in connection with elasticity is important because reversing it would invert the magnitude of the elasticity and produce a measure of what A. L. Moore calls the 'flexibility of prices.' In many cases the inverse relation would be the more natural one to consider." See the Encyclopaedia of the Social Sciences, Vol. V, p. 71.

elasticity in the formula $w = eu$.[1] In the argument given in detail above, the fraction u represents the relationship of the produce of land of the last quality brought under cultivation to the whole produce, so that

$$u = \frac{a_n r_n}{ar}$$

If, to use a modern notation, Q stands for the whole produce ar and ΔQ stands for the increment of product $a_n r_n$

$$u = \frac{\Delta Q}{Q}$$

In the passage which follows the one given above Whewell lets

$$p' = p\ (1 + w)$$

where p is the price before the change in quantity and p' is the price after the change.[2] Then

$$w = \frac{p' - p}{p}$$

Since $p' - p = \Delta p$, we have

$$w = \frac{\Delta p}{p}.$$

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1. Whewell, op.cit., Cambridge Philosophical Transactions, 1829, p. 211.
2. Idem.

Consequently, Whewell's formula $e = \frac{w}{u}$ can be written

$$e = \frac{\frac{\Delta p}{p}}{\frac{\Delta q}{q}}$$

where e is then the reciprocal of the elasticity of demand. Demand elasticity's reciprocal is, of course, the "flexibility of price." Oddly enough, throughout his analysis Whewell assumes the value of e to be approximately three wherever agricultural produce is being considered. In this connection he quotes the passage from Tooke, which was mentioned in Chapter II.[1]

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1. *Vide supra.*, p. 13.

7

Denis George Lubé (   )

This work begins with an "Appeal to Landlords" to resist the interests, which, from mistaken views, are opposed to agriculture; but after three chapters of polemic, the book takes on the aspect of a reasonably modern text book in economics. In outlining his theory of price, Lubé begins by saying that ". . . here the province of the political economist begins, whose business it is to find other data, by which to arrive at the same results, and to discover the means of ascertaining the relative values."[1] By value the author does not mean ideal or intrinsic value, nor does he mean cost of production. Value is exchangeable value estimated not by fancy nor utility nor cost, but by the quantity of other goods which a purchaser is willing to give in exchange. Value is purely relative, and at first the idea must be considered apart from money price.

Proceeding to inquire upon what elements value may be based, he finds that one of these is its "desirableness . . . or the demand in which it is held by persons who wish to become the possessors."[2] The extent of demand depends on

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1. D. G. Lubé, An Argument Against the Gold Standard, London: J. Ridgway, 1832, p. 59.
2. Ibid., p. 63.

whether or not the good is a necessity or a luxury, whether or not it is required by one or several classes of society, and so on. It is not supposed that relative value depends on demand alone, i.e., on the degree of estimation in which the good is held. Another factor is the variations in the quantity of the good offered--i.e., supply, which in turn depends on the time and the labor necessary to produce the goods. The time and labor necessary to produce the good constitute the natural cost of production. Supply varies inversely as the cost of the good; natural supply thus may be reduced to a definite quantity, and demand may also be reduced to a definite quantity as will be seen later on.

> "Now the value of an article is always ceteris paribus, in proportion to the demand; the greater the desire or necessity of obtaining it, the more will be given in exchange for it. On the other hand, it is obvious, that if of two given articles one could be supplied --for instance, in double the quantity of the other, at an equal expense of time and labor, two of the former would exchange against one of the latter, i.e., the greater the natural supply, the less the value. As then, exchangeable value varies directly as the demand, and inversely as the supply--that is, since the greater the demand, and the smaller the supply of a given article, the higher will be its value, and vice versa; --anyone acquainted with the first principles of arithmetic will know that it may be expressed by a fraction, whose

numerator shall be the relative demand, and denominator the natural supply; for the value of a fraction varies directly as the numerator, and inversely as the denominator. Thus, in general terms, making d stand for the demand, and s for the supply, the fraction d/s will represent the relative value of each commodity; it is only saying, in other words, that the value of anything depends upon its quantity, and the number of persons among whom it is to be distributed.

"If either of the terms of the fraction be wanting, there will be no exchangeable value. For though an article should be possessed of the greatest intrinsic utility, and therefore, the demand expressed by the numerator d very high, yet if it could be obtained without stint, and without trouble, and consequently, no expression could be found equivalent to the denominator s, it would be worth nothing in exchange; or, putting it more strictly, if the divisor be infinitely great, the quotient will be infinitely small. Thus, no article is of more general utility, and consequently, more in demand than water. Nature, however, has supplied it in such abundance, that, except under peculiar circumstances, nothing of value will be given in exchange for it, since everybody can supply himself without cost. In all great cities, water, being carried from a distance, costs both time and trouble, and hence immediately acquires an exchangeable value. In Paris, it is a regular article of trade, being sold at so much a bucket. In London, and other places, it is supplied to private families from great reservoirs by means of leaden pipes, for which a yearly tax is paid by the inhabitants. On the other hand, although the time and trouble in procuring or fashioning an article should be very great, and, consequently, the supply, within the given limit very small, yet, if there were no demand, if it were entirely useless, it would be worthless also, and nothing would be given in exchange for it. Many a man has employed months and years in modelling and constructing instruments, which have, in the end, turned out inadequate to the purposes intended, and, consequently, wholly unserviceable and valueless.

"The advantage both for clearness and precision, of using this fractional symbol to denote relative value, will be more apparent as we proceed. When we wish to compare the value of one article with another, we say, the value of the first is to that of the second, as d/s to d'/s', that is, as the relative demand and supply of the former to the relative demand and supply of the latter. Of this relation there are three cases."[1]

The author then proceeds to discuss the cases supposed. In the first instance he assumes the cost of production (i.e., the supply) of two goods equal. Relative value then is proportional to demand; if s equals s', then the value of the first is to the value of the second as d is to d'. Second, he supposes demand equal; relative value will depend altogether on cost of production--i.e., value ". . . will be inversely as the quantity of each that can be produced with a given degree of time and labor."[2] If d and d' are equal,

1. *Ibid.*, p. 167.
2. *Ibid.*, p. 168.

then the value of the first good is to the value of the second as s is to s'. This is the case to which Smith and others referred when they said that labour is the true measure of value; such a statement can only be true when the demands for two or more goods are equal.

Finally, if the demand for and the supply of two commodities are both unequal the relative value ". . . is neither in the proportion of the demand nor of the supply, but in a ratio compounded of both."[1] Relative value then ". . . is as the product of d and s' to the product of d' and s."[2]

Lubé is now ready to distinguish between natural cost, which is the time and labor necessary to bring an article to market, and artificial cost, which is this cost expressed in terms of money. Thus artifical supply is the ". . . quantity (of a good) that can be produced at a given money cost . . . ."[3] Similarly demand measured in money is the artificial demand.[4] Finally, there is actual supply which is

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1. Ibid., p. 69.
2. Idem.
3. Ibid., p. 77.
4. There follows at this point a most interesting discussion of the interaction of demand and supply which seems clearer to me than anything written on the subject to this date (1832). Since one would have to "read in" to the discussion the functional relations involved, this section has been omitted for the sake of brevity. Ibid., pp. 78-80.

apparently the physical number of units of a good for sale in the market. There are, then, two kinds of supply--artificial and actual. If we divide artificial demand by artificial supply we get relative value; if we divide artificial demand by actual supply, we get market price.[1]

Only one other matter need be noted to make the last part of the work understandable. It is a general principle that

> ". . . in barter . . . the supply of one commodity is the demand of the other, and vice versa. Applying this principle to the circulating medium, and amount of goods in the market, . . . the goods form the demand for the money, and the money for the goods."[2]

Now all the different goods are summed, b + c + f + g, etc., = a, and the amount of the currency equals c; the value of money is then expressed by a/c, and the value of goods in terms of money by c/a.[3] This latter value is called by "political economists" the nominal price.[4] If trade, a, increases, the value of money increases; if c increases nominal prices increase or the value of money decreases. It should be noted that general prices in one country are to

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1. Ibid., p. 78.
2. Ibid., p. 95.
3. This naive effort to give so simple an expression for the general level of prices makes the going difficult from now on.
4. Idem.

general prices in another country as c/a is to c'/a'. In preserving the ratio between trade and the quantity of money, the latter should always be adjusted to accommodate the former. Finally,

> "If to the aggregate b + d + f + g, etc., etc., of merchandise, a new commodity, or source of trade, x, were added, unless the currency were increased in the same proportion, the effectual demand for the original commodities b + d + f + g etc., would be diminished by so much of the existing currency as x would absorb, and, consequently, their actual supply must be diminished; and, hence, it is evident, that general improvement depends upon the currency being kept up in proportion to trade."[1]

Lubé is now ready to analyze the evil effects on an economy of the gold standard. He wishes to show two things: first, that the quantity of money is too small, given present taxes, to allow remunerative prices and second, that under a gold standard the circulation can increase by only a very small amount before reaching a fixed limit. This he does in a rather lengthy appendix, which I am reluctantly quoting in great part only because any considerable deletion of material leaves the argument unintelligible.

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1. Ibid., p. 96.

"Nominal value or price is that which a commodity has, in common with all other articles, expressed in currency, and, therefore, depends upon the relation which the entire trade of the country bears to its currency. General nominal value being $\frac{c}{a}$, and the demand in money, for the whole amount of merchandise (a), being (c), --the proportionate nominal demand, for any particular commodity as (b) is $\frac{bc}{a}$; which, being divided by the supply (b), whether artificial or actual, gives the common value $\frac{c}{a}$. A change in the value $\frac{c}{a}$ denotes, therefore, a rise or fall in the nominal price or value of every particular commodity, as well as in general prices; --but always, on the supposition, that the supply and demand of each commodity bear the same ratio to each other as before; --for if there be a disturbance of that ratio, then the effectual value is altered also. By nominal value, we compare the price of each article with itself, on the expansion or contraction of the currency; but, by the effectual demand, we estimate the relative value of commodities one with the other. The effectual demand is that proportion of the currency which forms the demand for a particular commodity; --the effectual, or relative value is, that demand divided by the artificial supply, --the actual supply has no influence on relative exchangeable value. . . .

"Let us now apply this doctrine to the case of gold bullion, which will also serve for an example to the foregoing reasoning; the only peculiarity, with respect to gold, being its having a standard mint price, and its being used for currency. It has, therefore, two characters--as coin and as bullion; as the former, it is a portion of the numerator (c) in the general fraction $\frac{c}{a}$, --as the latter, it is a part of the common stock of merchandise (a), --whence, it follows, that the market and the mint price of bullion must be equal; for if the mint price were higher, the bullion in (a) would be transferred to (c), --or, if the market price were higher, the bullion in (a) would be transferred to (c), --or, if the market price were higher, the coin in (c) would be melted into (a). Now, let us suppose, that when the mint and market prices are alike, its proportionate nominal demand is $\frac{bc}{a}$, and, of course, its actual supply (b): its nominal or market price then will be $\frac{c}{a}$, and its mint price the same. Then, if there were an increase made to the currency, which call (x), the new amount c' would be c+x, and the nominal demand for (b), would be $\frac{b\cdot\overline{c+x}}{a}$, which would give for the nominal or market price $\frac{c}{a}+\frac{x}{a}$, --that is, the market price of bullion would be higher than the mint price, in the proportion of $\frac{x}{a}$, and, therefore, coin would be melted into bullion, until (x) were again absorbed by (a). But, at the same time, that bullion advances in nominal price, every other article advances in the same proportion, so that relative value is not affected by the variation of currency.

"But, next, let us suppose, that the market price of gold advances, not in consequence of the increase of currency, but on account of an augmentation of the effectual demand with an inadequate supply, or, at the point where the actual supply has reached the artificial supply, and cannot be increased with the demand. Call the increase of effectual demand (y), the new demand will be $\frac{bc}{a} + y$, and dividing this demand by the supply (b) for the relative value, which is also the market price, we have $\frac{c}{a} + \frac{y}{b}$. Thus, an increase of currency, increases the value of gold in the ratio $\frac{x}{a}$, while the increase, in its effectual demand, causes its relative value to rise in the proportion $\frac{y}{b}$. But (b) is contained in (a), and forms but a small portion of it; consequently, a slight difference in the demand for gold will affect its value much more sensibly than a variation in the amount of currency. Hence, Mr. Ricardo's mistake, in estimating the depreciation of the currency during the war, by the rise in the market price of gold; because, although, in common with all other articles, the nominal price of gold increased in the proportion

$\frac{x}{a}$, yet the effectual demand for it was diminished by the substitution of a paper medium. Since the nominal demand is $\frac{b \cdot \overline{c+x}}{a}$, the <u>effectual</u> demand being diminished is $\frac{b \cdot \overline{c+x}}{a} - y$, and the true value $\frac{c}{x} + \frac{x}{a} - \frac{y}{b}$. Mr. Ricardo should, therefore, have added the value of $\frac{y}{b}$ to the nominal or market price of gold, in order to form a correct estimate of the rise in the nominal prices of general commodities.

"I will now endeavour to shew, upon similar principles, that a <u>gold standard</u> puts an artificial <u>limit</u> to currency, and, consequently, to improvement. We have already seen, that nominal prices being expressed by $\frac{c}{a}$, decline with general improvement, or an increase in the amount of trade (a). Call the increase (x), then nominal prices, after the increase, will be to nominal prices before as a : a + x, or as 1 : 1 + $\frac{x}{a}$. The amount of the decline, therefore, in nominal prices, is $\frac{x}{a}$. The <u>market price</u> of gold then, in common with all other articles, falls in the proportion of $\frac{x}{a}$; and if we suppose a gold circulation, and the <u>mint</u> price to be the same as the nominal or market price before the decline, bullion would be carried to the Mint, until the currency was increased in the proportion $\frac{x}{a}$, --viz. until currency was increased in the same ratio as trade -- c : c + $\frac{cx}{a}$. In this way, if the supply of gold could always be kept adequate to the increasing demand for currency in an improving country, prices would continue uniform, and no check would be imposed on the progress of trade. But gold is a natural production, and no art can increase the supply beyond the limit which nature has set. With an increasing demand, therefore, the supply must continually decline. Call the decrease of its supply (y), and the exchangeable value will be equivalent to the increased demand, --i.e., the relative value, after the decrease of supply, will be to the value before as b:b-y, or as 1:1- $\frac{y}{b}$. The amount of increase, in the relative value of gold, therefore is $\frac{y}{b}$; and supposing the market and mint prices were alike before the increase, the currency would be lowered in the proportion of $\frac{y}{b}$ by the melting of coin into bullion. Since then, with an advancement of trade, a gold currency has a tendency to increase in the ratio $\frac{x}{a}$, and with the consequent rise in the relative value of gold the currency has a tendency to decrease in the ratio $\frac{y}{b}$, the real variation, consequent upon an increase of trade, will be the difference between those two quantities, or $\frac{x}{a} - \frac{y}{b}$, i.e., the currency can only increase by a constantly decreasing series, and, therefore, has a fixed limit Q. E. D. That limit, is, when the relative value of gold becomes equal to the standard price, i.e., when the increase in relative value $\frac{y}{b}$ becomes equivalent to the decline in nominal value $\frac{x}{a}$, --when y:b::x:a. But (y) has a tendency to increase in the proportion of $\frac{x}{a}$, viz. as nominal prices decline, while (b) is stationary; and a small increase of (y) is equivalent to a large one of (x), for y:x::b:a, and (b) is but a small part of (a), therefore, $\frac{y}{b}$ increases much faster than $\frac{x}{a}$, and if $\frac{x}{a}$ increases $\frac{y}{b}$ will inevitably overtake it, and then all increase to the currency must stop."[1]

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1. Ibid., pp. 187-188, 189-192.

CHAPTER VII

THE SIGNIFICANT CONTRIBUTIONS

1

Up to this point we have been concerned only with the presentation of summaries of the known writings in mathematical economics which appeared before Cournot. An important question now arises. Which works among those presented are significant?

In a broad sense, all the writers included in Chapters III, IV, V, and VI are important in that they made a fundamental contribution to economic methodology. All of them have two characteristics in common. First, and most obvious, they used mathematical notation. Second, every author shows a consciousness of the necessity of considering an economic problem in terms of the dependent and independent variables of that problem, putting these variables into functional relation by means of some principle thought to be applicable. As we have seen, the relationships so expressed are frequently put in a very crude form. In the case, however, of each of our mathematical writers, there is conscious functional thinking.

It is probably not an overstatement to say that functional thinking was essential to the development of a pure science of economics. Professor Knight has hinted at this in his comment on the methodological deficiencies of the early English school.

> "The classical writers had no clear conception of causality in the mechanistic or 'positive' sense of function and variable, or, in less technical language, the relation between a continuously varying cause and its effect varying (quantitatively) in some corresponding way. This failure is manifest . . . in the failure to apply the 'law of diminishing utility'."[1]

Now, of course, it is a great help in thinking of this sort to be equipped with the tool of the calculus. It was not essential, however, that one think in terms of increments or in terms of rate of change in order to make the initial contribution to the development of ideas in the field of mathematical economics. Credit for this first step in economic analysis must, I think, be given to this group of men who, however haltingly, tried to "treat quantitative matters quantitatively."

In a broad sense, of course, everyone thinks functionally. Today great numbers of people are so exposed at one time or another to the language of mathematics that certain mathematical expressions have crept into every-day speech. It is not

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1. Frank Knight, "The Ricardian Theory of Production and Distribution," The Canadian Journal of Economics and Political Science, February, May, 1935, pp. 5-6.

uncommon to hear someone who is not technically trained speak of one quantity as being "a function of" another quantity. But such a person is a long way from a real consciousness of a relation "between a continuously varying cause and . . . its effect varying. . . in some quantitative way."

In this same broad sense it must be admitted that every writer in political economy has had to think functionally. But the mass of them were nevertheless wholly unconscious of the possibilities of mathematics in their methodology. Jevons makes this point when he says:

> "We might . . . discover that even the father of the science, as he is often considered, is thoroughly mathematical. In the fifth chapter of the First Book of the Wealth of Nations, for instance, we find Adam Smith continually arguing about 'quantities of labour,' 'measures of value,' 'measures of hardship,' 'proportion', 'equality,' etc; the whole of the ideas in fact are mathematical. The same might be said of almost any other passages from the scientific parts of the treatise, as distinguished from the historical parts. . . . And every use of the word equal or equality implies the existence of a mathematical equation; an equation is simply an equality; and every use of the word proportion implies a ratio expressible in the form of an equation. . . . But it is one thing to argue and another thing to understand and to recognize explicitly the method of the argument. As there are so many who talk prose without knowing it, or,

again, who syllogise without having the least idea what a syllogism is, so economists have long been mathematicians without being aware of the fact. The unfortunate result is that they have generally been bad mathematicians, and their works must fall. *Hence, the explicit recognition of the mathematical character of the science was an almost necessary condition of any real improvement of the theory.*"[1]

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1. Jevons, *op.cit.*, pp. xxii and xxiii. The italics are my own.

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It would have been disappointing, however, had we been unable to find some more specific contribution to the history of thought than the two general ones given above. From a methodological point of view, three original and important additions must already have been discerned by the reader. First, in three separate instances there was before Cournot a use of the calculus in economic analysis. In each case the employment of this mathematical tool corresponded closely to present-day usage. Bernoulli, Thompson, and Buquoy treated a problem of maximization in such a way that their method even now could be used with inconsequential additions. Buquoy's solution to the problem of determining the optimum depth of plowing a field is especially striking in this connection. It is a matter of agreement that ". . . the theory of maximization is the foundation of the marginal analysis--and therefore of the larger part of present-day micro-economics."[1] Whether or not the marginal analysis is in fact as important as we have supposed it to be is another matter. Taking the present state of economic theory as it is, we must admit the great importance in the history of economics of those who were first occupied with the problem.

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1. Kenneth E. Boulding, "Samuelson's Foundations: The Role of Mathematics in Economics," Journal of Political Economy, 56:194, June, 1948.

Second, we find three different attempts to make relationships graphic by the use of geometry. In two of these the geometric device was clearly subordinate to the algebraic one. It will be recalled that Thompson and Beccaria placed their geometric figures at the close of their treatments and apparently attached no great importance to them. Bernoulli, on the other hand, based his whole analysis on his geometric figure. In view of the present-day ubiquity of geometric presentation in economic theory, it is worthy of comment that the early writers thought more in terms of analytical (algebraic) method than they did in terms of the geometric.

In the third place, there was before Cournot a clear statement of the proper use of implicit functions in analysis. Buquoy's expression, while not modern in every respect, was certainly adequate, and his argument for the use of this kind of algebraic expression was exceptionally clear. It will be recalled that Lang, while not using a generally accepted notation, found it necessary to employ a single letter which would represent a varying portion of a particular quantity. The importance of this particular device is emphasized by the inaccuracies and even absurdities of those authors who, writing their equations only in the explicit form, were forced to state their "laws" in terms of strict proportionality. In this connection Canard, Fucco, and Major General Lloyd come to mind.

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Those who have followed this investigation in detail may have been disappointed in the fact that there was so little concern with general equilibrium analysis. Broadly speaking, the early mathematical writers were concerned with the relationships among a small number of variables. They were preoccupied, for the most part, with getting answers, and their comprehension of the essential unity of an economic system had not reached the point of considering one variable of the system as being functionally related to <u>all</u> the other variables in that system.

Isnard came as close to such a realization as anyone. In his consideration of the problem of the exchange values of goods, he emphasized the interdependence of the quantities involved in his highly simplified market for consumers' goods. Despite his awareness of the unity of his restricted system he only dimly foreshadowed the Lausanne school. He took two important steps forward, however. Having made his simplifying assumptions, he set up a hypothetical model which he used to get at the heart of the matter of value and price determination. He was consequently able to come to the view that value is not an absolute quantity "contained" in a good but a quantity which must vary as other quantities with which it is related vary. The fact that Isnard's contemporaries did not adopt this novel and important view does not diminish its signficiance as an early contribution to economics.

4

Several names ought to be selected for special mention even though their contributions are not so basic as those of Bernoulli, Thompson, Buquoy, and Isnard. One is struck, for example with the analysis of the anonymous English writer who indentifies himself as E. R. As did Beccaria, he took a practical problem of considerable complexity, defined his terms, reasoned upon "hypothoses framed. . . like what we have reason to believe is the actual state of things," and found an answer. E. R. was perhaps too naive in his belief that, once general formulas had been evolved, statistical information as needed sould be available for application. Nevertheless, his method required him to simplify, to abstract temporarily from reality. A comparison of his analysis of the effects of commutation of tithes with Ricardo's wanderings on the subject justifies the mathematical method.

The treatment of aggregative economic problems by Sismondi and by Lang suggests the usefulness of mathematics in this type of analysis. Sismondi's discussion is marred by such misconceptions as his concept of the "necessary wage." Lang's insistence upon dividing the population into classes and treating them separately was unnecessary and led him to an artificial separation of what we should today consider the important aggregates. When he came finally to make his summations, however, he arrived at expressions which evidenced a preoccupation with the same totals that were to be considered significant well over a century later. And it will be recalled that his formula $yZ = Px$ contained all the elements of the later "equation of exchange."

A special place in the history of the mathematical literature must be assigned to Canard. Whatever may be said of him in the way of adverse criticism, he must be given credit for the first comprehensive mathematical statement of a theory of price. Cournot states that Canard was the only one of the early mathematical writers whom he had read. But he goes on to say that his

> ". . . pretended principles are so radically at fault, and the application of them is so erroneous . . . that one can easily see why essays of this nature should not incline such economists as Say and Ricardo to algebra."[1]

1. Cournot, op.cit., p. 2.

Joseph Bertrand agrees with Cournot and is more specific in his criticism.

> "Le citoyen Canard, quoique professeur de mathématiques, ignore ou oublie les éléments du calcul du fonctions. Sachant que le prix d'une denrée s'accroît avec le nombre des acheteurs, avec leurs besoins, et avec les revenus dont ils disposent et qu'il diminue avec le nombre et l'empressement des vendeurs, la traduction dans la langue algébrique est pour lui immédiate; B+Ax est, en effet, suivant Canard, le type de toute fonction croissante de la variable x, et B' - A'x celui des fonctions décroissantes; tel est le point de départ et la base de sa théorie. Comment devint-il laureat de l' Institut? Sur le rapport de quelle commission? Je n'ai pas eu l'indiscrétion de la chercher."[1]

But Allix comments that if Canard is without excuse among the mathematicians perhaps he deserve indulgence on the part of economists. Largely self-taught, having in all probability read only Smith and Malthus, he had ". . . the unrewarded role and the merit of a precursor."[2] To Canard belongs the credit for first attempting a comprehensive treatment of the fundamental problem of price determination as a study in equilibria of forces. To Canard likewise must be attributed the first unequivocal attempt at supply and demand analysis. Whatever his errors may have been,

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1. Quoted by Edgar Allix, "Un Précurseur de l'École Mathématique: Nicolas-François Canard," Revue d'Histoire Économique et Sociale, Huitième Année, 1920, No.1., p. 39.
2. Ibid., p. 40.

there was a sufficient contribution in his book to have enabled a reader with great technical training and superior intellectual gifts to achieve a clearer and more accurate exposition of the matters which he considered.

Finally, a word should be said regarding William Whewell. Comment on his work has been, on the whole, unfavorable and similar to that regarding Canard. Moret expresses the opinion that Whewell's articles, like the writings of Canard, are only ". . . translations into algebraic symbols, under the pretext that they treat quantitatively preexisting economic theories."[1] The opinion that Whewell was simply a "translator" is prevalent, and there is some reason for this criticism as we have seen.

It has been pointed out, however, that Whewell had a clear conception of the usefulness of mathematics in economics and that he was far from taking an uncritical view of the doctrines which he put into mathematical form.[2] Whewell held that

> ". . . we are not at all justified in asserting that the principles

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1. Jacques Moret, _L'emploi des mathématiques en économie politique_, Paris: Giard et Brière, 1915, p. 75.
2. _Vide supra._, pp.

> which form the basis of Mr. Ricardo's system either to be steady and universal in their operation, or to be of such paramount and predominant influence, that other principles which oppose or control them, may be neglected in comparison. Some of them appear to be absolutely false in general, and others to be inapplicable in almost all particular cases."[1]

The refutation of Ricardian wage theory which follows is proof that Whewell is no mere "translator" but an economic thinker who might have made a great contribution.

It will be recalled that Whewell was the only author considered who formulated a means of measuring elasticity of demand, and this alone should assure him a place in the history of thought. Had this author become seriously interested in economics he might have written the first great work in pure economic theory. A highly competent mathematician and a professor of moral science at Cambridge, he was equipped by training and temperament to become one of the heroic figures in economics.[2] That he did not become such a figure can be

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1. William Whewell, "Mathematical Exposition of some of the Leading Doctrines in Mr. Ricardo's 'Principles of Political Economy and Taxation'," Cambridge Philosophical Transactions, vol. IV, 1831, pp. 156-157.
2. There are some interesting parallels in the lives of Whewell and Alfred Marshall. A mathematician before becoming interested in the moral sciences, he was, like Marshall, second wrangler (1816) and began his pursuits in economics as did Marshall by putting Ricardo into mathematical (and more general) form.

partly attributed to his lack of an abiding interest in economics. With him it appears to have been a subject which provided a satisfying mental exercise. He simply was not disposed to direct his investigations "architectonically."

CHAPTER VIII

SOME REFLECTIONS ON THE DEVELOPMENT OF ECONOMIC DOCTRINE

1

It has been my hope that this study should prove fruitful by making available materials which otherwise might have lain inaccessible. But it would be disappointing indeed if work of this sort failed to inform our thoughts concerning some aspects of the development of economic theory.

Generalizations rarely stand the test of scholarly examination, yet I am tempted to sum up the widely prevalent attitude toward the "true" line of development of our science as follows. Adam Smith was chosen as the prophet who revealed the nature, the scope, and the methodology of political economy. A host of writers before Smith had dabbled, clearly to no good purpose, and with no lasting results. A group of contemporary Frenchmen, who called themselves economists, devised an odd way of analyzing the working of a national economy together with a catchy phrase which has been widely mispronounced and misinterpreted. Adam Smith was influenced slightly by this group. A few gifted writers made minor additions and corrections to Smith, but only one, a highly successful businessman, contributed substantially to human enlightment by his cogent remarks regarding the way the income of the economy was "distributed" among the major classes of participants in production. The Frenchman Say had interpreted Smith at the turn of the century and his book became a classic. The Englishman Senior put classical writings in better form and cleared up some of the massive obscurities, but it remained for the great logician and political thinker, John Stuart Mill, to recast classical doctrine and put it in a final, lucid form. By his own admission in the subject of value nothing remained to be done, and if the popularity of his book in the English and American universities may be taken as a criterion, there was marked agreement on this score.

Suddenly in the early 1870's an Englishman, a Frenchman (who taught and wrote in Switzerland), and a German published works which emphasized the subjective evaluations of individuals as determinants of value, and the English and Frenchman applied the principle of the differential calculus to their analyses. Later Leon Walras and his successor at Lausanne, Vilfredo Pareto, were to call attention to the essential unity of the economic system by insisting upon the interdependence of economic quantities, and the latter developed a logically consistent method of analyzing subjective evaluations of consumers. The Germans made a contribution of the first order with their doctrine of "imputation." It remained, however, for Alfred Marshall to make the synthesis which was to give economic theory its "modern" form. While recognizing explicitly the mutual interdependence of the economic variables in an economy, he cast up the theory in terms of partial equilibria and for forty years it remained essentially unchanged. In the 1930's a group of bright young people made a considerable addition in the development of the economics of imperfect competition. Late in the 1930's we find the beginning of a real interest in general equilibrium models, and it appears today that a great advance has been made in the present preoccupation with economic aggregates now being studied within the framework developed by the great Lord Keynes.

It must be admitted that a number of well-informed economists, from Jevons and L. Walras to the present have insisted upon some correction of the development sketched so briefly, but to little avail. To be sure there has been a general recognition of the fact that Cournot, and to a lesser extent, Gossen, Dupuit, von Thünen, and Jenkins, had contributed something to the content of economic thought and even more to the <u>method</u> of analysis. But the "neglected"

British economists which Seligman mentioned so long ago have, so far as I know, remained neglected. Those infected with a national enthusiasm may insist on the importance of the early French writers or on the significance of the American "institutionalists," but essentially the orthodox view of what constitutes the "current" of economic thought remains unchanged.

2

Let us consider the hiatus in the orthodox view of economic development which appears between the publication of J. S. Mill's Principles of Political Economy and the appearance of the first major works of Jevons, of L. Walras, and of Menger. This is perhaps nowhere so clearly evident as in Mithcell's delightful presentation.[1] He argues effectively that ". . . in very large measure the important departures in economic theory have been intellectual responses to changing current problems."[2] Smith rationalized the trend of contemporary businessmen away from the restrictions of the State imposed by the Mercantilists, and Ricardo reflected the great concern of practical men in the England of his day with the problem

1. W. C. Mitchell, Lectures on Current Types of Economic Theory, unpublished lectures delivered at Columbia University, 1935.
2. Ibid., p. 1.

of sharing the national product between the landed aristocracy and the vigorous and growing business class.

Now this goes well until we observe even casually the trends in economic theory after Mill. As Mitchell puts it,

> "After John Stuart Mill's time economics seemed for a considerable while to lose a large measure of its contemporary interest. It retired more or less into academic halls. It was cultivated primarily by professors. It became more and more a specialty with its own body of erudition with which anyone who wanted to discuss economic problems had to make himself familiar before he ventured to express an opinion in public. And as economics took on this character it seemed in good part at least to change from being a vigorous criticism of current economic practices, as it had been notably in the cases of Adam Smith, of Malthus, of Ricardo, of John Stuart Mill, and to be in danger of becoming a defense in some sense at least, of the existing situation."[1]

Mitchell goes on to say that the economics of the latter half of the nineteenth century and the early part of the twentieth century is an intellectual reaction which reflects, on the whole, a satisfaction with the economic system, at any rate in the west of Europe and in America. As we pass from volume one to volume two of Mitchell's lectures, we find his insistence upon the close relationship between trends in economic theory and vital matters of public policy disappearing altogether.

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1. <u>Ibid.</u>, p. 4.

A better case can no doubt be made than the one which Mitchell makes for such a connection after 1860. It has been suggested (partly in jest) that the interest which economists of the first rank showed after 1870 in problems of subjective evaluations of consumers and in problems of welfare could be interpreted as a reflection of the growing preoccupation of men of practical affairs in "selling" their product. Similarly, we can interpret recent developments in "macro-economic" studies as the theoretical efforts of academicians to solve the very practical problem of unemployment for public administrators and corporation executives. Somehow Mitchell's thesis has a ring of truth when applied to the early study of "political economy." It sounds far-fetched when applied to the somewhat formalized, more or less rigorous analysis of recent times. Let us inquire briefly why this may be.

3

The subject matter of our field changed at about the time people stopped calling it political economy and started calling it economics. This change in names was really an explicit recognition of what henceforth was to be the primary concern of the economist--the study of pure theory as distinguished from "applied" economics. This is not to say that the theorist was not, on the one hand, to theorize after careful observation of pertinent phenomena, nor did it mean, on the other hand, that theoretical conclusions were not to be used for practical guidance in affairs. This is not to imply, either, that the economist from, say, 1870 on was to spend the greater part of his efforts in "pure" as distinguished from "applied" economics.[1] But when we have a clear recognition of the importance of defining the "economic problem" as distinguished from more or less well-ordered speculation as to the causes of the wealth of nations (or of particular social classes within nations) there has been, to say the least, a marked change in emphasis.

It was perhaps the great contribution of the English classical writers that they _tried_ for the first time to create a systematic body of principles, a true theoretical framework, which would enable them to predict scientifically. That they failed can scarcely be doubted. How often in a course in modern theory does a professor of economics refer to Smith or Ricardo or Mill? That they asked at least some of the right questions cannot be denied. That they furnished but few of the right answers must be admitted.

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1. That there can never be any carefully drawn line is obvious. See in this connection Boulding, _op.cit._, p. 190.

Their failure must be attributed, as has been indicated previously, to a refusal to adopt a proper methodology. Because they at no time borrowed from the field which might have enabled them to consider quantitative relationships adequately they never found it necessary to state explicitly their assumptions, to define their terms, or even to indicate in unequivocal terms what were the relationships between variables in their basic concepts. Thus we have no single instance of a statement of the concept of demand which would be expected today of the most disinterested student in a first course in economics. The question naturally arises as to why the Classical School so carefully avoided a mathematical method.

4

Speculation as to what might have been is ordinarily neither a very satisfying nor a very enlightening activity. There is some purpose, however, in an inquiry as to why those who dominated the field of political economy for nearly a century failed to use a technique which today is considered indispensable. Had mathematics not yet reached a stage where it could be put to practical use in economic study? Or did those whose interests lay within the broad province of "moral philosophy" have no inclination toward the more rigorous discipline of mathematics? Let us answer the latter question first.

Whatever may be our private views as to who made the first "systematic inquiry" in our field it must, I think, be admitted that Adam Smith's constitutes the first one universally considered important. The method which he used was to be the only orthodox one for a hundred years. Now Adam Smith was apparently well trained in mathematics, and he maintained some interest in the subject throughout his lifetime. "A college friend, Dr. Archibald Maclaine, Minister of the Scottish Church at the Hague, told Dugald Stewart that Adam Smith's favorite pursuits at the University (of Glasgow) were Mathematics and Natural Philosophy, and Dugald Stewart himself remembered his father, who was Professor of Mathematics at Edinburgh, discussing a particular problem which had been proposed to Simson when their acquaintance began."[1] As Dean of the Faculty at the University of Glasgow, Smith was deputed for the task of clearing accounts. "There were several distinguished mathematicians amongst the professors, but a special technique was required which Adam Smith had mastered."[2] Dugald Stewart has called attention to his remarkable retentiveness of memory,[3]

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1. William Robert Scott, Adam Smith as Student and Professor, Glasgow: Jackson, Son and Company, 1937, p. 34.
2. Ibid., p. 75.
3. Ibid., p. 77.

and it is hardly likely that any subject which he ever seriously pursued would be soon forgotten.

An inspection of the catalog of Smith's library reveals something over a score of works on mathematical subjects, and these included Maclaurin's Treatise on Fluxions, Newton's Treatise on the Method of Fluxions and Infinite Series, Thomas Simpson's Doctrine and Application of Fluxions, and Matthew Stewart's Some General Theorems of Considerable Use in the Higher Parts of Mathematics.[1] It is known that he was a great admirer of Newton and that he considered his Principia a high mark in human thinking. Smith's own early works, The History of Ancient Physics and History of Astronomy, the latter running through Descartes and Newton, suggest more than a casual familiarity with mathematical method.[2]

Perhaps even more significant is the fact that several of the works of the mathematical writers mentioned in previous chapters were in Smith's library. Cesare Beccaria's chief work Traité des delits et des peines, translated from the Italian by the Abbé André Morellet and probably a gift from Morellet, is included.[3] It contained no mathematical notation, but I mention

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1. James Bonar, A Catalog of the Library of Adam Smith, London: Macmillan and Co., 1932, pp. 107, 122, 167, and 180.
2. See Francis W. Hirst, Adam Smith, New York: The Macmillan Co., 1904, pp. 16-18.
3. Bonar, op.cit., p. 21.

it to suggest the possibility of Smith's acquaintance with other of Beccaria's writings. The second edition of Forbonnais' Élémens du commerce was in the library, though the entry suggests that the second part which contained the simple algebraic of money and the determination of foreign exchange rates was missing.[1] But Samuel Gale's Essay on the Nature and Principles of Public Credit was owned by Smith together with both volumes of Achille Nicolas Isnard's Traité des richesses, and it will be recalled that these two works were outstanding among mathematical treatments before Cournot.[2]

We cannot be sure that Smith had in fact read either Gale or Isnard. We can be certain, however, that he had been exposed to everything that Francis Hutcheson wrote, for as everyone knows Hutcheson was Smith's predecessor in the chair of Moral Philosophy at Glasgow from 1730-1746 and Smith's own much-admired teacher. One of the works mentioned in the brief section on Hutcheson in Chapter VII, An Essay on the Nature and Conduct of the Passions and Affections, with Illustrations on the Moral Sense, is listed in the Smith catalog.[3] An Inquiry into the Original of Our Ideas of Beauty and Virtue

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1. Ibid., p. 70.
2. Ibid., p. 72 and p. 157.
3. Ibid., p. 92.

does not appear in the catalog, but we know that Smith was quite familiar with its contents from his references to it in The Theory of Moral Sentiments.[1]

In short it seems altogether probable that Adam Smith was not only sufficiently well trained to carry on independently a mathematical analysis at the level required in economic reasoning but that such a method had almost certainly been suggested to him in his general reading. And what is true of Smith seems equally true with respect to Ricardo, Bentham, and the two Mills, who might have successfully introduced a more rigorous methodology because of their vast prestige. These men were not mathematicians. We do know, however, that both Bentham and Ricardo were sufficiently schooled in mathematics to be capable of carrying on independent researches in the physical sciences.[2] John Stuart Mill began the study of

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1. Adam Smith, The Theory of Moral Sentiments, London: G. Bell and Sons, Ltd., 1911, p. 472.
2. See Charles Warren Everett, The Education of Jeremy Bentham, New York: Columbia University Press, 1931, pp. 46, 49, 59, and 62-63. See also David Ricardo, Works, edited by J. R. McCulloch, p. xvii. Bentham came very close, of course, to a mathematical method in the development of his felicific calculus. In the work on Bentham just cited, p. 49, we are told that as a young man he ". . . determined to do for human society what Newton had done for natural science," and that he wanted to ". . . apply to legislation scientific principles, and discover for social engineering a mathematical calculus. . . ."

algebra at the age of eight. He sums up his mathematical training in these words.

> "I learnt elementary geometry and algebra thoroughly, the differential calculus, and other portions of the higher mathematics far from thoroughly: for my father, not having kept up this part of his early acquired knowledge, could not spare time to qualify himself for removing my difficulties, and left me to deal with them, with little other aid than that of books: while I was continually incurring his displeasure by my inability to solve difficult problems for which he did not see that I had not the necessary previous knowledge." [1]

It appears certain then that there were no insuperable obstacles to the use of mathematical method by Smith, or for that matter by Ricardo or the Mills, though when Smith adopted his literary, philosophical method it was foreordained that his followers should. The possibility of another kind of approach was apparently suggested. But Smith was interested primarily in practical affairs. He was essentially a commentator rather than a theorist. Political arithmetic had proved of little use, and the method of the speculative philosopher was more than adequate to a man whose training had been in philosophy and whose aim was to propound a philosophy of government and to describe in general terms the way in which nations became wealthy.

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1. J. S. Mill, Autobiography, New York: Henry Holt and Co., 1873, p. 12. Mill remarks later (p. 113) that he was inspired by certain readings in biography, "above all" by Condorcet's Life of Turgot which, it will be recalled, contained a mathematical analysis of Turgot's theories of taxation.

5

At the beginning of the last section the question was raised as to when mathematics may have reached a stage where it could be put to use in economic study. Now two basic concepts were necessary to modern analysis in pure economic theory. Economists had to have the idea of the calculus and they needed a clear concept of what constituted a function.

We are not interested in any disputation regarding the real discoverer of the calculus, though it does appear that the influence of the Newtonian concept of "fluxions" and his dot-notation did obscure some aspects of the study for Englishmen. The basic idea, however, was made available to mathematicians in general as early as 1696 with the appearance in Paris of Guillaume l'Hospital's _Analyse des infinement petits._[1] Prior to this year only four men had a knowledge of the calculus--Newton, Leibniz, Jacques Bernoulli, and Jean Bernoulli.[2] After the publication of l'Hospital's _Analyse_ the subject was widely studied, and by the middle of the eighteenth century the basic principles

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1. Florian Cajori, _A History of Mathematics_, New York: The Macmillan Co., 1919, p. 224.
2. Vera Sanford, _A Short History of Mathematics_, Boston: Houghton Mifflin Co., 1930, p. 321.

were well established, although confusion as to notation persisted into the nineteenth century.[1]

There seems to be general agreement that Leibniz was the first to use the word "function" although the function concept is implicit in the works of Descartes and Fermat.[2] The credit for introducing the familiar f (x) notation goes to Euler, who used it as early as 1734.[3] In 1748 Euler defined a function as follows: A function of a variable quantity is an analytical expression composed in some way of that variable quantity and of numbers or constant quantities.[4] "Lagrange in his Theorie des functions analytiques (1797) extended Euler's notation by using f, F, ϕ, X, etc., followed by parentheses to designate functions. He defines a function as 'a property of a series of powers of the independent variable'."[5]

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1. It will be recalled that Peronnet Thompson used the dot-notation in his work rather than the d-notation, which by the time of his writing was fairly generally used.
2. Herbert R. Hamley, Relational and Functional Thinking in Mathematics, New York: Bureau of Publications, Teachers College, Columbia University, 1934, p. 13.
3. Cajori, op.cit., p. 234.
4. Hamley, op. cit., p. 13. This definition appeared in Euler's Introduction in Analyson Infinitorum.
5. Ibid., p. 14.

It is interesting to note that further developments in the subject came from physicists who found that "while working with physical facts, . . . existing mathematical tools were inadequate for their purposes."[1] Apparently the physicists needed a more "general type of correspondence than any previously propounded."[2] But it was G. Lejeune Dirichlet, who, for the first time ". . . gave the word function a significance independent of any assumption of the possibility of an analytical representation."[3] Or to put the matter in other words, ". . . a definition was finally formed that entirely cast aside the old trammels under which a functional relation was dependent for existence on a mathematical formula."[4]

At the risk of making this section unduly long I shall quote the Dirichlet definition, for its importance to economists is considerable.

> "Let by a and b be understood two fixed values and by x a variable quantity, which gradually assumes all values lying between a and b. Now, if a single finite y correspond to every x, in such a way that while x continuously passes through the interval a to b,

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1. Idem.
2. Idem.
3. Ibid., p. 15. Quoted from Dini, U., Journal fur Mathematischen, Vol. IV, p. 157.
4. R. P. Richardson and E. H. Landis, Fundamental Conceptions of Modern Mathematics--Variables and Quantities, Chicago and London: The Open Court Publishing Co., 1916, pp. 181-182.

> y = f (X) likewise varies gradually, then y is called a continuous function for this interval. It is quite unnecessary that y in this entire interval should be dependent upon x according to the same law; indeed, we need never think of a dependence expressed in terms of mathematical operations."[1]

And giving the function a graphical interpretation he continues:

> "Considering x and y as abscissae and ordinates, a continuous function appears as a connected curve in which only one point corresponds to every abscissa between a and b. This definition does not attribute to the various parts of the curve a common law. We can think of the curve as made up of heterogeneous parts or as described entirely without law. Thus a function is to be regarded as completely determined for an interval, only if it is defined graphically for the whole extent of the interval or is subjected to mathematical laws valid for the several parts of the interval."[2]

Mathematicians have criticized the definitions adversely on grounds which need not be examined here. It is sufficient for our purposes to note that this definition of a function has been accepted with only minor modifications ever since.

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Thus, while the calculus had been available to economic thinkers since late in the seventeenth century, a definitive

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1. Hamley, <u>op. cit.</u>, p. 15.
2. <u>Idem.</u>

concept of what constituted a function was not presented until well into the nineteenth century. As we have seen there were three attempts to apply the calculus to economic analysis prior to 1838 and one clear use of implicit functions, the latter appearing some thirteen years before the publication of Dirichlet's definition of a function in 1829. Important as these contributions may be, taken as isolated instances, there still remained the task of incorporating the ideas in a logical and comprehensive framework.

Looking back it does not seem strange that the first great book in pure economic theory should have appeared after the relevant mathematical questions had been well settled. Nor is it surprizing that such a work should have come from the mind of a man highly competent as a mathematician who had received a training which was superb. After a year at the École normale Cournot carried on his studies at the Sorbonne where he, together with his close friend Dirichlet, was a favorite student of Lacroix and Hachette, who were, respectively, disciples of Condorcet and Monge. Both Cournot and Dirichlet became acquainted with the principal French scholars of the time, among them the greatest mathematician in France, Laplace.[1]

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1. H. L. Moore, "Antoine-Augustin Cournot," Revue de Métaphysique et de Morale, treizième année, 1905, pp. 528-532.

We should indeed like to find some close connection between Cournot and the economists whose works have been treated in the preceding pages. Only two connections can be made. He was a student at the École normale in 1821 and 1822 with Auguste Walras. It has already been mentioned that he had read Canard's Principes. Aupetit states that

> ". . . neither the unpublished memoirs of Augustin Cournot nor his prefaces tell us by what avenues his thought was directed toward economic problems . . . . Incidentally, the author informs us that he had read Smith and some others, but truly his conception is so different, his manner so original, some of his results so new, that he owes singularly more to himself than to his predecessors."[1]

With the materials at present available it seems impossible to say more regarding Cournot's debt to his predecessors. The three essentials of his analysis--the calculus, the use of implicit functions in the statement of economic relations, and the use of geometry to express two-variable functions--had appeared before Cournot. It is more than possible that of all the authors mentioned he had read only Canard, from whom he could have gotten the direction of thought which led to the Recherches.

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1. Albert Aupetit, "L'oeuvre économique de Cournot," Revue de Métaphysique et de Morale, treizième année, 1905, p. 377.

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What place in the history of economics may we assign to these men? They are almost unknown. Their names do not appear in the general textbooks in the history of thought. Years of attention to their lives and works may gradually yield facts which will link certain of them to the better-known figures, and for that matter to each other. They stand now, however, as more or less isolated figures who cannot be said to have contributed to a current of thought because there is no discernible flow.

Yet the sporadic nature of their contributions does not distract from their importance as precursors of a pure science. In order to make our textbooks and our classroom lectures unfold smoothly we have emphasized continuity at the sacrifice of what is significant. We should probably find half a dozen pages of Buquoy more helpful to an understanding of modern economics than nearly the whole of Ricardo's *Principles*. And the justification for the study of the development of economics must lie in the light thrown on the subject as it stands today. There cannot be profit in a study of matters which are "of historical interest only."

As it was once necessary to recognize the contribution of Cournot and Gossen, it now appears proper to give to the earliest mathematical economists the credit which is due them. They saw more or less clearly what their more distinguished contemporaries could not see at all--the need for a method which would make possible a true theoretical analysis. But more than this they provided the ideas for that analysis.

It has been well remarked that ". . . most ideas when they become propelling ideas are not actually new."[1] Regarding economic doctrines, Frank Fetter has noted that various ideas reappear ". . . like the pieces of colored glass in a kaleidoscope, the same essentially in detail, but in ever-changing settings."[2] But the kaleidoscopic nature of the early development of ideas must not cause us to overlook the importance of the first contribution, however tentative and uncertain. By studying ideas at their inception we are most helpfully instructed. The magnitude of the contribution of the thinker who makes the strategic synthesis, the final successful innovation, must not cause us to forget what lesser men, taken together, have accomplished.

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1. Fritz Redlich, The Molding of American Banking, Men and Ideas, New York: Hafner Publishing Co., Inc., 1947, p. 2.
2. Frank Fetter, "Lauderdale's Oversaving Theory," The American Economic Review, vol. 35, p. 283.

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APPENDIX A

The Jevons Bibliography

LIST OF MATHEMATICO-ECONOMIC BOOKS, MEMOIRS, AND OTHER PUBLISHED WRITINGS

Remarks upon the purpose and some of the contents of this list will be found in the preface to the second edition of this book.

Works unknown to the Author are placed in square brackets.

The criticisms and summaries in small type are by the author unless the contrary is indicated, thus (Ed.). At the pages indicated will be found notices of the particular works to which the references are attached.

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Brisson was a profound mathematician as well as an eminent engineer. He appears to have presented to the Institut in this year, in conjunction with his friend Dupuis de Torcy, an Essai, part of which has been printed in the 14e Cahier du Journal de l'Ecole Polytechnique. Other portions of the memoir, however, are stated never to have been published, including the first part, which contained "des considérations d'économie publique appliquée aux projets de communications nouvelles." See "Essai sur le systeme general de navigation interieure de la France," by B. Brisson, Paris, 1829, p. ii.
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APPENDIX B

The Fisher Bibliography

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APPENDIX C