THE CHARACTERIZATIONS OF A CLASS OF TRANSFORMATIONS AND OF A CLASS OF DIFFERENTIABLE FUNCTIONS

by

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INTRODUCTION

It is the purpose of this paper to present the notion of a certain generalized derivative which has been defined by Professor G. B. Price, and to characterize the class of mapping transformations which possess a non-zero generalized derivative of this type at all points of an open, connected set, E.

In the theory of functions of a single real variable, one of the basic concepts is that of the derivative of a function f(x) with respect to the variable x. When discussing a function of several real variables, $f(x^{(1)}, \ldots, x^{(n)})$, the notion of a partial derivative of f with respect to one of the variables, $x^{(i)}$, is fundamental.

If one thinks of the function f(x) as mapping a line segment (x values) onto another line segment with values f(x), a geometric interpretation of the derivative of the function f(x) with respect to the variable x at a point x_0 is that of the limit of the ratios of the signed lengths $f(x)-f(x_0)$ and $x - x_0$ as x is allowed to become arbitrarily close to x_0.

With this interpretation of the derivative in mind, Professor G. B. Price has defined a generalized derivative of a mapping function

$F(x)$: $f^{(i)}(x)$, $(i = 1, \ldots, n;\ x = (x^{(1)}, \ldots, x^{(n)}))$, where F is defined in n-dimensional Euclidean space, $R^{(n)}$. In the n-dimensional case, the increments considered are those oriented n-cells, $\Delta(x_0 x_1 \ldots x_n)$ determined by the n + 1 points, $x_0, x_1, \ldots, x_n$. (For n = 2, the 2-cells are triangles.) The volume of such an n-cell is given by

$$\Delta(x_0 x_1 \ldots x_n) = \frac{1}{n!}\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & \ldots & x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n^{(1)} & x_n^{(2)} & \ldots & x_n^{(n)} & 1 \end{vmatrix}.$$

Under the mapping F, the vertices, $x_0, x_1, \ldots, x_n$, are transformed into the points $F(x_0), F(x_1), \ldots, F(x_n)$, which also form the vertices of an n-cell, $\Delta(F{:}x_0 x_1 \ldots x_n)$, with volume given by the expression

$$\Delta F({:}x_0 x_1 \ldots x_n) = \frac{1}{n!}\begin{vmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & \ldots & f^{(n)}(x_0) & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{(1)}(x_n) & f^{(2)}(x_n) & \ldots & f^{(n)}(x_n) & 1 \end{vmatrix}.$$

As in the one-dimensional case, the ratios of the signed volumes of the two increments

$$\frac{\Delta(F{:}x_0 x_1 \ldots x_n)}{\Delta(x_0 x_1 \ldots x_n)} = \frac{\begin{vmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & \ldots & f^{(n)}(x_0) & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{(1)}(x_n) & f^{(2)}(x_n) & \ldots & f^{(n)}(x_n) & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & \ldots & x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n^{(1)} & x_n^{(2)} & \ldots & x_n^{(n)} & 1 \end{vmatrix}}$$

are considered and the limit is taken as the points $x_1, x_2,$

. . . , x_n are allowed to become arbitrarily close to x_0, with certain restrictions on the points x_1, x_2, . . . , x_n. If this limit exists and is finite, then the derivative of F: $f^{(i)}(x)$, $(i = 1, . . . , n)$ with respect to $x = (x^{(1)}, . . . , x^{(n)})$ is said to exist at the point x_0. This derivative is denoted by $D_x F|_{x_0}$.

One restriction on the points x_1, . . . , x_n is of course that $\Delta(x_0 x_1 . . . x_n) \neq 0$. Another restriction on the points x_1, . . . , x_n is that these points together with the point x_0 must always form an increment of a designated class while the limit is being taken. It happens that $D_x F|_{x_0}$ may exist with respect to one class of increments but not with respect to another. Three classes of increments, denoted by I_1, I_2, and I_3, are defined, but only one is the object of discussion in the paper. That is the class of increments I_1, which is composed of all the increments $\Delta(x_0 x_1 . . . x_n)$ which have n-dimensional volume not equal to zero. Thus, if the class I_1 is being considered, then the points x_0, x_1, . . . , x_n must always form an n-cell whose n-dimensional volume is not zero as x_1, x_2, . . . , x_n are allowed to become arbitrarily close to x_0.

A precise definition of the generalized derivative, $D_x F$, of a mapping function F (defined on a region E of $R^{(n)}$) at a point x_0 with respect to a certain class of increments I is the following:

Let F be a mapping function defined on a region E of $R^{(n)}$. The derivative of F with respect to the class of increments I exists at a point x_o of E and equals d if for every sufficiently small $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{\Delta(F:x_o x_1 \ldots x_n)}{\Delta(x_o x_1 \ldots x_n)} - d \right| < \epsilon$$

for all $\Delta(x_o x_1 \ldots x_n)$ in I such that $\|x_o x_i\| < \delta$, $(i = 1, \ldots, n)$, where the symbol $\|x_o x_i\|$ denotes the distance between the points x_o and x_i.

It is the purpose of this paper to characterize the class of mapping functions, $F: f^{(i)}(x)$, $(i = 1, \ldots, n)$, defined on a region E of $R^{(n)}$, which possess a non-zero derivative, $D_x F$, with respect to the class of increments I_1, at each point of E.

In Chapter I the above definition of the generalized derivative is given and three classes of increments are defined. A special case of the generalized derivative is found by setting $f^{(k)}(x) = x^{(k)}$, $(k = 1, \ldots, i-1, i+1, \ldots, n)$, and letting $f^{(i)}(x) = f(x)$. Then the ratios

$$\frac{\begin{vmatrix} x_0^{(1)} & \cdots & x_0^{(i-1)} & f(x_0) & x_0^{(i+1)} & \cdots & x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n^{(1)} & \cdots & x_n^{(i-1)} & f(x_n) & x_n^{(i+1)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n^{(1)} & x_n^{(2)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}}$$

are considered. The limit is taken with respect to a certain class of increments I. If this limit exists at x_0 then it is called the derivative of f(x) with respect to $x^{(i)}$ at the point x_0 with respect to the class of increments I and is denoted by $D_{x^{(i)}} f \big|_{x_0}$.

The classes of functions, $f(x)$, $(x = (x^{(1)}, \ldots, x^{(n)}))$ which have generalized derivatives $D_{x^{(i)}} f$ with respect to the classes of increments I_2 and I_3 are discussed. These classes of functions have been characterized by G. B. Price. The class of functions, $f(x^{(1)}, \ldots, x^{(n)})$, which have derivatives, $D_{x^{(i)}} f$ with respect to the class I_2 is the class of Stolz differentiable functions. A function f(x) is Stolz differentiable at a point x_0 if there exist constants a_i $(i = 1, \ldots, n)$, such that

$$f(x^{(1)}, \ldots, x^{(n)}) - f(x_0^{(1)}, \ldots, x_0^{(n)}) = \sum_{i=1}^{n} a_i (x^{(i)} - x_0^{(i)}) + r\left[\epsilon(r)\right],$$

where $r = \sqrt{\sum_{i=1}^{n} (x^{(i)} - x_0^{(i)})^2}$ and $\epsilon(r)$ is a function of r such that $\lim_{r \to 0} \epsilon(r) = 0$.

The class of functions $f(x^{(1)}, \ldots, x^{(n)})$ which have derivatives $D_{x^{(i)}}f$ with respect to the class of increments I_3 is the class of those functions which have ordinary partial derivatives.

The important Moore-Smith Limit, due to E. H. Moore and H. L. Smith, is discussed and it is shown that the generalized derivative $D_x F$ of a mapping function F is a Moore-Smith Limit.

To conclude Chapter I, a useful theorem concerning the derivative is proved. This theorem is a generalization of a theorem of Stieltjes for a function of a single variable, which states that if $\left.\frac{df}{dx}\right|_{x_0}$ exists, then,

$$\left.\frac{df}{dx}\right|_{x_0} = \lim_{\alpha, \beta \to x_0} \frac{f(\alpha) - f(\beta)}{\alpha - \beta}$$

where x_0 is always between α and β.

Chapter II is not concerned with generalized derivatives. The main purpose of this chapter is to show that the precise class of transformations,
$F: f^{(i)}(x)$, $(i = 1, \ldots, n)$, which are continuous, one-to-one and map points of a straight line into points of a straight line are the linear fractional transformations:

$$F: f^{(i)}(x) = \frac{a_{i,1} x^{(1)} + \ldots + a_{i,n} x^{(n)} + a_{i,n+1}}{a_{n+1,1} x^{(1)} + \ldots + a_{n+1,n} x^{(n)} + a_{n+1,n+1}}, \quad (i = 1, \ldots, n),$$

where

$$\begin{vmatrix} a_{1,1} & \cdots & a_{1,n+1} \\ \cdot & \cdots & \cdot \\ a_{n+1,1} & \cdots & a_{n+1,n+1} \end{vmatrix} \neq 0.$$

In order to prove this characterization for n-dimensional Euclidean space, the notions of linearly independent points and p-flats are important. The points $x_k = (x_k^{(1)}, \ldots, x_k^{(n)})$ of $R^{(n)}$, $(k = 0, \ldots, p; p \leq n)$ are said to be linearly independent if at least one of the $C(n,p)$ determinates of the form

$$\begin{vmatrix} x_0^{(i_1)} & \cdots & x_0^{(i_p)} & 1 \\ \cdot & \cdots & \cdot & \cdot \\ x_p^{(i_1)} & \cdots & x_p^{(i_p)} & 1 \end{vmatrix}$$

is different from zero, where $(i_1, \ldots, i_p)$ represents one of the $C(n,p)$ possible selections of p of the n columns of the matrix

$$\begin{pmatrix} x_0^{(1)} & \cdots & x_0^{(n)} \\ \cdot & \cdots & \cdot \\ x_p^{(1)} & \cdots & x_p^{(n)} \end{pmatrix},$$

Geometrically, this means that the p-cell determined by the points $x_0, \ldots, x_p$ has p-dimensional volume different from zero.

An equivalent definition is the following: The $p + 1$ points, $x_0, \ldots, x_p$, are said to be linearly independent if the vectors, $(x_i - x_0)$, $(i = 1, \ldots, p)$, are linearly independent in the ordinary sense.

The notion of a p-flat in Euclidean n-dimensional space has been used by many authors. Let $x_0, \ldots, x_p$ be p + 1 linearly independent points of $R^{(n)}$. By the p-flat, S_p, determined by these points is meant the set of points $x \in R^{(n)}$ such that

$$x = \sum_{i=0}^{p} \alpha_i x_i, \qquad \sum_{i=0}^{p} \alpha_i = 1.$$

An equivalent definition is the following:
Let $x_0, \ldots, x_p$ be p + 1 linearly independent points of $R^{(n)}$. By the p-flat, S_p, determined by these p + 1 linearly independent points is meant the set of all points x of $R^{(n)}$ such that the vectors $(x - x_0)$ satisfy the relation

$$(x - x_0) = \sum_{i=1}^{p} \beta_i (x_i - x_0),$$

with no restrictions on the β's.

A p-flat is a direct generalization of a line and a plane in 3-dimensional space. The name, 'p-flat', was taken from D. M. Y. Sommerville's book, An Introduction to the Geometry of N-Dimensions. (See the Bibliography at the end of this paper.) Alexandroff and Hopf, and Birkhoff and MacLane are other authors who discuss p-flats, although not under the name p-flat.

If $x_0, \ldots, x_p$ are any p + 1 linearly independent points of $R^{(n)}$, the set of points x such that

$$x = \sum_{i=0}^{p} \alpha_i x_i, \qquad \sum_{i=0}^{p} \alpha_i = 1, \qquad \alpha_i \geq 0,$$

is the p-cell, $\Delta(x_0 x_1 \ldots x_p) = \Delta x_p$.

Among the properties of p-flats and p-cells which are presented, the following are perhaps the most important to this paper:

1. A p-flat is isometric to the Euclidean space, $R^{(p)}$, and hence is p-dimensional.

2. If x is any interior point of a p-cell, Δx_p (relative to the p-flat, S_p, in which Δx_p lies), then a straight line through x, lying in S_p, intersects the boundary of Δx_p in precisely two points.

3. If F is a continuous, one-to-one transformation defined on a convex region E of $R^{(n)}$ which takes straight lines into straight lines and if $\Delta(x_0 x_1 \ldots x_p) = \Delta x_p$ is any p-cell of E, then F maps the k-dimensional faces of Δx_p, $(k \leq p)$, into distinct k-dimensional faces of a p-cell, ΔF_p, where by the k-dimensional face of Δx_p, determined by the points $x_0, \ldots, x_k$ chosen from the p + 1 vertices of Δx_p, is meant the set of all points x such that

$$x = \sum_{i=0}^{k} \beta_i x_i, \qquad \sum_{i=0}^{k} \beta_i = 1, \qquad \beta_i \geq 0.$$

4. If F is a continuous, one-to-one transformation defined on a convex region E of $R^{(n)}$, the necessary and sufficient condition that p-flats map into p-flats (p fixed; $1 \leq p \leq n-1$) is that straight lines map into straight lines.

After the properties of p-flats and p-cells are presented, one important lemma is proved before the characterization can be completed. This lemma, suggested by W. Kaplan of the University of Michigan, is the following:

Lemma. Let $x_0, \ldots, x_n$ be $n + 1$ linearly independent points in a convex region E of $R^{(n)}$, which form the vertices of an n-cell, Δx_n. Let x^* be the intersection of the medians of Δx_n. Let G: $g^{(i)}(x)$, $(i = 1, \ldots, n)$, be a transformation defined on E which is continuous, one-to-one and carries p-flats into p-flats (p fixed; $1 \leq p \leq n-1$), and which leaves the points $x_0, \ldots, x_n$, x^* fixed. Then G is the identity transformation.

Using this lemma, the main theorem of the chapter is proved; namely, that the class of transformations which are continuous, one-to-one, and map p-flats into p-flats (p fixed; $1 \leq p \leq n-1$) is the class of linear fractional transformations.

In Chapter III the generalized derivatives, $D_x F$, of a transformation F with respect to the class of increments I_1 is once again the topic of discussion. It is shown that if F: $f^{(i)}(x)$, $(i = 1, \ldots, n)$, is a transformation defined on a convex region E of $R^{(n)}$, such that $D_x F$ exists with respect to the class of increments I_1 and is different from zero at each point of E, then F is continuous, maps points of a straight line into points of a straight line and is one-to-one. It is concluded from the results of Chapter II that F must be linear fractional. The results are extended to the case in which E is any open connected set. It is also shown that if F is linear fractional, then $D_x F$ exists with respect to the class of increments I_1 and is different from zero. Hence, the precise class of transformations F defined on a region E of $R^{(n)}$ for which the generalized derivative, $D_x F$, exists with respect to the class of increments I_1 and is different from zero at each point of E, is the class of linear fractional transformations.

In carrying through the characterization, the results are first obtained for two dimensions and then extended to the n-dimensional case. While this is unnecessary in most cases, it is felt that a clearer understanding is obtained by organizing the developments in this way.

The similarity between generalized derivatives and Jacobians of a mapping function should be noted. Expecially prominent in this similarity is Theorem I.4.14, which states that if F: $f^{(i)}(x)$, $(i = 1, \ldots, n)$ is a transformation such that $D_{x^{(i)}} f^{(j)}$, $(i, j = 1, \ldots, n)$ exist at a point x_0 with respect to a class of non-zero increments I, then $D_x F|_{x_0}$ exists with respect to the class of increments I and equals

$$\begin{vmatrix} D_{X^{(1)}} f^{(1)}\Big|_{X_0} & \cdots & D_{X^{(n)}} f^{(1)}\Big|_{X_0} \\ \cdots & \cdots & \cdots \\ D_{X^{(1)}} f^{(n)}\Big|_{X_0} & \cdots & D_{X^{(n)}} f^{(n)}\Big|_{X_0} \end{vmatrix}$$

There have been other generalizations of the derivative. The generalization given in this paper can be compared with the generalized Jacobian introduced by Banach and with Burkill's modified Jacobian (see the Bibliography). However, in Banach's generalization (for the plane) squares are considered as increments, while in Burkill's generalization, four points in the plane form the vertices of the increments considered. It seems a more natural generalization to consider triangles in the plane as increments. In the one-dimensional case, (the ordinary case of the derivative of a function of a single variable), the increments considered are determined by two points -- one more than the dimension of the space. It seems natural then to consider triangles in the plane, tetrahedra in 3-dimensional space, and in general, n-cells in n-dimensional space as increments. Theorems in the theory of determinants can also be readily used in such a generalization.

For more complete information concerning p-flats, p-cells and their properties, one should study the references to Lefschetz, Alexandroff-Hopf, and Kerékjártó which are given in the bibliography. For more complete information concerning the theory of determinants, see the references to Kowalewski, Aitken and Price.

CHAPTER I

GENERALIZED DERIVATIVES AND THE MOORE-SMITH LIMIT

I.1. DEFINITIONS OF GENERALIZED DERIVATIVES AND CLASSES OF INCREMENTS

I.1.1. In the theory of functions of a single real variable, the derivative of a function, $f(x)$, with respect to the variable x at a point x_o is defined to be

$$\lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o} = \lim_{x \to x_o} \frac{\begin{vmatrix} f(x_o) & 1 \\ f(x) & 1 \end{vmatrix}}{\begin{vmatrix} x_o & 1 \\ x & 1 \end{vmatrix}}$$

provided this limit exists. One interpretation of the derivative of $f(x)$ at the point x_o is that of the limiting position of the secant line through $f(x)$ and $f(x_o)$, the limit being taken as x approaches x_o. (See Fig. 1.)

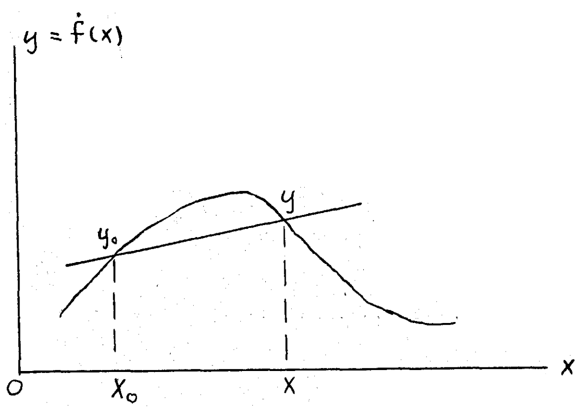

Fig. 1

A natural generalization of the derivative, in the light of this geometric interpretation is the following: Consider the function $z = f(x)$, where $x = (x^{(1)}, x^{(2)})$, the superscripts denoting coordinates of the point x. Let this function be defined on some set E in $R^{(2)}$, for example, on an open, convex set. The locus of $z = f(x)$ is a surface in $R^{(3)}$. Let $x_0 = (x_0^{(1)}, x_0^{(2)})$ be a fixed point of E, and $x_1 = (x_1^{(1)}, x_1^{(2)})$ and $x_2 = (x_2^{(1)}, x_2^{(2)})$ be two nearby points of E. The points $\left(x_0^{(1)}, x_0^{(2)}, f(x_0)\right)$, $\left(x_1^{(1)}, x_1^{(2)}, f(x_0)\right)$, and $\left(x_2^{(1)}, x_2^{(2)}, f(x_2)\right)$ lie on the surface, $z = f(x)$. The equation of the secant plane through these three points is given by

$$\text{(I.1.2)} \qquad \begin{vmatrix} x^{(1)} & x^{(2)} & f(x) & 1 \\ x_0^{(1)} & x_0^{(2)} & f(x_0) & 1 \\ x_1^{(1)} & x_1^{(2)} & f(x_1) & 1 \\ x_2^{(1)} & x_2^{(2)} & f(x_2) & 1 \end{vmatrix} = 0.$$

Expanding and solving for $f(x) - f(x\ )$, one obtains

$$\text{(I.1.3)} \quad f(x)-f(x_0) = \frac{\begin{vmatrix} f(x_0) & x_0^{(2)} & 1 \\ f(x_1) & x_1^{(2)} & 1 \\ f(x_2) & x_2^{(2)} & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_1^{(1)} & x_1^{(2)} & 1 \\ x_2^{(1)} & x_2^{(2)} & 1 \end{vmatrix}} (x^{(1)} - x_0^{(1)}) + \frac{\begin{vmatrix} x_0^{(1)} & f(x\ ) & 1 \\ x_1^{(1)} & f(x\ ) & 1 \\ x_2^{(1)} & f(x\ ) & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_1^{(1)} & x_1^{(2)} & 1 \\ x_2^{(1)} & x_2^{(2)} & 1 \end{vmatrix}} (x^{(2)} - x_0^{(2)}).$$

Allowing x_1 and x_2 to 'approach' x_0 in some manner, this secant plane approaches the tangent plane at $\left(x_0^{(1)}, x_0^{(2)}, f(x_0)\right)$, under certain conditions at least. The equation of the tangent plane at $\left(x_0^{(1)}, x_0^{(2)}, f(x_0)\right)$ is given by the expression

$$\text{(I.1.4)} \qquad f(x) - f(x_0) = \left.\frac{\partial f}{\partial x^{(1)}}\right|_{x_0}(x^{(1)} - x_0^{(1)}) + \left.\frac{\partial f}{\partial x^{(2)}}\right|_{x_0}(x^{(2)} - x_0^{(2)}).$$

Hence, it is logical to conclude that, under certain conditions at least,

$$\text{(I.1.5)} \qquad \lim_{x_1, x_2 \to x_0} \frac{\begin{vmatrix} f(x_0) & x_0^{(2)} & 1 \\ f(x_1) & x_1^{(2)} & 1 \\ f(x_2) & x_2^{(2)} & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_1^{(1)} & x_1^{(2)} & 1 \\ x_2^{(1)} & x_2^{(2)} & 1 \end{vmatrix}}$$

and

$$\text{(I.1.6)} \qquad \lim_{x_1, x_2 \to x_0} \frac{\begin{vmatrix} x_0^{(1)} & f(x_0) & 1 \\ x_1^{(1)} & f(x_1) & 1 \\ x_2^{(1)} & f(x_2) & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_1^{(1)} & x_1^{(2)} & 1 \\ x_2^{(1)} & x_2^{(2)} & 1 \end{vmatrix}}$$

are derivatives of a sort. They will be denoted by $D_{x^{(1)}} f\big|_{x_0}$ and $D_{x^{(2)}} f\big|_{x_0}$ respectively.

I.1.7. Another geometric interpretation of the definition of the derivative of a function of a single real variable is the following: The function $y = f(x)$ may be regarded as the mapping of a straight line (x values) onto another straight line (y values). The image of x_0 is y_0 and the image of a variable point x is y. The derivative of $f(x)$ at x_0 is the limit of the ratios of the signed lengths, $(y-y_0)$ and $(x-x_0)$, the limit being taken as x approaches x_0.

As a generalization of this interpretation, consider the mapping function $F: f^{(1)}(x), f^{(2)}(x)$, where $x = (x^{(1)}, x^{(2)})$*, defined on the oriented Euclidean plane. As increments in this case, one considers oriented triangles, denoted by $\Delta(x_0 x_1 x_2)$, or Δx, with vertices x_0, x_1, and x_2. These three points map into three points, $F(x_0) = \left(f^{(1)}(x_0), f^{(2)}(x_0)\right)$, $F(x_1) = \left(f^{(1)}(x_1), f^{(2)}(x_1)\right)$, and $F(x_2) = \left(f^{(1)}(x_2), f^{(2)}(x_2)\right)$, which also form the vertices of an oriented triangle, denoted by $\Delta(F: x_0 x_1 x_2)$ or by ΔF. The areas of these triangles are given by the expressions

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* Throughout the remainder of this paper, the n coordinates of a point x in n-dimensional Euclidean space, $R^{(n)}$, (n any positive integer), will be represented by superscripts; $x = (x^{(1)}, \ldots, x^{(n)})$, or as $x^{(j)}$, $(j = 1, \ldots, n)$. Two distinct points of $R^{(n)}$ will be distinguished by subscripts, as x_0 and x_1. If x_0 only is written, it is understood that $x_0 = (x_0^{(1)}, \ldots, x_0^{(n)}) = x_0^{(j)}$, $(j = 1, \ldots, n)$. The notations $(x_0^{(1)}, \ldots, x_0^{(n)})$ or $x_0^{(j)}$, $(j = 1, \ldots, n)$, will be used only when it is necessary to use the coordinates of a point in the proof of a theorem or to make the meaning of a statement more lucid. Otherwise, the notation x_0 will be used.

$$\text{(I.1.8)} \qquad \Delta(x_0 x_1 x_2) = \Delta x = \frac{1}{2!}\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_1^{(1)} & x_1^{(2)} & 1 \\ x_2^{(1)} & x_2^{(2)} & 1 \end{vmatrix}$$

and

$$\text{(I.1.9)} \qquad \Delta(F{:}x_0 x_1 x_2) = \Delta F = \frac{1}{2!}\begin{vmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & 1 \\ f^{(1)}(x_1) & f^{(2)}(x_1) & 1 \\ f^{(1)}(x_2) & f^{(2)}(x_2) & 1 \end{vmatrix} .$$

As in the one-dimensional case, the ratios of the signed areas of the two increments are examined and the limit is taken as x_1 and x_2 'approach' the fixed point x_0. If this limit exists and is finite, it is called the derivative of $F = (f^{(1)}, f^{(2)})$ at x_0 with respect to $x = (x^{(1)}, x^{(2)})$, and is denoted by $D_x F|_{x_0}$.

I.1.10. <u>Remark</u>. The generalized derivatives $D_{x^{(1)}}f$ and $D_{x^{(2)}}f$ of (I.1.5) and (I.1.6) are only special cases of the generalized derivative $D_x F$, for if one sets $f^{(2)}(x) = x^{(2)}$ in (I.1.9) and takes the limit of the ratios of (I.1.9) and (I.1.8), then (I.1.5) is obtained; and if one sets $f^{(1)}(x) = x^{(1)}$ in (I.1.9) and takes the limit of the ratios of (I.1.9) and (I.1.8), then (I.1.6) is obtained.

It should be further noted that in mapping the points of the plane onto another plane, it is not asserted that a triangle, Δx, is mapped into a triangle, ΔF, but only that the vertices of a triangle Δx are mapped into points which are the vertices of a triangle, denoted by ΔF. The value of ΔF depends entirely on the images of the vertices of Δx. However, in taking the limit, the ratio of the signed areas of the two triangles is considered.

I.1.11. The word 'approach' as used in the two generalizations must now be clarified. In the single variable case there is only one way in which x can approach x_0 and that is along a straight line. However, in the plane, when dealing with $z = f(x)$ and with $F: f^{(1)}(x), f^{(2)}(x)$, there are infinitely many ways in which x_1 and x_2 can become close to x_0. In the single variable case the precise definition of the derivative of $f(x)$ with respect to x at a point x_0 is the following:

The derivative of f with respect to x at x_0 exists and equals d if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(x)-f(x_0)}{x - x_0} - d \right| = \left| \frac{\Delta f}{\Delta x} - d \right| < \epsilon$$

whenever $|x - x_0| < \delta$.

This means that the difference quotient, $\frac{\Delta f}{\Delta x}$, gets as close to d as one chooses for <u>all</u> increments whose maximum length is less than a certain number, δ, as long as the increment is different from zero.

It is this idea of 'approach' which will be applied to the generalized derivatives. The derivative of F(x) with respect to $x = (x^{(1)}, x^{(2)})$ at a point x_0 is said to exist and equal d there if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{\Delta F}{\Delta x} - d \right| < \epsilon$$

for all increments $\Delta x \neq 0$ (with certain requirements which will be discussed next) such that $\|x_0 x_i\| < \delta$, $(i = 1, 2)$. The symbol $\|x_0 x_i\|$ denotes the distance between the points x_0 and x_i. Unless otherwise stated, Δx is always understood to have the fixed point x_0 as a vertex.

This interpretation of the word 'approach' will be used throughout the remainder of the paper. However, one must be careful, for, while in the one dimensional case there is only one possible type of increment--the length of the segment $\overline{x_0 x}$--in the plane one is confronted with all types of increments. In order to make the above generalizations meaningful, classes of increments will be defined. Once a class of increments has been designated for a particular problem, the points x_0, x_1, and x_2 must remain in the class while the limit is being taken. This particular point is important, for it turns out that some functions have a derivative with respect to one class of increments but not with respect to another.

Obviously, any number of classes of increments could be defined by making special requirements of the relative position of the vertices of the increments. However, only three important classes of increments will be defined here, and of these three, only one will be used in the remainder of the paper. The three classes of increments are:

I_1: The class of increments Δx, such that $\Delta x \neq 0$. This is the most general class of increments.

I_2: The class of increments Δx such that

$$\Delta x \geq \frac{1}{2!}\rho \|x_0 x_1\| \cdot \|x_0 x_2\| > 0, \quad \rho \text{ fixed}, \; 0 < \rho \leq 1,$$

where $\|x_0 x_i\|$ denotes the distance between the points x_0 and x_i.

I_3: The class of increments Δx such that

$$\Delta x = \frac{1}{2!}\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_0^{(1)}+\Delta x^{(1)} & x_0^{(2)} & 1 \\ x_0^{(1)} & x_0^{(2)}+\Delta x^{(2)} & 1 \end{vmatrix}, \quad \Delta x^{(1)} \neq 0, \quad \Delta x^{(2)} \neq 0.$$

It is easily seen that these classes of increments have the following inclusion property:

$$I_3 \subset I_2 \subset I_1 .$$

To verify, for example, that $I_3 \subset I_2$, one notices that any increment of I_3 satisfies the conditions of an increment in I_2 with $\rho = 1$. Furthermore, an increment Δx in I_2 is certainly also an increment of I_1, since $\Delta x \neq 0$.

I.1.12. <u>Remark</u>. There is one requirement that must be made. It is, that increments of the class in question must appear in every sufficiently small neighborhood of the fixed point, x_0, at which the derivative is being taken. This will certainly be the case if the set E containing x_0 is chosen properly; for example, if E is an open set, then increments of all three types will appear in every sufficiently small neighborhood of x_0.

Now that classes of increments have been defined, a precise definition of the derivative of F: $f^{(1)}(x)$, $f^{(2)}(x)$ with respect to $x = (x^{(1)}, x^{(2)})$ at a point x_0 can be made.

I.1.13. <u>Definition</u>. Let F: $f^{(i)}(x)$, $(i = 1, 2)$, be a mapping function defined on a region* E of $R^{(2)}$. The derivative of F with respect to x with respect to the class of increments I exists at a point x_0 of E and equals d if for every sufficiently small $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{\Delta F}{\Delta x} - d \right| < \epsilon$$

for all increments Δx in the class I such that $\|x_0 x_i\| < \delta$, $(i = 1, 2)$.

I.1.14. <u>Remark</u>. The ϵ, δ relationship of Definition I.1.13 must hold for <u>all</u> increments Δx in the class I which appear

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* The term 'region' is understood to mean an open, connected set of Euclidean space.

in $N_\delta(x_o)$, the δ-neighborhood of x_o. Otherwise the derivative at x_o is not d, but is either something else or does not exist.

I.1.15. Remark. The derivative, if it exists, is unique. Suppose there are two numbers, d and d', such that for every $\epsilon > 0$ there exists a $\delta_1 > 0$ such that

$$\left| \frac{\Delta F}{\Delta x} - d \right| < \epsilon/2$$

for all Δx in I in $N_{\delta_1}(x_o)$, and also a $\delta_2 > 0$ such that

$$\left| \frac{\Delta F}{\Delta x} - d' \right| < \epsilon/2$$

for all Δx in I in $N_{\delta_2}(x_o)$. Then if $\delta' = \min(\delta_1, \delta_2)$, one would have,

$$|d - d'| = \left| d - \frac{\Delta F}{\Delta x} + \frac{\Delta F}{\Delta x} - d' \right| \leq \left| \frac{\Delta F}{\Delta x} - d \right| + \left| \frac{\Delta F}{\Delta x} - d' \right| < \epsilon$$

whenever Δx is in I in $N_{\delta'}(x_o)$. This implies that d = d'.

I.1.16. The functions which have a derivative, $D_x F$, with respect to the three classes of increments defined above have the following inclusion property:

$$(\text{Class } I_1) \subset (\text{Class } I_2) \subset (\text{Class } I_3),$$

where (Class I_1) designates the class of functions having a derivative with respect to the class of increments I_1, etc.

That this is true is verified by noticing that if Definition I.1.13 holds for all increments of I_1 in $N_\delta(x_o)$, it will certainly hold for the increments in the subclasses I_2 and I_3. Similarly, if Definition I.1.13 holds for the class I_2, it will clearly hold for the subclass I_3.

I.2. GENERALIZATION FOR n-DIMENSIONAL SPACE

I.2.1. Definition. By a p-cell, Δx_p, in $R^{(n)}$, $(p \leq n)$ with the p + 1 vertices $x_o, x_1, \ldots, x_p$, is meant the set of points x of $R^{(n)}$ which can be represented as

$$\text{(I.2.2)} \qquad x = \sum_{i=0}^{p} \alpha_i x_i; \qquad \sum_{i=0}^{p} \alpha_i = 1, \quad \alpha_i > 0, \text{ all } i.$$

I.2.3. Remark. Further properties of p-cells will be developed in Chapter II.

I.2.4. Definition I.1.13 is readily extended to n-dimensions. In n-dimensional Euclidean space, $R^{(n)}$, the mapping function

$$F: f^{(i)}(x), \ (i = 1, \ldots, n),$$

is considered, where $x = (x^{(1)}, \ldots, x^{(n)})$*. The increments to be considered are those n-dimensional oriented n-cells, Δx_n, with vertices $x_o, x_1, \ldots, x_n$. The volume of such an n-cell is given by the expression [Aitken (1), pp. 42-44]**:

* See the footnote at the bottom of page 4.
** Names and numbers in brackets refer to the bibliography at the end of this paper.

$$\text{(I.2.5)} \qquad \Delta x_n = \frac{1}{n!}\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n^{(1)} & x_n^{(2)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}.$$

The mapping function F maps these n + 1 points into n + 1 points, $F(x_0), F(x_1), \ldots, F(x_n)$, where $F(x_j) = \{f^{(i)}(x_j)\}$, $(i = 1, \ldots, n)$. The volume of the cell with these points as vertices is given by the expression

$$\text{(I.2.6)} \qquad \Delta F_n = \frac{1}{n!}\begin{vmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & \cdots & f^{(n)}(x_0) & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{(1)}(x_n) & f^{(2)}(x_n) & \cdots & f^{(n)}(x_n) & 1 \end{vmatrix}.$$

As before classes of increments are defined, the definitions being precisely analogous to those given in Section I.1.11. The ratio of the signed volumes of the two cells is examined and the limit is taken as $x_1, x_2, \ldots, x_n$ 'approach' x_0, in the sense discussed in Section I.1.11. It is understood, of course, that the cell Δx_n, with the fixed vertex x_0, remains in the designated class of increments while the limit is being taken. If this limit exists, it is defined to be the derivative of F with respect to x at the point x_0 with respect to the class of increments in question. It is likewise designated by $D_x F|_{x_0}$. To put this in precise form for the n-dimensional case, the following definition is given:

I.2.7. Definition. Let F: $f^{(i)}(x\ )$, $(i = 1, \ldots, n)$, be a mapping function defined on a region E of $R^{(n)}$. The derivative of F with respect to x, with respect to a class I of increments, exists at a point x_o of E and equals d if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{\Delta F_n}{\Delta x_n} - d \right| < \epsilon$$

for all Δx_n in I such that $\| x_o x_i \| < \delta$, $(i = 1, \ldots, n)$. Δx_n is always understood to possess the fixed point x_o as a vertex.

I.2.8. Remark. By choosing

$$f^{(i)}(x) = x^{(i)},\ (i = 1, \ldots, k-1, k+1, \ldots, n),$$

a generalization of Definitions I.1.5 and I.1.6 will be obtained. For these choices of $f^{(i)}(x)$, the difference quotient, $\frac{\Delta F_n}{\Delta x_n}$, becomes

$$\text{(I.2.9)}\quad \frac{\begin{vmatrix} x_o^{(1)} \ldots x_o^{(k-1)} & f^{(k)}(x_o) & x_o^{(k+1)} \ldots x_o^{(n)} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ x_n^{(1)} \ldots x_n^{(k-1)} & f^{(k)}(x_n) & x_n^{(k+1)} \ldots x_n^{(n)} & 1 \end{vmatrix}}{\begin{vmatrix} x_o^{(1)} & \cdots & x_o^{(n)} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ x_n^{(1)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}}.$$

If the limit of this difference quotient exists as $x_1, \ldots, x_n$ 'approach' x_o, remaining in the designated class of increments, it will be called the derivative of $f^{(k)}$ with respect to $x^{(k)}$, and will be denoted by $D_{x^{(k)}}f^{(k)}$. Clearly, all the derivatives, $D_{x^{(i)}}f^{(j)}$, $(i,j = 1, \ldots, n)$, can be thus defined.

I.3. THE MOORE-SMITH LIMIT AND GENERALIZED DERIVATIVES

I.3.1. E. H. Moore and H. L. Smith have defined the limit which bears their name as follows [Moore-Smith, (1)]:

I.3.2. Definition. Consider a class P of elements p and a binary operation R defined on the class P, subject to the following conditions:

(1) R is transitive. That is, if $p_1 R p_2$ and $p_2 R p_3$, then $p_1 R p_3$.

(2) R has the combination property. That is, if p_1 and p_2 are any two elements of P, there is an element p_3 of P such that $p_3 R p_1$ and $p_3 R p_2$.

(The notation $p_1 R p_2$ means that p_1 is in the relation R to p_2.) Let $\alpha(p)$ be a numerically valued function defined on P. Then $\alpha(p)$ is said to converge to a limit d, with respect to the relation R, if for every $\epsilon > 0$ there exists an element p_ϵ such that

$$|\alpha(p) - d| < \epsilon \qquad \text{for all } p R p_\epsilon.$$

I.3.3. Examples of the Moore-Smith Limit.

Example (1). Let P be the set of all positive integers. Let $\alpha(p) = a_p$, $(p = 1, 2, \ldots)$, be an infinite sequence of real or complex numbers. Let the relation $p_1 R p_2$ mean that $p_1 > p_2$. Then the ordinary limit of a sequence of real or complex numbers is a Moore-Smith limit. For a sequence of real or complex numbers, a_p, $(p = 1, 2, \ldots)$, converges to a limit, a, if for every $\epsilon > 0$ there is a positive number p_ϵ, such that

$$|a_p - a| < \epsilon \quad \text{for all } p > p_\epsilon.$$

That is, the numerically-valued function, $\alpha(p) = a_p$, has a limit a if for every $\epsilon > 0$ there exists a p_ϵ, such that

$$|\alpha(p) - a| < \epsilon \quad \text{for all } p \; R \; p_\epsilon.$$

Example (2). Consider a function g(x) defined on a closed interval I: $a \leq x \leq b$. Let π be a subdivision of $a \leq x \leq b$ by a certain number of intermediate points. Let $P = \{\pi\}$ be the class of all subdivisions of I.

Let π_1 and π_2 be two subdivisions of I. Then the binary operation R is defined as follows: $\pi_2 R \pi_1$, if π_2 is a refinement of π_1; that is, if π_2 is obtained from π_1 by adding points of division to π_1. The operation R is clearly transitive. R also has the combination property; for if π_1 and π_2 are two subdivisions of I, let them be superimposed. This subdivision,

Π_3, is a refinement of both Π_1 and Π_2.

Define $\alpha(\Pi)$ as follows:

$$\alpha(\Pi) = \sum_{i=1}^{n} M[g(\xi_i)] \cdot (x_i - x_{i-1}),$$

where $M[g(\xi_i)] = \sup \{ g(\xi_i) \mid x_{i-1} \leq \xi_i \leq x_i \}$. Clearly, for

$$\Pi_2 \ R \ \Pi_1$$

it follows that

$$\alpha(\Pi_2) \leq \alpha(\Pi_1)$$

since sup f(x) on a finer subdivision of an interval is always less than or equal to sup f(x) on a coarser subdivision, for any function f(x).

Define $$\inf_{\Pi} \alpha(\Pi) = \overline{\int_a^b} g(x)\, dx.$$

From the properties of the infimum of a function it follows that for every $\epsilon > 0$ there is at least one Π_ϵ such that

$$\alpha(\Pi_\epsilon) < \overline{\int_a^b} g(x)\, dx + \epsilon .$$

Since $\alpha(\Pi) \leq \alpha(\Pi_\epsilon)$ if $\Pi \ R \ \Pi_\epsilon$, then

$$\left| \alpha(\Pi) - \overline{\int_a^b} g(x)\, dx \right| < \epsilon$$

for all $\Pi \ R \ \Pi_\epsilon$. That is, $\overline{\int_a^b} g(x)\, dx$ is the Moore-Smith limit of $\alpha(\Pi)$. If g(x) is Riemann integrable, then

$$\lim \alpha(\pi) = \int_a^b g(x)\, dx.$$

The Riemann integral of g(x) is also defined to be

$$\lim_{n(\pi) \to 0} \alpha(\pi),$$

where $n(\pi)$ is the length of the longest subinterval of π. The well-known lemma of Darboux states that the two definitions are equivalent.

Example 3. It will be shown that the generalized derivative defined in Definition I.2.7 is a Moore-Smith Limit. Let the class P be the class of increments I_1. For a given increment Δx_n of P, define $\alpha(\Delta x)_n$ to be the numerically-valued function $\frac{\Delta F_n}{\Delta x_n}$. Let $(\Delta x)_n'$ with vertices $x_0, x_1', \ldots, x_n'$, and $(\Delta x)_n''$ with vertices $x_0, x_1'', \ldots, x_n''$, be two increments of P. The increment $(\Delta x)_n'$ will be said to be in the relation R with $(\Delta x)_n''$ (written $(\Delta x)_n' \; R \; (\Delta x)_n''$) if

$$\max \left\{ \|x_0 x_i'\| \right\} \leq \max \left\{ \|x_0 x_i''\| \right\}, \quad i = 1, \ldots, n,$$

where x_0 is a common fixed vertex of both $(\Delta x)_n'$ and $(\Delta x)_n''$.

The relation R is clearly transitive. It also has the combination property. For if $(\Delta x)_n'$ and $(\Delta x)_n''$ are two increments of P, since it is assumed that increments of P appear in every neighborhood of x_0, then there exists an increment $(\Delta \bar{x})$ with vertices $x_0, \bar{x}_1, \ldots, \bar{x}_n$, such that max $\{\|x_o \bar{x}_i\|\} \leq$ max $\{\|x_o x_i'\|\}$ and max $\{\|x_o \bar{x}_i\|\} \leq$ max $\{\|x_o x_i''\|\}$,

where ($i = 1, \ldots, n$). That is, $(\Delta \bar{x})_n$ R $(\Delta x)_n'$ and $(\Delta \bar{x})_n$ R $(\Delta x)_n''$.

Now using the terminology of the Moore-Smith Limit, the function $\alpha(\Delta x)_n = \frac{\Delta F_n}{\Delta x_n}$ has the limit d at x_o if for every $\epsilon > 0$ there exists a $(\Delta x)_{n,\epsilon}$ such that

$$\left| \frac{\Delta F_n}{\Delta x_n} - d \right| < \epsilon$$

for all $(\Delta x)_n$ R $(\Delta x)_{n,\epsilon}$. This clearly coincides with the definition of the generalized derivative of F(x) given in Definition I.2.7, showing that the generalized derivative is a Moore-Smith Limit.

I.4. FUNCTIONS WHICH HAVE GENERALIZED DERIVATIVES

I.4.1. A natural question to ask is the following: Which classes of functions possess derivatives of the type given in Definition I.2.7, and which classes possess derivatives of the type described in paragraph I.2.8, with respect to the various classes of increments?

It is the purpose of this paper to answer the question as to which class of functions possess non-zero derivatives of the two types described with respect to the class I_1.

G. B. Price has shown that the precise class of functions $f(x^{(1)}, \ldots, x^{(n)})$ which possess a derivative of the type $D_{x^{(i)}}f$, $(i = 1, \ldots, n)$, with respect to the class of increments I_2 is the class of Stolz differentiable functions. A function $f(x^{(1)}, \ldots, x^{(n)})$ is said to be Stolz differentiable at a point $x_0 = (x_0^{(1)}, \ldots, x_0^{(n)})$ if there exist constants a_i, $(i = 1, \ldots, n)$, such that

$$f(x^{(1)}, \ldots, x^{(n)}) - f(x_0^{(1)}, \ldots, x_0^{(n)}) = \sum_{i=1}^{n} a_i (x^{(i)}-x_0^{(i)}) + r\left[\epsilon(r)\right],$$

where $r = \sqrt{\sum_{i=1}^{n}(x^{(i)}-x_0^{(i)})^2}$, and where $\epsilon(r)$ is a function of r such that $\lim_{r \to 0} \epsilon(r) = 0$. For a treatment of Stolz differentiable functions, see [Radamacher, (1)].

It is easily shown that the class of functions $f(x^{(1)}, \ldots, x^{(n)})$ which have a derivative of the type $D_{x^{(i)}}f$ with respect to the class of increments I_3 is the class of functions which are differentiable in the ordinary sense. To show this, suppose that $D_{x^{(k)}}f$ exists at a point x_0 with respect to the class of increments I_3. That is, the limit of the following difference quotient exists as the points $x_1, \ldots, x_n$ approach x_0, remaining in the class I_3:

$$
(I.4.2) \qquad \frac{\begin{vmatrix}
x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(k-1)} & f & x_0^{(k+1)} & \cdots & x_0^{(n)} & 1 \\
x_0^{(1)}+\Delta x^{(1)} & x_0^{(2)} & \cdots & x_0^{(k-1)} & f(\Delta x^{(1)}) & x_0^{(k+1)} & \cdots & x_0^{(n)} & 1 \\
x_0^{(1)} & x_0^{(2)}+\Delta x^{(2)} & \cdots & x_0^{(k-1)} & f(\Delta x^{(2)}) & x_0^{(k+1)} & \cdots & x_0^{(n)} & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(k-1)} & f(\Delta x^{(k)}) & x_0^{(k+1)} & \cdots & x_0^{(n)} & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(k-1)} & f(\Delta x^{(n)}) & x_0^{(k+1)} & \cdots & x_0^{(n)}+\Delta x^{(n)} & 1
\end{vmatrix}}{\begin{vmatrix}
x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(n)} & 1 \\
x_0^{(1)}+\Delta x^{(1)} & x_0^{(2)} & \cdots & x_0^{(n)} & 1 \\
x_0^{(1)} & x_0^{(2)}+\Delta x^{(2)} & \cdots & x_0^{(n)} & 1 \\
\cdots & \cdots & \cdots & \cdots & \\
x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(n)}+\Delta x^{(n)} & 1
\end{vmatrix}}
$$

where $f = f(x_0^{(1)}, \ldots, x_0^{(n)})$ and

$f(\Delta x^{(i)}) = f(x_0^{(1)}, \ldots, x_0^{(i-1)}, x_0^{(i)}+\Delta x^{(i)}, x_0^{(i+1)}, \ldots, x_0^{(n)})$,

$(i = 1, \ldots, n)$.

Subtract the first row from each of the remaining rows in both the numerator and denominator of (I.4.2), obtaining

$$(I.4.3)\qquad \frac{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(K-1)} & f & x_0^{(K+1)} & \cdots & x_0^{(n)} & 1 \\ \Delta x^{(1)} & 0 & \cdots & 0 & f(\Delta x^{(1)})-f & 0 & \cdots & 0 & 0 \\ 0 & \Delta x^{(2)} & \cdots & 0 & f(\Delta x^{(2)})-f & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & f(\Delta x^{(K)})-f & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & f(\Delta x^{(n)})-f & 0 & \cdots & \Delta x^{(n)} & 0 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(n)} & 1 \\ \Delta x^{(1)} & 0 & \cdots & 0 & 0 \\ 0 & \Delta x^{(2)} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \Delta x^{(n)} & 0 \end{vmatrix}}.$$

Expand both numerator and denominator of (I.4.3) by the last column and obtain

$$(I.4.4)\qquad \frac{\begin{vmatrix} \Delta x^{(1)} & 0 & \cdots & 0 & f(\Delta x^{(1)})-f & 0 & \cdots & 0 \\ 0 & \Delta x^{(2)} & \cdots & 0 & f(\Delta x^{(2)})-f & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & f(\Delta x^{(K)})-f & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & f(\Delta x^{(n)})-f & 0 & \cdots & \Delta x^{(n)} \end{vmatrix}}{\Delta x^{(1)}\Delta x^{(2)}\cdots\Delta x^{(n)}}.$$

Expand the numerator of (I.4.4) by the first row and obtain

$$(I.4.5)\qquad \Delta x^{(1)}\cdot\Delta x^{(2)}\cdot\ldots\cdot\Delta x^{(K-1)}\cdot\left(f(\Delta x^{(K)})-f\right)\cdot\Delta x^{(K+1)}\cdot\ldots\cdot\Delta x^{(n)}.$$

[The minor of $f(\Delta x^{(1)})-f$ is 0 since the kth row of this minor contains all zeros.] Hence, the difference quotient (I.4.4) reduces to

$$\frac{\left(f(\Delta x^{(k)})-f\right)\Delta x^{(1)}\Delta x^{(2)}\ldots\Delta x^{(k-1)}\Delta x^{(k+1)}\ldots\Delta x^{(n)}}{\Delta x^{(1)}\ldots\Delta x^{(n)}} =$$

$$\frac{f(\Delta x^{(k)})-f}{\Delta x^{(k)}}$$

Since it was assumed that the limit of the difference quotient (I.4.2) with respect to the class I exists, then

$$\lim_{\Delta x^{(k)}\to 0}\frac{f(\Delta x^{(k)})-f}{\Delta x^{(k)}}$$

exists. But this is the ordinary partial derivative of f with respect to $x^{(k)}$. This proves the statement.

The classes of functions, $f(x^{(1)}, \ldots, x^{(n)})$, having derivatives of the type $D_{x^{(i)}}f$ with respect to the classes of increments I_2 and I_3 have thus been determined. When one considers the classes of functions $F: f^{(i)}(x)$, $(i = 1, \ldots, n)$, which have derivatives of the type $D_x F$ with respect to the classes I_2 and I_3, the answer is not so clear. The following illustration will show what one is up against in dealing with $D_x F$ with respect to the class of increments I_3.

Consider the mapping function

$$(\text{I.4.6}) \qquad F: \begin{array}{l} f^{(1)}(x) = x^{(1)} - x^{(2)} - g(x^{(1)}) \\ f^{(2)}(x) = x^{(1)} + x^{(2)} + g(x^{(1)}) \end{array}, \quad x = (x^{(1)}, x^{(2)}),$$

where $g(x^{(1)})$ is defined for all $x^{(1)}$, and otherwise is completely arbitrary. Clearly, the ordinary partials of $f^{(1)}(x)$ and $f^{(2)}(x)$ cannot exist with respect to the variable $x^{(1)}$. However, $D_X F$ exists everywhere with respect to the class I_3. To show this, consider the difference quotient

$$(\text{I.4.7}) \quad \frac{\begin{vmatrix} (x_0^{(1)})-(x_0^{(2)})-g(x_0^{(1)}) & (x_0^{(1)})+(x_0^{(2)})+g(x_0^{(1)}) & 1 \\ (x_0^{(1)}+\Delta x^{(1)})-(x_0^{(2)})-g(x_0^{(1)}+\Delta x^{(1)}) & (x_0^{(1)}+\Delta x^{(1)})+(x_0^{(2)})+g(x_0^{(1)}+\Delta x^{(1)}) & 1 \\ (x_0^{(1)})-(x_0^{(2)}+\Delta x^{(2)})-g(x_0^{(1)}) & (x_0^{(1)})+(x_0^{(2)}+\Delta x^{(2)})+g(x_0^{(1)}) & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_0^{(1)}+\Delta x^{(1)} & x_0^{(2)} & 1 \\ x_0^{(1)} & x_0^{(2)}+\Delta x^{(2)} & 1 \end{vmatrix}}.$$

Clearly, the denominator reduces to $\Delta x^{(1)}\Delta x^{(2)}$, by the same procedure used in the last example. In the numerator, add the second column to the first column and obtain

$$(\text{I.4.8}) \qquad 2\begin{vmatrix} x_0^{(1)} & x_0^{(1)}+x_0^{(2)}+g(x_0^{(1)}) & 1 \\ x_0^{(1)}+\Delta x^{(1)} & x_0^{(1)}+\Delta x^{(1)}+x_0^{(2)}+g(x_0^{(1)}+\Delta x^{(1)}) & 1 \\ x_0^{(1)} & x_0^{(1)}+x_0^{(2)}+\Delta x^{(2)}+g(x_0^{(1)}) & 1 \end{vmatrix}.$$

Subtract the first column of (I.4.8) from the second column, obtaining

$$\text{(I.4.9)} \qquad 2\begin{vmatrix} x_0^{(1)} & x_0^{(2)} + g(x_0^{(2)}) & 1 \\ x_0^{(1)}+\Delta x^{(1)} & x_0^{(2)}+g(x_0^{(1)}+\Delta x^{(1)}) & 1 \\ x_0^{(1)} & x_0^{(2)}+\Delta x^{(2)}+g(x^{(1)}) & 1 \end{vmatrix}.$$

Multiply the last column by $x_0^{(1)}$ and $x_0^{(2)}$ and subtract from columns one and two respectively and (I.4.9) becomes

$$\text{(I.4.10)} \qquad 2\begin{vmatrix} 0 & g(x_0^{(1)}) & 1 \\ \Delta x^{(1)} & g(x_0^{(1)}+\Delta x^{(1)}) & 1 \\ 0 & \Delta x_0^{(2)}+g(x_0^{(1)}) & 1 \end{vmatrix}.$$

Subtract the first row from the last row and get

$$\text{(I.4.11)} \qquad 2\begin{vmatrix} 0 & g(x_0^{(1)}) & 1 \\ \Delta x^{(1)} & g(x_0^{(1)}+\Delta x^{(1)}) & 1 \\ 0 & \Delta x^{(2)} & 0 \end{vmatrix}.$$

Expanding (I.4.11) by the first column, the numerator of (I.4.7) finally becomes

$$\text{(I.4.12)} \qquad 2(-\Delta x^{(1)})(-\Delta x^{(2)}) = 2\,\Delta x^{(1)}\Delta x^{(2)}.$$

Hence, the difference quotient (I.4.7) becomes

$$\text{(I.4.13)} \qquad \frac{2\,\Delta x^{(1)}\Delta x^{(2)}}{\Delta x^{(1)}\Delta x^{(2)}} = 2.$$

Clearly, the derivative D_xF with respect to I_3 exists everywhere in the plane and equals 2.

Since $g(x^{(1)})$ was arbitrary one can see that the task of finding out more about the kinds of functions F having a derivative $D_x F$ with respect to the class of increments I_3 is not easy. Similar statements apply to the functions having a derivative with respect to the class I_2.

One additional fact can be proved concerning the functions $F: f^{(i)}(x)$, $(i = 1, \ldots, n)$; this is expressed in the following theorem.

I.4.14. Theorem. Let $F: f^{(i)}(x)$, $(i = 1, \ldots, n)$, be a mapping function such that $D_{x^{(j)}} f^{(i)}$, $(i,j = 1, \ldots, n)$, exist at a point x_0 with respect to any class of non-zero increments. Then $D_x F|_{x_0}$ exists and equals

$$\begin{vmatrix} D_{x^{(1)}} f^{(1)} & \ldots & D_{x^{(n)}} f^{(1)} \\ \ldots & \ldots & \ldots \\ D_{x^{(1)}} f^{(n)} & \ldots & D_{x^{(n)}} f^{(n)} \end{vmatrix}.$$

Proof. The proof of this theorem depends on the Bazin-Picquet-Reiss Theorem on determinants [Price, (1)], which states that if A and B are two n x n matrices, then

$$|A|^{C(n-1,k-1)} \cdot |B|^{C(n-1,k)}$$

$$= \left| \left| B\left[A(J_j^{(k)})/B(J_i^{(k)})\right] \right| \right|, \quad (i,j = 1, \ldots, C(n,k))^{*},$$

where $\left|B\left[A(J_j^{(k)})/B(J_i^{(k)})\right]\right|$ stands for the determinant

---

* The vertical bars here are all determinant signs.

obtained when one of the C(n,k) possible choices of k columns of B has been replaced by one of the C(n,k) possible choices of k columns of A, and where the symbol, C(n,k) stands for the number of combinations of n things taken k at a time. For each choice of k columns of B, all C(n,k) choices of k columns of A are substituted successively, forming a row of the determinant $\left\| B\left[A(J_j^{(k)})/B(J_i^{(k)}) \right] \right\|$. Since there are C(n,k) choices of k columns of B, the resulting determinant, $\left\| B\left[A(J_j^{(k)})/B(J_i^{(k)}) \right] \right\|$, is a C(n,k) x C(n,k) determinant.

Consider the following product of determinants:

$$(\text{I.4.15}) \quad \begin{vmatrix} f^{(1)}(x_o) & f^{(2)}(x_o) & \cdot\ \cdot & f^{(n)}(x_o) & 1 \\ \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ f^{(1)}(x_n) & f^{(2)}(x_n) & \cdot\ \cdot & f^{(n)}(x_n) & 1 \end{vmatrix} \cdot \begin{vmatrix} x_o^{(1)} & x_o^{(2)} & \cdot\ \cdot & x_o^{(n)} & 1 \\ \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\ x_n^{(1)} & x_n^{(2)} & \cdot\ \cdot & x_n^{(n)} & 1 \end{vmatrix}^{n} .$$

If one applies the Bazin-Picquet-Reiss Theorem with k = 1, one finds that (I.4.15) is equal to (I.4.16):

$$\begin{vmatrix}
\begin{vmatrix} f^{(1)}(x_0) & x_0^{(2)} \cdot & \cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ f^{(1)}(x_n) & x_n^{(2)} \cdot & \cdot\, x_n^{(n)} & 1 \end{vmatrix} &
\begin{vmatrix} x_0^{(1)} & f^{(1)}(x_0) \cdot & \cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ x_n^{(1)} & f^{(1)}(x_n) \cdot & \cdot\, x_n^{(n)} & 1 \end{vmatrix} & \cdot\cdot &
\begin{vmatrix} x_0^{(1)} \cdot & \cdot\, x_0^{(n)} & f^{(1)}(x_0) \\ \cdot & \cdot & \cdot \\ x_n^{(1)} \cdot & \cdot\, x_n^{(n)} & f^{(1)}(x_n) \end{vmatrix} \\
\begin{vmatrix} f^{(2)}(x_0) & x_0^{(2)} \cdot & \cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ f^{(2)}(x_n) & x_n^{(2)} \cdot & \cdot\, x_n^{(n)} & 1 \end{vmatrix} &
\begin{vmatrix} x_0^{(1)} & f^{(2)}(x_0) \cdot & \cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ x_n^{(1)} & f^{(2)}(x_n) \cdot & \cdot\, x_n^{(n)} & 1 \end{vmatrix} & \cdot\cdot &
\begin{vmatrix} x_0^{(1)} \cdot & \cdot\, x_0^{(n)} & f^{(2)}(x_0) \\ \cdot & \cdot & \cdot \\ x_n^{(1)} \cdot & \cdot\, x_n^{(n)} & f^{(2)}(x_n) \end{vmatrix} \\
\cdot & \cdot & \cdot & \cdot \\
\begin{vmatrix} f^{(n)}(x_0) & x_0^{(2)} \cdot & \cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ f^{(n)}(x_n) & x_n^{(2)} \cdot & \cdot\, x_n^{(n)} & 1 \end{vmatrix} &
\begin{vmatrix} x_0^{(1)} & f^{(n)}(x_0) \cdot & \cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ x_n^{(1)} & f^{(n)}(x_n) \cdot & \cdot\, x_n^{(n)} & 1 \end{vmatrix} & \cdot\cdot &
\begin{vmatrix} x_0^{(1)} \cdot & \cdot\, x_0^{(n)} & f^{(n)}(x_0) \\ \cdot & \cdot & \cdot \\ x_n^{(1)} \cdot & \cdot\, x_n^{(n)} & f^{(n)}(x_n) \end{vmatrix} \\
\begin{vmatrix} 1 & x_0^{(2)} \cdot & \cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & x_n^{(2)} \cdot & \cdot\, x_n^{(n)} & 1 \end{vmatrix} &
\begin{vmatrix} x_0^{(1)} & 1 & \cdot\,\cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ x_n^{(1)} & 1 & \cdot\,\cdot\, x_n^{(n)} & 1 \end{vmatrix} & \cdot\cdot &
\begin{vmatrix} x_0^{(1)} \cdot & \cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot \\ x_n^{(1)} \cdot & \cdot\, x_n^{(n)} & 1 \end{vmatrix}
\end{vmatrix} .$$

Since all the elements of the last row of (I.4.16) are zero except the last element, expanding by the last row, (I.4.16) becomes equal to

$$\begin{vmatrix} x_0^{(1)} \cdot & \cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot \\ x_n^{(1)} \cdot & \cdot\, x_n^{(n)} & 1 \end{vmatrix} \cdot
\begin{vmatrix}
\begin{vmatrix} f^{(1)}(x_0) & x_0^{(2)} \cdot & \cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ f^{(1)}(x_n) & x_n^{(2)} \cdot & \cdot\, x_n^{(n)} & 1 \end{vmatrix} & \cdots &
\begin{vmatrix} x_0^{(1)} \cdot & \cdot\, x_0^{(n-1)} & f^{(1)}(x_0) & 1 \\ \cdot & \cdot & \cdot & \cdot \\ x_n^{(1)} \cdot & \cdot\, x_n^{(n-1)} & f^{(1)}(x_n) & 1 \end{vmatrix} \\
\cdot & \cdot & \cdot \\
\begin{vmatrix} f^{(n)}(x_0) & x_0^{(2)} \cdot & \cdot\, x_0^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ f^{(n)}(x_n) & x_n^{(2)} \cdot & \cdot\, x_n^{(n)} & 1 \end{vmatrix} & \cdots &
\begin{vmatrix} x_0^{(1)} \cdot & \cdot\, x_0^{(n-1)} & f^{(n)}(x_0) & 1 \\ \cdot & \cdot & \cdot & \cdot \\ x_n^{(1)} \cdot & \cdot\, x_n^{(n-1)} & f^{(n)}(x_n) & 1 \end{vmatrix}
\end{vmatrix} .$$

Dividing both (I.4.15) and the determinant product above by

$$\begin{vmatrix} x_0^{(1)} & \cdots & x_0^{(n)} & 1 \\ \cdot & \cdots & \cdot & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}^{n+1},$$

one gets the equality

$$\text{(I.4.17)} \qquad \frac{\begin{vmatrix} f^{(1)}(x_0) & \cdots & f^{(n)}(x_0) & 1 \\ \cdot & \cdots & \cdot & \cdot \\ f^{(1)}(x_n) & \cdots & f^{(n)}(x_n) & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & \cdots & x_0^{(n)} & 1 \\ \cdot & \cdots & \cdot & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}} =$$

$$\begin{vmatrix} \dfrac{\begin{vmatrix} f^{(1)}(x_0) & x_0^{(2)} & \cdots & x_0^{(n)} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ f^{(1)}(x_n) & x_n^{(2)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & \cdots & x_0^{(n)} & 1 \\ \cdot & \cdots & \cdot & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}} & \cdots & \dfrac{\begin{vmatrix} x_0^{(1)} & \cdots & x_0^{(n-1)} & f^{(1)}(x_0) & 1 \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n-1)} & f^{(1)}(x_n) & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & \cdots & x_0^{(n)} & 1 \\ \cdot & \cdots & \cdot & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}} \\ \cdot & \cdots & \cdot \\ \dfrac{\begin{vmatrix} f^{(n)}(x_0) & x_0^{(2)} & \cdots & x_0^{(n)} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ f^{(n)}(x_n) & x_n^{(2)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & \cdots & x_0^{(n)} & 1 \\ \cdot & \cdots & \cdot & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}} & \cdots & \dfrac{\begin{vmatrix} x_0^{(1)} & \cdots & x_0^{(n-1)} & f^{(n)}(x_0) & 1 \\ \cdot & \cdots & \cdot & \cdot & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n-1)} & f^{(n)}(x_n) & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & \cdots & x_0^{(n)} & 1 \\ \cdot & \cdots & \cdot & \cdot \\ x_n^{(1)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}} \end{vmatrix} .$$

Taking the limit with respect to the class of increments in question, since it has been assumed that all the derivatives, $D_{x^{(j)}}f^{(i)}$, $(i,j = 1, \ldots, n)$, exist, then the right side of (I.4.17) has the limit

(I.4.18) $$\begin{vmatrix} D_{x^{(1)}} f^{(1)}\big|_{x_0} & \cdots & D_{x^{(n)}} f^{(1)}\big|_{x_0} \\ \cdots & \cdots & \cdots \\ D_{x^{(1)}} f^{(n)}\big|_{x_0} & \cdots & D_{x^{(n)}} f^{(n)}\big|_{x_0} \end{vmatrix}$$

Hence the left side of (I.4.17) must also have a limit and this limit is, by definition, $D_xF\big|_{x_0}$ with respect to the class of increments in question. This proves the theorem.

I.5. A USEFUL THEOREM CONCERNING THE GENERALIZED DERIVATIVE

I.5.1. In the remainder of this paper, the only class of increments which will be considered is the class I_1, the most general class. It will be unnecessary to refer to this fact again, for it will be understood that when D_xF appears in the discussion, it is always the derivative of F with respect to x with respect to the class of increments I_1.

To conclude this chapter a useful theorem concerning the method of taking the derivative at a point will be stated and proved. Before doing this, however, a lemma must be proved.

I.5.2. Lemma. Let F: $f^{(i)}(x)$, $(i = 1, \ldots, n)$, be an arbitrary mapping function defined on a set E in $R^{(n)}$. Let

$x_0, x_1, \ldots, x_n, x^*$ be any $n + 2$ points of E. Then

$$\Delta(F:x_0x_1 \ldots x_n) = \Delta(F:x^*x_1x_2 \ldots x_n) + \Delta(F:x_0x^*x_2 \ldots x_n) + \ldots + \Delta(F:x_0x_1 \ldots x_{n-1}x^*).$$

Proof.

$$\text{(I.5.3)} \quad \Delta(F:x_0x_1 \ldots x_n) = \frac{1}{n!}\begin{vmatrix} f^{(1)}(x_0) & \ldots & f^{(n)}(x_0) & 1 \\ \cdot & \cdot & \cdot & \cdot \\ f^{(1)}(x_n) & \ldots & f^{(n)}(x_n) & 1 \end{vmatrix} =$$

$$\frac{1}{n!}\begin{vmatrix} 0 & \ldots & 0 & 1 \\ f^{(1)}(x_1) & \ldots & f^{(n)}(x_1) & 1 \\ \cdot & \cdot & \cdot & \cdot \\ f^{(1)}(x_n) & \ldots & f^{(n)}(x_n) & 1 \end{vmatrix} + \frac{1}{n!}\begin{vmatrix} f^{(1)}(x_0) & \ldots & f^{(n)}(x_0) & 1 \\ 0 & \ldots & 0 & 1 \\ f^{(1)}(x_2) & \ldots & f^{(n)}(x_2) & 1 \\ \cdot & \cdot & \cdot & \cdot \\ f^{(1)}(x_n) & \ldots & f^{(n)}(x_n) & 1 \end{vmatrix} +$$

$$\ldots + \frac{1}{n!}\begin{vmatrix} f^{(1)}(x_0) & \ldots & f^{(n)}(x_0) & 1 \\ \cdot & \cdot & \cdot & \cdot \\ f^{(1)}(x_{n-1}) & \ldots & f^{(n)}(x_{n-1}) & 1 \\ 0 & \ldots & 0 & 1 \end{vmatrix}.$$

This is verified by expanding the ith determinant in the sum above by the ith row ($i = 1, \ldots, n + 1$) and adding the terms together. The result is the same as if the determinant

$$\frac{1}{n!}\begin{vmatrix} f^{(1)}(x_0) & \ldots & f^{(n)}(x_0) & 1 \\ \cdot & \cdot & \cdot & \cdot \\ f^{(1)}(x_n) & \ldots & f^{(n)}(x_n) & 1 \end{vmatrix}$$

were expanded by elements of the last column.

Now if $u^{(1)}, \ldots, u^{(n)}$ are arbitrary functions, one has, after multiplying the last column by $u^{(i)}$ and subtracting this from the ith column ($i = 1, \ldots, n$):

$$\Delta(F:x_0 x_1 \ldots x_n) = \frac{1}{n!}\begin{vmatrix} f^{(1)}(x_0)-u^{(1)} & \ldots & f^{(n)}(x_0)-u^{(n)} & 1 \\ f^{(1)}(x_1)-u^{(1)} & \ldots & f^{(n)}(x_1)-u^{(n)} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ f^{(1)}(x_n)-u^{(1)} & \ldots & f^{(n)}(x_n)-u^{(n)} & 1 \end{vmatrix}.$$

Applying the results of (I.5.3) to this expression, the following equality is obtained:

(I.5.4)

$$\Delta(F:x_0 x_1 \ldots x_n) = \frac{1}{n!}\begin{vmatrix} 0 & \ldots & 0 & 1 \\ f^{(1)}(x_1)-u^{(1)} & \ldots & f^{(n)}(x_1)-u^{(n)} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ f^{(1)}(x_n)-u^{(1)} & \ldots & f^{(n)}(x_n)-u^{(n)} & 1 \end{vmatrix} +$$

$$\frac{1}{n!}\begin{vmatrix} f^{(1)}(x_0)-u^{(1)} & \ldots & f^{(n)}(x_0)-u^{(n)} & 1 \\ 0 & \ldots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ f^{(1)}(x_n)-u^{(1)} & \ldots & f^{(n)}(x_n)-u^{(n)} & 1 \end{vmatrix} + \cdots + \frac{1}{n!}\begin{vmatrix} f^{(1)}(x_0)-u^{(1)} & \ldots & f^{(n)}(x_0)-u^{(n)} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ f^{(1)}(x_{n-1})-u^{(1)} & \ldots & f^{(n)}(x_{n-1})-u^{(n)} & 1 \\ 0 & \ldots & 0 & 1 \end{vmatrix}.$$

Multiplying the last column of each term on the right by $u^{(i)}$ and adding to the ith column ($i = 1, \ldots, n$), (I.5.4) becomes

$$(I.5.5) \quad \Delta(F:x_0x_1 \ldots x_n) = \frac{1}{n!}\begin{vmatrix} u^{(1)} & \ldots & u^{(n)} & 1 \\ f^{(1)}(x_1) & \ldots & f^{(n)}(x_1) & 1 \\ \cdots & \cdots & \cdots & \cdots \\ f^{(1)}(x_n) & \ldots & f^{(n)}(x_n) & 1 \end{vmatrix} +$$

$$\frac{1}{n!}\begin{vmatrix} f^{(1)}(x_0) & \ldots & f^{(n)}(x_0) & 1 \\ u^{(1)} & \ldots & u^{(n)} & 1 \\ f^{(1)}(x_2) & \ldots & f^{(n)}(x_2) & 1 \\ \cdots & \cdots & \cdots & \cdots \\ f^{(1)}(x_n) & \ldots & f^{(n)}(x_n) & 1 \end{vmatrix} + \cdots + \frac{1}{n!}\begin{vmatrix} f^{(1)}(x_0) & \ldots & f^{(n)}(x_0) & 1 \\ \cdots & \cdots & \cdots & \cdots \\ f^{(1)}(x_{n-1}) & \ldots & f^{(n)}(x_{n-1}) & 1 \\ u^{(1)} & \ldots & u^{(n)} & 1 \end{vmatrix}.$$

Since $u^{(1)}, \ldots, u^{(n)}$ are arbitrary, choose them to be $f^{(1)}(x^*), f^{(2)}(x^*), \ldots, f^{(n)}(x^*)$, where x^* is the point in the hypothesis of the lemma. Then (I.5.5) becomes

$$(I.5.6) \quad \Delta(F:x_0 \ldots x_n) = \Delta(F:x^*x_1 \ldots x_n) + \ldots + \Delta(F:x_0 \ldots x_{n-1}x^*).$$

This proves the Lemma.

I.5.7. <u>Remark</u>. A particular case of F is the identity mapping. Hence

$$\Delta(x_0x_1 \ldots x_n) = \Delta(x^*x_1 \ldots x_n) + \ldots + \Delta(x_0x_1 \ldots x_{n-1}x^*).$$

I.5.8. <u>Theorem</u>. Let $F: f^{(i)}(x\ )$, $(i = 1, \ldots, n)$, be a mapping function defined on a region E in $R^{(n)}$. Let D_xF exist at a point x_0 of E and have the value d there. Let $x_1, \ldots, x_n, x_{n+1}$ be n + 1 variable points of E which are always such that x_0 together with any n of the n + 1 points $x_1, \ldots, x_n, x_{n+1}$ form an increment of I_1 and where

$$(I.5.9)\quad \left|\frac{\Delta(x_1 x_2 \ldots x_{i-1} x_0 x_{i+1} \ldots x_{n+1})}{\Delta(x_1 x_2 \ldots x_{n+1})}\right| < M, \quad (i=1, \ldots, n+1)$$

for some fixed positive number M. Then

$$d = \lim_{x_i \to x_0} \frac{\Delta(F:x_1 x_2 \ldots x_{n+1})}{\Delta(x_1 x_2 \ldots x_{n+1})}, \quad (i=1, \ldots, n+1).$$

Proof. It must be shown that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$(I.5.10)\quad \left|\frac{\Delta(F:x_1 x_2 \ldots x_{n+1})}{\Delta(x_1 x_2 \ldots x_{n+1})} - d\right| < \epsilon$$

whenever $\|x_0 x_i\| < \delta$, $(i = 1, \ldots, n+1)$, and where the x_i are such that the conditions of the hypothesis are satisfied.

From Lemma I.5.2,

$$\Delta(x_1 x_2 \ldots x_{n+1}) = \Delta(x_0 x_2 \ldots x_{n+1}) + \ldots + \Delta(x_1 x_2 \ldots x_n x_0),$$

and

$$\Delta(F:x_1 x_2 \ldots x_{n+1}) = \Delta(F:x_0 x_2 \ldots x_{n+1}) + \ldots + \Delta(F:x_1 x_2 \ldots x_n x_0).$$

Hence

$$\left|\frac{\Delta(F:x_1x_2\ldots x_{n+1})}{\Delta(x_1x_2\ldots x_{n+1})} - d\right| =$$

$$\left|\frac{\Delta(F:x_1x_2\ldots x_{n+1})}{\Delta(x_1x_2\ldots x_{n+1})} - \left\{\frac{\Delta(x_0x_2\ldots x_{n+1}) + \ldots + \Delta(x_1x_2\ldots x_nx_0)}{\Delta(x_1x_2\ldots x_{n+1})}\right\}d\right| =$$

$$\left|\frac{\Delta(x_0x_2\ldots x_{n+1})}{\Delta(x_1x_2\ldots x_{n+1})}\left\{\frac{\Delta(F:x_0x_2\ldots x_{n+1})}{\Delta(x_0x_2\ldots x_{n+1})} - d\right\} + \cdots + \frac{\Delta(x_1\ldots x_nx_0)}{\Delta(x_1\ldots x_{n+1})}\left\{\frac{\Delta(F:x_1\ldots x_nx_0)}{\Delta(x_1\ldots x_nx_0)} - d\right\}\right|.$$

Since it has been assumed that $D_xF = d$ at x_0, then for the given $\epsilon > 0$, there exists a $\delta_i > 0$ such that

$$\left|\frac{\Delta(F:x_1\ldots x_{i-1}x_0x_{i+1}\ldots x_{n+1})}{\Delta(x_1\ldots x_{i-1}x_0x_{i+1}\ldots x_{n+1})} - d\right| < \epsilon/M(n+1)$$

whenever $\|x_0x_j\| < \delta_i$, $(j = 1, \ldots, n+1;\ j \neq i)$, $\Delta(x_1\ldots x_{i-1}x_0x_{i+1}\ldots x_{n+1}) \in I_1$. This holds for $i = 1, \ldots, n+1$. Hence, choosing

$$\delta = \min\{\delta_i\}, \quad (i = 1, \ldots, n+1),$$

$$\left|\frac{\Delta(F:x_1x_2\ldots x_{n+1})}{\Delta(x_1x_2\ldots x_{n+1})} - d\right| <$$

$$\left|\frac{\Delta(x_0x_2\ldots x_{n+1})}{\Delta(x_1x_2\ldots x_{n+1})}\right|\cdot\epsilon/M(n+1) + \ldots + \left|\frac{\Delta(x_1\ldots x_nx_0)}{\Delta(x_1x_2\ldots x_{n+1})}\right|\cdot\epsilon/M(n+1) < \epsilon$$

whenever $\|x_0x_j\| < \delta$, $(j = 1, \ldots, n+1)$. Hence

$$D_xF\big|_{x_0} = \lim_{x_1,\ldots x_{n+1}\to x_0}\frac{\Delta(F:x_1x_2\ldots x_{n+1})}{\Delta(x_1x_2\ldots x_{n+1})}.$$

This proves the theorem.

I.5 11. <u>Remark</u>. $\left|\dfrac{\Delta(x_1 \cdot \cdot x_{i-1} x_0 x_{i+1} \cdot \cdot x_{n+1})}{\Delta(x_1 x_2 \cdot \cdot \cdot x_n x_{n+1})}\right|$, $(i = 1, \ldots, n+1)$

is always bounded if x_0 is interior to $\Delta(x_1 x_2 \cdot \cdot x_{n+1})$.

<u>Proof</u>. If x_0 is interior to $\Delta(x_1 x_2 \cdot \cdot x_{n+1})$, then the coordinates of x_0 can be expressed as

$$x_0^{(\ell)} = \sum_{j=1}^{n+1} \alpha_j x_j^{(\ell)}, \quad (\ell = 1, \ldots, n)$$

where $\sum_{j=1}^{n+1} \alpha_j = 1$, and where $\alpha_j \geqslant 0$, all j. (This statement is proved in Chapter II.) Now,

$$\Delta(x_1 \cdot \cdot x_{i-1} x_0 x_{i+1} \cdot \cdot x_{n+1}) = \frac{1}{n!} \begin{vmatrix} x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(n)} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{i-1}^{(1)} & x_{i-1}^{(2)} & \cdots & x_{i-1}^{(n)} & 1 \\ x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(n)} & 1 \\ x_{i+1}^{(1)} & x_{i+1}^{(2)} & \cdots & x_{i+1}^{(n)} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n+1}^{(1)} & x_{n+1}^{(2)} & \cdots & x_{n+1}^{(n)} & 1 \end{vmatrix} =$$

$$\frac{1}{n!} \begin{vmatrix} x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(n)} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{i-1}^{(1)} & x_{i-1}^{(2)} & \cdots & x_{i-1}^{(n)} & 1 \\ \sum_{j=1}^{n+1} \alpha_j x_j^{(1)} & \sum_{j=1}^{n+1} \alpha_j x_j^{(2)} & \cdots & \sum_{j=1}^{n+1} \alpha_j x_j^{(n)} & 1 \\ x_{i+1}^{(1)} & x_{i+1}^{(2)} & \cdots & x_{i+1}^{(n)} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n+1}^{(1)} & x_{n+1}^{(2)} & \cdots & x_{n+1}^{(n)} & 1 \end{vmatrix}.$$

Multiplying the jth row by α_j and subtracting this from the ith row, $(j = 1, \ldots, i-1, i+1, \ldots, n+1)$, this determinant becomes

$$\frac{1}{n!}\begin{vmatrix} x_1^{(1)} & x_1^{(2)} & \ldots & x_1^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{i-1}^{(1)} & x_{i-1}^{(2)} & \ldots & x_{i-1}^{(n)} & 1 \\ \alpha_i x_i^{(1)} & \alpha_i x_i^{(2)} & \ldots & \alpha_i x_i^{(n)} & \alpha_i \\ x_{i+1}^{(1)} & x_{i+1}^{(2)} & \ldots & x_{i+1}^{(n)} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_{n+1}^{(1)} & x_{n+1}^{(2)} & \ldots & x_{n+1}^{(n)} & 1 \end{vmatrix}$$

which is equal to

$$\alpha_i \, \Delta(x_1 x_2 \ldots x_{n+1}).$$

Hence,

$$\left| \frac{\Delta(x_1 \ldots x_{i-1} x_0 x_{i+1} \ldots x_{n+1})}{\Delta(x_1 x_2 \ldots x_{n+1})} \right| = \alpha_i < 1.$$

This proves the statement.

I.5.12. <u>Remark</u>. Theorem I.5.8 is a generalization of a theorem due to Stieltjes concerning the derivative of a function of a single real variable. See [McShane, (1), p. 223], and [Stieltjes, (1)].

CHAPTER II

THE CHARACTERIZATION OF A CLASS OF TRANSFORMATIONS

II.1. INTRODUCTION

II.1.1. In this chapter the generalized derivatives which were defined and discussed in Chapter I will play no role. However, the results of this chapter will be directly applicable to the problem of characterizing the class of functions which have non-zero derivatives of the types discussed in Chapter I with respect to the class of increments I_1. This characterization will be the main purpose of Chapter III. Chapter II will be concerned with the characterization of the class of functions which are continuous, one-to-one, and which have the additional property that they map straight lines into straight lines.

A word of explanation should be stated concerning the statement that the mapping function maps straight lines into straight lines. What is meant by this statement is that all the points on a straight line are mapped by the function into points which lie on a straight line. Nothing is stated about the distribution of the image points, except that they lie on a straight line.

Another statement which is used frequently in the following pages must be explained also. Suppose a mapping function is such that it carries straight lines into straight lines and which leaves two distinct points on a straight line fixed. Then the statement is made that this straight line remains fixed. It is not implied in this statement that each individual point of the line remains fixed, only that each point of the straight line maps into some point on the same straight line. Clearly if two such fixed lines intersect, then that point of intersection must map into a point which is on both lines, and hence it must remain fixed in the strict sense.

These notions are extended to higher dimensions. When the statement is made that a certain function carries faces of an n-cell into faces of an n-cell, it means that every point of the n-cell maps into some point on the face of an n-cell, with nothing further implied. When the statement is made that a mapping function leaves the faces of an n-cell fixed, it means that every point of that face maps into some point on the same face. If every point remains fixed, it will be definitely stated as such.

It will be shown in this chapter that the precise class of mapping functions which are one-to-one, continuous, and which map straight lines into straight lines, is the class of functions of the form

$$F: f^{(i)}(x) = \frac{a_{i,1}x^{(1)} + \cdot \cdot \cdot + a_{i,n}x^{(n)} + a_{i,n+1}}{a_{n+1,1}x^{(1)} + \cdot \cdot \cdot + a_{n+1,n}x^{(n)} + a_{n+1,n+1}}, \ (i = 1, \ldots, n),$$

where

$$\begin{vmatrix} a_{1,1} & \cdot & \cdot & \cdot & \cdot & a_{1,n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n+1,1} & \cdot & \cdot & \cdot & \cdot & a_{n+1,n+1} \end{vmatrix} \neq 0.$$

II.2. HOMOGENEOUS COORDINATES AND LINEAR TRANSFORMATIONS*

II.2.1. Homogeneous Coordinates. Although any point in the Euclidean plane is uniquely determined by two quantities, it is sometimes convenient to use three quantities to locate the point. If this is done, the precise values of the three quantities are not important, but it is their ratios which are of value. Let $(X^{(1)}, X^{(2)}, X^{(3)})$ be the three quantities describing a point and define the ratios

$$\frac{X^{(1)}}{X^{(3)}} = x^{(1)}, \qquad \frac{X^{(2)}}{X^{(3)}} = x^{(2)},$$

where $(x^{(1)}, x^{(2)})$ are the cartesian coordinates of the point in the plane. It is clear that any three numbers which are proportional to $(X^{(1)}, X^{(2)}, X^{(3)})$ will represent the same point, $(x^{(1)}, x^{(2)})$. Hence, to any set of three numbers, $(X^{(1)}, X^{(2)}, X^{(3)})$, will correspond one and only one point, $(x^{(1)}, x^{(2)})$; but to each point, $(x^{(1)}, x^{(2)})$, there will correspond an infinite number of sets of three numbers, all of which are proportional.

The set of numbers, (0, 0, 0) will not describe a point at all, since the homogeneous coordinates of any

* For a more complete discussion, see [(Bocher, (1), Chapters I and VI.].

point may be made as small as one pleases; hence (0, 0, 0) may be regarded as the limits of the homogeneous coordinates of any point.

What has been said above is true for n dimensions. In Euclidean n-dimensional space, $R^{(n)}$, the n + 1 numbers, $(X^{(1)}, \ldots, X^{(n+1)})$ will determine the point

$$\frac{X^{(1)}}{X^{(n+1)}} = x^{(1)}, \ldots\ldots, \frac{X^{(n)}}{X^{(n+1)}} = x^{(n)},$$

where $(x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ are the cartesian coordinates of the point in $R^{(n)}$. Any set of n + 1 quantities proportional to these will determine the same point of $R^{(n)}$. The set of numbers, (0, 0, . . . , 0) will represent no point at all.

II.2.2. Linear Transformations. The equations

$$\text{(II.2.3)} \quad \begin{aligned} \rho\overline{X}^{(1)} &= a_{1,1} X^{(1)} + a_{1,2} X^{(2)} + a_{1,3} X^{(3)} \\ \rho\overline{X}^{(2)} &= a_{2,1} X^{(1)} + a_{2,2} X^{(2)} + a_{2,3} X^{(3)} \\ \rho\overline{X}^{(3)} &= a_{3,1} X^{(1)} + a_{3,2} X^{(2)} + a_{3,3} X^{(3)} \end{aligned}$$

define a linear, homogeneous transformation of the Euclidean plane into itself. That is, if $(X^{(1)}, X^{(2)}, X^{(3)})$ are the homogeneous coordinates of any point in the plane, a second point, $(\rho\overline{X}^{(1)}, \rho\overline{X}^{(2)}, \rho\overline{X}^{(3)})$, where ρ is any constant $\neq 0$, will be determined by (II.2.3), the value of the coordinates, $(\rho\overline{X}^{(1)}, \rho\overline{X}^{(2)}, \rho\overline{X}^{(3)})$ depending on the coefficients,

$a_{i,j}$. If $(\rho\overline{X}^{(1)}, \rho\overline{X}^{(2)}, \rho\overline{X}^{(3)}) = (0, 0, 0)$, then the point $(X^{(1)}, X^{(2)}, X^{(3)})$ is not transformed into any point at all. This will happen only when the determinant of the coefficients is equal to zero. To insure that this never happens, only the case where the determinant of the transformation is not equal to zero will be considered. Such transformations are called non-singular. In this case, to every point $(X^{(1)}, X^{(2)}, X^{(3)})$ will correspond a definite point $(\rho\overline{X}^{(1)}, \rho\overline{X}^{(2)}, \rho\overline{X}^{(3)})$ and conversely.

A non-singular transformation such as (II.2.3) is continuous, one-to-one and transforms points on a line into points on a line.

What has been said concerning linear, homogeneous transformations in the plane can be extended easily to n dimensions. In this case, the following set of equations is considered:

(II.2.4) $$\rho\overline{X}^{(i)} = a_{i,1} X^{(1)} + \ldots + a_{i,n+1} X^{(n+1)}, \quad (i = 1, \ldots, n + 1),$$

where

$$\begin{vmatrix} a_{1,1} & \cdot & \cdot & \cdot & a_{1,n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n+1,1} & \cdot & \cdot & \cdot & a_{n+1,n+1} \end{vmatrix} \neq 0.$$

This transformation is one-to-one, continuous, and maps points of an n-1 dimensional hyperplane into points of an n-1 dimensional hyperplane.

In terms of cartesian coordinates, the transformation (II.2.4) is of the form

$$\text{(II.2.5)} \qquad \overline{x}^{(i)} = \frac{a_{i,1}x^{(1)} + \ldots a_{i,n}x^{(n)} + a_{i,n+1}}{a_{n+1,1}x^{(1)} + \ldots a_{n+1,n}x^{(n)} + a_{n+1,n+1}}, \quad (i = 1, \ldots, n).$$

The following theorems proved in homogeneous coordinates will be useful later on.

II.2.6. Theorem. Every set of values of $x_1, \ldots, x_n$, which satisifes a system of n-1 linearly independent homogeneous linear equations in n unknowns is proportional to the set of (n-1)-rowed determinants obtained by striking out from the matrix of the coefficients first the first column, then the second, etc.

Proof. Denote by $a^{(i)}$ the (n-1)-rowed determinant obtained by striking out the ith column from the matrix of the equations. Since the equations are linearly independent, at least one of the determinants, $a^{(1)}, a^{(2)}, \ldots, a^{(n)}$, does not vanish. Let it be $a^{(i)}$. Assign to x_i any fixed value, c. Then

$$\begin{array}{l} a_{1,1}x_1 + \ldots + a_{1,i-1}x_{i-1} + a_{1,i+1}x_{i+1} + \ldots + a_{1,n}x_n = -a_{1,i}c \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n-1,1}x_1 + \ldots + a_{n-1,i-1}x_{i-1} + a_{n-1,i+1}x_{i+1} + \ldots + a_{n-1,n}x_n = -a_{n-1,i}c \end{array}$$

This is a system of n linear non-homogeneous equations in n unknowns. Using Cramer's Rule, there is one and only one solution for each x_k.

Solving for x_k, using Cramer's Rule:

$$x_k = \frac{(-1)^{i-k} \; c \; a^{(k)}}{a^{(i)}}, \quad (k = 1, \ldots, n).$$

Hence, x_k is proportional to $a^{(k)}$, $(k = 1, \ldots, n)$.

II.2.7. Remark. If two or more of the determinants, $a^{(1)}, \ldots, a^{(n)}$, do not vanish (for example, $a^{(i)}$ and $a^{(j)}$), then one can assign any value to x_i and get a set of values for the remaining x's, as above. If one uses x_j instead (assigning any value to x_j), a different set of values for the x's will in general result. But once an x_i is picked and a value assigned, the remaining x's are uniquely determined by Cramer's Rule. In either case x_k will be proportional to $a^{(k)}$.

II.2.8. Theorem. Any four coplanar points, no three of which are collinear, may be carried over into any four coplanar points, no three of which are collinear, by one and only one transformation of the type

$$\bar{x}^{(1)} = \frac{a_{1,1} x^{(1)} + a_{1,2} x^{(2)} + a_{1,3}}{a_{3,1} x^{(1)} + a_{3,2} x^{(2)} + a_{3,3}}$$

(II.2.9)

$$\bar{x}^{(2)} = \frac{a_{2,1} x^{(1)} + a_{2,2} x^{(2)} + a_{2,3}}{a_{3,1} x^{(1)} + a_{3,2} x^{(2)} + a_{3,3}}$$

where

$$\begin{vmatrix} a_{1,1} & \cdot & \cdot & \cdot & a_{1,3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{3,1} & \cdot & \cdot & \cdot & a_{3,3} \end{vmatrix} \neq 0.$$

<u>Proof</u>. The theorem will be proved using homogeneous coordinates. The transformation will then be of the form

$$\text{(II.2.10)} \qquad \begin{aligned} \rho\overline{X}^{(1)} &= a_{1,1}\, X^{(1)} + a_{1,2}\, X^{(2)} + a_{1,3}\, X^{(3)} \\ \rho\overline{X}^{(2)} &= a_{2,1}\, X^{(1)} + a_{2,2}\, X^{(2)} + a_{2,3}\, X^{(3)} \\ \rho\overline{X}^{(3)} &= a_{3,1}\, X^{(1)} + a_{3,2}\, X^{(2)} + a_{3,3}\, X^{(3)} \end{aligned} .$$

Let $x_1 = (X_1^{(1)}, X_1^{(2)}, X_1^{(3)})$, $x_2 = (X_2^{(1)}, X_2^{(2)}, X_2^{(3)})$, $x_3 = (X_3^{(1)}, X_3^{(2)}, X_3^{(3)})$, and $x_4 = (X_4^{(1)}, X_4^{(2)}, X_4^{(3)})$ be the four initial points. Let $\overline{x}_1 = (\rho_1\overline{X}_1^{(1)}, \rho_1\overline{X}_1^{(2)}, \rho_1\overline{X}_1^{(3)})$, $\overline{x}_2 = (\rho_2\overline{X}_2^{(1)}, \rho_2\overline{X}_2^{(2)}, \rho_2\overline{X}_2^{(3)})$, $\overline{x}_3 = (\rho_3\overline{X}_3^{(1)}, \rho_3\overline{X}_3^{(2)}, \rho_3\overline{X}_3^{(3)})$, and $\overline{x}_4 = (\rho_4\overline{X}_4^{(1)}, \rho_4\overline{X}_4^{(2)}, \rho_4\overline{X}_4^{(3)})$ be the points into which the initial points are to be transformed.

The transformation (II.2.10) carries any point, $x = (X^{(1)}, X^{(2)}, X^{(3)})$, into a point, $\overline{x} = (\rho\overline{X}^{(1)}, \rho\overline{X}^{(2)}, \rho\overline{X}^{(3)})$, whose position depends on the values of the constants, $a_{i,j}$, $(i,j = 1, 2, 3)$. If it is possible to find one and only one (except for a constant factor which may be introduced throughout) set of thirteen constants (the nine above and four others -- ρ_1, ρ_2, ρ_3, ρ_4 ,none of which is zero) which satisfy the twelve equations

$$\text{(II.2.11)}\quad \begin{aligned} \rho_i \overline{X}_i^{(1)} &= a_{1,1} X_i^{(1)} + a_{1,2} X_i^{(2)} + a_{1,3} X_i^{(3)} \\ \rho_i \overline{X}_i^{(2)} &= a_{2,1} X_i^{(1)} + a_{2,2} X_i^{(2)} + a_{2,3} X_i^{(3)} \quad (i = 1, 2, 3, 4), \\ \rho_i \overline{X}_i^{3} &= a_{3,1} X_i^{(1)} + a_{3,2} X_i^{(2)} + a_{3,3} X_i^{(3)} \end{aligned}$$

the theorem will be proved.

Since the X's and the $\overline{X}$'s are all known, (II.2.11) represents a system of twelve homogeneous linear equations in thirteen unknowns. Hence, there are always solutions other than zero, the number of independent ones depending on the rank of the matrix of the coefficients. It will be shown that the rank of the matrix is twelve, and that the ρ's are all different from zero. Since the rank of the matrix is twelve, there will be only one independent solution and the theorem will be proved.

Transposing and rearranging the above twelve equations, one obtains (II.2.12):

$$X_1^{(1)}a_{1,1}+X_1^{(2)}a_{1,2}+X_1^{(3)}a_{1,3}+0+0+0+0+0+0-\rho_1\overline{X}_1^{(1)}+0+0+0=0,$$
$$X_2^{(1)}a_{1,1}+X_2^{(2)}a_{1,2}+X_2^{(3)}a_{1,3}+0+0+0+0+0+0+0-\rho_2\overline{X}_2^{(1)}+0+0=0,$$
$$X_3^{(1)}a_{1,1}+X_3^{(2)}a_{1,1}+X_3^{(3)}a_{1,3}+0+0+0+0+0+0+0+0-\rho_3\overline{X}_3^{(1)}+0=0,$$
$$0+0+0+X_1^{(1)}a_{2,1}+X_1^{(2)}a_{2,2}+X_1^{(3)}a_{2,3}+0+0+0-\rho_1\overline{X}_1^{(2)}\ 0+0+0=0,$$
$$0+0+0+X_2^{(1)}a_{2,1}+X_2^{(2)}a_{2,2}+X_2^{(3)}a_{2,3}+0+0+0+0-\rho_2\overline{X}_2^{(2)}+0+0=0,$$
$$0+0+0+X_3^{(1)}a_{2,1}+X_3^{(2)}a_{2,2}+X_3^{(3)}a_{2,3}+0+0+0+0+0-\rho_3\overline{X}_3^{(2)}+0=0,$$
$$0+0+0+0+0+0+X_1^{(1)}a_{3,1}+X_1^{(2)}a_{3,2}+X_1^{(3)}a_{3,3}-\rho_1\overline{X}_1^{(3)}+0+0+0=0,$$
$$0+0+0+0+0+0+X_2^{(1)}a_{3,1}+X_2^{(2)}a_{3,2}+X_2^{(3)}a_{3,3}+0-\rho_2\overline{X}_2^{(3)}+0+0=0,$$
$$0+0+0+0+0+0+X_3^{(1)}a_{3,1}+X_3^{(2)}a_{3,2}+X_3^{(3)}a_{3,3}+0+0-\rho_3\overline{X}_3^{(3)}+0=0,$$
$$X_4^{(1)}a_{1,1}+X_4^{(2)}a_{1,2}+X_4^{(3)}a_{1,3}+0+0+0+0+0+0+0+0+0-\rho_4\overline{X}_4^{(1)}=0,$$
$$0+0+0+X_4^{(1)}a_{2,1}+X_4^{(2)}a_{2,2}+X_4^{(3)}a_{2,3}+0+0+0+0+0+0-\rho_4\overline{X}_4^{(2)}=0,$$
$$0+0+0+0+0+0+X_4^{(1)}a_{3,1}+X_4^{(2)}a_{3,2}+X_4^{(3)}a_{3,3}+0+0+0-\rho_4\overline{X}_4^{(3)}=0.$$

The matrix of these equations is (II.2.13):

$$\begin{pmatrix}
X_1^{(1)} & X_1^{(2)} & X_1^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{X}_1^{(1)} & 0 & 0 & 0 \\
X_2^{(1)} & X_2^{(2)} & X_2^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{X}_2^{(1)} & 0 & 0 \\
X_3^{(1)} & X_3^{(2)} & X_3^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{X}_3^{(1)} & 0 \\
0 & 0 & 0 & X_1^{(1)} & X_1^{(2)} & X_1^{(3)} & 0 & 0 & 0 & -\overline{X}_1^{(2)} & 0 & 0 & 0 \\
0 & 0 & 0 & X_2^{(1)} & X_2^{(2)} & X_2^{(3)} & 0 & 0 & 0 & 0 & -\overline{X}_2^{(2)} & 0 & 0 \\
0 & 0 & 0 & X_3^{(1)} & X_3^{(2)} & X_3^{(3)} & 0 & 0 & 0 & 0 & 0 & -\overline{X}_3^{(2)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_1^{(1)} & X_1^{(2)} & X_1^{(3)} & -\overline{X}_1^{(3)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_2^{(1)} & X_2^{(2)} & X_2^{(3)} & 0 & -\overline{X}_2^{(3)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_3^{(1)} & X_3^{(2)} & X_3^{(3)} & 0 & 0 & -\overline{X}_3^{(3)} & 0 \\
X_4^{(1)} & X_4^{(2)} & X_4^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{X}_4^{(1)} \\
0 & 0 & 0 & X_4^{(1)} & X_4^{(2)} & X_4^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{X}_4^{(2)} \\
0 & 0 & 0 & 0 & 0 & 0 & X_4^{(1)} & X_4^{(2)} & X_4^{(3)} & 0 & 0 & 0 & -\overline{X}_4^{(3)}
\end{pmatrix}.$$

Since x_1, x_2, x_3, and x_4 are distinct, coplanar and no three lie on a line, there exist non-zero constants, c_1, c_2, c_3, such that

$$
\begin{aligned}
c_1 X_1^{(1)} + c_2 X_2^{(1)} + c_3 X_3^{(1)} + X_4^{(1)} &= 0, \\
c_1 X_1^{(2)} + c_2 X_2^{(2)} + c_3 X_3^{(2)} + X_4^{(2)} &= 0, \\
c_1 X_1^{(3)} + c_2 X_2^{(3)} + c_3 X_3^{(3)} + X_4^{(3)} &= 0.
\end{aligned}
$$

Adding c_1, c_2, c_3 times the first, second and third rows respectively to the tenth row; c_1, c_2, c_3 times the fourth, fifth and sixth rows respectively to the eleventh row; and c_1, c_2, c_3 times the seventh, eighth and ninth rows respectively to the twelfth row, (II.2.13) becomes (II.2.14):

$$
\begin{pmatrix}
X_1^{(1)} & X_1^{(2)} & X_1^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{X}_1^{(1)} & 0 & 0 & 0 \\
X_2^{(1)} & X_2^{(2)} & X_2^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{X}_2^{(1)} & 0 & 0 \\
X_3^{(1)} & X_3^{(2)} & X_3^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{X}_3^{(1)} & 0 \\
0 & 0 & 0 & X_1^{(1)} & X_1^{(2)} & X_1^{(3)} & 0 & 0 & 0 & -\bar{X}_1^{(2)} & 0 & 0 & 0 \\
0 & 0 & 0 & X_2^{(1)} & X_2^{(2)} & X_2^{(3)} & 0 & 0 & 0 & 0 & -\bar{X}_2^{(2)} & 0 & 0 \\
0 & 0 & 0 & X_3^{(1)} & X_3^{(2)} & X_3^{(3)} & 0 & 0 & 0 & 0 & 0 & -\bar{X}_3^{(2)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_1^{(1)} & X_1^{(2)} & X_1^{(3)} & -\bar{X}_1^{(3)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_2^{(1)} & X_2^{(2)} & X_2^{(3)} & 0 & -\bar{X}_2^{(3)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_3^{(1)} & X_3^{(2)} & X_3^{(3)} & 0 & 0 & -\bar{X}_3^{(3)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_1\bar{X}_1^{(1)} & -c_2\bar{X}_2^{(1)} & -c_3\bar{X}_3^{(1)} & -\bar{X}_4^{(1)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_1\bar{X}_1^{(2)} & -c_2\bar{X}_2^{(2)} & -c_3\bar{X}_3^{(2)} & -\bar{X}_4^{(2)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_1\bar{X}_1^{(3)} & -c_2\bar{X}_2^{(3)} & -c_3\bar{X}_3^{(3)} & -\bar{X}_4^{(3)}
\end{pmatrix}.
$$

The rank of the matrix is unchanged by the above operations.

If the thirteenth column is deleted, the determinant of the resulting matrix is easily calculated to be

$$D_{(13)} = -c_1 c_2 c_3 \begin{vmatrix} X_1^{(1)} & X_1^{(2)} & X_1^{(3)} \\ X_2^{(1)} & X_2^{(2)} & X_2^{(3)} \\ X_3^{(1)} & X_3^{(2)} & X_3^{(3)} \end{vmatrix} \quad \begin{vmatrix} \overline{X}_1^{(1)} & \overline{X}_1^{(2)} & \overline{X}_1^{(3)} \\ \overline{X}_2^{(1)} & \overline{X}_2^{(2)} & \overline{X}_2^{(3)} \\ \overline{X}_3^{(1)} & \overline{X}_3^{(2)} & \overline{X}_3^{(3)} \end{vmatrix} \neq 0,$$

since x_1, x_2, x_3, and $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$ are distinct points and neither set is collinear. So the rank of the matrix (II.2.14) is twelve and there is only one linearly independent solution for the thirteen constants. Furthermore, by Theorem II.2.6, $\rho_4 \neq 0$, since it is proportional to $D_{(13)} \neq 0$.

In a similar fashion, $D_{(12)}$, $D_{(11)}$, and $D_{(10)}$ are all different from zero. Hence, ρ_3, ρ_2, and ρ_1 are all different from zero, by Theorem II.2.6. Thus, precisely one linearly independent solution for the thirteen constants can be found such that ρ_1, ρ_2, ρ_3, and ρ_4 are all different from zero, and the theorem is proved.

II.3. p-FLATS AND THEIR PROPERTIES

II.3.1. In Chapter I, the definition of a p-cell in $R^{(n)}$ was given. The volume of a p-cell in $R^{(n)}$, determined by the p + 1 points, x_0, x_1, . . . , x_p, $(0 \leq p \leq n)$, is proportional to

$$\begin{vmatrix} x_0^{(i_1)} & x_0^{(i_2)} & \cdots & x_0^{(i_p)} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_p^{(i_1)} & x_p^{(i_2)} & \cdots & x_p^{(i_p)} & 1 \end{vmatrix}$$

where the set $(i_1, i_2, \ldots, i_3)$ represents a selection of p of the possible n columns of the matrix

$$\begin{pmatrix} x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ x_p^{(1)} & x_p^{(2)} & \cdots & x_p^{(n)} \end{pmatrix}$$

and where the symbol $\sum_{C(n,p)}$ signifies that the sum extends over all the C(n,p) possible selections for $(i_1, i_2, \ldots, i_p)$ [Birkhoff-MacLane, (1), pp. 293-296] and [Price, (1), pp. 77-78]. The p-cell will have p-dimensional volume equal to zero only if each of the determinants in the sum is equal to zero. If this occurs, then one of the points, $x_0, x_1, \ldots, x_p$, can be represented as a linear combination of the other points. This gives rise to the following definition:

II.3.2. <u>Definition</u>. Let $x_0, x_1, \ldots, x_p$, $(0 \leq p \leq n)$, be p + 1 points of $R^{(n)}$. These points are said to be <u>linearly independent</u> if at least one of the C(n,p) determinants of the form

$$\text{(II.3.3)} \qquad \begin{vmatrix} x_0^{(i_1)} & \cdots & x_0^{(i_p)} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ x_p^{(i_1)} & \cdots & x_p^{(i_p)} & 1 \end{vmatrix}$$

is different from zero, where the set of superscripts $(i_1, i_2, \ldots, i_p)$ represents one of the $C(n,p)$ possible selections of p of the n columns of the matrix

$$\begin{pmatrix} x_0^{(1)} & \cdots & x_0^{(n)} \\ \cdots & \cdots & \cdots \\ x_p^{(1)} & \cdots & x_p^{(n)} \end{pmatrix}.$$

Otherwise, the $p + 1$ points are said to be linearly dependent.

II.3.4. <u>Definition</u>. Let $x_0, x_1, \ldots, x_p$, $(0 \le p \le n)$, be $p + 1$ points of $R^{(n)}$. These points are said to be <u>linearly independent</u> if the vectors, $(x_i - x_0)$, $(i = 1, \ldots, p)$ are linearly independent in the ordinary sense; that is, if there are no constants, a_i, $(i = 1, \ldots, p)$, except all zeros, such that

$$\sum_{i=1}^{p} a_i (x_i^{(j)} - x_0^{(j)}) = \theta, \quad (j = 1, \ldots, n),$$

where θ represents the zero vector.

II.3.5. <u>Theorem</u>. Definition II.3.2. and Definition II.3.4. are equivalent.

<u>Proof</u>. The quantities

$$(x_1^{(j)} - x_0^{(j)}), \ldots, (x_p^{(j)} - x_0^{(j)}), \ (j = 1, \ldots, n),$$

are linearly dependent if and only if all the p-rowed minors of the matrix

$$\text{(II.3.6)} \quad \begin{pmatrix} (x_1^{(1)} - x_0^{(1)}) & (x_1^{(2)} - x_0^{(2)}) & \dots & (x_1^{(n)} - x_0^{(n)}) \\ \dots & \dots & \dots & \dots \\ (x_p^{(1)} - x_0^{(1)}) & (x_p^{(2)} - x_0^{(2)}) & \dots & (x_p^{(n)} - x_0^{(n)}) \end{pmatrix}$$

are equal to zero; that is, if and only if all the C(n,p) determinants of the type

$$\text{(II.3.7)} \quad \begin{vmatrix} (x_1^{(i_1)} - x_0^{(i_1)}) & (x_1^{(i_2)} - x_0^{(i_2)}) & \dots & (x_1^{(i_p)} - x_0^{(i_p)}) \\ \dots & \dots & \dots & \dots \\ (x_p^{(i_1)} - x_0^{(i_1)}) & (x_p^{(i_2)} - x_0^{(i_2)}) & \dots & (x_p^{(i_p)} - x_0^{(i_p)}) \end{vmatrix}$$

are equal to zero, where the superscripts have the same meaning and range as in Definition II.3.2 [Bocher, (1), p. 36]. But the determinant (II.3.7) is equal to zero if and only if

$$\text{(II.3.8)} \quad \begin{vmatrix} x_0^{(i_1)} & \dots & x_0^{(i_p)} & 1 \\ x_1^{(i_1)} & \dots & x_1^{(i_p)} & 1 \\ \dots & \dots & \dots & \dots \\ x_p^{(i_1)} & \dots & x_p^{(i_p)} & 1 \end{vmatrix}$$

is equal to zero, since (II.3.7) and (II.3.8) are equal except possibly for sign.

This is precisely the definition of linear dependence of the p + 1 points according to Definition II.3.2. Consequently, the p + 1 points are linearly independent according to Definition II.3.2. if and only if they are linearly independent according to Definition II.3.4.

II.3.9. <u>Remark</u>. In Definition II.3.4, the point x_0 has, at first glance, been given a preferred position. However, the vectors $(x_i^{(j)} - x_k^{(j)})$, (k fixed; $i = 0, 1, \ldots, p$; $i \neq k$), are linearly independent if and only if $(x_i^{(j)} - x_o^{(j)})$, $(i = 1, \ldots, p)$, are linearly independent.

Proof. The necessary and sufficient condition that the vectors $(x_i^{(j)} - x_k^{(j)})$, (k fixed; $i = 0, 1, \ldots, p$; $i \neq k$), be linearly dependent is that the determinant of every p-rowed minor of the matrix

$$\text{(II.3.10)} \quad \begin{pmatrix} (x_0^{(1)} - x_k^{(1)}) & \ldots & (x_0^{(n)} - x_k^{(n)}) \\ \cdots & \cdots & \cdots \\ (x_{k-1}^{(1)} - x_k^{(1)}) & \ldots & (x_{k-1}^{(n)} - x_k^{(n)}) \\ (x_{k+1}^{(1)} - x_k^{(1)}) & \ldots & (x_{k+1}^{(n)} - x_k^{(n)}) \\ \cdots & \cdots & \cdots \\ (x_p^{(1)} - x_k^{(1)}) & \ldots & (x_p^{(n)} - x_k^{(n)}) \end{pmatrix}$$

vanish. But these p-rowed determinants are

$$\text{(II.3.11)} \quad \begin{vmatrix} (x_0^{(i_1)} - x_k^{(i_1)}) & \ldots & (x_0^{(i_p)} - x_k^{(i_p)}) \\ \cdots & \cdots & \cdots \\ (x_{k-1}^{(i_1)} - x_k^{(i_1)}) & \ldots & (x_{k-1}^{(i_p)} - x_k^{(i_p)}) \\ (x_{k+1}^{(i_1)} - x_k^{(i_1)}) & \ldots & (x_{k+1}^{(i_p)} - x_k^{(i_p)}) \\ \cdots & \cdots & \cdots \\ (x_p^{(i_1)} - x_k^{(i_1)}) & \ldots & (x_p^{(i_p)} - x_k^{(i_p)}) \end{vmatrix}$$

where the superscripts have the same meaning and range as in Definition II.3.2. But after the proper expansion each of these determinants is the same as one of those of the type (II.3.8), except possibly for sign. Each of the determinants of the type (II.3.8) is the same as one of the p-rowed minors of determinant (II.3.6), where x_o is given the preferred position.

Hence, all the p-rowed minors of (II.3.10) vanish if and only if all the p-rowed minors of (II.3.6) vanish, and the vectors, $(x_i^{(j)} - x_k^{(j)})$, (k fixed; $i = 0, 1, \ldots, p$; $i \neq k$), are linearly dependent if and only if the vectors $(x_i^{(j)} - x_o^{(j)})$, $(i = 1, \ldots, p)$, are linearly dependent, proving the statement.

The result of this remark is that the point x_o can always be put in the preferred position without any loss of generality, and with more convenience.

II.3.12. <u>Theorem</u>. Let $x_o, \ldots, x_p$ be a set of p + 1 linearly independent points of $R^{(n)}$. Then any subset of those points is linearly independent.

<u>Proof</u>. Suppose there is a subset of $x_o, \ldots, x_p$ which forms a linearly dependent set of points. Assume, without any loss of generality, that the points are arranged so that the points, $x_o, \ldots, x_j$, $(j < p)$, form the linearly dependent set. Consider the vectors, $(x_i - x_o)$, $(i = 1, \ldots, j)$. These are linearly dependent; hence, there are constants, a_i, $(i = 1, \ldots, j)$ not all zero, such that

$$\sum_{i=1}^{j} a_i (x_i - x_o) = \theta.$$

But then,

$$\sum_{i=1}^{j} a_i (x_i - x_o) + \sum_{i=j+1}^{p} 0 (x_i - x_o) = \theta$$

where not all the a_i's are zero. This contradicts the assumption that $x_o, \ldots, x_p$ are linearly independent. Hence it must be concluded that any subset of $x_o, \ldots, x_p$ is also linearly independent.

II.3.13. Definition. Let $x_o, \ldots, x_p$ $(p \leq n)$, be any $p + 1$ fixed linearly independent points of $R^{(n)}$. By the p-flat, S_p, of $R^{(n)}$, determined by $x_o, \ldots, x_p$, is meant the set of all points x of $R^{(n)}$ which can be represented as

$$x^{(j)} = \sum_{i=0}^{p} \alpha_i x_i^{(j)} \quad (j = 1, \ldots, n),$$

where $\sum_{i=0}^{p} \alpha_i = 1$.

II.3.14. Remark. The numbers, α_i, $(i = 0, 1, \ldots, p)$, are sometimes known as the barycentric coordinates of the point x. See [Alexandroff-Hopf, (1), p. 595].

II.3.15. Definition. Let $x_o, \ldots, x_p$, $(p \leq n)$, be $p + 1$ linearly independent points of $R^{(n)}$. By the p-flat, S_p, of $R^{(n)}$, determined by the points $x_o, \ldots, x_p$, is meant the set of all points, x, of $R^{(n)}$ for which the vectors, $(x^{(j)} - x_o^{(j)})$, satisfy the relation

$$(x^{(j)} - x_o^{(j)}) = \sum_{i=1}^{p} \beta_i (x_i^{(j)} - x_i^{(j)}), \quad (j = 1, \ldots, n),$$

with no restrictions on the β's.

II.3.16. <u>Theorem</u>. Definition II.3.13 and Definition II.3.15 are equivalent.

<u>Proof</u>. Let $x_0, \ldots, x_p$ be a set of p + 1 linearly independent points of $R^{(n)}$. They determine a p-flat, S_p, according to Definition II.3.13 and according to Definition II.3.15. Let x be a point of S_p according to Definition II.3.13. Then

$$x = \alpha_0 x_0 + \alpha_1 x_1 + \ldots + \alpha_p x_p, \quad \sum_{i=0}^{p} \alpha_i = 1.$$

Since $\sum_{i=0}^{p} \alpha_i = 1$, then $\alpha_0 = 1 - \sum_{i=1}^{p} \alpha_i$. Hence

$$x = (1 - \sum_{i=1}^{p} \alpha_i) x_0 + \alpha_1 x_1 + \ldots + \alpha_p x_p.$$

Collecting terms,

$$(x - x_0) = \alpha_1(x_1 - x_0) + \ldots + \alpha_p(x_p - x_0)$$
$$= \sum_{i=1}^{p} \alpha_i (x_i - x_0),$$

with no restrictions on α_i, (i = 1, . . . , p). Hence, x is a point of S_p according to Definition II.3.15.

Now suppose x is a point of S_p according to Definition II.3.15. Then

$$(x - x_0) = \sum_{i=1}^{p} \beta_i (x_i - x_0),$$

with no restrictions on the β's. Rearranging terms, this becomes

$$x = x_0 + \sum_{i=1}^{p} \beta_i (x_i - x_0) = x_0 - \sum_{i=1}^{p} \beta_i x_0 + \sum_{i=1}^{p} \beta_i x_i$$

$$= (1 - \sum_{i=1}^{p} \beta_i) x_0 + \sum_{i=1}^{p} \beta_i x_i .$$

Setting $(1 - \sum_{i=1}^{p} \beta_i) = \beta_0$, then $\sum_{i=0}^{p} \beta_i = 1$, and

$$x = \sum_{i=0}^{p} \beta_i x_i , \quad \text{where} \quad \sum_{i=0}^{p} \beta_i = 1.$$

That is, x is a point of S_p according to Definition II.3.13. Hence, the two definitions produce the same set of points, and are equivalent.

II.3.17. Remark. The last theorem permits one to use either Definition II.3.13 or Definition II.3.15 in discussing a p-flat. Sometimes it is more convenient to use the one definition; sometimes it is more convenient to use the other. In the following pages, both definitions will be used interchangeably.

II.3.18. Theorem. Let $x_0, x_1, \ldots, x_p$ be p + 1 linearly independent points of $R^{(n)}$ which determine a p-flat, S_p. Then every point x of S_p has a unique representation of the form

$$x = \sum_{i=0}^{p} \alpha_i x_i , \qquad \sum_{i=0}^{p} \alpha_i = 1.$$

Proof. Suppose also that

$$x = \sum_{i=0}^{p} \beta_i x_i , \qquad \sum_{i=0}^{p} \beta_i = 1.$$

Then,

$$(x - x_0) = \alpha_1(x_1 - x_0) + \cdots + \alpha_p(x_p - x_0),$$

and

$$(x - x_0) = \beta_1(x_1 - x_0) + \cdots + \beta_p(x_p - x_0).$$

But these are vectors. Subtracting, the result is

$$\theta = (\alpha_1 - \beta_1)(x_1 - x_0) + \cdots + (\alpha_p - \beta_p)(x_p - x_0).$$

Since $x_0, \ldots, x_p$ are linearly independent, then so are $(x_1 - x_0), \ldots, (x_p - x_0)$. Hence, one concludes that $(\alpha_i - \beta_i) = 0$, $(i = 1, \ldots, p)$. That is, $\alpha_i = \beta_i$ $(i = 1, \ldots, p)$. Therefore, $\alpha_0 = \beta_0$, also, and the representation is unique, proving the theorem.

II.3.19. Theorem. Let S_p be a p-flat of $R^{(n)}$, determined by the $p + 1$ linearly independent points $x_0, x_1, \ldots, x_p$, and let $y_0, \ldots, y_p$ be any other set of $p + 1$ linearly independent points of S_p. Then S_p can be determined by the $p + 1$ points, $y_0, \ldots, y_p$.

Proof. Since y_p is a point of S_p, then

$$y_p = \sum_{i=0}^{p} \alpha_i x_i, \qquad \sum_{i=0}^{p} \alpha_i = 1.$$

At least one of the coefficients is not zero. Suppose $\alpha_K \neq 0$.

Consider the set $y_p, x_0, \ldots, x_{K-1}, x_{K+1}, \ldots, x_p$. This set of points is linearly independent. For suppose it is linearly dependent. Then there exist constants a_i, $(i = 1, \ldots, p;\ i \neq k)$, not all zero, and a b_p, such that

$$\sum_{\substack{i=1\\ i\neq k}}^{p} a_i(x_i - x_o) + b_p\,(y_p - x_o) = \theta.$$

The number b_p cannot be zero, since then $x_o, \ldots, x_{k-1}, x_{k+1}, \ldots, x_p$ would be linearly dependent, which is impossible, by Theorem II.3.12. Hence

$$-b_p(y_p - x_o) = \sum_{\substack{i=1\\ i\neq k}}^{p} a_i(x_i - x_o).$$

Dividing by $-b_p$, this becomes

$$(y_p - x_o) = \sum_{\substack{i=1\\ i\neq k}}^{p} c_i(x_i - x_o), \qquad c_i = a_i/-b_p.$$

Then

$$y_p = (1 - \sum_{\substack{i=1\\ i\neq k}}^{p} c_i)x_o + \sum_{\substack{i=1\\ i\neq k}}^{p} c_i x_i.$$

Place $(1 - \sum_{\substack{i=1\\ i\neq k}}^{p} c_i) = c_o$. Then $\sum_{\substack{i=1\\ i\neq k}}^{p} c_i = 1$, and

$$y_p = \sum_{\substack{i=1\\ i\neq k}}^{p} c_i x_i, \qquad \sum_{\substack{i=1\\ i\neq k}}^{p} c_i = 1.$$

But then y_p is a linear combination of $x_o, \ldots, x_{k-1}, x_{k+1}, \ldots, x_p$. Since the representation is unique, this contradicts the assumption that $\alpha_k \neq 0$. Hence, the points $y_p, x_o, \ldots, x_{k-1}, x_{k+1}, \ldots, x_p$ are linearly independent.

Now since α_k is different from zero, x_k can be solved for, as follows:

(II.3.20 $$x_K = \frac{y_\rho - \sum_{\substack{i=0 \\ i \neq K}}^{\rho} \alpha_i x_i}{\alpha_K} .$$

Since $\sum_{i=0}^{\rho} \alpha_i = 1$, then $\alpha_K = 1 - \sum_{\substack{i=0 \\ i \neq K}}^{\rho} \alpha_i$. Hence, $-\sum_{\substack{i=0 \\ i \neq K}}^{\rho} \alpha_i = \alpha_K - 1$.

Therefore, the sum of the coefficients in (II.3.20) is

$$\frac{1 - \sum_{\substack{i=0 \\ i \neq K}}^{\rho} \alpha_i}{\alpha_K} = \frac{1 + (\alpha_K - 1)}{\alpha_K} = \frac{\alpha_K}{\alpha_K} = 1.$$

It follows from the above statements that the set of points y_ρ, x_0, . . . , x_{K-1}, x_{K+1}, . . . , x_ρ determine S_ρ. For let $\overline{x}$ be any point in S_ρ. Then

$$\overline{x} = \beta_0 x_0 + \cdot \cdot \cdot + \beta_{K-1} x_{K-1} + \beta_K x_K + \beta_{K+1} x_{K+1} + \cdot \cdot \cdot + \beta_\rho x_\rho$$

where $\sum_{i=0}^{\rho} \beta_i = 1$. Taking into account (II.3.20), this becomes

$$\overline{x} = \beta_0 x_0 + \cdot \cdot + \beta_{K-1} x_{K-1} + \beta_K \left(\frac{y_\rho - \sum_{\substack{i=0 \\ i \neq K}}^{\rho} \alpha_i x_i}{\alpha_K} \right) + \beta_{K+1} x_{K+1} + \cdot \cdot \cdot + \beta_\rho x_\rho$$

$$= \left(\beta_0 - \frac{\beta_K \alpha_0}{\alpha_K}\right) x_0 + \cdot \cdot + \left(\beta_{K-1} - \frac{\beta_K \alpha_{K-1}}{\alpha_K}\right) x_{K-1} + \frac{\beta_K}{\alpha_K} y_\rho + \left(\beta_{K+1} - \frac{\beta_K \alpha_{K+1}}{\alpha_K}\right) x_{K+1} + \cdot \cdot + \left(\beta_\rho - \frac{\beta_K \alpha_\rho}{\alpha_K}\right) x_\rho .$$

Consider the sum of the coefficients in the above expression:

$$(\beta_0 - \frac{\beta_K \alpha_0}{\alpha_K}) + \cdot \cdot + (\beta_{K-1} - \frac{\beta_K \alpha_{K-1}}{\alpha_K}) + \frac{\beta_K}{\alpha_K} + (\beta_{K+1} - \frac{\beta_K \alpha_{K+1}}{\alpha_K}) + \cdot \cdot \cdot + (\beta_p - \frac{\beta_K \alpha_p}{\alpha_K})$$

$$= \beta_0 + \cdot \cdot \cdot + \beta_{K-1} + \frac{\beta_K}{\alpha_K} + \beta_{K+1} + \cdot \cdot + \beta_p - \frac{\beta_K}{\alpha_K}(\alpha_0 + \cdot \cdot + \alpha_{K-1} + \alpha_{K+1} + \cdot \cdot \cdot + \alpha_p)$$

$$= \beta_0 + \cdot \cdot \cdot + \beta_{K-1} + \beta_K \left(\frac{1 - \alpha_0 - \cdot \cdot \cdot - \alpha_{K-1} - \alpha_{K+1} - \cdot \cdot \cdot - \alpha_p}{\alpha_K}\right) + \beta_{K+1} + \cdot \cdot + \beta_p$$

$$= \beta_0 + \cdot \cdot \cdot + \beta_{K-1} + \beta_K + \beta_{K+1} + \cdot \cdot \cdot + \beta_p = 1,$$

since $1 - \sum_{\substack{i=0 \\ i \neq K}}^{p} \alpha_i = 1$. Hence $\overline{x}$ is a linear combination of the linearly independent points $y_p, x_o, \ldots, x_{K-1}, x_{K+1}, \ldots, x_p$, with the sum of the coefficients equal to 1. Therefore, this set of points determines S_p.

Now further,

$$y_{p-1} = \delta_p y_p + \sum_{\substack{i=0 \\ i \neq K}}^{p} \gamma_i x_i,$$

where $\delta_p + \sum_{\substack{i=0 \\ i \neq K}}^{p} \gamma_i = 1$. All the γ_i cannot be equal to zero, for then y_{p-1} would be equal to y_p, contradicting the assumption that the y's are linearly independent. Hence there must be a γ_i (say γ_ℓ) which is different from zero. As before, one proves that x_ℓ is a linear combination of the set of points

(II.3.21) $y_{p-1}, y_p, x_o, \ldots, x_{K-1}, x_{K+1}, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_p,$

with the sum of the coefficients equal to one; that the set (II.3.21) is linearly independent; and that the set (II.3.21) determines S .

Repeating the same argument for each of the p + 1 y's in succession, it is finally shown that every point of S_ρ can be written as a linear combination of the y's with the sum of the coefficients equal to one. Hence, since $y_o, \dots, y_\rho$ was <u>any</u> set of p + 1 linearly independent points of S_ρ, the Theorem is proved.

II.3.22. <u>Theorem</u>. If $y_o, \dots, y_r$, $(r < p)$, are r + 1 linearly independent points in a p-flat, S_ρ, of $R^{(n)}$, then it is always possible to find p - r more points, $x_{r+1}, \dots, x_\rho$, of S_ρ, so that the points

$$y_o, \dots, y_r, x_{r+1}, \dots, x_\rho$$

form a set of p + 1 linearly independent points which determine S_ρ.

<u>Proof</u>. There are p + 1 linearly independent points, $x_o, \dots x_\rho$, which determine S_ρ. Carrying through the same procedure as in Theorem II.3.19, it is found, after r + 1 steps, that $y_o, \dots, y_r$, plus p - r of the set of x's, form a set of p + 1 linearly independent points which determine the p-flats, S_ρ.

II.3.23. <u>Lemma</u>. Let $x_o, \dots, x_\rho$ be a set of p + 1 linearly independent points lying in a p-flat, S_ρ, $(p \leq n-1)$. Let x_{p+1} be a point of $R^{(n)}$ which is not in S_ρ. Then $x_o, \dots, x_\rho, x_{\rho+1}$ form a set of p + 2 linearly

independent points.

Proof. Suppose the points, $x_0, \ldots, x_p, x_{p+1}$, are linearly dependent. Then the vectors, $(x_1 - x_0), \ldots, (x_{p+1} - x_0)$, are linearly dependent. That is, there exist constants, a_i, $(i = 1, \ldots, p + 1)$, not all zero, such that

$$\sum_{i=1}^{p+1} a_i(x_i - x_0) = \theta.$$

Now a_{p+1} is different from zero, since it has been assumed that the points $x_0, \ldots, x_p$ are linearly independent. Hence,

$$-a_{p+1}(x_{p+1} - x_0) = \sum_{i=1}^{p} a_i(x_i - x_0).$$

Dividing by $-a_{p+1}$, one obtains

$$(x_{p+1} - x_0) = \sum_{i=1}^{p} b_i(x_i - x_0),$$

where $b_i = a_i/-a_{p+1}$. Hence

$$x_{p+1} = (1 - \sum_{i=1}^{p} b_i)x_0 + \sum_{i=1}^{p} b_i x_i.$$

Setting $(1 - \sum_{i=1}^{p} b_i) = b_0$, then $\sum_{i=0}^{p} b_i = 1$, and

$$x_{p+1} = \sum_{i=0}^{p} b_i x_i, \qquad \sum_{i=0}^{p} b_i = 1.$$

This implies that x_{p+1} is in S_p, contradicting the assumption that x_{p+1} is not in S_p. Hence the points $x_0, \ldots, x_p, x_{p+1}$ must be linearly independent.

II.3.24. <u>Theorem</u>. If a p-flat, S_p, and a q-flat, S_q, $(p,q < n)$, have an r-flat, S_r, $(r \leq \min(p,q))$, in common, then the whole configuration lies in a $(p + q - r)$-flat.

<u>Proof</u>. S_r is common to both S_p and S_q. S_r is determined by $r + 1$ linearly independent points, $x_0, \ldots, x_r$. By Theorem II.3.22, since these points lie in S_p, $p - r$ other points, $y_{r+1}, \ldots, y_p$, all lying in S_p and not in S_r, can be found so that

$$x_0, \ldots, x_r, y_{r+1}, \ldots, y_p$$

form a set of $p + 1$ linearly independent points which determine S_p. Also, since $x_0, \ldots, x_r$ all lie in S_q, $q - r$ other points, $z_{r+1}, \ldots, z_q$, all lying in S_q and not in S_r, can be found so that

$$x_0, \ldots, x_r, z_{r+1}, \ldots, z_q$$

form a set of $q + 1$ linearly independent points which determine S_q. The points $y_{r+1}, \ldots, y_p$ are linearly independent with the points determining S_q, by Lemma II.3.23, and the points $z_{r+1}, \ldots, z_q$ are linearly independent with the points determining S_p, by the same lemma. Hence, the total number of linearly independent points in the configuration is

$$(r + 1) + (p - r) + (q - r) = (p + q - r + 1).$$

This is the number of linearly independent points required to determine a (p + q - r)-flat. Clearly, this (p + q - r)-flat contains all the points of both S_p and S_q, and the theorem is proved.

II.4. DIMENSION OF A p-FLAT

II.4.1. Some mathematicians feel that one of the most important theories in analysis is that of dimension. There have been many definitions of dimension, the early ones being quite vague and intuitive. Such men as Cantor and Peano first made it clear that precise definitions of dimension were needed when they produced examples contradicting some of the beliefs concerning dimension. These examples showed that the dimension of a space can be changed by either a one-to-one transformation or by a continuous transformation.

The question as to whether a one-to-one and continuous transformation can change the dimension of a space was answered (in the case of Euclidean space) by Brouwer in 1911 [Brouwer, (1)], when he showed that m-dimensional Euclidean space cannot be the continuous and one-to-one image of n-dimensional Euclidean space, unless m = n. In other words, dimension is a topological property of Euclidean space. Brouwer further showed [Brouwer, (3)], that n-dimensional Euclidean space is precisely n-dimensional.

In 1922 Menger and Urysohn independently gave a definition of dimension which is applicable to very general sets of points in a <u>metric</u> <u>separable</u> space, [Menger, (1) and (2)] and [Urysohn, (1) and (2)]. This work was independent of Brouwer's work and, while it closely followed the work of Brouwer, there were improvements as well.

Hurewicz and Wallman use the definition of Menger and Urysohn to prove that n-dimensional Euclidean space is precisely n-dimensional [Hurewicz-Wallman, (1), Chapters II, III, and IV]. This definition is as follows:

II.4.2. <u>Definition</u>

1. The empty set and only the empty set has dimension -1.

2. A space X has dimension $\leq n$ ($n \geq 0$) at a point p if p has arbitrarily small neighborhoods whose boundaries have dimension $\leq n - 1$.

3. X has dimension $\leq n$ if X has dimension $\leq n$ at each point.

4. X has dimension n at a point p if it is true that X has dimension $\leq n$ at p and it is false that X has dimension $\leq n - 1$ at p.

5. X has dimension n if $\dim X \leq n$ is true and $\dim X \leq n - 1$ is false.

6. X has dimension ∞ if $\dim X \leq n$ is false for each n.

The proof that $R^{(n)}$ has dimension $\leq n$ is by induction. The proof that dim $R^{(n)} \geq n$ requires the use of the Brouwer fixed point theorem, the notion of separation of sets, and the fact that a subspace of a space of dimension $\leq n$ has dimension $\leq n$.

II.4.3. Theorem. Any p-flat, S_p, $(0 < p \leq n-1)$, in $R^{(n)}$, is isometric to $R^{(p)}$, and hence is p-dimensional.

Proof. Let S_p be a p-flat in $R^{(n)}$, and let $x_0, x_1, \ldots, x_p$, be p + 1 linearly independent points which determine S_p. Every point x in S_p can be uniquely represented as

(II.4.4) $$x = \sum_{i=0}^{p} \alpha_i x_i, \qquad \sum_{i=0}^{p} \alpha_i = 1.$$

Rearranging terms and remembering that $\alpha_0 = (1 - \sum_{i=1}^{p} \alpha_i)$, (II.4.4) becomes

(II.4.5) $$x = x_0 + \sum_{i=1}^{p} \alpha_i (x_i - x_0).$$

Consider a new set of coordinates for $R^{(n)}$, obtained by a translation, with the new origin at the point x_0. Then the vectors $(x_1 - x_0), \ldots, (x_p - x_0)$ will be p linearly independent vectors with origin at x_0. Denote these vectors, for the sake of clarity, by $y_1, \ldots, y_p$. With respect to the new coordinate system of $R^{(n)}$, these vectors evidently form a basis for S_p, since every point x in S_p can be expressed uniquely as a linear combination of these p linearly independent vectors. With respect to the new coordinate system the point x of S can be written

$$x' = \alpha_1 y_1 + \ldots + \alpha_p y_p.$$

(The point x' is the same point as the point x, but the coordinate system has just been changed.)

Since a Euclidean space is being considered, an inner product, $(y', y'') = \sum_{i=1}^{n} y^{(j)'} \cdot y^{(j)''}$, and a norm, $\|y'\| = \left[\sum_{i=1}^{n} (y^{(j)'})^2\right]^{\frac{1}{2}}$ are defined for all points y' and y" in $R^{(n)}$.

Now by the Gram-Schmidt orthogonalization process, from the set of p linearly independent vectors, $y_1, \ldots, y_p$, one can construct a set of p orthonormal vectors as follows [Halmos, (1), p. 98]*:

Set

$$g_1 = y_1, \qquad \varphi_1 = g_1 / \|g_1\|.$$

By induction, set

(II.4.6) $$g_{k+1} = y_{k+1} - \sum_{i=1}^{k} (y_{k+1}, \varphi_i)\varphi_i, \qquad \varphi_{k+1} = g_{k+1} / \|g_{k+1}\|.$$

Now, $$(g_{k+1}, \varphi_j) = (y_{k+1} - \sum_{i=1}^{k} (y_{k+1}, \varphi_i)\varphi_i, \varphi_j)$$
$$= (y_{k+1}, \varphi_j) - (y_{k+1}, \varphi_j)$$
$$= 0, \quad (j = 1, \ldots, k).$$

---

* For original papers on this topic, see [Gram, (1)] and [Schmidt, (1), p. 442].

Hence, g_{k+1} is orthogonal to φ_j, $(j = 1, \ldots, k)$. Consequently, φ_{k+1} is orthonormal to φ_j, $(j = 1, \ldots, k)$.

Continuing in this manner until the y's are exhausted, p orthonormal vectors,

$$\varphi_1, \varphi_2, \ldots, \varphi_p,$$

will be obtained. Each φ_i is a unique linear combination of $y_1, \ldots, y_i$, $(i = 1, \ldots, p)$. Therefore, any linear combination of $\varphi_1, \varphi_2, \ldots, \varphi_p$ is also a linear combination of $y_1, \ldots, y_p$, and hence is a point of S_p. Conversely, if one solves the set of equation (II.4.6) for y_j, one sees that y_j is a unique linear combination of $\varphi_1, \ldots, \varphi_j$, $(j = 1, \ldots, p)$. Therefore, since every point in S_p is a unique linear combination of $y_1, \ldots, y_p$, it is also a unique linear combination of $\varphi_1, \ldots, \varphi_p$. This means that $\varphi_1, \ldots, \varphi_p$ form an orthonormal basis for S_p. Consequently, if x is any point in S_p, with respect to this coordinate system,

$$x = \beta_1 \varphi_1 + \ldots + \beta_p \varphi_p.$$

One can extend this basis to be an orthonormal basis for $R^{(n)}$. For $x \in S_p$, the components, $\varphi_{p+1}, \ldots, \varphi_n$, will all be zero.

Now to each point x in S_p with components $(\beta_1, \ldots, \beta_p)$, make correspond the point x' in $R^{(p)}$ with components $(\beta_1, \ldots, \beta_p)$. This correspondence is one-to-one and distance is preserved, since only Euclidean spaces are being considered, and since the distance between points x and x^* of S_p with components $(\beta_1, \ldots, \beta_p)$ and $(\beta_1^*, \ldots, \beta_p^*)$ respectively is $\left(\sum_{i=1}^{p} (\beta_i - \beta_i^*)^2\right)^{\frac{1}{2}}$. With the law of correspondence stated above, the distance between the image points x' and $x^{*'}$ in $R^{(p)}$ is precisely the same. Hence, one concludes that S_p and $R^{(p)}$ are isometric.

Now a metric space is a topological space. If two metric spaces are isometric they are certainly homeomorphic as topological spaces. For if x is a limit point of one space, since distances are preserved, its image will certainly be a limit point of the other space.

Since Euclidean spaces, which are separable metric spaces, are being considered, it can only be concluded that a p-flat, S_p, has dimension p, since it is homeomorphic with $R^{(p)}$, and since the dimension of a Euclidean space is invariant under a one-to-one and continuous transformation, and therefore certainly under a homeomorphism.

II.4.7. Remark. In speaking of a p-flat, S_p, in $R^{(n)}$, $(p < n)$, one would like to be able to speak of open sets of S_p and interior points of a set in S_p. With respect to $R^{(n)}$, no set of S_p can be open, since every n-dimensional neighborhood of a point in S_ρ contains points of $R^{(n)}$ which are not in S_ρ. That is, no point of a set E in S_ρ can be an interior point of E relative to $R^{(n)}$.

It is convenient to consider sets which are open relative to S_ρ. Let U be an n-dimensional open set. Then the set $U \cap S_\rho$ is called open relative to S_ρ. Similarly, let E be a set in S_ρ. If x_o is a point of E such that a neighborhood, $N_\delta(x_o) \cap S_\rho$, is completely contained in E, then x_o is called an interior point of E relative to S_ρ.

II.4.8. Remark. Let F be a continuous, one-to-one, transformation defined on a convex region E of $R^{(n)}$, with nothing said concerning its values outside the region E. A question which one would logically ask is: Does the image of E under the transformation F still have dimension n? The answer was given by Brouwer who showed that the continuous, one-to-one image of an n-dimensional region is also n-dimensional [Brouwer, (2)]. Let S_ρ be a p-flat in $R^{(n)}$ which passes through E. Then $S_\rho \cap E$ is an open set relative to S_ρ, and hence is p-dimensional. Therefore, according to Brouwer, a continuous and one-to-one image of $S_\rho \cap E$ is also p-dimensional. It can never happen, therefore, that the image of $S_\rho \cap E$ will be of different dimension than S_ρ under a continuous, one-to-one transformation.

II.5. FURTHER PROPERTIES OF p-FLATS

II.5.1. <u>Theorem</u>. Two distinct p-flats, $S_p^{(1)}$ and $S_p^{(2)}$, which both lie in a (p + 1)-flat, S_{p+1}, must intersect in a (p - 1)-flat, if they intersect at all.

<u>Proof</u>. Without loss of generality, the coordinate system of $R^{(n)}$ can be assumed to be such that any point x_i in S_{p+1} has the coordinates

$$x_i^{(1)}, \; . \; . \; . \; , x_i^{(p+1)}, 0, \; . \; . \; . \; , 0.$$

This is a consequence of Theorem II.4.3. Then with respect to S_{p+1}, the p-flats, $S_p^{(1)}$ and $S_p^{(2)}$, will be p-dimensional hyperplanes. Each can be represented as a single equation in the p+1 variables, $x^{(1)}, \; . \; . \; . \; , x^{(p+1)}$:

$$\text{(II.5.2)} \qquad \begin{aligned} S_p^{(1)}:&\quad a_1 x^{(1)} + \; . \; . \; . \; + a_{p+1} x^{(p+1)} + a_{p+2} = 0, \\ S_p^{(2)}:&\quad b_1 x^{(1)} + \; . \; . \; . \; + b_{p+1} x^{(p+1)} + b_{p+2} = 0. \end{aligned}$$

If the two hyperplanes intersect at the point $x_0 = (x_0^{(1)}, \; . \; . \; . \; , x_0^{(p+1)}, 0, \; . \; . \; . \; , 0)$, the two equations of (II.5.2) will then take the form

$$\text{(II.5.3)} \qquad \begin{aligned} a_1 (x^{(1)} - x_0^{(1)}) + \; . \; . \; . \; + a_{p+1} (x^{(p+1)} - x_0^{(p+1)}) &= 0, \\ b_1 (x^{(1)} - x_0^{(1)}) + \; . \; . \; . \; + b_{p+1} (x^{(p+1)} - x_0^{(p+1)}) &= 0. \end{aligned}$$

This system of equations has exactly p - 1 linearly independent, non-zero solutions, [Bocher, (1), pp. 49-52],

$$\left((x_i^{(1)} - x_o^{(1)}), \ldots (x_i^{p+1} - x_o^{(p+1)}\right), \quad (i = 1, \ldots, p-1).$$

Since x_o was fixed, this means that there are exactly $p - 1$ linearly independent points, x_i, $(i=1, \ldots, p-1)$, $x_i \neq x_o$, which satisfy (II.5.3). Hence,

$$a_1 x_i^{(1)} + \ldots + a_{p+1} x_i^{(p+1)} = a_1 x_o^{(1)} + \ldots + a_{p+1} x_o^{(p+1)} = -a_{p+2}$$

$$b_1 x_i^{(1)} + \ldots + b_{p+1} x_i^{(p+1)} = b_1 x_o^{(1)} + \ldots + b_{p+1} x_o^{(p+1)} = -b_{p+2}$$

$(i = 1, \ldots, p-1)$. This means that the points, x_i, $(i = 1, \ldots, p-1)$ satisfy (II.5.2).

Since $\left((x_i^{(1)} - x_o^{(1)}), \ldots, (x_i^{(p+1)} - x_o^{(p+1)})\right)$, $(i = 1, \ldots, p-1)$, are linearly independent, then the points $x_o, x_1, \ldots, x_{p-1}$ are linearly independent and hence determine a $(p-1)$-flat. The points $x_o, x_1, \ldots, x_{p-1}$ are all common to both $S_p^{(1)}$ and $S_p^{(2)}$ and there are no more linearly independent points common to both flats. Hence, it must be concluded that the two p-flats $S_p^{(1)}$ and $S_p^{(2)}$ intersect in a $(p-1)$-flat, proving the theorem.

II.5.4. <u>Theorem</u>. Let F: $f^{(i)}(x)$, $(i = 1, \ldots, n)$, be a continuous, one-to-one transformation defined on a convex region E of $R^{(n)}$, such that p-flats map into p-flats (p fixed; $1 \leq p \leq n-1$). Then distinct p-flats map into distinct p-flats for points of E.

<u>Proof</u>. Suppose the theorem is false. Let $S_p^{(1)}$ and $S_p^{(2)}$

be two distinct p-flats, containing points of E, such that $S_p^{(1)} \cap E$ and $S_p^{(2)} \cap E$ map into the same p-flat, T_p. There are two cases.

1. $S_p^{(1)}$ and $S_p^{(2)}$ intersect in E in a (p-1)-flat, S_{p-1}. Since dimension is preserved by a one-to-one, continuous transformation, the image of $S_{p-1} \cap E$ must be a (p-1)-dimensional region.

The point sets $S_p^{(1)} \cap E$ and $S_p^{(2)} \cap E$ are p-dimensional regions for they are open, connected sets relative to $S_p^{(1)}$ and $S_p^{(2)}$ respectively. Hence, the images of $S_p^{(1)} \cap E$ and $S_p^{(2)} \cap E$ must both be regions in T_p. Since $S_p^{(1)}$ and $S_p^{(2)}$ intersect, then their images must also have points in common. In fact, the image sets must have a whole p-dimensional region, G', in common. Since the image of S_{p-1} must be (p-1)-dimensional, then there are points of G' which must be the images of two distinct points, one in $S_p^{(1)} \cap E$ and the other in $S_p^{(2)} \cap E$. This contradicts the assumption that the mapping is one-to-one. Hence, case 1 cannot occur.

2. The p-flats, $S_p^{(1)}$ and $S_p^{(2)}$, do not intersect in a (p-1)-flat in E. In this case, choose p linearly independent points, $x_1, \dots, x_p$, of $S_p^{(1)} \cap E$, and choose x_0 to be a point of $S_p^{(2)} \cap E$ which is not in $S_p^{(1)}$. Then $x_0, x_1, \dots, x_p$ form a set of p + 1 linearly independent points and hence determine a p-flat, $S_p^{(3)}$, which intersects $S_p^{(1)}$ in E in a (p-1)-flat, S_{p-1}. By

case 1, $S_p^{(1)} \cap E$ and $S_p^{(2)} \cap E$ must map into distinct p-flats, T_p and $T_p^{(3)}$. The two p-flats, T_p and $T_p^{(3)}$, must contain a (p-1)-dimensional region in common, the image of S_{p-1}.

Consider the point x_o. By hypothesis, $F(x_o)$ is a point of T_p. But since x_o is also a point of $S_p^{(3)}$, then $F(x_o)$ must belong to $T_p^{(3)}$. Hence, $F(x_o)$ belongs to the intersection of T_p and $T_p^{(3)}$. This means that $F(x_o)$ belongs to the images of both $S_p^{(1)} \cap E$ and $S_p^{(2)} \cap E$. This means that $F(x_o)$ is the image of two distinct points, one in $S_p^{(1)} \cap E$ and the other in $S_p^{(2)} \cap E$. This contradicts the assumption that the mapping is one-to-one. Hence case 2 cannot occur.

In either case a contradiction has been reached. Hence, one must conclude that for points of E, distinct p-flats map into distinct p-flats.

II.6. p-CELLS AND THEIR PROPERTIES

Recall the definition of a p-cell, Δx_p, as given in Definition I.2.1. Some fundamental properties of p-cells will now be developed.

II.6.1. Definition. Let Δx_p be a p-cell with vertices $x_o, x_1, \ldots, x_p$. If one chooses from this set of points, a subset of k + 1 points $(-1 \leq k \leq p)$, then the k-cell determined by these k + 1 points is called a k-dimensional face of the p-cell, Δx_p. If $-1 < k < p$, then the k-cell is called a proper face of the p-cell; otherwise it is called an improper face.

Clearly, a point

$$x = \sum_{i=0}^{p} \alpha_i x_i, \qquad \sum_{i=0}^{p} \alpha_i = 1, \qquad \alpha_i \geqslant 0,$$

of a p-cell, Δx_p, is on a proper face of the p-cell if and only if at least one of the α's is equal to zero.

By the definition of a 1-flat (straight line), the straight line through two distinct points, x_1 and x_2, is the set of all points

$$\alpha_1 x_1 + \alpha_2 x_2,$$

where $\alpha_1 + \alpha_2 = 1$.

II.6.2. <u>Definition</u>. The subset of the line,

$$\alpha_1 x_1 + \alpha_2 x_2, \qquad \alpha_1 + \alpha_2 = 1,$$

for which $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$, is called the segment, $\overline{x_1 x_2}$. (From Definition I.2.1., it is also the 1-cell determined by x_1 and x_2.)

II.6.3. <u>Definition</u>. A point set E of $R^{(n)}$ is called <u>convex</u> if for $x_1 \in E$ and $x_2 \in E$ it follows that $\overline{x_1 x_2}$ belongs to E.

II.6.4. <u>Theorem</u>. A p-cell is a convex set.

<u>Proof</u>. Let $x_0, x_1, \ldots, x_p$ be the vertices of the p-cell, Δx_p. Let x' and x" be any two points of Δx_p. Then

$$x' = \sum_{i=0}^{p} \alpha_i x_i, \qquad \sum_{i=0}^{p} \alpha_i = 1, \qquad \alpha_i \geq 0, \text{ all } i,$$

and

$$x'' = \sum_{i=0}^{p} \beta_i x_i, \qquad \sum_{i=0}^{p} \beta_i = 1, \qquad \beta_i \geq 0, \text{ all } i.$$

Every point x of $\overline{x'x''}$ can be expressed as

$$\text{(II.6.4)} \quad x = \theta x' + (1 - \theta)x''$$

$$= \theta(\alpha_0 x_0 + \alpha_1 x_1 + \ldots + \alpha_p x_p) + (1-\theta)(\beta_0 x_0 + \ldots + \beta_p x_p)$$

$$= [\theta \alpha_0 + (1-\theta)\beta_0] x_0 + \ldots + [\theta \alpha_p + (1-\theta)\beta_p] x_p,$$

where $0 \leq \theta \leq 1$. All the coefficients of (II.6.4) are clearly ≥ 0.

Consider the sum of the coefficients of (II.6.4):

$$\text{(II.6.5)} \quad [\theta \alpha_0 + (1-\theta)\beta_0] + \ldots + [\theta \alpha_p + (1-\theta)\beta_p]$$

$$= \theta(\alpha_0 + \ldots + \alpha_p) + (1-\theta)(\beta_0 + \ldots + \beta_p)$$

$$= \theta + (1 - \theta) = 1.$$

Hence x belongs to Δx_p. Therefore the p-cell, Δx_p, is a convex set, proving the theorem.

II.6.6. Theorem. Let $x_0, x_1, \ldots, x_p$, $(p \leq n)$, be p + 1 linearly independent points of $R^{(n)}$ which determine a p-flat, S_p, and a p-cell, Δx_p. Then a point X^* of Δx_p which can be represented as

$$x^* = \sum_{i=0}^{p} \alpha_i x_i, \qquad \sum_{i=0}^{p} \alpha_i = 1, \qquad \alpha_i > 0, \text{ all } i,$$

is an interior point of Δx_p relative to S_p.

Proof. Any point $x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ of S_p can be represented uniquely as

$$x^{(j)} = \sum_{i=0}^{p} \beta_i x_i^{(j)}, \qquad \sum_{i=0}^{p} \beta_i = 1, \ (j = 1, \ldots, n).$$

For any given x of S_p the β's are uniquely determined. It will be shown that the β's are continuous functions of x. Consider the equations

$$\text{(II.6.7)} \qquad \begin{aligned} x^{(1)} &= \beta_0 x_0^{(1)} + \beta_1 x_1^{(1)} + \cdots + \beta_p x_p^{(1)}, \\ &\cdots\cdots\cdots\cdots \\ x^{(p)} &= \beta_0 x_0^{(p)} + \beta_1 x_1^{(p)} + \cdots + \beta_p x_p^{(p)}, \\ &\cdots\cdots\cdots\cdots \\ x^{(n)} &= \beta_0 x_0^{(n)} + \beta_1 x_1^{(n)} + \cdots + \beta_p x_p^{(n)}, \\ 1 &= \beta_0 + \beta_1 + \cdots + \beta_p . \end{aligned}$$

Since the points $x_0, x_1, \ldots, x_p$ are linearly independent, the rank of the matrix of the coefficients of the β's is p + 1 (See Definition II.3.2.) Furthermore, since the β's are uniquely determined for a given x, the augmented matrix also has rank p + 1. That is, all the C(n + 1, p + 2) determinants of the form

$$\text{(II.6.8)} \qquad \begin{vmatrix} x^{(j_1)} & x_0^{(j_1)} & \cdots & x_p^{(j_1)} \\ \cdots & \cdots & \cdots & \cdots \\ x^{(j_{p+2})} & x_0^{(j_{p+2})} & \cdots & x_p^{(j_{p+2})} \end{vmatrix}$$

vanish, where $x^{(j_{p+2})}$, $x_0^{(j_{p+2})}$, . . . , $x_p^{(j_{p+2})}$ may all be ones, and where the set of superscripts $(j_1, \ldots, j_{p+2})$ represents one of the $C(n+1, p+2)$ possible choices of $p + 2$ of the $n + 1$ rows of the augmented matrix of the set of equations (II.6.7).

The rank of the matrix of the equations (II.6.7) is therefore equal to the number of unknowns. Hence, picking out a set of $p + 1$ of the $n + 1$ equations of (II.6.7), such that the determinant of the coefficients of the β's is different from zero, the β's can be solved for uniquely by Cramer's Rule. The set of β's thus determined will be the same as those determined if any other set of $p + 1$ of the equations of (II.6.7) is chosen such that the determinant of the coefficients is different from zero. [Bocher, (1), p. 46.] Suppose the equations chosen are

$$
\text{(II.6.9)} \qquad
\begin{aligned}
x^{(j_1)} &= \beta_0 x_0^{(j_1)} + \beta_1 x_1^{(j_1)} + \cdots + \beta_p x_p^{(j_1)}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
x^{(j_{p+1})} &= \beta_0 x_0^{(j_{p+1})} + \beta_1 x_1^{(j_{p+1})} + \cdots + \beta_p x_p^{(j_{p+1})},
\end{aligned}
$$

the set of superscripts representing the proper choice so that the determinant of the coefficients of the β's is different from zero.

Solving for β_i:

$$\text{(II.6.10)} \quad \beta_i = \frac{\begin{vmatrix} x_0^{(j_1)} & \cdots & x_{i-1}^{(j_1)} & x^{(j_1)} & x_{i+1}^{(j_1)} & \cdots & x_p^{(j_1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_0^{(j_{p+1})} & \cdots & x_{i-1}^{(j_{p+1})} & x^{(j_{p+1})} & x_{i+1}^{(j_{p+1})} & \cdots & x_p^{(j_{p+1})} \end{vmatrix}}{\begin{vmatrix} x_0^{(j_1)} & \cdots & x_p^{(j_1)} \\ \cdots & \cdots & \cdots \\ x_0^{(j_{p+1})} & \cdots & x_p^{(j_{p+1})} \end{vmatrix}}, \quad (i = 0, \ldots, p).$$

Expanding (II.6.10) by the ith column:

$$\beta_i = A_{i,1} x^{(j_1)} + \cdots + A_{i,p+1} x^{(j_{p+1})}, \quad (i = 0, \ldots, p),$$

where the numbers $A_{i,k}$ are constants defined by

$$A_{i,k} = \frac{\begin{vmatrix} x_0^{(j_1)} & \cdots & x_{i-1}^{(j_1)} & x_{i+1}^{(j_1)} & \cdots & x_p^{(j_1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_0^{(j_{k-1})} & \cdots & x_{i-1}^{(j_{k-1})} & x_{i+1}^{(j_{k-1})} & \cdots & x_p^{(j_{k-1})} \\ x_0^{(j_{k+1})} & \cdots & x_{i-1}^{(j_{k+1})} & x_{i+1}^{(j_{k+1})} & \cdots & x_p^{(j_{k+1})} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_0^{(j_{p+1})} & \cdots & x_{i-1}^{(j_{p+1})} & x_{i+1}^{(j_{p+1})} & \cdots & x_p^{(j_{p+1})} \end{vmatrix}}{\begin{vmatrix} x_0^{(j_1)} & \cdots & x_p^{(j_1)} \\ \cdots & \cdots & \cdots \\ x_0^{(j_{p+1})} & \cdots & x_p^{(j_{p+1})} \end{vmatrix}}, \quad (k = 1, \ldots, p+1).$$

Hence, β_i is a continuous function of x $(i = 0, \ldots, p)$.

When $x = x^*$, $\beta_i = \alpha_i$, $(i = 0, 1, \ldots, p)$, and $\alpha_i > 0$. In fact, there are numbers $\epsilon_i > 0$ such that $\alpha_i > \epsilon_i > 0$, $(i = 0, 1, \ldots, p)$. Let $\epsilon' = \min \{\epsilon_i\}$, all i. Then $\alpha_i > \epsilon' > 0$. Since β_i is a continuous function of x, then for every sufficiently small $\epsilon > 0$ (in particular, for $\epsilon' > 0$), there is a $\delta_i > 0$ such that

$$|\beta_i - \alpha_i| < \epsilon'$$

whenever $\|x^* x\| < \delta_i$, $x \in N_{\delta_i}(x^*) \cap S_p$, $(i = 0, 1, \ldots, p)$. Let $\delta' = \min\{\delta_i\}$, $(i = 0, 1, \ldots, p)$. Then

$$|\beta_i - \alpha_i| < \epsilon'$$

whenever $x \in N_{\delta'}(x^*) \cap S_p$, $(i = 0, 1, \ldots, p)$. That is, for all points $x \in N_{\delta'}(x^*) \cap S_p$,

$$\beta_i > \alpha_i - \epsilon' > 0, \qquad (i = 0, \ldots, p).$$

Hence, in a sufficiently small neighborhood of x^* (relative to S_p), all points x can be represented as

$$x = \sum_{i=0}^{p} \beta_i x_i \qquad \sum_{i=0}^{p} \beta_i = 1, \qquad \beta_i > 0, \text{ all } i,$$

and these points belong to Δx_p. Hence, x^* is an interior point of Δx_p relative to S_p.

II.6.11. <u>Corollary</u>. It follows from Theorem II.6.6 that if x^* is on a proper face of Δx_p, then it is a boundary point of Δx_p relative to S_p. For if x^* is on a proper face of Δx_p, then at least one barycentric coordinate of x^*, say α_i, is equal to zero. Since the α's are continuous functions of x and since the representation of a point of S_p is unique, then in every small neighborhood of x^* (relative to S_p) there are points x of S_p such that the corresponding α_i is < 0, and hence such points do not belong to Δx_p. Hence, x^* is a boundary point of Δx_p.

II.6.12. Remark. It might be well to mention here another characterization of a p-flat in $R^{(n)}$. By Definition II.3.13, a p-flat, S_p, is the set of all points x of $R^{(n)}$ such that

$$x = \sum_{i=0}^{p} \alpha_i x_i , \qquad \sum_{i=0}^{p} \alpha_i = 1, \quad (j = 1, \ldots , n),$$

where $x_0, \ldots , x_p$ form a set of p + 1 linearly independent points of $R^{(n)}$. It is further shown that the representation is unique. That is, the equations

$$\text{(II.6.13)} \qquad \begin{array}{l} x^{(1)} = \alpha_0 x_0^{(1)} + \ldots + \alpha_p x_p^{(1)} \\ \ldots\ldots\ldots\ldots\ldots \\ x^{(n)} = \alpha_0 x_0^{(n)} + \ldots + \alpha_p x_p^{(n)} \\ 1 = \alpha_0 + \ldots + \alpha_p \end{array}$$

have precisely one solution for the α's. From the work in the last theorem, it follows that the rank of the augmented matrix

$$\text{(II.6.14)} \qquad \begin{pmatrix} x^{(1)} & x_0^{(1)} & \ldots & x_p^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ x^{(n)} & x_0^{(n)} & \ldots & x_p^{(n)} \\ 1 & 1 & \ldots & 1 \end{pmatrix}$$

is p + 1. That is, all the p + 2 rowed determinants formed from the matrix (II.6.14) must vanish. This is expressed as

$$(II.6.15)\qquad \begin{vmatrix} x^{(j_1)} & x_0^{(j_1)} & \cdots & x_p^{(j_1)} \\ \cdots & \cdots & \cdots & \cdots \\ x^{(j_{p+2})} & x_0^{(j_{p+2})} & \cdots & x_p^{(j_{p+2})} \end{vmatrix} = 0,$$

where the set of superscripts, $(j_1, \ldots j_{p+2})$, represents one of the $C(n+1, p+2)$ possible choices of $p + 2$ of the $n + 1$ rows of the matrix (II.6.14). (The numbers, $x^{(j_{p+2})}$, $x_0^{(j_{p+2})}$, . . . , $x_p^{(j_{p+2})}$ may all be ones.)

Therefore, the points x of S_p must satisfy the $C(n+1, p+2)$ equations of the type (II.6.15).

Conversely, consider the set of equations

$$(II.6.16)\qquad \begin{aligned} x^{(1)} &= \beta_0 x_0^{(1)} + \cdots + \beta_p x_p^{(1)} \\ &\cdots\cdots\cdots\cdots \\ x^{(n)} &= \beta_0 x_0^{(n)} + \cdots + \beta_p x_p^{(n)} \\ 1 &= \beta_0 \quad + \cdots + \beta_p \end{aligned}$$

where the points, x_0, x_1, . . . , x_p are linearly independent. Suppose x is such that all the $p + 2$-rowed determinant minors of the augmented matrix of the equations (II.6.16) vanish. That is, suppose that x satisfies all $C(n+1, p+2)$ equations of the type

$$(II.6.17)\qquad \begin{vmatrix} x^{(j_1)} & x_0^{(j_1)} & \cdots & x_p^{(j_1)} \\ \cdots & \cdots & \cdots & \cdots \\ x^{(j_{p+2})} & x_0^{(j_{p+2})} & \cdots & x_p^{(j_{p+2})} \end{vmatrix} = 0$$

where the superscripts $(j_1, \ldots, j_{p+2})$ represent one of the $C(n + 1, p + 2)$ possible selections of $p + 2$ of the $n + 1$ rows of the augmented matrix of the equations (II.6.16). Then there is a unique solution for the β's and by the definition of a p-flat, x lies in the p-flat determined by the $p + 1$ linearly independent points, $x_0, x_1, \ldots, x_p$.

The conclusion is that the p-flat, S_p, is composed precisely of the set of all points x of $R^{(n)}$ which satisfy the $C(n + 1, p + 2)$ equations of the type (II.6.17). That is, S_p is characterized by this set of equations.

II.6.18. Theorem. Let $x_0, \ldots, x_p$ be $p + 1$ linearly independent points of $R^{(n)}$ which determine a p-flat, S_p, and a p-cell, Δx_p. Let x^* be an interior point of Δx_p, with respect to S_p. Then every straight line through x^*, lying in S_p, intersects the boundary of Δx_p in exactly two points.

Proof. The p-cell Δx_p is a closed and bounded convex set with respect to S_p. It can be shown [Alexandroff-Hopf, (1), pp. 599-600] that if M is any closed and bounded convex set of $R^{(p)}$ and if x^* is interior to M with respect to $R^{(p)}$, then a straight line through x^* intersects the boundary of M in precisely two points. It is first proved that a ray drawn from an interior point of a convex set intersects the boundary in at most one point. If a set M is closed and bounded, then any ray from an interior point of M intersects the boundary of M in at least one point. Hence, if M is a closed, bounded, convex set, a ray from an interior point intersects the boundary in exactly one point. Therefore, any straight line through an interior point of M intersects the boundary in exactly two points.

For the purposes of this paper the following theorem, although not so strong as Theorem II.6.18, is sufficient.

II.6.19. <u>Theorem</u>. Let $x_0, \ldots, x_p$ be p + 1 linearly independent points of $R^{(n)}$, which determine a p-cell, Δx_p, and a p-flat, S_p. Let x' be any interior point of Δx_p, relative to S_p. Let x" be any other point of S_p. Then the straight line through x' and x" intersects the boundary of Δx_p in exactly two points.

<u>Proof</u>. The proof is an immediate consequence of the following lemma:

II.6.20. <u>Lemma</u>. Let $x_0, \ldots, x_p$ be p + 1 linearly independent points of $R^{(n)}$ which determine a p-cell, Δx_p and a p-flat, S_p. Let x' and x" be any two points of Δx_p, at least one of which is interior to Δx_p, relative to S_p. Then the straight line through x' and x" intersects the boundary of Δx_p in exactly two points.

<u>Proof</u>. Suppose x" is interior to Δx_p relative to S_p, and x' is either interior to Δx_p or is a boundary point. Then

$$x' = \sum_{i=0}^{p} \alpha_i x_i, \qquad \sum_{i=0}^{p} \alpha_i = 1, \qquad \alpha_i \geq 0, \text{ all } i,$$

and

$$x'' = \sum_{i=0}^{p} \beta_i x_i, \qquad \sum_{i=0}^{p} \beta_i = 1, \qquad \beta_i > 0, \text{ all } i,$$

where not every α_i is equal to the corresponding β_i. In fact, since x' is distinct from x'', then at least two α's are different from the corresponding β's. For if p of the α's are equal to the p corresponding β's, then since the sum of the α's is one and since the sum of the β's is one, the remaining α is equal to its corresponding β. Hence, each α is equal to its corresponding β, and the two points are not distinct, contrary to assumption.

All the points on the stright line through x' and x'' can be expressed as

$$\begin{aligned} x &= \theta x' + (1 - \theta)x'' \\ &= \theta\left(\sum_{i=0}^{p} \alpha_i x_i\right) + (1 - \theta)\left(\sum_{i=0}^{p} \beta_i x_i\right) \\ &= [\theta\alpha_0 + (1 - \theta)\beta_0]x_0 + \ldots + [\theta\alpha_p + (1 - \theta)\beta_p]x_p. \end{aligned}$$

For any choice of θ, the sum of these coefficients is equal to one, for

$$\begin{aligned} &[\theta\alpha_0 + (1 - \theta)\beta_0] + \ldots + [\theta\alpha_p + (1 - \theta)\beta_p] \\ &= \theta(\alpha_0 + \ldots + \alpha_p) + (1 - \theta)(\beta_0 + \ldots + \beta_p) \\ &= \theta + (1 - \theta) = 1. \end{aligned}$$

The problem is to find exactly two distinct values of θ such that for each of these two values, at least one of the coefficients, $[\theta\alpha_i + (1 - \theta)\beta_i]$, $(i = 0, 1, . . . , p)$, is equal to zero, and such that the remaining coefficients are ≥ 0. Clearly, all such possibilities for θ are found by setting each coefficient, $[\theta\alpha_i + (1 - \theta)\beta_i]$, $(i = 0, 1, . . . , p)$, equal to zero and solving for θ. This cannot always be done, since if $\alpha_K = \beta_K$, for some k, then the coefficient of x_K is $\theta\alpha_K + (1 - \theta)\beta_K = \theta\beta_K + \beta_K - \theta\beta_K = \beta_K$, which clearly cannot be set equal to zero. However, by a previous remark, there are at least two α's which are not equal to their corresponding β's. Hence, one can always find at least two possibilities for θ. These possibilities for θ are found to be

$$\theta_i = - \frac{\beta_i}{\alpha_i - \beta_i}$$

for all i such that $\alpha_i \neq \beta_i$.

The following is a table of values of the coefficients of x_i corresponding to the possible values for θ:

	X_0	X_1	. . . . .	X_p
θ_0	0	$\dfrac{\alpha_0\beta_1 - \alpha_1\beta_0}{\alpha_0 - \beta_0}$	. . . . .	$\dfrac{\alpha_0\beta_p - \alpha_p\beta_0}{\alpha_0 - \beta_0}$
θ_1	$\dfrac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_1 - \beta_1}$	0	. . . . .	$\dfrac{\alpha_1\beta_p - \alpha_p\beta_1}{\alpha_1 - \beta_1}$
. .	. . . . .	. . . . .	. . . . .	. . . . .
θ_p	$\dfrac{\alpha_p\beta_0 - \alpha_0\beta_p}{\alpha_p - \beta_p}$	$\dfrac{\alpha_p\beta_1 - \alpha_1\beta_p}{\alpha_p - \beta_p}$	. . . . .	0

Only values of θ_i will appear for those i for which $\alpha_i \neq \beta_i$.

The lemma will be proved if precisely two distinct choices of θ in the table will produce coefficients which are all non-negative. The points corresponding to these choices of θ will satisfy the requirements for being on the boundary of Δx_p.

Since $\sum_{i=0}^{p} \alpha_i = 1$ and $\sum_{i=0}^{p} \beta_i = 1$, and since at least two of the α's are different from the corresponding β's, then $\alpha_j > \beta_j$ for at least one j and $\alpha_k < \beta_k$ for at least one k. Consider the ratios

$$\frac{\alpha_0}{\beta_0}, \frac{\alpha_1}{\beta_1}, \ldots \frac{\alpha_p}{\beta_p}.$$

Since there are only a finite number of these ratios, there must be at least one which is smallest and at least one which is largest. Suppose α_k / β_k is the smallest ratio. Since $\alpha_k < \beta_k$ then $\alpha_k / \beta_k < 1$. Suppose

α_k/β_k is the largest ratio; then $\alpha_k/\beta_k > 1$. Arrange the ratios in order of increasing size;

$$0 \leqslant \frac{\alpha_h}{\beta_h} \leqslant \cdot \cdot \cdot \leqslant \frac{\alpha_k}{\beta_k} .$$

Consider θ_h and θ_k. (They exist since $\alpha_h \neq \beta_h$ and $\alpha_k \neq \beta_k$). Now $\theta_h \neq \theta_k$, since

$$\theta_h = - \frac{\beta_h}{\alpha_h - \beta_h} > 0$$

and

$$\theta_k = - \frac{\beta_k}{\alpha_k - \beta_k} < 0.$$

Since $\frac{\alpha_h}{\beta_h} \leqslant \frac{\alpha_i}{\beta_i}$, all i, then

$$\alpha_h \beta_i - \alpha_i \beta_h \leq 0, \qquad \text{all } i.$$

Since $\alpha_h - \beta_h < 0$, then the ratios

$$\frac{\alpha_h \beta_i - \alpha_i \beta_h}{\alpha_h - \beta_h} \geqslant 0, \qquad \text{all } i.$$

These are the coefficients of x_i corresponding to θ_h in the table; hence θ_h is one of the desired choices.

Similarly, since $\frac{\alpha_k}{\beta_k} \geqslant \frac{\alpha_i}{\beta_i}$, all i, then

$$\alpha_k \beta_i - \alpha_i \beta_k \geqslant 0, \qquad \text{all } i.$$

Since $\alpha_k - \beta_k < 0$, then the ratios

$$\frac{\alpha_K \beta_i - \alpha_i \beta_K}{\alpha_K - \beta_K} \geq 0, \qquad \text{all } i.$$

These are the coefficients corresponding to θ_K in the table. Hence, θ_K is also a desired choice.

It remains to be shown that no other distinct choice of θ in the table yields a point of the line through x' and x'' which is on the boundary of Δx_p. Consider θ_ℓ, where $\ell \neq h, k$. (Then $\alpha_\ell \neq \beta_\ell$, since otherwise there would be no θ_ℓ in the table). Suppose $\frac{\alpha_\ell}{\beta_\ell} = \frac{\alpha_h}{\beta_h}$. Then

$$\theta_\ell = -\frac{\beta_\ell}{\alpha_\ell - \beta_\ell} = -\frac{\alpha_\ell \beta_h}{\alpha_h\left(\alpha_\ell - \frac{\alpha_\ell \beta_h}{\alpha_h}\right)} = -\frac{\alpha_\ell \beta_h}{\alpha_h \frac{(\alpha_\ell \alpha_h - \alpha_\ell \beta_h)}{\alpha_h}} = -\frac{\beta_h}{\alpha_h - \beta_h} = \theta_h .$$

In this case, the points corresponding to θ_h and θ_ℓ are not distinct. A similar situation occurs if $\frac{\alpha_\ell}{\beta_\ell} = \frac{\alpha_K}{\beta_K}$.

Suppose $\frac{\alpha_h}{\beta_h} < \frac{\alpha_\ell}{\beta_\ell} < \frac{\alpha_K}{\beta_K}$, and suppose that $\alpha_\ell < \beta_\ell$. Then, since $\alpha_\ell \beta_h - \alpha_h \beta_\ell > 0$, the ratio

$$\frac{\alpha_\ell \beta_h - \alpha_h \beta_\ell}{\alpha_\ell - \beta_\ell} < 0,$$

and hence there is a coefficient corresponding to θ_ℓ which is negative and this point is not in Δx_p.

Similarly, if $\alpha_\ell > \beta_\ell$, then

$$\frac{\alpha_\ell \beta_K - \alpha_K \beta_\ell}{\alpha_\ell - \beta_\ell} < 0,$$

and again the point corresponding to θ_ℓ is not in Δx_ρ.

All possible cases have been exhausted. Hence there are precisely two choices of θ, in particular θ_ℓ and θ_K, which yield points of the line through x' and x'' which are on the boundary of Δx_ρ, proving the lemma.

From this lemma, the proof of the theorem easily follows. Let Δx_ρ be the p-cell with vertices $x_0, \ldots, x_\rho$, and let x' be the interior point of Δx_ρ, relative to S_ρ. If x'' is any other point of Δx_ρ, then by the lemma, the theorem is true. Let x'' be a point of S not in Δx_ρ. Construct the straight line, L, through x' and x''. Since x' is interior to Δx_ρ relative to S_ρ, then there is a neighborhood, $N_\delta(x')$, such that $N_\delta(x') \cap S_\rho$ is completely contained in Δx_ρ. Choose x''' to be a point of $L \cap N_\delta(x')$ different from x'. Since x' and x''' are distinct, they are linearly independent and therefore determine L. Applying the lemma, using x' and x''', it is seen that L intersects the boundary of Δx_ρ in precisely two points, proving the theorem.

II.6.21. <u>Theorem</u>. Let x^* be an interior point of a p-cell, Δx_ρ, relative to S_p, $(2 \le p \le n)$. Let $U \cap S_\rho$ be any spherical neighborhood of x^* relative to S_p, contained in Δx_ρ. Let L be any straight line through x^*. Then there is a point of $U \cap S_p$ (Hence an interior point of Δx_p) which does not lie on L.

Proof. The neighborhood $U \cap S_p$ is homeomorphic to S_p. Hence, p + 1 linearly independent points (or what is the same, p linearly independent vectors) can be chosen in $U \cap S_p$.

Now L intersects the boundary of $U \cap S_p$ in exactly two points, y_1 and y_2, by Theorem II.6.18. Consider the points x^* and $x' = \frac{1}{2}x^* + \frac{1}{2}y_1$. These two points are linearly independent since they are distinct, they both lie on L, and they both belong to $U \cap S_p$. Let x^{**} be another point of $U \cap S_p$, chosen to be linearly independent with x^* and x'. Then the vectors $(x^{**}-x^*)$ and $(x'-x^*)$ are linearly independent.

It follows that the point x^{**} does not lie on L, for if it did, then

$$x^{**} = (1 - \theta)x^* + \theta x',$$

and

$$(x^{**}-x^*) = \theta(x' - x^*).$$

But this means that the vectors $(x^{**}-x^*)$ and $(x'-x^*)$ are linearly dependent. This is a contradiction. Hence, x^{**} does not lie on L, and the theorem is proved.

II.6.22. Theorem. Let $x_0, x_1, \ldots, x_p$ be n + 1 linearly independent points of $R^{(n)}$ which form the vertices of an n-cell, Δx_n. Let F be a one-to-one, continuous transformation defined on a convex region E which contains the n-cell, Δx_n, and let F be such that it maps straight lines into straight lines. Then all the p-cell faces of Δx_n map into distinct p-cell faces of an n-cell, ΔF_n, under the transformation F, $(0 \leq p \leq n)$.

Proof. The proof is by induction on p. Let p = 0. Then since the transformation is one-to-one, all the 0-cell faces (vertices) of Δx_n map into distinct 0-cell faces of an n-cell, ΔF_n, which turns out to be non-degenerate.

Let p = 1. By hypothesis straight lines map into straight lines. By Theorem II.5.4 distinct lines map into distinct lines. Since F is one-to-one and continuous, then each of the 1-cell faces of Δx_n, formed by joining any two of the vertices of Δx_n, map into a 1-cell, formed by joining the corresponding vertices of ΔF_n. Since these 1-cells must be distinct, this means that all the image points $F(x_0)$, . . ., $F(x_0)$, taken three at a time, are linearly independent.

Let p = 2. Let $\Delta(x_{i_0}\, x_{i_1}\, x_{i_2})$ be any 2-cell of Δx_n, where (i_0, i_1, i_2) represents a choice of any three of the n + 1 vertices of Δx_n. The points x_{i_0}, x_{i_1}, and x_{i_2} map into linearly independent points $F(x_{i_0})$, $F(x_{i_1})$, and $F(x_{i_2})$ by the statement above. Hence $\Delta(F{:}x_{i_0}\, x_{i_2}\, x_{i_3})$ is a non-degenerate 2-cell, and $F(x_{i_0})$, $F(x_{i_1})$ and $F(x_{i_2})$

determine a 2-flat, T_2. By the induction hypothesis the boundary of $\Delta(x_{i_0} x_{i_1} x_{i_2})$ maps into the boundary of $\Delta(F{:}x_{i_0} x_{i_1} x_{i_2})$. Let x^* be any interior point of $\Delta(x_{i_0} x_{i_1} x_{i_2})$ relative to S_2, the 2-flat determined by x_{i_0}, x_{i_1}, and x_{i_2}. Let x' be a boundary point of $\Delta(x_{i_0} x_{i_1} x_{i_2})$. Let L be the straight line through x^* and x'. Then L intersects the boundary of $\Delta(x_{i_0} x_{i_1} x_{i_2})$ in exactly two distinct points, x' and x'', by Lemma II.6.20. The points x' and x'' map into $F(x')$ and $F(x'')$ on the boundary of $\Delta(F{:}x_{i_0} x_{i_1} x_{i_2})$, by the induction hypothesis. Hence,

$$F(x') = \sum_{j=0}^{2} \alpha_j F(x_{i_j}), \qquad \sum_{j=0}^{2} \alpha_j = 1, \quad \alpha_j \geqslant 0, \quad \text{all } j,$$

and

$$F(x'') = \sum_{j=0}^{2} \beta_j F(x_{i_j}), \qquad \sum_{j=0}^{2} \beta_j = 1, \quad \beta_j \geqslant 0, \quad \text{all } j.$$

Since straight lines map into straight lines and since F is continuous and one-to-one, then x^* on L between x' and x'' maps into $F(x^*)$ on the line segment $\overline{F(x')F(x'')}$, and

$$F(x^*) = \theta F(x') + (1 - \theta)F(x''), \qquad 0 < \theta < 1.$$

Hence,

$$F(x^*) = \theta\Big(\sum_{j=0}^{2} \alpha_j F(x_{i_j})\Big) + (1 - \theta)\Big(\sum_{j=0}^{2} \beta_j F(x_{i_j})\Big)$$

$$= [\theta\alpha_0 + (1-\theta)\beta_0]F(x_{i_0}) + \dots + [\theta\alpha_2 + (1-\theta)\beta_2]F(x_{i_2}).$$

Clearly, all the coefficients are $\geqslant 0$ since $0 < \theta < 1$, and the sum of the coefficients is equal to one by the work in Lemma II.6.20. Hence, $F(x^*)$ is in the 2-cell, $\Delta(F:x_{i_0}\,x_{i_1}\,x_{i_2})$. Since x^* was any interior point of $\Delta(x_{i_0}\,x_{i_1}\,x_{i_2})$, then the 2-cell $\Delta(x_{i_0}\,x_{i_1}\,x_{i_2})$ maps into $\Delta(F:x_{i_0}x_{i_1}\,x_{i_2})$.

Let x be any point of $S_2 \cap E$ not in $\Delta(x_{i_0}x_{i_1}\,x_{i_2})$. Since $S_2 \cap E$ is convex, x can be joined by a straight line L' to a point x^*, interior to $\Delta(x_{i_0}\,x_{i_1}\,x_{i_2})$. By Theorem II.6.19, L' intersects the boundary of $\Delta(x_{i_0}\,x_{i_1}\,x_{i_2})$ in exactly two distinct points, x' and x". The transformation F carries x' and x" into F(x') and F(x") on the boundary of $\Delta(F:x_{i_0}\,x_{i_1}\,x_{i_2})$. Hence,

$$F(x') = \sum_{j=0}^{2} \gamma_j F(x_{i_j}), \qquad \sum_{j=0}^{2} \gamma_j = 1, \qquad \gamma_j \geqslant 0, \text{ all } j,$$

and

$$F(x'') = \sum_{j=0}^{2} \delta_j F(x_{i_j}), \qquad \sum_{j=0}^{2} \delta_j = 1, \qquad \delta_j \geqslant 0, \text{ all } j.$$

Since x is on the line through x' and x" and since straight lines map into straight lines, then F(x) is on the line through F(x') and F(x"). Therefore,

$$\begin{aligned} F(x) &= \varphi F(x') = (1 - \varphi)F(x'') \\ &= \varphi\Big(\sum_{j=0}^{2} \gamma_j F(x_{i_j})\Big) + (1 - \varphi)\Big(\sum_{j=0}^{2} \delta_j F(x_{i_j})\Big) \\ &= \sum_{j=0}^{2} \big[\varphi\,\gamma_j + (1 - \varphi)\,\delta_j\big] F(x_{i_j}), \end{aligned}$$

and the sum of the coefficients is one. Hence, F(x) is in the 2-flat, T_2. Since x was any point of $S_2 \cap E$ not in

$\Delta(x_{i_0} x_{i_1} x_{i_2})$, it has been shown that $S_2 \cap E$ maps into T_2.

The same argument holds for each 2-cell face of Δx_n. By Theorem II.5.4, the distinct 2-flats determined by the vertices of all the 2-cell faces of Δx_n must map into distinct 2-flats, determined by the vertices of the corresponding image 2-cells. Hence, distinct 2-cell faces of Δx_n must map into distinct 2-cells, since they lie in distinct 2-flats. This means that the points $F(x_0)$, ... , $F(x_n)$, taken four at a time, are linearly independent.

Suppose it has been shown in this manner for $1 \leq k \leq n-1$, that all the k-cell faces of Δx_n map into k-cell faces and that all the k-flats determined by the vertices of each k-cell face map into k-flats, which by Theorem II.5.4 must be distinct; then distinct k-cell faces of Δx_n map into distinct k-cell faces. It follows that all the points $F(x_0)$, ... , $F(x_n)$, taken k + 2 at a time, are linearly independent.

Consider the n-cell, Δx_n. By the induction hypothesis, the points x_0, ... , x_n map into linearly independent points, $F(x_0)$, ... , $F(x_n)$, (hence $\Delta(F{:}x_0 \ldots x_n)$ is non-degenerate), and the boundary of Δx_n maps into the boundary of $\Delta(F{:}x_0 \ldots x_n)$. Let x^* be any interior point of Δx_n. Let x' be a boundary point of Δx_n. Then the line L through x' and x^* intersects the boundary of Δx_n in exactly two distinct points, x' and x'', by Lemma II.6.20. By the induction hypothesis, x' and x''

map into F(x') and F(x") on the boundary of $\Delta(F:x_0 . . . x_n)$. That is,

$$F(x') = \sum_{i=0}^{n} \alpha_i F(x_i), \qquad \sum_{i=0}^{n} \alpha_i = 1, \quad \alpha_i \geq 0, \text{ all } i,$$

and

$$F(x'') = \sum_{i=0}^{n} \beta_i F(x_i), \qquad \sum_{i=0}^{n} \beta_i = 1, \quad \beta_i \geq 0, \text{ all } i.$$

Since straight lines map into straight lines, and since F is one-to-one and continuous, then x^* maps into $F(x^*)$ on the line segment $\overline{F(x')F(x'')}$. Hence,

$$F(x^*) = \theta F(x') + (1 - \theta)F(x''), \qquad 0 < \theta < 1.$$

Therefore,

$$F(x^*) = \sum_{i=0}^{n} \left[\theta \alpha_i + (1 - \theta)\beta_i\right] F(x_i),$$

where all the coefficients are ≥ 0 and where the sum of the coefficients is equal to one. Therefore, $F(x^*)$ is in $\Delta(F:x_0 . . x_n)$. Since x^* was any interior point of Δx_n, the induction is complete and the theorem is proved.

II.6.23. Remark. It has actually been shown in the proof of Theorem II.6.22 that if F is continuous and one-to-one and maps straight lines into straight lines, then p-flats map into p-flats, $(1 \leq p \leq n - 1)$.

II.6.24. Theorem. Let F be a continuous, one-to-one transformation defined on a convex region E in $R^{(n)}$, such that p-flats map into p-flats for points in E (p fixed; $1 \leq p \leq n-1$). Then straight lines map into straight lines.

Proof. The proof is by induction on p. If p = 1, then straight lines map into straight lines by hypothesis.

Let p = 2. By Theorem II.5.4, distinct 2-flats map into distinct 2-flats for points in E. Let L be any straight line in E. Through L can be constructed two distinct 2-flats. This is easily done, since in E there will be a total of n + 1 linearly independent points. The line L is determined by only two linearly independent points. These two, together with one more not on L, will determine one of the required 2-flats, $S_2^{(1)}$. The same two points together with one point not on $S_2^{(1)}$ will determine the other required 2-flat, $S_2^{(2)}$. Since L is common to both 2-flats, and since the mapping is one-to-one and continuous, then L', the image of $L \cap E$ is common to $T_2^{(1)}$ and $T_2^{(2)}$, the images of $S_2^{(1)} \cap E$ and $S_2^{(2)} \cap E$ respectively. Since L' must be 1-dimensional, it must then be a straight line, as the intersection of two planes. Hence, the theorem is proved for p = 2.

Suppose it has been proved in this manner for p = 1, 2, . . . , n - 2, that if p-flats map into p-flats then straight lines map into straight lines. It will be shown that if (n-1)-flats map into (n-1)-flats, then straight lines map into straight lines. In this case, by Theorem II.5.4, distinct (n-1)-flats map into distinct (n-1)-flats.

Let S_{n-2} be an (n-2)-flat with points in E. As in the case of p = 2, two distinct (n-1)-flats, $S_{n-1}^{(1)}$ and $S_{n-1}^{(2)}$, having S_{n-2} in common can be constructed. Under the one-to-one and continuous transformation F, the two (n-1)-flats, $S_{n-1}^{(1)} \cap E$ and $S_{n-1}^{(2)} \cap E$ map into distinct (n-1)-flats, $T_{n-1}^{(1)}$ and $T_{n-1}^{(2)}$, and S_{n-2} must map into the intersection of $T_{n-1}^{(1)}$ and $T_{n-1}^{(2)}$ and hence the image of S_{n-2} must be an (n-2)-flat, by Theorem II.5.1. Since S_{n-2} was an arbitrary (n-2)-flat with points in E, then it must be concluded that (n-2)-flats map into (n-2)-flats. This puts the situation back in the previous case, and by the induction hypothesis it is immediately concluded that straight lines map into straight lines for points in E. This completes the induction and the proof of the theorem.

II.6.25. <u>Remark</u>. From Remark II.6.23 and Theorem II.6.24 it follows that the necessary and sufficient condition that a continuous, one-to-one mapping defined in a convex region E of $R^{(n)}$ take p-flats into p-flats, (p fixed; $1 \leq p \leq n-1$), is that the mapping take straight lines into straight lines.

II.7. THE CHARACTERIZATION FOR THE 2-DIMENSIONAL CASE

II.7.1. All the material is now at hand to prove the main theorem of this chapter for the case n = 2, except the following important lemma, which was suggested by W. Kaplan of the University of Michigan [Kaplan, (1)].

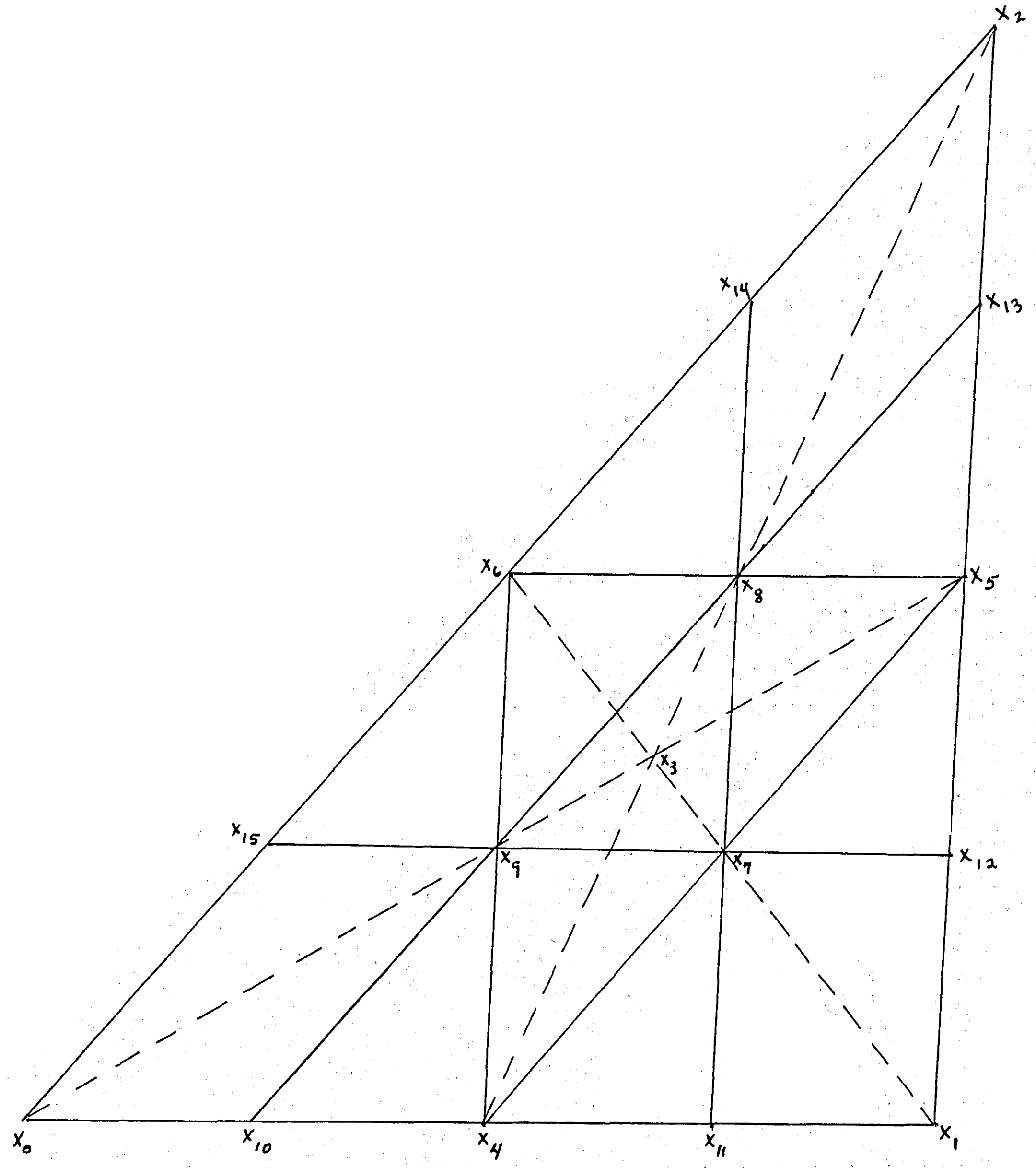

Fig. 2

II.7.2. Lemma. Let $\Delta(x_0 x_1 x_2)$ be any 2-cell of a convex region E in $R^{(2)}$. Let x_3 be the intersection of the medians of the triangle. Let $G:g^{(i)}(x)$, $(i = 1, 2)$, be a continuous, single valued transformation defined on E such that straight lines map into straight lines and such that x_0, x_1, x_2, and x_3 remain fixed. Then G is the identity transformation.
Proof. Since straight lines map into straight lines and since x_0, x_1, and x_2 remain fixed, then the sides of the triangle map into themselves. Furthermore, since x_3 remains fixed, the medians remain fixed. Therefore, the midpoints of the sides, x_4, x_5, and x_6 remain fixed as the intersection of fixed lines. (See Fig. 2).

Joining the midpoints of the sides, it is seen that the lines $\overline{x_4 x_5}$, $\overline{x_5 x_6}$, and $\overline{x_4 x_6}$ map into themselves. The points x_7, x_8, and x_9 remain fixed as intersections of fixed lines. The segments $\overline{x_5 x_6}$, $\overline{x_4 x_5}$, and $\overline{x_4 x_6}$ are parallel to $\overline{x_0 x_1}$, $\overline{x_0 x_2}$, and $\overline{x_1 x_2}$ respectively, since they divide the sides of $\Delta(x_0 x_1 x_2)$ in half. Therefore, $\Delta(x_0 x_4 x_6)$, $\Delta(x_4 x_1 x_5)$, $\Delta(x_5 x_2 x_6)$, and $\Delta(x_4 x_5 x_6)$ are all similar to $\Delta(x_0 x_1 x_2)$. Furthermore, the points x_7, x_8, and x_9 are the midpoints of $\overline{x_4 x_5}$, $\overline{x_5 x_6}$, and $\overline{x_4 x_6}$ respectively. To prove, for example, that x_8 is the midpoint of $\overline{x_5 x_6}$, notice first that since $\Delta(x_5 x_2 x_8)$ and $\Delta(x_4 x_1 x_2)$ are similar, then

$$\frac{\overline{x_2 x_8}}{\overline{x_2 x_4}} = \frac{\overline{x_5 x_8}}{\overline{x_1 x_4}}.$$

Also, since $\Delta(x_4 x_2 x_0)$ and $\Delta(x_8 x_2 x_6)$ are similar,

$$\frac{\overline{x_2 x_8}}{\overline{x_2 x_4}} = \frac{\overline{x_8 x_6}}{\overline{x_4 x_0}}.$$

Hence,

$$\frac{\overline{x_5 x_8}}{\overline{x_1 x_4}} = \frac{\overline{x_8 x_6}}{\overline{x_4 x_0}}.$$

But $\overline{x_4 x_0} = \overline{x_1 x_4}$. Hence, $\overline{x_5 x_8} = \overline{x_8 x_6}$.

To prove that the other two points mentioned are midpoints of the respective lines above, the same procedure is used.

The lines containing $\overline{x_7 x_8}$, $\overline{x_8 x_9}$, and $\overline{x_7 x_9}$ remain fixed. Hence, the points x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, and x_{15} remain fixed as the intersection of fixed lines.

The points x_{13} and x_{14} are the midpoints of the sides of $\Delta(x_5 x_2 x_6)$. To prove, for example, that x_{13} is the midpoint of $\overline{x_5 x_2}$, notice first that $\overline{x_8 x_{13}}$ is parallel to $\overline{x_6 x_2}$ since $\overline{x_{10} x_{13}}$ is parallel to $\overline{x_4 x_5}$ (since x_8 is the midpoint of $\overline{x_5 x_6}$ and x_9 is the midpoint of $\overline{x_4 x_6}$), which in turn is parallel to $\overline{x_6 x_2}$, since x_4 and x_5 are midpoints of the sides of the triangle, $\Delta(x_0 x_1 x_2)$. Therefore, $\Delta(x_5 x_2 x_6)$ and $\Delta(x_5 x_{13} x_8)$ are similar and

$$\frac{\overline{x_5 x_8}}{\overline{x_5 x_6}} = \frac{\overline{x_5 x_{13}}}{\overline{x_5 x_2}}.$$

But x_8 is the midpoint of $\overline{x_5x_6}$; hence,

$$\frac{\overline{x_5x_8}}{\overline{x_5x_6}} = \frac{1}{2}.$$

Consequently,

$$\frac{\overline{x_5x_{13}}}{\overline{x_5x_2}} = \frac{1}{2},$$

and x_{13} is the midpoint of $\overline{x_5x_2}$. In the same manner, x_{10} and x_{15} are the midpoints of the sides of $\Delta(x_0x_4x_6)$, and x_{11} and x_{12} are the midpoints of the sides of $\Delta(x_4x_1x_5)$.

Four small triangles, $\Delta(x_0x_4x_6)$, $\Delta(x_4x_1x_5)$, $\Delta(x_5x_2x_6)$, and $\Delta(x_4x_5x_6)$, have been constructed which are all similar to $\Delta(x_0x_1x_2)$ and each of which has its vertices and the midpoints of its sides, hence the intersection of its medians, fixed under the transformation G. It will be shown that there is a set of points dense in the perimeter of $\Delta(x_0x_1x_2)$ which remain fixed under the transformation G.

Let x^* be any point on the perimeter of $\Delta(x_0x_1x_2)$. It will be contained in one of the four smaller triangles which are similar to $\Delta(x_0x_1x_2)$. Choose this one and by a construction analogous to the preceding one, divide this triangle into four similar triangles, each of which has its vertices and the midpoints of its sides, hence the intersection of its medians, fixed under the transformation G, and one of which contains x^*. Choose the one containing x^* and repeat the construction. Continuing in this manner, a sequence of nested triangles is obtained, each of which has its vertices and the midpoints of its sides, hence the intersection of its medians, fixed under the transformation G, and each of which contains x^*. Eventually, a point on the perimeter of $\Delta(x_0 x_1 x_2)$, fixed under the transformation G, will be obtained which is as close to x^* as one chooses. That is, the set of points fixed under the transformation G is dense in the perimeter of $\Delta(x_0 x_1 x_2)$.

Since G is continuous, it follows that each point of the perimeter is fixed under G. Consider any point x of E which is an interior point of $\Delta(x_0 x_1 x_2)$. It also remains fixed under G; for let x' and x'' be two boundary points of $\Delta(x_0 x_1 x_2)$, not collinear with x. Each of the two distinct straight lines through $\overline{x'x}$ and $\overline{x''x}$ intersects the boundary of the triangle in two fixed points, by Lemma II.6.20. Hence, the lines must map into themselves and therefore, their intersection, x, must map into itself. Since x was any interior point of $\Delta(x_0 x_1 x_2)$ then the whole 2-cell, $\Delta(x_0 x_1 x_2)$, maps into itself.

Let x^* be any point of E which is not in the 2-cell, $\Delta(x_0 x_1 x_2)$. Let x' and x'' be two interior points of $\Delta(x_0 x_1 x_2)$ which are not collinear with x^*. Two such points exist by Theorem II.6.21. Since E is convex, x' and x'' can each be joined to x^* by a straight line. Since x' and x''

were interior points of $\Delta(x_0 x_1 x_2)$, each of these two straight lines must contain at least two fixed points of $\Delta(x_0 x_1 x_2)$, and hence the lines must map into themselves. Consequently, x^* must map into itself, as the intersection of two fixed lines. Since x^* was any point of E not in $\Delta(x_0 x_1 x_2)$, it has been shown that each point of E remains fixed under G and hence G must be the identity transformation.

II.7.3. <u>Theorem</u>. Let F: $f^{(i)}(x)$, $(i = 1, 2)$, be a continuous, one-to-one transformation defined on a convex region E in $R^{(2)}$ such that straight lines map into straight lines. Then F is of the form

$$\text{(II.7.4)} \quad \begin{aligned} f^{(1)}(x) &= \frac{a_{1,1} x^{(1)} + a_{1,2} x^{(2)} + a_{1,3}}{a_{3,1} x^{(1)} + a_{3,2} x^{(2)} + a_{3,3}}, \\ f^{(2)}(x) &= \frac{a_{2,1} x^{(1)} + a_{2,2} x^{(2)} + a_{2,3}}{a_{3,1} x^{(1)} + a_{3,2} x^{(2)} + a_{3,3}} \end{aligned}$$

where

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} \neq 0.$$

<u>Proof</u>: Let x_0, x_1, and x_2 be three linearly independent points of E. They determine a 2-cell of E. Let x_3 and x_4 be the midpoints of $\overline{x_0 x_1}$ and $\overline{x_0 x_2}$ respectively. Let x_5 be the intersection of the medians $\overline{x_2 x_3}$ and $\overline{x_1 x_4}$. F

carries x_0, x_1, x_2, into three points $F(x_0)$, $F(x_1)$, and $F(x_2)$ which are distinct and not collinear since F is one-to-one, continuous, and maps straight lines into straight lines. The point x_3 goes into $F(x_3)$ on $\overline{F(x_0)F(x_1)}$ and x_4 goes into $F(x_4)$ on $\overline{F(x_0)F(x_2)}$. The points $F(x_0)$, $F(x_1)$, $F(x_2)$, $F(x_3)$, and $F(x_4)$ are distinct because of one-to-oneness. The segments $\overline{x_2 x_3}$ and $\overline{x_1 x_4}$ map into $\overline{F(x_2)F(x_3)}$ and $\overline{F(x_1)F(x_4)}$ respectively. Hence, x_5 maps into $F(x_5)$ on the intersection of $\overline{F(x_2)F(x_3)}$ and $\overline{F(x_1)F(x_4)}$ and $F(x_5)$ is not on the sides of $\Delta(F: x_0 x_1 x_2)$, since $F(x_3)$ and $F(x_4)$ are distinct from $F(x_0)$, $F(x_1)$, and $F(x_2)$.

By Theorem II.2.8, there is one and only one transformation

$$F_1: \quad \begin{aligned} x^{(1)} &= \frac{\alpha_{1,1} f^{(1)}(x) + \alpha_{1,2} f^{(2)}(x) + \alpha_{1,3}}{\alpha_{3,1} f^{(1)}(x) + \alpha_{3,2} f^{(2)}(x) + \alpha_{3,3}} \\ x^{(2)} &= \frac{\alpha_{2,1} f^{(1)}(x) + \alpha_{2,2} f^{(2)}(x) + \alpha_{2,3}}{\alpha_{3,1} f^{(1)}(x) + \alpha_{3,2} f^{(2)}(x) + \alpha_{3,3}} \end{aligned}$$

where

$$\begin{vmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{vmatrix} \neq 0,$$

which carries the points $F(x_0)$, $F(x_1)$, $F(x_2)$, and $F(x_5)$ into the points x_0, x_1, x_2, and x_5 respectively. The transformation F_1 is continuous, one-to-one, and maps straight lines into straight lines. Consider the transformation F_1F. This transformation is continuous, one-to-one and carries straight lines into straight lines. Furthermore, the points x_0, x_1, x_2, and x_3 remain fixed under F_1F. By Lemma II.7.2, F_1F is the identity transformation. Hence $F = F_1^{-1}$, which is of the form (II.7.4), and the theorem is proved.

The characterization of the class of transformations F: $f^{(i)}(x)$, $(i = 1, 2)$, which are continuous, one-to-one, and map straight lines into straight lines is now complete. The next section of Chapter II extends the characterization to mapping functions F, which are continuous, one-to-one and map straight lines into straight lines, where F is defined in n-dimensions.

II.8. THE CHARACTERIZATION FOR THE n-DIMENSIONAL CASE.

II.8.1. In a triangle, the straight line which joins a vertex with the midpoint of the opposite side is called a median of the triangle. As a generalization of this concept, the following definition is given:

II.8.2. Definition. By a median of a tetrahedron is meant the straight line which joins a vertex with the intersection of the medians of the opposite face. In general, by a median of a p-cell, $(1 < p \leq n)$, is meant the straight line which joins a vertex of the p-cell with the intersection of the medians of the (p-1)-cell determined by the remaining p vertices of the p-cell.

This definition will be meaningful once it has been established that the medians of any p-cell intersect in a common point.

II.8.3. <u>Theorem</u>. Let $x_0 = (x_0^{(1)}, \ldots, x_0^{(n)}), \ldots,$ $x_p = (x_p^{(1)}, \ldots, x_p^{(n)})$, be the vertices of a p-cell, $(1 < p \leq n)$. Then the point

$$x_p^* = \left(\frac{x_0^{(1)} + \ldots + x_p^{(1)}}{p+1}, \ldots, \frac{x_0^{(n)} + \ldots + x_p^{(n)}}{p+1} \right)$$

is common to all the medians of the p-cell; that is, the medians of a p-cell intersect in a common point.

<u>Proof</u>. The proof is by induction on p. Let $x_0 = (x_0^{(1)}, \ldots, x_0^{(n)})$, $x_1 = (x_1^{(1)}, \ldots, x_1^{(n)})$, and $x_2 = (x_2^{(1)}, \ldots, x_2^{(n)})$ be the vertices of a triangle in $R^{(n)}$. The median from x_2 meets the opposite side of the triangle at the point

$$x_1^* = \left(\frac{x_0^{(1)} + x_1^{(1)})}{2}, \frac{x_0^{(2)} + x_1^{(2)}}{2}, \ldots, \frac{x_0^{(n)} + x_1^{(n)}}{2} \right).$$

The point x_2^*, which divides the median $x\,x^*$ into the ratio

$$\frac{\overline{x_2 x_2^*}}{\overline{x_2^* x_1^*}} = \frac{2}{1}$$

has the coordinates

$$x_2^* = \left(\frac{x_0^{(1)} + x_1^{(1)} + x_2^{(1)}}{3}, \ldots, \frac{x_0^{(n)} + x_1^{(n)} + x_2^{(n)}}{3} \right).$$

The same argument shows that x_2^* also divides the other two medians in the ratio 2:1. Hence, x_2^* lies on all the medians and the theorem is proved for the triangle.

Suppose it has been shown that the medians of the (p-1)-cell $\Delta(x_0 x_1 \ldots x_{p-1})$ meet in the common point

$$x_{p-1}^* = \left(\frac{x_0^{(1)} + x_1^{(1)} + \ldots + x_{p-1}^{(1)}}{p}, \ldots, \frac{x_0^{(n)} + x_1^{(n)} + \ldots + x_{p-1}^{(n)}}{p} \right).$$

Then, if $\Delta(x_0 x_1 \ldots x_p)$ is a p-cell, the median from x_p meets the opposite (p-1)-cell at the point x_{p-1}^*. The point

$$x_p^* = \left(\frac{x_0^{(1)} + x_1^{(1)} + \ldots + x_p^{(1)}}{p+1}, \ldots, \frac{x_0^{(n)} + x_1^{(n)} + \ldots + x_p^{(n)}}{p+1} \right),$$

divides the median $x_p x_{p-1}^*$ into the ratio

$$\frac{\overline{x_p x_p^*}}{\overline{x_p^* x_{p-1}^*}} = \frac{p}{1}.$$

The same argument shows that x_p^* divides the medians from the remaining p vertices of the p-cell in the same ratio. Hence, x_p^* lies on all the medians of the p-cell and the theorem is proved.

II.8.4. The two-dimensional case of the following theorem was proved on pages 43 - 48. The generalization to n-dimensions is analogous to the two-dimensional case, but the proof is given here for completeness.

II.8.5. Theorem. Any n + 2 points, $x_1, x_2, \ldots, x_{n+2}$ in n-dimensional Euclidean space, no n + 1 of which lie in an (n-1)-flat, may be carried over into any n + 2 points, $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{n+2}$, no n + 1 of which lie in an (n-1)-flat, by one and only one transformation of the form

$$\text{(II.8.6)} \qquad \overline{x}^{(j)} = \frac{a_{j,1}\, x^{(1)} + a_{j,2}\, x^{(2)} + \cdots + a_{j,n}\, x^{(n)} + a_{j,n+1}}{a_{n+1,1}\, x^{(1)} + a_{n+1,2}\, x^{(2)} + \cdots + a_{n+1,n}\, x^{(n)} + a_{n+1,n+1}}$$

$(j = 1, \ldots, n)$, where

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n+1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n+1} \end{vmatrix} \neq 0.$$

Proof. The proof will be carried through using homogeneous coordinates, as before. The transformation (II.8.6) will then be of the form

$$\rho \overline{x}^{(j)} = \sum_{k=1}^{n+1} a_{j,k} x^{(k)}, \qquad (j = 1, \ldots, n+1),$$

where the homogeneous coordinates of the point x are $(x^{(1)}, x^{(2)}, \ldots, x^{(n+1)})$.

The projective transformation

$\rho \overline{X}^{(j)} = \sum_{K=1}^{n+1} a_{j,K} X^{(K)}$, $(j = 1, \ldots, n + 1)$, carries over any given point x into a point $\overline{x}$, the position of $\overline{x}$ depending on the values of $a_{j,K}$. The proof of the theorem will be complete if it is possible to find one and only one (except for a constant factor which may be introduced throughout) set of $n^2 + 3n + 3$ constants, (the $a_{j,K}$'s being (n + 1) of them, and the n + 2 others being $\rho_1, \rho_2, \ldots, \rho_{n+2}$ --none of which is zero) which satisfy the $n^2 + 3n + 2$ equations

$$\rho_i \overline{X}_i^{(j)} = \sum_{K=1}^{n+1} a_{j,K} X_i^{(K)}, \quad (i = 1, \ldots, n + 2;\ j = 1, \ldots, n + 1).$$

Since all the X's and X's are known, there are $n^2 + 3n + 2$ homogeneous linear equations in $n^2 + 3n + 3$ unknowns. Hence, there are always solutions different from zero, the number of independent ones depending on the rank of the coefficients of the unknowns. Transposing and re-arranging the equations, the matrix of these equations becomes

$$
\text{(II.8.7)} \quad \begin{pmatrix}
(X_i^{(K)}) & (0) & \ldots & (0) & (-\delta_{i,K}\overline{X}_i^{(1)}) & (0) \\
(0) & (X_i^{(K)}) & \ldots & (0) & (-\delta_{i,K}\overline{X}_i^{(2)}) & (0) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(0) & (0) & \ldots & (X_i^{(K)}) & (-\delta_{i,K}\overline{X}_i^{(n+1)}) & (0) \\
(X_{n+2}^{(K)}) & (0) & \ldots & (0) & (0) & -\overline{X}_{n+2}^{(1)} \\
(0) & (X_{n+2}^{(K)}) & \ldots & (0) & (0) & -\overline{X}_{n+2}^{(2)} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(0) & (0) & \ldots & (X_{n+2}^{(K)}) & (0) & -\overline{X}_{n+2}^{(n+1)}
\end{pmatrix}
$$

$(i = 1, \ldots, n + 1; k = 1, \ldots, n + 1)$, where the k's are column numbers and the i's are row numbers of the submatrices. Notice that $(X_{n+2}^{(K)})$ is a row matrix with n + 1 elements.

Since the points $x_1, \ldots, x_{n+2}$ are all distinct and no n + 1 lie in an (n-1)-flat, there are n + 1 constants, c_i, none of which is zero, such that

$$\sum_{i=1}^{n+1} c_i X_i^{(K)} + X_{n+2}^{(K)} = 0, \qquad (k = 1, \ldots, n + 1).$$

Adding to the $(n^2 + 2n + 2)$th row c_i times the ith row, $(i = 1, \ldots, n + 1)$; adding to the $(n^2 + 2n + 3)$th row c_i times the $(n + 1 + i)$th row, $(i = 1, \ldots, n + 1)$; etc.; until finally, adding to the last (the $(n^2 + 3n + 2)$th) row c_i times the $(n^2 + n + 1)$th row, $(i = 1, \ldots, n + 1)$, (II.8.7) becomes

$$\text{(II.8.8)} \quad \begin{bmatrix} (X_i^{(K)}) & (0) & \cdots & (0) & (-\delta_{i,K}\overline{X}_i^{(1)}) & (0) \\ (0) & (X_i^{(K)}) & \cdots & (0) & (-\delta_{i,K}\overline{X}_i^{(2)}) & (0) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (0) & (0) & \cdots & (X_i^{(K)}) & (-\delta_{i,K}\overline{X}_i^{(n+1)}) & (0) \\ (0) & (0) & \cdots & (0) & (-c_K\overline{X}_K^{(i)}) & (-\overline{X}_{n+2}^{(i)}) \end{bmatrix}$$

($i = 1, \ldots, n + 1$; $k = 1, \ldots, n + 1$), where the k's stand for the column numbers and the i's stand for the row numbers. (Notice that $(-X_{n+2}^{(i)})$ is a column matrix of $n + 1$ elements.)

Deleting the last column, the determinant of the matrix of the remaining columns is easily calculated to be

$$D_{(n^2+3n+3)} = (-1)^{(n+1} \left(\prod_{i=1}^{n+1} c_i \right) \left| X_i^{(K)} \right|^{n+1} \left| \overline{X}_i^{(k)} \right| .$$

This is different from zero since the x's and the $\overline{x}$'s are distinct and no $n + 1$ lie in an $(n-1)$-flat. Also, by Theorem II.2.6, ρ_{n+2} is proportional to $D_{(n^2+3n+3)}$, and hence is different from zero.

A similar situation is found to be true for $D_{(n^2+3n+2)}, \ldots, D_{(n^2+2n+2)}$.

Hence, $\rho_i = 0$, ($i = 1, \ldots, n + 2$); therefore, one solution to the equations has been found and it is the only independent one since the rank of the matrix of the equations is $(n^2 + 3n + 2)$, one less than the number of unknowns. The theorem is therefore proved.

II.8.9. <u>Lemma</u>. Let $x_0, \ldots, x_n$ be n + 1 linearly independent points in a convex region E of $R^{(n)}$, $(n \geq 2)$, which form the vertices of an n-cell. Let x^* be the intersection of the medians of the n-cell. Let $G: g^{(i)}(x)$, $(i = 1, \ldots, n)$, be a transformation defined in E which is continuous, one-to-one and carries p-flats into p-flats (p fixed; $1 \leq p \leq n-1$), and which furthermore leaves the points $x_0, x_1, \ldots, x_n, x^*$ fixed. Then G is the identity transformation.

<u>Proof</u>. It should first be noted that since p-flats map into p-flats (p fixed; $1 \leq p \leq n-1$), then p-flats map into p-flats for all $p(1 \leq p \leq n-1)$ by Remark II.6.25.

The proof of the lemma is by induction. The lemma has already been proved for n = 2. (Lemma II.7.2).

Suppose n = 3. Let x_0, x_1, x_2, and x_3 be the vertices of a tetrahedron, $\Delta(x_0 x_1 x_2 x_3)$, in E and let x^* be the intersection of its medians. Since these points remain fixed and since straight lines map into straight lines, (hence faces of $\Delta(x_0 x_1 x_2 x_3)$ map into faces of $\Delta(x_0 x_1 x_2 x_3)$, by Theorem II.6.22), then the intersection of the median from any vertex with the opposite face must also remain fixed. This point is the intersection of the medians of that face. Since the theorem is true for n = 2, all the points of that face remain fixed under G. The same argument applied to the remaining faces shows that every point on the boundary of $\Delta(x_0 x_1 x_2 x_3)$ remains fixed. Let x be any point interior to $\Delta(x_0x_1x_2x_3)$. Let x' and x" be two points of the boundary of $\Delta(x_0x_1x_2x_3)$ not collinear with x. Each of the two distinct lines through $\overline{x'x}$ and $\overline{x''x}$ intersects the boundary of $\Delta(x_0x_1x_2x_3)$ in two fixed points, by Theorem II.6.19, and therefore must map into themselves. Hence their intersection x, must remain fixed as the intersection of two fixed lines. Hence, since x was any point on the interior of $\Delta(x_0x_1x_2x_3)$, G maps every point of $\Delta(x_0x_1x_2x_3)$ into itself. Let x** be any point of E not in $\Delta(x_0x_1x_2x_3)$ and let x' and x" be two points interior to $\Delta(x_0x_1x_2x_3)$ which are not collinear with x**. Two such points exist by Theorem II.6.21. Since E is convex, the points x' and x" can be joined to x** by two distinct straight lines, each of which must contain at least two fixed points of $\Delta(x_0x_1x_2x_3)$. Hence, these two lines must map into themselves. Consequently, x** must map into itself as the intersection of two fixed lines. Since x** was any point of E not in $\Delta(x_0x_1x_2x_3)$, then it has been shown that every point of E maps into itself under G, and hence G is the identity transformation, proving the lemma for n = 3.

Suppose the lemma is true for $n \leq k$. Let $n = k + 1$. Let $x_0, x_1, \ldots, x_{k+1}$ be k + 2 linearly independent points of E in $R^{(k+1)}$ which form the vertices of a (k + 1)-cell, Δx_{k+1}. Let x* be the intersection of the medians of Δx_{k+1}. By hypothesis all these points remain fixed under G. Since straight lines map into straight lines, then all the m-cell faces ($0 \leq m \leq k$) must map into themselves by Theorem II.6.22. Hence the point of intersection of the median from any vertex of Δx_{k+1} to the opposite k-cell must remain fixed under G. But this point of intersection is the intersection of the medians of that k-cell face. Since the lemma is true for n = k, by the induction hypothesis, every point of that k-cell face remains fixed under G. Repeating the argument for the remaining k-cell faces of the (k + 1)-cell, it is seen that every point of the boundary of the (k + 1)-cell, Δx_{k+1}, remains fixed under G.

Let x be any point interior to Δx_{k+1}. Let x' and x" be any two boundary points of Δx_{k+1} not collinear with x. Each of the two distinct lines through $\overline{x'x}$ and $\overline{x''x}$ intersect the boundary of Δx_{k+1} in exactly two fixed points, and therefore must remain fixed. Hence, x remains fixed as the intersection of two fixed lines. Since x was any point interior to Δx_{k+1}, then every point of the (k + 1)-cell, Δx_{k+1}, remains fixed under the transformation G.

Let x^{**} be any point of E not in Δx_{k+1}, and let x' and x" be two points interior to Δx_{k+1}, which are not collinear with x^{**}. This is possible by Theorem II.6.21. Since E is convex, the points x' and x" can be joined to x^{**} by two distinct lines, each of which must contain at least two fixed points of Δx_{k+1}. Hence, these two lines must map into themselves. Consequently, the point x^{**} must map into itself as the intersection of two fixed lines. Since x^{**} was any point of E not in Δx_{k+1}, it has been proved that every point of E maps into itself, and G is the identity transformation. This completes the induction and the proof of the theorem.

II.8.10. Theorem. Let $F: f^{(i)}(x)$, $(i = 1, \ldots, n)$, be a continuous, one-to-one mapping defined on a convex region E in $R^{(n)}$, which is such that p-flats map into p-flats, (p fixed; $0 < p \leq n-1$). Then F is of the form

$$(\text{II.8.11}) \qquad F: f^{(i)}(x) = \frac{a_{i,1}\, x^{(1)} + \cdots + a_{i,n}\, x^{(n)} + a_{i,n+1}}{a_{n+1,1}\, x^{(1)} + \cdots + a_{n+1,n}\, x^{(n)} + a_{n+1,n+1}},$$

$$(i = 1, \ldots, n),$$

where

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n+1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n+1} \end{vmatrix} \neq 0.$$

Proof. By Remark II.6.25, p-flats map into p-flats for all $p(1 \leq p \leq n-1)$. Let $x_0, x_1, \ldots, x_n$ be n + 1 linearly independent points of E which form the vertices of an n-cell, Δx_n. Let x^* be the intersection of the medians of Δx_n. Under the mapping F, the vertices of Δx_n map into the n + 1 linearly independent points, $F(x_0), \ldots, F(x_n)$, which form the vertices of an n-cell, ΔF_n. This is true because F takes k-cell faces of Δx_n into distinct k-cell faces of ΔF_n $(0 \leq k \leq n-1)$, by Theorem II.6.22. The point $F(x^*)$, the image of x^*, does not lie in any k-cell face of ΔF_n, $(0 \leq k \leq n-1)$, since if it did, the mapping would not be one-to-one.

By Theorem II.8.5 there is one and only one transformation of the type

$$\text{(II.8.12)} \qquad F_1 : x^{(i)} = \frac{\alpha_{i,1} f^{(1)}(x) + \dots + \alpha_{i,n} f^{(n)}(x) + \alpha_{i,n+1}}{\alpha_{n+1,1} f^{(1)}(x) + \dots + \alpha_{n+1,n} f^{(n)}(x) + \alpha_{n+1,n+1}},$$

$$(i = 1, \dots, n),$$

where

$$\begin{vmatrix} \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,n+1} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{n+1,1} & \alpha_{n+1,2} & \dots & \alpha_{n+1,n+1} \end{vmatrix} \neq 0,$$

which carries the points $F(x_0), \dots, F(x_n)$, and $F(x^*)$ into the points $x_0, x_1, \dots, x_n$ and x^* respectively. The transformation F_1 is continuous, one-to-one and carries straight lines into straight lines. Consider the transformation $F_1 F$. This transformation is continuous, one-to-one, carries straight lines into straight lines, and furthermore leaves the points $x_0, \dots, x_1$, and x^* fixed. Hence, by Lemma II.8.9, $F_1 F$ is the identity transformation. Therefore $F = F_1^{-1}$, which is of the form

(II.8.11). This proves the theorem.

II.8.13. Remark. It has been pointed out several times before that the transformations of the form (II.8.11) are one-to-one, continuous, and carry straight lines into straight lines (hence p-flats into p-flats, $(0 < p \leq n-1)$). Conversely, it has been shown that the class of transformations which are continuous, one-to-one, and carry p-flats into p-flats (p fixed; $1 \leq p \leq n-1$), are the linear fractional transformations of the form (II.8.11). Thus, one must conclude that the precise class of transformations which are continuous, one-to-one and map p-flats into p-flats (p fixed; $1 \leq p \leq n-1$) are the linear fractional transformations.

CHAPTER III

THE CHARACTERIZATION OF A CLASS OF DIFFERENTIABLE FUNCTIONS

III.1. INTRODUCTION

III.1.1. In this chapter the generalized derivatives defined and discussed in Chapter I will again be the main topic of discussion. It will be shown that the precise class of transformations, F: f (x), (i = 1, . . . , n), which have a non-zero derivative, D F, with respect to the class of increments I , is the class of linear fractional transformations:

$$\text{(III.1.2)} \qquad F\colon f^{(i)}(x) = \frac{a_{i,1}\, x^{(1)} + \dots + a_{i,n}\, x^{(n)} + a_{i,n+1}}{a_{n+1,1}\, x^{(1)} + \dots + a_{n+1,n}\, x^{(n)} + a_{n+1,n+1}},$$

$$(i = 1, \dots, n),$$

where

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n+1} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n+1} \end{vmatrix} \neq 0.$$

This will follow from the results of Chapter II when it has been shown that the transformation F, having a non-zero derivative, is continuous, one-to-one, and takes straight lines into straight lines.

The two-dimensional case will be discussed first to give a clearer understanding of what is taking place. The results will then be extended to n-dimensions. In the next section $x = (x^{(1)}, x^{(2)})$.

III.2. THE CHARACTERIZATION FOR THE 2-DIMENSIONAL CASE

III.2.1. Before the main theorem of this section can be proved, several preliminary theorems must be proved. These theorems give some important properties of the generalized derivatives with respect to the class of increments I .

III.2.2. <u>Theorem.</u> Let $F \neq f^{(i)}(x)$, $(i = 1, 2)$ be defined on an open set E in $R^{(2)}$ and let $D_X F$ exist and have the value $d \neq 0$ at a point x_0 in E. The F is continuous at x_0.

<u>Proof.</u> Since it has been assumed that $D_X F|_{x_0} = d$, then for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{(III.2.3)} \qquad \left| \frac{\Delta(F : x_0 x_1 x_2)}{\Delta(x_0 x_1 x_2)} - d \right| = \left| \frac{\Delta F}{\Delta x} - d \right| < \epsilon/3$$

whenever $\|x_0 x_i\| < \delta$, $(i = 1, 2)$. In particular, choose $\epsilon = \epsilon^*$ so that $\epsilon^*/3 < |d|$. Then there is a δ^* such that inequality (III.2.3) holds.

By theorem I.5.8, $D_X F|_{x_0}$ can be calculated by taking the limit of the ratios, $\Delta(F : x_1 x_2 x_3)/\Delta(x_1 x_2 x_3)$, where $\Delta(x_1 x_2 x_3)$ is chosen to satisfy the conditions of the hypotheses of Theorem I.5.8. Choose three points, x_1, x_2, and x_3 in $N_{\delta^*}(x_0)$ so that x_0 is interior to

$\Delta(x_1 x_2 x_3)$ and keep these points fixed. For these three points

$$\left| \frac{\Delta(F:x_1 x_2 x_3)}{\Delta(x_1 x_2 x_3)} - d \right| < \epsilon^*$$

since these three points were chosen to satisfy the hypotheses of Theorem I.5.8.

The 2-cells, $\Delta(x_0 x_1 x_2)$, $\Delta(x_1 x_0 x_3)$, and $\Delta(x_0 x_2 x_3)$, all have two-dimensional volume different from zero. Furthermore, since relation (III.2.3) holds for each of these increments with $\epsilon^*/3 < |d|$, and since $d \neq 0$, then $\Delta(F:x_0 x_1 x_2)$, $\Delta(F:x_1 x_0 x_3)$, and $\Delta(F:x_0 x_2 x_3)$ must all be different from zero.

Consider the quantities

$$\left| \frac{f^{(i)}(x_j) - f^{(i)}(x_0)}{\Delta(F:x_0 x_1 x_2)} \right|,$$

$$\left| \frac{f^{(i)}(x_k) - f^{(i)}(x_0)}{\Delta(F:x_1 x_0 x_3)} \right|,$$

$$\left| \frac{f^{(i)}(x_\ell) - f^{(i)}(x_0)}{\Delta(F:x_0 x_2 x_3)} \right|,$$

where ($i = 1, 2$), ($j = 1, 2$), ($k = 1, 3$), and ($\ell = 2, 3$). These quantities are all fixed, since all the points involved are fixed points. Hence, there is a largest one, which will be denoted by S.

Let x' be a variable point of $N_{\delta^*}(x_0)$, which for the moment, is restricted to lie off the lines containing the segments $\overline{x_0x_1}$ and $\overline{x_0x_2}$. Relation (III.2.3) holds for $\Delta(x_0x'x_2)$ and $\Delta(x_0x_1x')$ and their images.

Now given a sufficiently small $\epsilon' > 0$ (in particular, for $\epsilon' \leq \epsilon^*$), there exists a $\delta_1 > 0$ such that

$$\left|\Delta(F:x_0x'x_2)\right| < \epsilon'/2S$$

whenever $\|x_0x'\| < \delta_1$. For suppose this assertion is false. Then for fixed $\epsilon' \leq \epsilon^*$ and for <u>every</u> $\delta > 0$, there is at least one point $x^* \in N_\delta(x_0)$ such that

$$\left|\Delta(F:x_0x^*x_2)\right| \geqq \epsilon'/2S.$$

As δ is allowed to approach zero, $\Delta(x_0x'x_2)$ approaches zero, since $\Delta(x_0x'x_2)$ varies directly as $\|x_0x'\|$, x_0 and x_2 being fixed points. Then, as δ approaches zero, the difference quotient

$$\frac{\Delta(F:x_0x'x_2)}{\Delta(x_0x'x_2)}$$

becomes arbitrarily large for the points x' in $N_\delta(x_0)$ such that $\left|\Delta(F:x_0x'x_2)\right| \geqq \epsilon'/2S$. For such points, relation (III.2.3), with $\epsilon' \leq \epsilon^*$, cannot hold, contradicting the assumption that $D_xF|_{x_0} = d$.

Similarly, if ϵ' is any fixed positive number less than or equal to ϵ^*, there must exist a $\delta_2 > 0$ such that

$$\left|\Delta(F:x_0x_1x')\right| < \epsilon'/2S$$

whenever $\|x_0x'\| < \delta_2$, x' in the restricted region. Let $\delta' = \min(\delta_1, \delta_2)$. Then

$$\left|\Delta(F:x_0x'x_2)\right| < \epsilon'/2S \quad \text{and} \quad \left|\Delta(F:x_0x_1x')\right| < \epsilon'/2S$$

whenever $\|x_0x'\| < \delta'$, x' remaining in the restricted region.

Now

$$\text{(III.2.4)} \quad \Delta(F:x_0x'x_2) = \frac{1}{2!}\begin{vmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & 1 \\ f^{(1)}(x') & f^{(2)}(x') & 1 \\ f^{(1)}(x_2) & f^{(2)}(x_2) & 1 \end{vmatrix}$$

and

$$\text{(III.2.5)} \quad \Delta(F:x_0x_1x') = \frac{1}{2!}\begin{vmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & 1 \\ f^{(1)}(x_1) & f^{(2)}(x_1) & 1 \\ f^{(1)}(x') & f^{(2)}(x') & 1 \end{vmatrix} .$$

Subtracting the first row from the remaining rows and expanding by the last column in each case, (III.2.4) and (III.2.5) become

$$\text{(III.2.6)} \quad \Delta(F:x_0x'x_2) = \frac{1}{2!}\begin{vmatrix} f^{(1)}(x') - f^{(1)}(x_0) & f^{(2)}(x') - f^{(2)}(x_0) \\ f^{(1)}(x_2) - f^{(1)}(x_0) & f^{(2)}(x_2) - f^{(2)}(x_0) \end{vmatrix}$$

and

$$\text{(III.2.7)} \quad \Delta(F:x_0x_1x') = \frac{1}{2!}\begin{vmatrix} f^{(1)}(x_1) - f^{(1)}(x_0) & f^{(2)}(x_1) - f^{(2)}(x_0) \\ f^{(1)}(x') - f^{(1)}(x_0) & f^{(2)}(x') - f^{(2)}(x_0) \end{vmatrix}$$

respectively. Expanding, (III.2.6) and (III.2.7) become

$$2!\ \Delta(F:x_0x'x_2) =$$

$$\left[f^{(1)}(x')-f^{(1)}(x_0)\right]\left[f^{(2)}(x_2)-f^{(2)}(x_0)\right]-\left[f^{(2)}(x')-f^{(2)}(x_0)\right]\left[f^{(1)}(x_2)-f^{(1)}(x_0)\right]$$

and

$$2!\ \Delta(F:x_0x_1x') =$$

$$-\left[f^{(1)}(x')-f^{(1)}(x_0)\right]\left[f^{(2)}(x_1)-f^{(2)}(x_0)\right]+\left[f^{(2)}(x')-f^{(2)}(x_0)\right]\left[f^{(1)}(x_1)-f^{(1)}(x_0)\right]$$

respectively.

It is possible to solve for $f^{(1)}(x') - f^{(1)}(x_0)$ and $f^{(2)}(x') - f^{(2)}(x_0)$ provided the determinant of their coefficients in the above two equations is not zero. This determinant is

$$D = \begin{vmatrix} f^{(2)}(x_2) - f^{(2)}(x_0) & -\left[f^{(1)}(x_2) - f^{(1)}(x_0)\right] \\ -\left[f^{(2)}(x_1) - f^{(2)}(x_0)\right] & f^{(1)}(x_1) - f^{(1)}(x_0) \end{vmatrix}$$

$$= \begin{vmatrix} f^{(2)}(x_2) - f^{(2)}(x_0) & f^{(1)}(x_2) - f^{(1)}(x_0) \\ f^{(2)}(x_1) - f^{(2)}(x_0) & f^{(1)}(x_1) - f^{(1)}(x_0) \end{vmatrix}$$

$$= \begin{vmatrix} f^{(2)}(x_0) & f^{(1)}(x_0) & 1 \\ f^{(2)}(x_2) & f^{(1)}(x_2) & 1 \\ f^{(2)}(x_1) & f^{(1)}(x_1) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & 1 \\ f^{(1)}(x_1) & f^{(2)}(x_1) & 1 \\ f^{(1)}(x_2) & f^{(2)}(x_2) & 1 \end{vmatrix}$$

$$= 2!\ \Delta(F:x_0x_1x_2) \neq 0.$$

Hence it is possible to solve for $f^{(1)}(x') - f^{(1)}(x_0)$ and $f^{(2)}(x') - f^{(2)}(x_0)$ in the expansions of (III.2.6) and (III.2.7). Solving these two equations, one obtains

$$f^{(1)}(x') - f^{(1)}(x_0) = \frac{2! \begin{vmatrix} \Delta(F:x_0 x' x_2) & -[f^{(1)}(x_2) - f^{(1)}(x_0)] \\ \Delta(F:x_0 x_1 x') & f^{(1)}(x_1) - f^{(1)}(x_0) \end{vmatrix}}{2!\ \Delta(F:x_0 x_1 x_2)}$$

and

$$f^{(2)}(x') - f^{(2)}(x_0) = \frac{2! \begin{vmatrix} f^{(2)}(x_2) - f^{(2)}(x_0) & \Delta(F:x_0 x' x_2) \\ -[f^{(2)}(x_1) - f^{(2)}(x_2)] & \Delta(F:x_0 x_1 x') \end{vmatrix}}{2!\ \Delta(F:x_0 x_1 x_2)}.$$

Now if $\|x_0 x'\| < \delta'$, x' in the restricted region, then

$$\left|f^{(1)}(x') - f^{(1)}(x_0)\right| <$$
$$\frac{\left|f^{(1)}(x_1) - f^{(1)}(x_0)\right| \cdot \epsilon'/2S + \left|f^{(1)}(x_2) - f^{(1)}(x_0)\right| \cdot \epsilon'/2S}{\left|\Delta(F:x_0 x_1 x_2)\right|}$$
$$< \frac{2L_1 \epsilon'}{2S} = L_1 \epsilon'/S,$$

and

$$|f^{(2)}(x') - f^{(2)}(x_0)| < \frac{|f^{(2)}(x_2) - f^{(2)}(x_0)| \cdot \epsilon'/2S + |f^{(2)}(x_1) - f^{(2)}(x_0)| \cdot \epsilon'/2S}{|\Delta(F: x_0 x_1 x_2)|}$$

$$< \frac{2L_2 \epsilon'}{2S} = L_2 \epsilon'/S,$$

where

$$L_1 = \max \left\{ \frac{|f^{(1)}(x_i) - f^{(1)}(x_0)|}{|\Delta(F: x_0 x_1 x_2)|} \right\}$$

and

$$L_2 = \max \left\{ \frac{|f^{(2)}(x_i) - f^{(2)}(x_0)|}{|\Delta(F: x_0 x_1 x_2)|} \right\},$$

$(i = 1, 2)$.

In exactly the same way, restricting x' to remain off the lines containing $\overline{x_0 x_1}$ and $\overline{x_0 x_3}$, it can be shown that for every sufficiently small $\epsilon' > 0$ there exists a $\delta'' > 0$ such that

$$|f^{(1)}(x') - f^{(1)}(x_0)| < L_3 \epsilon'/S$$

and

$$|f^{(2)}(x') - f^{(2)}(x_0)| < L_4 \epsilon'/S,$$

whenever $\|x_0 x'\| < \delta''$, x' in this restricted region, and where

$$L_3 = \max \left\{ \frac{|f^{(1)}(x_j) - f^{(1)}(x_0)|}{|\Delta(F: x_1 x_0 x_3)|} \right\}$$

and

$$L_4 = \max\left\{ \frac{\left|f^{(2)}(x_j) - f^{(2)}(x_0)\right|}{\left|\Delta(F:x_1 x_0 x_3)\right|} \right\},$$

$(j = 1, 3)$.

Finally, letting x' remain off the lines containing $\overline{x_0 x_2}$ and $\overline{x_0 x_3}$, for every sufficiently small $\epsilon' > 0$ there exists a $\delta''' > 0$ such that

$$\left| f^{(1)}(x') - f^{(1)}(x_0) \right| < L_5 \, \epsilon'/S$$

and

$$\left| f^{(2)}(x') - f^{(2)}(x_0) \right| < L_6 \, \epsilon'/S,$$

whenever $\|x_0 x'\| < \delta'''$, x' in this restricted region, and where

$$L_5 = \max\left\{ \frac{\left|f^{(1)}(x_k) - f^{(1)}(x_0)\right|}{\left|\Delta(F:x_0 x_2 x_3)\right|} \right\}$$

and

$$L_6 = \max\left\{ \frac{\left|f^{(2)}(x_k) - f^{(2)}(x_0)\right|}{\left|\Delta(F:x_0 x_2 x_3)\right|} \right\},$$

$(k = 2, 3)$.

Choose $\delta^* = \min(\delta', \delta'', \delta''')$. Since $S \geq L_i$, $(i = 1, \ldots, 6)$, it follows that for every sufficiently small $\epsilon' > 0$ there is a $\delta^* > 0$ such that

$$\left| f^{(1)}(x') - f^{(1)}(x_0) \right| < \epsilon'$$

and

$$\left| f^{(2)}(x') - f^{(2)}(x_o) \right| < \epsilon'$$

whenever $\|x_o x'\| < \delta^*$, with no other restriction on x'. Hence $f^{(1)}(x)$ and $f^{(2)}(x)$ are both continuous at x_o, proving the theorem.

III.2.8. <u>Lemma</u>. Let $F: f^{(i)}(x)$, $(i = 1, 2)$, be defined on a convex region E in $R^{(2)}$ and let $D_x F$ exist and be different from zero in E. Let x_o be a point of E. Then in a sufficiently small neighborhood of x_o straight lines through x_o map into straight lines through $F(x_o)$, the image of x_o under F.

<u>Proof</u>. The transformation F is continuous in E by Theorem III.2.2, since $D_x F \neq 0$ at each point of E. Let $D_x F|_{x_o} = d$. Let $\epsilon > 0$ be given such that $\epsilon < |d|$. Since $D_x F|_{x_o} = d \neq 0$, there exists a $\delta_o > 0$ such that

$$\left| \frac{\Delta(F: x_o x_1 x_2)}{\Delta(x_o x_1 x_2)} - d \right| < \epsilon \qquad \text{(III.2.9)}$$

whenever $\|x_o x_i\| < \delta_o$, $(i = 1, 2)$, $\Delta(x_o x_1 x_2)$ in the class I_1. It will be shown that in $N_{\delta_o}(x_o)$, straight lines through x_o map into straight lines.

Suppose the theorem is false. Then there is a straight line, L, through x_o such that $L \cap N_{\delta_o}(x_o)$ does not map into a straight line.

In $N_{\delta_0}(x_0)$, no point other than x_0 maps into $F(x_0)$; for if $x_1' \neq x_0$ maps into $F(x_0)$, then x_1' and x_0 together with a suitably chosen point x_2' would map into an increment of zero area. For $\Delta(x_0 x_1' x_2')$, relation (III.2.9), with $\epsilon < |d|$, would not hold, contradicting the assumption that $D_x F|_{x_0} = d$.

Let $x_1 \neq x_0$ be a point on L in $N_{\delta_0}(x_0)$. The point x_1 maps into $F(x_1) \neq F(x_0)$. Since the theorem is false there is a point $F(x_2)$, not on the line containing the segment $\overline{F(x_0)F(x_1)}$, which is the image of at least one point x_2 on L in $N_{\delta_0}(x_0)$. Let x be a variable point of $N_{\delta_0}(x_0)$, which, together with x_0 and x_1, forms an increment of I_1. Let x approach x_2. Since F is continuous in $N_{\delta_0}(x_0)$, F(x) approaches $F(x_2)$. Hence, $\Delta(x_0 x_1 x)$ approaches $\Delta(x_0 x_1 x_2) = 0$ as x approaches x_2, but $\Delta(F{:}x_0 x_1 x)$ approaches $\Delta(F{:}x_0 x_1 x_2) \neq 0$, since $F(x_0)$, $F(x_1)$ and $F(x_2)$ are not collinear. Hence the ratio

$$\frac{\Delta(F{:}x_0 x_1 x)}{\Delta(x_0 x_1 x)}$$

becomes arbitrarily large as x approaches x_2. Relation (III.2.9) does not then hold, contradicting the assumption that $D_x F|_{x_0} = d$.

Hence, one must conclude that in $N_{\delta_0}(x_0)$, straight lines through x_0 map into straight lines through $F(x_0)$.

III.2.10. Theorem. Let F: $f^{(i)}(x)$, $(i = 1, 2)$, be defined on a convex region E in $R^{(2)}$ and let $D_x F$ exist and be different from zero in E. Then straight lines in E map into straight lines.

Proof. Let L be a line defined in E and let x_0 be a point of $L \cap E$. By Lemma III.2.8 there is a $\delta_0 > 0$ such that $L \cap N_{\delta_0}(x_0)$ maps into a straight line, L'. Let x^* be any other point on $L \cap E$. If it can be shown that x^* maps into L', the theorem will be proved.

If $\|x_0 x^*\| < \delta_0$, the theorem is already proved.

If $\|x_0 x^*\| = \delta_0$, since x^* is in E, Lemma III.2.8 applies to x^* and there is a $\delta^* > 0$ such that $L \cap N_{\delta^*}(x^*)$ maps into a straight line. It must map into L' since $N_{\delta_0}(x_0)$ and $N_{\delta^*}(x^*)$ have points of L in common. In this case the theorem is proved.

If $\|x_0 x^*\| > \delta_0$, let x_1 be the point of L between x_0 and x^* such that $\|x_0 x_1\| = \delta_0$. The point x_1 is a point of E and hence, by Lemma III.2.8, there is a $\delta_1 > 0$ such that $L \cap N_{\delta_1}(x_1)$ maps into a straight line, which is L' since $N_{\delta_0}(x_0)$ and $N_{\delta_1}(x_1)$ contain common points of L. If x^* is in $N_{\delta_1}(x_1)$ the theorem is proved. If not, then repeat the above argument, choosing x_2 to be the point of L between x_1 and x^* such that $\|x_1 x_2\| = \delta_1$. Then Lemma III.2.8 can be applied to x_2, and there is a $\delta_2 > 0$ such that $L \cap N_{\delta_2}(x_2)$ maps into a straight line, which must be L' since $N_{\delta_2}(x_2)$ and $N_{\delta_1}(x_1)$ contain common points of L. If

x^* is in $N_{\delta_2}(x_2)$ the theorem is proved. If not, continue in this manner until finally an x_r on L between x_0 and x^* is reached such that there is a $\delta_r > 0$ such that $L \cap N_{\delta_r}(x_r)$ maps into L' and x^* is in $N_{\delta_r}(x_r)$.

It seems possible that the δ_i's might become increasingly smaller and the chosen centers of the $N_{\delta_i}(x_i)$'s might approach a limit point x^{**} of L, before x^* is reached. Conceivably, the above extension process could not be carried past x^{**}. But since x^{**} is in E, Lemma III.2.8 applies to x^{**}, and there is a $\delta^{**} > 0$ such that $L \cap N_{\delta^{**}}(x^{**})$ maps into L'. This neighborhood includes points of L which are beyond x^{**} (that is, between x^{**} and x^*); hence, the extension process can be carried beyond x^{**}, and eventually x^* is reached and maps into L'.

III.2.11. Lemma. Let F: $f^{(i)}(x)$, $(i = 1, 2)$, be a mapping function defined on a convex region E in $R^{(2)}$, such that $D_x F$ exists and is different from zero in E. Let x_0 be a fixed point of E and let x_1, x_2, and x_3 be three variable points of E such that $\Delta(x_1 x_2 x_3)$ is always in I_1 and such that x_0 is always on the line joining x_1 and x_2. Then

$$D_x F\Big|_{x_0} = \lim_{\substack{x_i \to x_0 \\ i=1,2,3}} \frac{\Delta(F: x_1 x_2 x_3)}{\Delta(x_1 x_2 x_3)} .$$

Proof. Since $D_x F$ exists and is different from zero in E, the mapping is continuous and takes straight lines into straight lines by Theorems III.2.2 and III.2.10.

Let $D_x F|_{x_0} = d$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left| \frac{\Delta(F:x_0 x_1 x_2)}{\Delta(x_0 x_1 x_2)} - d \right| < \epsilon/2$$

whenever $\|x_0 x_i\| < \delta$, $(i = 1, 2)$, $\Delta(x_0 x_1 x_2)$ in I_1.

It must be shown that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{\Delta(F:x_1 x_2 x_3)}{\Delta(x_1 x_2 x_3)} - d \right| < \epsilon$$

whenever $\|x_0 x_i\| < \delta$, $(i = 1, 2, 3)$, where $\Delta(x_1 x_2 x_3)$ is in I_1, and where x_0 is always on the line between x_1 and x_2.

Let x_1, x_2, and x_3 be variable points such that the conditions of the hypothesis are satisfied. In this case, $\Delta(x_0 x_2 x_3)$ and $\Delta(x_1 x_0 x_3)$ are in I_1. By Lemma I.5.2

$$\Delta(F:x_1 x_2 x_3) = \Delta(F:x_0 x_2 x_3) + \Delta(F:x_1 x_0 x_3) + \Delta(F:x_1 x_2 x_0)$$

and

$$\Delta(x_1 x_2 x_3) = \Delta(x_0 x_2 x_3) + \Delta(x_1 x_0 x_3) + \Delta(x_1 x_2 x_0).$$

Since x_0, x_1, and x_2 are collinear, then $\Delta(x_1 x_2 x_0) = 0$. Also, since straight lines map into straight lines, then $\Delta(F:x_1 x_2 x_0) = 0$. Hence,

$$\Delta(F:x_1 x_2 x_3) = \Delta(F:x_0 x_2 x_3) + \Delta(F:x_1 x_0 x_3)$$

and

$$\Delta(x_1 x_2 x_3) = \Delta(x_0 x_2 x_3) + \Delta(x_1 x_0 x_3).$$

Let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$\left| \frac{\Delta(F:x_1 x_2 x_3)}{\Delta(x_1 x_2 x_3)} - d \right| =$$

$$\left| \frac{\Delta(x_0 x_2 x_3)}{\Delta(x_1 x_2 x_3)} \left\{ \frac{\Delta(F:x_0 x_2 x_3)}{\Delta(x_0 x_2 x_3)} - d \right\} + \frac{\Delta(x_1 x_0 x_3)}{\Delta(x_1 x_2 x_3)} \left\{ \frac{\Delta(F:x_1 x_0 x_3)}{\Delta(x_1 x_0 x_3)} - d \right\} \right| <$$

$$\left| \frac{\Delta(x_0 x_2 x_3)}{\Delta(x_1 x_2 x_3)} \right| \epsilon/2 + \left| \frac{\Delta(x_1 x_0 x_3)}{\Delta(x_1 x_2 x_3)} \right| \epsilon/2 < \epsilon,$$

whenever $\|x_0 x_i\| < \delta$, $(i = 1, 2, 3)$, and whenever the points x_1, x_2, and x_3 satisfy the conditions of the hypothesis. This proves the lemma.

III.2.12. <u>Remark</u>. Since x_0 is always on the line between x_1 and x_2, then

$$\left| \frac{\Delta(x_0 x_2 x_3)}{\Delta(x_1 x_2 x_3)} \right| \quad \text{and} \quad \left| \frac{\Delta(x_1 x_0 x_3)}{\Delta(x_1 x_2 x_3)} \right|$$

are both less than or equal to one.

III.2.13. <u>Remark</u>. This lemma seems to be almost a special case of Theorem I.5.8. However, although the steps in the two proofs are similar, the hypotheses are not quite the same. The hypothesis that $D_\chi F \neq 0$ is important to the last lemma, for this fact implies that straight lines map into straight lines. From this fact, it follows that since $\Delta(x_1 x_2 x_0) = 0$, then also $\Delta(F:x_1 x_2 x_0) = 0$. Without this knowledge, the proof of Lemma III.2.11 would not be possible.

III.2.14. Theorem. Let $F: f^{(i)}(x)$, $(i = 1, 2)$, be a mapping function defined on a convex region E in $R^{(2)}$. Let $D_x F$ exist and be different from zero everywhere in E. Then the mapping in one-to-one.

Proof. By Theorems III.2.2 and III.2.10, F is continuous and maps straight lines into straight lines. It will be shown that every image point, $F(x_0)$, is the image of precisely one point, x_0, under the mapping F.

Suppose, on the contrary, that there is a point, $F(x_0)$, which is the image of two distinct points, x_0 and x_1. Two situations may occur:

Case 1. The segment $\overline{x_0 x_1}$ maps into the single point, $F(x_0)$. Suppose $D_x F|_{x_0} = d$. Choose $\epsilon < |d|$. Then there exists a $\delta > 0$ such that

$$\text{(III.2.15)} \qquad \left| \frac{\Delta(F:x_0 x_1 x_2)}{\Delta(x_0 x_1 x_2)} - d \right| < \epsilon$$

whenever $\|x_0 x_i\| < \delta$, $(i = 1, 2)$, $\Delta(x_0 x_1 x_2)$ in I_1. Let x_2 be a point of E not on $\overline{x_0 x_1}$. In every small neighborhood, $N_\delta(x_0)$, of x_0, the increment $\Delta(x_0 x_1^* x_2^*)$, where x_1^* is on $\overline{x_0 x_1}$, and x_2^* is on $\overline{x_0 x_2}$, and both points are in $N_\delta(x_0)$, will map into an increment of zero area. Then relation (III.2.15), with $\epsilon < |d|$, cannot hold. This is a contradiction. Hence, it must be concluded that case 1 cannot occur.

Case 2. The segment $\overline{x_0 x_1}$ maps into a line segment $\overline{F(x')F(x'')}$, where $F(x')$ is the image of at least one point x' on $\overline{x_0 x_1}$ and where $F(x'')$ is the image of at least one point x'' on $\overline{x_0 x_1}$, and where $x' \neq x''$. Without loss of generality, one may assume that x' is between x_0 and x''. $F(x_0) = F(x_1)$ is either an interior point of $\overline{F(x')F(x'')}$ or is an end point. Suppose $F(x_0) \neq F(x'')$. Then either $F(x_0) = F(x')$ or $F(x_0)$ is an interior point of the interval.

Now every point x on $\overline{x_0 x_1}$ can be written as

$$x = \theta x_1 + (1 - \theta)x_0,$$

and x is a continuous, one-to-one function of θ. When $\theta = 0$, $x = x_0$, and when $\theta = 1$, $x = x_1$.

Let θ' be the value of θ which yields x' and let θ'' be the value of θ which yields x''. Then $\theta' < \theta''$, since x' is between x_0 and x''.

Since $F(x)$ is a continuous function of x, it is also a continuous function of θ.

Every point $F(x)$ on $\overline{F(x')F(x'')}$ can be expressed as

$$F(x) = \varphi F(x'') + (1 - \varphi)F(x_0),$$

and $F(x)$ is a continuous, one-to-one function of φ. When $\varphi = 0$, $F(x) = F(x_0)$, and when $\varphi = 1$, $F(x) = F(x'')$, and conversely. By the work in Theorem II.6.6, φ is also a continuous function of F(x). Hence, φ is a continuous function of θ, say $\varphi = \Phi(\theta)$.

When $\theta = 0$, $F(x) = F(x_0)$, and hence $\varphi = \Phi(0) = 0$. When $\theta = 1$, $x = x_1$, $F(x) = F(x_1) = F(x_0)$, and $\varphi = 0$ again. When $\theta = \theta'$, $F(x) = F(x')$, and $\varphi = \Phi(\theta') = \varphi'$. When $\theta = \theta''$, $F(x) = F(x'')$, and $\varphi = \Phi(\theta'') = \varphi'' = 1$.

Consider the closed interval $[x_0, x']$. This interval corresponds in a one-to-one manner with the closed interval $[0, \theta']$. Since $\Phi(\theta)$ is a continuous function of θ, φ takes on every value between 0 and φ' at least once, as θ moves from 0 to θ'. Hence F(x) takes on every value between $F(x_0)$ and $F(x')$ at least once, as x goes from x_0 to x'.

Consider the closed interval $[x', x'']$. This corresponds in a one-to-one manner to the interval $[\theta, \theta'']$. Again, since $\Phi(\theta)$ is a continuous function of θ, φ takes one every value between φ' and φ'' at least once, as θ goes from θ' to θ''. That is, F(x) must take on every value on $F(x')F(x'')$ at least once as x goes from x' to x''.

Finally, consider the closed interval $[x'', x_1]$. It corresponds in a one-to-one manner with the closed interval $[\theta'', 1]$. Since $\Phi(\theta)$ is a continuous function of θ, φ must take on every value between $\varphi'' = 1$ and 0 at least once, as θ goes from θ'' to 1. That is, F(x) must take on every value between F(x'') and $F(x_1) = F(x_0)$, at least once, as x goes from x'' to x_1.

It is concluded that every point $F(x)$ between $F(x_0)$ and $F(x')$ is the image of at least two points on $\overline{x_0x_1}$, one of which is between x_0 and x', the other between x' and x''. Similarly, every point $F(x)$ between $F(x_0)$ and $F(x'')$ is the image of at least two points on $\overline{x_0x_1}$, one of which lies between x' and x'' and the other between x'' and x_1.

Consider the point x''. Let $D_xF\big|_{x''} = d''$. Choose $\epsilon'' < |d''|$. Then there is a $\delta'' > 0$ such that

(III.2.16) $$\left| \frac{\Delta(F:x''x_1''x_2'')}{\Delta(x''x_1''x_2'')} - d'' \right| < \epsilon''$$

whenever $\|x''x_i''\| < \delta''$, $(i = 1, 2)$, $\Delta(x''x_1''x_2'')$ in I_1. In every neighborhood of x'' there is a point x_1'' on $\overline{x_0x_1}$ between x' and x'' and a point x_2'' on $\overline{x_0x_1}$ between x'' and x_1, both of which map into the same image point.

By Lemma III.2.11, in taking the derivative at x'', the increments formed by two points, x_1'' and x_2'' on $\overline{x_0x_1}$, with x'' on $\overline{x_0x_1}$ between them, and another point x_3'', not on $\overline{x_0x_1}$, may be considered. For these increments relation (III.2.16) must hold with $\epsilon'' < |d''|$. But among these increments will be found, in every neighborhood of x'', those for which the points x_1'' and x_2'' map into a single point. But in these cases the increments map into increments of zero area. Hence, for these increments the relation

(III.2.16), with $\epsilon'' < |d''|$, cannot hold. This contradicts the assumption that $D_x F|_{x''} = d''$. Hence it must be concluded that case 2 cannot arise.

In either case, a contradiction has been reached. Hence, every image point is the image of precisely one point of E, proving the theorem.

III.2.17. If F: $f^{(i)}(x)$, $(i = 1, 2)$, is a mapping function defined on a convex region E in $R^{(2)}$, and if $D_x F$ exists everywhere in E and is different from zero, then F is continuous, one-to-one and maps straight lines into straight lines, by the theorems just proved. Now using the results of Chapter II, in particular, Theorem II.7.3, the following theorem has already been proved:

II.2.18. <u>Theorem</u>. Let F: $f^{(i)}(x)$, $(i = 1, 2)$, be a mapping function defined on a convex region E in $R^{(2)}$ and let $D_x F$ exist everywhere in E and be different from zero there. Then F is of the form

$$\text{(III.2.19)} \qquad F:\ f^{(i)}(x) = \frac{a_{i,1}\, x^{(1)} + a_{i,2}\, x^{(2)} + a_{i,3}}{a_{3,1}\, x^{(1)} + a_{3,2}\, x^{(2)} + a_{3,3}}\ , \ (i = 1, 2),$$

where

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} \neq 0.$$

The following theorem is in the nature of a converse.

III.2.20. <u>Theorem.</u> Let

$$\text{(III.2.21)} \quad F: f^{(i)}(x) = \frac{a_{i,1}x^{(1)} + a_{i,2}x^{(2)} + a_{i,3}}{a_{3,1}x^{(1)} + a_{3,2}x^{(2)} + a_{3,3}}, \quad (i = 1, 2),$$

where

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} \neq 0$$

be defined on a region E of $R^{(2)}$ which does not contain points of the line $a_{3,1}x^{(1)} + a_{3,2}x^{(2)} + a_{3,3} = 0$. Then $D_x F$ exists and is different from zero in E.

<u>Proof.</u> Let x_0 be any point in E and let x_1 and x_2 be two variable points of E so that $\Delta(x_0 x_1 x_2)$ is in I_1. Examine the difference quotient

$$\text{(III.2.22)} \quad \frac{\begin{vmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & 1 \\ f^{(1)}(x_1) & f^{(2)}(x_1) & 1 \\ f^{(1)}(x_2) & f^{(2)}(x_2) & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_1^{(1)} & x_1^{(2)} & 1 \\ x_2^{(1)} & x_2^{(2)} & 1 \end{vmatrix}}.$$

The numerator of this difference quotient is equal to

$$\begin{vmatrix} \dfrac{a_{1,1}x_0^{(1)}+a_{1,2}x_0^{(2)}+a_{1,3}}{a_{3,1}x_0^{(1)}+a_{3,2}x_0^{(2)}+a_{3,3}} & \dfrac{a_{2,1}x_0^{(1)}+a_{2,2}x_0^{(2)}+a_{2,3}}{a_{3,1}x_0^{(1)}+a_{3,2}x_0^{(2)}+a_{3,3}} & 1 \\ \dfrac{a_{1,1}x_1^{(1)}+a_{1,2}x_1^{(2)}+a_{1,3}}{a_{3,1}x_1^{(1)}+a_{3,2}x_1^{(2)}+a_{3,3}} & \dfrac{a_{2,1}x_1^{(1)}+a_{2,2}x_1^{(2)}+a_{2,3}}{a_{3,1}x_1^{(1)}+a_{3,2}x_1^{(2)}+a_{3,3}} & 1 \\ \dfrac{a_{1,1}x_2^{(1)}+a_{1,2}x_2^{(2)}+a_{1,3}}{a_{3,1}x_2^{(1)}+a_{3,2}x_2^{(2)}+a_{3,3}} & \dfrac{a_{2,1}x_2^{(1)}+a_{2,2}x_2^{(2)}+a_{2,3}}{a_{3,1}x_2^{(1)}+a_{3,2}x_2^{(2)}+a_{3,3}} & 1 \end{vmatrix} =$$

$$\frac{1}{\prod_{i=0}^{2}(a_{3,1}x_i^{(1)}+a_{3,2}x_i^{(2)}+a_{3,3})} \begin{vmatrix} a_{1,1}x_0^{(1)}+a_{1,2}x_0^{(2)}+a_{1,3} & a_{2,1}x_0^{(1)}+a_{2,2}x_0^{(2)}+a_{2,3} & a_{3,1}x_0^{(1)}+a_{3,2}x_0^{(2)}+a_{3,3} \\ a_{1,1}x_1^{(1)}+a_{1,2}x_1^{(2)}+a_{1,3} & a_{2,1}x_1^{(1)}+a_{2,2}x_1^{(2)}+a_{2,3} & a_{3,1}x_1^{(1)}+a_{3,2}x_1^{(2)}+a_{3,3} \\ a_{1,1}x_2^{(1)}+a_{1,2}x_2^{(2)}+a_{1,3} & a_{2,1}x_2^{(1)}+a_{2,2}x_2^{(2)}+a_{2,3} & a_{3,1}x_2^{(1)}+a_{3,2}x_2^{(2)}+a_{3,3} \end{vmatrix}.$$

By the multiplication theorem for determinants [Kowalewski, (1), pp. 66 ff.] (III.2.22) finally becomes equal to

$$\frac{1}{\prod_{i=0}^{2}(a_{3,1}x_i^{(1)}+a_{3,2}x_i^{(2)}+a_{3,3})} \frac{\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} \begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_1^{(1)} & x_1^{(2)} & 1 \\ x_2^{(1)} & x_2^{(2)} & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_1^{(1)} & x_1^{(2)} & 1 \\ x_2^{(1)} & x_2^{(2)} & 1 \end{vmatrix}} =$$

$$\frac{\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}}{(a_{3,1}x_0^{(1)}+a_{3,2}x_0^{(2)}+a_{3,3})(a_{3,1}x_1^{(1)}+a_{3,2}x_1^{(2)}+a_{3,3})(a_{3,1}x_2^{(1)}+a_{3,2}x_2^{(2)}+a_{3,3})}.$$

By hypothesis, the numerator is different from zero. Letting x_1 and x_2 approach x_0, $\Delta(x_0 x_1 x_2)$ remaining in the class I_1, it is seen that $D_x F|_{x_0}$ exists and is equal to

$$\frac{\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}}{(a_{3,1} x_0^{(1)} + a_{3,2} x_0^{(2)} + a_{3,3})^3} \neq 0.$$

Since x_0 was any point in E, the derivative is different from zero everywhere in E, proving the theorem.

III.2.23. <u>Remark</u>. It should be noted that since the determinants

$$\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_1^{(1)} & x_1^{(2)} & 1 \\ x_2^{(1)} & x_2^{(2)} & 1 \end{vmatrix}$$

in (III.2.22) cancel out, it really does not make any difference if the points x_0, x_1, and x_2 remain in the class I_1, or even that they approach x_0. If x_0, x_1, and x_2 approach any point x^* of E in any manner at all, the derivative $D_x F$ exists at x^* and is equal to

$$\frac{\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}}{(a_{3,1} x^{*} + a_{3,2} x^{*} + a_{3,3})^3} \neq 0.$$

The linear fractional transformations have a generalized derivative under the most general conditions.

III.2.24. Remark. Theorems III.2.18 and III.2.20 together show that the precise class of mapping functions, defined on a convex region E of $R^{(2)}$, which have a non-zero derivative, $D_x F$, in E, is the class of linear fractional transformations.

III.2.25. Remark. The generalized derivatives $D_{x^{(1)}} f^{(1)}$ and $D_{x^{(2)}} f^{(2)}$ are only special cases of the generalized derivative $D_x F$, according to Remark I.1.10. It follows that if $D_{x^{(1)}} f^{(1)}$ exists and is different from zero in a convex region E then $f^{(1)}(x)$ is of the form (III.2.26)

$$f^{(1)}(x) = a_{1,1} x^{(1)} + a_{1,2} x^{(2)} + a_{1,3}.$$

For if one sets $f^{(2)}(x) = x^{(2)}$ in the difference quotient

$$\frac{\begin{vmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & 1 \\ f^{(1)}(x_1) & f^{(2)}(x_1) & 1 \\ f^{(1)}(x_2) & f^{(2)}(x_2) & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & 1 \\ x_1^{(1)} & x_1^{(2)} & 1 \\ x_2^{(1)} & x_2^{(2)} & 1 \end{vmatrix}}$$

and if the limit is taken with respect to the class I_1, then $D_x F\big|_{x_0} = D_{x^{(1)}} f^{(1)}\big|_{x_0}$. Since $f^{(2)}(x)$ and $f^{(1)}(x)$ must have the same denominators, then $f^{(1)}(x)$ must be of the form (III.2.26).

Similarly, if $D_{x^{(2)}} f^{(2)}$ exists and is different from zero in a convex region E, then $f^{(2)}(x)$ must be of the form

(III.2.27) $$f^{(2)}(x) = a_{2,1}\, x^{(1)} + a_{2,2} x^{(2)} + a_{2,3}.$$

III.3. THE CHARACTERIZATION FOR THE n-DIMENSIONAL CASE

III.3.1. The results obtained in Section III.2 will now be generalized to the n-dimensional case. The procedure is the same, but certain difficulties arise in the generalization which did not occur in the 2-dimensional case.

In this section, $x = (x^{(1)}, x^{(2)}, \ldots, x^{(3)})$.

III.3.2. <u>Theorem</u>. Let F: $f^{(i)}(x)$, $(i = 1, \ldots, n)$, be a mapping function defined on an open set E in $R^{(n)}$. Let $D_x F$ exist and have the value d different from zero at a point x_0 of E. Then F is continuous at x_0.

<u>Proof</u>. For every $\epsilon > 0$ there is a $\delta > 0$ such that

(III.3.3) $$\left| \frac{\Delta(F{:}x_0 x_1 \ldots x_n)}{\Delta(x_0 x_1 \ldots x_n)} - d \right| < \epsilon/(n+1)$$

whenever $\|x_0 x_i\| < \delta$, $(i = 1, \ldots, n)$, $\Delta(x_0 x_1 \ldots x_n)$ in I_1. This is true since the derivative has been assumed to exist and equal d at x_0. In particular, for $\epsilon = \epsilon^*$, such that $\epsilon^*/(n + 1) < |d|$, there is a δ^* such that the above inequality holds.

By Theorem I.5.8, $D_x F|_{x_0}$ can be calculated by taking the limit of the ratios

$$\frac{\Delta(F:x_1 x_2 \cdot \cdot \cdot x_{n+1})}{\Delta(x_1 x_2 \cdot \cdot \cdot x_{n+1})},$$

where $x_1, x_2, \ldots, x_{n+1}$ are chosen to satisfy the conditions of the hypothesis I.5.8. In particular, if $x_1, x_2, \ldots, x_{n+1}$ in $N_{\delta^*}(x_0)$ are chosen to form an increment of I_1 with x_0 interior to $\Delta(x_1 x_2, \ldots x_{n+1})$, then Theorem I.5.8 can be applied and for these chosen points.

$$\left| \frac{\Delta(F:x_1 x_2 \cdot \cdot \cdot x_{n+1})}{\Delta(x_1 x_2 \cdot \cdot \cdot x_{n+1})} - d \right| < \epsilon^*.$$

Keep $x_1, x_2, \ldots, x_{n+1}$ fixed. The increments $\Delta(x_0 x_2 \cdot \cdot \cdot x_{n+1})$, $\Delta(x_1 x_0 x_3 \cdot \cdot \cdot x_{n+1})$, . . . , and $\Delta(x_1 x_2 \cdot \cdot \cdot x_n \; x_0)$ are all in the class I_1. Furthermore, since relation (III.3.3) must hold, with $\epsilon = \epsilon^*$, the image increments, $\Delta(F:x_0 x_2 \cdot \cdot \cdot x_{n+1})$, $\Delta(F:x_1 x_0 x_3 \cdot \cdot \cdot x_{n+1})$, . . . , and $\Delta(F:x_1 x_2 \cdot \cdot \cdot x_n \; x_0)$ must all be different from zero, hence are in the class I_1.

Let x' be a variable point of $N_*(x\ )$, which for the moment is required to remain off the (n - 1)-flats determined by the sets of points $(x_0, x_1, \ldots, x_n)$, $(x_0, x_2, \ldots, x_{n-1}, x_{n+1})$, . . . , and $(x_0, x_3, \ldots, x_{n+1})$.

There are $C(n, n-1) = n$ of these (n-1)-flats. (Notice that the point x_1 is in none of these (n-1)-flats).

Consider the increments $\Delta(x_0x_2 \ldots x_nx')$, $\Delta(x_0x_2 \ldots x_{n-1}x'x_{n+1})$, . . . , and $\Delta(x_0x'x_3 \ldots x_{n+1})$. These increments are in the class I_1 in $N_{\delta^*}(x_0)$, for x' in the restricted region, and hence relation (III.3.3) holds for these increments and their images, with $\epsilon = \epsilon^*$. It must follow that for every sufficiently small $\epsilon' > 0$ (in particular for $\epsilon' \leq \epsilon^*$), there exists a $\delta^{(i)} > 0$ such that

$$\text{(III.3.4)} \qquad \left|\Delta(F:x_0x_2 \ldots x_{i-1}x'x_{i+1} \ldots x_{n+1})\right| < \epsilon'/S$$

whenever $\|x_0x'\| < \delta^{(i)}$, $(i = 2, 3, \ldots, n+1)$, where S is an absolute constant which will be chosen later. Suppose this assertion is not true. Then for fixed $\epsilon' \leq \epsilon^*$, and for every $\delta > 0$, there is at least one point $x^* \in N_\delta(x_0)$ such that

$$\left|\Delta(F:x_0x_2 \ldots x_{i-1}x^*x_{i+1} \ldots x_{n+1})\right| \geq \epsilon'/S, \ (i = 2, 3, \ldots, n+1).$$

As δ is allowed to approach zero, $\Delta(x_0x_2 \ldots x_{i-1}x^*x_{i+1} \ldots x_{n+1})$ also approaches zero, since $\Delta(x_0x_2 \ldots x_{i-1}x^*x_{i+1} \ldots x_{n+1})$ varies directly as $\|x_0x^*\|$, $x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}$ being fixed points. Then, as δ approaches zero, the difference quotient

$$\frac{\Delta(F:x_0x_2 \ldots x_{i-1}x^*x_{i+1} \ldots x_{n+1})}{\Delta(x_0x_2 \ldots x_{i-1}x^*x_{i+1} \ldots x_{n+1})}$$

becomes arbitrarily large for the points x^* in $N_\delta(x_o)$ such that $\Delta(F:x_o x_2 . . . x_{i-1} x^* x_{i+1} . . . x_{n+1}) \geq \epsilon'/S$. For such points, which are also in $N_{\delta^*}(x_o)$, the ϵ, δ relation (III.3.3) cannot hold, contradicting the assumption that $D_x F|_{x_o} = d$. Hence, for every $\epsilon' \leq \epsilon^*$ there is a $\delta^{(i)} > 0$ such that (III.3.4) holds, $(i = 2, 3, . . . , n + 1)$.

Choose $\delta_1 = \min \{\delta^{(i)}\}$, $(i = 2, 3, . . . , n + 1)$. Then

$$\left| \Delta(F:x_o x_2 . . . x_{i-1} x' x_{i+1} . . . x_{n+1}) \right| < \epsilon'/S$$

whenever $\|x_o x'\| < \delta_1$, x' remaining in the restricted region, $(i = 2, 3, . . . , n + 1)$.

Now

$$n!\ \Delta(F:x_0x_2\ldots x_{i-1}x'x_{i+1}\ldots x_{n+1}) =$$

$$\begin{vmatrix} f^{(1)}(x_0) & \cdots & f^{(n)}(x_0) & 1 \\ f^{(1)}(x_2) & \cdots & f^{(n)}(x_2) & 1 \\ \cdots & \cdots & \cdots & \cdots \\ f^{(1)}(x_{i-1}) & \cdots & f^{(n)}(x_{i-1}) & 1 \\ f^{(1)}(x') & \cdots & f^{(n)}(x') & 1 \\ f^{(1)}(x_{i+1}) & \cdots & f^{(n)}(x_{i+1}) & 1 \\ \cdots & \cdots & \cdots & \cdots \\ f^{(1)}(x_{n+1}) & \cdots & f^{(n)}(x_{n+1}) & 1 \end{vmatrix} =$$

$$(-1)^n \begin{vmatrix} f^{(1)}(x_2) - f^{(1)}(x_0) & \cdots & f^{(n)}(x_2) - f^{(n)}(x_0) \\ \cdots & \cdots & \cdots \\ f^{(1)}(x_{i-1}) - f^{(1)}(x_0) & \cdots & f^{(n)}(x_{i-1}) - f^{(n)}(x_0) \\ f^{(1)}(x') - f^{(1)}(x_0) & \cdots & f^{(n)}(x') - f^{(n)}(x_0) \\ f^{(1)}(x_{i+1}) - f^{(1)}(x_0) & \cdots & f^{(n)}(x_{i+1}) - f^{(n)}(x_0) \\ \cdots & \cdots & \cdots \\ f^{(1)}(x_{n+1}) - f^{(1)}(x_0) & \cdots & f^{(n)}(x_{n+1}) - f^{(n)}(x_0) \end{vmatrix}$$

$(i = 2, 3, \ldots, n + 1)$. Expanding by the ith row, the above equation becomes (III.3.5):

$$(-1)^{(n)} n!\,\Delta(F:x_0x_2\ldots x_{i-1}x'x_{i+1}\ldots x_{n+1}) = \sum_{j=1}^{n} A_{i,j}\, f^{(j)}(x') - f^{(j)}(x_0),$$

$(i = 2, 3, \ldots, n + 1)$, where

$A_{i,j}$, $(i = 2, \ldots, n + 1;\ j = 1, \ldots, n)$, is the co-factor of $f^{(j)}(x_i) - f^{(j)}(x_0)$ in the determinant

$$A = \begin{vmatrix} f^{(1)}(x_2) - f^{(1)}(x_0) & \cdots & f^{(n)}(x_2) - f^{(n)}(x_0) \\ \cdots & \cdots & \cdots \\ f^{(1)}(x_{n+1}) - f^{(1)}(x_0) & \cdots & f^{(n)}(x_{n+1}) - f^{(n)}(x_0) \end{vmatrix}.$$

($A = n!\ \Delta(F{:}x_0 x_2 \ldots x_{n+1})$, except possibly for sign. Hence, $A \neq 0$).

Equation (III.3.5) represents a system of n equations in the n unknowns, $f^{(j)}(x') - f^{(j)}(x_0)$ ($j = 1, \ldots, n$). There will be a solution if the determinant of the coefficients is different from zero. This determinant is

$$D = \begin{vmatrix} A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n+1,1} & A_{n+1,2} & \cdots & A_{n+1,n} \end{vmatrix} = \text{adj}\, A.$$

By a corollary to the Sylvester-Franke Theorem on determinants [Price, (1), p. 82],

$$D = \text{adj}\ A = A^{n-1} = [n!\ \Delta(F{:}x_0 x_2 \ldots x_{n+1})]^{n-1},$$

except possibly for sign. Since $\Delta(F{:}x_0 x_2 \ldots x_{n+1})$ is different from zero, there is a solution for $f^{(j)}(x') - f^{(j)}(x_0)$, ($j = 1, \ldots, n$). Solving for $f^{(j)}(x') - f^{(j)}(x_0)$:

$$f^{(j)}(x') - f^{(j)}(x_0) =$$

$$(-1)^n n! \frac{\begin{vmatrix} A_{2,1} \cdots A_{2,j-1} & \Delta(F:x_0x'x_3 \cdots x_{n+1}) & A_{2,j+1} \cdots A_{2,n} \\ \cdots & \cdots & \cdots \\ A_{i,1} \cdots A_{i,j-1} & \Delta(F:x_0x_2 \cdots x_{i-1}x'x_{i+1} \cdots x_{n+1}) & A_{i,j+1} \cdots A_{i,n} \\ \cdots & \cdots & \cdots \\ A_{n+1,1} \cdots A_{n+1,j-1} & \Delta(F:x_0x_2 \cdots x_nx') & A_{n+1,j+1} \cdots A_{n+1,1} \end{vmatrix}}{[n!\,\Delta(F:x_0x_2 \cdots x_{n+1})]^{n-1}}$$

$(j = 1, \ldots, n)$, except possibly for sign.

All the terms on the right are constants except the elements of the jth column. Expanding by elements of this column, and remembering (III.3.4), it follows that for every sufficiently small ϵ' (in particular for $\epsilon' \leq \epsilon^{*}$) there is a $\delta_1 > 0$ such that

$$|f^{(j)}(x') - f^{(j)}(x_0)| < M_{1,j} \cdot \epsilon'/S, \quad (j = 1, \ldots, n),$$

whenever $\|x_0x'\| < \delta_1$, (x' remaining in the restricted region), where $M_{1,j}$ is a constant which is equal to the sum of the absolute values of the minors of $\Delta(F:x_0x_2 \ldots x_{i-1}x'x_{i+1} \ldots x_{n+1})$ in the above expansion all divided by $|n!^{n-2}[\Delta(F:x_0x_2 \ldots x_{n+1})]^{n-1}|$, $(i = 2, \ldots, n + 1)$.

In general, requiring x' to remain off the (n-1)-flats determined by x_0 together with any n-1 of the points $x_1, x_2, \ldots, x_{k-1}, x_{k+1} \ldots, x_{n+1}$, it is found in exactly the same manner for each K, $(k = 1, \ldots, n + 1)$, that for every sufficiently small number $\epsilon' > 0$ (in particular for $\epsilon' \leq \epsilon^*$), there is a $\delta_k > 0$ such that

$$\left| f^{(j)}(x') - f^{(j)}(x_0) \right| < M_{k,j} \, \epsilon'/S \quad (j = 1, \ldots, n),$$

whenever $\|x_0 x'\| < \delta_k$, x' remaining in the restricted region, where again $M_{k,j}$ is a constant which is equal to the sum of the absolute values of the minors of $\Delta(F{:}x_0 \ldots x_{i-1} x' x_{i+1} \ldots x_{n+1})$ in the expansion corresponding to that on the previous page, all divided by $\left| n!^{\,n-2} [\Delta(F{:}x_0 \ldots x_{k-1} \, x_{k+1} \ldots x_{n+1})]^{n-1} \right|$, $(i = 1, \ldots, k-1, k+1, \ldots, n+1)$.

The constants, $M_{k,j}$, depend on $\Delta(F{:}x_0 x_2 \ldots x_{k-1} \, x_{k+1} \ldots x_{n+1})$ and on column j of adj A. They are all absolute constants since they ultimately depend upon only the fixed numbers $F(x_0), \ldots F(x_{n+1})$.

Choose $S = \max \{M_{k,j}\}$, all k and j, and choose $\delta' = \min \{\delta_k\}$, all k. Then

$$\left| f^{(j)}(x') - f^{(j)}(x_0) \right| < \epsilon', \quad (j = 1, \ldots, n),$$

whenever $\|x_0 x'\| < \delta'$, with no other restriction on x'. Hence, F is continuous at x_0, proving the theorem.

III.3.6. <u>Theorem</u>. Let $F{:}\ f^{(i)}(x)$, $(i = 1, \ldots, n)$, be a mapping function defined on a convex region E in $R^{(n)}$ and let $D_x F$ exist and be different from zero in E. Let x_0 be a point of E. Then in a sufficiently small neighborhood of x_0, (n-1)-flats containing x_0 map into (n-1)-flats.

<u>Proof</u>. Let $D_x F\big|_{x_0} = d$. Let $\epsilon > 0$ be given such that $\epsilon < |d|$. Then there exists a $\delta_0 > 0$ such that

$$\text{(III.3.7)} \qquad \left| \frac{\Delta(F: x_0 x_1 \ldots x_n)}{\Delta(x_0 x_1 \ldots x_n)} - d \right| < \epsilon$$

whenever $\|x_0 x_i\| < \delta_0$, $(i = 1, \ldots, n)$, $\Delta(x_0 x_1 \ldots x_n)$ in I_1. It will be shown that in $N_{\delta_0}(x_0)$, (n-1)-flats containing x_0 map into (n-1)-flats.

Suppose the theorem is false. Then there is an (n-1)-flat, S_{n-1}, through x_0 such that $S_{n-1} \cap N_{\delta_0}(x_0)$ does not map into an (n-1)-flat.

Let $x_1, x_2, \ldots, x_{n-1}$ be n-1 points of $S_{n-1} \cap N_{\delta_0}(x_0)$ which with x_0 form a set of n linearly independent points. These n points will determine S_{n-1}. The points $x_0, x_1, \ldots x_{n-1}$ map into linearly independent points, $F(x_0), F(x_1), \ldots, F(x_{n-1})$; otherwise $x_0, x_1, \ldots, x_{n-1}$ together with a suitable chosen point x_n of $N_{\delta_0}(x_0)$ would form an increment of I_1 which would map into an increment of n-dimensional volume zero, contradicting the assumption that relation (III.3.7) holds, with $\epsilon < |d|$, for all increments in I_1 in $N_{\delta_0}(x_0)$.

The points $F(x_0), F(x_1), \ldots, F(x_{n-1})$ determine an (n-1)-flat, T_{n-1}. Let $F(x^*)$ be a point, not in T_{n-1}, which is the image of at least one point x^* of $S_{n-1} \cap N_{\delta_0}(x_0)$. Such a point exists, otherwise the theorem is already true. The increment $\Delta(x_0 \ldots x_{n-1}\ x^*) = 0$, since $x_0, x_1, \ldots, x_{n-1}, x^*$ are linearly dependent.

But $\Delta(F:x_0 \ldots x_{n-1}x^*) \neq 0$, since $F(x_0)$, $F(x_1), \ldots, F(x_{n-1})$, and $F(x^*)$ are linearly independent.

Let x be a variable point of $N_{\delta_0}(x_0)$ which together with $x_0, \ldots, x_{n-1}$ always forms an increment of I_1. Let $F(x)$ be its image. For the points $x_0, x_1, \ldots, x_{n-1}, x$, the ϵ, δ relation of (III.3.7), with $\epsilon < |d|$, must hold. As x approaches x^*, $F(x)$ approaches $F(x^*)$, since the mapping is continuous by Theorem III.3.2. Now $\Delta(x_0 \ldots x_{n-1}\, x)$ approaches $\Delta(x_0 \ldots x_{n-1}x^*) = 0$, while $\Delta(F:x_0 \ldots x_{n-1}x)$ approaches $\Delta(F:x_0 \ldots x_{n-1}\, x^*) \neq 0$. Hence, the difference quotient

$$\frac{\Delta(F:x_0 \ldots x_{n-1}\, x)}{\Delta(x_0 \ldots x_{n-1}x)}$$

becomes arbitrarily large, contradicting the assumption that relation (III.3.7) holds for all increments of I_1 in $N_{\delta_0}(x_0)$. Therefore, the (n-1)-flat, $S_{n-1} \cap N_{\delta_0}(x_0)$, must map into an (n-1)-flat, and the theorem is proved.

III.3.8. Corollary. Let F: $f^{(i)}(x)$, $(i = 1, \ldots, n)$, be a mapping function defined in a convex region E of $R^{(n)}$ and let D_xF exist and be different from zero in E. Then, if S_{n-1} is an (n-1)-flat with points in E, $S_{n-1} \cap E$ maps into an (n-1)-flat.

Proof. Let x_0 be a point of $S_{n-1} \cap E$. By Theorem III.3.6, in a sufficiently small neighborhood, $N_{\delta_0}(x_0)$, of x_0, $S_{n-1} \cap N_{\delta_0}(x_0)$ maps into an (n-1)-flat, T_{n-1}. Let x^* be any other point of $S_{n-1} \cap E$. It will be shown that x^* also maps into T_{n-1}.

If $\|x_0 x^*\| < \delta_0$, the corollary is already proved.

If $\|x_0 x^*\| = \delta_0$, then since x^* is in $S_{n-1} \cap E$, Theorem III.3.6 applies to x^* and there is a $\delta^* > 0$ such that $S_{n-1} \cap N_{\delta^*}(x^*)$ maps into an (n-1)-flat, which must be T_{n-1}, since $S_{n-1} \cap N_{\delta_0}(x_0)$ and $S_{n-1} \cap N_{\delta^*}(x^*)$ have points of S_{n-1} in common.

Suppose $\|x_0 x^*\| > \delta_0$. Since E is convex, x_0 and x^* can be joined by a straight line segment, $\overline{x_0 x^*}$, which lies entirely in E, and also in S_{n-1}. Let x_1 be the point of $\overline{x_0 x^*}$ between x_0 and x^* such that $\|x_0 x_1\| = \delta_0$. The point x_1 is in $S_{n-1} \cap E$ and Theorem III.3.6 can be applied. Then there is a $\delta_1 > 0$ such that $S_{n-1} \cap N_{\delta_1}(x_1)$ maps into an (n-1)-flat, which must be T_{n-1}, since $N_{\delta_0}(x_0)$ and $N_{\delta_1}(x_1)$ have points of S_{n-1} in common. If x^* is in $N_{\delta_1}(x_1)$, the corollary is proved.

If x^* is not in $N_{\delta_1}(x_1)$, denote by x_2 the point of $\overline{x_1 x^*}$ between x_1 and x^* such that $\|x_1 x_2\| = \delta_1$. Theorem III.3.6 applies to x_2 and there is a $\delta_2 > 0$ such that $S_{n-1} \cap N_{\delta_2}(x_2)$ maps into an (n-1)-flat which must be T_{n-1}, since $N_{\delta_2}(x_2)$ and $N_{\delta_1}(x_1)$ have points of S_{n-1} in common. If x^* is in $N_{\delta_2}(x_2)$, the corollary is proved.

If x^* is not in $N_{\delta_2}(x_2)$, continue in this manner until an x_r on $\overline{x_0 x^*}$ is reached for which there is a $\delta_r > 0$ such that $S_{n-1} \cap N_{\delta_r}(x_r)$ maps into T_{n-1} and such that x^* is in $N_{\delta_r}(x_r)$. Then x^* also maps into T_{n-1}.

It is conceivable that the δ_i-neighborhoods considered become smaller and smaller with the centers, x_i, approaching a limit point, $\bar{x}$, on $\overline{x_o x^*}$. Then possibly the extension of the argument could not be carried past $\bar{x}$. However, $\bar{x}$ is a point of $S_{n-1} \cap E$ and Theorem III.3.6 applies to $\bar{x}$. Hence there is a $\bar{\delta} > 0$ such that $S_{n-1} \cap N_{\bar{\delta}}(\bar{x})$ maps into T_{n-1}, and the inclusion of points of $\overline{x_o x^*}$ beyond $\bar{x}$ which map into T_{n-1} has been accomplished. (Beyond means between $\bar{x}$ and x^*.) Therefore, the argument can be continued until x^* is found to be a point of $S_{n-1} \cap E$ which maps into T_{n-1}. Since x^* was any point of $S_{n-1} \cap E$, it must be concluded that every point of $S_{n-1} \cap E$ maps into T_{n-1} and the corollary is proved.

III.3.9. <u>Theorem</u>. Let F: $f^{(i)}(x)$, $(i = 1, \ldots, n)$, be a mapping function defined on a convex region E of $R^{(n)}$ and let $D_x F$ exist and be different from zero in E. Then, if L is a straight line passing through E, the segment $L \cap E$ maps into a straight line.

<u>Proof</u>. Let L be a straight line passing through E. Let x_o be a point of $L \cap E$. Let $D_x F|_{x_o} = d_o$. Let $\epsilon > 0$ be chosen so that $\epsilon < |d_o|$. Then there exists a $\delta_o > 0$ such that

$$\text{(III.3.10)} \qquad \left| \frac{\Delta(F: x_o x_1 \ldots x_n)}{\Delta(x_o x_1 \ldots x_n)} - d_o \right| < \epsilon$$

whenever $\|x_o x_i\| < \delta_o$, $(i = 1, \ldots, n)$, $\Delta(x_o x_1 \ldots x_n)$ in I_1.

Let $x_1 \neq x_0$ be any other point of $L \cap N_{\delta_0}(x_0)$. Let $x_2, \ldots, x_n$ be n-1 points of $N_{\delta_0}(x_0)$, not on L, which together with x_0 and x_1 form a set of n + 1 linearly independent points. These points form the vertices of an increment, $\Delta(x_0 x_1 \ldots x_n)$ which has n-dimensional volume different from zero. The image increment, $\Delta(F:x_0 x_1 \ldots x_n)$, must also have n-dimensional volume different from zero, since otherwise relation (III.3.10), with $\epsilon < |d_0|$, would not hold. That is, $F(x_0)$, $F(x_1), \ldots, F(x_n)$ form a set of n + 1 linearly independent points.

Consider the (n-1)-flats, $S_{n-1}^{(j)}$, determined by x_0, x_1 and the n-2 other points, $x_2, \ldots, x_{j-1}$, $x_{j+1}, \ldots, x_n$, $(j = 2, \ldots, n)$. Since L is completely determined by x_0 and x_1, then L must be common to all $S_{n-1}^{(j)}$. By Corollary III.3.8 each (n-1)-flat, $S_{n-1}^{(j)}$, maps into an (n-1)-flat, $T_{n-1}^{(j)}$, which is determined by $F(x_0)$, $F(x_1)$, and $F(x_2), \ldots, F(x_{j-1}), F(x_{j+1}), \ldots, F(x_n)$, $(j = 2, \ldots, n)$, since for each j, the set of points $F(x_0), F(x_1), F(x_2), \ldots, F(x_{j-1}), F(x_{j+1}), \ldots$, $F(x_n)$, forms a set of n linearly independent points of $T_{n-1}^{(j)}$. The image of L must be common to each (n-1)-flat, $T_{n-1}^{(j)}$. Denote this image by L'.

Now each (n-1)-flat, $T_{n-1}^{(j)}$, can be represented as a single equation in the unknowns $f^{(1)}(x), \ldots, f^{(n)}(x)$, as follows:

(III.3.11) $T_{n-1}^{(j)}: a_{j,1} f^{(1)}(x) + \dots + a_{j,n} f^{(n)}(x) + a_{j,n+1} = 0,$

$(j = 2, \dots, n)$. It will be shown that the (n-1)-flats, $T_{n-1}^{(j)}$, $(j = 2, \dots, n)$, intersect in a straight line which contains the points $F(x_0)$ and $F(x_1)$.

Since $F(x_0)$ and $F(x_1)$ are common to all the (n-1)-flats, $T_{n-1}^{(j)}$, then clearly they are both solutions of the set of equations (III.3.11), and they are linearly independent solutions, since $F(x_0) \neq F(x_1)$. The equations (III.3.11) may be written as

(III.3.12) $$a_{j,1}\left(f^{(1)}(x)-f^{(1)}(x_0)\right) + \dots + a_{j,n}\left(f^{(n)}(x)-f^{(n)}(x_0)\right) = 0,$$

$(j = 2, \dots, n)$, since $F(x_0)$ is a solution of (III.3.11). This is a system of n-1 homogeneous equations in n unknowns. There is only one non-zero linearly independent solution of this system of equations [Bocher, (1), pp. 49-52]. Clearly, this solution is $F(x_1)-F(x_0)$ $= \left\{f^{(i)}(x_1)-f^{(i)}(x_0)\right\}$, $(i = 1, \dots, n)$. All the remaining solutions are linearly dependent on $F(x_1)-F(x_0)$, and hence all the points in common to all the (n-1)-flats, $T_{n-1}^{(j)}$, must lie on the straight line through $F(x_1)$ and $F(x_0)$. Hence, since L' is common to $T_{n-1}^{(j)}$, $(j = 2, \dots, n)$, then it must be contained in this straight line, and hence points of $L \cap E$ map into points on a straight line, which is the fact that was to be proved.

III.3.13. <u>Lemma</u>. Let F: $f^{(i)}(x)$, $(i = 1, \ldots, n)$, be a mapping function defined on a convex region E in $R^{(n)}$ and let $D_x F$ exist and be different from zero in E. Let x_0 be a point of E, and let $x_1, \ldots, x_{n+1}$ be n + 1 variable points of E such that $\Delta(x_1 x_2 \ldots x_{n+1})$ is always in I_1 and such that x_0 is always on the line between x_1 and x_2. Then

$$D_x F\Big|_{x_0} = \lim_{\substack{x_i \to x_0 \\ i=1,\ldots,n+1}} \frac{\Delta(F:x_1 x_2 \ldots x_{n+1})}{\Delta(x_1 x_2 \ldots x_{n+1})}.$$

<u>Proof</u>. Since $D_x F$ exists and is different from zero in E, then F is continuous and maps straight lines into straight lines. Let $D_x F|_{x_0} = d$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\text{(III.3.14)} \qquad \left| \frac{\Delta(F:x_0 x_1 \ldots x_n)}{\Delta(x_0 x_1 \ldots x_n)} - d \right| < \epsilon/2$$

whenever $\|x_0 x_i\| < \delta$, $(i = 1, \ldots, n)$, $\Delta(x_0 x_1 \ldots x_n)$ in I_1.

It must be shown that for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\text{(III.3.15)} \qquad \left| \frac{\Delta(F:x_1 x_2 \ldots x_{n+1})}{\Delta(x_1 x_2 \ldots x_{n+1})} - d \right| < \epsilon$$

whenever $\|x_0 x_i\| < \delta$, $(i = 1, \ldots, n + 1)$, and where $\Delta(x_1 x_2 \ldots x_{n+1})$ is always in I_1 and x_0 is on the line between x_1 and x_2.

Let $x_1, x_2, \ldots, x_{n+1}$ be n + 1 variable points such that the conditions of the hypotheses of the theorem are satisfied. The increments $\Delta(x_0 x_2 \ldots x_{n+1})$ and $\Delta(x_1 x_0 x_3 \ldots x_{n+1})$ are in I_1, but $\Delta(x_1 x_2 \ldots x_{i-1} x_0 x_{i+1} \ldots x_{n+1})$, $(i = 3, \ldots, n + 1)$, all have n-dimensional volume zero since x_0, x_1, and x_2 are collinear.

Since straight lines map into straight lines, $F(x_0)$, $F(x_1)$, and $F(x_2)$ are collinear, and all the increments $\Delta(F{:}x_1 x_2 \ldots x_{i-1} x_0 x_{i+1} \ldots x_{n+1})$, $(i = 3, \ldots, n + 1)$, have n-dimensional volume zero.

By Lemma I.5.1 and Remark I.5.7, and from the above statement,

$$\Delta(F{:}x_1 \ldots x_{n+1}) = \Delta(F{:}x_0 x_2 \ldots x_{n+1}) + \Delta(F{:}x_1 x_0 x_3 \ldots x_{n+1})$$

and

$$\Delta(x_1 \ldots x_{n+1}) = \Delta(x_0 x_2 \ldots x_{n+1}) + \Delta(x_1 x_0 x_3 \ldots x_{n+1}).$$

Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$\left|\frac{\Delta(F:x_1x_2\cdot\cdot\cdot x_{n+1})}{\Delta(x_1x_2\cdot\cdot\cdot x_{n+1})}-d\right| =$$

$$\left|\frac{\Delta(x_0x_2\cdot\cdot x_{n+1})}{\Delta(x_1x_2\cdot\cdot x_{n+1})}\left\{\frac{\Delta(F:x_0x_2\cdot\cdot x_{n+1})}{\Delta(x_0x_2\cdot\cdot x_{n+1})}-d\right\}+\frac{\Delta(x_1x_0x_3\cdot\cdot x_{n+1})}{\Delta(x_1x_2\cdot\cdot\cdot x_{n+1})}\left\{\frac{\Delta(F:x_1x_0x_3\cdot\cdot x_{n+1})}{\Delta(x_1x_0x_3\cdot\cdot x_{n+1})}-d\right\}\right|$$

$$<\left|\frac{\Delta(x_0x_2\cdot\cdot\cdot x_{n+1})}{\Delta(x_1x_2\cdot\cdot\cdot x_{n+1})}\right|\cdot\epsilon/2+\left|\frac{\Delta(x_1x_0x_3\cdot\cdot\cdot x_{n+1})}{\Delta(x_1x_2x_3\cdot\cdot\cdot x_{n+1})}\right|\cdot\epsilon/2 ,$$

whenever $\|x_0x_i\| < \delta$, $(i = 1, \ldots, n + 1)$, and where the points $x_1, \ldots, x_{n+1}$ satisfy the requirements of the hypotheses. This proves the theorem.

(III.3.16) Remark. The quantities

$$\left|\frac{\Delta(x_0x_2\cdot\cdot\cdot x_{n+1})}{\Delta(x_1x_2\cdot\cdot\cdot x_{n+1})}\right| \quad \text{and} \quad \left|\frac{\Delta(x_1x_0x_3\cdot\cdot\cdot x_{n+1})}{\Delta(x_1x_2\cdot\cdot\cdot x_{n+1})}\right|$$

are both ≤ 1, since x_0 is on the line between x_1 and x_2.

III.3.17. Theorem. Let $F: f^{(i)}(x)$, $(i = 1, \ldots, n)$, be a mapping function defined on a convex region E in $R^{(n)}$ and let D_xF exist and be different from zero in E. Then the mapping is one-to-one.

Proof. By Theorems III.3.2 and III.3.9, F is continuous and maps straight lines into straight lines. It will be shown that every image point, F(x), is the image of precisely one point of E under the mapping F.

Suppose on the contrary, that there is a point $F(x_0)$ which is the image of at least two distinct points, x_0 and x_1. Since E is convex, x_0 and x_1 can be joined by a straight line, $\overline{x_0x_1}$. Two situations may occur.

Case 1. The segment $\overline{x_0x_1}$ maps into the single point $F(x_0)$. Let $D_xF|_{x_0} = d$. Let $0 < \epsilon < |d|$, be given. Then there exists a $\delta > 0$ such that

$$\text{(III.3.18)} \qquad \left| \frac{\Delta(F:x_0x_1 \ldots x_n)}{\Delta(x_0x_1 \ldots x_n)} - d \right| < \epsilon$$

whenever $\|x_0x_i\| < \delta$, $(i = 1, \ldots, n)$, $\Delta(x_0x_1 \ldots x_n)$ in I_1.

Since $\overline{x_0x_1}$ maps into the single point $F(x_0)$, then in every neighborhood of x_0, one can find an increment, $\Delta(x_0x^*x_2 \ldots x_n)$ of I_1, where x^* is on $\overline{x_0x_1}$, such that for this increment, $\Delta(F:x_0x^*x_2 \ldots x_n)$ has n-dimensional volume zero since $F(x^*) = F(x_0)$. For such increments, relation (III.3.18) does not hold, for $\epsilon < |d|$, contradicting the assumption that $D_xF|_{x_0} = d$. Since a contradiction has been reached, it follows that case 1 cannot occur.

Case 2. The segment $\overline{x_0x_1}$ maps into the segment $F(x')F(x'')$, where $F(x')$ is the image of at least one point x' on $\overline{x_0x_1}$, and $F(x'')$ is the image of at least one point x'' on $\overline{x_0x_1}$, and $x' \neq x''$. Without loss of generality, one may assume x' to be between x_0 and x''. The point $F(x_0) = F(x_1)$ is either an interior point of $\overline{F(x')F(x'')}$, or else is one end point. Assume that $F(x_0) = F(x_1) \neq F(x')$.

By the same argument as in Theorem III.2.14, the following statements are true:

Every point $F(x)$ between $F(x_0)$ and $F(x')$ is the image of at least one point x between x_0 and x'. Every point $F(x)$ between $F(x')$ and $F(x'')$ is the image of at least one point x between x' and x''. Every point $F(x)$ between $F(x'')$ and $F(x_1) = F(x_0)$ is the image of at least one point x between x'' and x_1.

Consider the point x'. Let $D_x F|_{x'} = d'$. Choose a fixed positive $\epsilon' < |d'|$. Then there exists a $\delta' > 0$ such that

(III.3.19) $$\left| \frac{\Delta(F\colon x'x_1 \ldots x_n)}{\Delta(x'x_1 \ldots x_n)} - d' \right| < \epsilon'$$

whenever $\|x'x_i\| < \delta'$, $(i = 1, \ldots, n)$, $\Delta(x'x_1 \ldots x_n)$ in I_1. In every sufficiently small neighborhood of x' there is a point on $\overline{x_0 x_1}$ between x_0 and x' and a point on $\overline{x_0 x_1}$ between x' and x'', both of which map into the same point.

By Lemma III.3.13, in taking the derivative at x', the increments of I_1 formed by taking two points, x_1' and x_2' (with x' on the line between them) and n-1 other points, $x_3, \ldots, x_{n+1}$, none of which is on $\overline{x_1' x_2'}$, may be used. For these increments, relation (III.3.19) must hold, with $\epsilon' < |d'|$. But among these increments, in every neighborhood of x', those for which the points x_1'

and x_2' map into the same point will be found. For such increments, relation (III.3.19), with $\epsilon' < |d'|$, will not hold, since the image increment has n-dimensional volume zero. This contradicts the assumption that $D_X F|_{X_0} = d'$. Hence, case 2 cannot occur.

In either case, a contradiction has been reached. It is concluded that the mapping is one-to-one.

III.3.20. <u>Remark</u>. It has been shown that if F is defined on a convex region E in $R^{(n)}$ and if $D_X F$ exists and is different from zero in E, then F is continuous, one-to-one and maps straight lines into straight lines. Hence, from Remark II.6.25, p-flats map into p-flats, $(1 \leq p \leq n-1)$.

III.3.21. The main theorem of this chapter has now in effect been proved. For since F is continuous, one-to-one and takes straight lines into straight lines, Theorem II.8.10 can be applied and the following theorem is true:

III.3.22. <u>Theorem</u>. Let F: $f^{(i)}(x)$, $(i = 1, \ldots, n)$, be a mapping function defined on a convex region E in $R^{(n)}$ such that $D_X F$ exists and is different from zero in E. Then F is of the form

$$\text{(III.3.23)} \qquad F\colon f^{(i)}(x) = \frac{a_{i,1} x^{(1)} + \ldots + a_{i,n} + x^{(n)} + a_{i,n+1}}{a_{n+1,1} x^{(1)} + \ldots + a_{n+1,n} + x^{(n)} + a_{n+1,n+1}},$$

$(i = 1, \ldots, n)$, where

$$\begin{vmatrix} a_{1,1} & \cdots & a_{1,n+1} \\ \cdots & \cdots & \cdots \\ a_{n+1,1} & \cdots & a_{n+1,n+1} \end{vmatrix} \neq 0.$$

III.3.24. Remark. The theorems above have been proved for convex regions in $R^{(n)}$. The results can be extended to include any open, connected set E of $R^{(n)}$. This is done as follows:

III.3.25. Theorem. Let F: $f^{(i)}(x)$, $(i = 1, \ldots, n)$, be a mapping function defined on an open, connected set E in $R^{(n)}$, such that $D_x F$ exists and is different from zero at every point of E. The F is of the form (III.3.23).

Proof. Let x_o and x^* be any two points of E. It must be shown that F is of the form (III.3.23) at x_o and x^*, with the same constants, $a_{i,j}$.

The points x_o and x^* can be joined by a path C lying entirely in E since E is open and connected. The path C is a closed and bounded set in E. Hence there is a $\rho > 0$ such that every point of C is at a distance $\geqslant \rho$ from the boundary of E [Knopp, (1), p. 19]. Divide C by a finite number of points of division, $x_o, x_1, \ldots,$ $x_k = x^*$, such that $\| x_i x_{i+1} \| < \rho$, $(i = 0, 1, \ldots, k-1)$. Around each point of division, x_i, construct a sphere, T_i, lying entirely in E, with x_i as center and with radius $r_i \geqslant \rho$. Every point of C is in at least one of the spheres and adjoining spheres have points of C in common. Each

T_i is a convex region. Hence, Theorem III.3.22 can be applied to each sphere. In each T_i, F is of the form (III.3.23). Since the spheres have points in common, it must be concluded that the coefficients, $a_{i,j}$, must be the same for each sphere, and hence, the $a_{i,j}$ are the same at x_o as at x^*. This proves the theorem.

The following theorem is in the nature of a converse to Theorem III.3.25.

III.3.26. <u>Theorem</u>. Let

$$F: f^{(i)}(x) = \frac{a_{i,1} x^{(1)} + a_{i,2} x^{(2)} + \ldots + a_{i,n} x^{(n)} + a_{i,n+1}}{a_{n+1,1} x^{(1)} + a_{n+1,2} x^{(2)} + \ldots + a_{n+1,n} x^{(n)} + a_{n+1,n+1}},$$

$(i = 1, \ldots, n)$, where

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n+1} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n+1,1} & a_{n+1,2} & \ldots & a_{n+1,n+1} \end{vmatrix} \neq 0,$$

be defined in a region E of $R^{(n)}$ which does not contain the line $a_{n+1,1}x^{(1)} + \ldots + a_{n+1,n}x^{(n)} + a_{n+1,n+1} = 0$. Then $D_x F$ exists and is different from zero at each point of E.

<u>Proof</u>. Let x_o be any fixed point in E and let $x_1, x_2, \ldots x_n$ be n variable points of E so that $\Delta(x_o x_1 \ldots x_n)$ is in I_1. Examine the difference quotient,

$$(\text{III.3.27}) \qquad \frac{\begin{vmatrix} f^{(1)}(x_0) & f^{(2)}(x_0) & \cdots & f^{(n)}(x_0) & 1 \\ f^{(1)}(x_1) & f^{(2)}(x_1) & \cdots & f^{(n)}(x_1) & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f^{(1)}(x_n) & f^{(2)}(x_n) & \cdots & f^{(n)}(x_n) & 1 \end{vmatrix}}{\begin{vmatrix} x_0^{(1)} & x_0^{(2)} & \cdots & x_0^{(n)} & 1 \\ x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(n)} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n^{(1)} & x_n^{(2)} & \cdots & x_n^{(n)} & 1 \end{vmatrix}} .$$

The numerator of this difference quotient is equal to

$$\begin{vmatrix} \dfrac{a_{1,1}x_0^{(1)}+\cdots+a_{1,n}x_0^{(n)}+a_{1,n+1}}{a_{n+1,1}x_0^{(1)}+\cdots+a_{n+1,n}x_0^{(n)}+a_{n+1,n+1}} & \cdots & \dfrac{a_{n,1}x_0^{(1)}+\cdots+a_{n,n}x_0^{(n)}+a_{n,n+1}}{a_{n+1,1}x_0^{(1)}+\cdots+a_{n+1,n}x_0^{(n)}+a_{n+1,n+1}} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{a_{1,1}x_n^{(1)}+\cdots+a_{1,n}x_n^{(n)}+a_{1,n+1}}{a_{n+1,1}x_n^{(1)}+\cdots+a_{n+1,n}x_n^{(n)}+a_{n+1,n+1}} & \cdots & \dfrac{a_{n,1}x_n^{(1)}+\cdots+a_{n,n}x_n^{(n)}+a_{n,n+1}}{a_{n+1,1}x_n^{(1)}+\cdots+a_{n+1,n}x_n^{(n)}+a_{n+1,n+1}} & 1 \end{vmatrix} =$$

$$\frac{1}{\prod_{i=0}^{n}(a_{n+1,1}x_i^{(1)}+\cdots+a_{n+1,n}x_i^{(n)}+a_{n+1,n+1})} \begin{vmatrix} (a_{1,1}x_0^{(1)}+\cdots+a_{1,n}x_0^{(n)}+a_{1,n+1}) & \cdots & (a_{n+1,1}x_0^{(1)}+\cdots+a_{n+1,n}x_0^{(n)}+a_{n+1,n+1}) \\ \cdots & \cdots & \cdots \\ (a_{1,1}x_n^{(1)}+\cdots+a_{1,n}x_n^{(n)}+a_{1,n+1}) & \cdots & (a_{n+1,1}x_n^{(1)}+\cdots+a_{n+1,n}x_n^{(n)}+a_{n+1,n+1}) \end{vmatrix} .$$

Using the multiplication theorem for determinants, the above product becomes

$$\frac{1}{\prod_{i=0}^{n}(a_{n+1,1}x_i^{(1)}+\cdots+a_{n+1,n}x_i^{(n)}+a_{n+1,n+1})} \begin{vmatrix} a_{1,1} & \cdots & a_{1,n+1} \\ \cdots & \cdots & \cdots \\ a_{n+1,1} & \cdots & a_{n+1,n+1} \end{vmatrix} \begin{vmatrix} x_0^{(1)} & \cdots & x_0^{(n)} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ x_n^{(1)} & \cdots & x_n^{(n)} & 1 \end{vmatrix} .$$

Hence, the difference quotient, (III.3.27) becomes

$$\frac{\begin{vmatrix} a_{1,1} & \cdots & a_{1,n+1} \\ \cdots & \cdots & \cdots \\ a_{n+1,1} & \cdots & a_{n+1,n+1} \end{vmatrix}}{\prod_{i=0}^{n} (a_{n+1,1} \; x_i^{(1)} + \cdots a_{n+1,n} \; x_i^{(n)} + a_{n+1,n+1})}.$$

By hypothesis the numerator is different from zero. Allowing x_i to approach x_0, $(i = 1, \ldots, n)$, $\Delta(x_0 x_1 \ldots x_n)$ remaining in the class I_1, it is concluded that $D_x F|_{x_0}$ exists and equals

$$\frac{\begin{vmatrix} a_{1,1} & \cdots & a_{1,n+1} \\ \cdots & \cdots & \cdots \\ a_{n+1} & \cdots & a_{n+1,n+1} \end{vmatrix}}{(a_{n+1,1} \; x_0^{(1)} + \cdots + a_{n+1,1} \; x_0^{(n)} + a_{n+1,n+1})^{n+1}} \neq 0.$$

III.3.28. <u>Remark</u>. As in the 2-dimensional case, one sees that since the determinant $\Delta(x_0 x_1 \ldots x_n)$ cancels out and does not enter in the difference quotient while the limit is being taken, then $D_x F$ exists at any point x^* of E, with the limit being taken as $x_0, x_1, \ldots, x_n$ approach x^* in any manner whatever. In the limit

$$D_x F|_{x^*} = \frac{\begin{vmatrix} a_{1,1} & \cdots & a_{1,n+1} \\ \cdots & \cdots & \cdots \\ a_{n+1,1} & \cdots & a_{n+1,n+1} \end{vmatrix}}{(a_{n+1,1} \; x^{(1)*} + \cdots + a_{n+1,n} \; x^{(n)*} + a_{n+1,n+1})^{n+1}}.$$

III.3.29. Remark. Consider the function F, defined on a region E of $R^{(n)}$, and assume that $D_x F$ exists and is not zero everywhere in E. Then F must be an affine transformation. That is, finite points must go into finite points. Otherwise, if some point, say x^*, mapped into an infinite point, then $D_x F|_{x^*}$ would not exist, since all the image increments with one vertex at x^* would be infinite, and the difference quotient considered would be infinitely large. Then, for every $\epsilon > 0$, there would be no $\delta > 0$ such that the usual ϵ, δ relation for the difference quotient would hold.

It follows that if it assumed that F is defined on the whole Euclidean space $R^{(n)}$ and if $D_x F$ exists everywhere and is not zero, then F must not only be linear fractional, but must be linear:

$$F: f^{(i)}(x) = a_{i,1} x^{(1)} + \ldots + a_{i,n} x^{(n)} + a_{i,n+1} \quad (i = 1, \ldots, n).$$

Otherwise, there would be some finite points which would map into infinite points. This would be impossible, since it has been assumed that $D_x F$ exists and is different from zero everywhere.

III.3.30. Remark. Theorems III.3.25 and III.3.26 together show that the precise class of mapping functions, F, defined on a connected, open set E of $R^{(n)}$ which have a non-zero derivative $D_x F$ at each point of E is the class of linear fractional transformations. If the set E is the whole space, $R^{(n)}$, then F is linear.

In the special case where F is of the form

$$F:\ f^{(i)}(x) = x^{(i)},\ (i = 1, \ldots k-1, k+1, \ldots n),$$

then, as in the 2-dimensional case mentioned in Remark III.2.25, $f^{(k)}(x)$ must be linear. That is, if $D_{x^{(k)}}f^{(k)}$ exists and is different from zero at each point of E, then $f^{(k)}$ is linear.

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