NORMAL DETERMINANTS AND EXPANSIONS IN MODIFIED SEQUENCES

by

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Submitted to the Department of Mathematics and the Faculty of the Graduate School of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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May, 1952

ACKNOWLEDGEMENTS

1. I wish to extend my sincere thanks to the Office of Naval Research for its support. Some of the results in this dissertation were obtained while I was a research assistant from September, 1949 to September, 1951 on Project NR 043 093, Contract N9 onr-81300 of the Office of Naval Research on which Professor G. B. Price was principal investigator.

2. I wish to extend my sincere thanks to the University of Kansas for its support. I was employed as an assistant instructor in the years 1948-1949 and was the holder of a University Fellowship in the year 1951-1952 at the University of Kansas.

TABLE OF CONTENTS

CHAPTER FOUR - COMPLETENESS OF MODIFIED SEQUENCES

Introduction

The ensuing work is concerned with various phases of a general theory of series expansions that has been investigated [15]* by G. B. Price and others. Let the functions $f(x)$ and $u_k(x)$, $k = 0, 1, \ldots,$ have either real or complex values, while x may be an element of an abstract set E. The points $x_{n0}, x_{n1}, \ldots, x_{nn}$ are assumed to be in E. Briefly, this general theory treats the series expansions

$$f(x) = d_0 u_0(x) + d_1 u_1(x) + \ldots + d_k u_k(x) + \ldots ,$$

where

$$\text{(A)} \qquad d_k = \lim_{n \to \infty} d_k^{(n)}$$

and

$$\text{(B)} \quad d_k^n = \frac{\begin{vmatrix} u_0(x_{n0}) & u_1(x_{n0}) & \ldots & u_{k-1}(x_{n0}) & f(x_{n0}) & u_{k+1}(x_{n0}) & \ldots & u_n(x_{n0}) \\ u_0(x_{n1}) & u_1(x_{n1}) & \ldots & u_{k-1}(x_{n1}) & f(x_{n1}) & u_{k+1}(x_{n1}) & \ldots & u_n(x_{n1}) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_0(x_{nn}) & u_1(x_{nn}) & \ldots & u_{k-1}(x_{nn}) & f(x_{nn}) & u_{k+1}(x_{nn}) & \ldots & u_n(x_{nn}) \end{vmatrix}}{\begin{vmatrix} u_0(x_{n0}) & u_1(x_{n0}) & \ldots & u_{k-1}(x_{n0}) & u_k(x_{n0}) & u_{k+1}(x_{n0}) & \ldots & u_n(x_{n0}) \\ u_0(x_{n1}) & u_1(x_{n1}) & \ldots & u_{k-1}(x_{n1}) & u_k(x_{n1}) & u_{k+1}(x_{n1}) & \ldots & u_n(x_{n1}) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_0(x_{nn}) & u_1(x_{nn}) & \ldots & u_{k-1}(x_{nn}) & u_k(x_{nn}) & u_{k+1}(x_{nn}) & \ldots & u_n(x_{nn}) \end{vmatrix}}$$

* Numbers in brackets refer to the bibliography at the end of the paper.

The denominator of $d_{k}^{(n)}$ is assumed to be different from zero. This represents a restriction on the functions $u_{k}(x)$ and on the points $\{x_{ni}\}$. The specific problems to be treated in this work can now be stated.

Problem 1. Study the additive homogeneous functional $d_{k}(f)$; in particular, determine (a) the class of functions f for which it is defined, (b) its continuity properties, and (c) its representations by means of integrals, derivatives, infinite determinants, and so on.

In the treatment of Problem 1, certain modified sequences are brought under the general theory indicated above. By a modified $u_{k}(x)$-sequence is meant a sequence of functions of the form $u_{k}(x)+\phi_{k}(x)$ where the functions $\phi_{k}(x)$ must satisfy certain conditions. This suggests another task.

Problem 2. Examine the functionals d_{k} and the expansions $\sum_{k=0}^{\infty} d_{k}\left[u_{k}(x)+\phi_{k}(x)\right]$ in the modified sequences encountered in Problem 1 as well as other modified sequences.

In working on Problem 1, the expression $d_{k}^{(n)}$ in (B) arises from the interpolation of a function f(x) defined in E by the interpolating polynomial

$$d_{0}^{(n)} u_{0}(x) + d_{1}^{(n)} u_{1}(x) + \dots + d_{n}^{(n)} u_{n}(x)$$

on the n + 1 points $x_{n0}, x_{n1}, \dots, x_{nn}$. Solving the system of equations

$$d_{0}^{(n)} u_{0}(x_{i}) + d_{1}^{(n)} u_{1}(x_{i}) + \dots + d_{n}^{(n)} u_{n}(x_{i}) = f(x_{i}),$$

$i = 0, 1, \ldots, n$, by Cramer's rule, we obtain $d_k^{(n)}$ as given in (B).

It is found that a useful device in the simplification of $d_k^{(n)}$ is the notion of finite orthogonality. Thus multiply both numerator and denominator of $d_k^{(n)}$ by the transpose of the denominator. It is found that in the resulting determinants, except in the column that involves $f(x)$, all elements will be of the form

$$\text{(C)} \qquad \sum_{i=0}^{n} u_k(x_{ni})\, u_l(x_{ni}) \, .$$

Now, briefly, we say that the functions $u_k(x)$, $k = 0, 1, \ldots, n$, are finite orthogonal over the points $\{x_{ni}\}$ if the expression in (C) vanishes when k is not equal to l and is equal to a non-zero constant, $\alpha_k^{(n)}$ when k is equal to l. $\alpha_k^{(n)}$ depends on k and n in general. If the functions $u_k(x)$ are finite orthogonal over the given points, it is easily seen that the new expression for $d_k^{(n)}$ reduces to

$$\text{(D)} \qquad \frac{\sum_{i=0}^{n} f(x_{ni})\, u_k(x_{ni})}{\sum_{i=0}^{n} u_k^2(x_i)} \, , \quad k = 0, 1, \ldots, n \, .$$

If the points x_{ni}, $i = 0, \ldots, n$, are properly chosen points in an interval $[a, b]$ of the real line and $f(x)$ is Riemann integrable, it has been shown in some cases [15] that the limit of the expression in (D) is a Riemann integral.

In addition to finite orthogonality, a more general notion, that of finite biorthogonality, is useful in the same way for the simplification of $d_k^{(n)}$.

Using these notions it has already been shown that the Fourier series [16, p. 114], the Fourier sine series [8, pp. 30-35] and the Taylor series in the complex plane [15] come under the general theory indicated above. In all of these cases d_κ turns out to be an integral. The Taylor series for real x has also been considered [15] with d_κ being a derivative in this case.

In Chapter One it is shown that the double Fourier series comes under the general theory. Also, using the notion of finite biorthogonality, the Laurent series is brought under this framework.

In connection with Problem 1, a form of Bessel's inequality is proven for finite orthogonal functions which is of interest. It shows that the notion of Bessel's inequality is so strong that its skeleton appears when only the values of functions on a finite set of points of an interval of the real line are considered rather than the values on the whole interval.

Certain other more artificial functions are brought under the general theory in Chapter One.

The work in Chapter One is concerned solely with sequences of functions which are finite orthogonal or are associated with another finite biorthogonal sequence. In contrast, in Chapter Two the sequences of functions considered are no longer finite orthogonal. However the notion of finite orthogonality is used to study $d_\kappa^{(n)}$ and the sequences are again brought under the general theory. It is found that d_κ, in this case, is the ratio of two infinite determinants. The sequences considered there are of the type $1 + \phi_0(x)$, $\cos x + \phi_1(x)$, $\sin x + \phi_2(x)$, ..., $\cos kx + \phi_{2k-1}(x)$, $\sin kx + \phi_{2k}(x)$, ... Certain more general types of modified sequences are also considered. An extended definition of finite biorthogonal functions is given and a theorem concerning such functions is proven.

In working on Problem 2 several results are obtained. First, a rather abstract situation in regard to modified sequences is considered. Let $\{u_k(x)\}_0^\infty$ be a sequence of functions defined on some interval of the real line or in some region of the complex plane. Let $f(x)$ be defined on the same interval or in the region. Let $\{O_k\}_0^\infty$ be a sequence of additive, homogeneous functionals such that $O_k[u_1(x)] = \delta_{k1}$. Then expansions $f(x) = \sum_{i=0}^{\infty} a_i(f)[u_i(x) + \phi_i(x)]$ are investigated where $\phi_i(x)$ are some properly restricted sequence of functions able to be expanded in a series of the $u_i(x)$. The functionals a_i are also investigated and found explicitly.

It is seen that, because of the nature of the operators O_k, modified orthogonal sequences, modified Taylor sequences and modified biorthogonal sequences as well as their associated expansions are to be investigated.

It might appear that if from a sequence $\{u_k(x)\}_0^\infty$ a modified sequence, $\{u_k(x) + \phi_k(x)\}_0^\infty$ is obtained which has certain properties, then a further modification to a sequence $\{u_k(x) + \phi_k(x) + \Theta_k(x)\}_0^\infty$ might be very fruitful. However in Chapter Three it is shown that under reasonable hypotheses the result of making two modifications in succession is completely equivalent to making one two-step modification.

An example of a modification and an extension of the main result to a Banach space are included in Chapter Three.

In addition to the topics already indicated the important question of completeness of modified sequences is discussed. The notion of the completeness of a certain set of operators is also used. A necessary and sufficient condition for the completeness of modified sequences $\left\{ P_k(x) + \phi_k(x) \right\}_0^\infty$, providing the sequence $\left\{ P_k(x) \right\}_0^\infty$ is complete is given as the non-vanishing of a certain infinite determinant. A criterion for the non-vanishing of some of these determinants is presented that is independent of the problem considered here. Certain other properties of modified sequences are also discussed.

Some remarks about the methods are appropriate. Heavy use was made of interpolation in the first two chapters, but in the treatment of Problem 2 it was found that interpolation could be abandoned as a tool, thus enabling some of the heavier restrictions on the problem to be removed. The role played by normal determinants in the entire paper is a considerable one. Normal determinants were first discovered by Von Koch in 1892 and most of the work done on them (see [28] [29]) is due to him. A brief account of normal determinants is also given by F. Riesz [25] , who in addition presents the notion of an absolutely convergent determinant. It might be remarked that the extension of the results of this paper to the case where the determinants involved are absolutely convergent rather than normal seems to present no real difficulty. Only a few new results concerning normal determinants are proven here. Theorem 19 represents one such result that seems to have been overlooked. Theorem 21 was proven by F. Riesz in a much more general fashion, using different methods. However, the proof given in Theorem 21 uses only the properties of normal determinants. Another result that deals purely with normal determinants is Theorem 31. This theorem is a simple extension of one proven by G. B. Price [24] . Now, the fact that a normal determinant will be different from zero if the elements satisfy certain conditions was proven [29] by Von Koch. Theorem 31 does not handle any more cases but the bounds given there are at times sharper than Von Koch's bounds.

Problem 1 has an interesting history that has been pointed out [15] before. Problem 2 lies in a field that was opened by G. D. Birkhoff [3] in 1917. At that time he laid down a guiding principle that is of interest and which certainly holds true in this paper. "Any set of vectors in a functional space lying near enough to a complete set of vectors admitting a reciprocal set is itself complete and admits a reciprocal set." The reciprocal sets here can be thought of as biorthogonal sets.

Birkhoff's work on modified sequences was continued by J. L. Walsh [30] who in 1922 discussed expansions in sequences which were near orthogonal sequences. Paley and Weiner in 1934 [23] gave an important criterion having to do with expansions in a sequence of functions that were "near" an orthonormal sequence. Most of the work done since that time has been of the Paley-Weiner type. For some of these results one should consult the bibliography. In this later work, a modified expansion is obtained if $\{P_n(x)\}$ is a sequence of orthonormal functions and $\{g_n(x)\}$ is any sequence of functions close to $\{P_n(x)\}$ in some sense. The modified expansion is in terms of the $g_n(x)$. Being close in some sense involves a certain numerical restriction on the difference of $P_n(x)$ and $g_n(x)$. In this dissertation it is seen that if, instead of taking any set of functions $g_n(x)$ near $P_n(x)$, we take only certain $g_n(x)$, this numerical restriction can be discarded.

One unsolved problem in connection with this work deserves further mention. It has been found [8, pp. 30-35] [16, p. 114] [27, p. 150] that the sequence 1, cos x, sin x, ..., cos kx, sin kx, ... is finite orthogonal over a certain set of points. In addition, the sequences

a) $1, z, z^2, \ldots$,

b) $\ldots, z^{-2}, z^{-1}, 1, z, z^2, \ldots$,

c) 1, sin x, sin 2x, ...

d) the Haar sequence of functions,

have been shown [15, p. 13] to be finite biorthogonal over certain sets of points. Other more artificial functions are shown in Chapter One to possess this property. But these are all the sequences with this property that have been found. The locating of other such sequences or the establishing of the fact that there are no more must remain as intriguing problems at this writing.

In addition it is suggested that there is a continuing need for more knowledge about infinite determinants. (For a different class of infinite determinants then those studied here, see [15, pp. 100-123])

After the completion of this thesis it was found that Theorem 15 together with Theorem 28 had been proven by N. Bary [2] and even for the case where the infinite determinants involved were absolutely convergent determinants rather than normal determinants. Bary considered only the integral case however. Theorem 29 which is a generalization of Theorem 28 was not considered by Bary.

As far as the mechanics of the thesis are concerned, chapters, sections of chapters and certain items in the sections are indicated numerically. Thus an entry such as 4.3.2 refers to the second designated item of the third section of Chapter Four.

In conclusion, the author wishes to extend his sincere thanks to Professor G. B. Price whose encouragement, aid and inspiration have made this thesis possible.

CHAPTER ONE

FINITE ORTHOGONALITY

1.1. Applications of Finite Orthogonality.

A general theory of series expansions has been pointed out by G. B. Price [15]. It will be described briefly below, and the application of finite orthogonality to this general theory will be discussed.

Consider the series expansion

1.1.1 $$f(x) = a_0P_0(x) + a_1P_1(x) + \cdots + a_kP_k(x) + \cdots ,$$

where

1.1.2 $$a_k = \lim_{n\to\infty} c_k^{(n)},$$

and

1.1.3 $$c_k^{(n)} = \frac{\begin{vmatrix} P_0(x_0^{(n)}) & \cdots & P_{k-1}(x_0^{(n)}) & f(x_0^{(n)}) & P_{k+1}(x_0^{(n)}) & \cdots & P_n(x_0^{(n)}) \\ P_0(x_1^{(n)}) & \cdots & P_{k-1}(x_1^{(n)}) & f(x_1^{(n)}) & P_{k+1}(x_1^{(n)}) & \cdots & P_n(x_1^{(n)}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P_0(x_n^{(n)}) & \cdots & P_{k-1}(x_n^{(n)}) & f(x_n^{(n)}) & P_{k+1}(x_n^{(n)}) & \cdots & P_n(x_n^{(n)}) \end{vmatrix}}{\begin{vmatrix} P_0(x_0^{(n)}) & \cdots & P_{k-1}(x_0^{(n)}) & P_k(x_0^{(n)}) & P_{k+1}(x_0^{(n)}) & \cdots & P_n(x_0^{(n)}) \\ P_0(x_1^{(n)}) & \cdots & P_{k-1}(x_1^{(n)}) & P_k(x_1^{(n)}) & P_{k+1}(x_1^{(n)}) & \cdots & P_n(x_1^{(n)}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P_0(x_n^{(n)}) & \cdots & P_{k-1}(x_n^{(n)}) & P_k(x_n^{(n)}) & P_{k+1}(x_n^{(n)}) & \cdots & P_n(x_n^{(n)}) \end{vmatrix}}.$$

Here the denominator of $c_k^{(n)}$ is assumed to be different from zero for all n. This represents a restriction on the sequence of functions $P_0(x), P_1(x), \ldots$. This assumption will be ever present in this work, although in special cases the non-zero value of the denominator of $c_k^{(n)}$ will be calculated.

The general theory arises from the study of interpolation of an arbitrary function $f(x)$ by a finite linear combination of the sequence $P_o(x), P_1(x), \ldots$. Thus let $x_i^{(n)}$, $i = 0, 1, \ldots, n$, be $n+1$ points lying in an interval $[a, b]$ of the real axis, or in a region of the complex plane. Let $f(x)$ be defined on $[a, b]$. Set

$$f(x_i^{(n)}) = a_o^{(n)}P_o(x_i^{(n)}) + a_1^{(n)}P_1(x_i^{(n)}) + \cdots + a_n^{(n)}P_n(x_i^{(n)}) \ , \quad i = 0, 1, \ldots, n.$$

Solving this set of equations in the usual way, bearing in mind the restrictions on the $P_j(x)$ stated above, we find that $a_k^{(n)} = c_k^{(n)}$, $k = 0, 1, \ldots, n$, as given in 1.1.3. Now 1.1.3 is rather unwieldy in general. Therefore it would be interesting to obtain an explicit evaluation of the operator $c_k^{(n)}$. This has been done in two special cases

Let $P_j(x) = \sin jx$ and $x_i^{(n)} = \frac{i\pi}{n+1}$. It is known [8, pp. 30-35] that

$$a_k^{(n)} = c_k^{(n)} = \frac{2}{n+1} \sum_{i=1}^{n} f(x_i) \sin kx_i \ .$$

Let $P_o(x) = 1, \ldots, P_{2k-1}(x) = \cos kx, P_{2k}(x) = \sin kx, \ldots$. Here let $x_i^{(n)} = \frac{i2\pi}{2n\ 1}$, $i = 0, 1, \ldots, 2n$. It is known [16, p. 115] that

$$c_o^{(n)} = \frac{1}{2n+1} \sum_{i=0}^{2n} f(x_i) \ ,$$

$$c_{2k-1}^{(n)} = \frac{2}{2n+1} \sum_{i=0}^{2n} f(x_i) \cos kx_i \quad , \quad k = 1, \ldots, n,$$

$$c_{2k}^{(n)} = \frac{2}{2n+1} \sum_{i=0}^{2n} f(x_i) \sin kx_i \quad , \quad k = 1, \ldots, n.$$

It is appropriate to look briefly at the calculation of the operator $c_j^{(n)}$, $j = 0, 1, \ldots, 2n$, in this last case as it will bring to light an interesting property of the sine functions and of the sine and cosine functions which will be examined in the case of other sequences of functions.

Therefore consider 1.1.3 with $P_0(x) = 1, \cdots, P_{2k-1}(x) = \cos kx$, $P_{2k}(x) = \sin kx, \cdots$. Set $x_i^{(n)} = \frac{i2\pi}{2n+1}$. Hereafter the superscript (n) will be omitted from $x_i^{(n)}$ but it is to be borne in mind that the set of interpolation points varies with n.

Multiply both the numerator and denominator of c_k in 1.1.3 from the left by the transpose of the denominator. In the resulting ratio of two determinants, except in the column that involves the function f(x), all of the elements will be of the form $\sum_{i=0}^{2n} \cos kx_i \cos lx_i$, $\sum_{i=0}^{2n} \sin kx_i \sin lx_i$, or $\sum_{i=0}^{2n} \sin kx_i \cos lx_i$, $0 \leqq k, l \leqq n$. We use the following property of the sines and cosines. If $x_i = \frac{i2\pi}{2n+1}$, $i = 0, 1, \cdots, 2n$, then

$$\text{a)} \quad \sum_{i=0}^{2n} \cos kx_i \cos lx_i = \frac{2n+1}{2}\,\delta_{kl} \quad , \quad 1 \leqq k, l \leqq n,$$

1.1.4
$$\text{b)} \quad \sum_{i=0}^{2n} \cos kx_i \sin lx_i = 0 \quad , \quad 0 \leqq k, l \leqq n,$$

$$\text{c)} \quad \sum_{i=0}^{2n} \sin kx_i \sin lx_i = \frac{2n+1}{2}\,\delta_{kl} \quad , \quad 1 \leqq k, l \leqq n.$$

a) and c) may be verified by expressing the terms on the left as cosine series and using the formula for the sum of a cosine series. b) may be verified by using the symmetry of the points x_i about π. In Chapter Two formulae including the above for more general values of k and l will be derived.

It is clear that the expression obtained for c_k may be evaluated immediately when the relations of 1.1.4 are used. We find

$$c_0 = \frac{1}{2n+1} \sum_{i=0}^{2n} f(x_i) \quad .$$

$$c_{2k-1} = \frac{2}{2n+1} \sum_{i=0}^{2n} f(x_i) \cos kx_i \quad , \quad k = 1, \ldots, n,$$

$$c_{2k} = \frac{2}{2n+1} \sum_{i=0}^{2n} f(x_i) \sin kx_i \quad , \quad k = 1, \ldots, n.$$

c_{2k-1} can be written as

$$c_{2k-1} = \frac{1}{\pi} \sum_{i=0}^{2n} f(x_i) \cos kx_i \frac{2\pi}{2n+1} .$$

If $f(x)$ is Riemann integrable, it is found that

$$\lim_{n\to\infty} c_{2k-1} = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\,dx \quad ,$$

which is the customary Fourier coefficient. The same argument applies to c_{2k} and c_o.

When given an arbitrary sequence of functions, a set of relations analogous to 1.1.4 would be of use then, as this would enable us to calculate c_k readily. These considerations lead naturally to the following definition.

Definition. The sequence of functions $\phi_o(x)$, $\phi_1(x)$, $\phi_2(x)$, ... is said to be finite orthogonal over the set of points $\{x_i^{(n)}\}$, $i = 0, 1, \ldots, n$, if and only if

$$\sum_{i=0}^{n} \phi_k(x_i^{(n)})\ \phi_l(x_i^{(n)}) = d_k^{(n)}\, \delta_{kl},\ d_k^{(n)} \neq 0,\ k,l = 0, 1, \ldots, n,$$

where $d_k^{(n)}$ is a constant depending on k and n in general. The problem of finding sequences of functions which satisfy this property of finite orthogonality seems to be a difficult one in general. Some sequences which satisfy this property, or another quite similar one, in addition to the sequences seen above will be pointed out below. Also it has been found that the Haar functions [15, p. 52] satisfy this property over a certain set of points.

1.2 Double Fourier Series.

An obvious extension of the work in the preceding paragraph to double Fourier series is possible. Consider first the sequence of functions $\{\sin px \sin qy\}$, $p = 1, \ldots, n$, $q = 1, \ldots, n$. Choose the double sequence of numbers $x_i = \frac{i\pi}{n+1}$, $i = 1, \ldots, n$, $y_i = \frac{i\pi}{n+1}$, $i = 1, \ldots, n$. Now take the points (x_i, y_j) in the plane. The given sequence of functions is finite orthogonal over these n^2 points for every n. For $0 < m, l, p, q \leqq n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sin mx_i \sin ly_j \sin px_i \sin qy_j$$

$$= \sum_{i=1}^{n} \sin mx_i \sin px_i \sum_{j=1}^{n} \sin ly_j \sin qy_j = \left(\frac{n+1}{2}\right)\delta_{mp}\left(\frac{n+1}{2}\right)\delta_{lq}$$

$$= \begin{cases} 0, & m \neq p \text{ or } l \neq q \\ \left(\frac{n+1}{2}\right)^2, & m = p \text{ and } l = q. \end{cases}$$

Then we have from 1.1.3

$$c_{kl} = \left(\frac{2}{n+1}\right)\left(\frac{2}{n+1}\right) \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j) \sin kx_i \sin ly_j$$

The limiting process may be applied and the traditional Fourier double-sine coefficient is obtained :

$$c_{kl} = \left(\frac{2}{\pi}\right)^2 \int_0^{\pi}\int_0^{\pi} f(x, y) \sin kx \sin ly \, dx\, dy \quad , k, l, \text{ integers.}$$

The double Fourier series may be handled in an analogous manner.

1.3 Evaluation of the Operator for Laurent Series.

For simplicity, let us consider an annulus about the origin in the z- plane with radii R and r, $r < R$. It is to be noted that the following result will hold for an annulus about the point $z = a$.

Let f(z) be a function that is analytic in the annulus. Let

$$a_{-n} z^{-n} + a_{-n+1} z^{-n+1} + \dots + a_0 + \dots + a_{n-1} z^{n-1} + a_n z^n$$

be an interpolating polynomial that interpolates the given function on 2n + 1 points lying on the circle $|z| = \rho$, $r \leqq \rho \leqq R$. More compactly let $z_j = \rho e^{i\theta_j}$ where $\theta_j = j\left(\frac{2\pi}{2n+1}\right)$, $j = 1, 2, \dots,$ $2n+1$. We have the system of equations

$$\begin{aligned} f(z_1) &= a_{-n} z_1^{-n} + a_{-n+1} z_1^{-n+1} + \dots + a_0 + \dots + a_n z_1^n, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ f(z_{2n+1}) &= a_{-n} z_{2n+1}^{-n} + a_{-n+1} z_{2n+1}^{-n+1} + \dots + a_0 + \dots + a_n z_{2n+1}^n . \end{aligned} \tag{1.3.1}$$

Letting $z_j = \rho e^{i\theta_j}$, we have by Cramer's rule

$$a_k = \frac{\left|\rho^l e^{li\theta_j}\right|(k, f(\rho e^{i\theta_s}))}{\left|\rho^l e^{li\theta_j}\right|}, \quad l = -n, \dots, n;$$

$j = 1, \dots, 2n+1$; $k = -n, \dots, n$. Here $\left|\rho^l e^{li\theta_j}\right|(k, f(\rho e^{i\theta_s}))$ indicates that the column with elements $\rho^k e^{ki\theta_j}$ is replaced by the column of elements $f(\rho e^{i\theta_s})$, $s = 1, \dots, 2n+1$.

We may factor out the ρ's from all of the columns leaving a multiplier of $\frac{1}{\rho^K}$. We multiply the remaining expression in the numerator and denominator from the left by the conjugate transpose of the denominator.

Lemma.

1.3.2

(1) $$\sum_{j=1}^{2n+1} e^{ri\theta_j} e^{si\theta_j} = 2n+1, \quad -r \equiv s \mod (2n+1)$$

(2) $$\sum_{j=1}^{2n+1} e^{ri\theta_j} e^{si\theta_j} = 0, \quad -r \not\equiv s \mod (2n+1)$$

Proof: The first identity is obvious. In (2)

$$\sum_{j=1}^{2n+1} e^{ri\theta_j} e^{si\theta_j} = \sum_{j=1}^{2n+1} e^{(r+s)i\theta_j} = \sum_{j=1}^{2n+1} e^{(r+s)ij\frac{2\pi}{2n+1}}$$

The last sum is merely a geometric progression, its sum being

$$e^{(r+s)i\frac{2\pi}{2n+1}} \frac{1 - \left[e^{(r+s)i\frac{2\pi}{2n+1}}\right]^{2n+1}}{1 - e^{(r+s)i\frac{2\pi}{2n+1}}} = 0$$

The denominator is never equal to zero since $r + s \neq k(2n+1)$, k an integer.

Using these identities, the transformed expression for a_K is clearly

1.3.3 $$a_K = \frac{1}{\rho^K} \frac{1}{2\pi} \sum_{j=1}^{2n+1} e^{-Ki\theta_j} f(\rho e^{i\theta_j}) \frac{2\pi}{2n+1}.$$

Now as $n \to \infty$, since f(z) is analytic in the annulus and $\frac{2\pi}{2n+1} = \Delta\theta$, the above sum is merely a Cauchy-Riemann sum and

$$\lim_{n\to\infty} a_k = \frac{1}{\rho^k}\frac{1}{2\pi}\int_0^{2\pi} e^{-ki\theta} f(\rho e^{i\theta})\, d\theta .$$

Now $z = \rho e^{i\theta}$, $d\theta = \frac{1}{i}\frac{dz}{z}$ so that

$$\lim_{n\to\infty} a_k = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(z)}{z^{k+1}}\, dz .$$

This is the usual Laurent coefficient. Of course, a further study can be made of the convergence of the interpolating polynomial

$$a_{-n} z^{-n} + a_{-n+1} z^{-n+1} + \dots + a_0 + \dots + a_{n-1} z^{n-1} + a_n z^n ,$$

where a_j is now given by 1.3.3 to f(z) in the annulus.

It has also been shown [15, p.6] that the functions 1, z, z^2, ..., z^n are such that on the points $z_j = \rho e^{\frac{j i 2\pi}{n+1}}$ $j = 0, 1, \dots, n$,

$$\sum_{j=0}^{n} z_j^{-k} z_j^{l} = (n+1)\,\delta_{kl}$$

The proof is similar to that used for the Laurent series.

1.4 Another Set of Finite Orthogonal Functions.

Let $\{x_n\}$, $n = 1, 2, \dots$, be a sequence of distinct points in the interval $[0,1]$ that have the point 0 as an accumulation point. We can assume the sequence is ordered in descending magnitude.

Define a sequence of functions $\phi_n(x)$: $\phi_n(x_j) = 0$, $j \neq n$, $\phi_n(x_n)$ is not zero but otherwise arbitrary. Then it is clear that for $1 \leqq k$, $1 \leqq n$

$$\sum_{j=1}^{n} \phi_k(x_j)\,\phi_1(x_j) = 0 \quad , \quad k \neq 1$$

$$= \left[\phi_k(x_k)\right]^2, \quad k = 1 .$$

The process used in paragraph 1.1 may be carried out simply. We obtain

$$c_k = \frac{f(x_k)}{\phi_k(x_k)} \quad , \qquad k = 0, 1, \ldots, n.$$

Here it is assumed that the functions f(x) is defined and finite throughout the interval $[0,1]$. Now, if the resulting series is to represent f(x) at each of the points x_n, then obviously

$$\text{1.4.1} \qquad f(x) = \frac{f(x_1)}{\phi_1(x_1)}\,\phi_1(x) + \frac{f(x_2)}{\phi_2(x_2)}\,\phi_2(x) + \cdots + \frac{f(x_n)}{\phi_n(x_n)}\,\phi_n(x) + \cdots .$$

Here it is seen that the interpolating coefficient is actually the same as the series coefficient without using the limiting process. The series 1.4.1 is the same for all functions which have the same value at the points $\{x_n\}$, $n = 1, 2, \ldots$. Therefore at other points in $[0,1]$ it can represent only one of the functions from this set.

It is clear that many such sets of finite orthogonal functions can be constructed.

1.5 "Bessel's Inequality" for Finite Orthogonal Functions.

Suppose the set of functions $\phi_0(x)$, $\phi_1(x)$, ..., $\phi_n(x)$, is finite orthogonal over the points $x_0, x_1, \ldots, x_n$, contained in some interval $[a,b]$. Then in the usual way

$$c_k^{(n)} = a_k' = \frac{1}{d_k}\sum_{i=0}^{n} f(x_i)\,\phi_k(x_i)$$

where $d_k = \sum_{i=0}^{n}\phi_k(x_i)^2$ providing $f(x)$ is defined and finite on $[a,b]$.

Theorem 1. $$\sum_{k=0}^{n}(a_k')^2 d_k = \sum_{i=0}^{n}[f(x_i)]^2.$$

In order to prove this theorem, consider the expression

$$\left(f(x) - \sum_{k=0}^{n} a_k'\,\phi_k(x)\right)^2$$

$$= [f(x)]^2 - 2\sum_{k=0}^{n} a_k' f(x)\,\phi_k(x) + \sum_{j=0}^{n}\sum_{k=0}^{n} a_k' a_j'\,\phi_k(x)\,\phi_j(x).$$

Therefore

$$\sum_{i=0}^{n}\left[f(x_i) - \sum_{k=0}^{n} a_k'\,\phi_k(x_i)\right]^2 = \sum_{i=0}^{n}[f(x_i)]^2 - 2\sum_{i=0}^{n}\sum_{k=0}^{n} a_k' f(x_i)\,\phi_k(x_i)$$

$$+\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} a_k' a_j'\,\phi_k(x_i)\,\phi_j(x_i).$$

Interchange summations and sum over i first. Using the finite orthogonality property of $\phi_0(x), \ldots, \phi_n(x)$, we find

$$\sum_{i=0}^{n}\left[f(x_i) - \sum_{k=0}^{n} a'_k \, \phi_k(x_i)\right]^2 = \sum_{i=0}^{n}[f(x_i)]^2 - 2\sum_{k=0}^{n}(a'_k)^2 d_k$$

$$+\sum_{j=0}^{n}\sum_{k=0}^{n} a'_k a_j d_k \delta_{kj} \quad ,$$

$$= \sum_{i=0}^{n}[f(x_i)]^2 - 2\sum_{k=0}^{n}(a'_k)^2 d_k + \sum_{k=0}^{n}(a'_k)^2 d_k$$

$$= \sum_{i=0}^{n}[f(x_i)]^2 - \sum_{k=0}^{n}(a'_k)^2 d_k .$$

Examining the expression on the left, it is seen that it is equal to zero for each i, since the interpolating polynomial is equal to f(x) at the points x_0 , x_1 , ..., x_n . Therefore

1.5.1
$$\sum_{k=0}^{n}(a'_k)^2 d_k = \sum_{i=0}^{n}[f(x_i)]^2$$

This is true for every n. This completes the proof.

If the set $\{\phi_n(x)\}$, n = 0, 1, ..., is for example 1, cos x, sin x, ..., and the points x_i are $x_i = \frac{i\,2\pi}{2n+1}$, then 1.5.1 becomes, with a trivial change in the range of summation,

$$2{a'_0}^2 + \sum_{m=1}^{n}\left({a'_m}^2 + {b'_m}^2\right) = \frac{2}{2n+1}\sum_{i=1}^{2n+1}[f(x_i)]^2$$

This is obviously the point analogue of Bessel's inequality for the Fourier series but with equality occurring instead.

If we consider the functions z^{-n}, ..., 1, ..., z^n, and instead of calculating $\sum_{i=0}^{n}\left[f(x_i) - \sum_{k=0}^{n} a'_k \, \phi_k(x_i)\right]^2$ we calculate $\sum_{j=1}^{2n+1} \left| f(z_j) - \sum_{k=-n}^{+n} b_k' \rho^k e^{ki\theta_j} \right|^2$ where $b_k' = \frac{1}{\rho^k} \frac{1}{2n+1} \sum_{j=1}^{2n+1} e^{-ki\theta_j} f(\rho e^{i\theta_j})$, we obtain in a manner completely analogous to the above

$$\sum_{k=-n}^{+n} |b_k'|^2 \rho^{2k} = \frac{1}{2n+1} \sum_{i=1}^{2n+1} |f(z_i)|^2.$$

This is the point analogue of an equality used in the proof [26, p. 84] of Cauchy's inequality.

Corollary. Suppose $d_k = d_j = \alpha$, all k, j. Then let $f(x) \equiv 1$. We have from 1.5.1

$$\sum_{k=0}^{n} d_k \left(\frac{1}{d_k} \sum_{i=0}^{n} \phi_k(x_i) \right)^2 = n+1,$$

or

1.5.2 $$\sum_{k=0}^{n} \left(\sum_{i=0}^{n} \phi_k(x_i) \right)^2 = \alpha(n+1)$$

1.5.2 is a necessary condition that the functions $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_n(x)$ be finite orthogonal over the points x_0, x_1, x_2, ..., x_n. It should be noted that α depends on n in general. 1.5.2 is a little disappointing in many cases. If the function 1 (or if a constant function c) is one of the functions $\phi_j(x)$, say $\phi_0(x) = 1$, then

$$\sum_{i=0}^{n} \phi_k(x_i)\phi_0(x_i) = \sum_{i=0}^{n} \phi_k(x_i) = 0, \quad k \neq 0,$$

by finite orthogonality. Therefore 1.5.2 reduces to

$$\left[\sum_{i=0}^{n} \phi_0(x_i) \right]^2 = \alpha(n+1)$$

which is trivial, since α must be n + 1.

CHAPTER TWO

FINITE ORTHOGONALITY AND MODIFIED SERIES EXPANSIONS

In this chapter we will still be concerned with the evaluation of a_κ in 1.1.2. However, the expression $c_\kappa^{(n)}$ in 1.1.3 will no longer reduce to a Riemann sum as it has in the other cases. Here, $c_\kappa^{(n)}$ will keep its form as the ratio of two determinants and we will be concerned with proving the existence of $\lim_{n\to\infty} c_\kappa^{(n)}$ and the identification of the limit. Before proceeding further in this direction it is pertinent to review some elementary notions about normal determinants which will be used rather heavily in the remainder of this work. A rather complete discussion of normal determinants will be found in Kowalewski's book, EINFÜHRUNG IN DIE DETERMINANTENTHEORIE [20].

2.1. Normal Determinants.

Assume that the series $s = c_0^0 + c_0^1 + c_1^0 + c_0^2 + c_1^1 + c_2^0 + c_0^3 + \ldots$ converges absolutely. Then the infinite determinant

2.1.1 $$\left| \delta_{ij} + c_i^j \right| \quad , \quad i, j = 0, 1, \ldots ,$$

has meaning and is called a normal determinant. Its value is taken as $\lim_{n\to\infty} D_n$ where D_n is the $(n + 1)x(n + 1)$ determinant in the upper left hand corner of 2.1.1.

Consider the matrix

2.1.2 $$\left\| c_i^j \right\| \quad , \quad i, j = 0, 1, \ldots .$$

We shall consider minors of 2.1.2 and these are formed in the usual way, for example

$$\left| C_{\beta_j}^{\alpha_i} \right| \qquad , \quad i, j = 1, \ldots, n.$$

Theorem 2. The series formed by taking all minors of 2.1.2 converges absolutely.

This is proven by showing that the series of absolute values of all products appearing in all n-rowed minors of 2.1.2 is less than or equal to $\frac{t^n}{n!}$, where

$$t = |c_0^0| + |c_0^1| + |c_1^0| + |c_0^2| + |c_1^1| + |c_2^0| + \cdots .$$

The result follows from the convergence of $\sum_{n=1}^{\infty} \frac{t^n}{n!}$.

Theorem 3. If one column of 2.1.1 is replaced by a bounded number sequence, the determinant still has meaning in the sense that $\lim_{n\to\infty} D_n$ exists.

The proof of this theorem may be found in Kowalewski's book.

Now consider the following set of equations (see [20, p. 383]) in the unknowns x_i :

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots = b_1 ,$$

2.1.3
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots = b_2 ,$$
$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + \ldots = b_3 ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Assume that $A = |a_{ij}|$, $i, j = 1, 2, \ldots,$ is a non-vanishing normal determinant. Assume $|b_n| < M$, $n = 0, 1, \ldots,$ where M is a constant independent of n.

Theorem 4. There is one and only one bounded solution of 2.1.3, namely

$$x_k = \frac{|a_{ij}|(k, b_1)}{|a_{ij}|}, \quad i, j = 1, 2, \ldots; k = 1, 2, \ldots$$

using the notation of paragraph 1.3.

Theorem 5. The product of two normal determinants is a normal determinant.

2.2 Modifications of Fourier Series.

After pointing out a few important theorems concerning normal determinants we are now ready to discuss the modification of the Fourier series. This will be done from the standpoint of interpolation. By a modification of the Fourier series is meant a change in the basic sequence 1, cos x, sin x, ..., cos nx, sin nx, ..., to a new set of functions $1 + \phi_0(x)$, $\cos x + \phi_1(x)$, $\sin x + \phi_2(x)$, ..., $\cos nx + \phi_{2n-1}(x)$, $\sin nx + \phi_{2n}(x)$, ... Here only the evaluation of the coefficients in the new series will be considered. This involves the solution of a typical double-limit problem. The convergence of the given interpolating polynomial to a given function remains to be discussed. However the convergence of the resulting series in a special case will be discussed in the next chapter from a different viewpoint than interpolation. One word of caution is needed. Here and in the next chapter all theorems will be stated under the tacit assumption that the determinant appearing in the denominator of the coefficients is non-vanishing. This assumption is always present except where the question is specifically treated.

Theorem 6. Let f(x) be a continuous function defined on the interval $[0, 2\pi]$ and defined by periodicity (period 2π) over the real line. Further assume that the Fourier series of f(x) converges absolutely in $[0, 2\pi]$. Let f(x) be interpolated on the points $j\left(\frac{2\pi}{2n+1}\right)$, $j = 0, 1, \ldots, 2n$, by the interpolating polynomial

$$a_0(1 + \phi_0(x)) + a_1(\cos x + \phi_1(x)) + b_1(\sin x + \phi_2(x)) + \cdots$$

$$+ a_n(\cos nx + \phi_{2n-1}(x)) + b_n(\sin nx + \phi_{2n}(x)).$$

Also let $\phi_j(x)$ be continuous on $[0, 2\pi]$, periodic of period 2π and let the totality of Fourier coefficients of $\phi_0(x), \phi_1(x), \ldots$ taken in series form converge absolutely. Then $\lim_{n\to\infty} a_K$, $\lim_{n\to\infty} b_K$ exist and their precise values can be stated.

Interpolate f(x) on the 2n + 1 points $j\left(\frac{2\pi}{2n+1}\right)$, $j = 0, 1, \ldots, 2n$, as described in the theorem. We obtain the set of equations

$$a_0(1 + \phi_0(x_j)) + a_1(\cos x_j + \phi_1(x_j)) + b_1(\sin x_j + \phi_2(x_j))$$

$$+ \cdots + a_n(\cos nx_j + \phi_{2n-1}(x_j)) + b_n(\sin nx_j + \phi_{2n}(x_j)) = f(x_j)$$

$j = 0, 1, \ldots, 2n.$

We obtain by Cramer's rule, under the assumption that the determinant of the coefficients does not vanish,

$$
2.2.1 \quad a_k = \frac{\begin{vmatrix} 1+\phi_0(x_0) & \cos x_0 + \phi_1(x_0) & \sin x_0 + \phi_2(x_0) \cdots f(x_0) \cdots \sin n x_0 + \phi_{2n}(x_0) \\ \cdots & \cdots & \cdots \\ 1+\phi_0(x_j) & \cos x_j + \phi_1(x_j) & \sin x_j + \phi_2(x_j) \cdots f(x_j) \cdots \sin n x_j + \phi_{2n}(x_j) \\ \cdots & \cdots & \cdots \\ 1+\phi_0(x_{2n}) & \cos x_{2n} + \phi_1(x_{2n}) & \sin x_{2n} + \phi_2(x_{2n}) \cdots f(x_{2n}) \cdots \sin n x_{2n} + \phi_{2n}(x_{2n}) \end{vmatrix}}{\begin{vmatrix} & \cos kx_0 + \phi_{2k-1}(x_0) & \\ & \vdots & \\ \text{"Same"} & \cos kx_j + \phi_{2k-1}(x_j) & \text{"Same"} \\ & \vdots & \\ & \cos kx_{2n} + \phi_{2k-1}(x_{2n}) & \end{vmatrix}}
$$

$k = 1, 2, \ldots, n$. Analogous expressions are obtained for a_o and b_k, $k = 1, \ldots, n$. The proof given below needs only slight modification for a_o and b_k.

Multiply both numerator and denominator from the left by the transpose of the denominator of a_k with all $\phi_j(x)$ set equal to zero. (This non-zero multiplier can be readily evaluated when it is multiplied by itself and the relations 1.1.4 are used). If the relations 1.1.4 are applied to the resulting expression for a_k and if every row in the numerator and the denominator is divided by $\frac{2n+1}{2}$ except the first rows and those by $2n+1$ we find (all sums run from $j = 0$ to $j = 2n$):

$$
2.2.2 \quad a_k = \frac{\begin{vmatrix} 1+\frac{1}{2n+1}\sum \phi_0(x_j) & \cdots & \sum \frac{f(x_j)}{2n+1} & \cdots & \sum \phi_{2n}(x_j)\frac{1}{2n+1} \\ \sum \phi_0(x_j)\cos x_j \frac{2}{2n+1} & \cdots & \sum f(x_j)\cos x_j \frac{2}{2n+1} & \cdots & \sum \phi_{2n}(x_j)\cos x_j \frac{2}{2n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \sum \phi_0(x_j)\sin nx_j \frac{2}{2n+1} & \cdots & \sum f(x_j)\sin nx_j \frac{2}{2n+1} & \cdots & 1+\sum \phi_{2n}(x_j)\sin nx_j \frac{2}{2n+1} \end{vmatrix}}{\begin{vmatrix} & \sum \phi_{2k-1}(x_j)\frac{1}{2n+1} & \\ & \sum \phi_{2k-1}(x_j)\cos x_j \frac{2}{2n+1} & \\ \text{"Same"} & \vdots & \text{"Same"} \\ & \sum \phi_{2k-1}(x_j)\sin nx_j \frac{2}{2n+1} & \end{vmatrix}}
$$

We wish to prove that $\lim_{n\to\infty} a_k$ exists. First since the $\phi_j(x)$ and $f(x)$ are integrable over $[0, 2\pi]$, each element in the two determinants above approaches a limit. If we temporarily ignore the double limit problem involved we would expect $\lim_{n\to\infty} a_k$ to be

$$
2.2.3 \quad \bar{a}_k = \frac{\begin{vmatrix} 1+\frac{1}{2\pi}\int_0^{2\pi}\phi_0(x)\,dx & \frac{1}{2\pi}\int_0^{2\pi}\phi_1(x)\,dx & \cdots & \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx & \cdots \\ \frac{1}{\pi}\int_0^{2\pi}\phi_0(x)\cos x\,dx & 1+\frac{1}{\pi}\int_0^{2\pi}\phi_1(x)\cos x\,dx & \cdots & \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}}{\begin{vmatrix} & \frac{1}{2\pi}\int_0^{2\pi}\phi_{2k-1}(x)\,dx & \\ \text{"Same"} & \frac{1}{\pi}\int_0^{2\pi}\phi_{2k-1}(x)\cos x\,dx & \text{"Same"} \\ & \vdots & \\ & 1+\frac{1}{\pi}\int_0^{2\pi}\phi_{2k-1}(x)\cos kx\,dx & \\ & \vdots & \end{vmatrix}}
$$

The expression has the form of a ratio of two normal determinants. By hypotheses, the totality of elements in each determinant (excluding the numbers one on the main diagonal) taken in series form converges absolutely. Thus the proposed a_k has meaning.

We have yet to show that $\lim_{n\to\infty} a_k = \bar{a}_k$. For this purpose, let $\bar{P}_{2n+1}$ denote the $(2n+1)x(2n+1)$ principal minor in the upper left hand corner of the denominator of $\bar{a}_k$. We know that $\lim_{n\to\infty} \bar{P}_{2n+1}$ exists and is equal to the denominator of $\bar{a}_k$. Let P_{2n+1} denote the $(2n+1)x(2n+1)$ determinant in the denominator of a_k in 2.2.2. We will prove that $\lim_{n\to\infty} P_{2n+1} = \lim_{n\to\infty} \bar{P}_{2n+1}$. Since the Fourier series of $\phi_i(x)$ converges uniformly to $\phi_i(x)$ we may write

$$\phi_i(x) = a_0^i + a_1^i \cos x + b_1^i \sin x + \ldots$$

where a_k^i is the Fourier cosine coefficient of $\phi_i(x)$ and b_k^i is the Fourier sine coefficient of $\phi_i(x)$. We use the elementary facts concerning finite orthogonality to write P_{2n+1} in the following form:

$$P_{2n+1} = \begin{vmatrix} 1+a_0^0+\frac{1}{2n+1}\sum\limits_{i=0}^{2n}\sum\limits_{k=n+1}^{\infty}(a_k^0\cos kx_i+b_k^0\sin kx_i) & \cdots & a_0^{2n}+\frac{1}{2n+1}\sum\limits_{i=0}^{2n}\sum\limits_{k=n+1}^{\infty}(a_k^{2n}\cos kx_i+b_k^{2n}\sin kx_i) \\ a_1^0+\frac{2}{2n+1}\sum\limits_{i=0}^{2n}\sum\limits_{k=n+1}^{\infty}(a_k^0\cos kx_i+b_k^0\sin kx_i)\cos x_i & \cdots & a_1^{2n}+\frac{2}{2n+1}\sum\limits_{i=0}^{2n}\sum\limits_{k=n+1}^{\infty}(a_k^{2n}\cos kx_i+b_k^{2n}\sin kx_i)\cos x_i \\ \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \\ b_n^0+\frac{2}{2n+1}\sum\limits_{i=0}^{2n}\sum\limits_{k=n+1}^{\infty}(a_k^0\cos kx_i+b_k^0\sin kx_i)\sin nx_i & \cdots & 1+b_n^{2n}+\frac{2}{2n+1}\sum\limits_{i=0}^{2n}\sum\limits_{k=n+1}^{\infty}(a_k^{2n}\cos kx_i+b_k^{2n}\sin kx_i)\sin nx_i \end{vmatrix}.$$

For brevity we will set

$$\overline{P}_{2n+1} = \left|\overline{P}_{ij}\right| , i, j = 0, 1, \ldots 2n , \quad P_{2n+1} = \left|P_{ij}\right|,$$

$i, j = 0, 1, \ldots 2n$. We may also write

$$P_{2n+1} = \left|\overline{P}_{ij} + r_{ij}\right| , \quad i, j = 0, \ldots, 2n,$$

where the r_{ij} may be identified easily in the determinant P_{2n+1} above. Let R_k be a k-rowed minor of $R^{(2n+1)} = \left|r_{ij}\right|$, $i, j = 0, 1, \ldots, 2n$. Let s_k denote the corresponding minor of $\overline{P}_{2n+1}$, and T_k the algebraic complement of s_k. (A complete demonstration of this method, applied to another problem, can be found in Kowalewski's book [20, pp. 388-389]). Then

$$P_{2n+1} = \overline{P}_{2n+1} + \sum_1 R_1 T_{2n} + \cdots + \sum_{2n} R_{2n} T_1 + R_{2n+1}$$

where $\sum_k$ denotes summation over all k-rowed minors of $R^{(2n+1)}$.

This expression for P_{2n+1} is obtained by expressing P_{2n+1} as the sum of 2^{2n+1} determinants and expanding each term by means of the minors involving only r_{ij} terms. Letting $r_{2n+1} = \sum_1 \left|R_1\right|$, we see that

$$\left|P_{2n+1} - \overline{P}_{2n+1}\right| \leqq \sum_1 \left|R_1 \; T_{2n}\right| + \cdots + \sum_{2n} \left|R_{2n} \; T_1\right| + \left|R_{2n+1}\right| ,$$

and

$$\sum_1 \left|R_1\right| \leqq \frac{r_{2n+1}}{1!} , \qquad \sum_2 \left|R_2\right| \leqq \frac{r_{2n+1}^2}{2!}, \ldots, \qquad \left|R_{2n+1}\right| \leqq \frac{r_{2n+1}^{2n+1}}{(2n+1)!}$$

where $r_{2n+1} = \sum_{i,j=0}^{2n} \left|r_{ij}\right|$. This last statement can be verified by considering r_{2n+1}^k. Each term in $\sum_k \left|R_k\right|$ appears in the expansion of r_{2n+1}^{k} k! times. Now as $n \to \infty$, the T_i are minors of a normal determinant. Therefore they are uniformly bounded [20, p. 375]. Let P be this bound. Then

$$\left| P_{2n+1} - \overline{P}_{2n+1} \right| \leq P\left(\frac{r_{2n+1}}{1!} + \frac{r_{2n+1}^{2}}{2!} + \cdots + \frac{r_{2n+1}^{2n+1}}{(2n+1)!} + \cdots \right) = P\left(e^{r_{2n+1}} - 1\right)$$

Therefore if we prove that $\lim_{n\to\infty} r_{2n+1} = 0$, we will have shown the existence of the $\lim_{n\to\infty} P_{2n+1}$ and that it equals $\lim_{n\to\infty} \overline{P}_{2n+1}$. However

$$r_{2n+1} = \sum_{1} |R_1| = \sum_{i,j=0}^{2n} |r_{ij}| = \sum_{s=0}^{2n} \left\{ \frac{1}{2n+1} \left| \sum_{i=0}^{2n} \sum_{k=n+1}^{\infty} \left(a_k^s \cos kx_i + b_k^s \sin kx_i \right) \right| \right.$$

$$+ \frac{2}{2n+2} \left| \sum_{i=0}^{2n} \sum_{k=n+1}^{\infty} \left(a_k^s \cos kx_i + b_k^s \sin kx_i \right) \cos x_i \right| + \cdots$$

$$\left. + \frac{2}{2n+1} \left| \sum_{i=0}^{2n} \sum_{k=n+1}^{\infty} \left(a_k^s \cos kx_i + b_k^s \sin kx_i \right) \sin nx_i \right| \right\} .$$

Lemma:

a) $\sum_{i=0}^{2n} \cos kx_i \cos rx_i = 0$, $k+r \not\equiv 0(2n+1)$ and $k-r \not\equiv 0(2n+1)$,

$= \frac{2n+1}{2}$, $k+r \equiv 0(2n+1)$ and $k-r \not\equiv 0(2n+1)$,

, or $k+r \not\equiv 0(2n+1)$ and $k-r \equiv 0(2n+1)$,

$= 2n+1$, $k+r \equiv 0(2n+1)$ and $k-r \equiv 0(2n+1)$,

2.2.4

b) $\sum_{i=0}^{2n} \cos kx_i \sin rx_i = 0$; all r and k ;

c) $\sum_{i=0}^{2n} \sin kx_i \sin rx_i = 0$, $k+r \not\equiv 0(2n+1)$ and $k-r \not\equiv 0(2n+1)$,

$= -\left(\frac{2n+1}{2}\right)$ $k+r \equiv 0(2n+1)$ and $k-r \not\equiv 0(2n+1)$,

$$= \frac{2n+1}{2} \quad , \; k-r \equiv 0(2n+1) \text{ and } k+r \not\equiv 0(2n+1),$$

$$= 0 \qquad , \; k+r \equiv 0(2n+1) \text{ and } k-r \equiv 0(2n+1).$$

Proof. We shall prove only the statement given in a). b) follows from the fact that the set of points $j\,\frac{2\pi}{2n+1}$, $j = 0, 1, \ldots, 2n$ are equally spaced about π while the sine and cosine functions are odd and even about π respectively. The proof of c) is analogous to a). Now

$$2.2.5 \quad \sum_{i=0}^{2n} \cos kx_i \cos rx_i = \frac{1}{2}\left\{ \sum_{i=0}^{2n} \cos(k+r)x_i + \sum_{i=0}^{2n} \cos(k-r)x_i \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{2} + \frac{\sin\left(2n+\frac{1}{2}\right)\frac{(k+r)2\pi}{2n+1}}{2\sin\frac{(k+r)2\pi}{2(2n+1)}} + \frac{1}{2} + \frac{\sin\left(2n+\frac{1}{2}\right)\frac{(k-r)2\pi}{2n+1}}{2\sin\frac{(k-r)2\pi}{2(2n+1)}} \right\},$$

substituting the proper value for x_i and using the formula for the sum of a cosine series. Now we can write

$$\left(2n + \frac{1}{2}\right)\frac{(k \pm r)2\pi}{2n+1} = (k \pm r)2\pi - \frac{1}{2}\,\frac{(k \pm r)2\pi}{2n+1}$$

Then clearly

$$\sum_{i=0}^{2n} \cos kx_i \cos rx_i = \frac{1}{2}\left\{ 1 - \frac{\sin\frac{1}{2}\frac{(k+r)2\pi}{2n+1}}{2\sin\frac{(k+r)2\pi}{2(2n+1)}} - \frac{\sin\frac{(k-r)2\pi}{2(2n+1)}}{2\sin\frac{(k-r)2\pi}{2(2n+1)}} \right\}.$$

The first case of a) follows immediately. For the other three cases one or both of the sums in 2.2.5 can be evaluated immediately while the treatment given above is applied to the remaining sum.

The relations 2.2.4 are not new. They can be found, for example in Tonelli's book [27, p. 150] .

Applying the lemma to r_{2n+1} after a change in summation, we see that

$$r_{2n+1} = \sum_{s=0}^{2n} \Big\{ \Big[a^s_{2n+1} + a^s_{2(2n+1)} + \cdots + a^s_{\lambda(2n+1)} + \cdots \Big] + \Big[a^s_{2n} + a^s_{4n+1} + a^s_{6n+2} + \cdots + a^s_{\lambda(2n+1)-1} + \cdots + a^s_{2n+2} + a^s_{4n+3} + a^s_{6n+4} + \cdots + a^s_{\lambda(2n+1)+1} + \cdots \Big] + \cdots +$$

$$\Big[a^s_{n+1} + a^s_{3n+2} + a^s_{5n+3} + \cdots + a^s_{\lambda(2n+1)-n} + \cdots + a^s_{3n+1} + a^s_{5n+2} + a^s_{7n+3} + \cdots + a^s_{\lambda(2n+1)+n} \cdots \Big]$$

$$+ \Big[-\big(b^s_{2n} + b^s_{4n+1} + b^s_{6n+2} + \cdots + b^s_{\lambda(2n+1)-1} + \cdots \big) + \big(b^s_{2n+2} + b^s_{4n+3} + b^s_{6n+4} + \cdots + b^s_{\lambda(2n+1)+1} + \cdots \big) \Big]$$

$$+ \Big[-\big(b^s_{2n-1} + b^s_{4n} + b^s_{6n+1} + \cdots + b^s_{\lambda(2n+1)-2} + \cdots \big) + \big(b^s_{2n+3} + b^s_{4n+4} + b^s_{6n+5} + \cdots + b^s_{\lambda(2n+1)+2} + \cdots \big) \Big]$$

$$+ \cdots + \Big[-\big(b^s_{n+1} + b^s_{3n+2} + b^s_{5n+3} + \cdots + b^s_{\lambda(2n+1)-n} \big) + \big(b^s_{3n+1} + b^s_{5n+2} + b^s_{7n+3} + \cdots + b^s_{\lambda(2n+1)+n} + \cdots \big) \Big] \Big\} ,$$

λ a positive integer.

Consider the set of b^s_j first. Clearly each b^s_j from b^s_{n+1}, b^s_{n+2}, ... onward appears once and only once except for the $b^s_{\lambda(2n+1)}$ which do not appear in this set. Consider the set of a^s_j . Clearly each a^s_j in the sequence a^s_{n+1}, a^s_{n+2}, ... appears once and only once.

Therefore it is seen that r_{2n+1} is a series of elements taken from a subset of the set of Fourier coefficients of $\phi_i(x)$, $i = 0, 1,$

..., 2n. The series formed by adding all elements of the set of Fourier coefficients of $\phi_i(x)$, $i = 0, 1, \ldots$ was assumed to converge absolutely. This series might be written in block fashion as

$$\begin{array}{l} a_0^0 + a_1^0 + b_1^0 + a_2^0 + b_2^0 + \ldots + a_n^0 + b_n^0 + \ldots \\ + a_0^1 + a_1^1 + b_1^1 + a_2^1 + b_2^1 + \ldots + a_n^1 + b_n^1 + \ldots \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + a_0^{2n} + a_1^{2n} + b_1^{2n} + a_2^{2n} + b_2^{2n} + \ldots + a_n^{2n} + b_n^{2n} + \ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

It may be written in conventional series form by starting at the upper left hand corner and proceeding as

$$a_0^0 + a_0^1 + a_1^1 + a_1^0 + b_1^0 + b_1^1 + b_1^2 + a_1^2 + a_0^2 + \ldots .$$

It is clear that for an arbitrary $\epsilon > 0$ we can find an N: for $2n > N$ all terms taken in absolute value in the above series appearing outside of the above box have a sum less than ϵ. But r_{2n+1} for these n is less than or equal to the sum of a subseries of this series. Hence $r_{2n+1} < \epsilon$. Since ϵ is arbitrary, we have $\lim_{n\to\infty} r_{2n+1} = 0$.

Therefore it has been proven that $\lim_{n\to\infty} P_n = \lim_{n\to\infty} \overline{P}_n = \overline{a}_k^{(D}$ where the superscript D denotes the denominator of $\overline{a}_k$. The proof for the numerator of $\overline{a}_k$ is analogous if the Fourier coefficients of f(x) converge absolutely.

The convergence of the resulting series will be discussed in the next chapter.

2.3 A General Theorem on Modifications.

The theorem in the preceding paragraph suggests a possible generalization. To obtain this generalization an extended definition of finite biorthogonality will now be given.

Definition: Let $\{\phi_i(x)\}$, $i = 0, 1, \ldots$ be defined on an interval $[a, b]$ of the real line or on a region R of the complex plane. Let $x_0^{(n)}, x_1^{(n)}, \ldots, x_n^{(n)}$ be a certain set of points in the interval $[a, b]$ (or in R). Let $\{\Theta_i(x)\}$, $i = 0, 1, \ldots$ be another sequence of functions defined in $[a, b]$ such that

1) $\left| \Theta_s(x_i^{(n)}) \right| \leq M_s$, $i = 0, 1, \ldots, n.$

2) $\sum_{i=0}^{n} \phi_k(x_i)\Theta_r(x_i) = c_{nk}\delta_{kr}$, $c_{nk} \neq 0$, $k, r = 0, 1, \ldots, n.$

3) $\sum_{i=0}^{n} \phi_k(x_i)\,\Theta_r(x_i) = 0$, $k = n+1, n+2, \ldots$, $r = 0, 1, \ldots, n,$

except for $k = \beta_1^{(r)}, \beta_2^{(r)}, \ldots$. Here $\beta_1^{(0)}, \beta_2^{(0)}, \ldots, \beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_1^{(n)}, \beta_2^{(n)}, \ldots$ is a set of integers greater than or equal to $n + 1$ in which each integer appears at most M times, M is independent of n and is the same finite number for each integer in question.

4) $\sum_{i=0}^{n} \phi_k(x_i)\Theta_r(x_i) = c_{nkr} \neq 0$ for the $\beta_i^{(j)}$ set.

5) $\left| \dfrac{c_{n,\beta_j^{(l)},l}}{c_{nl}} \right| \leq P$, P finite $j = 0, 1, \ldots, l = 1, \ldots, n$ all n.

Then we say that the two sequences $\{\phi_i(x)\}$ and $\{\Theta_i(x)\}$ are finite biorthogonal over $\{x_j^{(n)}\}$, $j = 0, 1, \ldots, n$, all n.

Example 1. Let $\{x_j^{(n)}\} = \dfrac{j\,2\pi}{2n+1}$. Let $\{\phi_i(x)\}$ and $\{\Theta_i(x)\}$

be the sequence 1, cos x, sin x, cos 2x, sin 2x, ... Then 1) is true where $M_s = 1$, all s and n. 2) and 3) are true by the previous theorem where $c_{nk} = \frac{2n+1}{2}$, $k = 1, \ldots, 2n$. $c_{n0} = 2n + 1$; $M = 1$. 4) is clearly true as a consequence of 3). 5) is true where we take $P = 2$, since by 2.2.4 $c_{n,\beta_j^{(1)},1}$ is $2n+1$ or $\pm\frac{2n+1}{2}$. Since the two sequences of functions are identical in this case we call them finite orthogonal.

Example 2. Let $\{x_j^n\} = \rho e^{ij\frac{2\pi}{2n+1}}$. Let $\{\phi_i(z)\} = 1$, $z, z^{-1}, z^2, z^{-2}, \ldots$. Let $\{\Theta_i(z)\} = 1, z^{-1}, z^1, z^{-2}, z^2, \ldots$. Then 1) is true taking M_s to be $\rho^{(-1)^s\left[\frac{s+1}{2}\right]}$, where the bracket means the greatest integer contained in $\frac{s+1}{2}$. 2) is true by 1.3.2. To show that 3) is true use 1.3.2. Then $\sum_{j=0}^{2n} e^{ki\theta_j} e^{li\theta_j} = 0$ except for $k = -l + \lambda(2n+1)$, λ is each and every integer in turn, $l = -n, \ldots, 1, \ldots, n$. We can take $M = 1$. 4) is a consequence of 3). 5) is true with $P = 1$.

It is possible to give other examples, namely the functions in 1.4 with some obvious restrictions imposed, and the non-negative powers of z.

Theorem 7. Let f(x) be a function able to be expanded in an absolutely convergent $\phi_j(x)$-series in $[a, b]$. Assume that the set of series coefficients taken in series form converges absolutely. Let f(x) be interpolated on the points $x_0, x_1, \ldots, x_n$ above by the interpolating polynomial

$$a_0(\phi_0(x) + \psi_0(x)) + a_1(\phi_1(x) + \psi_1(x)) + \cdots + a_n(\phi_n(x) + \psi_n(x))$$

Assume that the determinant of the coefficients is non-vanishing for all n. Further, require that the $\psi_j(x)$ are able to be expanded in absolutely convergent $\phi_j(x)$-series and that the totality of series coefficients taken in series form converges absolutely. Then $\lim_{n\to\infty} a_k$ exists and its precise value can be stated. The notation of the theorem was chosen in accordance with the definition given at the first of the paragraph.

Interpolate f(x) as described above. We obtain the set of equations:

$$a_0(\phi_0(x_i)+\psi_0(x_i)) + a_1(\phi_1(x_i)+\psi_1(x_i))+\cdots+ a_n(\phi_n(x_i)+\psi_n(x_i)) = f(x_i)$$

$i = 0, 1, \ldots, n.$

By assumption the determinant of the coefficients is not equal to zero, then

$$a_k^{(n)} = \frac{|\phi_j(x_i)+\psi_j(x_i)|(k, f(x_i))}{|\phi_j(x_i)+\psi_j(x_i)|}, \quad i, j = 0, 1, \ldots, n,$$

$k = 0, 1, \ldots, n.$

Multiply both numerator and denominator by $|\theta_i(x_j)|$, $i, j = 0, 1, \ldots, n$, which is not equal to zero, (multiply it by $|\phi_i(x_j)|$ $i, j = 0, 1, \ldots, n$ and use orthogonality relations). Now divide the i-th row in numerator and denominator by c_{ni} to obtain

$$a_k^{(n)} = \frac{\left|\delta_{lj} + \sum_{i=0}^{n} \frac{\theta_l(x_i)\psi_j(x_i)}{c_{nl}}\right|\left(k, \sum_{i=0}^{n}\theta_l(x_i)f(x_i)\right)}{\left|\delta_{lj} + \sum_{i=0}^{n} \frac{\theta_l(x_i)\psi_j(x_i)}{c_{nl}}\right|}, \quad l, j = 0, 1, \cdots, n,$$

$k = 0, 1, \cdots, n.$

Now by hypotheses $\psi_1(x) = \sum_{j=0}^{\infty} \alpha_j^1 \phi_j(x)$ where $\sum_{1,j} |\alpha_j^1| < \infty$.

Then for $0 \leqq s \leqq n$,

$$\sum_{i=0}^{n} \theta_s(x_i)\psi_1(x_i) = \sum_{i=0}^{n} \theta_s(x_i) \sum_{j=0}^{\infty} \alpha_j^1 \phi_j(x_i)$$

$$= \sum_{i=0}^{n} \theta_s(x_i) \sum_{j=0}^{n} \alpha_j^1 \phi_j(x_i) + \sum_{i=0}^{n} \theta_s(x_i) \sum_{j=n+1}^{\infty} \alpha_j^1 \phi_j(x_i)$$

$$= C_{ns}\alpha_s^1 + \sum_{i=0}^{n} \theta_s(x_i) \sum_{j=n+1}^{\infty} \alpha_j^1 \phi_j(x_i) ,$$

using the absolute convergence of the series and the fact that $|\theta_s(x_i)| < M_s$, all i. Using this relation, it is possible to write

$$a_k^{(n)} = \frac{|\delta_s^j + \alpha_s^j + r_s^j| (k, \alpha_s^f + r_s^f)}{|\delta_s^j + \alpha_s^j + r_s^j|}, \quad s, j = 0, 1, \ldots, n,$$

$k = 0, 1, \ldots, n$.

Here $r_s^j = \dfrac{\sum_{i=0}^{n} \theta_s(x_i) \sum_{h=n+1}^{\infty} \alpha_h^j \phi_h(x_i)}{C_{ns}}$. We wish to prove that

$$\lim_{n\to\infty} a_k^{(n)} = \frac{|\delta_s^1 + \alpha_s^1| (k, \alpha_s^f)}{|\delta_s^1 + \alpha_s^1|}, \quad s, 1 = 0, 1, \ldots,$$

all k. To do this we proceed as in 2.2. Consider the denominator. (The proof is analogous for the numerator.) Using the notation of 2.2,

$$r_n = \sum_{s=0}^{n} \sum_{t=0}^{n} \left(\left| \frac{\sum_{i=0}^{n} \theta_t(x_i) \sum_{j=n+1}^{\infty} \alpha_j^s \phi_j(x_i)}{C_{nt}} \right| \right).$$

Now by the hypothesis, $|\,\Theta_t(x_i)\,| < M_t$, $i = 1, \ldots, n$. Since $\sum_{j=n+1}^{\infty} \alpha_j^s \;\phi_j(x_i)$ converges absolutely for each i, interchange summations and obtain

$$r_n = \sum_{s=0}^{n} \sum_{t=0}^{n} \left(\left| \frac{\sum_{j=n+1}^{\infty} \alpha_j^s \sum_{i=0}^{n} \Theta_t(x_i)\,\phi_j(x_i)}{C_{nt}} \right| \right) ,$$

$$= \sum_{s=0}^{n} \sum_{t=0}^{n} \left| \frac{\sum_{j=n+1}^{\infty} \alpha^s_{\beta_j^{(t)}} \; C_{n\beta_j^{(t)}t}}{C_{nt}} \right| ,$$

where $\beta_j^{(t)} \geqq n+1$. Now by hypothesis $\left| \frac{C_{n\beta_j^{(t)}t}}{C_{nt}} \right| < P$, $j = 0, 1, \ldots, t = 1, \ldots, n$. Then

$$r_n \leqq P \sum_{s=0}^{n} \sum_{t=0}^{n} \sum_{j=n+1}^{\infty} \left| \alpha^s_{\beta_j^{(t)}} \right|$$

As the t-summation is performed, any integer greater than $n + 1$ appears in the sequence $\beta_0^{(0)}, \beta_1^{(0)}, \ldots, \beta_0^{(1)}, \beta_1^{(1)}, \ldots, \beta_0^{(i)}, \beta_1^{(i)}, \ldots$ at most M times. Then

$$r_n \leqq PM \sum_{s=0}^{n} \sum_{j=n+1}^{\infty} |\,\alpha_j^s\,| .$$

But by hypothesis $\sum_{s,j=0}^{\infty} |\,\alpha_j^s\,| < \infty$. Hence $\lim_{n\to\infty} \sum_{s=0}^{n} \sum_{j=n+1}^{\infty} |\alpha_j^s| = 0$. Therefore $\lim_{n\to\infty} r_n = 0$. The theorem is proven.

CHAPTER THREE

MODIFIED SERIES EXPANSIONS

The question of modifying some well-known sequences of functions used in expansions will now be discussed. In contrast to the last chapter, interpolation will not be used, but the results here are suggested by the interpolation procedures already discussed. Indeed, if the results in this chapter are compared with those of the last, it is seen that they are more general. The property of finite orthogonality that was so essential in the proofs of the last chapter is here abandoned. On the other hand, the sequences considered in this chapter have not in general been brought under the general theory indicated in Chapter One.

Theorem 8 and its generalizations are the main results. Theorem 8 itself is quite general as is pointed out in the applications. For any special sequence of functions, the hypotheses are rather severe, but the advantage lies in the generality.

It should be again noted that the assumption that the determinant of the coefficients is non-vanishing is ever present, except when it is stated to be lacking.

3.1 An Abstract Modification Theorem.

Theorem 8. Let $P_j(x)$, $j = 0, 1, \ldots$, denote a sequence of functions defined for $a \leqq x \leqq b$. Let O_i, $i = 0, 1, 2, \ldots$, denote a set of additive, homogeneous functionals such that $O_i[P_j(x)] = \delta_{ij}$, $i = 0, 1, \ldots, j = 0, 1, \ldots$. Let (R) denote the space on which

Θ_i , $i = 0, 1, \ldots,$ is defined. Suppose $f(x) \in (R)$ and can be expanded in a series of the $P_j(x)$: $f(x) = \sum_{n=0}^{\infty} d_n P_n(x)$, where $d_n = O_n[f(x)]$ and $|d_n| < M$, $n = 0, 1, \ldots,$ and the series converges absolutely. Consider another sequence of functions $\phi_n(x)$ such that $\phi_n(x) \in (R)$ and $\phi_n(x) = \sum_{i=0}^{\infty} c_i^n P_i(x)$ in $[a, b]$, $n = 0, 1, \ldots,$ where $c_i^n = O_i[\phi_n(x)]$ and $\sum_{i,n} |c_i^n| < \infty$, and the series $\sum_{\substack{i=0 \\ n=0}}^{\infty} c_i^n P_i(x)$ converges absolutely in $[a, b]$. Then $f(x)$ can be expanded in a series of the functions $P_n(x) + \phi_n(x)$. That is,

$$3.1.1 \qquad f(x) = \sum_{n=0}^{\infty} a_n \left(P_n(x) + \phi_n(x)\right) .$$

The series on the right of 3.1.1 converges absolutely to $f(x)$. Here

$$3.1.2 \qquad a_n = \frac{|\delta_{ij} + c_i^j|\,(n, O_k(f))}{|\delta_{ij} + c_i^j|} \quad , \; i, j = 0, 1, \ldots .$$

Here $|\delta_{ij} + c_i^j|(n, O_k(f))$ indicates that the column with elements $\delta_{in} + c_i^n$ is replaced by the column of elements $O_k[f(x)]$, $k = 0, 1, \ldots$.

Remark 1. In the above and ensuing, x may be a complex variable. All functions will be then functions of a complex variable. In this event, replace $[a, b]$ by some region in the complex plane.

Remark 2. It is possible to alter the hypotheses of this theorem slightly. If $|P_j(x)| < M$, $j = 0, 1, \ldots,$ where M is a constant, we may of course drop the hypothesis: $\sum_{\substack{i=0 \\ n=0}}^{\infty} c_i^n P_i(x)$ converges absolutely.

Consider the expression 3.1.1. If the series on the right converges in such a way that it is susceptible to O_i operation we have

$$O_i\left[f(x)\right] = \sum_{n=0}^{\infty} a_n \; O_i\left[P_n(x) + \phi_n(x)\right] \quad , \; i = 0, 1, \ldots .$$

But since $O_i\left[P_n(x)\right] = \delta_{in}$ and $O_i\left[P_n(x) + \phi_n(x)\right] = O_i\left[P_n(x)\right] + O_i\left[\phi_n(x)\right]$, then

$$\begin{aligned} O_0\left[f(x)\right] &= a_0 + \sum_{n=0}^{\infty} a_n \; O_0\left[\phi_n(x)\right] , \\ O_1\left[f(x)\right] &= a_1 + \sum_{n=0}^{\infty} a_n \; O_1\left[\phi_n(x)\right] , \\ O_2\left[f(x)\right] &= a_2 + \sum_{n=0}^{\infty} a_n \; O_2\left[\phi_n(x)\right] , \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

By hypothesis, since $O_i\left[\phi_n(x)\right] = c_i^n$, the above set of equations becomes

$$\begin{aligned} O_0\left[f(x)\right] &= a_0 + \sum_{n=0}^{\infty} a_n \, c_0^n , \\ O_1\left[f(x)\right] &= a_1 + \sum_{n=0}^{\infty} a_n \, c_1^n , \\ O_2\left[f(x)\right] &= a_2 + \sum_{n=0}^{\infty} a_n \, c_2^n , \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{3.1.3}$$

3.1.3 consists of an infinite number of equations in an infinite number of unknowns $a_0, a_1, \ldots$. Since $\left|O_n(f(x))\right| < M$, $n = 0, 1, \ldots$ and $\sum_{i,n=0}^{\infty} \left|c_i^n\right| < \infty$, we can solve 3.1.3 by the obvious extension of Cramer's rule [see Theorem 4 in 2.1] and we obtain a unique bounded solution: a_n, $n = 0, 1, \ldots$ as given in 3.1.2.

The permanent assumption, $\left|\delta_{ij} + c_i^j\right| \neq 0$, was made in accordance with previous remarks, and was used above.

From the hypotheses, this is the ratio of a determinant, that is a normal determinant with one column replaced by a bounded number sequence, to a normal determinant. a_n therefore has meaning [see Theorem 3 of 2.1].

There is however no a *priori* reason for assuming the convergence of the series in 3.1.1 so that the process used above is of questionable validity. Therefore consider the system of equations 3.1.3 outright. a_n, $n = 0, 1, \ldots$ form the one and only bounded solution of this system. We can transform 3.1.3 into

3.1.4 $$0_j[f(x)]\, P_j(x) = a_j P_j(x) + \sum_{n=0}^{\infty} a_n c_j^n P_j(x), \quad j = 0, 1, \ldots$$

It is desired to add these equations. We notice that

$$\sum_{k=0}^{\infty} \left| a_k P_k(x) \right| = \sum_{k=0}^{\infty} \left| 0_k[f(x)]\, P_k(x) - \sum_{n=0}^{\infty} (a_n c_k^n P_k(x)) \right|$$

$$\leqq \sum_{k=0}^{\infty} \left| 0_k[f(x)]\, P_k(x) \right| + \sum_{\substack{k=0 \\ n=0}}^{\infty} \left| a_n c_k^n P_k(x) \right| .$$

Now, since the first series on the right converges by hypotheses, and $\{a_n\}$, $n = 0, 1, \ldots$, forms a bounded number sequence, and $\sum_{\substack{k=0 \\ n=0}}^{\infty} |c_k^n P_k(x)|$ converges, then we have $\sum_{k=0}^{\infty} \left| a_k P_k(x) \right| < \infty$.

It is easily seen that the same statement is true under the altered hypotheses mentioned after the statement of the theorem.

In either case, the equations 3.1.4 may be added and become

$$\sum_{n=0}^{\infty} d_n P_n(x) = \sum_{n=0}^{\infty} a_n P_n(x) + \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k c_n^k P_n(x) .$$

Since the series on the right converges absolutely, interchange summation

$$\sum_{n=0}^{\infty} d_n P_n(x) = \sum_{n=0}^{\infty} a_n P_n(x) + \sum_{k=0}^{\infty} a_k \sum_{n=0}^{\infty} c_n^k P_n(x)$$

$$= \sum_{n=0}^{\infty} a_n P_n(x) + \sum_{K=0}^{\infty} a_K \phi_K(x)$$

Thus

$$f(x) = \sum_{n=0}^{\infty} d_n P_n(x) = \sum_{n=0}^{\infty} a_n\big(P_n(x) + \phi_n(x)\big) .$$

The conclusion is then that the series $\sum_{n=0}^{\infty} a_n(P_n(x) + \phi_n(x))$, using the stated value of a_n converges absolutely to f(x). If this series converges to f(x) in such a manner that it yields to an O_i - operation, then the coefficients a_n can be obtained by a term by term operation on 3.1.1.

In Theorem 8, if we omit all hypotheses concerning operators and consider only a set of numbers d_n, $n = 0, 1, \ldots$, a certain generalization can be made.

Theorem 9. Let $f(x) = \sum_{n=0}^{\infty} d_n P_n(x)$ where $|d_n| < M$, $n = 0, 1, \ldots$, and the series for f(x) converges absolutely. Consider the sequence of functions $\phi_n(x) = \sum_{i=0}^{\infty} c_i^n P_i(x)$ in $[a, b]$ for $n = 0, 1, \ldots$, where $\sum_{i,n} |c_i^n| < \infty$ and the series $\sum_{\substack{i=0 \\ n=0}}^{\infty} c_i^n P_i(x)$ converges absolutely in $[a, b]$. Then f(x) has an expansion:
$f(x) = \sum_{n=0}^{\infty} a_n\big(P_n(x) + \phi_n(x)\big)$, which converges absolutely and where the appropriate modification is made in the a_n of 3.1.2.

This may be proven in a fashion similar to Theorem 8.

Remark 1. In Theorem 9, x can be taken as a complex variable. All functions are then functions of a complex variable. Replace $[a, b]$ by a region in the complex plane.

Remark 2. If $|P_i(x)| < M$, $i = 0, 1, \ldots$, then the hypothesis that $\sum_{i,n=0}^{\infty} c_i^n P_i(x)$ converges absolutely may be removed.

3.2 The Group Property Applied to Modification.

If it is desired to modify the basic sequence $P_0(x)$, $P_1(x)$, $\ldots$, by two sequences of functions, that is to $P_0(x) + \phi_0(x) + \Theta_0(x)$, $P_1(x) + \phi_1(x) + \Theta_1(x)$, $\ldots$, there seem to be two ways to proceed. The first method is clear.

In Theorem 8 keep all hypotheses on the functions $\{P_j(x)\}$ and $\{\phi_j(x)\}$. Impose on the new sequence $\{\phi_n(x)\}$ the restrictions:

1. $\Theta_n(x) \in (R)$ and $\Theta_n(x) = \sum_{i=0}^{\infty} b_i^n P_i(x)$ in $[a, b]$, $n = 0, 1, \ldots$.

2. $b_i^n = 0_i[\Theta_n(x)]$

3. $\sum_{i,n=0}^{\infty} |b_i^n| < \infty$

4. $\sum_{i,n=0}^{\infty} b_i^n P_i(x)$ converges absolutely in $[a, b]$.

Then the desired modification can be made. Thus we must check the hypotheses of Theorem 8 for the functions $\{\phi_n(x) + \Theta_n(x)\}$, $n = 0, 1, \ldots$.

1. $\phi_n(x) + \Theta_n(x) \in (R)$, since $\phi_n(x), \Theta_n(x) \in (R)$ and 0_i is additive, for all i, n. Also

$$\phi_n(x) + \Theta_n(x) = \sum_{i=0}^{\infty} c_i^n P_i(x) + \sum_{i=0}^{\infty} b_i^n P_i(x).$$

Since both series converge absolutely, they may be added term by term. Thus $\phi_n(x) + \Theta_n(x) = \sum_{i=0}^{\infty} (c_i^n + b_i^n) P_i(x)$.

2. $c_i^n + b_i^n = O_i\left[\phi_n(x)\right] + O_i\left[\theta_n(x)\right] = O_i\left[\phi_n(x) + \theta_n(x)\right]$.

3. $$\sum_{i,n=0}^{\infty} \left|c_i^n + b_i^n\right| \leqq \sum_{i,n} |c_i^n| + \sum_{i,n} |b_i^n| < \infty .$$

4. $$\sum_{i,n=0}^{\infty} \left|b_i^n + c_i^n\right| \; \left|P_i(x)\right| \leqq \sum_{i,n=0}^{\infty} |b_i^n| \, |P_i(x)| + \sum_{\substack{i=0\\n=0}}^{\infty} |c_i^n| \, |P_i(x)| < \infty .$$

Therefore all hypotheses are checked for the functions $\phi_n(x) + \theta_n(x)$. Then by Theorem 8, we can obtain an absolutely convergent expansion for $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n' \left[P_n(x) + \phi_n(x) + \theta_n(x)\right]$$

where

$$3.2.1 \qquad a_n' = \frac{\left|\delta_{ij} + c_i^j + b_i^j\right| (n, O_i[f])}{\left|\delta_{ij} + c_i^j + b_i^j\right|} \quad , \; i, j = 0, 1, \ldots .$$

$n = 0, 1, \ldots$. Therefore the first type of two-step modification is clear. It is apparent that the result of making a two-step modification is completely equivalent to making a single one-step modification.

The second type is just slightly more subtle. It is interesting to make the following transformation in Theorem 8. Let $P_n(x)$ transform into $P_n(x) + \phi_n(x)$. Let O_n transform into a_n, considered as an operator. This last statement requires some explanation since a_n; $n = 0, 1, \ldots$, as given in 3.1.2, are simply numbers. But if we consider a_n with $f(x)$ removed from the $(n+1)$st column of the determinant in the numerator, it is clear that we now have an operator. It is this operator which is meant in the above statement. We wish to check the hypotheses of Theorem 8 using the above transformation.

First, $P_j(x) + \phi_j(x)$, $j = 0, 1, \ldots$ denotes a sequence of functions defined for $a \leqq x \leqq b$. Next, a_i, $i = 0, 1, 2, \ldots$ is obviously an additive and homogeneous functional. It is also true that $a_i[P_j(x) + \phi_j(x)] = \delta_{ij}$, $i = 0, 1, \ldots, j = 0, 1, \ldots$. To see this, replace $f(x)$ in the $(n+1)$st column of a_n in 3.1.2 by $P_j(x) + \phi_j(x)$. We remember that $O_i[P_j(x) + \phi_j(x)] = \delta_{ij} + c_i^j$. Now if $j = n$, the determinants in the numerator and denominator are identical. If $j \neq n$, then in the determinant in the numerator, two columns are identical. Hence $a_i[P_j(x) + \phi_j(x)] = \delta_{ij}$.

Now, by Theorem 8 we already know that

$$f(x) = \sum_{n=0}^{\infty} a_n[f(x)]\,[P_n(x) + \phi_n(x)]$$

where the convergence is absolute. As a further result, we know that there exists a number $M : |a_n[f(x)]| < M$, $n = 0, 1, \ldots$. Then if $\zeta_n(x) = \sum_{i=0}^{\infty} c_i'^n \left(P_i(x) + \phi_i(x)\right)$ where $c_i'^n = a_i[\zeta_n(x)]$ and $\sum_{i,n=0}^{\infty} |c_i'^n| < \infty$, and $\sum_{i,n=0}^{\infty} |c_i'^n|\,|P_i(x) + \phi_i(x)| < \infty$, all x, we may apply Theorem 8 again to obtain

$$3.2.2 \qquad f(x) = \sum_{n=0}^{\infty} b_n\left(P_n(x) + \phi_n(x) + \zeta_n(x)\right)$$

where

$$3.2.3 \qquad b_n = \frac{|\delta_{ij} + c_i'^j|\,(n, a_k[f(x)])}{|\delta_{ij} + c_i'^j|}, \quad i, j = 0, 1, \ldots$$

$n = 0, 1, \ldots$. Here $c_i'^j$ and a_i are infinite determinants for all i, j. Moreover the series in 3.2.2 converges absolutely. This process may be kept up indefinitely but there is hardly any point in doing this, as will be seen shortly.

Theorem 10. If it is known that

$$\text{A)}\quad f(x) = \sum_{n=0}^{\infty} d_n P_n(x)$$

with the same hypotheses as in Theorem 8, and as before that $\phi_n(x) = \sum_{i=0}^{\infty} c_i^n P_i(x)$ with the same hypotheses as in Theorem 8, then

$$\text{B)}\quad f(x) = \sum_{n=0}^{\infty} a_n\left(P_n(x) + \phi_n(x)\right)$$

where the $\{a_n\}$, $n = 0, 1, \ldots$ are given in Theorem 8. Then if $\xi_n(x) = \sum_{i=0}^{\infty} c_i'^n \left(P_i(x) + \phi_i(x)\right)$ where $\sum_{i,n} |c_i'^n| < \infty$ and $c_i'^n = a_i[\xi_n(x)]$ and $\sum_{i,n=0}^{\infty} |c_i'^n| \, |P_i(x) + \phi_i(x)| < \infty$, all x, it is known from the foregoing remark that

$$\text{C)}\quad f(x) = \sum_{n=0}^{\infty} a_n'\left[P_n(x) + \phi_n(x) + \xi_n(x)\right]$$

for some numbers a_n'. Then the expansion C) may be obtained from A) ignoring expansion B) providing

$$3.2.4\qquad O_j[\xi_n(x)] = \sum_{i=0}^{\infty} c_i'^n \, O_j\left[P_j(x) + \phi_j(x)\right]$$

$j = 0, 1, \ldots, n = 0, 1, \ldots$.

To show this we need only to conclude that the functions $\phi_n(x) + \xi_n(x)$ satisfy the hypotheses of Theorem 8, in virtue of the restrictions in this theorem.

1. $\phi_n(x) + \xi_n(x) \in (R)$, since $\phi_n(x) \in (R)$ and $\xi_n(x) = \sum_{i=0}^{\infty} c_i'^n [P_i(x) + \phi_i(x)]$ where $c_i'^n = a_i[\xi_n(x)]$. This implies that $O_i[\xi_n(x)]$ has meaning for all i, n. Then $\xi_n(x) \in (R)$. Also

$$3.2.5\qquad \phi_n(x) + \xi_n(x) = \sum_{i=0}^{\infty} c_i^n P_i(x) + \sum_{i=0}^{\infty} c_i'^n \left[P_i(x) + \phi_i(x)\right]$$

$$= \sum_{i=0}^{\infty} c_i^n P_i(x) + \sum_{i=0}^{\infty} c_i'^n \left[P_i(x) + \sum_{k=0}^{\infty} c_k^i P_k(x) \right]$$

Now the double series on the right converges absolutely, since $\sum_{i,k=0}^{\infty} c_k^i P_k(x)$ converges absolutely and $|c_i'^n|$ is less than a constant M for all i, n. For

$$\sum_{i=0}^{\infty} |c_i'^n| \left(|P_i(x)| + \sum_{k=0}^{\infty} |c_k^i P_k(x)| \right) = \sum_{i=0}^{\infty} |c_i'^n| |P_i(x)| + \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} |c_i'^n| \, |c_k^i| \, |P_k(x)| \, .$$

$$\leqq \sum_{i=0}^{\infty} |c_i'^n| \, |P_i(x)| + M \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} |c_k^i| \, |P_k(x)| \, .$$

Now the second term on the right is finite for all x. Consider the first term

$$\sum_{i=0}^{\infty} |c_i'^n| \, |P_i(x)| = \sum_{i=0}^{\infty} |c_i'^n| \, |P_i(x) - \phi_i(x) + \phi_i(x)|$$

$$\leqq \sum_{i=0}^{\infty} |c_i'^n| \, |P_i(x) + \phi_i(x)| + \sum_{i=0}^{\infty} |c_i'^n| \, \left| \sum_{k=0}^{\infty} c_k^i P_k(x) \right|$$

Both terms on the right are finite for all x.

Therefore in 3.2.5 we can interchange summations and obtain

$$\phi_n(x) + \xi_n(x) = \sum_{i=0}^{\infty} (c_i^n + c_i'^n) P_i(x) + \sum_{j=0}^{\infty} c_j'^n \sum_{i=0}^{\infty} c_i^j P_i(x)$$

$$= \sum_{i=0}^{\infty} \left(c_i^n + c_i'^n + \left(\sum_{j=0}^{\infty} c_j'^n c_i^j \right) \right) P_i(x) \, .$$

2. Now Theorem 8 requires that the coefficient of $P_i(x)$ be of the form $0_i\left[\phi_n(x) + \xi_n(x)\right]$. But

$$c_i^n + c_i'^n + \left(\sum_{j=0}^{\infty} c_j'^n c_i^j \right) = c_i^n + \sum_j c_j'^n (\delta_{ij} + c_i^j)$$

$$= c_i^n + 0_i\left[\xi_n(x)\right] ,$$

by 3.2.4. Then

$$c_i^n + c_i'^n + \left(\sum_{j=0}^{\infty} c_j'^n \quad c_i^j\right) = c_i\left[\phi_n(x)\right] + c_i\left[\xi_n(x)\right]$$

$$= c_i\left[\phi_n(x) + \xi_n(x)\right],$$

as required.

3. Next it must be shown that

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\left|c_j^n + c_j'^n + \left(\sum_{i=0}^{\infty} c_i'^n \quad c_j^i\right)\right| < \infty.$$

But this quantity is less than or equal to

$$\sum_{j,n=0}^{\infty}|c_j^n| + \sum_{j,n=0}^{\infty}|c_j'^n| + \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\left|\left(\sum_{i=0}^{\infty} c_i'^n \; c_j^i\right)\right|$$

Because the first two series converge by hypotheses and $\left|\sum_{i=0}^{\infty} c_i'^n \quad c_j^i\right|$ $\leq\left(\sum_{i=0}^{\infty}|c_i'^n|\right)\left(\sum_{i=0}^{\infty}|c_j^i|\right)$, which implies that the third series converges, then the series in question converges.

4. To show that $\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\left|\left(c_i^n + c_i'^n + \left(\sum_{j=0}^{\infty} c_j'^n \quad c_i^j\right)\right)\right|\left|P_i(x)\right| < \infty$ for all x, is a straight forward task using the same ideas as in the preceding work.

Therefore all hypotheses of Theorem 8 are satisfied relevant to the functions $\phi_n(x) + \xi_n(x)$. We can then apply Theorem 8 directly to obtain

$$\text{c)} \quad f(x) = \sum_{n=0}^{\infty} a_n'' \left[P_n(x) + \phi_n(x) + \xi_n(x)\right]$$

where

3.2.6 $$a_n'' = \frac{|\delta_{ij} + \alpha_i^j|\,(n, C_n[f])}{|\delta_{ij} + \alpha_i^j|}, \quad i, j = 0, 1, \ldots$$

$n = 0, 1, \ldots$ and $\alpha_i^j = 0_i\left[\phi_j(x) + \xi_j(x)\right] = c_i^j + c_i'^j + \sum_{r=0}^{\infty} c_r'^j \, c_j^r$.

Then Theorem 10 is proven.

Theorem 11. In Theorem 10, assume that the functions $P_n(x) + \phi_n(x) + \xi_n(x)$ are such that

$$\sum_{r=0}^{\infty} a_r\left[P_r(x) + \phi_r(x) + \xi_r(x)\right] = 0$$

implies $a_r = 0$, $r = 0, 1, \ldots$. Then a_n' of 3.2.1, replacing $\theta_n(x)$ by $\xi_n(x)$ is equal to b_n of 3.2.3 and they are both equal to a_n'' of 3.2.6 providing 3.2.4 holds.

The proof of this theorem is clear, as we have obtained three expansions for $f(x)$ in terms of the functions $P_n(x) + \phi_n(x) + \xi_n(x)$, namely 3.2.2 where the elements of b_n are infinite determinants, C) of Theorem 10 by means of a single jump from A), and the expansion preceding 3.2.1 with a_n' as the coefficients. All three of these expansions converge absolutely to $f(x)$. Subtracting the expansions from each other and using the hypothesis, we obtain equality as demanded in the theorem.

The fact that a_n' of 3.2.1 is equal to a_n'' of 3.2.6 is actually trivial when 3.2.4 holds. For then $0_i\left[\xi_j(x)\right] = b_i^j = c_i'^j + \sum_r c_r'^j c_j^r$.

According to Theorem 10, the effect of performing two transformations in succession on the $P_j(x)$ and then on the $P_j(x) + \phi_j(x)$ into $P_j(x) + \phi_j(x) + \xi_j(x)$ is completely equivalent to performing one transformation of the type described in Theorem 8 from the $P_j(x)$ to the $P_j(x) + \phi_j(x) + \xi_j(x)$, providing 3.2.4 holds. Since, in the hypothesis $\xi_n(x) = \sum_{i=0}^{\infty} c_i'^n \left[P_i(x) + \phi_i(x)\right]$ where $\sum_{i,n} |c_i'^n| < \infty$

To satisfy the hypotheses of Theorem 8, we must require that $\left|\frac{f^{(k)}(0)}{k!}\right| < M$, M some constant, for $k = 0, 1, \ldots$. Obviously O_i , in this case, is an additive, homogeneous functional for all i. We require that the series 3.3.1 converge absolutely in some interval. Now if 3.3.1 converges in $[-b, b]$, end-points excluded or not, then it converges absolutely in any interval $[-b + \epsilon , b - \epsilon]$ where ϵ is any arbitrarily small positive number less than 2b. (See [11, p. 535] for example). The modifying functions $\phi_n(x)$ must be analytic in an interval containing $[-b, b]$. Thus

$$\phi_n(x) = \sum_{k=0}^{\infty} \frac{\phi_n^{(k)}(0)}{k!} x^k$$

where it is required that $\sum_{n,k=0}^{\infty} \frac{|\phi_n^{(k)}(0)|}{k!} < \infty$ and $\sum_{n,k=0}^{\infty} \phi_n^{(k)}(0) \frac{x^k}{k!}$ converges absolutely in $-b + \epsilon \leqq x \leqq b - \epsilon$. All hypotheses of Theorem 8 are satisfied. The conclusion is that f(x) has an absolutely convergent expansion in terms of the functions $x^n + \phi_n(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n\left(x^n + \phi_n(x)\right)$$

where a_n, $n = 0, 1, \ldots$ is given by 3.1.2 with the proper expressions substituted for c_i^j and $O_i[f(x)]$. The resulting series converges absolutely to f(x) in $-b + \epsilon \leqq x \leqq b - \epsilon$. Care must be taken that the determinant in the denominator of 3.1.2 does not vanish.

(b) The Taylor series for functions of a complex variable. Let $P_j(z) = z^j$. Let f(z) belong to (R) the space of functions analytic in a circle $|z| \leqq k$. A discussion for a circle about an arbitrary point $z = a$ would be analogous. Let

$$O_n[f(z)] = \frac{1}{2\pi i} \int \frac{f(w)}{w^{n+1}} \, dw \qquad , \; n = 0, 1, \ldots$$

and the series for $\xi_n(x)$, $\sum_{i=0}^{\infty} c_i'^n [P_i(x) + \phi_i(x)]$, is assumed to be absolutely convergent to $\xi_n(x)$, it is seen that there is little room for a continuation process such as is described above, in other words for a special case of the operator O, 3.2.4 will almost always hold.

3.3 Applications of Theorem 8.

(a) The Taylor series for functions of a real variable.

Let $P_j(x) = x^j$ and $O_i[f] = \dfrac{f^{(i)}(0)}{i!}$. Then

$$O_i[P_j(x)] = \frac{d^i}{dx^i} \frac{x^j}{i!}\bigg|_{x=0}.$$

Now if $i > j$, then $O_i[P_j(x)] = 0$ for any x. If $i < j$, then

$$\frac{d^i}{dx^i}\frac{x^j}{i!}\bigg|_{x=0} = \frac{(j)(j-1)(j-2)\cdots(j-(i-1))}{i!}\, x^{j-i}\bigg|_{x=0} = 0$$

If $i = j$, then from the last statement

$$\frac{d^i}{dx^i}\frac{x^j}{i!}\bigg|_{x=0} = \frac{(i)(i-1)(i-2)\cdots(1)}{i!}\bigg|_{x=0} = 1.$$

Hence

$$\frac{d^i}{dx^i}\frac{x^j}{i!}\bigg|_{x=0} = \delta_{ij}$$

as required. Let (R) denote the space of analytic functions. Assume that $f(x)$ has a Taylor expansion about the point 0. A discussion for an arbitrary point a would be analogous. Then

3.3.1 $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Then

$$O_n\left[P_j(z)\right] = \frac{1}{2\pi i}\int_{|w|=k}\frac{w^j}{w^{n+1}}\,dw = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{k^{n-j}e^{(n-j)i\theta}} = \delta_{jn}$$

Obviously O_n is additive and homogeneous for all n. We can express f(z) as

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{|w|=k}\frac{f(w)}{w^{n+1}}\,dw\; z^n.$$

Since f(z) is analytic in $|z| \leqq k$, then this series converges absolutely in $|z| \leqq k-\epsilon$, where ϵ is an arbitrarily small positive number less than k, by Abel's Lemma [11, p. 535]. We must require that

$$\left|\frac{1}{2\pi i}\int_{|w|=k}\frac{f(w)}{w^{n+1}}\,dw\right| < M \quad, n = 0, 1, \ldots .$$

Then the modifying functions are so chosen that they are analytic in regions containing the circle $|z| \leqq k$:

$$\phi_n(z) = \sum_{j=0}^{\infty} c_j^n z^j$$

where $\sum_{n,j}|c_j^n| < \infty$ and $\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} c_j^n z^j$ converges absolutely in a region containing the circle $|z| \leqq k-\epsilon$.

Here

$$c_j^n = \frac{1}{2\pi i}\int_{|w|=k}\frac{\phi_n(w)}{w^{j+1}}\,dw$$

Then all hypotheses of Theorem 8 are satisfied. f(z) can thus be expanded in a series of the analytic functions $z^n + \phi_n(z)$:

$$f(z) = \sum_{n=0}^{\infty} a_n\left(z^n + \phi_n(z)\right)$$

where a_n, $n = 0, 1, \ldots$ are given by 3.1.2 with the appropriate modification of the elements. The resulting series converges absolutely in $|z| \leqq k - \epsilon$.

(c) The Laurent series for functions of a complex variable. Let the sequence $\{P_j(z)\}$ be $1, z, \frac{1}{z}, z^2, \frac{1}{z^2}, \ldots, z^n, \frac{1}{z^n}, \ldots$.

Let the sequence O_i be $\frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(w)}{w}\,dw$, $\frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(w)}{w^2}\,dw$,

$$\frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(w)}{w^0}\,dw, \ldots, \frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(w)}{w^{n+1}}\,dw, \frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(w)}{w^{-n+1}}\,dw, \ldots$$

Let $f(z)$ be analytic in an annulus about the origin with circles of radius r and R where $r \leqq \rho \leqq R$ say, and $r < R$. Then $O_n[P_j(z)] = \delta_{jn}$, in a similar fashion to (b). O_n is obviously additive and homogeneous for all n. Then we can express $f(z)$:

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n' z^n$$

where

$$a_n' = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(w)}{w^{n+1}}\,dw.$$

We require that $|a_n'| < M$, all n, for some constant M. Now this series for f(z) converges absolutely in any annulus $r + \epsilon \leqq |z| \leqq R - \eta$, where ϵ, η are arbitrarily small positive numbers. Let $\phi_n(z)$ be analytic in annuli containing $|r| \leqq |z| \leqq R$. As usual require that, where

$$\phi_n(z) = \sum_{j=-\infty}^{+\infty} a_j^n z^j ,$$

then $\sum_{n,j} |a_j^n| < \infty$ and $\sum_{n=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} a_j^n z^j$ converges absolutely in $r \leqq |z| \leqq R$. Here

$$a_j^n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{\phi_n(w)}{w^{j+1}} dw$$

Then all hypotheses of Theorem 8 are satisfied. f(z) has an expansion in terms of the functions $z^n + \phi_n(z)$ $n = \ldots -1, 0, 1, \ldots$ where the coefficients are determined in the usual way. The resulting series converges absolutely to f(z) in the annulus $r + \epsilon \leqq |z| \leqq R - \eta$.

(d) Orthonormal functions.

Let $\{P_j(x)\}$ be a sequence of functions orthonormal with a weight function w(x) over some interval $[a, b]$. Let $O_i[f] = \int_a^b w(x)\, f(x)\, P_i(x)\, dx$ $P_i(x)\, dx$; for all f(x) for which this has meaning and for all i. Then

$$O_i[P_j(x)] = \int_a^b w(x)\, P_j(x)\, P_i(x)\, dx = \delta_{ij}$$

by hypothesis. If $w(x) \equiv 1$, this is simply an orthonormal sequence of functions. Suppose f(x) can be expanded in a series of the $P_j(x)$:

$$f(x) = \sum_{j=0}^{\infty} d_j P_j(x) , \quad d_j = \int_a^b w(x)\, f(x)\, P_j(x)\, dx.$$

By Bessel's inequality for orthonormal functions, it is here true an M can be found: such that $|d_j| < M$, M a constant, all j. We ask that this series converge absolutely in $[a, b]$ or in some subinterval of $[a, b]$. Let $\phi_n(x) = \sum_{j=0}^{\infty} c_j^n P_j(x)$, where $c_j^n = \int_a^b w(x)\phi_n(x)P_j(x)\,dx$. We ask that $\sum_{n,j} |c_j^n| < \infty$ and that $\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} c_j^n P_j(x)$ shall converge absolutely for all x in $[a, b]$ or in the subinterval. Then as before f(x) has an expansion :

$$f(x) = \sum_{n=0}^{\infty} a_n\left(P_n(x) + \phi_n(x)\right)$$

where a_n, $n = 0, 1, \ldots$ are given by 3.1.2 with the appropriate specialization of c_i^j and $O_i[f(x)]$.

It seems pertinent to remark that if $w(x) \equiv 1$ and the $P_j(x)$ are the normalized Fourier sines and cosines, then a_n above is the same coefficient as was obtained in Chapter Two, 2.2.3. There the process used was interpolation. No mention was made of the convergence of the resulting series to the function. It is clear that the work in this chapter is a natural generalization of the work done in Chapter Two. However many of the sequences of functions treated in the present chapter have not been brought under the general theory indicated in Chapter One and which is exhibited for some special cases in Chapter Two.

(e) Biorthogonal Functions.

Definition. Given two sequences of functions $P_j(x)$, $j = 0, 1, \ldots$ and $V_j(x)$, $j = 0, 1, \ldots$ defined on an interval $[a, b]$, we say that the sequences are biorthonormal over $[a, b]$ if

$$\int_a^b P_i(x)\, v_j(x)\, dx = \delta_{ij} \quad i = 0, 1, \ldots, \quad j = 0, 1, \ldots .$$

Therefore in Theorem 8, let $P_j(x)$ be the given sequence and let $O_i[f(x)] = \int_a^b f(x)\, v_i(x)\, dx$, all i, for all f(x) for which this has meaning. Then

$$O_i\left[P_j(x)\right] = \int_a^b P_j(x)\, v_i(x)\, dx = \delta_{ij} \quad i = 0, 1, \ldots,$$

$j = 0, 1, \ldots$. O_i is obviously an additive and homogeneous functional. Assume f(x) can be expanded in a series of the $P_j(x)$:

$$f(x) = \sum_{k=0}^{\infty} d_k P_k(x) \quad , \quad d_k = \int_a^b f(x)\, v_k(x)\, dx ,$$

where $|d_k| < M$, some constant M, all k, and the series for f(x) converges absolutely in $[a, b]$ or in some subinterval. If

$$\phi_n(x) = \sum_{k=0}^{\infty} c_k^n P_k(x), \quad c_k^n = \int_a^b \phi_n(x)\, v_k(x)\, dx , \text{ and}$$

$\sum_{n,k} |c_k^n| < \infty$, while $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_k^n P_k(x)$ converges absolutely in $[a, b]$ or in the subinterval, then the hypotheses of Theorem 8 are satisfied. Then f(x) has an expansion in terms of the functions $P_n(x) + \phi_n(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n \left(P_n(x) + \phi_n(x)\right)$$

where a_n, $n = 0, 1, \ldots,$ are given by 3.1.2. The resulting series converges absolutely to f(x) in $[a, b]$ or in the subinterval.

3.4 An Example of a Modification.

At this point, a concrete example would help to illustrate the theory. Therefore in Theorem 8, let the sequence $P_j(x)$ be the sequence $\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots$. With O_i being the operator indicated in 3.3 (d) with $w(x) \equiv 1$, this sequence satisfies the hypotheses of Theorem 8 with interval $[0, 2\pi]$. Let $f(x)$ be any function with an absolutely convergent Fourier series :

$$f(x) = \frac{a_0'}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \left(a_n' \frac{\cos nx}{\sqrt{\pi}} + b_n' \frac{\sin nx}{\sqrt{\pi}} \right)$$

where a_n' and b_n' are the usual Fourier coefficients each multiplied by a factor $\sqrt{\pi}$, while a_0' is the usual coefficient multiplied by $\sqrt{\frac{\pi}{2}}$. Since the series for $f(x)$ converges absolutely, it is clear that the coefficients are bounded. Consider the following sequence of functions $\{ \phi_n(x) \}$:

$$\phi_0(x) = \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}}\right),$$

$$\phi_1(x) = \frac{1}{4}\frac{\cos x}{\sqrt{\pi}} + \frac{1}{4}\frac{\sin x}{\sqrt{\pi}},$$

$$\phi_2(x) = \frac{1}{4}\frac{\cos x}{\sqrt{\pi}} + \frac{1}{4}\frac{\sin x}{\sqrt{\pi}},$$

$$\phi_3(x) = \frac{1}{8}\frac{\cos 2x}{\sqrt{\pi}} + \frac{1}{8}\frac{\sin 2x}{\sqrt{\pi}} + \frac{1}{8}\frac{\cos 3x}{\sqrt{\pi}},$$

$$\phi_4(x) = \frac{1}{8}\frac{\cos 2x}{\sqrt{\pi}} + \frac{1}{8}\frac{\sin 2x}{\sqrt{\pi}} + \frac{1}{8}\frac{\cos 3x}{\sqrt{\pi}},$$

$$\phi_5(x) = \frac{1}{8}\frac{\cos 2x}{\sqrt{\pi}} + \frac{1}{8}\frac{\sin 2x}{\sqrt{\pi}} + \frac{1}{8}\frac{\cos 3x}{\sqrt{\pi}},$$

$$\phi_6(x) = \frac{1}{16}\frac{\sin 3x}{\sqrt{\pi}} + \frac{1}{16}\frac{\cos 4x}{\sqrt{\pi}} + \frac{1}{16}\frac{\sin 4x}{\sqrt{\pi}} + \frac{1}{16}\frac{\cos 5x}{\sqrt{\pi}},$$

$$\phi_7(x) = \frac{1}{16}\frac{\sin 3x}{\sqrt{\pi}} + \frac{1}{16}\frac{\cos 4x}{\sqrt{\pi}} + \frac{1}{16}\frac{\sin 4x}{\sqrt{\pi}} + \frac{1}{16}\frac{\cos 5x}{\sqrt{\pi}},$$

$$\phi_8(x) = \frac{1}{16}\frac{\sin 3x}{\sqrt{\pi}} + \frac{1}{16}\frac{\cos 4x}{\sqrt{\pi}} + \frac{1}{16}\frac{\sin 4x}{\sqrt{\pi}} + \frac{1}{16}\frac{\cos 5x}{\sqrt{\pi}},$$

$$\phi_9(x) = \frac{1}{16}\frac{\sin 3x}{\sqrt{\pi}} + \frac{1}{16}\frac{\cos 4x}{\sqrt{\pi}} + \frac{1}{16}\frac{\sin 4x}{\sqrt{\pi}} + \frac{1}{16}\frac{\cos 5x}{\sqrt{\pi}},$$

. . . .

To satisfy the hypotheses of Theorem 8 we must have $\sum_{i,j} |c_j^i| < \infty$.
In this case, when the elements are summed in an obvious manner,

$$\sum_{i,j} |c_j^i| = \tfrac{1}{2} + 4(\tfrac{1}{4}) + 9(\tfrac{1}{8}) + 16(\tfrac{1}{16}) + 25(\tfrac{1}{32}) + \dots + n^2(\tfrac{1}{2^n}) + \dots$$

To show that this series converges use the simple ratio test.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2}$$

Since the limit is less than 1, $\sum_{i,j} |c_j^i|$ is convergent. It is a trivial fact that $\sum_{n=0}^{\infty} \phi_n(x)$, in series form, converges absolutely.
The only stipulation left to be checked is that $|c_i^j + \delta_{ij}| \neq 0$.

$$|\delta_{ij} + c_i^j| = \begin{vmatrix} 1+\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1+\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{4} & 1+\frac{1}{4} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1+\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{8} & 1+\frac{1}{8} & \frac{1}{8} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & 1+\frac{1}{8} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+\frac{1}{16} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{vmatrix}$$

Clearly, since the existence of the determinant is already established,

$$|\delta_{ij} + c_{ij}| = \prod_{n=1}^{\infty} \overbrace{\begin{vmatrix} 1+\frac{1}{2^n} & \frac{1}{2^n} & \cdots & \frac{1}{2^n} \\ \frac{1}{2^n} & 1+\frac{1}{2^n} & \cdots & \frac{1}{2^n} \\ \vdots & \vdots & & \vdots \\ \frac{1}{2^n} & \frac{1}{2^n} & \cdots & 1+\frac{1}{2^n} \end{vmatrix}}^{\leftarrow n \text{ columns} \rightarrow} = \prod_{n=1}^{\infty} D^{(n)}$$

Lemma :

$$A^{(n)} = \begin{array}{c} \leftarrow n \text{ columns} \rightarrow \\ \begin{vmatrix} 1+a & a & \dots & a \\ a & 1+a & \dots & a \\ \vdots & \vdots & & \vdots \\ a & a & & 1+a \end{vmatrix} \end{array} = 1+ na$$

Proof. Subtract the last column from each other column. Then

$$A^{(n)} = \begin{vmatrix} 1 & 0 & \dots & 0 & a \\ 0 & 1 & \dots & 0 & a \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & a \\ -1 & -1 & \dots & -1 & 1+a \end{vmatrix}$$

Add the first, second, ... $(n - 1)$st row to the last row.

$$A^{(n)} = \begin{vmatrix} 1 & 0 & \dots & 0 & a \\ 0 & 1 & \dots & 0 & a \\ \vdots & \vdots & & & \\ 0 & 0 & \dots & 1 & a \\ 0 & 0 & \dots & 0 & (1+a) + \overbrace{(a+\dots+a)}^{(n-1)\text{ times}} \end{vmatrix} = 1+na.$$

Letting $a = \frac{1}{2^n}$, $D^{(n)} = 1 + \frac{n}{2^n}$. Therefore $\left|\delta_{ij}+c_i^j\right| = \prod_{n=1}^{\infty}\left(1+\frac{n}{2^n}\right)$

Since the determinant $\left|\delta_{ij}+c_i^j\right|$ is already known to be a normal determinant, this infinite product exists. An alternative proof of existence follows. The infinite product exists (see [26, p. 15]) if $\sum_{n=1}^{\infty} \frac{n}{2^n} < \infty$. Use the ratio test

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}.$$

Therefore the series and also the product converges. Since all the factors in the product are greater than 1, the product is not zero. Hence $\left|\delta_{ij} + c_i^j\right| \neq 0$.

We can now apply Theorem 8. The conclusion is that f(x) has an absolutely convergent expansion in $[0, 2\pi]$:

$$f(x) = a_0\left[\frac{1}{\sqrt{2\pi}} + \phi_0(x)\right] + \sum_{k=1}^{\infty}\left(a_{2k-1}\left[\cos kx + \phi_{2k-1}(x)\right] + a_{2k}\left[\sin kx + \phi_{2k}(x)\right]\right)$$

where

$$a_k = \frac{1}{\prod_{n=1}^{\infty}\left(1+\frac{n}{2^n}\right)} \begin{array}{c} \\ \begin{vmatrix} 1+\frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & a_0' & 0 & \cdots \\ 0 & 1+\frac{1}{4} & \frac{1}{4} & 0 & \cdots & 0 & a_1' & 0 & \cdots \\ 0 & \frac{1}{4} & 1+\frac{1}{4} & 0 & \cdots & 0 & b_1' & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \end{vmatrix} \end{array}$$

with $a_0', a_1', b_1', \ldots$ in the $(k+1)$st column.

To simplify $a_{(k-1)}$ locate k between successive values of $\frac{n(n+1)}{2}$. Thus $\frac{n(n+1)}{2} < k \leq \frac{(n+1)(n+2)}{2}$. Then

$$\begin{array}{c} \leftarrow (n+1) \text{ columns} \rightarrow \\ a_{k-1} = \dfrac{1}{1+\dfrac{(n+1)}{2^{n+1}}} \begin{vmatrix} 1+\frac{1}{2^{n+1}} & \frac{1}{2^{n+1}} & \cdots & \alpha_{\frac{n(n+1)}{2}} & \cdots & \frac{1}{2^{n+1}} \\ \frac{1}{2^{n+1}} & 1+\frac{1}{2^{n+1}} & \cdots & \alpha_{\frac{n(n+1)}{2}+1} & \cdots & \frac{1}{2^{n+1}} \\ \vdots & \vdots & & \vdots & & \\ \frac{1}{2^{n+1}} & \frac{1}{2^{n+1}} & \cdots & \alpha_{\frac{(n+1)(n+2)}{2}} & \cdots & 1+\frac{1}{2^{n+1}} \end{vmatrix} \end{array}$$

Where the α_i are a set of the a_i' and b_i' whose identities are clear.

3.5 Modified Expansions in a Banach Space.

First, some elementary notions concerning Banach spaces will be reviewed. Briefly, a Banach space (B) [1, p. 53] is a non empty set of elements which comprise a normed, complete, linear vector space. Complete, in this sense means that any Cauchy sequence of elements of the space converges to an element in the space.

We say that O is an additive functional defined on (B) if for every $x \in (B)$, $O[x]$ is defined and satisfies

$$O[x+y] = O[x] + O[y], \quad \text{all } x, y \in (B).$$

(B) may be said to have a biorthogonal system $\{x_i, O_i\}$, $i = 0, 1, \ldots$, when $x_i \in (B)$, all i, and O_i is an additive homogeneous functional defined on (B), while $O_i[x_j] = \delta_{ij}$. For every $x \in (B)$ we thus have an expansion $\sum_{i=0}^{\infty} O_i[x]\, x_i$, convergent or not.

We are now in a position to generalize Theorem 8, using the notion of a Banach space.

Theorem 12. Let (B) be a Banach space with a biorthogonal system $\{x_i, O_i\}$, $i = 0, 1, \ldots$. Suppose $x \in (B)$, $x = \sum_{i=0}^{\infty} O_i[x]\, x_i$, and the convergence is such that $\sum_{i=0}^{\infty} |O_i[x]|\, \|x_i\| < \infty$, while there exists a number M: $|O_i(x)| < M$, $i = 0, 1, \ldots$. Consider a sequence of elements $\{y_n\}$, $n = 0, 1, \ldots$ where $y_n \in (B)$ and $y_n = \sum_{i=0}^{\infty} O_i[y_n]\, x_i$ and such that

1. $\sum_{i,j=0}^{\infty} |O_i[y_j]| < \infty$,

2. $\sum_{i,n=0}^{\infty} |O_i[y_n]|\, \|x_i\| < \infty$,

3. $|\delta_{ij} + O_i[y_j]|$, $i, j = 0, 1, \ldots$ is different from zero.

Then x can be represented as

$$x = \sum_{n=0}^{\infty} a_n[x]\ [x_n + y_n]$$

where $\sum_{n=0}^{\infty} |a_n(x)|\ \|x_n + y_n\| < \infty$, $|a_n(x)| < \overline{M}$,

$n = 0, 1, \ldots$ where $\bar{M}$ is some constant and furthermore $\{x_n + y_n, a_n\}$ $n = 0, 1, \ldots$ is a biorthogonal system. Here

3.5.1 $$a_n = \frac{|\delta_{ij} + O_i[y_j]|\,(n, O_n[x])}{|\delta_{ij} + O_i[y_j]|} \quad , i, j = 0, 1, \ldots ,$$

$n = 0, 1, \ldots$.

Consider the system of equations

3.5.2 $$\begin{aligned} O_0[x] &= a_0 + \sum_{n=0}^{\infty} a_n\, O_0[y_n] , \\ O_1[x] &= a_1 + \sum_{n=0}^{\infty} a_n\, O_1[y_n] , \\ &\cdots\cdots\cdots\cdots , \\ O_j[x] &= a_j + \sum_{n=0}^{\infty} a_n\, O_j[y_n] \end{aligned}$$

in the unknowns a_n, $n = 0, 1, \ldots$. Since $|O_j(x)| < M$, $j = 0, 1, \ldots$ and the determinant of the coefficients is normal and different from zero, there is one and only one bounded solution, namely 3.5.1.

From 3.5.2 we obtain

3.5.3 $$O_j[x]\,x_j = a_j\,x_j + \sum_{n=0}^{\infty} a_n\, O_j[y_n]\,x_j \quad , j = 0, 1, \ldots$$

To see this, for any $\epsilon > 0$, choose K, a positive integer, such that for $k > K$, then

$$\left| O_j[x] - \left(a_j + \sum_{n=0}^{\infty} a_n\, O_j[y_n]\right)\right| < \frac{\epsilon}{\|x_j\|} .$$

($\|x_j\| \neq 0$, since $0_j[x_j] = 1$.) Therefore

$$\Big\| 0_j[x]\, x_j - \Big(a_j x_j + \sum_{n=0}^{K} a_n 0_j[y_n]\, x_j\Big\|$$
$$= \Big| 0_j(x) - \Big(a_j + \sum_{n=0}^{K} a_n 0_j[y_n]\Big| \; \|x_j\| < \frac{\epsilon}{\|x_j\|} \cdot \|x_j\| = \epsilon.$$

This is valid for all j. Since ϵ is arbitrary, 3.5.3 holds.

We wish to sum over j in 3.5.3. Let k_1, k_2 be positive integers with $k_2 > k_1$, say. Then

$$\Big\| \sum_{j=0}^{K_2} 0_j[x]\, x_j - \sum_{j=0}^{K_2} \sum_{n=0}^{\infty} a_n 0_j[y_n]\, x_j - \sum_{j=0}^{K_1} 0_j[x]\, x_j$$
$$+ \sum_{j=0}^{K_1} \sum_{n=0}^{\infty} a_n 0_j[y_n]\, x_j \Big\| = \Big\| \sum_{j=K_1+1}^{K_2} 0_j[x]\, x_j - \sum_{j=K_1+1}^{K_2} \sum_{n=0}^{\infty} a_n 0_j[y_n]\, x_j \Big\|$$
$$\leq \sum_{j=K_1+1}^{K_2} |0_j[x]|\, \|x_j\| + \sum_{j=K_1+1}^{K_2} \sum_{n=0}^{\infty} |a_n\, 0_j[y_n]|\, \|x_j\|.$$

By hypotheses, and since $|a_n| < \overline{M}$, $n = 0, 1, \ldots,$ both of these expressions go to zero as $k_1, k_2 \to \infty$. Thus

$$\{s_K(x)\} = \Big\{ \sum_{j=0}^{K} \Big(0_j[x]\, x_j - \sum_{n=0}^{\infty} a_n 0_j[y_n]\, x_j \Big\},$$

$k = 0, 1, \ldots$ is a Cauchy sequence. Its limit is the obvious one. Thus we can sum over j in 3.5.3 and obtain

$$\sum_{j=0}^{\infty} 0_j[x]\, x_j = \sum_{j=0}^{\infty} a_j x_j + \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} a_n 0_j[y_n]\, x_j$$

or

$$x = \sum_{j=0}^{\infty} a_j x_j + \sum_{n=0}^{\infty} a_n \sum_{j=0}^{\infty} 0_j[y_n]\, x_j$$
$$= \sum_{j=0}^{\infty} a_j x_j + \sum_{n=0}^{\infty} a_n y_n$$
$$= \sum_{n=0}^{\infty} a_n (x_n + y_n).$$

The series converges as demanded. Also $a_n[x_j + y_j] = \delta_{nj}$, by inspection of 3.5.1. Since a_n is additive and homogeneous for all n, $\{x_n + y_n, a_n\}$, $n = 0, 1, \ldots$ is a biorthogonal system for (B).

The next theorem is the obvious generalization of Theorem 9, and is proven in a similar fashion to the above.

Theorem 13. Let $x \in (B)$ and $x = \sum_{n=0}^{\infty} d_n x_n$ where $\{x_n\}$, $n = 0, 1, \ldots$ is a basis in (B). Assume that $\sum_{n=0}^{\infty} |d_n| \, \|x_n\| < \infty$. Let $y_n = \sum_{i=0}^{\infty} c_i^n x_i$, $n = 0, 1, \ldots$ where $\sum_{i,n=0}^{\infty} |c_i^n| < \infty$ and $\sum_{i=0}^{\infty} \sum_{n=0}^{\infty} |c_i^n| \, \|x_i\| < \infty$. Then $x = \sum_{n=0}^{\infty} a_n (x_n + y_n)$ where $\sum_{n=0}^{\infty} |a_n| \, \|x_n + y_n\| < \infty$ where the appropriate modification is made in the a_n of 3.5.1.

Corollary to Theorem 12. Let (B) be a Banach space with a biorthogonal system $\{x_i ; 0_i\}$. For any sequence of elements $\{y_n\} \in (B)$: $\sum_{i,j} |0_i[y_j]| < \infty$, there is a set of additive functionals a_i, $i = 0, 1, \ldots$: $a_i[x_n + y_n] = \delta_{in}$, provided $|\delta_{ij} + 0_i[y_j]|$, $i, j = 0, 1, \ldots$, is different from zero.

3.6 Some Results on Biorthogonality.

In 3.3 e), it was seen that we can consider the modification of a sequence of functions which is biorthogonal to another sequence of functions. In the last section, we saw that we were dealing with a biorthogonal system in general. Now it can be shown that the coefficients a_n, $n = 0, 1, \ldots$, of 3.1.2 can be reduced to integrals in a great many cases.

Consider a_n of 3.1.2. Then a_n can be written as

$$3.6.1 \qquad a_n = \sum_{r=0}^{\infty} \frac{O_r[f(x)]\ A_{rn}}{A} \qquad , n = 0, 1, \ldots,$$

where A_{rn} is the cofactor of $c_r^n + \delta_{rn}$ in $A = |\delta_{ij} + c_i^j|$, $i = 0, 1, \ldots, j = 0, 1, \ldots$. From the theory of normal determinants [20, p. 375] we know that $|A_{rn}| < M$, M a constant, for all r, n and that $\sum_{n=0}^{\infty} |A_{rn}| < \infty$.

Theorem 14. Let $\{p_j(x), l_j(x)\}$, $j = 0, 1, \ldots$ be a biorthonormal system of functions in L_2 such that each set is uniformly bounded. Let $\phi_j(x) = \sum_{i=0}^{\infty} c_i^j\ p_i(x)$, $j = 0, 1, \ldots,$ where $\sum_{i,j} |c_i^j| < \infty$. Let $O_r[f(x)] = \int_a^b f(x)\, l_r(x)\, dx$, where $f(x)$ belongs to L_2. There exists a sequence of functions $\{v_n(x)\}$, $n = 0, 1, \ldots$ such that

$$a_n = \int_a^b f(x)\, v_n(x)\, dx \quad , \quad n = 0, 1, \ldots$$

Furthermore,

$$\int_a^b v_n(x) \left[\phi_j(x) + p_j(x) \right] dx = \delta_{nj} \qquad , n = 0, 1, \ldots, \quad j = 0, 1, \ldots$$

We first note that the numbers $\int_a^b f(x)\, l_r(x)\, dx$, $r = 0, 1, \ldots$ have a uniform bound. This follows from a straight forward application of Schwarz's inequality and as a consequence of the uniform bound of $\{l_r(x)\}$, $r = 0, 1, \ldots$. Hence, from 3.6.1,

$$a_n = \frac{1}{A} \sum_{r=0}^{\infty} O_r[f(x)]\ A_{rn} = \frac{1}{A} \sum_{r=0}^{\infty} A_{rn} \int_a^b f(x)\, l_r(x)\, dx$$

$$= \frac{1}{A} \sum_{r=0}^{\infty} \int_a^b f(x)\, A_{rn}\, l_r(x)\, dx .$$

Since $\sum_{r=0}^{\infty} |A_{rn}| < \infty$, the $l_r(x)$ are uniformly bounded, and $f(x)$ belongs to L_2, then the series $\frac{1}{A} \sum_{r=0}^{\infty} A_{rn}\, l_r(x)$ converges absolutely and uniformly to a function $v_n(x)$ in $[a, b]$. Therefore integrating term by term,

$$\int_a^b f(x)\, v_n(x)\, dx = \frac{1}{A} \sum_{r=0}^{\infty} \int_a^b f(x)\, l_r(x)\, A_{rn}\, dx = a_n,$$

$n = 0, 1, \ldots$. Thus

$$a_n = \int_a^b f(x)\, v_n(x)\, dx \quad , \quad n = 0, 1, \ldots .$$

Moreover,

$$\int_a^b v_n(x) \left\{ \phi_j(x) + p_j(x) \right\} dx$$

$$= \int_a^b \left(\frac{1}{A} \sum_{r=0}^{\infty} A_{rn}\, l_r(x) \right) \left(\sum_{i=0}^{\infty} (c_i^j + \delta_{ij})\, p_i(x) \right) dx \; ,$$

$$= \sum_{i,r=0}^{\infty} \frac{1}{A} A_{rn} \left(c_i^j + \delta_{ij} \right) \int_a^b l_r(x)\, p_i(x)\, dx$$

$$= \sum_{i,r=0}^{\infty} \left[\frac{1}{A} A_{rn} \left(c_i^j + \delta_{ij} \right) \right] \delta_{ri} = \sum_{r=0}^{\infty} \frac{1}{A} A_{rn} \left(c_r^j + \delta_{rj} \right)$$

$$= \frac{1}{A} \left[A\, \delta_{nj} \right] = \delta_{nj} .$$

In the above calculation, the fact that the two series in the integrand converge absolutely and that the $l_r(x)$, $p_r(x)$ are uniformly bounded was used. This implied uniform convergence of the resulting double series, and term by term integration was permissible.

Corollary. If $\{l_r(x)\}$, $r = 0, 1, \ldots$, is a sequence of continuous functions, then the sequence $\{v_n(x)\}$, $n = 0, 1, \ldots$, is a sequence of continuous functions.

This is clear since the defining series for $v_n(x)$, $n = 0, 1, \ldots$, is then a uniformly convergent series of continuous functions. By a well-known theorem, the limit is continuous.

3.7 Comparison With Known Results.

As mentioned in the introduction, Theorem 8 and its consequences can be compared with the work of J. L. Walsh [30] and that of Paley and Weiner [23, p. 100].

In case Theorem 8 is specialized so that the sequence $\{P_j(x)\}$ is a sequence of orthonormal, uniformly bounded function over some interval $[a, b]$ and $O_j[f(x)] = \int_a^b f(x) P_j(x)\,dx$, the foregoing results are quite close to those of J. L. Walsh [30]. Using our previous notation, he achieves expansions in terms of functions $\{P_j(x) + \phi_j(x)\}$ for any function $f(x)$ that is integrable and with an integrable square over $[a, b]$, although, of course, the resulting series is not always convergent to $f(x)$. The hypotheses in Walsh's paper among others are that the three series

$$\text{(a)} \quad \sum_{i,k=0}^{\infty} (c_k^i)^2 \qquad \text{(b)} \quad \sum_{i=0}^{\infty} \left(\sum_{k=0}^{\infty} (c_k^i)^2 \right)^{\frac{1}{2}}, \qquad \text{(c)} \quad \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} (c_k^j)^2 \right)^{\frac{1}{2}}$$

converge and that the value of the series in (a) is less than one. These hypotheses are to be compared with the hypothesis $\sum_{i,k=0}^{\infty} |c_k^i| < \infty$ This implies (b) and (c) converges for

$$\sum_{i=0}^{\infty} \left(\sum_{k=0}^{\infty} |c_k^i|^2 \right)^{\frac{1}{2}} \leqq \sum_{i=0}^{\infty} \sqrt{\left(\sum_{k=0}^{\infty} |c_k^i| \right) \left(\sum_{j=0}^{\infty} |c_j^i| \right)} = \sum_{i,k=0}^{\infty} |c_k^i| < \infty,$$

and similarly for (c). However the value of (a) is not less than 1 if any one of the c_k^i is greater than or equal to 1 in absolute value. Walsh's method of proof is rather different than that of Theorem 8, but results are obtained for functions whose $P_j(x)$-series does not converge. In Theorem 8, attention was restricted to those functions with absolutely convergent $P_j(x)$-series.

With the same specialization of Theorem 8 as in the above, it can be compared with the results of Paley and Weiner [23, p. 100]. The hypothesis $\sum_{i,k} |c_i^k| < \infty$ is here to be compared with the hypothesis

$$\int_a^b \left| \sum_{n=0}^{N} a_n \left(P_n(x) - g_n(x) \right) \right|^2 dx \leqq \Theta^2 \sum_{n=0}^{N} |a_n|^2$$

where $\Theta < 1$, and where Θ is independent of N and $\{a_n\}$. The $\{a_n\}$ is any set of numbers and $\{g_n(x)\}$ is the sequence in which it is desired to obtain expansions. An expansion in terms of the $g_n(x)$ is obtained for all functions f(x) that belong to L_2 over $[a, b]$. In some cases, the $g_n(x)$ are not asymptotic to the $P_n(x)$, whereas in Theorem 8, the $P_j(x) + \phi_j(x)$ are asymptotic to the $P_j(x)$. If the $g_n(x)$ are of the form $P_n(x) + \phi_n(x)$, the above condition becomes

$$\int_a^b \left| \sum_{n=0}^{N} a_n \phi_n(x) \right|^2 dx \leqq \Theta^2 \sum_{n=0}^{N} |a_n|^2$$

In Theorem 8, we had $\phi_n(x) = \sum_{i=0}^{\infty} c_i^n P(x)$, $n = 0, 1, \ldots$ where for each n the convergence is absolute and uniform in $[a, b]$ due to the uniform boundness of $P_i(x)$, $i = 0, 1, \ldots$. Thus

$$\int_a^b \left| \sum_{n=0}^{N} a_n \sum_{i=0}^{\infty} c_i^n P_i(x) \right|^2 dx \leqq \Theta^2 \sum_{n=0}^{N} |a_n|^2 .$$

Since $a_0, \ldots, a_N$ forms a bounded set of numbers, we can interchange summation and obtain

$$\int_a^b \left| \sum_{i=0}^{\infty} \left(\sum_{n=0}^{N} a_n \, c_i^n \right) P_i(x) \right|^2 dx \leqq \theta^2 \sum_{n=0}^{N} |a_n|^2$$

or

3.7.1 $$\sum_{i=0}^{\infty} \left| \sum_{n=0}^{N} a_n \, c_i^n \right|^2 \leqq \theta^2 \sum_{n=0}^{N} |a_n|^2$$

Let $a_0, a_1, \ldots, a_{N-1}$ be equal to 1, $a_N = 0$. The condition must be satisfied for this set. That is

3.7.2 $$\sum_{i=0}^{\infty} \left| \sum_{n=0}^{N-1} c_i^n \right|^2 \leqq N \, \theta^2$$

Now with N fixed, choose $c_0^0, c_0^1, c_0^2, \ldots, c_0^{N-1}$ to be any set of numbers greater than or equal to 1, and choose the other c_j^i so that $\sum_{i,j} |c_j^i| < \infty$, and the determinant $|\delta_{ij} + c_i^j| \neq 0$.

Then

$$\sum_{i=0}^{\infty} \left| \sum_{n=0}^{N-1} c_i^n \right|^2 \geqq N^2$$

Therefore a $\theta < 1$ cannot be found for which 3.7.2 is satisfied. However Theorem 8 is applicable.

For example, let $N = 2$, $c_0^0 = 2$, $c_0^1 = 2$, all other $c_i^j = 0$. Then $\sum_{i,j} |c_j^i| < \infty$ trivially and $|\delta_{ij} + c_i^j| = 3$, as may be seen easily. Using this set of c_i^j, the sequence $\{P_j(x) + \phi_j(x)\}$ becomes $3P_0(x)$; $P_1(x) + 2P_0(x)$, $P_2(x)$, $P_3(x)$, $\ldots$. The sequence $\{v_j(x)\}$ becomes $\frac{1}{3} P_0(x) - \frac{2}{3} P_1(x)$, $P_1(x)$, $P_2(x)$, $\ldots$. It may be verified quite easily that the two sequences are biorthogonal.

On the other hand, let $c_0^0 = \frac{1}{2}$, $c_1^1 = \frac{1}{2}, \ldots, c_j^j = \frac{1}{2}, \ldots$

$c^i_j = 0$, $i \neq j$, then 3.7.1 becomes

$$\sum_{i=0}^{\infty} \left| \sum_{n=0}^{N} a_n \tfrac{1}{2} \, \delta_{in} \right|^2 \leqq \theta^2 \sum_{n=0}^{N} |a_n|^2$$

This will be satisfied if we ask that

$$\tfrac{1}{4} |a_0|^2 + \tfrac{1}{4} |a_1|^2 + \dots + \tfrac{1}{4} \, |a_N|^2 \leqq \theta^2 \sum_{n=0}^{N} |a_n|^2$$

or

$$\frac{1}{4} \sum_{n=0}^{N} |a_n|^2 \leqq \theta^2 \sum_{n=0}^{N} |a_n|^2 .$$

Then choose θ such that $\frac{1}{4} < \theta^2 < 1$. The condition is satisfied. However $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)$ does not converge. Therefore for this set of c^i_j Theorem 8 does not apply, but the results of Paley and Weiner hold.

Again, if Theorem 8 is specialized so that the sequence $\{P_j(x)\}$ is $\{z^j\}$, the results obtained can be compared with those [3] of G. D. Birkhoff. Let

$$O_n\left[f(z)\right] = \frac{1}{2\pi i} \int_{|w|=k} \frac{f(w)}{w^{n+1}} \, dw \, , \quad n = 0, 1, \dots,$$

where $f(z)$ is analytic in $|z| \leqq k$. It will be found that the results overlap as in the last two cases. For example, using our previous notation, let $\phi_j(z) = 0$, $j > 0$ and let $\phi_0(z) = 2$. Then the modified sequence becomes $3, z, z^2, \dots$. It is easily seen that Birkhoff's theorem does not cover this sequence whereas it is included in the foregoing work. Conversely let $\phi_j(z) = 0$, $j > 0$, and let $\phi_0(z) = \sum_{n=1}^{\infty} \frac{1}{4n} z^n$. Then the modified sequence becomes $1 + \sum_{n=1}^{\infty} \frac{1}{4n} z^n$, z, z^2, z^3, $\dots$. Birkhoff's theorem includes this sequence, whereas the foregoing work does not include it since $\sum_{n=1}^{\infty} \frac{1}{4n}$ diverges.

It should be emphasized again that the present results concern an abstract operator 0_K. The work done by these other authors considered only the case where the operator 0_K was a certain integral.

CHAPTER FOUR

COMPLETENESS OF MODIFIED SEQUENCES

4.1 The Completeness of $\{P_k(x) + \phi_k(x)\}$.

Definition. A sequence of functions $\{f_n(x)\}$, $n = 0, 1, \dots$, defined over $[a, b]$, where $f_n(x)$ belongs to L_2, is said to be complete if

$$\int_a^b f(x)\, f_n(x)\, dx = 0 \quad ; \quad n = 0, 1, \dots$$

implies $f(x) = 0$, except possibly on a set of measure zero. [17, p. 45]. Here $f(x) \in L_2$.

Theorem 15. Let the $P_j(x)$ of Theorem 8 denote a sequence of complete, orthonormal, uniformly bounded functions belonging to L_2 on $[a, b]$. Let $C_i[f] = \int_a^b f(x)\, P_i(x)\, dx$. As before, let $\phi_n(x) = \sum_{i=0}^{\infty} c_i^n\, P_i(x)$, $n = 0, 1, \dots$, where $\sum_{i,n} |c_i^n| < \infty$ and $|c_i^n + \delta_i^n| \neq 0$. Then $\{P_n(x) + \phi_n(x)\}$, $n = 0, 1, \dots$, is a complete sequence of functions.

To prove the theorem, assume that where $f(x) \in L_2$, that

$$\int_a^b f(x) \left\{ P_n(x) + \phi_n(x) \right\} dx = 0 \quad , n = 0, 1, \dots .$$

Then from the hypotheses,

$$\int_a^b f(x) \left\{ P_n(x) + \sum_{i=0}^{\infty} c_i^n\, P_i(x) \right\} dx = 0 \quad , n = 0, 1, \dots .$$

Since the convergence of the series in the integrand is uniform, and $f(x) \in L_2$, we can integrate term by term. Thus

$$\sum_{i=0}^{\infty} (\delta_{in} + c_i^n) \int_a^b f(x)\, P_i(x)\, dx = 0, \quad n = 0, 1, \dots .$$

This constitutes an infinite number of linear equations in an infinite number of unknowns $\int_a^b f(x)\, P_i\, (x)\, dx$, $i = 0, 1, \ldots$. The determinant of the coefficients $|\delta_{in} + c_i^n|$ is normal and assumed different from zero. Now the sequence $\left\{ \int_a^b f(x)\, P_i\, (x)\, dx \right\}$, $i = 0, 1, \ldots$ is known to be bounded from Bessel's inequality for orthonormal functions. Hence, we must look for a bounded solution of this system of equations. However, there is one and only one bounded solution, namely the trivial one

$$\int_a^b f(x)\ P_i\,(x)\,dx = 0 \quad , \quad i = 0, 1, \ldots .$$

But since $\{P_i\,(x)\}$ was taken to be a complete sequence, this relation implies that $f(x) = 0$, except possibly on a set of measure zero. Then the sequence $\{P_i\,(x) + \phi_i(x)\}$ is complete.

Definition. The sequence $\{f_n(x)\}$, $n = 0, 1, \ldots$, is closed with respect to L_2 if for any $f(x) \in L_2$ and any positive ϵ there is a linear form

$$P_n\,(x) = \sum_{k=0}^{n} a_k\, f_k\,(x)$$

such that

$$\int_a^b |P_n(x) - f(x)|^2\, dx < \epsilon .$$

The following theorem is known to be true. The proof presented below is taken from Paley and Weiner's book [23, p. 26] .

Theorem 16. A set of functions $\{\phi_n(x)$, $n = 0, 1, \ldots$, where $\phi_n(x) \in L_2$, is complete when and only when it is closed.

The proof will be presented only in the case where $\{\phi_n(x)\}$ represents a sequence of orthonormal functions. The more general proof consists of orthogonalizing the given sequence and using the proof for orthonormal sequences.

Let f(x) be any function in L_2. Then, in the usual way,

$$\int_a^b \left| f(x) - \sum_{k=0}^{n} a_k \phi_k(x) \right|^2 dx = \int_a^b |f(x)|^2 dx - \sum_{k=0}^{n} \left| \int_a^b f(x)\phi_k(x)dx \right|^2$$

where $a_k = \int_a^b f(x) \phi_k(x)dx$, $k = 0, 1, \ldots, n$, and this choice for a_k renders the expression on the left a minimum. Now if the sequence $\{\phi_k(x)$ is closed, the expression on the left may be made arbitrarily small. Thus

$$\int_a^b |f(x)|^2 dx = \sum_{k=0}^{\infty} \left| \int_a^b f(x) \phi_k(x) dx \right|^2.$$

Now, if f(x) is orthogonal to every $\phi_k(x)$, the sum on the right vanishes which implies that $\int_a^b |f(x)|^2 dx = 0$ which implies [21, p. 130] that f(x) is equivalent to zero. Hence a closed sequence of functions is complete.

Now, assume that the sequence $\{\phi_k(x)\}$, $k = 0, 1, \ldots,$ is complete. If it is not closed, there exists $f(x) \in L_2$:

$$\lim_{n\to\infty} \int_a^b \left| f(x) - \sum_{k=1}^{n} \phi_k(x) \int_a^b f(x) \phi_k(x) dx \right|^2 dx > 0.$$

But as above, this implies

4.1.1 $$\int_a^b |f(x)|^2 dx - \sum_{k=1}^{\infty} \left| \int_a^b f(x) \phi_k(x) dx \right|^2 > 0.$$

Let

$$g_n(x) = \sum_{k=1}^{n} \phi_k(x) \int_a^b f(x)\, \phi_k(x)\, dx.$$

Then

$$g(x) = \underset{n \to \infty}{\text{limit in mean}} \; (g_n(x))$$

exists by the Riesz-Fischer theorem. Also

4.1.2 $$\int_a^b |g(x)|^2 dx = \lim_{n\to\infty} \int_a^b |g_n(x)|^2 dx = \sum_{k=1}^{\infty} \left| \int_a^b f(x)\, \phi_k(x)\, dx \right|^2$$

But

$$\int_a^b \big(f(x) - g(x)\big)\, \phi_n(x)\, dx = \lim_{l\to\infty} \int_a^b \big[f(x) - g_l(x)\big]\, \phi_n(x)\, dx = 0,$$

$n = 0, 1, \ldots$.

Since the sequence $\{\phi_n(x)\}$ is complete, then $f(x) - g(x)$ is equivalent to zero. But by 4.1.1 and 4.1.2

$$\int_a^b |f(x)|^2 dx - \int_a^b |g(x)|^2 dx > 0$$

Then f cannot be equivalent to g. This is a contrediction. Therefore no such f(x) exists and the sequence $\{\phi_k(x)\}$, $k = 0, 1, \ldots$ is closed.

Corollary to Theorem 15. Given any $f(x) \in L$ and any positive ϵ, there is a linear form $Q_n(x) = \sum_{k=0}^{n} a_k\big[P_k(x) + \phi_k(x)\big]$ such that

4.1.3 $$\int_a^b |f(x) - Q_n(x)|^2 dx < \epsilon$$

This follows immediately upon application of Theorem 16 to Theorem 15 to conclude that the sequence $\{P_K(x) + \phi_K(x)\}$ is closed.

4.2 The Coefficients of the Best Approximation.

A problem presents itself at this time. We know that the sequence $\{P_n(x) + \phi_n(x)\}$ is closed. Then what is the set of coefficients a_K of the linear form $\varphi_n(x)$ that render 4.1.3 its least value? In working with orthonormal sequences as was seen in 4.1, the question is easily answered. The traditional series coefficients are also those which give the analogous expression its minimum value. But in the biorthogonal case, a different result will be obtained. In order to answer this question, it may be observed that

4.2.1 $$\int_a^b \left[f(x) - \sum_{i=0}^n a_i\left(P_i(x) + \phi_i(x)\right)\right]^2 dx = \int_a^b [f(x)]^2 dx$$

$$-2\sum_{i=0}^n a_i \int_a^b f(x)\left\{P_i(x) + \phi_i(x)\right\}dx + \int_a^b\left(\sum_{i=0}^n a_i\left(P_i(x)+\phi_i(x)\right)\right)\left(\sum_{r=0}^n a_r\left(P_r(x)+\phi_r(x)\right)\right)dx$$

$$= \int_a^b [f(x)]^2 dx - \phi.$$

Here

$$-\phi = -2\sum_{i=0}^n a_i \int_a^b f(x)\left\{P_i(x) + \phi_i(x)\right\}dx + \sum_{i=0}^n a_i^2 + 2\sum_{\substack{i=0\\r=0}}^n a_i a_r \int_a^b P_i(x)\phi_r(x)\,dx$$

$$+ \sum_{\substack{i=0\\r=0}}^n a_i a_r \int_a^b \phi_i(x)\phi_r(x)\,dx.$$

In order that 4.2.1 have a least value, $-\phi$ must have a minimum value. For a set of a_i rendering $-\phi$ a minimum, $\frac{\partial\phi}{\partial a_i} = 0$, $i = 0, 1, \ldots, n$, necessarily. However

$$-\frac{\partial\phi}{\partial a_i} = -2\int_a^b f(x)\big(P_i(x)+\phi_i(x)\big)dx + 2a_i + 2\sum_{r=0}^{n} a_r\int_a^b P_i(x)\,\phi_r(x)dx + 2\sum_{r=0}^{n} a_r\int_a^b P_r(x)\,\phi_i(x)dx$$

$$+\sum_{r=0}^{n} a_r\int_a^b \phi_i(x)\,\phi_r(x)dx + \sum_{r=0}^{n} a_r\int_a^b \phi_r(x)\,\phi_i(x)\,dx.$$

Thus set

$$-\int_a^b f(x)\big(P_i(x)+\phi_i(x)\big)dx + \sum_{r=0}^{n} a_r\Big(\delta_{ir} + \int_a^b P_i(x)\,\phi_r(x)\,dx$$

$$+\int_a^b P_r(x)\,\phi_i(x)dx + \int_a^b \phi_i(x)\,\phi_r(x)\,dx\Big) = 0 \quad , \quad i = 0, 1, \cdots, n$$

or

$$-\int_a^b f(x)\big(P_i(x)+\phi_i(x)\big)dx + \sum_{r=0}^{n} a_r\Big(\delta_{ir} + c_i^r + c_r^i + \sum_{s=0}^{\infty} c_s^i\, c_s^r\Big) = 0,$$

$i = 0, 1, \ldots, n.$

Assuming temporarily that the determinant of the coefficients a_r does not vanish, this set of equations has the unique solution:

$$a_k^{(n)} = \frac{\left|\delta_{ij} + c_i^j + c_j^i + \sum_{s=0}^{\infty} c_s^i\, c_s^j\right| \left(k, \int_a^b f(x)\left[P_i(x)+\phi_i(x)\right]dx\right)}{\left|\delta_{ij} + c_i^j + c_j^i + \sum_{s=0}^{\infty} c_s^i\, c_s^j\right|}$$

$i, j = 0, 1, \ldots, n, \quad k = 0, 1, \ldots, n.$

Now as promised at the beginning of the paragraph, the set of coefficients furnishing the best approximation are not the series coefficients $\int_a^b f(x)\, v_i\,(x)\, dx,\ i = 0, 1, \ldots$. It would be desirable to relate this result to the determinants already encountered in the theory. This can be done rather easily. Consider the determinant $D = \left| \delta_{ik} + c_i^k \right|$ in the denominator of a_k of 3.1.2 in Theorem 8. Multiply it by itself with rows and columns interchanged. Then

$$D;D = \left| \delta_{ij} + c_i^j + c_j^i + \sum_{s=0}^{\infty} c_s^i\, c_s^j \right|$$

$i, j = 0, 1, \ldots$.

It is to be noted that the product of two normal determinants is a normal determinant [20, p. 387] .

Let $(D;D)_n$ denote the $(n+1)x(n+1)$ principal minor located in the upper left hand corner of this determinant. If it is noticed that

$$\int_a^b f(x) \left[P_k(x) + \phi_k(x) \right] dx = \int_a^b f(x) \left[\sum_{i=0}^{\infty} \left(\delta_{ik} + c_i^k \right) \right] P_i(x)\, dx$$
$$= \sum_{i=0}^{\infty} \left[\delta_{ik} + c_i^k \right] \int_a^b f(x)\, P_i\,(x)\, dx \quad ,$$

the expression for $a_k^{(n)}$ can be simplified. In a manner similar to the above compute $D;N$ where N is the numerator of a_k in 3.1.2. It is seen that $a_k^{(n)}$, the solution to the least square problem, can be written in the form

$$a_k^{(n)} = \frac{(D.N)_n}{(D.N)_n} \qquad , k = 0, 1, \ldots, n.$$

The superscript on a_k indicates that it is a function of $n+1$, the number of terms appearing in $Q_n(x)$. It is easy to see that

$$\lim_{n\to\infty} a_k^{(n)} = \lim_{n\to\infty} \frac{(D\cdot N)_n}{(D\cdot D)_n} = \frac{D\cdot N}{D\cdot D} = \frac{N}{D} = a_k$$

which is the actual series coefficient found in Theorem 8. Here the fact that the product of two normal determinants is a normal determinant is used.

There is yet some unfinished business in this discussion. It remains to be shown that the determinant in the denominator of a_k^n does not vanish. It could have been written as

$$4.2.2 \qquad \left| \int_a^b \left[P_i(x) + \phi_i(x)\right]\left[P_j(x) + \phi_j(x)\right] dx \right|, \quad i,j = 0, \ldots, n$$

Considering $\int_a^b \left[P_i(x) + \phi_i(x)\right]\left[P_j(x) + \phi_j(x)\right] dx$, as the inner product of $P_i(x) + \phi_i(x)$ and $P_j(x) + \phi_j(x)$ considered as elements of the space L_2, we can apply Gram's criterion [20, p. 321] to the determinant 4.2.2. Gram's criterion implies that 4.2.2 is greater than zero if the functions $P_i(x) + \phi_i(x)$, $i = 0, 1, \ldots, n$, are linearly independent. This can be shown easily.

Assume

$$\sum_{i=0}^{n} b_i\left(P_i(x) + \phi_i(x)\right) = 0$$

Then for $j = 0, 1, \ldots, n$

$$\sum_{i=0}^{n} b_i\left(P_i(x) + \phi_i(x)\right) v_j(x) = 0$$

or

$$\sum_{i=0}^{n} b_i \int_a^b \left[P_i(x) + \phi_i(x)\right] v_j(x)\, dx = 0 .$$

This implies $\sum_{i=0}^{n} b_i \delta_{ij} = b_j = 0$, $j = 0, 1, \ldots, n$. Thus the given functions are linearly independent and the determinant 4.2.2 is greater than zero for all n.

A remaining task is to prove that the integral 4.2.1 actually takes on a least value. However

$$\frac{1}{2}\frac{\partial}{\partial a_j}\left(\frac{\partial \phi}{\partial a_i}\right) = \delta_{ij} + c_i^j + c_j^i + \sum_{s=0}^{\infty}\left(c_s^i \, c_s^j\right).$$

Then

$$\left|\frac{\partial}{\partial a_j}\left(\frac{\partial \phi}{\partial a_i}\right)\right| , \quad i = 0, 1, \ldots, n, \quad j = 0, 1, \ldots, n,$$

is precisely the determinant multiplied by 2^{n+1} that has been shown to be greater than zero above. By the same reasoning then,

$$\left|\frac{\partial}{\partial a_j}\left(\frac{\partial \phi}{\partial a_i}\right)\right| > 0, \quad i = 0, \ldots, k, \quad j = 0, \ldots, k$$

This is true for $k = 0, \ldots, n$. But this is a sufficient condition [13, pp. 81-95] that the integral in question have a minimum at the point where $\frac{\partial \phi}{\partial a_j} = 0$, $j = 0, 1, \ldots, n$.

It is to be especially noted that the approximating coefficients vary with n, again, unlike the orthogonal case.

4.3 Approximation with $\{v_i(x)\}$.

It will be shown in this section that the same questions maybe asked about the sequence $\{v_i(x)\}$ as were asked about the sequence $\{P_i(x) + \phi_i(x)\}$ and analogous answers will be obtained.

Theorem 17. Use the hypotheses of Theorem 15. Let $v_i(x) = \sum_{s=0}^{\infty} \frac{A_{si}}{A} P_s(x)$, $P_i(x)$, $i = 0, 1, \ldots,$ as before. Then the sequence $\{v_i(x)\}$ is complete.

To prove this, assume that where $f(x) \in L_2$, then

$$\int_a^b f(x)\, v_i(x)\, dx = 0, \quad i = 0, 1, \ldots .$$

Then

$$\sum_{s=0}^{\infty} \frac{A_{si}}{A} \int_a^b f(x)\, P_s(x)\, dx = 0, \quad i = 0, 1, \ldots .$$

This implies that a_0, a_1, ..., a_κ, ... are all zero where a_κ is given by 3.1.2. This can be true when and only when $\int_a^b f(x)\, P_s(x)\, dx = 0$, $s = 0, 1, \ldots$. But the sequence $\{P_s(x)\}$ is complete. Hence $f(x) = 0$, except possibly on a set of measure zero. Hence the sequence $\{v_i(x)\}$ is complete.

While this is the result that will be of interest, Theorem 17 can be generalized.

Theorem 18. Use the notation of Theorem 12 in 3.5. If the O_n, $n = 0, 1, \ldots$ are a complete set of functionals, then the a_n, $n = 0, 1, \ldots$, of 3.5.1 are also complete.

Definition. A sequence of additive, homogeneous, functionals O_0, O_1, ..., defined on a Banach space is called complete if $O_n[x] = 0$, $n = 0, 1, \ldots$, implies that $x = \theta$, the zero element of the Banach space.

Consider the following set of equations under the assumption $a_n[x] = 0$, $n = 0, 1, \ldots$,

$$O_j[x] = a_j + \sum_{n=0}^{\infty} a_n O_j[y_n] \quad , \quad j = 0, 1, \ldots .$$

Since $a_k[x]$, $k = 0, 1, \ldots,$ as given in 3.5.1 is the unique bounded solution of this set and since $a_k[x] = 0$, all k, then $0_k[x] = 0$, $k = 0, 1, \ldots$. Since $0_0, 0_1, \ldots$ is a complete set, then $x = \theta$. Hence $a_0, a_1, \ldots$ is a complete set.

Corollary to Theorem 17. The sequence $v_i(x)$, $i = 0, 1, \ldots$ is closed.

We may now search for the set of coefficients b_k of the linear form $\sum_{k=0}^{n} b_k v_k(x)$ which renders the integral

4.3.1 $$\int_a^b \left| f(x) - \sum_{k=0}^{n} b_k v_k(x) \right|^2 dx$$

its least value. After the experience of the last paragraph, we do not expect to obtain the traditional series coefficient, namely $\int_a^b f(x)\left(P_k(x) + \phi_k(x)\right)dx$. Here our expectations are fulfilled. For

$$\int_a^b \left| f(x) - \sum_{k=0}^{n} b_k V_k(x) \right|^2 dx =$$

$$\int_a^b |f(x)|^2 dx - 2\sum_{k=0}^{n} b_k \int_a^b f(x) v_k(x) dx + \sum_{k=0}^{n} \sum_{r=0}^{n} b_k b_r \int_a^b v_k(x) v_r(x)\, dx.$$

Then in order that this expression have a minimum, we want $-\phi$ to have a minimum where

$$\phi = 2\sum_{k=0}^{n} b_k \int_a^b f(x) v_k(x)\, dx - \sum_{k=0}^{n} \sum_{r=0}^{n} b_k b_r \int_a^b v_k(x) v_r(x)\, dx.$$

Now

$$\frac{\partial\phi}{\partial b_k} = 2\int_a^b f(x)\, v_k(x)\, dx - 2\sum_{r=0}^{n} b_r \int_a^b v_k(x)\, v_r(x)\, dx$$

$k = 0, 1, \ldots, n$. Setting these expressions equal to zero, assuming temporarily that the determinant of the coefficients is non-vanishing, we have the unique solution

$$4.3.2 \qquad b_k^{(n)} = \frac{\left|\int_a^b v_i(x)\, v_j(x)\, dx\right| \left(k, \int_a^b f(x)\, v_i(x) dx\right)}{\left|\int_a^b v_i(x)\, v_j(x)\, dx\right|}$$

$i, j = 0, 1, \ldots, n, \quad k = 0, 1, \ldots, n.$

But the determinant in the denominator is non-vanishing. This is shown in a manner similar to the proof in the last paragraph. Assume that

$$c_0 v_0(x) + c_1 v_1(x) + \ldots + c_n v_n(x) \equiv 0 .$$

Multiply by $P_r(x) + \phi_r(x)$ and integrate over $[a, b]$. Then $c_r = 0$. This is true for all $r : 0 \leqq r \leqq n$. Hence the set $v_0(x), \ldots, v_n(x)$ is linearly independent for all n.

The proof that 4.3.1 actually assumes a minimum for this set of coefficients is completely analogous to the proof of the last paragraph.

It remains to relate 4.3.2 to some determinants already encountered and to find $\lim_{n\to\infty} b_k^{(n)}$. After past experiences we would expect $\lim_{n\to\infty} b_k^n$ to be $\int_a^b f(x)\left[P_k(x) + \phi_k(x)\right] dx.$

We know that $v_n(x) = \frac{1}{A}\sum_{r=0}^{\infty} A_{rn}\, P_r(x)$. Therefore

$$\int_a^b v_i(x)\, v_j(x)\, dx = \frac{1}{A^2}\sum_{r,k=0}^{\infty} A_{ri}\; A_{kj} \int_a^b P_r(x)\, P_k(x)\, dx,$$

where the term by term integration is permissible due to the absolute convergence of the two series. Therefore

$$\int_a^b v_i(x)\, v_j(x)\, dx = \frac{1}{A^2}\sum_{r,\kappa=0}^{\infty} A_{ri}\, A_{\kappa j}\, \delta_{r\kappa} = \frac{1}{A^2}\sum_{r=0}^{\infty} A_{ri}\, A_{rj}\,.$$

Then 4.3.2 can be written as

$$\text{4.3.3} \qquad b_\kappa^{(n)} = \frac{\left|\sum_{r=0}^{\infty} \frac{A_{ri}A_{rj}}{A^2}\right| \left(\kappa, \sum_{r=0}^{\infty} \frac{A_{ri}}{A}\int_a^b f(x)\, P_r(x)dx\right)}{\left|\sum_{r=0}^{\infty} \frac{A_{ri}A_{rj}}{A^2}\right|},$$

$i, j = 0, 1, \ldots, n;\ k = 0, 1, \ldots, n.$

At this point a digression will be made which will enable us to put 4.3.3 in a more pleasing form. However the following theorem has much interest in its own right.

Theorem 19. Let A_{ij} denote the cofactor of $\delta_{ij} + c_i^j$ in $A = |\delta_{ij} + c_i^j|$ $i = 0, 1, \ldots, j = 0, 1, \ldots$. Assume $A \neq 0$. Then $\left|\frac{A_{ij}}{A}\right|$, $i = 0, 1, \ldots,$ $j = 0, 1, \ldots,$ is a normal determinant and its value is $\frac{1}{A}$.

The given determinant may be written as

$$\left|\delta_{ij} + \frac{A_{ij}}{A} - \delta_{ij}\right| \quad ,\ i = 0, 1, \ldots, j = 0, 1, \ldots .$$

Consider the two series

$$\text{4.3.4} \qquad \sum_{i=0}^{\infty} \left|\frac{A_{ii}}{A} - 1\right| \quad ,$$

$$\text{4.3.5} \qquad \sum_{\substack{i,j=0 \\ i\neq j}}^{\infty} \left|\frac{A_{ij}}{A}\right|.$$

If both of these series converge, then the determinant in question is normal. Examine 4.3.4

$$\sum_{i=0}^{\infty}\left|\frac{A_{ii}}{A}-1\right| = \frac{1}{|A|}\sum_{i=0}^{\infty}\left|A_{ii}-A\right|$$
$$= \frac{1}{|A|}\sum_{i=0}^{\infty}\left|A_{ii} - \sum_{r=0}^{\infty}(\delta_{ri}+c_r^i)\quad A_{ri}\right|$$
$$= \frac{1}{|A|}\sum_{i=0}^{\infty}\left|-\sum_{r=0}^{\infty}(c_r^i)A_{ri}\right|$$

Now the absolute values of all the first minors of a normal determinant are uniformly bounded [20, p. 375] by some constant M. Hence

$$\sum_{i=0}^{\infty}\left|\frac{A_{ii}}{A}-1\right| \leqq \frac{M}{|A|}\sum_{i,r=0}^{\infty}|c_r^i| < \infty$$

Consider 4.3.5. Remove the factor $\frac{1}{|A|}$ from consideration. Let us examine A_{ij}, where $i \neq j$, a little more closely.

$$A_{ij} = \pm\begin{vmatrix}
c_0^0+1 & c_0^1 & c_0^2 & \cdots & c_0^{i-1} & c_0^i & c_0^{i+1} & \cdots & c_0^{j-1} & c_0^{j+1} & \cdots \\
c_1^0 & 1+c_1^1 & c_1^2 & \cdots & c_1^{i-1} & c_1^i & c_1^{i+1} & \cdots & c_1^{j-1} & c_1^{j+1} & \cdots \\
c_2^0 & c_2^1 & 1+c_2^2 & \cdots & c_2^{i-1} & c_2^i & c_2^{i+1} & \cdots & c_2^{j-1} & c_2^{j+1} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
c_{i-1}^0 & c_{i-1}^1 & c_{i-1}^2 & \cdots & 1+c_{i-1}^{i-1} & c_{i-1}^i & c_{i-1}^{i+1} & \cdots & c_{i-1}^{j-1} & c_{i-1}^{j+1} & \cdots \\
c_{i+1}^0 & c_{i+1}^1 & c_{i+1}^2 & \cdots & c_{i+1}^{i-1} & c_{i+1}^i & 1+c_{i+1}^{i+1} & \cdots & c_{i+1}^{j-1} & c_{i+1}^{j+1} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
c_{j-1}^0 & c_{j-1}^1 & c_{j-1}^2 & \cdots & c_{j-1}^{i-1} & c_{j-1}^i & c_{j-1}^{i+1} & \cdots & 1+c_{j-1}^{j-1} & c_{j-1}^{j+1} & \cdots \\
c_j^0 & c_j^1 & c_j^2 & \cdots & c_j^{i-1} & c_j^i & c_j^{i+1} & \cdots & c_j^{j-1} & c_j^{j+1} & \cdots \\
c_{j+1}^0 & c_{j+1}^1 & c_{j+1}^2 & \cdots & c_{j+1}^{i-1} & c_{j+1}^i & c_{j+1}^{i+1} & \cdots & c_{j+1}^{j-1} & 1+c_{j+1}^{j+1} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots &
\end{vmatrix}$$

Notice that there is precisely one column and one row of A_{ij} which does not contain an element $1 + c_k^k$ for some k. Expand the above determinant by the elements of the column of c^i_k , $k = 0, 1, \ldots$. Expand the resulting determinants by the elements of the row with c^1_j terms. We obtain

$$A_{ij} \leqq \sum_{k=0}^{\infty}{}'' |c^i_k| \left(\sum_{1=0}^{\infty}{}'' |c^1_j| \left| A\begin{pmatrix} i, & j, & 1 \\ i, & j, & k \end{pmatrix} \right| \right) + \left| A\begin{pmatrix} i,j \\ i,j \end{pmatrix} \right| |c^i_j|$$

The primes on the first summation indicate that k skips the values i, j. The double prime on the second summation indicates that 1 skips the values i and j. $A\begin{pmatrix} i,j,1 \\ i,j,k \end{pmatrix}$ is a determinant obtained by striking out the i, j, 1 columns and the i, j, k rows. As such it is itself a first minor of the normal determinant obtained from A when one strikes out the i, j rows and i, j columns. As such $\left| A\begin{pmatrix} i,j,1 \\ i,j,k \end{pmatrix} \right|$ has a uniform bound [20, p. 375] for all i, j, k, l. Call this bound P. Then in 4.3.5, letting $P' = \max(P\ \ M)$,

$$\sum_{\substack{i,j=0 \\ i \neq j}}^{\infty} \left| \frac{A_{ij}}{A} \right| \leqq \frac{P'}{|A|} \sum_{\substack{i,j=0 \\ i \neq j}}^{\infty} \left(\sum_{k=0}^{\infty}{}'' |c^i_k| \sum_{1=0}^{\infty}{}'' |c^1_j| + |c^i_j| \right)$$

$$\leqq \frac{P'}{|A|} \left[\sum_{\substack{i=0 \\ i \neq j}}^{\infty} \sum_{k=0}^{\infty}{}' |c^i_k| \sum_{\substack{j=0 \\ j \neq i}}^{\infty} \sum_{1=0}^{\infty}{}'' |c^1_j| + \sum_{\substack{i,j=0 \\ i \neq j}}^{\infty} |c^i_j| \right] \leqq \frac{P'}{|A|} \left[s^2 + s \right]$$

where $s = \sum_{i,j=0}^{\infty} |c^i_j| < \infty$, by definition of a normal determinant. Thus both 4.3.4 and 4.3.5 converge which implies that $\left| \frac{A_{ij}}{A} \right|$, $i = 0, 1, \ldots, j = 0, 1, \ldots$, is a normal determinant. Now, since we can multiply two normal determinants, multiply $\left| \frac{A_{ij}}{A} \right|$, $i = 0, 1, \ldots,$ $j = 0, 1, \ldots,$ from the left by the transpose of A. Use the relations

$$\text{4.3.6} \qquad \sum_{r=0}^{\infty} \frac{(\delta_{ir} + c^i_r) A_{rj}}{A} = \delta_{ij} \; .$$

We see that $A\left|\frac{A_{ij}}{A}\right| = 1$, or $\left|\frac{A_{ij}}{A}\right| = \frac{1}{A}$. The theorem is proven.

Theorem 20. The set A of all infinite matrices, the determinants of which are normal and different from zero, forms a group under ordinary matrix multiplication.

The identity element, which is the matrix $\|\delta_{ij}\|$, i, j = 1, 2, ..., is clearly such that its determinant is normal. Since the product of two normal determinants is normal, then the product of two matrices in A is again in A. Finally, every matrix whose determinant is normal and different from zero has an inverse given by Theorem 19. Hence A is a group.

Let us now reexamine 4.3.3. In view of Theorem 19 we can now write 4.3.3 as

$$b_K^{(n)} = \frac{\left(\left|\frac{A_{ij}}{A}\right| \cdot \left|\frac{A_{ij}}{A}\right| \left(K, \int_a^b f(x)\, P_1(x)\, dx\right)\right)_n}{\left(\left|\frac{A_{ij}}{A}\right| \cdot \left|\frac{A_{ij}}{A}\right|\right)_n}, \quad k = 0, 1, \ldots, n.$$

Therefore

4.3.7 $$\lim_{n\to\infty} b_K^{(n)} = \frac{\left|\frac{A_{ij}}{A}\right| \left(K, \int_a^b f(x)\, P_1(x)\, dx\right)}{\left|\frac{A_{ij}}{A}\right|}$$

To put this in a simpler form, multiply both numerator and denominator from the left by the transpose of A. Use the relations 4.3.6. It is found that

4.3.8 $$\lim_{n\to\infty} b_K^{(n)} = \int_a^b f(x)\, P_K(x)\, dx + \sum_{r=0}^{\infty} c_r^K \int_a^b f(x)\, P_r(x)\, dx$$
$$= \int_a^b f(x)\, P_K(x)\, dx + \int_a^b f(x)\, \phi_K(x)\, dx$$

$$= \int_a^b f(x) \left[P_k(x) + \phi_k(x)\right] dx \; .$$

It is to be recalled that this was the standard series coefficient.

4.4 Approximation Using the Series Coefficients.

It was found in 4.2 that we could find a set of coefficients b_i^n such that

4.4.1 $$\int_a^b \left| f(x) - \sum_{i=0}^{n} b_i^n \left(P_i(x) + \phi_i(x)\right) \right|^2 dx$$

would take on its minimum value. These coefficients were not the natural series coefficients $\int_a^b f(x)\, v_i(x)\, dx$ that we might have hoped for. We now wish to show that, if these series coefficients are used in 4.4.1, then that expression can still be made arbitrarily small. To do this we must first prove a theorem which has much interest itself.

Theorem 21. Assume that $\sum_{i=0}^{\infty} b_i^2 < \infty$, where $\{b_n\}$ is a sequence of real numbers, and that $\sum_{i,j=0}^{\infty} |c_i^j| < \infty$, while $A = |\delta_{ij} + c_i^j| \neq 0$. Then $\sum_{k=0}^{\infty} a_k^2 < \infty$ where

$$a_k = \frac{|\delta_{ij} + c_i^j|\,(k, b_i)}{|\delta_{ij} + c_i^j|} \quad , \quad i, j = 0, 1, \ldots, k = 0, 1, \ldots \; .$$

To prove this, notice that

$$a_k = \frac{1}{A} \sum_{j=0}^{\infty} A_{jk}\, b_j = \frac{1}{A}\Big(A_{kk}\, b_k + \sum_{\substack{j=0 \\ j \neq k}}^{\infty} A_{jk}\, b_j \Big)$$

Then

$$a_k^2 = \frac{1}{A^2}\left(A_{kk}^2\, b_k^2 + 2A_{kk}\, b_k \Big(\sum_{\substack{j=0 \\ j\neq k}}^{\infty} A_{jk}\, b_j \Big) + \Big(\sum_{\substack{j=0 \\ j\neq k}}^{\infty} A_{jk}\, b_j \Big)^2 \right)$$

Therefore

$$\sum_{k=0}^{\infty} a_k^2 \leqq \frac{1}{A^2}\left[\sum_{k=0}^{\infty} A_{kk}^2\, b_k^2 + 2\sum_{k=0}^{\infty} |A_{kk}||b_k| \left(\sum_{\substack{j=0\\ j\neq k}}^{\infty} |A_{jk}||b_j|\right.\right.$$

$$\left.\left. + \sum_{k=0}^{\infty}\left(\sum_{\substack{j=0\\ j\neq k}}^{\infty} |A_{jk}||b_j|\right)^2\right]$$

Let $P = \sum_{i=0}^{\infty} b_i^2 < \infty$. Let M be the uniform bound of $|A_{ii}|$, $i = 0, 1, \ldots$. Then

$$\sum_{k=0}^{\infty} a_k^2 \leqq \frac{1}{A^2}\left[M^2 \sum_{k=0}^{\infty} b_k^2 + 2MP^2 \sum_{k=0}^{\infty} \sum_{\substack{j=0\\ j\neq k}}^{\infty} |A_{jk}| + P^2 \sum_{k=0}^{\infty}\Big(\sum_{\substack{j=0\\ j\neq k}}^{\infty} |A_{jk}|\Big)^2\right]$$

The first expression in the bracket is finite by hypothesis. The second is finite by the proof in the last paragraph which showed that $\sum_{k=0}^{\infty} \sum_{\substack{j=0\\ j\neq k}}^{\infty} |A_{jk}| < \infty$. This same fact implies the finiteness of the third expression. The theorem is proven.

Corollary 1. $\lim_{i\to\infty} a_i = 0$.

Corollary 2. Where $v_n(x)$, $n = 0, 1, \ldots$, is given in Theorem 14 of 3.6 we have for any $f(x) \in L_2$

$$\sum_{n=0}^{\infty}\left[\int_a^b f(x)\, v_n(x)\, dx\right]^2 < \infty.$$

This follows easily from the fact that $\sum_{i=0}^{\infty}\left[\int_a^b f(x)\, P_i(x)\, dx\right]^2 < \infty$ for all $f(x) \in L_2$, and the previous theorem.

Theorem 22. With the notation and hypotheses of Theorem 14 of 3.6, then

$$\left|\sum_{i=0}^{n}\left[\int_a^b f(t)\, v_i(t)\, dt\right]\left\{\int_a^b g(t)\left[P_i(t) + \phi_i(t)\right] dt\right\}\right|$$

has an n-free upper bound for all $f(x)$, $g(x) \in L_2$.

For

$$\sum_{i=0}^{n}\left[\int_a^b f(t)\, v_i(t)\, dt\right]\left[\int_a^b g(t)\left[P_i(t)+\phi_i(t)\right] dt\right.$$

$$= \sum_{i=0}^{n} a_i \int_a^b g(t)\, P_i(t)\, dt + \sum_{i=0}^{n} a_i \int_a^b g(t)\, \phi_i(t)\, dt = A + B,$$

Where a_i is the expression used in Theorem 14 of 3.6. Now, since $|a_i| < M$, all i,

$$|B| \leqq M \sum_{i=0}^{n} \sum_{s=0}^{\infty} |c_s^i| \left| \int_a^b g(t)\, P_s(t)\, dt \right|$$

But since $\left|\int_a^b g(t)\, P_s(t)\, dt\right| < M'$, M' a constant, all s, then

$$|B| \leqq M\, M' \sum_{i=0}^{n} \sum_{s=0}^{\infty} |c_s^i| .$$

However $\sum_{i,s=0}^{\infty} |c_s^i| < \infty$. Hence $|B| < \alpha$, α, a constant independent of n. Now

$$|A| \leqq \sqrt{\sum_{i=0}^{n} a_i^2}\ \sqrt{\sum_{i=0}^{n}\left(\int_a^b g(t)\, P_i(t)\, dt\right)^2}$$

by the Cauchy inequality. Since the functions $\{P_i(t)\} \in L_2$ and are orthonormal, the right factor has an n-free upper bound. By the Corollary to Theorem 21, the left factor has an n-free upper bound. Hence $|A| < \beta$, β a constant independent of n. Therefore $|A+B| \leqq |A| + |B| \leqq \beta + \alpha$, where $\beta+\alpha$ does not depend on n. The theorem is proven.

The following theorem, except for a little reorganization and specialization pertinent to the foregoing work, may be found in Kaczmarz and Steinhaus' book [17, p. 273] .

Theorem 23. If

$$\left| \sum_{i=0}^{n} \left[\int_a^b f(t)\, v_i(t)\, dt \right] \left[\int_a^b g(t) \left[P_i(t) + \phi_i(t) \right] dt \right] \right|$$

$n = 0, 1, \ldots,$ for all $f(x), g(x) \in L_2$ has an n-free bound, then the biorthogonal set $\{v_i(x),\ P_i(x) + \phi_i(x)\}$, $i = 0, 1, \ldots,$ has the properties

$$1. \quad \lim_{n\to\infty} \int_a^b \left(f(t) - r_n(t) \right)^2 dt = 0,$$

$$2. \quad \lim_{n\to\infty} \int_a^b \left(g(t) - s_n(t) \right)^2 dt = 0,$$

$$3. \quad \lim_{n\to\infty} \int_a^b \left(f(t) - r_n(t) \right) \left(g(t) - s_n(t) \right) dt = 0,$$

where $r_n(t) = \sum_{i=0}^{n} \zeta_i \left(P_i(t) + \phi_i(t) \right)$, $\zeta_i = \int_a^b f(t)\, v_i(t)\, dt,$

$$s_n(t) = \sum_{i=0}^{n} \eta_i\, v_i(t), \quad \eta_i = \int_a^b g(t) \left[P_i(t) + \phi_i(t) \right] dt.$$

Thus the hypothesis may be written $\left| \sum_{i=0}^{n} \zeta_i \eta_i \right|$ has an n-free bound for all n. Now

$$\begin{aligned} \left| \sum_{i=0}^{n} \zeta_i \eta_i \right| &= \left| \sum_{i=0}^{n} \zeta_i \int_a^b g(t) \left[P_i(t) + \phi_i(t) \right] dt \right| \\ &= \left| \int_a^b g(t) \sum_{i=0}^{n} \zeta_i \left[P_i(t) + \phi_i(t) \right] dt \right| \\ &= \left| \int_a^b g(t)\, r_n(t)\, dt \right| \end{aligned}$$

Then the integrals on the right have an n-free bound. Since this is true for all $g(t) \in L_2$, then necessarily (see [17, p.21])

$$\int_a^b |r_n(t)|^2 dt < M \qquad n = 0, 1, \ldots .$$

In turn, this implies (see [17, p.20])

4.4.3 $$\int_a^b |r_n(f, t)|^2 \, dt \leqq \mu \int_a^b |f(t)|^2 \, dt$$

where r_n is considered as a linear operator acting on $f(x)$ and μ is a constant independent of n and $f(x)$.

We have already shown that $\{P_n(x) + \phi_n(x)\}$ is a closed sequence of functions in L_2. Then there exists a sequence of elements $\rho_1, \rho_2, \ldots, \rho_n, \ldots$ with $\rho_n = \sum_{k=0}^{n} \beta_{nk}\{P_k(x) + \phi_k(x)\}$:

4.4.4 $$\lim_{n\to\infty} \int_a^b |f(t) - \rho_n(t)|^2 \, dt = 0$$

However

$$\int_a^b |r_n(f - \rho_n)|^2 \, dt \leqq \mu \int_a^b |f(t) - \rho_n(t)|^2 \, dt$$

by 4.4.3. Since the right hand side goes to zero with n by 4.4.4, then the left hand side goes to zero. But

$$\int_a^b |r_n(f - \rho_n)|^2 \, dt = \int_a^b |r_n(f) - r_n(\rho_n)|^2 \, dt$$

However

$$\begin{aligned} r_n(\rho_n) &= \sum_{i=0}^{n}\left[\int_a^b \sum_{k=0}^{n} \beta_{nk}\{P_k(t) + \phi_k(t)\} v_i(t)\,dt\right](P_i(t) + \phi_i(t)) \\ &= \sum_{i=0}^{n}\left[\sum_{k=0}^{n} \beta_{nk} \int_a^b [P_k(t) + \phi_k(t)]\, v_i(t)\,dt\right](P_i(t) + \phi_i(t)) \\ &= \sum_{i=0}^{n}\left[\sum_{k=0}^{n} \beta_{nk}\,\delta_{ik}\right][P_i(t) + \phi_i(t)] \\ &= \sum_{i=0}^{n} \beta_{ni}[P_i(t) + \phi_i(t)] = \rho_n \end{aligned}$$

Hence

4.4.5 $$\int_a^b |r_n(f - \rho_n)|^2 \, dt = \int_a^b |r_n(f) - \rho_n|^2 \, dt$$

Since the left hand side goes to zero with n, so does the right.

Now

$$\int_a^b \left| f(t) - r_n(t) \right|^2 dt = \int_a^b \left| f(t) - P_n(t) + P_n(t) - r_n(t) \right|^2 dt$$
$$\leqq \left\{ \int_a^b \left| f(t) - P_n(t) \right|^2 dt \right\}^{\frac{1}{2}} + \left\{ \int_a^b \left| P_n(t) - r_n(t) \right|^2 dt \right\}^{\frac{1}{2}}$$

The first term on the right goes to zero by 4.4.4. The second goes to zero by 4.4.5. Hence

$$\lim_{n\to\infty} \int_a^b \left| f(t) - r_n(t) \right|^2 dt = 0 ,$$

and 1 is proven.

The proof of 2 is completely analogous to the preceding. To prove 3 notice that

$$\left| \int_a^b \left[f(t) - r_n(t) \right] \left[g(t) - s_n(t) \right] dt \right.$$
$$= \left| \int_a^b \left[f(t) - r_n(t) \right] g(t)\, dt - \int_a^b \left[f(t) - r_n(t) \right] \left[s_n(t) \right] dt \right|$$
$$\leqq \sqrt{\int_a^b \left| g(t) \right|^2 dt} \cdot \sqrt{\int_a^b \left(f(t) - r_n(t) \right)^2 dt} + \left| - \int_a^b f(t) s_n(t) dt + \int_a^b r_n(t) s_n(t) dt \right|$$

The first term goes to zero with n by 1. Now

$$\int_a^b f(t)\, s_n(t)\, dt = \int_a^b f(t) \sum_{i=0}^{n} \eta_i\, v_i(t)\, dt = \sum_{i=0}^{n} \eta_i \xi_i$$

and

$$\int_a^b r_n(t)\, s_n(t)\, dt = \int_a^b \left[\sum_{i=0}^{n} \eta_i\, v_i(t) \right] \left[\sum_{i=0}^{n} \xi_i \left(P_i(t) + \phi_i(t) \right) \right] dt$$
$$= \sum_{i=0}^{n} \xi_i \eta_i .$$

Therefore the last two terms cancel. Then 3 is proven.

1 is the result that we have been seeking, namely that we can approximate any function $f(x) \in L_2$ by the functions $\{P_i(x) + \phi_i(x)\}$ using the usual series coefficients. 2 is the analogous result for the sequence $\{v_i(x)\}$. 3 is obtained in the process and will furnish [17, p. 272] the analogue of Parseval's Theorem for orthogonal functions. It is to be noted that if we were working with orthogonal sequences that 1 and 2 coincide. Now by 3,

$$\int_a^b [f(t) - r_n(t)]\,[g(t) - s_n(t)]\,dt = \int_a^b f(t)\,g(t)\,dt + \sum_{i=0}^{n} \xi_i \eta_i - \sum_{i=0}^{n} \xi_i \eta_i - \sum_{i=0}^{n} \xi_i \eta_i .$$

By 3, $\sum_{i=0}^{\infty} \xi_i \eta_i$ converges and

$$\sum_{i=0}^{\infty} \xi_i \eta_i = \int_a^b f(t)\,g(t)\,dt ,$$

for all $f(x), g(x) \in L_2$. Clearly if $f(x) = g(x)$, then

$$\sum_{i=0}^{\infty} \xi_i \eta_i = \int_a^b (f(t))^2\,dt$$

or using our previous notation

$$\sum_{i=0}^{\infty} \int_a^b f(t)\,[P_i(t) + \phi_i(t)]\,dt \int_a^b f(t)\,v_i(t)\,dt = \int_a^b (f(t))^2\,dt.$$

The following theorem is an obvious corollary of Theorem 21.

Theorem 24. $$\sum_{K=0}^{\infty} \frac{\left|\frac{A_{ij}}{A}\right|^2 (K, Q_i[f])}{\left|\frac{A_{ij}}{A}\right|^2} < \infty \quad \text{if} \quad \sum_{l=0}^{\infty} [O_l[f]]^2 < \infty$$

This is true since $\left|\frac{A_{ij}}{A}\right|$, $i, j = 0, 1, \ldots$ is a normal determinant.

Apply Theorem 18.

Corollary to Theorem 24.

$$\lim_{k\to\infty} \frac{\left|\frac{A_{ij}}{A}\right| (k, O_1(f))}{\left|\frac{A_{ij}}{A}\right|} = 0$$

Theorem 25. Let $\sum_{1=0}^{\infty} [O_1(f)]^2 < \infty$. Then

$$\left|\sum_{k=0}^{\infty} a_k \alpha_k\right| < \infty .$$

where

$$a_k = \frac{|\delta_{ij} + c_i^j| \; (k, O_1(f))}{|\delta_{ij} + c_i^j|}, \quad \alpha_k = \frac{\left|\frac{A_{ij}}{A}\right| (k, O_1[f])}{\left|\frac{A_{ij}}{A}\right|} .$$

For

$$\left|\sum_{k=0}^{\infty} a_k \alpha_k\right| \leqq \sqrt{\sum_{k=0}^{\infty} a_k^2} \sqrt{\sum_{k=0}^{\infty} \alpha_k^2} < \infty,$$

by Theorem 21 and Theorem 24.

It is possible to obtain a simple expression for $a_k \alpha_k$ which will be of use when $\sum_{1=0}^{\infty} [O_1(f)]^2$ is not necessarily convergent.

$$a_k \alpha_k = \left[\sum_{s=0}^{\infty} (\delta_{ks} + c_s^k) O_s[f]\right]\left[\sum_{1=0}^{\infty} \frac{A_{1k}}{A} O_1[f]\right] .$$

Given a set of operators O_s, this expression can be used to study the convergence of $\sum_{k=0}^{\infty} a_k \alpha_k$.

4.5 Analogues of the Riesz-Fischer Theorem.

Theorem 25. Let $\xi_0, \xi_1, \cdots, \xi_n, \cdots$ be a set of real numbers such that $\sum_{i=0}^{\infty} \xi_i^2 < \infty$. Then there is a function $f(x) \in L_2$ such that

$$\int_a^b f(t)\, v_i(t)\, dt = \xi_i \qquad , \quad i = 0, 1, \ldots .$$

Let $r_n(t) = \sum_{i=0}^{n} \xi_i \left(P_i(t) + \phi_i(t) \right)$, where $\xi_0, \xi_1, \cdots$ are the given numbers. Then for $m < n$

$$\int_a^b \left[r_n(t) - r_m(t) \right]^2 dt = \int_a^b \left[\sum_{i=m+1}^{n} \xi_i \left(P_i(t) + \phi_i(t) \right) \right]^2 dt$$

$$= \sum_{i,j=m+1}^{n} \xi_i \xi_j \int_a^b P_i(t) P_j(t)\, dt + 2 \sum_{i,j=m+1}^{n} \xi_i \xi_j c_i^j + \sum_{i,j=m+1}^{n} \xi_i \xi_j \int_a^b \phi_i(t) \phi_j(t)\, dt$$

$$= \sum_{i=m+1}^{n} \xi_i^2 + 2 \sum_{i,j=m+1}^{n} \xi_i \xi_j c_i^j + \sum_{i,j=m+1}^{n} \xi_i \xi_j \sum_{s=0}^{\infty} c_s^i c_s^j$$

$$= A + B + C .$$

Now A can be made less than an arbitrary positive ϵ by hypothesis. Since $|\xi_i| < M$, $i = 0, 1, \ldots$, then $|B| \leqq 2M^2 \sum_{i,j=m+1}^{n} |c_i^j|$. Thus $|B|$ can be made arbitrarily small since $\sum_{i,j=0}^{\infty} |c_i^j| < \infty$. Consider C

$$|C| \leqq M^2 \sum_{i,j=m+1}^{n} \sum_{s=0}^{\infty} |c_s^i|\, |c_s^j| .$$

Now since $\sum_{i,j=0}^{\infty} |c_i^j| < \infty$, then given $\epsilon > 0$ we can find an N: $j \geqq N$ and $\sum_{\substack{i=0 \\ j=N+1}}^{\infty} |c_i^j| < \sqrt{\epsilon}$. Then for m, n > N

$$|C| \leqq M^2 \sum_{i,j=m+1}^{n} \sum_{s=0}^{\infty} |c_s^i| \, |c_s^j| \leqq \left(\sum_{i=N+1}^{\infty} \sum_{s=0}^{\infty} |c_s^i| \right)^2 M^2 < \epsilon M^2$$

Therefore

$$\lim_{n,m \to \infty} \int_a^b \left[r_n(t) - r_m(t) \right]^2 dt = 0 .$$

The sequence $r_1, r_2, \ldots, r_n, \ldots$ is thus a Cauchy sequence with respect to the metric of L_2 . Since L_2 is a complete space, there exists an element $f(x) \in L_2$ to which the sequence $\{r_n(x)\}$ converges in the metric. Now for $m \leqq n$,

$$\xi_m = \sum_{i=0}^{n} \xi_i \int_a^b \left[P_i(t) + \phi_i(t) \right] v_m(t) \, dt$$

$$= \int_a^b \sum_{i=0}^{n} \xi_i \left(P_i(t) + \phi_i(t) \right) v_m(t) \, dt = \int_a^b r_n(t) \, v_m(t) \, dt.$$

Then

$$\left| \int_a^b f(t) \, v_m(t) \, dt - \xi_m \right| = \left| \int_a^b \left[f(t) - r_n(t) \right] v_m(t) \, dt \right.$$

$$\leqq \sqrt{\int_a^b |f(t) - r_n(t)|^2 dt} \; \sqrt{\int_a^b v_m^2(t) \, dt} \; .$$

The first factor on the right goes to zero with n by the foregoing remarks. For a fixed m, the right factor is some finite number. Since the left hand side is independent of n, then

$$\xi_m = \int_a^b f(t) \, v_m(t) \, dt \qquad , m = 0, 1, \ldots ,$$

as was to be shown.

Theorem 27. Let $\eta_0, \eta_1, \cdots, \eta_n, \cdots$ be a set of real numbers such that $\sum_{i=0}^{\infty} \eta_i^2 < \infty$. Then there is a function f(x) in L_2 such that

$$\int_a^b f(t)\,[P_i(t) + \phi_i(t)]\,dt = \eta_i \qquad , i = 0, 1, \ldots .$$

Let $s_n(t) = \sum_{i=0}^{n} \eta_i\, v_i(t)$. For $m < n$,

$$\begin{aligned}
\int_a^b [s_n(t) - s_m(t)]^2\,dt &= \int_a^b \Big[\sum_{i=m+1}^{n} \eta_i\, v_i(t)\Big]^2 dt , \\
&= \sum_{i,j=m+1}^{n} \eta_i \eta_j \int_a^b v_i(t)\, v_j(t)\,dt , \\
&= \sum_{i,j=m+1}^{n} \eta_i \eta_j \sum_{r=0}^{\infty} \frac{A_{ri} A_{rj}}{A^2} \\
&= 2\sum_{i,j=m+1}^{n} \eta_i \eta_j \frac{A_{ii} A_{ij}}{A^2} + \sum_{i,j=m+1}^{n} \eta_i \eta_j \sum_{r=0}^{\infty}{}'' \frac{A_{ri} A_{rj}}{A^2} .
\end{aligned}$$

The double prime on the last summation indicates that r skips the values i, j. Now since $\sum_{\substack{i,j=0 \\ i \neq j}}^{\infty} |A_{ji}| < \infty$, the second factor will go to zero with m, n. Since A_{ii}, A_{ij}, have a uniform bound for all i, j and since $\sum_{i=0}^{\infty} \eta_i^2 < \infty$, the first expression will go to zero. Hence the sequence $s_0, s_1, \ldots$ is a Cauchy sequence in L_2. The remainder of the proof is completely analogous to the proof of the previous theorem.

4.6 The Non-vanishing of $|\delta_{ij} + c_i^j|$.

Throughout the previous work, we have been bothered by a seemingly independent assumption, that the determinants appearing in the denominators of the coefficients be different from zero. In this section, it will be seen that this assumption is an inherent part of the whole problem and not an accidental restriction on it. The following theorem will help to set off these remarks.

Theorem 28. With the notation used in previous theorems, let $\{P_j(x)\}$ denote a uniformly bounded, orthonormal sequence of functions in L_2 over $[a, b]$. Assume further that this sequence is complete in L_2. Assume $\{P_j(x) + \phi_j(x)\}$, with the same restrictions on the $\phi_j(x)$ as before, is complete in L_2. Then $|\delta_{ij} + c_i^j| \neq 0$.

For assume that $|\delta_{ij} + c_i^j| = 0$. Consider the set of equations

4.6.1 $$\sum_{s=0}^{\infty} (\delta_{is} + c_s^i)\, b_s = 0 \quad , i = 0, 1, \ldots .$$

Lemma 1. If the rank of $|\delta_{ij} + c_j^i|$ is r, then there exist r independent solutions of 4.6.1.

The proof of this lemma may be found in Kowalewski's book [20, pp. 405-406]. More than this is true. For all of these independent solutions, it is true that $\sum_{s=0}^{\infty} b_s^2 < \infty$. To see this, rearrange the equations 4.6.1 and the notation so that

$$|\delta_{ij} + c_i^j| \neq 0 \quad , \; i, j = r, r+1, \ldots .$$

This is possible since the rank of the determinant is r. Now consider

$$\sum_{s=0}^{\infty} (\delta_{is} + c_s^i)\, b_s = 0, \quad i = r, r+1, \ldots$$

or

$$\sum_{s=r}^{\infty} (\delta_{is} + c_s^i)\, b_s = -\sum_{s=0}^{r-1} c_s^i \; b_s \quad , i = r, r+1, \ldots$$

This set of equations has a unique bounded solution $b_r, b_{r+1}, \ldots$. Also $\sum_{s=r}^{\infty} b_s^2 < \infty$. This follows from the convergence of $\sum_{s,i=0}^{\infty} |c_s^i|$ which implies that $\sum_{i=r}^{\infty} \left(-\sum_{s=0}^{r-1} (c_s^i)\, b_s\right)^2 < \infty$. By

Theorem 21, $\sum_{s=r}^{\infty} b_s^2 < \infty$. Then also $\sum_{s=0}^{\infty} b_s^2 < \infty$. Now these numbers b_s, $s = 0, 1, \ldots$ also satisfy the equations 4.6.1 where $i = 0, \ldots, r$, since these equations are linear combinations of the others. The lemma is proven.

Then there is at least one solution of 4.6.1, $b_0, b_1, \ldots$, which has at least one b_i different from zero. For this set of b_s, consider the set of equations

4.6.2 $$\int_a^b f(x)\, P_s(x)\, dx = b_s \quad , \quad s = 0, 1, \ldots .$$

By the Riesz-Fischer theorem, since $\sum_{s=0}^{\infty} b_s^2 < \infty$, there exists a function $f(x)$ in L_2 over $[a, b]$ such that 4.6.2 is satisfied. This function is not equivalent to zero, for if it were, b_s would be zero for all s. Now consider the equations 4.6.1 with the appropriate substitution:

$$\sum_{s=0}^{\infty} (\delta_{is} + c_s^i) \int_a^b f(x)\, P_s(x)\, dx = 0 \quad , i = 0, 1, \ldots$$

or $$\int_a^b f(x) \sum_{s=0}^{\infty} (\delta_{is} + c_s^i)\, P_s(x)\, dx = 0, \; i = 0, 1, \ldots$$

Thus

$$\int_a^b f(x) \left[P_i(x) + \phi_i(x) \right] dx = 0, \quad i = 0, 1, \ldots,$$

although $f(x)$ is not equivalent to zero. This contradicts the completeness of $\{P_i(x) + \phi_i(x)\}$ in L_2. Then $|\delta_{ij} + c_i^j| \neq 0$. Theorem 28 in conjunction with Theorem 15 of 4.1, enables us to say that if $\{P_j(x)\}$ is a sequence of complete, orthonormal, uniformly bounded functions belonging to L_2 on $[a, b]$ and $\phi_n(x) = \sum_{i=0}^{\infty} c_i^n P_i(x)$

where $\sum_{i,n} |c_i^n| < \infty$ then the necessary and sufficient condition that $\{P_j(x) + \phi_j(x)\}$ be complete in L_2 over $[a, b]$ is that $|\delta_{ij} + c_i^j| \neq 0$.

In seeking to generalize Theorem 28 to cases included in Theorem 8, it is necessary to find operators which are generalizations of

$$\int_a^b f(x)\ P_k(x)\,dx \quad \text{and} \quad \int_a^b f(x)\ [P_k(x) + \phi_k(x)]\,dx,$$

operating on f(x). But the generalization of the first operation is clearly $O_k[f(x)]$. From 4.3.7 and 4.3.8 it is seen that (with a slight modification)

$$d_k = \frac{\left|\frac{A_{ij}}{A}\right| (k,\ O_i[f(x)])}{\left|\frac{A_{ij}}{A}\right|}$$

is a generalization of $\int_a^b f(x)\ [P_k(x) + \phi_k(x)]\,dx$. However, this operator is meaningless when A is zero. But it can be transformed in appearence so that it has meaning when A is zero. Multiply both numerator and denominator above formally from the left by the transpose of A. It becomes

4.6.3 $$d_k = \sum_{i=0}^{\infty} (\delta_{ik} + c_i^k)\ O_i\,[f(x)]$$

Now the set of operators $\{d_k\}$, $k = 0, 1, \ldots$, has meaning even when $A = 0$, and reduces to $\int_a^b f(x)\ [P_k(x) + \phi_k(x)]\,dx$ when $O_i[f] = \int_a^b f(x)\,P_i(x)\,dx$. d_k has meaning for every f(x): $|O_i\ [f(x)]| < M$, $i = 0, 1, \ldots$, for example. We are now in a position to generalize Theorem 28.

Theorem 29. Let the operators O_i, $i = 0, 1, \ldots$ be such that the equations

$$O_i[f(x)] = c_i \quad , \; i = 0, 1, \ldots$$

where $\sum_{i=0}^{\infty} c_i^2 < \infty$ implies the existence of an f(x) which satisfies the equations. Here the c_i are any set of numbers satisfying the given restriction . If the operators α_k, $k = 0, 1, \ldots$ form a complete set of operators, then $|\delta_{ij} + c_i^j| \neq 0$.

For assume that the determinant is zero. Then the equations 4.6.1 implies a non-zero solution such that the sum of the squares is convergent. By hypothesis, since the O_i are additive, this implies the existence of an f(x), not the Θ - element such that

$$\sum_{s=0}^{\infty} (\delta_{is} + c_s^i)\ O_s\,[f(x)] = 0, \quad i = 0, 1, \ldots .$$

But this contradicts the completeness of the operators α_i , $i = 0, 1, \ldots$. Hence $|\delta_{ij} + c_i^j| \neq 0$.

At this point a certain duality is becoming clear. If we replace the set of elements $\{\delta_{ij} + c_i^j \ , a_k , o_k\}$ by $\{A_{ij}, \alpha_k, O_k\}$, or if we make the substitution in reverse, analogous theorems are obtained. To complete this duality, an additional theorem will now be proven. Theorem 30 is the analogue of Theorem 18.

Theorem 30. With the notation of Theorem 12 in 3.5, let $y_n = \sum_{i=0}^{\infty} c_i^n\ x_i$, $n = 0, 1, \ldots$, where $|\delta_{ij} + c_i^j| \neq 0$, and $\sum_{i,n} |c_i^n| < \infty$. Assume $O_n[x] < M$, all n. If the O_n , $n = 0, 1, \ldots$, are a complete set of functionals then the set α_n , $n = 0, 1, \ldots$, is complete.

For assume that $\alpha_n[x] = 0$, all n. Then, by 4.6.3,

$$\alpha_k[x] = \sum_{i=0}^{\infty} (\delta_{ik} + c_i^k)\ O_i\ [x] = 0, \; k = 0, 1, \ldots .$$

Since the determinant of the coefficients is normal and different from zero, there is one and only one bounded solution, namely the trivial one. Since the O_n, $n = 0, 1, \ldots$ are a complete set of functionals, then $x = \Theta$. Then α_k, $k = 0, 1, \ldots$ constitutes a complete set of functionals.

4.7 A Sufficient Condition for Non-vanishing of a Normal Determinant.

In the previous section it was shown that the non-vanishing of a given normal determinant is an integral part of the problem. It is then desirable, when given a normal determinant, to have a criterion that it be non-zero. While a general criterion would be very difficult, some cases can be taken care of with comparative ease. The following theorem is an easy extension of a theorem proven by G. B. Price [24] for finite determinants. A similar theorem was proven by Von Koch [29] for normal determinants.

Let $((a_{ij})) = ((\delta_{ij} + c_{ij}))$ denote a matrix where i, j take on the values $1, 2, \ldots$. Let the elements a_{ij} be complex and assume that D, the determinant of the matrix, is normal. Thus $\sum_{i,j} |c_{ij}| < \infty$

Let $D^{(n)} = |\delta_{ij} + c_{ij}|$, $j, i = 1, 2, \ldots, n$. Set

$$m_i^{(n)} = |1 + c_{ii}| - \sum_{j=i+1}^{n} |c_{ij}| \,, \quad M_i^{(n)} = |1 + c_{ii}| + \sum_{j=i+1}^{n} |c_{ij}|.$$

Theorem 31. If $\sum_{\substack{j=1 \\ j\neq i}}^{\infty} |c_{ij}| < |1 + c_{ii}|$, $i = 1, 2, \ldots$, then

$$0 < \prod_{i=1}^{\infty} m_i^{(\infty)} \leqq |D| \leqq \prod_{i=1}^{\infty} M_i^{(\infty)}$$

First, we must show that $m_i^{(\infty)}$, $M_i^{(\infty)}$ have meaning. But $m_i^{(\infty)} =$

$|1 + c_{ii}| - \sum_{j=i+1}^{\infty} |c_{ij}|$. The infinite series here converges as a result of the normality of D. $m_i^{(\infty)}$ therefore has meaning, and it is positive for all i, by hypothesis. The same remarks hold for $M_i^{(\infty)}$, all i. Consider $D^{(n)}$. By the results of G. B. Price [24],

$$0 < m_1^{(n)} \; m_2^{(n)} \; \cdots \; m_n^{(n)} \leqq \left| D^{(n)} \right| \leqq M_1^{(n)} \; M_2^{(n)} \; \cdots \; M_n^{(n)}$$

Since $m_i^{(n+1)} \leqq m_i^{(n)}$ and $M_i^{(n+1)} \geqq M_i^{(n)}$, by hypothesis, then

$$0 < m_1^{(\infty)} \; m_2^{(\infty)} \; \cdots \; m_n^{(\infty)} \leqq \left| D^{(n)} \right| \leqq M_1^{(\infty)} \; M_2^{(\infty)} \; \cdots \; M_n^{(\infty)}$$

Since the determinant D is assumed normal, $\lim_{n \to \infty} D^{(n)}$ exists and is some finite number. We must show that $\prod_{i=1}^{\infty} m_i^{(\infty)}$ exists and does not diverge to zero. Also, it is to be shown that $\prod_{i=1}^{\infty} M_i^{(\infty)}$ exists. Now

$$\prod_{i=1}^{\infty} m_i^{(\infty)} = \prod_{i=1}^{\infty} \left(\left| 1 + c_{ii} \right| - \sum_{j=i+1}^{\infty} \left| c_{ij} \right| \right)$$

$$= \prod_{i=1}^{\infty} \left[1 + \left(\left| 1 + c_{ii} \right| - 1 - \sum_{j=i+1}^{\infty} \left| c_{ij} \right| \right) \right]$$

Consider that

$$\sum_{i=1}^{\infty} \left| \left| 1 + c_{ii} \right| - 1 - \sum_{j=i+1}^{\infty} \left| c_{ij} \right| \right.$$

$$\leqq \sum_{i=1}^{\infty} \left| \left| 1 + c_{ii} \right| - 1 \right. + \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \left| c_{ij} \right| .$$

The second expression is finite by hypothesis. Consider the first expression

$$\sum_{i=1}^{\infty} \left| \left| 1 + c_{ii} \right| - 1 \right| \leqq \sum_{i=1}^{\infty} \left| c_{ii} \right| < \infty .$$

Hence $\prod_{i=1}^{\infty} m_i^{(\infty)}$ converges absolutely. This implies [26, p. 14] that it is not zero. Since all of its factors are positive, it is actually greater than zero. The proof of the existence of $\prod_{i=1}^{\infty} M_i^{(\infty)}$ involves only a change in sign. Hence the theorem is proven.

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