Three Papers on Monetary Aggregation under Knightian Uncertainty, Kernel Estimation, and Dynamic Modelling

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Abstract

This dissertation considers monetary services aggregation theory in Macroeconomics, the nonparametric approaches in Econometrics with censoring data and endogenous variables, and the macroeconomic dynamic modelling. It contributes to the current literature in three ways. First, it extends the monetary aggregation theory to incorporate Knightian uncertainty by using a non-additive probability measure. Our aggregation theory under uncertainty nests the previous literature of the perfect certainty and/or risky cases. Second, we consider a nonparametric estimation of a censoring data model with endogenous variables and transform the problem into a nonparametric LAD additive model for estimation and testing. Third, in the open economy literature, the high consumption correlation among different countries is a stubborn anomaly. We establish an open economy Dynamic Stochastic General Equilibrium (DSGE) model that successfully solves this problem. The model must feature asymmetric preference, incomplete financial markets, and terms of trade shocks at the same time.

The first part considers monetary aggregation under uncertainty aversion (perhaps under risk aversion as well). The presence of uncertainty and the agent's attitude towards it are represented by a nonadditive probability measure. The major findings are three-fold: first, the user cost of monetary assets under uncertainty aversion produces useful boundaries. We no longer have covariances, instead, we have inequalities, and our model nests some of the previously derived results. Second, deviating from expected utility does not exclude the existence of a user-cost solution which is analogous to the expected utility representation, but that is only a special case. Third, under Choquet expectation the user costs have an interval within which no trade of monetary assets will occur, such an effect depends solely on uncertainty aversion, not on risk aversion.

The second paper deals with the problem of nonparametric estimation using censored data in a model that features endogeneity. Nonparametrics with endogenous variables is difficult to handle because of ill-posed inverse problem. Nonparametrics with censoring does not attract the attention as it deserves because people are inclined to resort to quantile estimation when data is censored. We stick to the nonparametric estimation under two mild conditions. It is the endogeneity that shapes the model to be additive, and it is because of censoring the model is reduced to a (nonparametric) LAD estimation under the assumption of conditional zero median of the error term. This paper therefore transforms the problem into a Nonparametric Additive Least Absolute Deviation estimation which is saliently more robust than L_2 norm estimation. We establish the asymptotic normality of the estimated unknown functions. The estimation and inference are easy to carry out.

The third paper establishes a dynamic stochastic general equilibrium model of the Chinese open real economy and aims to give a theoretical account of the empirical stylized facts of economic volatility. Specifically, we investigate two questions: first, what are the stylized facts of the Chinese open economy fluctuation? Is there anything that makes it different from other major economies? Secondly, could theoretical models reasonably explain and fit those facts well? To answer the first question, we use four different filters to extract volatility to contribute a robust summary of the stylized facts. As for the second question, we find that asymmetric preference, incomplete financial markets and terms of trade shocks significantly improve the model's prediction. Negative international co-movement of investment is the special feature of the Chinese economy, and our model caters for that well.

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This dissertation is dedicated to my beloved family. My parents did not only give me life, but also endless unconditional support so that I am free to pursuit the voice echoing from the deep recess of my heart.

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Chapter 1 Monetary Services Aggregation Theory under Knightian Uncertainty[1]

1 Introduction

In this paper we consider monetary aggregation theory under non-expected utility and derive the model implications for monetary asset user costs and optimal portfolio selection, when the agents are uncertainty averse (perhaps risk averse as well). Specifically, by non-expected utility, we mean for one thing that the utility deviates from linear probabilistic additivity, which is pervasively accepted as a representation of rational expectation models. For another, we allow the utility function, in a dynamical context, to exhibit a recursive structure: current period utility depends on expected future utility as well as on current consumption. Under the assumption that this recursive dependence is time-separable, the non-expected utility we use is expected utility under a nonadditive probability measure. The resulting model aims to separate the more subtle "uncertainties" from quantifiable "risk".

The objective of this paper is to unravel the implications of monetary aggregation when consumers' behavior deviates from expected utility. The literature on monetary services aggregation derived from aggregation theory began with Barnett (1978, 1980). Because simple sum monetary aggregates like M1, M2 are inconsistent with economic theory, the idea of separating investment motive from services motive of monetary assets if we want to measure money correctly (Divisia index) has been of central importance to economists and central banks. Development of these disciplines has been burgeoning fruitful results, including among many others, Barnett (1995) who considers monetary aggregation under risk; Barnett, Liu, and Jensen (1997) who connect Divisia to CAPM models, and this connection is further explored by Barnett and Wu (2005) who generate even larger CAPM user cost risk adjustment. Keating and Lee Smith (2018) dexterously test the usefulness of monetary aggregates in Taylor rules in the framework of a rational expectations model. Kelly, Barnett and Keating (2011) show how measurement error in money is associated with liquidity puzzle.

What is unknown in this realm, however, is what will become of the monetary aggregation theory, if people's behavior deviates from expected utility theory. Money assets are durable

[1] A slightly revised version of this paper is forthcoming in *Journal of Economic Behavior and Organization* under the title "Monetary services aggregation under uncertainty: A behavioral economics extension using Choquet expectation", coauthored with William A. Barnett and Jianbo Zhang.

goods and thus have user cost prices. Deriving those user costs is a fundamental step in producing monetary aggregates from economic aggregation theory. As a result, our research begins by determining the implications of non-expected utility for the user costs of monetary assets.

The expected utility of von Neumann and Morgenstern, which is further rationalized by Savage (1954) using a prior subjective probability that sums up to one, has been the building block of the largest amount of economic models. Yet Allais (1953) and Ellsberg (1961) paradox find that human being's behavior frequently falls outside the prediction of expected utility. One group of models, seeking to generalize expected utility theory, distinguishes between risk and uncertainty, as defined by Knight (1921) and further developed by Bewley (2002). In that literature, risk exists when economic agents know the objective probabilities, which do sum to one. Under uncertainty, the objective probabilities are not known to agents, and the resulting behavior of economic agents need not to be representable by a subjective probability distribution, having the same measure theoretic properties of the unknown probability distribution. Although the subjective joint probability of the union of all possible outcomes is necessarily one, the sum of the probabilities of each those independent, separate outcomes is not necessarily one.[2]

We follow that approach. In particular, the model we use is built on a nonadditive probability measure. This approach has its foundations in Schmeidler's findings: if probability reflects people's willingness to bet, this probability needs not be additive. An axiomatic treatment of nonadditive probability models can be found in Schmeidler (1986, 1989), Gilboa (1987, 2009), and Gilboa and Schmeidler (1989).

We find that nonadditive probability measure yields boundaries to the user cost of monetary assets, depending on whether the marginal utility and rate of return are comonotonic or countermonotonic. This does not mean, however, that user costs under nonadditive expectation are only subject to inequality solutions. If there exists an underlying probability measure to properly define the nonadditive probability from the subjective additive probability, we find that the user cost has a rank-dependent expected utility representation. This solution has an expected utility form, but uses transformed distorted additive probability as weights. The rank dependence is much less restrictive than might appear to be the case, since there is always a permutation to line up the objective function in an ascending/descending order.

[2] For a formal definition of nonadditive probability, see the first paragraph of section 2.1 below.

We also find that under optimality there is a user cost interval within which the agent will not hold any position in the monetary asset. When the user cost is below the lower limit of this interval, she will want to buy more of the monetary asset. When the user cost is above the upper limit of this interval, she will want to sell the monetary asset (short). The two limits of this interval constitute the reserve prices for transactions, if the agent's belief reflects uncertainty aversion. This result does not hinge on her attitude towards risk. Our model thus is capable of explaining why there are situations under which people are not active in changing their monetary asset portfolios. A reasonable individual may not behave consistently with Savage's model. Maximizing utility under nonadditive prior can provide a useful rationale for observed behavior in the market. When probability becomes additive, the model reduces to von Neumann-Morgenstern expected utility case. The existing publications on monetary aggregation under risk become special cases of our analysis and hence are fomally nested within our theory.

The rest of the paper is organized as follows. In section 2 we introduce the model and the associated nonadditive probability measure, solve for the user cost under uncertainty aversion, and derive the user cost boundaries. In section 3 we find the conditions under which the user cost has a rank-dependent expected utility representation. In section 4 we consider the consumer's problem from an asset pricing perspective and demonstrate our main theorem providing the user cost interval within which no trade will happen. In section 5 we conclude the paper. The appendix contains the mathematical proofs of theorems and useful lemmas.

2 The Model

2.1 Utility Function and Uncertainty Averse

When we say the probability is nonadditive, we mean that if A and B are two disjoint events in the sample space Ω, such that $A \cup B = \Omega$, their probabilities being $v(A)$ and $v(B)$ respectively, $v(A) + v(B) \neq 1$, although $v(A \cup B) = 1$. As explained below, uncertainty aversion will imply $v(A) + v(B) < 1$. Under a nonadditive probability measure, the proper way to define an integral is no longer Riemann but Chquet. Under these conditions, Riemann integration suffers from discontinuity, nonmonotonicity, and ambiguity (dependence upon the way we write utility functions). Suppose there is a function $f \geq 0$, then the Choquet (1954) integral integrates over rectangles horizontally:

$$\int f dv = \int_0^{+\infty} v\left(s\left|f\left(s\right) \geq t\right.\right) dt,$$

where the right hand side is a standard Riemann integral. Choquet integral has many attractive properties, such as reflecting linear translations multiplied by a positive coefficient. But generally it is not additive, unless the functions under evaluation are comonotonic, a property that will be relevant to some of our subsequent results.

Uncer uncertainty, the utility function under our consideration is in the form of nonexpected utility[3] as follows:

$$V_t = U\left(c_t, \mathbf{m}_t, E_t^C V_{t+1}\right) = u\left(c_t, \mathbf{m}_t\right) + \beta \int V_{t+1} dv, \tag{1}$$

where c_t is the date t consumption of goods, $\mathbf{m}_t$ is the vector of monetary assets, and $E_t^C V_{t+1}$ is the expected future utility, conditional on all information at time t. We use a superscript C on the expectation operator to denote Choquet expectation. In this uncertainty context, $U\left(\cdot\right)$ is the aggregator function through which current consumption, all monetary assets, and expected future utility are aggregated. We follow canonical Macroeconomic models to allow time separability, where β is the discount factor and V_{t+1} is tomorrow's utility in each of tomorrow's states. Without the separability assumption, the discount factor would be the derivative of $U\left(\cdot\right)$ with respect to its third argument.

We further assume there exists a linearly homogenous aggregator function $M_t = M\left(\mathbf{m}_t\right)$, such that:

$$u\left(c_t, \mathbf{m}_t\right) = F\left[c_t, M\left(\mathbf{m}_t\right)\right]. \tag{2}$$

In this paper, additive probability is denoted by P, while capacity (nonadditive probability or "charge") is denoted by v, so that $\int\left(\cdot\right) dv$ is Choquet integral.

More formally, suppose that S is a finite set of states of nature, and in every period there are a finite number of n different states. Let $\mathcal{F}$ be the σ-algebra generated by the events on S. Then capacity v on a measurable space $\left(S, \mathcal{F}\right)$ is a real-valued set function $v : \mathcal{F} \to [0, 1]$ such that $v\left(\phi\right) = 0$, $v\left(S\right) = 1$, and $v\left(A\right) \leq v\left(B\right)$ for all $A \subseteq B \in \mathcal{F}$. An

[3] Distinguishing attitudes towards risk from behavior towards intertemporal substitution is beyond the scope of this paper. Once we include monetary assets in the utility function, the effects of Epstein and Zin (1989) or Weil's (1990) generalized isoelastic utility are much harder to find. But it could be a topic worth pursuing.

example of capacity could be $v = P^{\alpha}$. In this case $\alpha \in (0,1) \cup \{1\} \cup (1,+\infty)$ measures the agent's attitude towards uncertainty. If $\alpha = 1$, then capacity reduces to an additive prior, and the probability measure is both concave and convex. We emphasize that the fact that probability is nonadditive itself represents both the presence of uncertainty and the agent's attitude towards it.

Using the example from the beginning of this section, $v(A) + v(B) < 1 = v(A \cup B)$ is equivalent to concluding that the agent's decisions reflect uncertainty aversion.[4] Schmeidler (1986, 1989) defines uncertainty aversion in terms of probability capacity by:

$$v(A) + v(B) \leq v(A \cup B) + v(A \cap B), \tag{3}$$

although that definition is not universally accepted. That condition is also known as supermodularity, convexity, or 2-monotonicity of v.

The states of nature are a natural partition of the sample space S. If today's nature is denoted by s, we denote the nature tomorrow by s'. With a somewhat informal notation for V, it can be useful to rewrite the utility function (1) in terms of states for any given sequence $\mathbf{X} = \{c_t, \mathbf{m}_t\}_{t=0}^{T}$, as

$$V(\mathbf{X}) = \lim_{T\to\infty} \left\{ u(c_0, \mathbf{m}_0) + \beta \int \left[u(c_1, \mathbf{m}_1) + \cdots + \beta \int u(c_T, \mathbf{m}_T)\, v_{s_{T-1}}(ds_T) \cdots \right] v_{s_0}(ds_1) \right\}. \tag{4}$$

This facilitates the calculation of Choquet integral using Riemann integrals.

2.2 Equilibrium

The agent holds two types of assets, monetary assets and nonmonetary assets. Nonmonetary assets provide only investment return, while monetary assets provide both investment return and monetary service flows, which we seek to measure. The budget constraints are:

$$W_t = p_t c_t + \sum_{i=1}^{L} p_t m_{it} + \sum_{j=1}^{K} p_t k_{jt} \tag{5}$$

$$W_{t+1} = \sum_{i=1}^{L} R_{i,t+1} p_t m_{it} + \sum_{j=1}^{K} \tilde{R}_{j,t+1} p_t k_{jt} + y_{t+1} \tag{6}$$

[4] We avoid use of the word "ambiguity", which is usually defined to mean that the agent vaguely perceives the probability of a particular state in a range. This possibility is out of the scope of this paper.

where W_t is the agent's wealth in period t, p_t is the true cost of living index, c_t is consumption of goods, and y_{t+1} is income from all other sources, received at the beginning of $t+1$. The variables, m_{it} and k_{jt}, denote the quantities of monetary asset i and nonmonetary asset j respectively. The interest rate $R_{i,t+1}$ is the gross rate of return of holding the monetary asset m_{it} between periods t and $t+1$, while the interest rate $\tilde{R}_{j,t+1}$ is the gross return of nonmonetary asset k_{jt} from t to $t+1$. Suppose L and K are the number of two types of assets in the agent's portfolio, since nonmonetary assets do not provide service flows, other than their investment rates of return, it follows that $\tilde{R}$ is higher than R. Combining equation (5) and (6) yields the following flow of funds equation:

$$p_t c_t = \sum_{i=1}^{L} \left[R_{it} p_{t-1} m_{i,t-1} - p_t m_{it}\right] + \sum_{j=1}^{K} \left[\tilde{R}_{jt} p_{t-1} k_{j,t-1} - p_t k_{jt}\right] + y_t. \tag{7}$$

Hence, the individual's consumption of goods is funded each period from the proceeds from rolling over the monetary assets and nonmonetary assets, and from all other income. Note that equation (7) is the one used in Barnett (1980) and Barnett, Liu and Jensen (1997) to facilitate comparison of our results with the existing literature.

The agent maximizes the lifetime discounted utility (4), subjects to the flow of funds constraint (7). The resulting Bellman equation is:

$$\begin{aligned} V_s(W_t) &= \sup_{\{c_t, \mathbf{m}_t, \mathbf{k}_t\}} \left\{ u(c_t, \mathbf{m}_t) + \beta \int V_{s'}(W_{t+1})\, v_s(ds') \right\} \\ s.t.\;\; p_t c_t &= \sum_{i=1}^{L} \left[R_{it} p_{t-1} m_{i,t-1} - p_t m_{it}\right] + \sum_{j=1}^{K} \left[\tilde{R}_{jt} p_{t-1} k_{j,t-1} - p_t k_{jt}\right] + y_t \end{aligned} \tag{8}$$

Here $V_s(W_t)$ denotes Bellman value function. The agent is also subject to the following transversality condition:

$$\lim_{t\to\infty} \beta^t \frac{\partial V^*}{\partial W_t^*} W_t^* = 0, \tag{9}$$

with $*$ denoting the solution value from the optimization.

After substituting from the Benveniste-Scheinkman equation, the first order conditions (Euler equations) with respect to consumption become:

$$\frac{\partial u}{\partial c_t} = \beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} \tilde{R}_{j,t+1} \frac{p_t}{p_{t+1}} \right], \tag{10}$$

while the first order conditions with respect to monetary assets become:

$$\frac{\partial u}{\partial m_{it}} = \frac{\partial u}{\partial c_t} - \beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} R_{i,t+1} \frac{p_t}{p_{t+1}} \right]. \tag{11}$$

The contemporaneous real user-cost price of the services of monetary asset i is the marginal rate of substitution between monetary asset and consumption,

$$\pi_{it} = \frac{\frac{\partial u}{\partial m_{it}}}{\frac{\partial u}{\partial c_t}} = \frac{\frac{\partial u}{\partial c_t} - \beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} R_{i,t+1} \frac{p_t}{p_{t+1}} \right]}{\beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} \tilde{R}_{j,t+1} \frac{p_t}{p_{t+1}} \right]} = \frac{\frac{\partial u}{\partial c_t} - \beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1} \right]}{\beta E_t^C \left[\frac{\partial u}{\partial c_{t+1}} \tilde{r}_{j,t+1} \right]}. \tag{12}$$

For notational convenience, we convert the nominal gross returns, $R_{i,t+1}$ and $\tilde{R}_{j,t+1}$, to the corresponding real gross rates of return, $r_{i,t+1} = R_{i,t+1} \frac{p_t}{p_{t+1}}$ and $\tilde{r}_{j,t+1} = \tilde{R}_{j,t+1} \frac{p_t}{p_{t+1}}$. Since the expectation $E_t^C(\cdot)$ is not additive, it is the Choquet integral.

Also note that under the weak separability condition (2), we have:

$$\frac{\partial u}{\partial m_{it}} = \frac{\partial F}{\partial M_t} \frac{\partial M_t}{\partial m_{it}}.$$

Substituting the definition of the user cost, we acquire:

$$\frac{\partial M_t}{\partial m_{it}} = \pi_{it} \frac{\frac{\partial u}{\partial c_t}}{\frac{\partial F}{\partial M_t}}.$$

Taking the total differential of the monetary aggregator function, $M_t = M(\mathbf{m}_t)$, yields:

$$dM_t = \sum_{i=1}^{L} \frac{\partial M_t}{\partial m_{it}} dm_{it} = \frac{\frac{\partial u}{\partial c_t}}{\frac{\partial F}{\partial M_t}} \sum_{i=1}^{L} \pi_{it} dm_{it} = \frac{\frac{\partial u}{\partial c_t}}{\frac{\partial F}{\partial M_t}} \sum_{i=1}^{L} \pi_{it} m_{it} d \log m_{it}. \tag{13}$$

Since $M(\mathbf{m}_t)$ is linearly homogenous of degree one, Euler theorem applies to M_t,

$$M_t = \sum_{i=1}^{L} \frac{\partial M_t}{\partial m_{it}} m_{it} = \frac{\frac{\partial u}{\partial c_t}}{\frac{\partial F}{\partial M_t}} \sum_{i=1}^{L} \pi_{it} m_{it}. \tag{14}$$

Dividing equation (13) by (14) yields the Divisia index

$$d \log M_t = \sum_{i=1}^{L} s_{it} d \log m_{it}, \tag{15}$$

where $s_{it} = \frac{\pi_{it} m_{it}}{\sum_{l=1}^{L} \pi_{lt} m_{lt}}$ is the user cost valued expenditure share. We conclude that the resulting Divisia quantity index is in exactly the same form as in Barnett (1980), with the only difference being that the user costs now are computed under a nonadditive probability measure.

2.3 User-Cost Boundaries

We now return to the user-cost, π_{it}, in equation (12). Because the expectation is non-additive, we no longer have $E_t\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right] = Cov\left(\frac{\partial u}{\partial c_{t+1}}, r_{i,t+1}\right) + E_t\left(\frac{\partial u}{\partial c_{t+1}}\right) E_t\left(r_{i,t+1}\right).$ Instead, we have the following theorem:

Theorem 1 *If* $\frac{\partial u}{\partial c_{t+1}}, r_{i,t+1} \geq 0$ *are comonotonic, then*

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right] \geq E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right) E_t^C\left(r_{i,t+1}\right). \tag{16}$$

If v *is submodular, while* $\frac{\partial u}{\partial c_{t+1}}$ *and* $r_{i,t+1}$ *are countermonotonic, then:*

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right] \leq E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right) E_t^C\left(r_{i,t+1}\right). \tag{17}$$

The proof of this theorem is in the appendix. Comonotonicity is defined as follows. For every pair of states $s_1', s_2' \in S$,

$$\left[\frac{\partial u}{\partial c_{t+1}}\left(s_1'\right) - \frac{\partial u}{\partial c_{t+1}}\left(s_2'\right)\right]\left[r_{i,t+1}\left(s_1'\right) - r_{i,t+1}\left(s_2'\right)\right] \geq 0. \tag{18}$$

That is, the marginal utility and the rate of return increase or decrease at the same time. Countermonotonicity just reverses the direction of the above inequality. Hence, under non-additive probabilities, we do not have covariances, but we have inequalities. Equation (17) corresponds to uncertainty loving which is unusual, but it is not as unlikely as it seems to be. If we think about gain-loss asymmetry, when people particularly hate to lose what they have already had, such an extreme loss aversion might lead people to behave in an uncertainty loving way in the domain of losses.

Barnett, Liu, and Jensen (1997) proved, in their Theorem 1, that the user cost of services of monetary assets under risk aversion has an additional adjustment term not appearing in the risk free user cost. That adjustment term is about covariances, as in all CCAPM risk adjustments. The Barnett, Liu, and Jensen's risk adjusted user cost is a special case of our result. If the probability measure is additive, so that uncertainty is removed, risk aversion is all that is left. Then the Choquet expectation in equation (12) becomes the linearly additive expectation, and covariances appear.

When the agent is not only risk averse but also uncertainty averse, then equation (12) cannot be further simplified by collecting covariances. We end up with inequalities giving rise to boundaries on user costs. In the next section, we will see that equality solutions do exist for $E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right]$, but those again hold as special cases of Choquet expectation. Our case nests Barnett, Liu and Jensen's (1997) result. If we further assume away both uncertainty aversion and risk aversion, we will have the perfect certainty case. Then equation (12) reduces to the user cost derived in Barnett (1980).

It is convenient to work on rates of returns, $r_{i,t+1}$, which are usually assumed to be stationary, so that taking averages is meaningful. But marginal utility, $\frac{\partial u}{\partial c_{t+1}}$, is not observed and difficult to estimate. Therefore we reinterpret equation (12) in terms of a stochastic discount factor, which, although still not observable, is much easier to estimate. We assume the agent has not passed the blissful point, so that $\frac{\partial u}{\partial c_t} > 0$. Given date t information, uncertainty at time t has been resolved, and $\frac{\partial u}{\partial c_t}$ can be treated as a constant. By the positive homogeneity of Choquet integral, equation (12) can be written as

$$\pi_{it} = \frac{1 - E_t^C\left[\beta \frac{\partial u/\partial c_{t+1}}{\partial u/\partial c_t} r_{i,t+1}\right]}{E_t^C\left[\beta \frac{\partial u/\partial c_{t+1}}{\partial u/\partial c_t} \tilde{r}_{j,t+1}\right]} = \frac{1 - E_t^C\left[Q_{t+1} r_{i,t+1}\right]}{E_t^C\left[Q_{t+1} \tilde{r}_{j,t+1}\right]}, \tag{19}$$

where we denote by $Q_{t+1} = \beta \frac{\partial u/\partial c_{t+1}}{\partial u/\partial c_t}$ the pricing kernel. Note that from equation (10) and (11) we have respectively

$$1 = E_t^C\left[Q_{t+1} \tilde{r}_{j,t+1}\right], \tag{20}$$

$$\pi_{it} = 1 - E_t^C\left[Q_{t+1} r_{i,t+1}\right]. \tag{21}$$

Based on equation (19), a reinterpretation of Theorem 1 is that, if $Q_{t+1}, r_{i,t+1} \geq 0$ and $Q_{t+1}, \tilde{r}_{j,t+1} \geq 0$ are both comonotonic, then:

$$\pi_{it} \leq \frac{1 - E_t^C\left(Q_{t+1}\right) E_t^C\left(r_{i,t+1}\right)}{E_t^C\left(Q_{t+1}\right) E_t^C\left(\tilde{r}_{j,t+1}\right)}. \tag{22}$$

If v is submodular, Q_{t+1} and the rate of return on both the monetary and non-monetary assets are countermonotonic, we have:

$$\pi_{it} \geq \frac{1 - E_t^C\left(Q_{t+1}\right) E_t^C\left(r_{i,t+1}\right)}{E_t^C\left(Q_{t+1}\right) E_t^C\left(\tilde{r}_{j,t+1}\right)}. \tag{23}$$

Since returns tend to move together, the dual satisfaction of comonotonic (or countermonotonic) with Q_{t+1} is not restrictive.

Therefore when the probability measure is nonadditive, Choquet expectation produces boundaries to the user cost of monetary assets. More specifically, assume the expected consumption is before the blissful point, the real rates of return on both types of assets are positive, and the substitution effect on intertemporal consumption dominates (so that the comonotonicity between Q_{t+1} and $r_{i,t+1}$ ($\tilde{r}_{j,t+1}$) is satisfied). Then the calculated user cost should be lower than $\frac{1-E_t^C(Q_{t+1})E_t^C(r_{i,t+1})}{E_t^C(Q_{t+1})E_t^C(\tilde{r}_{j,t+1})}$. On the other hand, if the agent were uncertainty loving, meanwhile the income effect wins out in the intertemporal allocation of consumption (so that the countermonotonicity between Q_{t+1} and $r_{i,t+1}$ ($\tilde{r}_{j,t+1}$) is satisfied). Then any calculated user cost would be incorrect, if it were lower than $\frac{1-E_t^C(Q_{t+1})E_t^C(r_{i,t+1})}{E_t^C(Q_{t+1})E_t^C(\tilde{r}_{j,t+1})}$.

3 Rank-Dependent Representation

The existence of derived boundaries is not the only result under nonadditive probabilities. In this section we show that under some circumstances, there exists a linear solution for equation (21). Suppose $\mathbf{P} = \left(P_1, P_2, \cdots, P_n\right)^T$ is an additive probability vector satisfying $\sum_{s=1}^n P_s = 1$, and suppose there is a probability measure μ such that for some nondecreasing function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$, the capacity $v = f(\mu)$ is well-defined. Then a new, additive, probability vector $\mathbf{P}^{\uparrow}$ is permissible to order events as follows:

$$\begin{aligned}\mathbf{P}^{\uparrow} &= \left(P_1^{\uparrow}, P_2^{\uparrow}, \cdots, P_{n-1}^{\uparrow}, P_n^{\uparrow}\right)^T \\ &= \left[1 - f\left(\sum_{s\geq 2} P_s\right), f\left(\sum_{s\geq 2} P_s\right) - f\left(\sum_{s\geq 3} P_s\right), \cdots, f\left(\sum_{s\geq n-1} P_s\right) - f\left(P_n\right), f\left(P_n\right)\right]^T.\end{aligned} \tag{24}$$

If the agent is uncertainty loving, $f(\cdot)$ should be convex, in this case higher states are weighted less. Such a transformed probability is tailored for accumulative lottery outcomes, where $x_1 \leq \cdots \leq x_n$ are in a lottery $(x_1, P_1; \cdots; x_n, P_n)$. This observation is a reason we choose the notation $\uparrow$ on the left side of equation (24). Take the distorted probability as an example, in which $f(\mu) = \mu^\alpha$, where $\alpha \in (0, +\infty)$, and μ is the probability measure relative to which the additive probabilities P_s are given. Then $P_t^\uparrow = \left(\sum_{s=t}^n P_s\right)^\alpha - \left(\sum_{s=t+1}^n P_s\right)^\alpha$, so that the above probability vector $\mathbf{P}^\uparrow$ becomes

$$\mathbf{P}^\uparrow = \left[1 - \left(\sum_{s\geq 2} P_s\right)^\alpha, \left(\sum_{s\geq 2} P_s\right)^\alpha - \left(\sum_{s\geq 3} P_s\right)^\alpha, \cdots, \left(\sum_{s\geq n-1} P_s\right)^\alpha - (P_n)^\alpha, (P_n)^\alpha\right]^T.$$

The higher states are weighted less in this example when, $\alpha > 1$.

Similarly, we define another probability vector, $\mathbf{P}^\downarrow$, for decumulative outcomes $x_1 \geq \cdots \geq x_n$ as follows:

$$\begin{aligned}\mathbf{P}^\downarrow &= \left(P_1^\downarrow, P_2^\downarrow, \cdots, P_{n-1}^\downarrow, P_n^\downarrow\right)^T \qquad (25)\\ &= \left[f(P_1), f\left(\sum_{s\leq 2} P_s\right) - f(P_1), \cdots, f\left(\sum_{s\leq n-1} P_s\right) - f\left(\sum_{s\leq n-2} P_s\right), 1 - f\left(\sum_{s\leq n-1} P_s\right)\right]^T\end{aligned}$$

If the agent is uncertainty loving, higher states are weighted more. This approach is also the method proposed by Yaari (1987) to deal with the violation of continuity and monotonicity in Kahneman and Tversky's (1979) prospect theory. We therefore have the following lemma showing that Choquet expectation has an expected utility solution, but with a transformed probability measure on ordered utilities.

Lemma 2 *Suppose P is an additive probability measure, for any capacity $v = f(\mu)$ that is well supported by the probability measure μ, and for any nonnegative function $u \in \mathbb{R}_+^n$, the Choquet integral has a rank-dependent expected utility representation:*

$$\int u_s v(ds) = \mathbf{u}^T \mathbf{P}^\uparrow = \sum_{s=1}^n u_s P_s^\uparrow \quad \textit{if } u \textit{ is weakly increasing in } s, \qquad (26)$$

$$\int u_s v(ds) = \mathbf{u}^T \mathbf{P}^\downarrow = \sum_{s=1}^n u_s P_s^\downarrow \quad \textit{if } u \textit{ is weakly decreasing in } s, \qquad (27)$$

where $\mathbf{P}^\uparrow$ and $\mathbf{P}^\downarrow$ are state-reweighted probability vectors defined above.

The proof of the lemma is in the appendix. With this result, if $Q_{t+1} r_{i,t+1}$ is weakly increasing in s', as can always be done by permutation, we have

$$\pi_{it} = 1 - E_t^C\left[Q_{t+1} r_{i,t+1}\right] = 1 - \sum_{s'=1}^{n} Q_{t+1} r_{i,t+1} P_{s'}^{\uparrow}. \tag{28}$$

If $Q_{t+1} r_{i,t+1}$ is weakly decreasing in s', then

$$\pi_{it} = 1 - E_t^C\left[Q_{t+1} r_{i,t+1}\right] = 1 - \sum_{s'=1}^{n} Q_{t+1} r_{i,t+1} P_{s'}^{\downarrow}. \tag{29}$$

Therefore, in addition to deriving inequality bounds, we also have an alternative solution. Choquet expectation relative to v coincides with an expected utility model defined by $f(\cdot)$. This expected utility requires rank dependence, so that the product $Q_{t+1}(s') \, r_{i,t+1}(s')$ must be either weakly increasing or weakly decreasing in s'. It's important to emphasize that the correspondence between Choquet expectation and the rank-dependent representation does not always exist. Rather, the rank-dependent expected utilities are a special case of Choquet expected utility, a case in which the underlying probability measure μ exists and contains sufficient information to define v. The Ellsberg paradox is a violation of this condition and therefore has no rank-dependence representation. In all those cases, there does not exist an underlying measure which provides us all we need to know about events. Potentially, Choquet expectation is more general, in that it allows us to work on scenarios during which our capabilities of defining probabilities are limited.

Note that equation (20) also features a similar rank-dependent solution:

$$\sum_{s'=1}^{n} Q_{t+1} \tilde{r}_{j,t+1} P_{s'}^{\uparrow} = 1 \quad \text{if } Q_{t+1} \tilde{r}_{j,t+1} \text{ is weakly increasing in } s', \tag{30}$$

$$\sum_{s'=1}^{n} Q_{t+1} \tilde{r}_{j,t+1} P_{s'}^{\downarrow} = 1 \quad \text{if } Q_{t+1} \tilde{r}_{j,t+1} \text{ is weakly decreasing in } s'. \tag{31}$$

These two equations provide a useful guidance for estimating the stochastic discount factor when uncertainty aversion is involved. We can compare equation (20) with the classical asset pricing theory under additive priors. In that case, returns should follow

$$1 = E_t\left[Q_{t+1} \tilde{r}_{t+1}\right]. \tag{32}$$

That is, one dollar paid today is weighted against how many dollars or units of consumption the agent will get in return tomorrow. Nevertheless, if the decision also involves attitude towards uncertainty, we now see that equation (32) becomes $1 = E_t^C\left[Q_{t+1}\tilde{r}_{t+1}\right]$. With the implication of Lemma 2, it becomes clear that even if people manage to evaluate this true equation, as in (30) and (31), the result would still be a special case of our more general theory.

4 Monetary Asset Choice under Uncertainty Aversion: A No Trade Interval

In this section we reverse the perspective by looking at the monetary asset portfolio choice problem. Given v is a probability measure, the value of expected discounted real rate of return exhibits linearity and translation invariance; that is, $E_t^C\left[\alpha Q_{t+1}r_{i,t+1} + \beta\right] = \alpha E_t^C\left[Q_{t+1}r_{i,t+1}\right] + \beta$ if $\alpha \geq 0$, $\beta \in \mathbb{R}$. But property does not hold when α is negative. Therefore, we consider $-E_t^C\left[-Q_{t+1}r_{i,t+1}\right]$ instead, giving rise to the following lemma:

Lemma 3 *If the agent is uncertainty averse, the Choquet expected value satisfies:*

$$-E_t^C\left[-Q_{t+1}r_{i,t+1}\right] > E_t^C\left[Q_{t+1}r_{i,t+1}\right]. \tag{33}$$

The proof of this lemma is relegated to the appendix. Intuitively, adding a constant to a random variable or multiplying a random variable by a positive number will linearly shift the Choquet expectation. This relationship does not hold for negative multipliers. The nonadditivity of the probability causes an asymmetric effect, it turns out $-E_t^C\left[-Q_{t+1}r_{i,t+1}\right] > E_t^C\left[Q_{t+1}r_{i,t+1}\right]$. It is this interval that leads to non-transaction of the monetary asset i. There will be a range of discounted returns from $E_t^C\left[Q_{t+1}r_{i,t+1}\right]$ to $-E_t^C\left[-Q_{t+1}r_{i,t+1}\right]$, within which the agent neither want to buy nor to sell the monetary asset. If the discounted return $E_t^C\left[Q_{t+1}r_{i,t+1}\right]$ is larger than 1, she will want to buy the monetary asset. If the discounted return $-E_t^C\left[-Q_{t+1}r_{i,t+1}\right]$ is lower than 1, she will want to sell this monetary asset (short).

To prove this result, we assume the utility function, $u \geq 0$, is twice continuously differentiable with $u' > 0$, $u'' < 0$. We use Jensen's inequality to prove this result. But first we need to verify whether Jensen's inequalities hold under a nonadditive probability measure.

Lemma 4 *Let $(S, \mathcal{F}, \mathcal{V})$ be a nonadditive probability space, $s' \in S$, and let $v \in \mathcal{V}$ be capacity.*

Suppose $x_{t+1}(s')$ is choquet integrable. If u is a concave function on $[0,+\infty)$, then Jensen's inequality follows:

$$u\left\{E_t^C\left[x_{t+1}(s')\right]\right\} \geq E_t^C\left\{u\left[x_{t+1}(s')\right]\right\}. \tag{34}$$

Proof: We produce a second order Taylor expansion of $u(x_{t+1}(s'))$ around $E_t^C(x_{t+1}(s'))$:

$$u(x) = u\left[E^C(x)\right] + u'\left[E^C(x)\right]\left[x - E^C(x)\right] + \frac{1}{2}u''(\xi)\left[x - E^C(x)\right]^2,$$

taking Choquet expectation on both sides, we have:

$$\begin{aligned}
E^C\left[u(x)\right] &= E^C\left\{u\left[E^C(x)\right] + u'\left[E^C(x)\right]\left[x - E^C(x)\right] + \frac{1}{2}u''(\xi)\left[x - E^C(x)\right]^2\right\} \\
&= u\left[E^C(x)\right] + E^C\left\{u'\left[E^C(x)\right]\left[x - E^C(x)\right] + \frac{1}{2}u''(\xi)\left[x - E^C(x)\right]^2\right\} \\
&\leq u\left[E^C(x)\right] + E^C\left\{u'\left[E^C(x)\right]\left[x - E^C(x)\right]\right\} \\
&= u\left[E^C(x)\right] + u'\left[E^C(x)\right]E^C\left[x - E^C(x)\right] \\
&= u\left[E^C(x)\right] + u'\left[E^C(x)\right]\left[E^C(x) - E^C(x)\|v\|\right] \\
&= u\left[E^C(x)\right],
\end{aligned}$$

where $\xi = \lambda x_{t+1}(s') + (1-\lambda)E_t^C\left[x_{t+1}(s')\right]$ with $\lambda \in [0,1]$. The convex case can be proved similarly. *Q.E.D.*

The second line holds because of comonotonic additivity, the constant is comonotonic with any variable. The third line holds because the Choquet integral is monotone and $u''(\xi) < 0$, $\left[x - E^C(x)\right]^2 \geq 0$. The fourth line is true because of positive homogeneity, $u'\left[E^C(x)\right]$ is a constant. The rest holds because of translation invariance.

So Jensen's equalities are still satisfied under nonadditive probabilities, and they are satisfied in multivariate case. We now have the main result, as follows:

Theorem 5 *Consider a risk neutral or risk averse agent with wealth W_t, who is considering investing m_{it} on a monetary asset, yielding a real rate of return $r_{i,t+1}$. Suppose the two conditions in Lemma 4 are satisfied. Denoting the Choquet expected discounted rate of return by $E_t^C\left[Q_{t+1}r_{i,t+1}\right]$, she will buy this monetary asset if $1 < E_t^C\left[Q_{t+1}r_{i,t+1}\right]$, or equivalently $\pi_{it} < 0$. She will sell the asset (short) if $1 > -E_t^C\left[-Q_{t+1}r_{i,t+1}\right]$, or equivalently $\pi_{it} > -E_t^C\left[-Q_{t+1}r_{i,t+1}\right] - E_t^C\left[Q_{t+1}r_{i,t+1}\right]$.*

We only sketch the proof. Suppose the agent spends m_{it} on this monetary asset. Then by Jensen's inequality

$$E_t^C \{u [W_t - m_{it} + m_{it} \cdot Q_{t+1} r_{i,t+1}]\} \leq u \left\{E_t^C [W_t - m_{it} + m_{it} \cdot Q_{t+1} r_{i,t+1}]\right\} \leq u (W_t) . \tag{35}$$

The last inequality holds, if $E_t^C [Q_{t+1} r_{i,t+1}] \leq 1$. Therefore the individual is at least as well off not buying anything as holding a positive position in monetary asset i. Analogous arguments give rise to selling the asset, if $1 > -E_t^C [-Q_{t+1} r_{i,t+1}]$. In this circumstance,

$$\begin{aligned} & 1 - \left[-E_t^C [-Q_{t+1} r_{i,t+1}]\right] \qquad (36) \\ = \quad & 1 - E_t^C [Q_{t+1} r_{i,t+1}] - \left[-E_t^C [-Q_{t+1} r_{i,t+1}]\right] + E_t^C [Q_{t+1} r_{i,t+1}] > 0, \end{aligned}$$

since $\pi_{it} = 1 - E_t^C [Q_{t+1} r_{i,t+1}]$, this condition is equal to $\pi_{it} > -E_t^C [-Q_{t+1} r_{i,t+1}] - E_t^C [Q_{t+1} r_{i,t+1}]$, and by Lemma 3 this difference is positive. *Q.E.D.*

Hence $\left[0, -E_t^C [-Q_{t+1} r_{i,t+1}] - E_t^C [Q_{t+1} r_{i,t+1}]\right]$ is a range of user costs with no trade under uncertainty aversion. If the user cost π_{it} is lower than zero, we conclude $1 < E_t^C [Q_{t+1} r_{i,t+1}]$, so the return tomorrow is larger than the one dollar spent on it today, and she will buy it. If π_{it} is larger than $-E_t^C [-Q_{t+1} r_{i,t+1}] - E_t^C [Q_{t+1} r_{i,t+1}]$, then $1 > -E_t^C [-Q_{t+1} r_{i,t+1}]$ and the uncertainty premium is not enough to compensate for the cost of holding the monetary asset. She will want to sell it. This range of user cost depends only on the beliefs and attitude towards uncertainty, not on the attitude towards risk.

5 Concluding Remarks

In this paper we consider the monetary services aggregation theory under uncertainty, as distinguished by Knight (1921) from risk. The presence of, and the agent's attitude towards uncertainty is represented by the nonadditivity of the probability measure. We acquire three primary conclusions. First, different from CCAPM risk adjusted user costs incorporating covariances and subject to the "risk premium puzzle" critique, we find that the uncertainty adjusted user cost, in its most general form, produces boundaries. The previously derived perfect certainty user cost and the risk adjusted user cost are special cases of ours, if the probability measure becomes additive. Second, we are able to derive an expected utility analogous solution using transformed additive probabilities, the result will be rank-dependent and will require the existence of an underlying probability measure that contains sufficient information to properly define the capacity. This is a special case of Choquet expectation. Third, user costs under uncertainty produce an interval within which no transactions of monetary assets will occur. This effect is brought about solely by uncertainty aversion, not by risk aversion captured by the utility function.

Appendix

Proof of Theorem 1:

Let $\frac{\partial u}{\partial c_{t+1}}$ and $r_{i,t+1}$ be comonotonic in the sense that for each pair of states, $s_1', s_2' \in S$,

$$\left[\frac{\partial u}{\partial c_{t+1}}(s_1') - \frac{\partial u}{\partial c_{t+1}}(s_2')\right]\left[r_{i,t+1}(s_1') - r_{i,t+1}(s_2')\right] \geq 0.$$

Suppose $E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right] < \infty$ and both $\frac{\partial u}{\partial c_{t+1}}, r_{i,t+1} \geq 0$. For any given s_0', we have by comonotonicity:

$$\left[\frac{\partial u}{\partial c_{t+1}} - \frac{\partial u}{\partial c_{t+1}}(s_0')\right]\left[r_{i,t+1} - r_{i,t+1}(s_0')\right] \geq 0.$$

That is:

$$\frac{\partial u}{\partial c_{t+1}} \cdot r_{i,t+1} + \frac{\partial u}{\partial c_{t+1}}(s_0') \cdot r_{i,t+1}(s_0') \geq \frac{\partial u}{\partial c_{t+1}}(s_0') \cdot r_{i,t+1} + \frac{\partial u}{\partial c_{t+1}} \cdot r_{i,t+1}(s_0').$$

Since Choquet Expectation is monotone, we have

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}} \cdot r_{i,t+1} + \frac{\partial u}{\partial c_{t+1}}(s_0') \cdot r_{i,t+1}(s_0')\right] \geq E_t^C\left[\frac{\partial u}{\partial c_{t+1}}(s_0') \cdot r_{i,t+1} + \frac{\partial u}{\partial c_{t+1}} \cdot r_{i,t+1}(s_0')\right].$$

Given s_0', $\frac{\partial u}{\partial c_{t+1}}(s_0')$ and $r_{i,t+1}(s_0')$ are constants, by translatability[5], we have

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}} \cdot r_{i,t+1}\right] + \frac{\partial u}{\partial c_{t+1}}(s_0') \cdot r_{i,t+1}(s_0') \|v\| \geq E_t^C\left[\frac{\partial u}{\partial c_{t+1}}(s_0') \cdot r_{i,t+1} + \frac{\partial u}{\partial c_{t+1}} \cdot r_{i,t+1}(s_0')\right].$$

By positive homogeneity, since $\frac{\partial u}{\partial c_{t+1}}(s_0')$ and $r_{i,t+1}(s_0') \geq 0$, we have

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right] + \frac{\partial u}{\partial c_{t+1}}(s_0') \cdot r_{i,t+1}(s_0') \|v\| \geq \frac{\partial u}{\partial c_{t+1}}(s_0') E_t^C(r_{i,t+1}) + r_{i,t+1}(s_0') E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right),$$

[5] Choquet integrals are translatable for any real number β, such that $E_t^C(X + \beta) = E_t^C(X) + \beta\|v\|$, if v is a monotone measure on the measurable space $(S, \mathcal{F})$.

and this holds for any $s_0' \in S$. That is,

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right] + \frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\|v\| \geq \frac{\partial u}{\partial c_{t+1}} E_t^C\left(r_{i,t+1}\right) + r_{i,t+1} E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right).$$

When both $E_t^C\left(r_{i,t+1}\right)$ and $E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right)$ are finite, apply translatability again to acquire

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right]\|v\| + E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right]\|v\| \geq E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right) E_t^C\left(r_{i,t+1}\right) + E_t^C\left(r_{i,t+1}\right) E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right).$$

Dividing the norm on both sides, we find

$$E_t^C\left[\frac{\partial u}{\partial c_{t+1}} r_{i,t+1}\right] \geq \frac{1}{\|v\|} E_t^C\left(\frac{\partial u}{\partial c_{t+1}}\right) E_t^C\left(r_{i,t+1}\right).$$

Since v is a probability measure, $\|v\| = 1$. This proves part 1 of the theorem.

If v is submodular and $\frac{\partial u}{\partial c_{t+1}}$ and $r_{i,t+1}$ are countermonotonic, the second part of the theorem follows from the same logic. This completes the proof.

Proof of Lemma 2:

Suppose u is weakly increasing in $s \in S$. Given a monotone measure space $(S, \mathcal{F}, \mathcal{V})$, we denote $\{u_s \geq t\} = \{s \,|u(s) \geq t\}$ for any $t > 0$. The Choquet integral of u over S with respect to a real monotone measure v is:

$$\begin{aligned}
\int u_s v(ds) &= \int_0^{+\infty} v\left(\{i \,|u_i \geq t\}\right) dt \\
&= \int_0^{u_1} v\left(\{i \,|u_i \geq t\}\right) dt + \sum_{s=2}^{n} \int_{u_{s-1}}^{u_s} v\left(\{i \,|u_i \geq t\}\right) dt + \int_{u_n}^{+\infty} v\left(\{i \,|u_i \geq t\}\right) dt \\
&= \int_0^{u_1} v\left(\{1, 2, \cdots, n\}\right) dt + \sum_{s=2}^{n} \int_{u_{s-1}}^{u_s} v\left(\{s, s+1, \cdots, n\}\right) dt + \int_{u_n}^{+\infty} v(\phi)\, dt \\
&= u_1 f\left(\sum_i P_i\right) + \sum_{s=2}^{n} \left(u_s - u_{s-1}\right) f\left(\sum_{i \geq s} P_i\right) \\
&= \sum_{s=1}^{n} u_s f\left(\sum_{i \geq s} P_i\right) - \sum_{s=2}^{n} u_{s-1} f\left(\sum_{i \geq s} P_i\right) \\
&= u_n f\left(P_n\right) + \sum_{s<n} u_s f\left(\sum_{i \geq s} P_i\right) - \sum_{s<n} u_s f\left(\sum_{i > s} P_i\right) \\
&= \sum_{s=1}^{n} u_s P_s^{\uparrow}.
\end{aligned}$$

The weakly decreasing case of u can be proven likewise. *Q.E.D.*

Proof of Lemma 3: Now we prove the fact that $-E_t^C\left[-Q_{t+1}r_{i,t+1}\right] > E_t^C\left[Q_{t+1}r_{i,t+1}\right].$ Denoting by $A(t) = \{s' \in S \,|Q_{t+1}(s')\, r_{i,t+1}(s') \geq t\}$, by definition

$$E_t^C\left[Q_{t+1}(s')\, r_{i,t+1}(s')\right] = \int_{-\infty}^{0}\left[v(A(t)) - 1\right]dt + \int_{0}^{+\infty} v(A(t))\,dt.$$

Base on the definition $A(t)$, consider the event $-Q_{t+1}(s')\, r_{i,t+1}(s') > t$:

$$\begin{aligned} & \{s' \in S \,|-Q_{t+1}(s')\, r_{i,t+1}(s') > t\} \\ = \;& \{s' \in S \,|Q_{t+1}(s')\, r_{i,t+1}(s') < -t\} \\ = \;& \Omega\backslash A(-t) \\ = \;& A(-t)^c. \end{aligned}$$

Here this superscript lower case c means complement of $A(-t)$, it should not be confused with the upper case C superscript notation for Choquet. We have therefore

$$\begin{aligned} E_t^C\left[-Q_{t+1}(s')\, r_{i,t+1}(s')\right] &= \int_{-\infty}^{0}\left[v\left(A(-t)^c\right) - 1\right]dt + \int_{0}^{+\infty} v\left(A(-t)^c\right)dt \\ &= -\int_{\infty}^{0}\left[v\left(A(z)^c\right) - 1\right]dz - \int_{0}^{-\infty} v\left(A(z)^c\right)dz \\ &= \int_{-\infty}^{0} v\left(A(t)^c\right)dt + \int_{0}^{+\infty}\left[v\left(A(t)^c\right) - 1\right]dt. \end{aligned}$$

Furthermore, $E_t^C\left[Q_{t+1}(s')\, r_{i,t+1}(s')\right] + E_t^C\left[-Q_{t+1}(s')\, r_{i,t+1}(s')\right]$ yields,

$$\begin{aligned} & E_t^C\left[Q_{t+1}(s')\, r_{i,t+1}(s')\right] + E_t^C\left[-Q_{t+1}(s')\, r_{i,t+1}(s')\right] \\ = \;& \int_{-\infty}^{+\infty}\left[v(A(t)) + v\left(A(t)^c\right) - 1\right]dt, \end{aligned}$$

by the fact that the probability is nonadditive, particularly the agent is uncertainty averse, $v(A) + v(A^c) < 1$, thus $\int_{-\infty}^{+\infty}\left[v(A(t)) + v\left(A(t)^c\right) - 1\right]dt < 0$. This proves $-E_t^C\left[-Qr\right] > E_t^C\left[Qr\right].$ *Q.E.D.*

Chapter 2 Nonparametric Estimation of Censored Regression with Endogeneity[6]

1 Introduction

It might be a good idea to start out with a motivating example. Consider the demand for and supply of labor:

$$\begin{cases} l^d = g_0\left(w, z_1\right) + \varepsilon, & \text{demand for labor} \\ w = h_0\left(l^s, z_2\right) + u, & \text{inverse labor supply} \\ l^s = L^* \cdot 1\left(L^* > 0\right), & l^s \text{ is censored} \end{cases}$$

We assume $E\left(\varepsilon \left| z_1, z_2\right.\right) = E\left(u \left| z_1, z_2\right.\right) = 0$. In equilibrium $l^d = l^s$. Let t be a percentage income tax that is paid by the working class. The equilibrium working hours is the solution $l^*\left(z_1, z_2, \varepsilon, u\right)$ to

$$\widetilde{l} = g_0\left[(1+t)\left(h_0\left(\widetilde{l}, z_2\right) + u\right), z_1\right] + \varepsilon.$$

The effect of tax on average working hours would be $E\left(\widetilde{l} - l^*\right)$. A successful evaluation of this effect requires us to identify the unknown function $g_0(\cdot)$ and $h_0(\cdot)$.

Endogeneity in nonparametrics itself is a question with debate. For example, consider the following model:

$$Y_t = g_0\left(X_t\right) + \varepsilon_t, \quad E\left(X_t \varepsilon_t\right) \neq 0, \; t = 1, \cdots, n.$$

where $g_0(\cdot)$ is the true unknown function to be estimated, but the error term ε_t is correlated with the independent variable resulting that $E\left(Y_t \left| X_t\right.\right) \neq g_0\left(X_t\right)$. Without the correct model specification $E\left(\varepsilon_t \left| X_t\right.\right) = 0$, the model we have no longer reveals the conditional mean of Y_t which is the center of interests. Further more, if we believe ε_t can be expressed as a nonparametric function of X_t plus an exogenous error disturbance,

$$\varepsilon_t = k_0\left(X_t\right) + v_t,$$

[6] This paper is under the remarkable guidance of Prof. Zongwu Cai.

then the original equation becomes:

$$Y_t = g_0(X_t) + k_0(X_t) + v_t, \quad E(v_t | X_t) = 0.$$

The problem is now $g_0(\cdot)$ and $k_0(\cdot)$ are both functions of the same argument, the model therefore could not be identified. From this simple example however, the insight we get is endogeneity in nonparametrics tends to transform the model to be additive. Therefore from the point of view of correct model specification, endogeneity in nonparametric models results in unidentification.

Nevertheless, it is worth noticing that endogeneity presents even if the model is well grounded by design. For instance, the problem is in the data, not in the model, data contains measurement error. So besides incorrect model specification, endogeneity arises in cases like structural breaks, error-in-variables, heterogeneous treatment effects, partial noncompliance, frailty, correlated random effect in panel data, and sample selection, etc. Nonparametrics IV regression alone is fraught with confusion and difficulty, adding censoring into consideration complicates the situation even more because when the data is censored, the model cannot be identified through moment restrictions, but could possibly be identified through quantile restrictions.

This paper tries to propose a nonparametric solution to censored model with instrumental variables. Under the mild assumptions of median zero error terms and strict exogeneity of the instruments, we transform the model into a Least Absolute Deviation of Noparametric Additive Local Linear estimation. Because of endogeneity the nonparametric model is additive, by the assumption of conditional median zero it is a L_1 norm distance, and our estimation strategy is local linear estimation. We prove that the estimated unknown functions are asymptotically normal, and needless to say our estimators are more robust to outliers and leverage points which severely distort nonparametric estimation. Chernozhukov *et al.* (2015) deal with the same issue of censoring and endogeneity using quantile regression approach with IV, and we offer a completely different approach to tackling with the same problem using nonparametrics. In their paper they use a trimming indicator to exclude extreme values so as to work on a benign trimmed supports, we are less worried with respect to that.

2 Literature Review

Denoting by Y_t^* the latent variable, and by Y_t the observable. The distribution of Y_t^* is nonparametric but we do not observe the full range of its support, so its moments cannot be used for estimation. Denoting by $Q_\tau(Y_t^*)$ and $Q_\tau(Y_t)$ the $\tau - th$ quantiles of Y_t^* and Y_t, then:

$$Q_\tau(Y_t) = \begin{cases} Q_\tau(Y_t^*) & Prob\{Y_t^* \leq 0\} < \tau \\ 0 & Prob\{Y_t^* \leq 0\} \geq \tau \end{cases}.$$

Without loss of generality we suppose the censoring happens at 0 from below, this assumption is not restrictive as censoring at other values or from another direction can be easily accommodated. If the probability of censoring is smaller than τ, the two quantiles are equal; if the probability of censoring is larger than τ, then quantile of Y_t is just the censored value. This means:

$$Q_\tau(Y_t) = \max\{Q_\tau(Y_t^*), 0\},$$

and particularly:

$$Med(Y_t) = \max\{Med(Y_t^*), 0\}$$

if the censoring rate is less than half. So censoring changes the modelling in a way that it makes moment restrictions fail to work, but some quantile restrictions can be used for identification. That's why most of the papers rely on quantile approach to tackling with censoring. Chernozhukov, Fernández-Val and Kowalski (2015) develop a censored quantile instrument variable estimator that combines Powell (1986a)'s censored quantile regression to deal with censoring and a control variable approach to deal with endogeneity.

Powell (1984)'s Censored Least Absolute Deviation (CLAD) and Powell (1986b)'s Symmetrically Censored Least Squares (SCLS) estimator, Newey and Powell (1990)'s weighted CLAD estimator, Khan and Powell (2001)'s two-step procedure, all of these estimators the conditional median is still linear in X_t, a rather parametric assumption.

Lewbel and Linton (2002) make a strong assumption that the error $e_t = Y_t^* - g(X_t)$ is independent of X_t, and develop kernel estimation that is essentially based on moment functions.

Newey (2013), Blundell and Powell (2004) investigate endogeneity in nonparametric models.

3 The Model Framework

3.1 The Model

Consider the following model:

$$Y_t^* = g_0(X_t) + \varepsilon_t, \qquad E(\varepsilon_t | Z_t) = 0,$$

$$Y_t = Y_t^* \cdot 1(Y_t^* > 0).$$

$$X_t = \pi_0(Z_t) + u_t, \qquad E(u_t | Z_t) = 0.$$

Suppose X is continuous, not censored, and is possibly correlated with ε, so $E(X_t \varepsilon_t) \neq 0$. Z is the instrument, $g_0(\cdot)$ is unknown and Y_t is censored from below at zero. $\pi_0(\cdot)$ is a reduced form function for X_t. $E(\varepsilon_t | Z_t) = 0$ can be implied by Economic theory. Without censoring we have:

$$E(Y_t^* | Z_t) = g_0(X_t).$$

Assuming $\varepsilon | Z$ has median 0,[7] so conditional median of $Y^* | Z$ equals the conditional mean:

$$Med(Y_t^* | Z_t) = g_0(X_t)$$

$$Med(Y_t | Z_t) = \max\{Med(Y_t^* | Z_t), 0\} = \max\{g_0(X_t), 0\}.$$

Consider the median of Y_t^* conditional on X_t and Z_t:

$$\begin{aligned}
Med(Y_t^* | X_t, Z_t) &= g_0(X_t) + Med(\varepsilon_t | X_t, Z_t) \\
&= g_0(X_t) + Med(\varepsilon_t | Z_t, u_t) \quad X_t \text{ is a fun of } Z_t, u_t \\
&= g_0(X_t) + Med(\varepsilon_t | u_t) \qquad \text{strong exo. } Z_t \\
&= g_0(X_t) + h_0(u_t)
\end{aligned}$$

With hindsight, under the assumption that $Med(\varepsilon_t | u_t) = E(\varepsilon_t | u_t) = h_0(u_t)$, we have the additive model for Y^*. Because of censoring, we have median; because of endogeneity, we have one more term $h_0(u_t)$. The true model for Y_t^* thus becomes:

$$Y_t^* = g_0(X_t) + h_0(u_t) + v_t$$

[7] We believe this is not a very restrictive assumption in light of the fact that ε is the error term that is not correlated with Z. And it does not necessarily imply that the conditional distribution of ε is symmetric.

$$E\left(v_t\left|X_t, u_t\right.\right) = 0$$

Assuming $v_t\left|X_t, u_t\right.$ is median 0, the conditional median of Y_t on X_t and u_t is

$$\begin{aligned} Med\left(Y_t\left|X_t, u_t\right.\right) &= \max\left\{Med\left(Y_t^*\left|X_t, u_t\right.\right), 0\right\} \\ &= \max\left\{g_0\left(X_t\right) + h_0\left(u_t\right), 0\right\}. \end{aligned}$$

The criterion function

$$\begin{aligned} S_n\left(g, h\right) &= \sum_{t=1}^{n}\left|Y_t - \max\left\{g_0\left(X_t\right) + h_0\left(u_t\right), 0\right\}\right| \\ &= \sum_{t=1}^{n}\mathbf{1}\left(g_0\left(X_t\right) + h_0\left(u_t\right) > 0\right)\cdot\left|Y_t - g_0\left(X_t\right) - h_0\left(u_t\right)\right| \end{aligned}$$

Our idea is to estimate the model nonparametrically, so the objective function further becomes:

$$Q_n\left(g, h\right) = \sum_{t=1}^{n}\mathbf{1}\left(g_0\left(X_t\right) + h_0\left(u_t\right) > 0\right)\cdot\left|Y_t - g_0\left(X_t\right) - h_0\left(u_t\right)\right|\cdot K_X\left(\frac{X_t - x_0}{h}\right)K_u\left(\frac{u_t - u_0}{h}\right).$$

where $K_X\left(\cdot\right)$ and $K_u\left(\cdot\right)$ are kernel functions for X and u respectively, h is the bandwidth. Currently we use the same bandwidth for both variables, but this can be further relaxed later on with minor modification to the regularity conditions.

Denote by $\tilde{X}_t = \left\{X_t, u_t\right\}^T$, $t = 1, 2, \cdots n$. So $\tilde{X}_t$ is the explanatory variables augmented by the residuals from the artificial/reduced regression. Suppose $\tilde{X}_t$ features an additive model such that:

$$m_0\left(\tilde{X}_t\right) = g_0\left(X_t\right) + h_0\left(u_t\right) + \eta_t \quad \text{and} \quad E\left(\eta_t\left|X_t, u_t\right.\right) = 0.$$

Then the indicator function $\mathbf{1}\left(g_0\left(X_t\right) + h_0\left(u_t\right) > 0\right)$ can be treated as a selector $\mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right) > 0\right)$ which basically predicts whether the t^{th} observation is τ^{th} quantile uncensored. $\hat{m}\left(\tilde{X}_t\right)$ is a consistent estimate of the unknown function $m_0\left(\tilde{X}_t\right)$.

It's clear from the objective function $Q_n\left(g, h\right)$ that this is a nonparametric Least Absolute Deviation estimation upon the uncensored sample. We will focus on the identification of the unknown funcions $g_0\left(X_t\right)$ and $h_0\left(u_t\right)$ by local linear estimation since it entwines with

LAD. Suppose $g_0(x)$ is approximated by:

$$g_0(x) \doteq g_0(x_0) + g_0'(x_0)(x - x_0)$$

where x is in the neighborhood of x_0 within a closed support of X; and $h_0(u_t)$ is approximated by:

$$h_0(u) \doteq h_0(u_0) + h_0'(u_0)(u - u_0)$$

where u is in the neighborhood of u_0 within a closed support of U. Denoting by a_ν the local linear LAD estimate for $g_0^{(\nu)}(x_0)$, and b_ν the local linear LAD estimate for $h_0^{(\nu)}(u_0)$, $\nu = 0, 1$. Let's further denote by $r = \{a_0, a_1, b_0, b_1\}^T$ Then the Local Linear Least Absolute Deviation estimator is the solution of the following minimization problem:

$$\begin{aligned} \hat{r} &= \arg\min_{r \in \Theta} \sum_{t=1}^{n} \mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right) > 0\right) \cdot |Y_t - a_0 - a_1(X_t - x_0) - b_0 - b_1(u_t - u_0)| \\ & \cdot K_X\left(\frac{X_t - x_0}{h}\right) K_u\left(\frac{u_t - u_0}{h}\right). \end{aligned}$$

By the idea of Wagner (1959), this L_1 norm minimization problem is equivalent to solving a linear programming problem as follows:

$$\min_{r \in \Theta} \sum_{t=1}^{n} \mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right) > 0\right) \cdot \left(\epsilon_t^+ + \epsilon_t^-\right) \cdot K_X\left(\frac{X_t - x_0}{h}\right) K_u\left(\frac{u_t - u_0}{h}\right)$$

$$s.t. \;\; a_0 + a_1(X_t - x_0) + b_0 + b_1(u_t - u_0) + \epsilon_t^+ - \epsilon_t^- = Y_t$$

$$\epsilon_t^+, \epsilon_t^- \geq 0.$$

Suppose ϵ_t is the difference between Y_t and its fitted value, ϵ_t^+ and ϵ_t^- are defined to be the positive and negative parts of ϵ_t for each data point t, corresponding to the vertical deviations "above" or "below" the regression line. $\epsilon_t = \epsilon_t^+ - \epsilon_t^-$ and $|\epsilon_t| = \epsilon_t^+ + \epsilon_t^-$.

The model is well structured as it arises naturally from Economic context, it is endogeneity that makes our model to be additive in the sense that $E[m_0(X_t, u_t)] = g_0(X_t) + h_0(u_t)$ meanwhile X_t and u_t are correlated. Our interests lie in $g_0(X_t)$, not $h_0(u_t)$, ingoring $h_0(u_t)$ nevertheless would result in inconsistency of $g_0(X_t)$. Therefore we will restrict ourselves on the asymptotic behavior of $\sqrt{nh}g(x_0)$ and $\sqrt{nh}h(u_0)$, not $\sqrt{nh^2}m(x_0, u_0)$. To be able to do that, by the projection method of Cai and Masry (2000), let's define:

$$\tilde{g}(x_0) = \frac{1}{n}\sum_{t=1}^{n} \hat{m}(x_0, u_t)$$

and

$$\tilde{h}(u_0) = \frac{1}{n}\sum_{t=1}^{n} \hat{m}(X_t, u_0).$$

The purpose of this paper is to investigate the asymptotic distribution of $\tilde{g}(x_0)$ and $\tilde{h}(u_0)$. Now we are ready to state the assumptions on which our main results are built.

3.2 Assumptions

The first assumption is i.i.d. random sample, but our frame work can easily accommodate martingale difference sequence of time series.

Assumption 1. $\{Y_t, X_t, Z_t\}^T$, $t = 1, 2, \cdots, n$ is an independent and identically distributed random sample.

Assumption 2. Bandwidth $h = O\left(n^{-\frac{1}{5}}\right)$.

Assumption 3. The error term ε_t is mean zero and median zero conditional on IV Z_t; the error term v_t is mean zero and median zero conditional on X_t and u_t.

Assumption 4. $g_0(x) \in C^2(B_{x_0}(r))$, and $h_0(u) \in C^2(B_{u_0}(r))$, where $B_{x_0}(r)$ and $B_{u_0}(r)$ are open balls centered at x_0 and u_0 with radius r respectively.

Assumption 5. The joint density function $f(x, u)$ together with the marginal density functions $f_X(x)$ and $f_u(u)$ are continuous in the neighborhood of (x_0, u_0), x_0, and u_0 respectively. And $f(x_0, u_0)$, $f_X(x_0)$ and $f_u(u_0)$ are all nonzero.

Assumption 6. The kernel functions $K_X(\cdot)$ and $K_u(\cdot)$ are symmetric, bounded, compactly supported.

Assumption 7. Denoting by $\phi(\epsilon\,|x, u)$ the conditional probability function of ϵ_t given X_t and u_t, and $\phi(\epsilon\,|x, u)$ is continuous in the neighborhood of 0. Furthermore, $\phi(0\,|x, u) > 0$.

Assumption 8. $\sup_{t\in[1,n]} \|X_t\| = o_p\left(n^{\frac{1}{10}}\right)$.

4 Asymptotic Normality

In this section we will derive the asymptotic distribution of the estimators. Our task is to estimate $g_0(X_t)$ and $h_0(u_t)$, by Taylor expansion around a given point x_0 and u_0, we have:

$$g_0\left(X_t\right)=g_0\left(x_0\right)+g_0'\left(x_0\right)\left(X_t-x_0\right)+\frac{1}{2}g_0''\left(\bar{x}\right)\left(X_t-x_0\right)^2,$$

and

$$h_0\left(u_t\right)=h_0\left(u_0\right)+h_0'\left(u_0\right)\left(u_t-u_0\right)+\frac{1}{2}h''\left(\bar{u}\right)\left(u_t-u_0\right)^2,$$

where $\bar{x}=\lambda_1 x_0+\left(1-\lambda_1\right)X_t$, and $\bar{u}=\lambda_2 u_0+\left(1-\lambda_2\right)u_t$, both $\lambda_1,\ \lambda_2\in[0,1]$.

$$\begin{aligned}
\hat{r} &= \arg\min_{r\in\Theta}\sum_{t=1}^{n}\mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right)>0\right)\cdot\left|Y_t-a_0-a_1\left(X_t-x_0\right)-b_0-b_1\left(u_t-u_0\right)\right| \\
&\quad \cdot K_X\left(\frac{X_t-x_0}{h}\right)K_u\left(\frac{u_t-u_0}{h}\right) \\
&= \arg\min_{r\in\Theta}\sum_{t=1}^{n}\mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right)>0\right)\cdot\left\{\begin{array}{c}\left|\begin{array}{c}\frac{\sqrt{nh^2}}{\sqrt{nh^2}}\left[\left(a_0-g_0\left(x_0\right)\right)+\left(a_1-g_0'\left(x_0\right)\right)\left(X_t-x_0\right)\right] \\ +\frac{\sqrt{nh^2}}{\sqrt{nh^2}}\left[\left(b_0-h_0\left(u_0\right)\right)+\left(b_1-h_0'\left(u_0\right)\right)\left(u_t-u_0\right)\right] \\ -\frac{1}{2}g_0''\left(\bar{x}\right)\left(X_t-x_0\right)^2-\frac{1}{2}h_0''\left(\bar{u}\right)\left(u_t-u_0\right)^2-\epsilon_t\end{array}\right| \\ -\left|\frac{1}{2}g_0''\left(\bar{x}\right)\left(X_t-x_0\right)^2+\frac{1}{2}h_0''\left(\bar{u}\right)\left(u_t-u_0\right)^2+\epsilon_t\right|\end{array}\right\} \\
&\quad \cdot K_X\left(\frac{X_t-x_0}{h}\right)K_u\left(\frac{u_t-u_0}{h}\right).
\end{aligned}$$

The appearance of the last part in the curly brackets is harmless because it is not entangled with the optimization variables. To facilitate derivation, we would like to transform the variables: $\alpha_0=\sqrt{nh^2}\left(a_0-g_0\left(x_0\right)\right)$, $\alpha_1=\sqrt{nh^2}\left[\left(a_1-g_0'\left(x_0\right)\right)h\right]$, $\beta_0=\sqrt{nh^2}\left(b_0-h_0\left(u_0\right)\right)$, $\beta_1=\sqrt{nh^2}\left[\left(b_1-h_0'\left(u_0\right)\right)h\right]$. And let's use $\gamma=\left\{\alpha_0,\alpha_1,\beta_0,\beta_1\right\}^T$ to denote the new variable of interest. The aforementioned problem therefore is equivalent to:

$$\begin{aligned}
\begin{pmatrix}\hat{\alpha}_0\\ \hat{\alpha}_1\\ \hat{\beta}_0\\ \hat{\beta}_1\end{pmatrix} &= \arg\min_{\gamma\in\Theta}\sum_{t=1}^{n}\mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right)>0\right)\cdot\left\{\begin{array}{c}\left|\begin{array}{c}\frac{1}{\sqrt{nh^2}}\left[\alpha_0+\alpha_1\left(\frac{X_t-x_0}{h}\right)\right] \\ +\frac{1}{\sqrt{nh^2}}\left[\beta_0+\beta_1\left(\frac{u_t-u_0}{h}\right)\right] \\ -\frac{1}{2}g_0''\left(\bar{x}\right)\left(X_t-x_0\right)^2-\frac{1}{2}h_0''\left(\bar{u}\right)\left(u_t-u_0\right)^2-\epsilon_t\end{array}\right| \\ -\left|\frac{1}{2}g_0''\left(\bar{x}\right)\left(X_t-x_0\right)^2+\frac{1}{2}h_0''\left(\bar{u}\right)\left(u_t-u_0\right)^2+\epsilon_t\right|\end{array}\right\} \\
&\quad \cdot K_X\left(\frac{X_t-x_0}{h}\right)K_u\left(\frac{u_t-u_0}{h}\right).
\end{aligned}$$

The objective function is not only highly nonlinear, but also non-differentiable, our quest for asymptotics starts from the properties of the criterion function first. As we will see in the last, the asymptotic distribution of the minimizer of the criterion function turns out to be the distribution of the minimizer of the limit function of the criterion. So define the criterion to be:

$$
\begin{aligned}
A_n \quad \equiv \quad & \sum_{t=1}^{n} \left\{ \begin{array}{c} \left| \begin{array}{c} \frac{1}{\sqrt{nh^2}} \left[\alpha_0 + \alpha_1 \left(\frac{X_t - x_0}{h}\right)\right] + \frac{1}{\sqrt{nh^2}} \left[\beta_0 + \beta_1 \left(\frac{u_t - u_0}{h}\right)\right] \\ -\frac{1}{2} g_0'' \left(\bar{x}\right) \left(X_t - x_0\right)^2 - \frac{1}{2} h_0'' \left(\bar{u}\right) \left(u_t - u_0\right)^2 - \epsilon_t \end{array} \right| \\ - \left| \frac{1}{2} g_0'' \left(\bar{x}\right) \left(X_t - x_0\right)^2 + \frac{1}{2} h_0'' \left(\bar{u}\right) \left(u_t - u_0\right)^2 + \epsilon_t \right| \end{array} \right\} \\
& \cdot K_X \left(\frac{X_t - x_0}{h}\right) K_u \left(\frac{u_t - u_0}{h}\right) \cdot \mathbf{1} \left(\hat{m}\left(\tilde{X}_t\right) > 0\right)
\end{aligned}
$$

and $F_n = \mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right) > 0\right) \cdot E\left(A_n \left|\mathbf{X}, \mathbf{u}\right.\right)$,

$$
\begin{aligned}
R_n \quad = \quad & A_n - F_n + \sum_{t=1}^{n} \left\{ \frac{1}{\sqrt{nh^2}} \left[\alpha_0 + \alpha_1 \left(\frac{X_t - x_0}{h}\right)\right] + \frac{1}{\sqrt{nh^2}} \left[\beta_0 + \beta_1 \left(\frac{u_t - u_0}{h}\right)\right] \right\} \\
& \cdot K_X \left(\frac{X_t - x_0}{h}\right) K_u \left(\frac{u_t - u_0}{h}\right) \cdot \mathbf{1} \left(\hat{m}\left(\tilde{X}_t\right) > 0\right) \cdot sgn\left(\epsilon_t\right).
\end{aligned}
$$

As we will see (proved in appendix), this function R_n converges to 0 in probability as $n \to \infty$.

Theorem 6 *Under Assumptions 1-7, for any fixed value* $\alpha_0, \alpha_1, \beta_0, \beta_1$*, we have:*

$$
\underset{n \to \infty}{plim} R_n = 0.
$$

Theorem 7 *Under Assumptions 1-7, for any fixed value* $\alpha_0, \alpha_1, \beta_0, \beta_1$*, the function* F_n *converges to* $F\left(\alpha_0, \alpha_1, \beta_0, \beta_1\right)$*:*

$$
F\left(\alpha_0, \alpha_1, \beta_0, \beta_1\right) = \phi\left(0 \left|x_0, u_0\right.\right) f\left(x_0, u_0\right) \left[\begin{array}{c} \alpha_0^2 + \alpha_1^2 \int z^2 K_X(z)\,dz + \beta_0^2 + \beta_1^2 \int z^2 K_u(z)\,dz + 2\alpha_0\beta_0 \\ -g''\left(x_0\right) \left[\alpha_0 \int z^2 K_X(z)\,dz + \beta_0 \int z^2 K_X(z)\,dz\right] \\ -h''\left(x_0\right) \left[\alpha_0 \int z^2 K_u(z)\,dz + \beta_0 \int z^2 K_u(z)\,dz\right] \end{array} \right].
$$

Under the assumptions we make, after proper scaling, the estimator $\tilde{g}\left(x_0\right) - g_0\left(x_0\right)$ and $\tilde{h}\left(u_0\right) - h_0\left(x_0\right)$ are asymptotically normal:

Theorem 8 *Under Assumptions 1-8, we have*

$$
\sqrt{nh}\left(\tilde{g}\left(x_0\right) - g_0\left(x_0\right) - \frac{1}{2} h^2 g''\left(x_0\right) \int z^2 K_X(z)\,dz\right) \xrightarrow{d} N\left(0, \frac{1}{4} \int K_X^2(z)\,dz \int f_u^2(s)\, f\left(x_0, s\right) ds\right),
$$

and

$$\sqrt{nh}\left(\tilde{h}\left(u_0\right)-h_0\left(x_0\right)-\frac{1}{2}h^2h''\left(u_0\right)\int z^2K_u\left(z\right)dz\right)\xrightarrow{d}N\left(0,\frac{1}{4}\int K_u^2\left(z\right)dz\int f_X^2\left(s\right)f\left(s,u_0\right)ds\right).$$

It's well known that the estimate $\hat{g}\left(\cdot\right)$ is subject to the ill-posed inverse problem, that is, $\hat{g}\left(\cdot\right)$ is not a continuous function of $\hat{\lambda}\left(\cdot\right)$ and $\hat{F}\left(dx\left|z\right.\right)$ where $\hat{\lambda}\left(\cdot\right)$ is the estimate of $E\left(y\left|z\right.\right)$. The question is, does local linear estimation of this version suffer from ill-posed inverse problem? Remember, the regularization methods used to overcome this problem is to restrict $g_0\left(\cdot\right)$ belongs to a compact set under the Sobolev norm. Obviously, local linear restricts the space of allowable functions $g_0\left(\cdot\right)$, nevertheless, $\hat{u}_i$ will be produced via sieve estimation.

5 Concluding Remarks

In this paper, we consider the problem of nonparametric estimation in a model that features censoring data and endogeneity. Nonparametrics with endogenous variables is difficult to handle because of ill-posed inverse problem. Nonparametrics with censoring does not attract the attention as it deserves because people are inclined to shift to quantile estimation when data is censored. We stick to the nonparametric estimation under two mild conditions and claim that endogeneity shapes the model to be additive, and censoring delivers a (nonparametric) LAD estimation under the assumption of conditional zero median of the error term. This paper therefore transforms the problem into a Nonparametric Additive Least Absolute Deviation estimation which is saliently robust than L_2 norm estimation. We establish the asymptotic normality of the estimated unknown functions, the estimation and inference are easy to carry out.

Appendix

In this appendix we will prove those three theorems in this chapter. Our main results are Theorem 3 which is the asymptotic normality of the estimators $\tilde{g}\left(x_0\right)$ and $\tilde{h}\left(u_0\right)$. The properties of the functions R_n and F_n which are stated respectively in the first two theorems play a crucial role leading to the final results.

Proof of Theorem 1. It is obvious that for any fixed $\alpha_0, \alpha_1, \beta_0$ and β_1, the following quantity converges to 0 in probability as $n \to \infty$:

$$\xi_n = \sup_{t \in [1,n]} \left\{ \begin{array}{c} \left| \frac{1}{\sqrt{nh^2}} \left[\alpha_0 + \alpha_1 \left(\frac{X_t - x_0}{h} \right) \right] + \frac{1}{\sqrt{nh^2}} \left[\beta_0 + \beta_1 \left(\frac{u_t - u_0}{h} \right) \right] \right| \\ + \left| \frac{1}{2} g_0'' \left(\bar{x} \right) \left(X_t - x_0 \right)^2 + \frac{1}{2} h'' \left(\bar{u} \right) \left(u_t - u_0 \right)^2 \right| \end{array} \right\} = o_p \left(1 \right).$$

Let

$$\begin{aligned} \Upsilon_{nt} &= \left\{ \frac{1}{\sqrt{nh^2}} \left[\alpha_0 + \alpha_1 \left(\frac{X_t - x_0}{h} \right) \right] + \frac{1}{\sqrt{nh^2}} \left[\beta_0 + \beta_1 \left(\frac{u_t - u_0}{h} \right) \right] \right\} \\ &\quad \cdot K_X \left(\frac{X_t - x_0}{h} \right) K_u \left(\frac{u_t - u_0}{h} \right) \cdot \mathbf{1} \left(\hat{m} \left(\tilde{X}_t \right) > 0 \right) \cdot sgn \left(\epsilon_t \right) \\ &\quad + \left\{ \begin{array}{c} \left| \begin{array}{c} \frac{1}{\sqrt{nh^2}} \left[\alpha_0 + \alpha_1 \left(\frac{X_t - x_0}{h} \right) \right] + \frac{1}{\sqrt{nh^2}} \left[\beta_0 + \beta_1 \left(\frac{u_t - u_0}{h} \right) \right] \\ -\frac{1}{2} g_0'' \left(\bar{x} \right) \left(X_t - x_0 \right)^2 - \frac{1}{2} h'' \left(\bar{u} \right) \left(u_t - u_0 \right)^2 - \epsilon_t \end{array} \right| \\ - \left| \frac{1}{2} g_0'' \left(\bar{x} \right) \left(X_t - x_0 \right)^2 + \frac{1}{2} h'' \left(\bar{u} \right) \left(u_t - u_0 \right)^2 + \epsilon_t \right| \end{array} \right\} \\ &\quad \cdot K_X \left(\frac{X_t - x_0}{h} \right) K_u \left(\frac{u_t - u_0}{h} \right) \cdot \mathbf{1} \left(\hat{m} \left(\tilde{X}_t \right) > 0 \right). \end{aligned}$$

If $|\epsilon_t| > \xi_n$, then $\Upsilon_{nt} = 0$. Au contraire, if $|\epsilon_t| \leq \xi_n$, then

$$\begin{aligned} \left| \Upsilon_{nt} \right| &\leq 2 \left| \frac{1}{\sqrt{nh^2}} \left[\alpha_0 + \alpha_1 \left(\frac{X_t - x_0}{h} \right) \right] + \frac{1}{\sqrt{nh^2}} \left[\beta_0 + \beta_1 \left(\frac{u_t - u_0}{h} \right) \right] \right| \\ &\quad \cdot K_X \left(\frac{X_t - x_0}{h} \right) K_u \left(\frac{u_t - u_0}{h} \right) \cdot \mathbf{1} \left(\hat{m} \left(\tilde{X}_t \right) > 0 \right). \end{aligned}$$

For $\forall \vartheta > 0$, $\delta > 0$, we have:

$$Prob \left\{ |R_n| > \vartheta \right\} \leq Prob \left\{ \xi_n \geq \delta \right\} + Prob \left\{ \mathbf{1} \left(\xi_n < \delta \right) \cdot |R_n| > \vartheta \right\}.$$

Denoting by $L_{nt} = \left\{ \frac{1}{\sqrt{nh^2}} \left[\alpha_0 + \alpha_1 \left(\frac{X_t - x_0}{h} \right) \right] + \frac{1}{\sqrt{nh^2}} \left[\beta_0 + \beta_1 \left(\frac{u_t - u_0}{h} \right) \right] \right\} \cdot K_X \left(\frac{X_t - x_0}{h} \right) K_u \left(\frac{u_t - u_0}{h} \right) \cdot \mathbf{1} \left(\hat{m} \left(\tilde{X}_t \right) > 0 \right) \cdot sgn \left(\epsilon_t \right)$, since $E \left[sgn \left(\epsilon_t \right) \right] = 0$, we have:

$$\begin{aligned} E \left[\mathbf{1} \left(\xi_n < \delta \right) \cdot R_n^2 \right] &= E \left[\mathbf{1} \left(\xi_n < \delta \right) \cdot \left(A_n - F_n + L_n \right)^2 \right] \\ &= E \left[\mathbf{1} \left(\xi_n < \delta \right) \cdot \left[\sum_{t=1}^{n} \left(\Upsilon_{nt} - E \left(\Upsilon_{nt} \right) \right) \right]^2 \right], \end{aligned}$$

where $L_n = \sum_{t=1}^n L_{ni} = \sum_{t=1}^n \left\{ \frac{1}{\sqrt{nh^2}} \left[\alpha_0 + \alpha_1 \left(\frac{X_t - x_0}{h} \right) \right] + \frac{1}{\sqrt{nh^2}} \left[\beta_0 + \beta_1 \left(\frac{u_t - u_0}{h} \right) \right] \right\} \cdot K_X \left(\frac{X_t - x_0}{h} \right)$

$K_u\left(\frac{u_t-u_0}{h}\right)\cdot \mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right)>0\right)\cdot sgn\left(\epsilon_t\right).$ Then by Law of Iterated Expectation:

$$
\begin{aligned}
& \lim_{n\to\infty} E\left[\mathbf{1}\left(\xi_n<\delta\right)\cdot R_n^2\right] \\
= \; & \lim_{n\to\infty} E\left\{\mathbf{1}\left(\xi_n<\delta\right)\cdot\left\{\begin{array}{c} \sum_{t=1}^n E\left[\left(\Upsilon_{nt}-E\left(\Upsilon_{nt}\right)\right)^2|\mathbf{X},\mathbf{u}\right] \\ +\sum_{t\neq s} E\left[\left(\Upsilon_{nt}-E\left(\Upsilon_{nt}\right)\right)\left(\Upsilon_{ns}-E\left(\Upsilon_{ns}\right)\right)|\mathbf{X},\mathbf{u}\right]\end{array}\right\}\right\} \\
\leq \; & 4\lim_{n\to\infty} E\left\{\begin{array}{c} \mathbf{1}\left(\xi_n<\delta\right)\cdot\sum_{t=1}^n\left[\frac{1}{\sqrt{nh^2}}\left[\alpha_0+\alpha_1\left(\frac{X_t-x_0}{h}\right)\right]+\frac{1}{\sqrt{nh^2}}\left[\beta_0+\beta_1\left(\frac{u_t-u_0}{h}\right)\right]\right]^2 \\ \cdot K_X^2\left(\frac{X_t-x_0}{h}\right)K_u^2\left(\frac{u_t-u_0}{h}\right)\cdot Prob\left\{|\epsilon_t|\leq\xi_n\,|\mathbf{X},\mathbf{u}\right\}\cdot\mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right)>0\right)\end{array}\right\} \\
\leq \; & 16\delta\sum_{t=1}^n E\left\{\begin{array}{c}\left[\frac{1}{\sqrt{nh^2}}\left[\alpha_0+\alpha_1\left(\frac{X_t-x_0}{h}\right)\right]+\frac{1}{\sqrt{nh^2}}\left[\beta_0+\beta_1\left(\frac{u_t-u_0}{h}\right)\right]\right]^2 \\ \cdot K_X^2\left(\frac{X_t-x_0}{h}\right)K_u^2\left(\frac{u_t-u_0}{h}\right)\cdot\mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right)>0\right)\cdot\phi\left(0\,|X_t,u_t\right)\end{array}\right\} \\
= \; & 16\delta\cdot\phi\left(0\,|x_0,u_0\right)f\left(x_0,u_0\right)\mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right)>0\right) \\
& \left\{\begin{array}{c}\int K_u^2(z)\,dz\left[\alpha_0^2\int K_X^2(z)\,dz+2\alpha_0\alpha_1\int zK_X^2(z)\,dz+\alpha_1^2\int z^2K_X^2(z)\,dz\right] \\ +\int K_X^2(z)\,dz\left[\beta_0^2\int K_u^2(z)\,dz+2\beta_0\beta_2\int zK_X^2(z)\,dz+\beta_1^2\int z^2K_u^2(z)\,dz\right]+o_p(1)\end{array}\right\} \\
\to \; & 0 \;\text{ as }\; \delta\to 0 \;\text{ and }\; n\to\infty.
\end{aligned}
$$

We have proven $\xi_n=o_p(1)$ and $E\left[\mathbf{1}\left(\xi_n<\delta\right)\cdot R_n^2\right]\to 0$ when $\delta\to 0$ and $n\to\infty$, under $Prob\left\{|R_n|>\vartheta\right\}\leq Prob\left\{\xi_n\geq\delta\right\}+Prob\left\{\mathbf{1}\left(\xi_n<\delta\right)\cdot|R_n|>\vartheta\right\}$, $R_n\to 0$ by Chebyshev Inequality. This completes the proof of Theorem 1. ■

Proof of Theorem 2. To prove $F_n\to F\left(\alpha_0,\alpha_1,\beta_0,\beta_1\right)$ pointwise, firstly notice that:

$$\frac{1}{\sqrt{nh^2}}\left[\alpha_0+\alpha_1\left(\frac{X_t-x_0}{h}\right)\right]+\frac{1}{\sqrt{nh^2}}\left[\beta_0+\beta_1\left(\frac{u_t-u_0}{h}\right)\right]\xrightarrow{P}0,$$

$$\frac{1}{2}g_0''\left(\bar{x}\right)\left(X_t-x_0\right)^2+\frac{1}{2}h''\left(\bar{u}\right)\left(u_t-u_0\right)^2\xrightarrow{P}0,$$

under Theorem 1 as $n\to\infty$. We might as well define:

$$\Lambda\left(X_t,u_t\right)=\frac{1}{\sqrt{nh^2}}\left[\alpha_0+\alpha_1\left(\frac{X_t-x_0}{h}\right)\right]+\frac{1}{\sqrt{nh^2}}\left[\beta_0+\beta_1\left(\frac{u_t-u_0}{h}\right)\right],$$

$$\Psi\left(X_t,u_t\right)=\frac{1}{2}g_0''\left(\bar{x}\right)\left(X_t-x_0\right)^2+\frac{1}{2}h''\left(\bar{u}\right)\left(u_t-u_0\right)^2.$$

Consider:

$$
\begin{aligned}
& \mathbf{1}\left(\xi_n<\delta\right) \cdot F_n \\
= & \mathbf{1}\left(\xi_n<\delta\right) E\left\{\sum_{t=1}^n\left|\Lambda\left(X_t, u_t\right)-\Psi\left(X_t, u_t\right)-\epsilon_t\right|-\left|\Psi\left(X_t, u_t\right)+\epsilon_t\right|\right\} \\
& \cdot K_X\left(\frac{X_t-x_0}{h}\right) K_u\left(\frac{u_t-u_0}{h}\right) \\
= & \mathbf{1}\left(\xi_n<\delta\right) \cdot n E\left\{\left|\Lambda\left(X_1, u_1\right)-\Psi\left(X_1, u_1\right)-\epsilon_1\right|-\left|\Psi\left(X_1, u_1\right)+\epsilon_1\right|\right\} \\
& \cdot K_X\left(\frac{X_1-x_0}{h}\right) K_u\left(\frac{u_1-u_0}{h}\right) \\
= & \mathbf{1}\left(\xi_n<\delta\right) \cdot n E\left\{E\left\{\begin{array}{c}
\left[\left|\Lambda\left(X_1, u_1\right)-\Psi\left(X_1, u_1\right)-\epsilon_1\right|-\left|\Psi\left(X_1, u_1\right)+\epsilon_1\right|\right] \\
\cdot K_X\left(\frac{X_1-x_0}{h}\right) K_u\left(\frac{u_1-u_0}{h}\right) \mid \mathbf{X}, \mathbf{u}
\end{array}\right\}\right\} \\
= & \mathbf{1}\left(\xi_n<\delta\right) \cdot n E\left\{\begin{array}{c}
\int_{\delta>s>\Lambda\left(X_1, u_1\right)-\Psi\left(X_1, u_1\right)}\left[s+\Psi\left(X_1, u_1\right)-\Lambda\left(X_1, u_1\right)\right] \phi\left(s \mid X_1, u_1\right) d s \\
+\int_{-\delta<s<\Lambda\left(X_1, u_1\right)-\Psi\left(X_1, u_1\right)}\left[\Lambda\left(X_1, u_1\right)-\Psi\left(X_1, u_1\right)-s\right] \phi\left(s \mid X_1, u_1\right) d s \\
-\int_{\delta>s>-\Psi\left(X_1, u_1\right)}\left[s+\Psi\left(X_1, u_1\right)\right] \phi\left(s \mid X_1, u_1\right) d s \\
+\int_{-\delta<s<-\Psi\left(X_1, u_1\right)}\left[s+\Psi\left(X_1, u_1\right)\right] \phi\left(s \mid X_1, u_1\right) d s \\
-\int_{s>\delta} \Lambda\left(X_1, u_1\right) \phi\left(s \mid X_1, u_1\right) d s \\
+\int_{s<-\delta} \Lambda\left(X_1, u_1\right) \phi\left(s \mid X_1, u_1\right) d s
\end{array}\right\} \\
& \cdot K_X\left(\frac{X_1-x_0}{h}\right) K_u\left(\frac{u_1-u_0}{h}\right) .
\end{aligned}
$$

Applying the Integral Mean Value Theorem[8] yields:

$$
\begin{aligned}
& \mathbf{1}\left(\xi_n<\delta\right) \cdot F_n \\
= & \mathbf{1}\left(\xi_n<\delta\right) \cdot n E\left\{\begin{array}{c}
\phi\left(\iota_1 \mid X_1, u_1\right)\left[\frac{\delta^2}{2}+\delta\left[\Psi\left(X_1, u_1\right)-\Lambda\left(X_1, u_1\right)\right]+\frac{\left[\Psi\left(X_1, u_1\right)-\Lambda\left(X_1, u_1\right)\right]^2}{2}\right] \\
+\phi\left(\iota_2 \mid X_1, u_1\right)\left[\frac{\delta^2}{2}+\delta\left[\Lambda\left(X_1, u_1\right)-\Psi\left(X_1, u_1\right)\right]+\frac{\left[\Lambda\left(X_1, u_1\right)-\Psi\left(X_1, u_1\right)\right]^2}{2}\right] \\
+\Lambda\left(X_1, u_1\right)\left[\Phi\left(\delta \mid X_1, u_1\right)+\Phi\left(-\delta \mid X_1, u_1\right)-1\right] \\
-\phi\left(\iota_3 \mid X_1, u_1\right)\left[\frac{\delta^2}{2}+\delta \Psi\left(X_1, u_1\right)+\frac{\left[\Psi\left(X_1, u_1\right)\right]^2}{2}\right] \\
-\phi\left(\iota_4 \mid X_1, u_1\right)\left[\frac{\delta^2}{2}-\delta \Psi\left(X_1, u_1\right)+\frac{\left[\Psi\left(X_1, u_1\right)\right]^2}{2}\right]
\end{array}\right\} \\
& \cdot K_X\left(\frac{X_1-x_0}{h}\right) K_u\left(\frac{u_1-u_0}{h}\right) .
\end{aligned}
$$

where $\Phi(\cdot|\cdot)$ is the conditional CDF of ϵ. Notice that all the mean values $\iota_1, \iota_2, \iota_3, \iota_4 \rightarrow 0$

[8] Suppose $f(\cdot)$ and $g(\cdot)$ are continuous functions on $[a, b]$, and $g(\cdot)$ could be $g(\cdot) \geq 0$ or $g(\cdot) \leq 0$, then there exists a number $\zeta \in [a, b]$ such that: $\int_a^b f(x) g(x) dx = f(\zeta) \int_a^b g(x) dx$.

as $\delta \to 0$ and $n \to \infty$. Then for $\forall \delta$ that is small enough, we have:

$$\begin{aligned}
& \mathbf{1}\left(\xi_n < \delta\right) \cdot F_n \\
= \;& \mathbf{1}\left(\xi_n < \delta\right) \cdot nE\left\{\left[\phi\left(0\left|X_1, u_1\right.\right) + o_p\left(1\right)\right]\left[\Lambda^2\left(X_1, u_1\right) - 2\Lambda\left(X_1, u_1\right)\Psi\left(X_1, u_1\right)\right]\right\} \\
& \cdot K_X\left(\frac{X_1 - x_0}{h}\right) K_u\left(\frac{u_1 - u_0}{h}\right) \\
= \;& \mathbf{1}\left(\xi_n < \delta\right) \cdot \left\{ \begin{array}{c} \phi\left(0\left|X_0, u_0\right.\right) f\left(x_0, u_0\right) \left[\begin{array}{c} \alpha_0^2 + \alpha_1^2 \int z^2 K_X\left(z\right) dz + \beta_0^2 + \beta_1^2 \int z^2 K_u\left(z\right) dz + 2\alpha_0\beta_0 \\ -g''\left(x_0\right)\left(\alpha_0 \int z^2 K_X\left(z\right) dz + \beta_0 \int z^2 K_X\left(z\right) dz\right) \\ -h''\left(x_0\right)\left(\alpha_0 \int z^2 K_u\left(z\right) + \beta_0 \int z^2 K_u\left(z\right) dz\right) \end{array} \right] \\ +o_p\left(1\right) \end{array} \right\} \\
= \;& \mathbf{1}\left(\xi_n < \delta\right) \cdot \left[F\left(\alpha_0, \alpha_1, \beta_0, \beta_1\right) + o_p\left(1\right)\right].
\end{aligned}$$

Since $F_n = \mathbf{1}\left(\xi_n < \delta\right) \cdot F_n + \mathbf{1}\left(\xi_n \geq \delta\right) \cdot F_n$, we have:

$$F_n = F\left(\alpha_0, \alpha_1, \beta_0, \beta_1\right) + o_p\left(1\right) - \mathbf{1}\left(\xi_n \geq \delta\right) \cdot \left[F\left(\alpha_0, \alpha_1, \beta_0, \beta_1\right) + o_p\left(1\right) - F_n\right].$$

For any fixed $\alpha_0, \alpha_1, \beta_0, \beta_1$, we know $\xi_n \to 0$ as $n \to \infty$, it hereby completes the proof of Theorem 2. ■

Proof of Theorem 3. Consider the function L_n :

$$\begin{aligned}
L_n \;=\; & \sum_{t=1}^{n} L_{ni} = \sum_{t=1}^{n} \left\{ \frac{1}{\sqrt{nh^2}} \left[\alpha_0 + \alpha_1\left(\frac{X_t - x_0}{h}\right)\right] + \frac{1}{\sqrt{nh^2}} \left[\beta_0 + \beta_1\left(\frac{u_t - u_0}{h}\right)\right] \right\} \\
& \cdot K_X\left(\frac{X_t - x_0}{h}\right) K_u\left(\frac{u_t - u_0}{h}\right) \cdot \mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right) > 0\right) \cdot sgn\left(\epsilon_t\right).
\end{aligned}$$

and $R_n = A_n - F_n + L_n$. Under Theorem 2 when $\alpha_0, \alpha_1, \beta_0, \beta_1$ are fixed, we have:

$$A_n + L_n = F\left(\alpha_0, \alpha_1, \beta_0, \beta_1\right) + o_p\left(1\right) + R_n \equiv F\left(\alpha_0, \alpha_1, \beta_0, \beta_1\right) + \tilde{R}_n.$$

Actually the second moment of L_n is bounded:

$$
\begin{aligned}
E\left(L_n^2\right) &= \sum_{t=1}^{n} E\left(L_{nt}^2\right) \\
&\leq 2nE\left\{\frac{1}{nh^2}\left[\alpha_0+\alpha_1\left(\frac{X_t-x_0}{h}\right)\right]^2+\frac{1}{nh^2}\left[\beta_0+\beta_1\left(\frac{u_t-u_0}{h}\right)\right]^2\right\} \\
&\quad \cdot K_X\left(\frac{X_t-x_0}{h}\right)K_u\left(\frac{u_t-u_0}{h}\right)\cdot\mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right)>0\right) \\
&= 2\left\{\begin{array}{c}\int K_u^2(z)\,dz\left[\alpha_0^2\int K_X^2(z)\,dz+2\alpha_0\alpha_1\int zK_X^2(z)\,dz+2\alpha_1\int z^2K_X^2(z)\,dz\right] \\ +\int K_X^2(z)\,dz\left[\beta_0^2+2\beta_0\beta_1\int zK_X^2(z)\,dz+2\beta_1^2\int z^2K_u^2(z)\,dz\right]\end{array}\right\} \\
&\quad \cdot\mathbf{1}\left(\hat{m}\left(\tilde{X}_t\right)>0\right).
\end{aligned}
$$

So L_n is bounded in probability for any fixed $\alpha_0,\alpha_1,\beta_0,\beta_1$, and A_n+L_n converges to $F\left(\alpha_0,\alpha_1,\beta_0,\beta_1\right)$ in probability. Since A_n+L_n is a convex fuction, suppose $\left\{\alpha_0,\alpha_1,\beta_0,\beta_1\right\}^T\in\Bbbk$ which is a compact subset of the convex open parameter space, by the Convexity Lemma of Pollard (1991):

$$
\sup_{\left\{\alpha_0,\alpha_1,\beta_0,\beta_1\right\}^T\in\Bbbk}\left|A_n+L_n-F\left(\alpha_0,\alpha_1,\beta_0,\beta_1\right)\right|\xrightarrow{P}0.
$$

And that implies:

$$
\sup_{\left\{\alpha_0,\alpha_1,\beta_0,\beta_1\right\}^T\in\Bbbk}\left|\tilde{R}_n\right|=o_p(1)\quad\text{as}\quad\delta\to0.
$$

According to the proof of Theorem 1 in Pollard (1991), the solution $\hat{\gamma}$ of minimizing A_n converges to the solution $\check{\gamma}$ of minimizing $F\left(\alpha_0,\alpha_1,\beta_0,\beta_1\right)-L_n$. And minimizing $F\left(\alpha_0,\alpha_1,\beta_0,\beta_1\right)-L_n$ yields the following estimated coefficients:

$$
\begin{aligned}
\check{\alpha}_0 &= \frac{1}{2}g''(x_0)\int z^2K_X(z)\,dz, \\
\check{\alpha}_1 &= \frac{\frac{1}{\sqrt{nh^2}}\sum_{t=1}^{n}\frac{X_t-x_0}{h}K_X\left(\frac{X_t-x_0}{h}\right)K_u\left(\frac{u_t-u_0}{h}\right)sgn\left(\epsilon_t\right)}{2\phi\left(0\left|x_0,u_0\right.\right)f\left(x_0,u_0\right)\int z^2K_X(z)\,dz}, \\
\check{\beta}_0 &= \frac{1}{2}h''(u_0)\int z^2K_u(z)\,dz, \\
\check{\beta}_1 &= \frac{\frac{1}{\sqrt{nh^2}}\sum_{t=1}^{n}\frac{u_t-u_0}{h}K_X\left(\frac{X_t-x_0}{h}\right)K_u\left(\frac{u_t-u_0}{h}\right)sgn\left(\epsilon_t\right)}{2\phi\left(0\left|x_0,u_0\right.\right)f\left(x_0,u_0\right)\int z^2K_u(z)\,dz}.
\end{aligned}
$$

To derive the asymptotic normality of $\hat{\gamma}$, it sufficies to derive the asymptotic normality of $\check{\gamma}\equiv\left\{\check{\alpha}_0,\check{\alpha}_1,\check{\beta}_0,\check{\beta}_1\right\}^T$. Obviously from the structure of the solution, such a task is reduced to finding the asymptotic distributions of $\frac{1}{\sqrt{nh^2}}\sum_{t=1}^{n}\frac{X_t-x_0}{h}K_X\left(\frac{X_t-x_0}{h}\right)K_u\left(\frac{u_t-u_0}{h}\right)sgn\left(\epsilon_t\right)$

and $\frac{1}{\sqrt{nh^2}}\sum_{t=1}^{n}\frac{u_t-u_0}{h}K_X\left(\frac{X_t-x_0}{h}\right)K_u\left(\frac{u_t-u_0}{h}\right)sgn\left(\epsilon_t\right)$. We will show the proof for $\frac{1}{\sqrt{nh^2}}\sum_{t=1}^{n}\frac{X_t-x_0}{h}K_X\left(\frac{X_t-x_0}{h}\right)K_u\left(\frac{u_t-u_0}{h}\right)sgn\left(\epsilon_t\right)$ only, since the proof for the other half just follows suit. Note that one can directly apply the Lindeberg CLT for triangular arrays under the Lindeberg condition if without $\frac{X_t-x_0}{h}$ and $sgn\left(\epsilon_t\right)$ in the expression.

Now let

$$\Gamma_n^2=\frac{1}{nh^2}\sum_{t=1}^{n}\left(\frac{X_t-x_0}{h}\right)^2K_X^2\left(\frac{X_t-x_0}{h}\right)K_u^2\left(\frac{u_t-u_0}{h}\right),$$

then:

$$\Gamma_n^2\xrightarrow{P}\Gamma^2=f\left(x_0,u_0\right)\int z^2K_X^2\left(z\right)dz\int K_u^2\left(z\right)dz.$$

Set

$$q_t=\frac{1}{\sqrt{nh^2}}\frac{X_t-x_0}{h}K_X\left(\frac{X_t-x_0}{h}\right)K_u\left(\frac{u_t-u_0}{h}\right),$$

Under the assumption that $\sup_{t\in[1,n]}\|X_t\|=o_p\left(n^{\frac{1}{10}}\right)$,

$$\sup_{t\in[1,n]}|q_t|\leq C\left(nh\right)^{-0.5}h^{-1}\sup_{t\in[1,n]}\|X_t\|=o_p\left(1\right)$$

in which C is some positive constant. Then for any $\vartheta_0>0$, there exists some $\delta_0>0$ such that:

$$\begin{aligned}&\left(\sum_{t=1}^{n}q_t^2\right)^{-1}\sum_{t=1}^{n}E\left[\left(q_tsgn\left(\epsilon_t\right)\right)^2\cdot\mathbf{1}\left(|q_tsgn\left(\epsilon_t\right)|\geq\vartheta_0\sum_{t=1}^{n}q_t^2\right)|\mathbf{X},\mathbf{u}\right]\\ \leq\ &\left[\left(\Gamma_n^2\right)^{-1}\sum_{t=1}^{n}q_t^2\mathbf{1}\left(|q_t|\geq\vartheta_0\Gamma_n^2\right)\right]\mathbf{1}\left(\Gamma_n^2\leq\delta_0\right)+\mathbf{1}\left(\sup|q_t|\geq\vartheta_0\Gamma_n^2\right)\cdot\mathbf{1}\left(\Gamma_n^2>\delta_0\right)=o_p\left(1\right).\end{aligned}$$

This guarantees that the Lindeberg-Feller condition holds, by applying the Cramér-Wold device, we have:

$$\frac{1}{\sqrt{nh^2}}\Gamma^{-1}\sum_{t=1}^{n}\frac{X_t-x_0}{h}K_X\left(\frac{X_t-x_0}{h}\right)K_u\left(\frac{u_t-u_0}{h}\right)sgn\left(\epsilon_t\right)\xrightarrow{d}N\left(0,1\right).$$

Then Theorem 3 follows. ■

Chapter 3 International Real Business Cycles: Asymmetric Preference, Incomplete Financial Market, and Terms of Trade Shocks[9]

1 Introduction

It is well known that the Dynamic Stochastic General Equilibrium (DSGE) paradigm encounters stubborn anomalies when it is extended to open economy, and those difficulties are not the result of international risk sharing (Backus *et al.*, 1992). Nonetheless after decades of exploration[10] those quantity anomalies are still not convincingly accredited, openness changes the behavior of the model but in a way that remains ambiguous to Macroeconomists. Personally we believe a successful resolution of the problem lies in the flexible functional form that is globally regular and preferably parsimonious,[11] and perhaps assuming away intertemporal preference independence (intertemporally strong separability), this is just way too complicated in a general equilibrium setting. It does not mean, however, that nothing can be learned from the representative consumer model. This paper tries to provide insights into Chinese open economy, the twofold questions in our mind are: first, above everything else what are the empirical stylized facts of the Chinese economy? Do any of them make it particularly different from the other major economies of the world? This is important not only because a useful model should be able to account for those facts well, but also we find in many cases it is the summary of the facts that is skewed or reckless. Second, can a theoretical general equilibrium model satisfactorily explain those open economy volatilities? If yes, what is the mechanism? If not, what is the problem?

In response to those questions, what this paper does is firstly, using four different kinds of high pass or band pass filters to extract volatility so that it contributes to a robust description of the Chinese open economy fluctuations. It is astonishing that different filters give rise to estimates that differ quite a lot in almost all the occasions, nonetheless those estimates constitute a possible range of the magnitude of fluctuations, and this range forms a natural gauge of how well the model explains the facts. Meanwhile as an incidental product, we believe this robust summary of the historical properties of Chinese business cycle contributes to the literature a work that is parallel and complementary to Backus and Kehoe (1992),

[9] This paper is published on *Emerging Market Finance and Trade*, Volume 55, Issue 9, 2019, pp. 1926-1953.

[10] For a great survey please refer to King and Rebelo (1999).

[11] Taking Cobb-Douglas utility for instance, every single elasticity of Cobb-Douglas is fixed and remains. All income elasticities are 1, all own price elasticities are -1, all cross price elasticities are 0, all elasticities of substitution are 1. Flexible functional form in contrast gives rise to meaningful estimation of elasticities.

in which China is missing in the international comparison. Secondly, the paper establishes a two country stochastic general equilibrium model which is thus calibrated according to the Chinese real economy. Interestingly, we find that China's investment is consistently negatively correlated with the investment of most of the economies in the world; and the model that features asymmetric preference, incomplete financial markets and terms of trade shocks explains the data well. It reveals that the core mechanism lies in the zero wealth effect of leisure which makes both of the countries' labor supply grow at the same time and so does the output.

The rest of the paper is arranged as follows: the next section reviews the literature and comments. Section 3 summarizes the empirical stylized facts which consist of the relative volatility and international co-movements of the major macro variables. Section 4 unfolds the theoretical model which starts with a benchmark and then is extended step by step. Section 5 concludes.

2 Literature Review

After Backus *et al.* (1992) extend Kydland and Prescott (1982)'s model which was built on Brock and Mirman (1972), it is wildly acknowledged that the open economy models fail to account for the real world for the following reasons: consumption is smoother in the model than reality, investment is way too volatile, the fluctuation of trade in theory is higher than what is suggested by the data, the contemporaneous correlation of investment and output is much smaller in the model, outputs are positively correlated in the data but the model suggests the other way around. Most of these problems can be tackled with parameter adjustment or model revision, but the high contemporaneous correlation of consumption which is even higher than the co-movement of output is the stubborn anomaly that is difficult to deal with. The literature that follows either tries to solve this anomaly but fails, or gets it on lock by causing other problems. Either way, possible refinements lie in two aspects: shock processes and transmission mechanisms.

2.1 Beyond technology shocks

Though macroeconomists get used to attributing the long-run growth to the increase of total factor productivity (TFP), very few of them tend to believe recessions are brought about by technological setbacks. Galí (1999) questions the propagation aroused by a positive technology shock in the competitive equilibrium of a neoclassical model, evidence from a structural VAR is at odds with the RBC interpretations of aggregate fluctuations, but is reconcilable with a monopolistic competitive model with demand shocks. Christiano and Eichenbaum (1992) look into shocks to government purchases. Bencivenga (1992) explores preference shocks, among others. Smets and Wouters (2003) even use more orthogonal structural shocks (ten) than observables (seven) to build a general equilibrium model for Euro area. Smets and Wouters (2007) generalize the processes followed by some of the exogenous shocks to fit a model with the US data in which there are seven shocks and seven observables. It is worthwhile pointing out that in almost every case of multiple shocks, though they help lower the co-movement of consumption a little bit (still excessively high), the tacit assumption that those shocks are orthogonal and policy invariant is a possible source of partial misspecification. When we seek for an explanation of severe recessions like the sub-prime crisis, what we need is a shock that is more powerful than the ones exemplified in Kydland and Prescott (1982) and its subsequent extensions. But none of the shocks are quantitatively large enough to explain the huge real economic volatility. So people naturally turn to the exploration of transmission mechanism that preferably has the effect of a multiplier.

2.2 Transmission Mechanism

Baxter and Crucini (1995) and Kollmann (1996) rely on incomplete market to reduce risk pooling and sharing, Heathcote and Perri (2002) compare a model that is completely incapable of international asset transaction with a model that has a complete market but one riskless bond. Kose and Yi (2002, 2006) extend Heathcote and Perri's comparison to a three country model and discover that a stronger trade linkage indeed helps to boost international co-movement, but doesn't help to eliminate the excessively high cross country correlation of consumption. Besides the usual tradable sector, Stockman and Tesar (1995) add to the model a non-tradable sector which partially solves the consumption anomaly though, also results in a complete negative correlation between price and consumption and extremely low estimate of the standard error of the trade balance.

There are papers that endogenize trade in a way that firms decide whether to enter an industry first, then they decide whether or not to become exporters. Alessandria and Choi (2004) partially solve the consumption/output anomaly by making domestic consumers dislike foreign products. Ghironi and Melitz (2005) vary each monopolistic competitive firm's productivity and there is a sunk cost for entering domestic markets, and there is a fixed cost and variable cost for being an exporter. Exogenous shocks would make firms decide whether to enter or export if any, so that it changes the consumption structure of a country. Farhat (2009) modifies a little on the capital accumulation and labor market which does not change the major result much.

2.3 Studies on Chinese Economy

If we thumb through the dynamic modelling literature on the Chinese business cycles, there are more papers focusing on estimation based upon some DESG models than tailoring a dynamic equilibrium model that helps explain the Chinese stylized facts. Dai *et al.* (2015) examine Smets and Wouters (2007) model using Chinese data, they estimate this model using three different methods: Bayesian, Maximum Likelihood, and indirect inference procedures. Their conclusion is that under MLE and indirect inference methods the model with a fair-sized competitive product market sector outperforms the New Keynesian model. But we know that Smets and Wouters (2007) is a closed economy model and exactly the same model is calibrated and used to match the US data. Song *et al.* (2011) elegantly build a closed economy Overlapping Generations (OLG) model that incorporates two types of firms with high and low productivity respectively. The high productivity firms are financially constrained, while the low productivity firms are endowed with better access to credit markets. This model successfully explains reallocations within the manufacturing sectors, but has a slight drawback in terms of high savings rate. They want to claim the high productivity firms are subject to financial imperfections so they finance investments through internal savings. Yet in an OLG model the high savings rate originates from the fact that old people have no income besides relying on the savings they made when they were young, so young generations just over save and the implication is that there are too many capital flopping around for the high productivity firms, which contradicts with the assumption that they are financially awkward. Wang *et al.* (2017) also consider the borrowing constraint of the Chinese economy, they establish a canonical New Keynesian model with heterogeneous production sectors with the private firms are credit constrained through a Kiyotaki-Moore (1997) type of borrowing restriction. Mehrotra *et al.* (2013) investigate the effect of rebalancing from the investment-led to consumption-led growth through a revised Christiano *et al.* (2005) model, they estimate the key parameters and calibrate the model with data for the Chinese economy. No matter how well these closed economy models explain the growth pattern of China, there is no guarantee these models could do an equally decent job in an open economy context. Just like the US case from Kydland and Prescott (1982) to Backus *et al.* (1992).

This paper emphasizes open economy modelling and tailors the model carefully so as to explore how a two-country DSGE model could explain Chinese economy fluctuations. It points out that a combination of asymmetric preference, incomplete financial markets and terms of trade fits China's open economy reasonably well. Current literature focuses models themselves and systematically ignores whether the stylized facts are robust. A careful summarization of the stylized facts offered as follows implies that the gap between theory and empirical facts exists but is not as big as we thought before.

3 Empirical Stylized Facts of Chinese Open Economy

The term business cycle describes the magnitude and correlation of the fluctuations of major macroeconomic variables. We single out and analyze aggregates like output, employment,[12] consumption expenditure, investment, capital stock and net export. Firstly we investigate the relative volatility and the cyclical properties of those observables; then we take a look at the co-movement based on an international comparison with China's major trade partners in the world.

3.1 Relative Volatility and Cyclical Properties

3.1.1 Data

Admittedly data credibility is a problem for studies about China, so we are cautious in data selection and try our best to make sure data reveal sense. All data are annual data, the reason for that is quarterly GDP starts from 1992, statistics during 1992-2007 serve as a reference and are not acknowledged by the authority; quarterly labor supply is even worse, the closest proxies for that is registered unemployment rate in cities and towns starting from 2000, or end of period number of employees in cities and towns. That's why we decide to use yearly data, the model thus is calibrated accordingly.

For investment, there is only Fixed Asset Investment available since 1980, before that the only legitimate statistics for investment is the Total Fixed Asset Investment of the Working Units Owned by All the Population.[13] It seems pertinent to start the data from 1980 for investment. Another problem is the Price Index of the Fixed Asset Investment, current index begins from 1991 before which the data is missing. In this paper we deflate the Fixed

[12] Employment here refers to the number of labor hired which is unfortunately a lousy measure. Data on working hours is unavailable.

[13] Change of the gauge makes a huge difference. Taking a two year subsample of 1980 and 1986 for instance, Fixed Asset Investment is 91.085 and 312.06 billion; whereas the Total Fixed Asset Investment of the Working Units Owned by All the Population is 74.59 and 197.85 billion respectively.

Asset Investment between 1980 and 1990 using Producer Price Index for Manufactured Products (PPI), which is less likely to cause a problem in that the two indexes always move together.[14] So the real investment spans from 1980 to 2009, it is obtained by deflating nominal Fixed Asset Investment using the Price Index of Fixed Asset Investment with 1991 as the base year, we use PPI based on the same year to replace the missing price index during 1980-1990.

Capital stock is calculated by perpetual inventory:

$$K_{t+1} = I_t + (1 - \delta) K_t,$$

where the depreciate rate δ is 10%. Several papers use this method to estimate China's capital stock (Chow (1993), Zhang and Zhang (2003), Zhang *et al.* (2004), Sun and Ren (2005)), and the difference can be huge in the resulting estimation due to the initial value employed. The initial capital stock used in this paper is 626.7 billion RMB in 1978 which is directly from the estimate of Zhang *et al.* (2004).[15]

Net export in this paper means the current value trade deficit over the nominal GDP, which naturally renders it into a real variable. Since sometimes this ratio is quite small, changes of its standard error and correlation against output attribute to the changes in the numerator.

All data related to national income are from Guo Tai An Data Base, only the Producer Price Index for Manufactured Products during 1980-1990 comes from *China Statistics Almanac*. All variables are rendered into real variables, specifically, we deflate output using CPI with 1978 as the base year, consumption expenditure is catered by People's Consumption Level Deflator with 1978 as the base year. Investment is deflated by a combination of Fixed Asset Investment Price Index and PPI based on 1990. Investment used in the estimation of capital stock is real investment, employment and net exports (trade balance/GDP) are real variables. Except for the net exports, all variables are logarithmically transformed.

3.1.2 Filters

Data contain growth and volatility among everything else, we need to separate growth from volatility which constitutes business cycle. The most acknowledged way of extracting

[14] Truncating a subsample from where both indexes are available, what we can find is when the price index of fixed asset investment is low (eg: 99.8 and 99.6 in the year 1998 and 1999) the PPI is also low (95.9 and 97.6 respectively); when the price of fixed asset investment is high (126.6 in 1993) PPI is also high (124.0).

[15] Zhang Jun, Guiying Wu and Jipeng Zhang. 2004. "Estimating the Provincial Capital Stock of China: 1952-2000." Economic Research Journal, (10): p. 43. (in Chinese).

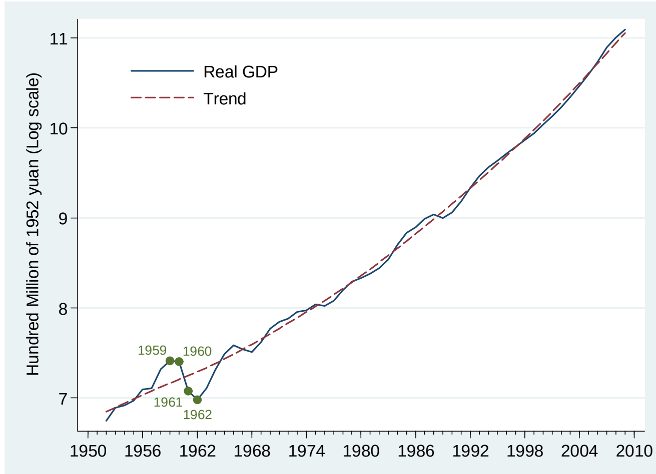

Figure 1: Real GDP and Its Trend

volatility is Hodrick-Prescott (HP) filter which is so widely used that the greatest majority of business cycle discussion is about HP volatility. It's worthwhile noticing that HP filter is a highpass filter which removes the low frequency behavior, only the high frequency components remain and are associated with business cycles. The problem is not all the high frequency contents are regarded as business cycles. If we believe cycles of two to eight years correspond to reasonable business cycle, frequencies lower than that refer to cycles longer than eight years and frequencies higher than that refers to annual cycle or even seasonality, none of both are desirable remanence in our definition of business cycles. The essence of filters is to ascertain business cycle frequencies in an opportune manner and get rid of the rest, so instead of focusing only on HP filter, this paper for the sake of robustness also employs Baxter-King filter (BK), Christiano-Fitzgerald random walk filter (CF), and Butterworth square wave filter (BW). Among them HP and BW are highpass filters, BK and CF are bandpass filters. As a demonstration figure 1 shows China's real GDP with its trend in dashed line, figure 2 compares the volatility extracted by using different kinds of filters.

Baxter and King (1999) believe, compared with HP filter, the NBER definition of business cycles coincides more properly with bandpass filters. In BK filter the optimal bandpass is an infinite order moving average for which we need to find a good approximation, and there will be a loss of observations on both ends of the sequence. Christiano and Fitzgerald (2003) find that the resulting bandpass is the closest to optimality when the DGP is assumed to be a random walk. Au contraire, CF filter does not sacrifice sample observations.

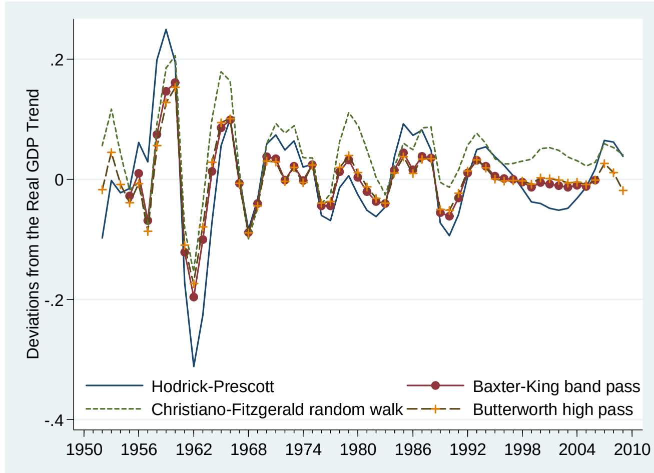

Figure 2: Real GDP Volatility Measured by Four Filters

The reason we employ BW filter is to take into account structural breaks which are rather common. As it can be seen from figure 1, the year 1959-1962 corresponds to a break which was the Great Leap Forward and the great famine afterwards. Figure 2 confirms an apparently increased volatility during this period. The aforementioned filters lack dexterity on this. Recall that HP filter uses a single parameter to control the smoothness, so this parameter both controls the cut-off point of frequency and the rate of transition between passband and stopband. Pollock (2000) presents a new technique for designing frequency-selective filter which is based on a rational function approximation to the ideal square wave, and the filter is also called Butterworth filter in engineering.

As we can see shortly, the differences under four filters can be substantial. The results from HP filter are generally larger than others in almost every occasion, a phenomenon leads us to ponder that it is the summary of facts that is biased rather than the model is absurd when we attempt to compare the model with data, though the model itself is bound to be highly abstract and patently unreal.

Figure 2 Comovement and Transnational Correlation

Country/Area	Filters	Correlations with the Corresponding Chinese Variables					Second Order Movement of Each Country	
		Output	Consumption	Investment	Gov. Expend.	NetExport/GDP	Corr(NX/GDP, Y)	Std Dev. of NX/GDP
	HP	0.17	0.34	0.02	-0.02	0.19	-0.49	0.73
US	BK	0.37	0.38	0.09	0.04	0.18	-0.44	0.33
(1952-2009)	CF	0.45	0.18	0.13	0.01	-0.12	-0.57	0.36
	BW	0.43	0.31	0.01	0.00	-0.05	-0.46	0.31
	HP	0.24	-0.19	-0.58	-0.29	-0.26	-0.04	0.48
Euro Zone	BK	-0.05	0.13	-0.22	0.07	-0.15	-0.17	0.28
(1998-2009)	CF	0.69	0.38	-0.41	-0.43	0.06	-0.04	0.41
	BW	0.75	0.21	-0.73	-0.27	0.03	-0.01	0.37
	HP	-0.07	-0.32	-0.39	-0.06	-0.31	-0.12	0.97
Japan	BK	-0.24	-0.07	-0.54	-0.24	-0.33	-0.39	0.68
(1955-2009)	CF	0.01	-0.19	-0.39	-0.20	0.04	-0.25	0.77
	BW	0.01	-0.08	-0.47	-0.17	-0.10	-0.34	0.68
	HP	0.12	0.30	-0.31	-0.22	0.15	-0.63	1.24
UK	BK	0.31	0.30	-0.22	0.00	0.11	-0.58	0.76
(1952-2009)	CF	0.43	0.20	-0.23	-0.22	0.09	-0.53	0.74
	BW	0.41	0.18	-0.25	-0.02	0.07	-0.51	0.66
	HP	0.13	0.15	-0.07	0.38	0.14	-0.08	1.10
Australia	BK	0.26	-0.03	-0.03	0.37	0.33	-0.28	0.89
(1959-2009)	CF	0.31	-0.12	0.00	0.08	0.17	-0.41	0.90
	BW	0.34	0.03	-0.04	0.06	0.23	-0.41	0.86
	HP	0.44	0.33	-0.06	0.05	-0.25	-0.10	0.78
India	BK	-0.07	0.20	-0.44	0.08	0.09	0.12	0.47
(1960-2009)	CF	-0.24	0.12	-0.55	0.05	0.31	0.33	0.48
	BW	-0.14	0.15	-0.47	-0.15	0.04	0.12	0.44

Figure 1 Volatility Characteristics of China's Major Macroeconomic Variables

Variables	Filters	Standard Deviation		Correlation with Output of a lag length of k, where k=										
		Percentage	Relative to Y	-5	-4	-3	-2	-1	0	1	2	3	4	5
	HP	9.03	1.00	-0.27	-0.49	-0.45	0.00	0.63	1.00	0.42	-0.29	-0.33	-0.09	0.05
Real Output	BK	5.88	1.00	0.08	-0.37	-0.69	-0.35	0.42	1.00	0.47	-0.25	-0.38	-0.08	0.16
(1952-2009)	CF	6.59	1.00	0.14	-0.49	-0.81	-0.40	0.44	1.00	0.52	-0.23	-0.38	-0.13	0.05
	BW	5.23	1.00	0.15	-0.37	-0.71	-0.40	0.38	1.00	0.55	-0.12	-0.29	-0.12	0.03
	HP	2.68	0.30	-0.25	-0.33	-0.20	0.17	0.42	0.45	0.26	0.09	0.10	0.14	0.13
Employment	BK	1.91	0.32	-0.07	-0.31	-0.32	0.11	0.31	0.14	0.17	0.12	0.14	0.20	0.19
(1952-2009)	CF	1.80	0.27	-0.03	-0.35	-0.36	0.08	0.33	0.27	0.37	0.28	0.21	0.21	0.19
	BW	1.74	0.33	-0.04	-0.30	-0.32	0.10	0.29	0.25	0.34	0.30	0.26	0.25	0.20
	HP	9.25	1.02	-0.23	-0.35	-0.42	-0.42	-0.29	-0.03/0.56	-0.38	-0.23	-0.02	-0.02	-0.06
Consumption	BK	3.68	0.63	0.15	0.07	-0.06	-0.35	-0.58	-0.17/0.74	-0.63	-0.32	-0.03	-0.02	-0.06
(1978-2009)	CF	3.25	0.49	0.47	0.38	0.06	-0.44	-0.71	-0.28/0.70	-0.67	-0.36	0.02	0.06	0.00
	BW	2.86	0.55	0.28	0.20	0.04	-0.35	-0.68	-0.24/0.73	-0.60	-0.35	-0.09	-0.04	-0.02
	HP	15.31	1.70	-0.46	-0.57	-0.39	-0.00	0.47	0.86	0.07	-0.47	-0.33	-0.14	-0.06
Investment	BK	8.04	1.37	-0.08	-0.40	-0.50	-0.42	-0.06	0.82	0.05	-0.47	-0.48	-0.37	-0.27
(1980-2009)	CF	6.71	1.02	0.18	-0.33	-0.52	-0.42	0.06	0.73	0.09	-0.49	-0.51	-0.30	-0.17
	BW	6.52	1.25	0.09	-0.32	-0.48	-0.44	-0.02	0.71	0.03	-0.44	-0.46	-0.32	-0.21
	HP	6.05	0.67	0.11	-0.12	-0.34	-0.43	-0.32	0.09	-0.50	-0.20	0.09	0.11	0.08
Capital Stock	BK	2.10	0.36	0.38	0.16	-0.27	-0.57	-0.66	-0.38	-0.68	-0.17	0.18	0.23	0.18
(1978-2009)	CF	1.30	0.20	0.52	0.41	-0.11	-0.51	-0.62	-0.30	-0.76	-0.36	0.25	0.43	0.34
	BW	1.42	0.27	0.42	0.27	-0.17	-0.50	-0.65	-0.41	-0.67	-0.25	0.16	0.28	0.25
	HP	1.30	...	-0.10	0.02	0.07	0.06	-0.07	-0.17	0.01	0.22	0.17	0.04	-0.02
NetExport/GDP	BK	0.91	...	-0.11	0.06	0.14	0.17	0.01	-0.27	0.04	0.30	0.28	0.13	0.01
(1952-2009)	CF	0.89	...	-0.06	0.12	0.18	0.17	-0.05	-0.31	-0.01	0.28	0.25	0.08	-0.03
	BW	0.84	...	-0.09	0.05	0.11	0.18	0.03	-0.20	0.07	0.25	0.20	0.07	-0.01

3.1.3 Results

Table 1 summarizes the cyclical properties of the Chinese economy. The left panel is the standard deviation of those major variables under four filters respectively, they are displayed in percentage as well as their relative value against the output. The right panel reports their correlation with output of a lag length of k where k takes an integer value from [-5, 5], so when k equals 0 it reveals contemporaneous correlation. The volatility of real output is 5-9 in percentage under different measure of filters, we normalize them to 1 when we talk about the relative volatility. Employment is one third as volatile as output. Consumption is almost twice as volatile as employment, that is, 50-60% of the output fluctuation. Only the measure under HP filter is a bit higher than that of the output. Investment is more volatile than output under all filters, the highest value is 1.7 times higher than the output fluctuation. The volatility of capital stock is rather mild, only 0.20-0.67 as volatile as the output. It's particularly interesting that the contemporaneous correlations of capital and output are mostly slightly negative, the coefficients lie between -0.3 to -0.4. Only the measure of HP filter indicates that capital is weakly pro-cyclical, with a correlation coefficient of 0.09. Taking into account that capital stock is negatively correlated with output of a lag/lead length of one and two, a robust conclusion would be that capital stock is counter-cyclical! Since the correlation between capital stock and output lagging 3-5 years are all positive, it takes roughly three or four years for capital to adjust. It sounds counter intuitive that capital is counter-cyclical, but the discussion is in terms of fluctuations. When output is volatile, capital is relatively stable. This result is robust under all filters used. It is investment that fluctuates at the same pace with output, not capital stock. Lastly, net export is a ratio thus is not logarithmically transformed, we only consider its volatility in percentage which varies between 0.84 and 1.30. There are mild values compared with output under each filter. Even though for a long time China was considered as an export driven economy, its net export volatility is counter-cyclical like most of the other economies in the world. The counter cyclicality of net export is robust when we focus on a subsample that starts from the year 1990. In a nutshell, China is not excluded from the regularity of the world.

Among all the stylized facts, the counter-cyclicality of consumption is the strangest phenomenon which does not make sense. The contemporaneous correlations of consumption and output (fluctuation) are between -0.03 to -0.28 under four filters, the correlations of consumption and output with lag orders of ± 1, ± 2 are all negative and are higher (in absolute value) than the contemporaneous coefficients. This leads us to question whether the data of consumption from Guo Tai An (GTA) is reliable, because if we measure the contemporaneous correlation of consumption and output using the data from the IMF, all coefficients are positive, indicating that consumption is pro-cyclical. We listed nonetheless all the coefficients in the intersection of the row of consumption and the vector of contemporaneous correlation in Table 1[16]. The behavior of other variables makes sense, investment

[16] Indeed China's consumption is taking a declining proportion in the consumption output ratio, which by

is strongly pro-cyclical, employment is slightly pro-cyclical, net export and capital stock are weakly counter-cyclical.

3.2 International Co-movements

Co-movements analyze how and to what extent China synchronizes with the rest of the world. Specifically, this paper calculates the correlations of expenditure components of China with the same components of six other countries which are the US, Euro zone, Japan, the UK, Australia and India. The US, Europe and Japan are China's top three trade partners, Australia and India are also among the top ten. All these internationally comparable data are from the IFS database. For the data about China, we use the data from IFS for the sake of comparison. Indeed IFS statistics differ from China's own release, those differences, as we believe after scrutinizing, should not result in contradictions. The data are all nominal data for all countries, we transform GDP, household consumption, government expenditure and fixed asset formation into real variables using GDP deflator with 2005 as the base year. Net export is defined as the ratio of trade balance over nominal GDP as before. All variables are transformed to logarithms except for the net export, and all the filters are applied for the sake of robustness.

Although all these nations differ dramatically in terms of institutions, monetary and fiscal policy, manufacturing industry, trade structure, and average growth rate, there are still common regularities to be inventoried. Table 2 summarizes in the left panel the contemporaneous correlations of the expenditure variable with the same variable of China for each country under different filters; the right panel is the correlations of net export and GDP as well as the standard deviation of net export. The transnational correlation of China's output volatility varies by countries and even by filters, other than with Japan and India, it's generally positive with the US, the Europe, the UK and Australia, among which the largest correlation is with the Europe, the correlation with America is roughly equal to the correlation with the Great Britain, and the correlation with Australia is obviously lower. The results are robust under respective filters. The co-movements with Japan either are negative or positive but quite close to zero, which somehow is related with the policy intervention of the Japanese government during the sample period (1955-2009). The correlations with India are mostly negative, which probably reveals the competition as developing countries. Generally, for the output co-movement, China is positively related with western countries

no means serves to justify that consumption is counter cyclical. From this part onwards, we use the data from the source of IMF wherever the property of consumption is call for.

but negatively related with Asian countries.

The second variable in the left panel is the transnational correlation of consumption. Except with Japan, China's consumption is positively correlated with the rest of the countries. By and large, consumption regularity of the Chinese economy complies with the international evidence with one exception: co-movements of consumption are not apparently lower than the co-movement of output. Among the consumption co-movements with four countries (the US, Euro zone, the UK and Australia), the correlations are lower than those of the output only under the measure of CF and BW; HP and BK filters suggest otherwise. Again we would conclude consumption co-movements are higher than output co-movements should we only focus on the HP filter. Au contraire, we are inclined to conclude that co-movements of consumption are generally positive, but there is no consistent evidence to claim that co-movements of consumption are higher or lower than those of output. In other words, it is acceptable to say that transnational correlations of consumption lie between 1/3 and equal proportion to the co-movements of output.

One aspect that deserves special attention is the transnational co-movement of investment. Except the correlations with America which are positive but fairly low, China's co-movements of investment are unanimously negatively correlated with the rest of the countries. Government expenditure does not exhibit much regularity. Co-movements of net exports are generally positive except with Japan. Although processing trade takes up more than half of the entire trade for decades, it does not change the fact that China's trade has long synced with the world trade. The right panel of Table 2 indicates that net exports are counter-cyclical for most of the countries.

3.3 Summary and Explanation

Based on Table 1 and 2, we briefly summarize the empirical stylized facts of China's open economy business cycles which serve both as a reference for the prediction of the theory and a source of calibration.

(1) Employment is only 1/3 as volatile as output, consumption is about 50-60% as volatile as output, investment is more volatile than both but its volatility does not exceed twice the volatility of output.

(2) Investment is strongly pro-cyclical, employment is weakly pro-cyclical, consumption (based on the IMF data) is also pro-cyclical. Capital stock and net export are counter-cyclical. The volatility of net export is higher than that of the other countries.

(3) The co-movements of output and consumption are all positive, but there is evidence that China's transnational correlation of consumption is close to the transnational correlation of output.

(4) Co-movements of investment are negative, co-movements of net export are positive, government expenditure has no tractable regularity. Negative investment co-movement is the particular feature that makes the Chinese economy stands differently.

The excess volatility in exports and negative co-movement in investment are the two China-specific stylized facts that warrant some explanation. China's net exports are more volatile because of the processing trade which takes approximately half of the entire trade volume during the sample period. Companies engaging in processing trade do not embark on product design or own the brands, what they do is to obtain the material and produce the final products that are labor-intensive. Therefore processing trade is vulnerable to foreign business cycles. It is booming when the foreign economy is strong, and it is plunging when the foreign economy is sluggish. Particularly, this line of business is hard to sustain even if the labor cost increases a bit. All these contribute to a high volatility of China's net exports.

Another puzzle is that China's investment is negatively correlated with that of the other countries, this is mostly because of government dominance in the investment behavior. It is only until very recent years that consumption becomes the driving force of economic growth in China, for a long time the number one driving force was investment, and investment was highly dominated by governments. State Owned Enterprises (SOEs) among which 33% are owned by the central government and the rest by local governments account for 30%-40% of total GDP. During economic downturns when the private investment from other countries is dampened, SOE investment from China is meant to increase so as to stop the erosion of confidence, thus results in a negative co-movement of investment which is unique as far as we see it.

4 The Model

In this section we firstly demonstrate a benchmark model, and then gradually expand it by introducing asymmetric preference, incomplete financial markets and terms of trade shocks. There are two countries and each country has an infinite number of homogenous normative representative consumers and a unique technology. Since terms of trade necessitate at least two commodities, we tackle with this in a simple way by assuming every country produces a single product using her specialized technology. Part of the products is for domestic purpose, the rest becomes exports. Every individual in both countries ultimately consumes a composite good that is aggregated by products from both countries, and this aggregated good can be used both for consumption and investment. Such a specification facilitates us for the investigation of terms of trade effects and exchange rates effects later on.

4.1 Benchmark: Complete Market and Symmetric Preferences

Preference symmetry refers that consumers of both countries maximize the same discounted life time utility under uncertainty:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) = E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left[C_t^{\nu}(1-N_t)^{1-\nu}\right]^{1-\sigma}}{1-\sigma},$$

where C is consumption, N is labor input and time is normalized to 1 so that $1-N$ stands for leisure. The utility index is CRRA if we treat the whole thing in the square brackets as a composite from the consumption of commodities and leisure, the reason that the composite takes a Cobb-Douglas (CD) form is to match the fact that the share for leisure is relatively stable in data, although real wage increases pretty much. We assume the coefficient of relative risk aversion $\sigma \geq 0$. Consumers of foreign country have the same preference, all foreign variables are denoted with an asterisk:

$$U(C_t^*, N_t^*) = \frac{\left[C_t^{*\nu}(1-N_t^*)^{1-\nu}\right]^{1-\sigma}}{1-\sigma}.$$

Two countries produce a different good each using their respective capital and labor, the technology is homogenous of degree one:

$$Y_t = F(K_t, N_t) = yd_t + EX_t = Z_t K_t^{\alpha} N_t^{1-\alpha},$$

$$Y_t^* = F(K_t^*, N_t^*) = yd_t^* + EX_t^* = Z_t^* K_t^{*\alpha} N_t^{*1-\alpha}.$$

where yd is for domestic usage and EX is export. Y and Y^* are aggregate output in both countries but are not final goods. The final goods are produced via CES aggregator:

$$G(yd_t, EX_t^*) = \left[\omega yd_t^{1-\varphi} + (1-\omega) EX_t^{*1-\varphi}\right]^{\frac{1}{1-\varphi}},$$

$$G\left(yd_t^*, EX_t\right) = \left[\omega^* yd_t^{*1-\varphi^*} + \left(1-\omega^*\right) EX_t^{1-\varphi^*}\right]^{\frac{1}{1-\varphi^*}}$$

where the elasticity of substitution between home and foreign product is $1/\varphi$. $0 < \omega < 1$ is sometimes called home bias, consumers are indifferent between home and foreign commodities when $\omega = 0.5$. Later in the extended model we want to use Terms of Trade shocks, to be able to do that, we need at least two prices: export price and import price. Therefore on the production side we let each country produce one country-specific good, and the final commodity which can be used to consume and invest is aggregated using both the home product and foreign product. National identity now becomes:

$$C_t + I_t + g_t + EX_t - EX_t^* = G\left(yd_t, EX_t^*\right),$$

$$C_t^* + I_t^* + g_t^* + EX_t^* - EX_t = G\left(yd_t^*, EX\right).$$

The law of motion for capital is specified as follows:

$$K_{t+1} = I_t + (1-\delta) K_t - \frac{\phi}{2}\left(K_{t+1} - K_t\right)^2,$$

$$K_{t+1}^* = I_t^* + (1-\delta^*) K_t^* - \frac{\phi}{2}\left(K_{t+1}^* - K_t^*\right)^2.$$

where $\frac{\phi}{2}\left(K_{t+1} - K_t\right)^2$ accounts for adjustment costs and δ is the depreciation rate. It turns out that adding adjustment cost effectively reduces the excessive volatility of capital. Lastly, $\mathbf{Z} = \left(Z, Z^*\right)^T$ is the source of stochasticity and we assume it follows a Markov process with a transition probability

$$Q\left(z', z\right) = Prob\left\{Z_{t+1} \left| Z_t, Z_{t-1}, \cdots, Z_{t-k}\right.\right\} = Prob\left\{Z_{t+1} \leq z' \left| Z_t = z\right.\right\}.$$

The simplest Markov is AR(1), thus we may as well set

$$\begin{pmatrix} Z_{t+1} \\ Z_{t+1}^* \end{pmatrix} = \begin{pmatrix} \rho_1 & \rho_{21} \\ \rho_{12} & \rho_2 \end{pmatrix} \begin{pmatrix} Z_t \\ Z_t^* \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ \varepsilon_t^* \end{pmatrix},$$

$$\boldsymbol{\varepsilon} \sim N\left[0, V\right], \quad V = \begin{pmatrix} \kappa^2 & \tau \\ \tau & \kappa^{*2} \end{pmatrix}.$$

ρ_1 and ρ_2 are coefficients of persistence, ρ_{21} and ρ_{12} are cross persistence or spillovers. V is the covariance matrix of the disturbance.

The above equations delineate the decentralized economy, since there is not any externalities or monopoly, we solve this problem via a central planer's problem. Supposedly the planer maximizes the following weighted expected utility:

$$\pi E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t\right) + \left(1-\pi\right) E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t^*, N_t^*\right)$$

where π is the weight of home country. The planner is also subject to a budget constraint that all expenditure should not exceed final world yields:

$$G\left(yd_t, EX_t^*\right) - C_t - I_t - g_t + G\left(yd_t^*, EX_t\right) - C_t^* - I_t^* - g_t^* \geq 0.$$

4.1.1 Solution

The Lagrangian of the central planner's problem is:

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \pi U\left(C_t, N_t\right) + \left(1-\pi\right) U\left(C_t^*, N_t^*\right) \\ +m_t \left[\begin{array}{c} \left(\omega y d_t^{1-\varphi} + \left(1-\omega\right) EX_t^{*1-\varphi}\right)^{\frac{1}{1-\varphi}} - C_t - I_t - g_t \\ + \left(\omega^* y d_t^{*1-\varphi^*} + \left(1-\omega^*\right) EX_t^{1-\varphi^*}\right)^{\frac{1}{1-\varphi^*}} - C_t^* - I_t^* - g_t^* \end{array} \right] \\ +\lambda_t \left[Z_t K_t^{\alpha} N_t^{1-\alpha} - yd_t - EX_t + Z_t^* K_t^{*\alpha} N_t^{*1-\alpha} - yd_t^* - EX_t^*\right] \end{array} \right\}$$

where m_t and λ_t are multipliers. The first order conditions (FOC) with respect to consumption are:

$$m_t = \pi \nu \left(C_t^{\nu} \left(1-N_t\right)^{1-\nu}\right)^{1-\sigma} \frac{1}{C_t},$$

$$m_t = \left(1-\pi\right) \nu \left(C_t^{*\nu} \left(1-N_t^*\right)^{1-\nu}\right)^{1-\sigma} \frac{1}{C_t^*}.$$

FOC with respect to labor:

$$\lambda_t \left(1-\alpha\right) Z_t \left(\frac{K_t}{N_t}\right)^{\alpha} = \pi \left(1-\nu\right) \left(C_t^{\nu} \left(1-N_t\right)^{1-\nu}\right)^{1-\sigma} \frac{1}{1-N_t},$$

$$\lambda_t \left(1-\alpha^*\right) Z_t^* \left(\frac{K_t^*}{N_t^*}\right)^{\alpha^*} = \left(1-\pi\right) \left(1-\nu\right) \left(C_t^{*\nu} \left(1-N_t^*\right)^{1-\nu}\right)^{1-\sigma} \frac{1}{1-N_t^*}.$$

The above two equations are labor supply functions. First order conditions with respect to capital stock are:

$$\nu \frac{\left(C_t^{\nu}\left(1-N_t\right)^{1-\nu}\right)^{1-\sigma}}{C_t}\left[1+\phi\left(K_{t+1}-K_t\right)\right]=\beta E_t\left(C_{t+1}^{\nu}\left(1-N_{t+1}\right)^{1-\nu}\right)^{1-\sigma}$$
$$\cdot\left\{\nu \frac{1}{C_{t+1}}\left[(1-\delta)+\phi\left(K_{t+2}-K_{t+1}\right)\right]+(1-\nu) \frac{\alpha}{1-\alpha} \frac{N_{t+1}}{1-N_{t+1}} \frac{1}{K_{t+1}}\right\}$$

$$\nu \frac{\left(C_t^{*\nu}\left(1-N_t^*\right)^{1-\nu}\right)^{1-\sigma}}{C_t^*}\left[1+\phi\left(K_{t+1}^*-K_t^*\right)\right]=\beta E_t\left(C_{t+1}^{*\nu}\left(1-N_{t+1}^*\right)^{1-\nu}\right)^{1-\sigma}$$
$$\cdot\left\{\nu \frac{1}{C_{t+1}^*}\left[\left(1-\delta^*\right)+\phi\left(K_{t+2}^*-K_{t+1}^*\right)\right]+(1-\nu) \frac{\alpha^*}{1-\alpha^*} \frac{N_{t+1}^*}{1-N_{t+1}^*} \frac{1}{K_{t+1}^*}\right\}$$

These two equations are consumption Euler equations, the marginal utility of increasing current consumption should equal the discounted marginal utility next period brought about by the weighted alternatively augmented capital taking into account adjustment cost, and the intertemporal substitution between consumption and labor. The first order conditions of domestic usage yd are:

$$\frac{C_t}{1-N_t}\left(\frac{N_t}{K_t}\right)^{\alpha}=\frac{\nu}{1-\nu}(1-\alpha) \omega Z_t\left(\omega y d_t^{1-\varphi}+(1-\omega) E X_t^{*1-\varphi}\right)^{\frac{\varphi}{1-\varphi}} y d_t^{-\varphi},$$

$$\frac{C_t^*}{1-N_t^*}\left(\frac{N_t^*}{K_t^*}\right)^{\alpha}=\frac{\nu}{1-\nu}\left(1-\alpha^*\right) \omega^* Z_t^*\left(\omega^* y d_t^{*1-\varphi^*}+\left(1-\omega^*\right) E X_t^{1-\varphi^*}\right)^{\frac{\varphi^*}{1-\varphi^*}} y d_t^{*-\varphi^*}.$$

And the FOC with respect to export are:

$$\frac{C_t}{1-N_t}\left(\frac{N_t}{K_t}\right)^{\alpha}=\frac{\nu}{1-\nu}(1-\alpha)(1-\omega) Z_t\left(\omega y d_t^{1-\varphi}+(1-\omega) E X_t^{*1-\varphi}\right)^{\frac{\varphi}{1-\varphi}} E X_t^{*-\varphi},$$

$$\frac{C_t^*}{1-N_t^*}\left(\frac{N_t^*}{K_t^*}\right)^{\alpha}=\frac{\nu}{1-\nu}\left(1-\alpha^*\right)\left(1-\omega^*\right) Z_t^*\left(\omega^* y d_t^{*1-\varphi^*}+\left(1-\omega^*\right) E X_t^{1-\varphi^*}\right)^{\frac{\varphi^*}{1-\varphi^*}} E X_t^{-\varphi^*}.$$

These four equations restrict the dynamics of domestic usage and export, the intuition is that the trade-off among capital, labor and consumption should satisfy the marginal product of domestic usage and export respectively.

Based on the specialization of both countries, if we denote the prices of domestic and foreign products as P and P^*, then the terms of trade $TOT=P^*/P$ becomes:

$$TOT_t=\frac{P^*}{P}=\frac{\partial G_t / \partial E X_t^*}{\partial G_t / \partial y d_t}=\frac{1-\omega}{\omega}\left(\frac{y d_t}{E X_t^*}\right)^{\varphi}.$$

In our context home country's export is EX while import from foreign country is EX^*, thus the trade balance of home country becomes $EX - TOT \cdot EX^*$, and the net export ratio is therefore:

$$NX_t = \frac{EX_t - TOT_t \cdot EX_t^*}{F(K_t, N_t)}.$$

This complete the description of this dynamic system, the solution is equivalent to the decentralized competitive equilibrium.

4.1.2 Steady State and Calibration

In steady state adjustment cost equals zero, so firstly we have:

$$\delta = \frac{Y}{K}\left(1 - \frac{C}{Y}\right).$$

Based on the data from GTA, the average value of the consumption output ratio (C/Y) during 1978-2009 is 0.47, the average capital output ratio (K/Y) is 3.26, which leads us a depreciation rate of 16.3%.[17] So we calibrate China's depreciation rate as $\delta = 15\%$, and the foreign country's depreciation rate as $\delta^* = 10\%$. The annual interest is 3% in data so the discount factor $\beta = 0.97$. The first order conditions of the profit maximization indicate that:

$$\frac{\alpha Y}{K} = r + \delta,$$

$$\frac{(1-\alpha)Y}{N} = w,$$

where $r + \delta$ is the real reward for capital and w is the steady state wage rate. Denoting leisure by $L = 1 - N$, from $U_L/U_C = w$ we have:

$$w = \frac{1-\nu}{\nu}\frac{C}{1-N}.$$

If the real reward for capital is 19.3% and capital output ratio is 3.26, then $\alpha = 0.63$. When we calibrate for the foreign country, $\alpha^* = 0.36$ which conforms to the evidence in Backus *et al.* (1992). Combining the above two equations yields:

$$1 - \alpha = \frac{1-\nu}{\nu}\frac{C}{Y}\frac{N}{1-N}.$$

[17] This depreciation rate seems a little bit high but China maintains a relatively high growth rate of real investment which is 15.8% on average.

If people devote 1/3 of the time to work, then $N/(1-N) = 1/2$, so $\nu = 0.39$. Curvature parameter σ determines the degree of relative risk aversion and the intertemporal substitutability. According to the neoclassical Euler equation:

$$\frac{\dot{C}}{C} = \frac{r-\rho}{\sigma}$$

The left hand side is the growth rate of consumption, r is the real interest rate and $\rho = (1-\beta)/\beta$ is the subjective discount parameter. According to the IFS data the annual growth rate of consumption in China is 3.1%. If the reward for capital after tax is 10%, the upper limit of the CRRA coefficient σ is 3.2 so as to guarantee ρ is positive. That is, σ is disinclined to take a high value, so we calibrate $\sigma = 3$. We use the same coefficient for the foreign country, because we do not believe people's attitude towards risk has systematic difference between China and the US.

In order to reflect that China produces to export, we lower home country's ω to be 0.4 but accentuate foreign country's ω^* to be 0.6, the purpose is to emphasize that China exports a larger portion of its own product. The estimation of elasticity between home and foreign countries' product $1/\varphi$ varies a lot from 0.2 to 3.5, and it can even be as high as 12 or 13 (Ruhl, 2008), but seldom exceeds 20. This paper uses 0.67 as the calibration of φ, the reason for this value is to make the import/GDP ratio to lie between 0.2 and 0.3 which is what China's import takes after becoming a member of WTO, the highest value approaches to 30% in 2005. Likewise, we calibrate φ^* in the way so that foreign country's import takes 15% which is consistent with the US data. The parameter of adjustment cost ϕ is set to be 0.028 for both countries, this calibration directly follows from Mendoza (1991, 1995).

To calibrate the spillovers we firstly estimate the Sino-US total factor productivity by Solow residuals:[18]

$$\ln Z_t = \ln Y_t - \alpha \ln K_t - (1-\alpha) \ln N_t.$$

Lacking of working hours from China creates once again a problem, in order to make the estimation comparable under the same standard, we use annual Labor Force for N as well when it comes to the estimation of the US productivity.[19] Labor Force is from BvD data base,[20] America's capital stock is from the US Bureau of Economic Analysis. China's capital

[18] It seems to be biased just to estimate Sino-US productivity instead of the Sino-World productivity. The difficulty again lies in the data. There is not capital stock of the rest of the world excluding China, only a few countries offer statistics of working hours though mostly reflect hours of physical labor only.

[19] Backus et al. (1992, 1994) are not absent from this also.

[20] BvD also offers labor force of china which is of course different from China's domestic data, such a

stock is calculated as aforementioned, the rest of the data used are from IMF. We normalize both countries' productivity to have mean 1 so that it directly measures deviation percentage from the steady state after a shock. The estimation of the coefficient is:

$$\rho = \begin{bmatrix} .962\,(.065) & .045\,(.065) \\ .006\,(.017) & .861\,(.121) \end{bmatrix}, \quad V = Var\,(\boldsymbol{\varepsilon}) = \begin{bmatrix} .0323^2 & 4.9 \times 10^{-5} \\ 4.9 \times 10^{-5} & .0066^2 \end{bmatrix},$$

where standard errors are in brackets. The persistency of China's TFP is 0.962 which is higher than that of the US' but is also more volatile. Meanwhile the spillover effect of the US technology to China is 0.045, which is approximately seven times more than China's spillover to the US. Two eigenvalues of matrix ρ are 0.9646 and 0.8548. From the covariance matrix V we know that the correlation between two countries' innovation is 0.231. We use result (27) to calibrate the shock process. Lastly, China's real GDP measured by purchasing power parity is roughly about 0.39 of the US GDP during 1980 to 2009, so π is calibrated as 0.35. The calibration of the benchmark model is thus summed up in Table 3:

Table 3 Calibration of the Benchmark Model Parameters

Preference	$\beta = 0.97$	$\nu = 0.34$	$\sigma = 3$	$\pi = 0.35$	
	$\beta^* = 0.97$	$\nu^* = 0.34$	$\sigma^* = 3$		
Technology	$\delta = 0.15$	$\alpha = 0.63$	$\phi = .028$	$\varphi = 0.67$	$\omega = 0.4$
	$\delta^* = 0.1$	$\alpha^* = 0.36$	$\phi^* = .028$	$\varphi^* = 0.67$	$\omega^* = 0.6$
Shocks	$\rho = \begin{bmatrix} 0.962 & 0.045 \\ 0.006 & 0.861 \end{bmatrix}$		$\kappa = .0323$	$\kappa^* = .0066$	$corr\,(\varepsilon, \varepsilon^*) = .231$

4.1.3 Results

We solve the model by second order Taylor approximation of the nonlinear system of equations around its steady state,[21] then generate the impulse response function by sampling from the Markov process that the technological shock follows. The number of replication is 1000. Please find in Table 4 a summary of the results.

Table 4 Solution of the Benchmark Model

difference however is not obvious after taking logs. For the sake of conformity we also use the data from BvD for China's labor force in this context.

[21] We expand the approximation up to the second order so as to avoid the certainty equivalence.

Variables	Standard Deviation percentage	relative to Y	Corr with Y	Auto Corr, $k=$ 1	2	3	4	5
Output	4.44	1.00	1.00	0.78	0.46	0.19	-0.01	-0.13
Labor	1.05	0.24	0.81	0.83	0.52	0.23	-0.01	-0.16
Consumption	0.31	0.07	0.97	0.75	0.47	0.24	0.04	-0.08
Investment	3.64	0.82	0.50	0.13	-0.01	-0.09	-0.13	-0.06
NX/GDP	4.55	...	0.79	0.72	0.43	0.16	-0.04	-0.16
$Corr(Y,Y^*)$ = -0.85, $Corr(C,C^*)$ = 0.38, $Corr(I,I^*)$ = -0.96, $Corr(NX,NX^*)$ = 0.44								

The standard deviation of the output in the model is 4.44%, which is 15% lower than the lowest measure in Table 1. For the other variables the gaps between theoretical volatility and real data are large. In reality employment is about 1/3 of the output volatility, while in theory it is 0.24 which is acceptable. But the volatility of consumption is rather low in the model both in percentage and relative to output. We have known from the literature that risking sharing of the complete market is only responsible for a tiny portion of this. Investment volatility is underestimated by 50%, and we cannot attribute this to the fact that capital takes a higher portion in the model. Actually even if we lower home country's α to 0.36 which is the same to the foreign country, lower the depreciation rate to foreign country's level (0.1), and further more increase both countries' adjustment cost ϕ, investment volatility is still smaller than reality. Nonetheless, we find that investment fluctuation will be higher than output fluctuation if both countries are producing a homogenous good, i.e., abandoning the CES aggregation. Thus low volatility of investment is brought about by the request of two commodities. Lastly, the volatility of net export is way too high, it is even higher than the volatility of output.

Consumption is pro-cyclical, but its correlation with output is higher than reality. Net export is not counter-cyclical which is contrary to the data, its correlation with output is positive and pretty high. The correlation between employment and output is overestimated, but the correlation between investment and output is underestimated. Most endogenous variables in the model feature a strong persistency, which is the main source of difficulty when we compare general equilibrium models with time series models. Of course we do not expect that the benchmark model accomplishes anything in explaining the business cycle properties. Later on we will extend the model, but to answer why the extended model is useful, it might serve a most natural reply to compare how well each model explains the data.

The benchmark model however, successfully predicts the positive co-movement of consumption and negative co-movement of investment. The correlation between home and foreign country's consumption is 0.38 in the model, which is quite close to the consumption co-movement between China and the US or the UK in Table 2. On the other hand, output co-movement is negative and relatively high, even though the co-movement of investment is negative which conforms to the stylized facts, the correlation coefficient is way too high in absolute value. Similar scenarios happen to net export as well, the positive co-movement reconciles with data, but the coefficient is higher in absolute value. These facts reveal the transmission mechanism of the benchmark model. The most fundamental source of fluctuation transmission is factors tend to move to the most productive place, it is not the reliance on trade that makes the difference. Factor productivity can only be marginally higher in one place after the shock, capital gathers in that place under free mobility and thus increases the yield there but the output of another country decreases due to the out flow of capital, which ends up with a negative co-movement of output and a negative co-movement of investment. Then the shocked place benefits the rest of the world through spillovers and international trade under complete markets. Everybody's consumption increases even though investment flows away from some countries. That makes a positive international correlation of net export, and a positive co-movement of consumption.

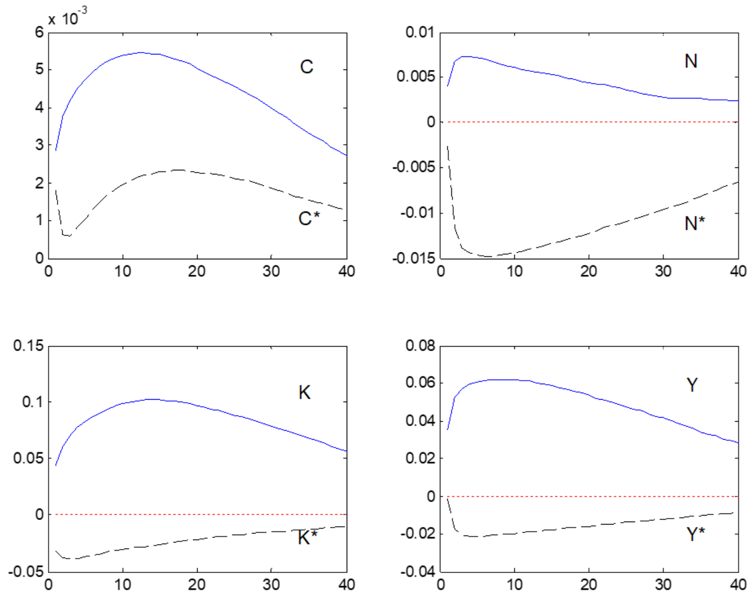

Figure 3a: IRF of Consumption, Employment, Capital Stock and Output

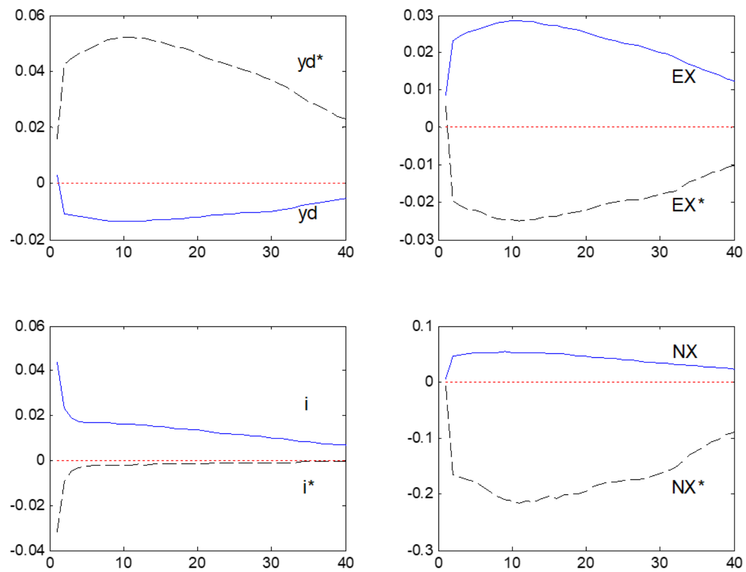

Figure 3b: IRF of Domestic Usage, Export, Investment and Net Export

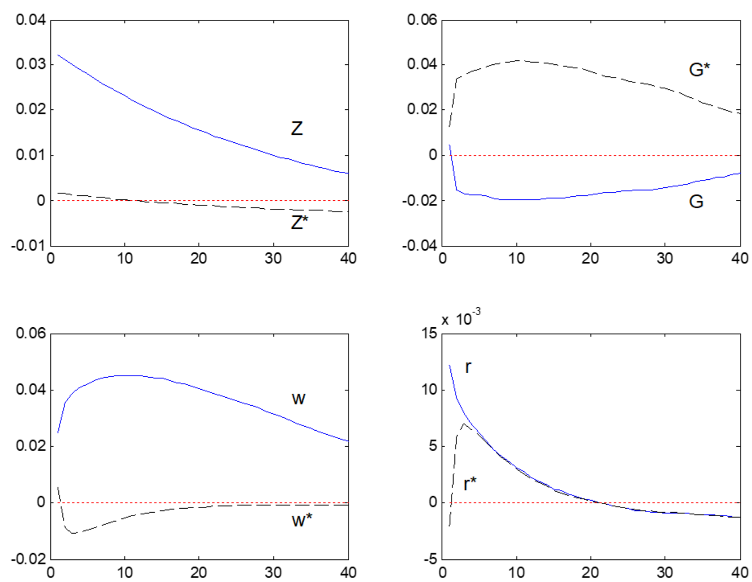

Figure 3c: IRF of Productivity, Composite Good, Wage and Interest Rate

Figures 3a-c plot the impulse response functions in the next forty periods after a one unit of standard deviation shock on the technology from its steady state, home country is in the solid line and foreign country is the dashed line. The productivity in the home country rises after a technological shock and then gradually decreases, foreign productivity also increases and converges to its steady state because of the spill over. Home country's consumption, labor, capital stock and output all feature a hump shape increase and a steady convergence, while investment increases dramatically first, then encounters a transient precipitous fall, and converges slowly. It is interesting that after the technological shock domestic usage decreases but export increases. In other words, because of consumption smoothing in the foreign country, raises in the capital and labor in the home country are devoted to the export, domestic absorption is actually decreasing. Since production shifts to home country which is marginally more productive now, foreign country's employment, investment, capital and output are all decreasing after the shock and then converging to the steady states. Foreign country loses some of her capital so the aggregation relies more on her domestic product, as a result domestic usage raises but export declines, which is just the reverse of the home country. Foreign country's consumption is nonetheless elevated, it is just a little lower than home country's level.

The silver side of the benchmark model is that it captures the negative co-movement of investment which is the distinct feature of the Chinese economy. On the other side, the model fits the data poorly in that the relative fluctuations of consumption and investment tend to be low, the co-movement of output has a wrong sign and the coefficient is large in absolute value. Mild attuning of the parameters can only lead to very weak refinements,[22] some substantial improvements are in order.

4.2 Extension: Asymmetric Preference, Incomplete Markets and Terms of Trade Shocks

The idea of general equilibrium paradigm is to attribute everything to tastes and technology, but somehow tastes were handled in a reckless way in the literature. Complete market is at least partially responsible for the anomalies of international co-movements. As we mentioned in the last subsection, the purpose of insisting on having two commodities is to give possibility for the discussion of the terms of trade which is the miniature of price changes of oil, future of bulk commodities, and resource goods, etc.

4.2.1 Asymmetric Preference

The extended model changes the home country's preference to GHH preference:

$$U(C_t, N_t) = \frac{\left[C_t - \eta N_t^{\mu}\right]^{1-\sigma}}{1-\sigma}$$

where $\mu > 1$ and $\eta > 0$, μ determines the intertemporal elasticity of substitution in labor

[22] For instance, the experiment we run includes increasing the degree of risk aversion σ, raising the adjustment cost ϕ, slightly adjusting ω and φ in the CES aggregation.

supply which is $1/(\mu-1)$, and η determines the working hours in steady state. The exterior is still structured as Constant Relative Risk Aversion (CRRA). As for the foreign country, it remains a Cobb-Douglas inside as in the benchmark. So home country is Greenwood-Hercowitz-Huffman (GHH) inside and foreign country is CD inside, that is what we mean by asymmetry.

The micro evidence that justifies GHH is that Chinese households feature a relatively higher substitution between consumption and nonmarket labor input than developed countries. In the data, the ratio that consumption takes in GDP stabilizes between 62%-69% in the US, the average value from 1952 is 65%. But in China this ratio is steadily dropping no matter whether the data source is IMF or GTA. The consumption output ratio was 50% in 1980s, and 40% in 1990s, then it further declined to 35%-38% after 2005. Since Cobb-Douglas requires consumption takes a stable portion, America meets this restriction but China does not. Consumption takes a declining portion meanwhile welfare is increasing, it is surmised that there must be substitutions between consumption and leisure. Although 40 hours a week is generally a nominal binding constraint for market labor input, but consumption has various ways to substitute non-market labor activities which consequently vary the quality of market labor input. Celebrated examples are cooking for yourself or go to the restaurants, take care of the children by yourself (or grandparents) or send to a nursery, clean the house yourself or hire a cleaner, etc. Under fixed time endowment there will be a trade-off between non-market labor input and the quality of labor market input, which connects consumption and effective labor. GHH utility is pertinent if the declining consumption ratio in China can be compensated by less effective labor supplied (thus more non-market labor devotion).

To calibrate the new parameters introduced we follow Greenwood *et al.* (1998) who set $\mu = 1.58$, it means the elasticity of labor is 1.72. Given this value we set η to be 3.24 so that labor takes 1/3 of the time endowment. Evidence from Microeconomics suggests that elasticity of intertemporal substitution of labor ranges 0.2 to 1.7 which means that μ could acceptably take value from 1.58 to 6. So the objective function of the central planner becomes to:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \pi \frac{\left[C_t - \eta N_t^{\mu}\right]^{1-\sigma}}{1-\sigma} + (1-\pi) \frac{\left[C_t^{*\nu} \left(1-N_t^*\right)^{1-\nu}\right]^{1-\sigma}}{1-\sigma} \right\}.$$

4.2.2 Bond Economy

To restrict risk sharing we confine each individual to the trade of country specific commodities and a one period non-state contingent bond. Suppose B_t is home country's period t purchase of the riskless bond that guarantees one unit of consumption in terms of foreign goods at date $t+1$ no matter which state will be realized, the price is q_t and obviously $q_t = (1+r_t^*)^{-1}$, where r_t^* is the foreign interest rate. The budget constraint of the home country thus becomes:

$$C_t + I_t + g_t + q_t B_{t+1} \leqq G(yd_t, EX_t^*) + B_t$$

Likewise, the budget constraint of foreign country becomes:

$$C_t^* + I_t^* + g_t^* + q_t B_{t+1}^* \leqq G(yd_t^*, EX_t) + B_t^*$$

Denoting by s_t and s_t^* the multiplier of the above two constraints, then the transversality conditions of the incomplete market are:

$$\lim_{t\to\infty} \beta^t s_t B_{t+1} = 0,$$

$$\lim_{t\to\infty} \beta^t s_t^* B_{t+1}^* = 0.$$

Net supply of the bonds sums up to zero, so the market cleaning condition is:

$$B_{t+1} + B_{t+1}^* = 0.$$

The implication is that only one asset is independent, so we keep home country's bond and eliminate the foreign country's budget constraint in the solution process. The first order condition with respect to bond becomes:

$$q_t s_t = \beta E_t s_{t+1}.$$

With the introduction of bond, state vector of the system enlarges to (K_t, K_t^*, B_t), co-state vector enlarges to (m_t, λ_t, s_t), i.e., we can still analyze the reallocation brought about by incomplete markets using the method of a complete market, but with an enlarged state space.

4.2.3 Terms of Trade Shocks

In this subsection we will expand the stochastic space by adding terms of trade (TOT) shocks under the convenience of two goods framework. Equation (23) is the terms of trade under competitive markets, now we make it an AR(1) process without a drift:

$$TOT_t = aTOT_{t-1} + u_t$$

$$TOT_t^* = a^* TOT_{t-1}^* + u_t^*$$

We calibrate the auto regressive parameters according to Sino-US terms of trade which are depicted in Figure 4.[23]

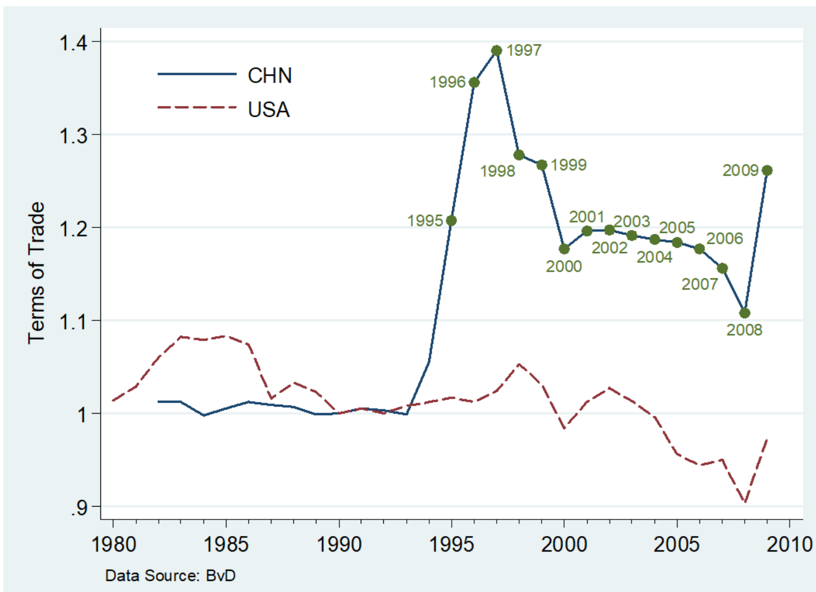

Figure 4: Sino-US Terms of Trade

The US terms of trade is relatively stationary but gets slightly worse from 2004. For China, this relative price drastically rises to around 1.2 in 1995 and stays pretty high in the following years. To account for the structural change on China's side we add a dummy variable in our effort of estimating the covariance matrix of error terms in the above two equations. The estimation results are $a = 0.9988$, $a^* = 0.9983$.

$$Var\left(\mathbf{u}\right) = \begin{bmatrix} .00339 & .00059 \\ .00059 & .00073 \end{bmatrix}.$$

[23] The terms of trade data are from BvD Macro data base.

It turns out that the AR(1) coefficients a and a^* are way too persistent, both of them are almost one, which is inclined to lead the endogenous variables in the system to a new steady states rather than coming back to the original ones. So we lower the coefficients to be $a = a^* = 0.95$, in calibration, but we still use the above variance-covariance as the estimate of the covariance matrix of the error terms. As a robustness check we range a and a^* from 0.9 to 0.98, the differences are on the second digit of the decimal, it won't exert a first order impact on the results.

Introducing TOT shocks enlarges the dimension of disturbances, we do not assume orthogonality of different shock processes. Instead, we estimate correlation of four shock processes from the data:

$$Var\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathbf{u} \end{pmatrix} = \begin{bmatrix} .001080 & .000046 & .000029 & .000130 \\ .000046 & .000030 & -.000079 & .000011 \\ .000029 & -.000079 & .003393 & .000585 \\ .000130 & .000011 & .000585 & .000729 \end{bmatrix}$$

4.2.4 Results

The extended model is solved under the benevolent central planner problem. All the parameter values are preserved from the benchmark to the extended model, except for the newly introduced parameters. We do not want re-parameterization to interfere with the transmission mechanism.[24] Table 5 reports the extended model results by sequentially as well as accumulatively introducing asymmetric preference, non-state contingent bonds and the terms of trade shocks. Namely in the second to the last column, "+Incomp Mkt" means adding incomplete market to the asymmetric preference model, the last column "+TOT Shocks" means adding TOT shocks to the model that already has asymmetric preferences and incomplete markets. For the sake of convenience data and benchmark results are also listed in Table 5. For the real world data, the panels of relative volatility and correlation with Y are summarized from Table 1, the panel of co-movements is from Table 2 where the contemporaneous correlations of various countries' (we mostly refer to the European and the American) aggregate variables with the same Chinese aggregates. Most entries in the "Data" column feature a range instead of a specific number because there are measures from four filters and there is a selection of reference countries when it comes to co-movements.

[24] We thank the referee for pointing this out.

We believe forming a range is more pertinent than just relying on the result of one particular filter. These ranges are not constructed strictly by the extreme points of Table 1 and 2, but they are subsets of the upper and lower bounds.

Table 5 Data, Benchmark and the Extensions

	Data	Benchmark	Asym Prefe.	Bond Econ	TOT Shock
Relative Volitility					
Real Output	1.00	1.00	1.00	1.00	1.00
Labor	0.27-0.33	0.24	0.27	0.25	0.27
Consumption	0.49-1.02	0.07	0.41	0.51	0.41
Investment	1.02-1.70	0.82	0.67	0.60	0.66
Net Export	0.84-1.03	4.55	0.04	0.96	0.93
Correlation with Output					
Real Output	1.00	1.00	1.00	1.00	1.00
Labor	0.14-0.45	0.81	0.15	0.19	0.19
Consumption	0.56-0.74	0.97	0.97	0.98	0.97
Investment	0.71-0.86	0.50	0.72	0.73	0.69
Net Export	-0.17–0.31	0.79	-0.75	0.50	-0.15
Comovement					
Real Output	0.17-0.45	-0.85	0.26	0.32	0.28
Labor	...	-0.77	0.29	0.34	0.35
Consumption	0.18-0.38	0.38	0.25	0.25	0.42
Investment	-0.22–0.73	-0.96	-0.94	-0.62	-0.69
Net Export	0.05-0.30	0.44	0.29	0.49	0.31

Note: (1) # means the standard deviation of net export GDP ratio, not the relative volatility denominated by output so that it is comparable with the corresponding data in Table 1. (2) Co-Movements of employment is left blank due to the lack of internationally comparable data on working hours.

The volatility of employment does not change much with the introduction of GHH preference. If we believe 0.27-0.33 is a reasonable range of the relative fluctuation of employment, the extended model with only asymmetric preference predicts it to be 0.27 which does not deviate much from 0.24 in the bench mark model. The relative volatility of employment is still robust even we further introduce incomplete market and terms of trade shocks. The benchmark model highly underestimates the consumption volatility, but the extended model fixes this problem and clearly the improvement is mostly accredited to asymmetric preference. Though it makes sense that incomplete market further increases consumption volatility a bit because the ability of risk sharing is under restriction, the improvement is a menial 0.1 comparing with a huge increase from benchmark's 0.07 to 0.41 under the sole asymmetric preference case. The extended model does not make better the investment volatility, on the contrary it makes it even smaller. But we have already known from the analysis of the benchmark model that the low investment volatility is brought about by CES aggregation, evidence from the large amount of single good models shows that mere mild adjustment of parameters will produce reasonable investment volatility. The overestimation of the relative fluctuation of net export is also rectified to a reasonable range suggested by data, the volatility of net export is low under mere asymmetric preference, but incomplete market and terms of trade shocks bring the ultimate refinements.

With the introduction of asymmetric preference, the correlation between output and employment greatly reduces. This makes sense because if we look at the logarithmic first order condition with respect to labor under asymmetric preference, it is more nonlinear than the logarithmic FOC with respect to labor. One side effect brought about by GHH preference is that consumption and output are highly correlated. This is a typical phenomenon of GHH preference which results in an effect quite close to log utility in a complete market under which international correlation of consumption is one (see Baxter, 1995). The fact that consumption is procyclical does not change much under all extended models, but the procyclicality of investment enhances.

The benchmark model underestimates the correlation between investment and output, but the extended model improves the prediction on this in all three cases. To see that clearly, let's suppose there is a positive innovation in the foreign country, then output and consumption should increase at least in the following several periods. Home country also benefits from this positive shock by having a higher welfare. In the benchmark model the utility is Cobb-Douglas inside which requires the shares of consumption and leisure held fixed. This gives rise to a force of consuming more leisure among other things, so it reduces the hours worked. Then on the production side is has a force to curb the investment a bit in conjunction with the reduction of the working hours. This force is not dominant of course, it won't make the home country worse off because the initial positive innovation will make the home investment jump to a high level. This force at best will make the investment gradually decline over the periods. Since both output and investment increase in the home country, they are positively correlated. But because of this curbing force on the investment, the correlation is a bit lower in the benchmark. Nonetheless in the extended model, there is not such a force to drag down the working hours and therefore the investment.

Asymmetric preference changes net exports from being pro-cyclical to countercyclical as it should be, but the absolute value of the correlation coefficient is still a bit higher. The reason for that is when the home country suddenly becomes better off (richer), it always has a desire to import more so as to consume more, and this is the income effect. The counter force is the need to produce more to export, and this originates from the wealth effect based on the former discussion. Therefore, when the net exports are counter cyclical just like in the asymmetric preference case, our conjecture is that the income effect is greater than the wealth effect. When the net exports are procyclical like in the benchmark model and incomplete market case, it is the wealth effect that dominates. And now it's clear why under the terms of trade shocks the degree of counter cyclicality of the net exports is smaller. The introduction of TOT shocks reinforces the substitution effect which works against the income effect. In our experiment, this substitution effect does not complete balance out the income effect, but together with the wealth effect, it offsets a majority of the income effect so that the correlation of the net export and output reduces (in absolute value) from -0.75 under asymmetric preference to -0.15 under TOT shocks. The interesting thing is that incomplete markets drag the net exports back to pro-cyclical. Intuitively, because the incomplete markets confine the mobility of international capital flow through non-state contingent bonds imposed on the financial markets, consumption smoothing relies more on international trade. The more risk averse the agent is (in the extended model $\sigma = 3$), the more eagerly the foreign consumers want to shun from consumption risk. In response to a positive technological shock with a higher national yield, foreign consumers import more to smooth consumption so that home country's output and net export are positively correlated. The logic is reducing the coefficient of relative risk aversion will help overturn the counter-fact that net export is pro-cyclical. In our experiment when we calibrate σ to be 2 in a model with asymmetric preference and incomplete market, the correlation of net export and output immediately becomes -0.007. Alternatively, we are capable of attaining the same effect by introducing terms of trade shocks without changing the CRRA coefficient. Even though σ is still fixed at 3, terms of trade shocks bring down the correlation between net export and GDP from 0.50 to -0.15 which is quite close to the upper limit of the data range. Although the error terms of TOT and technology are by assumption correlated and we estimate their interaction from the real data, terms of trade shocks seem only to exert an obvious effect on net export, changes of the other endogenous variables are not substantial.

As for the co-movements, labor inputs are internationally negatively correlated under CD preference under the benchmark model. Intuitively, when the foreign country benefits from a positive technological shock, capital flows in, employment rises accordingly. The higher output benefits both countries so home country's consumption also increases. Since consumption and leisure take a fixed share in CD preference, consumption of leisure also increases when consumption of goods increases, namely working hour declines. When both the labor input and capital decreases in the home country, the co-movement of output is also negatively correlated. Nonetheless, GHH preference which features zero leisure elasticity with respect to income alters the mechanism in a way that home country's labor/leisure does not respond much to a foreign positive shock, thus labor supply does not go down. Foreign country elevates its consumption by importing more from the home country and thus aggregating more final consumption goods. An export surge in the home country necessitates an increase in the labor input, so employments are positively correlated across countries. Since the mutual increases of the employment are induced by exports only, the correlation is lower in absolute value than the CD preference scenario. Though comparable data on working hours among different countries are not available, we surmise that a transnational correlation between 0.2-0.3 should not deviate far from the true situation. Since both countries increase their labor inputs, national yields also raise, which results in a positive co-movement of output. It was particularly pointed out in section 3 that negative transnational correlation of investment is the distinctive feature that makes China different from other economies. The extended model (as well as the benchmark) fits this stylized fact, though benchmark and asymmetric preference model overestimate the international correlation of investment (in absolute value), incomplete financial markets which limit international capital flow reduce this estimate to a safe ground. It is rather interesting to explore how the high correlation between consumption and output entwines the co-movements of investment. When the foreign country experiences a positive shock, capital flows in from the home country, thus investment in the foreign country increases and investment in the home country decreases, so international correlation of investment is negative. From the national identity we see that holding the government expenditure and net exports fixed, if consumption is more synchronized with output (high correlation between consumption and output), then investment is less sensitive to the changes of output! The intuition is that if consumption explains more of the variations of output, then investment explains less, ceteris paribus. As a result, co-movements of investment get weakened. If we add bonds to the model that make the market less efficient, then the international correlation of investment is further dampened. Adding terms of trade shocks makes the net export closer to reality, because they are innovations on the relative price of net exports, by perturbing the relative prices they reinforce the substitution effect on net exports. Net exports should be counter cyclical, but sometimes the models fail to do so. When we add incomplete market to the asymmetric preferences, net exports becomes procyclical. That is to say the wealth effect is way too strong. If the home country experiences a positive TOT shock then the export price is higher, it will reduce the net exports whatsoever, this can be seen from Figure 5.2. So output increases but net exports drop, they become counter cyclical again. Among all the refinements in the extended model, the vital significance is due to the zero wealth effect under GHH preference which is the central ingredient that makes the model predictions to be consistent with most of the facts.

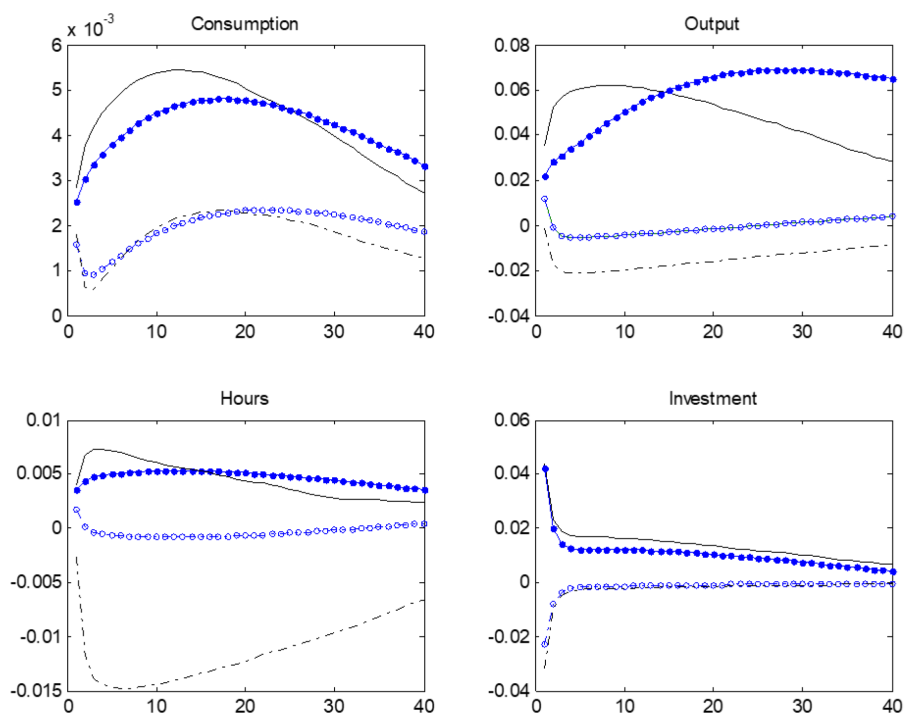

Figure 5a: Impulse Response Functions: Benchmark and Extended Models

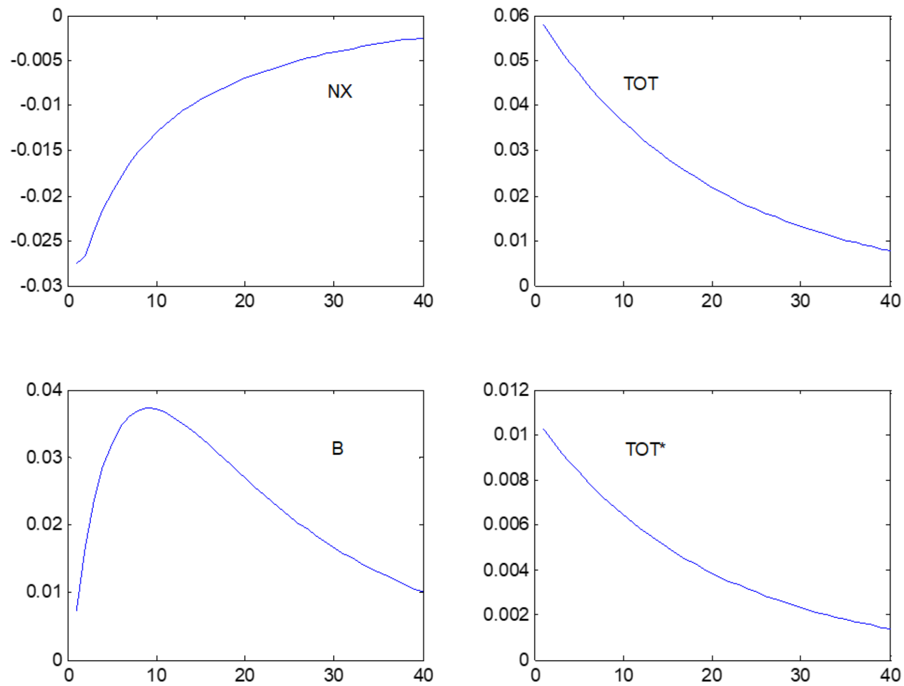

Figure 5b: In Response to (Home Country's) Terms of Trade Shocks

If we compare the impulse response functions between the benchmark and extended model (with asymmetric preference, incomplete markets and term of trade shocks all imposed on) as Figure 5a shows, working hours responds more sensitively in the benchmark especially in the foreign country (the dashed line). But in the extended model the response (hollowed dotted line) is less obvious but slightly positive. Likewise, the output response of the foreign country is weakened in the extended model compared with the benchmark case. The output of the home country no longer jumps up and then steadily declines, but gradually increases and then tends to converge, so that outputs are positively correlated. Home country's investment is less responsive in the extended model because of the constraint from incomplete markets, which explains how incomplete markets work in reducing the co-movements of investment. Figure 5b delineates how net export and bond holding react to a home country terms of trade shock. The covariance between terms of trade makes foreign TOT also increase and then decline, but the magnitude is weaker than home country. Home country's net export decreases first, then steadily converges. The impulse response function of the bond holding in the home country also helps explain the foreign debts that China is accumulating now.

5 Concluding Remarks

This paper tries to convey a better understanding of the real business cycles of the Chinese economy with an emphasis of addressing the well-known anomalies that are prevalent in an international version of general equilibrium models. The first thing we do is to give a robust snippet of the empirical stylized facts of the Chinese economy volatilities by using four bandpass or highpass filters, we find that the negative international correlation of investment is the stylized fact that makes the Chinese economy different from the others. Any model that is addressing China's open economy business cycles should be able to produce this distinct phenomenon. The model sets out from a simple benchmark, and is gradually expanded by adding asymmetric preference, incomplete financial markets and terms of trade shocks, the combination of which we believe explain the stylized facts well. As the model reveals, output and employment are counter-factually negatively correlated with the same variable of the foreign country in the benchmark model, and the coefficient of international correlation of investment is too high (in absolute value). To put it simply, these are all because factors tend to go to the marginally more productive area.

The extended model fits the data better because deviating from symmetric Cobb-Douglas preference is beneficial. For one thing, we see empirically that the consumption output ratio in China steadily declines from 50% in 1980s to 40% in 1990s, and to 35%-38% after 2005. Nonetheless the consumption output ratio in the US stabilizes around 62%-69% during the same periods. Then specifying the preferences to be symmetric for both the home and foreign countries meanwhile calibrating the model using Sino-US data is a huge mismatch. For the other, in terms of transmission mechanism, the GHH preference which features zero leisure elasticity helps weakly increase the home country's labor supply even when the foreign country has a positive shock. This helps the co-movements of both real GDP and employment become positive from the benchmark case. Introduction of the bonds effectively reduces the international correlation of investment, terms of trade shocks enable the prediction of net exports to be closer to reality. Admittedly these refinements arrive at a cost, consumption is way too procyclical, the correlation between consumption and output could approach almost one.[25] But besides that, all the predicted moments fall within the range suggested by the filters. It is in this sense that the model fits the data well.

[25] Compared with the excessively high co-movement of consumption, this problem is much easier to solve and is better understood in the literature. For example, habit formation is an effective cure for it. See Christiano et al. (2005). I am indebted to the referee for insightful comments on this.

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