Matroid Independence Polytopes and Their Ehrhart Theory

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Abstract

A *matroid* is a combinatorial structure that provides an abstract and flexible model for dependence relations between elements of a set. One way of studying matroids is via geometry: one associates a polytope to a matroid, then uses both combinatorics and geometry to understand the polytope and thereby the original matroid. By a *polytope*, we mean a bounded convex set in Euclidean space $\mathbb{R}^n$ defined by a finite list of linear equations and inequalities, or equivalently as the convex hull of a finite set of points. The best-known polytope associated with a matroid M is its *base polytope* $P(M)$, first introduced by Gel'fand, Goresky, Macpherson and Serganova in 1987 [9]. This dissertation focuses on a closely related construction, the *independence polytope* $Q(M)$, whose combinatorics is much less well understood. Both $P(M)$ and $Q(M)$ are defined as convex hulls of points corresponding to the bases or independence sets, respectively; defining equations and inequalities were given for $P(M)$ by Feichtner and Sturmfels [8] in terms of the "flacets" of M, and for $Q(M)$ by Schrijver [24]. One significant difference between the two constructions is that matroid basis polytopes are *generalized permutahedra* as introduced by Postnikov [23], but independence polytopes do not *a priori* share this structure, so that fewer tools are available in their study.

One of the fundamental questions about a polytope is to determine its combinatorial structure as a cell complex: what are its faces of each dimension and which faces contain others? In general it is a difficult problem to extract this combinatorial structure from a geometric description. For matroid base polytopes, the edges (one-dimensional faces) have a simple combinatorial descriptions in terms of the defining matroid, but faces of higher dimension are not understood in general. In Chapter 2 we give an exact combinatorial and geometric description of all the one- and two-dimensional faces of a matroid independence polytope (Theorems 2.9 and 2.11). One consequence (Proposition 2.10) is that matroid independence polytopes can be transformed into

generalized permutahedra with no loss of combinatorial structure (at the cost of making the geometry slightly more complicated), which may be of future use.

In Chapter 3 we consider polytopes arising from *shifted matroids*, which were first studied by Klivans [16, 15]. We describe additional combinatorial structures in shifted matroids, including their circuits, inseparable flats, and flacets, leading to an extremely concrete description of the defining equations and inequalities for both the base and independence polytopes (Theorem 3.12). As a side note, we observe that shifted matroids are in fact *positroids* in the sense of Postnikov [22], although we do not pursue this point of view further.

Chapter 4 considers an even more special class of matroids, the *uniform matroids* $U(r,n)$, whose independence polytopes $TC(r,n) = Q_U(r,n)$ are hypercubes in ^{n} truncated at "height" r. These polytopes are strongly enough constrained that we can study them from the point of view of Ehrhart theory. For a polytope P whose vertices have integer coordinates, the function $i(P,t) = |tP \cap^n|$ (that is, the number of integer points in the t^{th} dilate) is a polynomial in t [7], called the *Ehrhart polynomial*. We give two purely combinatorial formulas for the Ehrhart polynomial of $TC(r,n)$, one a reasonably simple summation formula (Theorem 4.9) and one a cruder recursive version (Theorem 4.6) that was nonetheless useful in conjecturing and proving the "nicer" Theorem 4.9. We observe that another fundamental Ehrhart-theoretic invariant, the h^*-*polynomial* of $TC(r,n)$, can easily be obtained from work of Li [18] on closely related polytopes called *hyperslabs*.

Having computed these Ehrhart polynomials, we consider the location of their complex roots. The integer roots of $i(Q_M,t)$ can be determined exactly even for arbitrary matroids (Theorem 4.17), and extensive experimentation using Sage leads us to the conjecture that for all r and n, all roots of $TC(r,n)$ have negative real parts. We prove this conjecture for the case $r = 2$ (Theorem 4.21), where the algebra is manageable, and present Sage data for other values in the form of plots at the end of Chapter 4.

Acknowledgements

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A collective thanks to all my drinking buddies: Mangles, Kevin, Lucian, Nick, and James and the R Bar Crew. "Because without beer, things do not seem to go as well" - Brother Epp Capuchin

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Chapter 1

Background

1.1 Matroids

Definition 1.1. A matroid, M, is an ordered pair $(E, \mathscr{I})$ where E is a finite set and $\mathscr{I}$ is a collection of subsets of E with the properties:

1. $\varnothing \in \mathscr{I}$.
2. If $I \in \mathscr{I}$ and $J \subseteq I$, then $J \in \mathscr{I}$.
3. If $I, J \in \mathscr{I}$ and $|J| < |I|$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathscr{I}$.

The set E is referred to as the *ground set* of M. The elements of $\mathscr{I}$ are called *independent sets*, and subsets of E not in $\mathscr{I}$ are called *dependent*.

Matroids arise naturally in a variety of contexts. Some common examples of matroids are as follows.

Definition 1.2 (Vector Matroids)**.** Let X be an $m \times n$ matrix over the field $\mathbb{F}$ and E be the set of columns of X. Define $\mathscr{I}$ to be set of subsets of E that are linearly independent. Then $M(X) = (E, \mathscr{I})$ is a matroid. We will refer to such matroids as *vector matroids*.

Example 1.3. Consider the real matrix

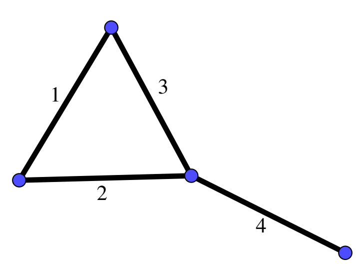
$$X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Label the columns $1,2,3,4$ from left to right. A is an independent set of $M(X)$ if A is contained in one of the following three sets: $\{1,2,4\},\{1,3,4\},\{2,3,4\}$.

Definition 1.4 (Uniform Matroids)**.** The *uniform matroid*, $U(r,n)$, of rank r of size n has as its ground set $[n]$ and its independent sets all subsets of $[n]$ of size at most r.

Definition 1.5 (Graphic Matroids)**.** Let G be a finite graph with edge set E. The *graphic matroid*, $M(G)$, is the ordered pair $(E,\mathscr{I})$ where $\mathscr{I}$ consists of sets of edges containing no cycles.

Example 1.6. Let G be the graph pictured below. Then the graphic matroid $M(G)$ has independent sets: $\varnothing,1,2,3,4,12,13,14,23,24,34,124,134,234$ where 234 is shorthand for $\{2,3,4\}$.



When there exists a bijection between the ground sets of two matroids that induces a bijection between their independent sets, we will say that the matroids are isomorphic. A matroid that is isomorphic to a vector matroid over $\mathbb{F}$ is said to be *representable* over $\mathbb{F}$. For example, the graphic matroid in Example 1.6 is representable over $\mathbb{R}$ since it is isomorphic to the matroid $M(X)$ in

Example 1.3.

The similarity between the words matroid and matrix is no accident. The term matroid was coined in 1935 by [27]. Whitney's definition only included properties 2 and 3 in Definition 1.1, allowing for a matroid with no independent sets. He notes that these two properties are shared by linear independent collections of vectors and acyclic edge sets of graphs. For this reason, much of the terminology used to describe matroids are terms from Linear Algebra or Graph Theory.

Let $M = (E, \mathscr{I})$ be a matroid.

- A *basis* of M is an independent set maximal with respect to inclusion. By property 3 of Definition 1.1, all bases have the same cardinality. For vector matroids, bases correspond to column bases of the corresponding matrix. For graphic matroids, bases correspond to spanning forests of the graph. Let $\mathscr{B}(M)$ denote the set of bases of M. Given distinct bases $A, B \in \mathscr{B}(M)$, and $a \in A \setminus B$, then there exists $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\} \in \mathscr{B}(M)$. This property is called the *basis exchange property* and will be used a number of times in this thesis.

- Every matroid comes equipped with a *rank function*, $r : 2^E \to \mathbb{Z}_{\geq 0}$ where $\mathrm{rk}(A) = \max\{I \in \mathscr{I} \mid I \subseteq A\}$. In terms of vector matroids, $\mathrm{rk}(A)$ is precisely the rank of the matrix whose columns are the elements of A. For a graphic matroid, $r(A) = |V(A)| - c(A)$ where $V(A)$ is the set of vertices which are incident to an edge in A and $c(A)$ is the number of connected components of the subgraph induced by A.

- The *closure* of $A \subseteq E$ is defined as $\overline{A} := \{x \in E \mid \mathrm{rk}(A \cup \{x\}) = \mathrm{rk}(A)\}$.

- A *circuit* of M is a minimal dependent set. In other words, a *circuit* is a dependent set for which every proper subset is independent. The terminology comes from the fact that circuits in graphic matroids correspond to cycles in the graph.

- The set $A \subset E$ is called a *flat* of M if for all $x \notin A$, we have that $\mathrm{rk}(A \cup \{x\}) > \mathrm{rk}(A)$. Alternatively, A is a *flat* if $\overline{A} = A$. In terms of vector matroids, you can think of a flat of M as the collection of all vectors contained in some subspace of the span of E.

The notions of bases, rank functions, closures, circuits, and flats can be axiomatized independently of each other. These are a few of the many cryptomorphic[1] ways of defining matroids.

The *dual matroid* of M, denoted M^*, is the matroid on ground set E whose bases are complements of bases of M.

Later on, we will be concerned with the *inseparable flats* of M. The set $A \subset E$ is called *separable* if there are disjoint subsets R and S such that $A = R \cup S$ and $\mathrm{rk}(A) = \mathrm{rk}(R) + \mathrm{rk}(S)$. The pair (S, T) is referred to a *separation*. Otherwise, A is called *inseparable*. Of particular interest for my purposes are certain inseparable flats that are referred to as *flacets*, following the terminology [8]. A flat $\varnothing \subsetneq A \subsetneq E$ is called a *flacet* if A is inseparable, and A^c is inseparable in M^*.

1.2 Simplicial Complexes

Simplicial complexes are interesting examples of highly combinatorial topological spaces. Singular homology of topological spaces is often very difficult to compute directly. However, with the tools of simplicial homology, the computations can be accomplished with fairly simple linear algebra [10]. Simplicial complexes also enjoy a deep and beautiful connection with commutative algebra via the Stanley-Reisner correspondence. A wealth of information can be gathered about a Stanley-Reisner ring by studying the combinatorial and topological properties of its corresponding simplicial complex, and vice versa [25].

Definition 1.7. A collection Δ of finite subsets of a set V is an *abstract simplicial complex* if for all $X \in \Delta$ and $Y \subseteq X$, $Y \in \Delta$.

We will often conflate the notion of an abstract simplicial complex with its geometric realization. The geometric realization is comprised of points, line segments, triangles, tetrahedra, etc... corre-

[1] Two systems of axioms are *cryptomorphic* if they are equivalent, but not obviously so. The term was coined by Birkhoff, and popularized by Rota in the context of matroid theory.

sponding to singletons, doubletons, triples, quadruples, etc... in Δ. For this reason, the elements of Δ are called *faces*, with singletons and doubletons referred to as *vertices* and *edges*, respectively.

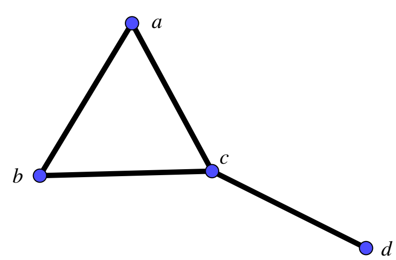
The faces of Δ that are maximal with respect to inclusion are called *facets*. A complex is determined by its facets and therefore we will write $\langle F_1, \ldots, F_s \rangle$ to be the simplicial complex whose facets are $F_1, \ldots, F_s$. A simplicial complex is *pure* if all facets have the same cardinality.

For $F \in \Delta$, the *dimension* of F is $\dim(F) = |F| - 1$. This definition of dimension aligns with the dimension of faces in the geometric realization. For example, the dimension of a tripleton (which corresponds to a triangle in the geometric realization) is two. The dimension of Δ is

$$\dim(\Delta) = \max\{\dim(F) \mid F \in \Delta\}.$$

The *f-vector* of Δ is the vector whose i^{th} entry is the number of faces of Δ of dimension i.

Example 1.8. One-dimensional simplicial complexes are precisely simple graphs. Consider the graph G from Example 1.6. As a simplicial complex, $G = \{\varnothing, a, b, c, d, ab, ac, bc, cd\} = \langle ab, ac, bc, cd \rangle$.



The facets of a graph are the edges and the isolated vertices. Since G has no isolated vertices, the facets of G are the four edges. Therefore G is pure.

1.3 Shifted Complexes

We will be concerned with a class of simplicial complexes whose structure depends highly on the ordering of the vertex set.

Definition 1.9. Let $n \in \mathbb{N}$ and let $\binom{[n]}{k}$ denote the subsets $[n]$ of size k. Define the *shifted ordering* $\preceq$ on $\binom{[n]}{k}$ by $S = \{s_1 < \cdots < s_k\} \preceq T = \{t_1 < \cdots < t_k\}$ if $s_i \leq t_i$ for all $i \in [k]$.

Definition 1.10. A simplicial complex Δ with vertices $[n]$ is said to be *shifted* if its facets form an order ideal of $2^{[n]}$ under $\preceq$.

This definition can be extended to non-pure complexes as well by suitably modifying $\preceq$. For example, the complex with $\langle 12, 3 \rangle$ is shifted. However, for our purposes, we will only be dealing with pure shifted complexes. This is because independence complexes of matroids are necessarily pure.

Shifted complexes are interesting objects that show up in a variety of contexts. In [12], Kalai defined the notion of *algebraic shifting* which takes a simplicial complex Δ and associates to it a shifted complex, $S(\Delta)$. $S(\Delta)$ shares many combinatorial and topological properties with the original complex such as Betti numbers and f-vector. The advantage of making this association is that $S(\Delta)$ is topologically and algebraicly simpler in some sense. For instance, $S(\Delta)$ is a wedge of spheres and non-pure shellable in the sense of [4]. Shifted complexes and matroid complexes share the property of being *Laplacian integral* [6], [17]. This means that their Laplacians have integer eigenvalues, which is an uncommon property. In the case of one-dimensional complexes, i.e., graphs, being shifted is equivalent to a number of properties such as *threshold* and *chordal*. In higher dimensions, threshold implies shifted, but they are not equivalent. For example, $\langle\langle 178, 239, 456 \rangle\rangle$ is not threshold [14].

Let $\langle\langle T_1, \ldots, T_s \rangle\rangle$ denote the shifted complex whose maximal facets with respect to the shifted ordering $\preceq$ are $T_1, \ldots, T_s$. A shifted complex is a matroid if and only if it has a unique facet maximal with respect to the shifted ordering [16]. The ordering $\preceq^*$ is defined by $S \preceq^* T$ if and only if $T \preceq S$.

Let $\langle\langle T_1, \ldots, T_s \rangle\rangle^*$ denote the shifted complex whose maximal facets with respect to the ordering $\preceq^*$ are $T_1, \ldots, T_s$. The complex $\langle\langle T_1, \ldots, T_s \rangle\rangle^*$ can be thought of as a shifted complex under the reverse ordering on $[n]$.

Example 1.11. $\langle\langle 35 \rangle\rangle$ is the matroid complex with facets/bases

$$35, 25, 15, 34, 24, 14, 24, 23, 13, 12.$$

$\langle\langle 35 \rangle\rangle^*$ is the matroid complex with facets/bases

$$35, 45.$$

Proposition 1.12. *Let $M = ([n], \langle\langle a_1 \ldots a_r \rangle\rangle)$ be a shifted matroid. Then $M^* = ([n], \langle\langle b_1 \ldots b_{n-r} \rangle\rangle^*)$ with $\{b_1, \ldots, b_{n-r}\} = [n] \backslash \{a_1, \ldots, a_r\}$.*

Proof. The bases of M^* are complements of bases of M. It is clear that for all the complements of bases of M, we have that $B = \{b_1, \ldots, b_{n-r}\}$ is the least under $\preceq$. Therefore under $\preceq^*$, B is the largest complement of a basis of M. $\square$

1.4 Polytopes

Definition 1.13. A *convex polytope*, P, is the convex hull of finitely many points. Alternatively, a polytope can be described as the bounded intersection of finitely many closed half-spaces.

Figure 1.1 shows a polytope in $\mathbb{R}^3$. Given a linear functional φ on $\mathbb{R}^n$, let

$$F = \{x \in P \mid \varphi(x) \geq \varphi(y) \text{ for all } y \in P\}.$$

Any set F of this form is called a *(closed) face* of P and we will say that φ is "maximized on F". The sets obtained in such a fashion are called the *faces* of P. The dimension of F, $\dim F$, is defined

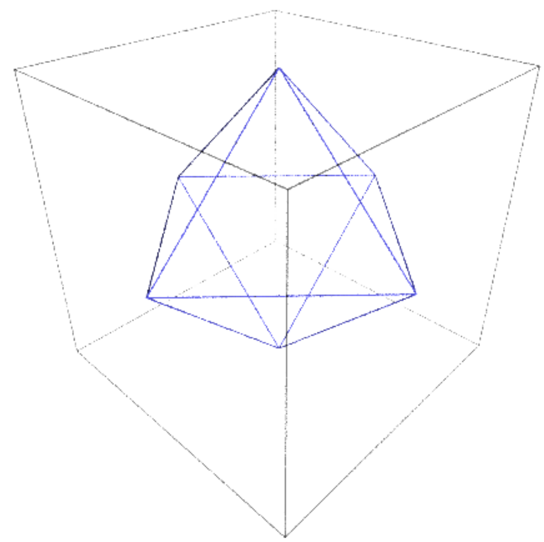


Figure 1.1: A polytope in $\mathbb{R}^3$

to be the dimension of the affine hull of F. The zero-dimensional faces of P are called *vertices*. Note that P is itself a face of P (take $\varphi = 0$).

We will now restrict our focus to *integral polytopes*, that is, polytopes whose vertices lie in $\mathbb{Z}^n$. The t^{th} dilate of P is

$$tP = \{tx \mid x \in P\}.$$

Definition 1.14. Given an integral convex polytope P, define the *lattice point enumerator* as

$$i(P,t) = |tP \cap \mathbb{Z}^n|.$$

Example 1.15. Let $P = [0,1]^n$ be the $0/1$ cube in $\mathbb{R}^n$. Then $tP = [0,t]^n$ and $i(P,t) = (t+1)^n$.

For integral convex polytopes, $i(P,t)$ is a polynomial [7]. Thus it is often referred to as the *Ehrhart Polynomial* of P.

A well known fact about Ehrhart polynomials is *Ehrhart reciprocity*.

Theorem 1.16 (Ehrhart Reciprocity). *[3] Suppose P is a convex integral polytope. Then*

$$i(P,-t) = (-1)^{\dim(P)} i(P^\circ, t)$$

where P° is the relative interior of the polytope P.

Example 1.17. Let $P = [0,1]^n$ be the $0/1$ cube in $\mathbb{R}^n$. Then $i(P,-t) = (1-t)^n = (-1)^n(t-1)^n$. When $n = 2$, we can see that for $t = 1,2,3$ we get $|i(P,-t)| = 0,1,4 = |i(P^\circ,t)|$ respectively.

Definition 1.18. The *Ehrhart Series* of P, $Ehr_P(x)$, is the generating function for the lattice point enumerator of P. That is,

$$Ehr_P(x) = \sum_{k=0} i(P,k)x^k.$$

The Ehrhart Series of integral convex polytopes bear a striking resemblance to the Hilbert series of Stanley-Reisner rings/ Simplicial Complexes. $Ehr_P(x)$ can be expressed as a rational function,

$$\frac{h_0^* + h_1^* x + \cdots + h_d^* x^d}{(1-x)^{d+1}}$$

where $d = dim(P)$. The vector $h^*(P) = (h_0^*, h_1^*, \ldots, h_d^*)$ is called the h^* vector of P (similar to the h-vector of a simplicial complex).

Example 1.19. Recall that for the $0/1$ cube in $\mathbb{R}^n$, $i(P,t) = (t+1)^n$. Then

$$Ehr_P(x) = \sum_{k=0}^{\infty}(k+1)^n x^k = \frac{1}{x}\sum_{j=1}^{\infty} j^n x^j = \frac{\sum_{m=0}^{n-1} A(n,m)x^{m+1}}{x(1-x)^{n+1}} = \frac{\sum_{m=0}^{n-1} A(n,m)x^m}{(1-x)^{n+1}}.$$

Here $A(n,m)$ are the *Eulerian Numbers* [26] which count the elements of the symmetric group S_n with m ascents.

For the $0/1$ cube in $\mathbb{R}^4$, $Ehr_P(x) = \dfrac{x^3+11x^2+11x+1}{(1-x)^5}$.

We will present some facts about the h^* vector of an integral convex polytope in the following theorem. The interested reader can consult [11] for more details.

Theorem 1.20. *[11] Suppose $P \subseteq \mathbb{R}^n$ is an integral convex polytope of dimension d. Then*

1. $h_0^* = 1$, $h_1^* = i(P,1) - (d+1)$
2. *Suppose* $h_{d-j}^* = 0$ *for* $j = 0,\ldots,k$. *Then* $i(P,-j) = 0$ *for* $j = 0,\ldots k$, *and* $h_{d-(k+1)}^* = i(P,-(k+1))$.
3. *(Stanley)* $h_i^* \geq 0$.
4. $\frac{h_0^* + \cdots + h_d^*}{n!} = vol(P)$.

Chapter 2

Independence Polytopes of Matroids

Let $[n]$. For $A \subseteq E$, let $\chi_A = \sum_{i \in A} \mathbf{e}_i$ where $\mathbf{e}_i$ is the i^{th} standard basis vector in $\mathbb{R}^n$.

Definition 2.1. Let $M = (E, \mathscr{I})$ be a matroid. The *Matroid Independence Polytope* of M is defined as

$$Q_M = \operatorname{conv}(\chi_I \mid I \in \mathscr{I}).$$

The main results of this section are characterizations of the $1-$skeleton 2.9 and $2-$skeleton 2.12 of Q_M. My study of the independence polytope was inspired by the study of *matroid base polytopes*. Therefore, before continuing with the proofs of the main results, we will enter into a brief discussion of the base polytope.

Definition 2.2. Let $M = (E = [n], \mathscr{I})$ be a matroid. The *Matroid Base Polytope* of M is defined as

$$P_M = \operatorname{conv}(\chi_B \mid \text{where } B \text{ is a basis of } M).$$

Matroid base polytopes are very well-studied objects with beautiful mathematics surrounding them. For example, base polytopes are generalized permutohedra in the sense of [23] and [21]. There are many results describing ways to decompose base polytopes into gluings of smaller base polytopes of associated matroids [5]. Additionally, every face of a base polytope is again a base polytope. Many of the facets of matroid base polytopes are known to be indexed by *flacets*.

Definition 2.3. A flat $\varnothing \subsetneq A \subsetneq E$ is a *flacet* if A is inseparable and A^c is inseparable in M^*.

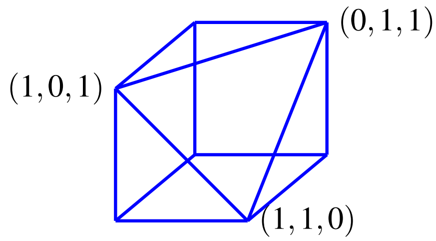
With this definition in hand, we present the following theorem:

Theorem 2.4. *[8] The following system of inequalities defines P_M:*

1. $x_1 + \cdots + x_n = rk(M)$.
2. $x_e \geq 0$ *for all* $e \in E$.
3. $\sum_{e \in F} x_e \leq rk(F)$ *where* F *is a* flacet *of* M.

It should be noted that the above system of inequalities is not minimal, in general. Often times some of the inequalities of the form $x_e \geq 0$ are redundant.

I like to think of Q_M as the "shadow" of the base polytope. Q_M is everything below (as in on the side of the origin) the base polytope and within the $0/1$ cube.



P_M (front triangle) and Q_M for $M = U(2,3)$.

Much less is known about matroid independence polytopes than base polytopes. In fact, many of the useful properties of base polytopes do not hold for independence polytopes. For instance, not every face of an independence polytope is an independence polytope. Also, independence polytopes are not generalized permutohedra. However, using Theorem 2.9 it can be shown that through a unimodular transformation, the independence polytope and be lifted to become a generalized permutohedra 2.10. This could perhaps provide a new approach to studying independence polytopes. On the other hand, a description of the facets of independence polytopes exists and is very similar to the result for base polytopes. In fact, this description is better than 2.4 in that it provides a minimal system of inequalities describing Q_M.

Theorem 2.5 (Theorem 40.5)**.** *[24] If* M *is loopless, the following is a minimal system of inequalities for* Q_M*:*

1. *$x_e \geq 0$ for all $e \in E$.*

2. *$\sum_{e \in F} x_e \leq rk(F)$ for each non-empty inseparable flat, F, of M.*

It is immediate from this theorem that independence polytopes are *geometrically shifted*: that is, for any point in Q_M decreasing a coordinate without making it negative does not leave the polytope.

For a general matroid, understanding its lattice of flats is a daunting task. In later sections, we will tackle the more tractable problem of understanding the inseparable flats of shifted matroids.

2.1 The 1-Skeleton of Q_M

The rest of this section will be dedicated to giving a complete characterization of the 1-skeleton and 2-skeleton of Q_M.

Lemma 2.6. *If $A \subseteq B \in \mathscr{I}$, then $F := conv(\{\chi_C | A \subseteq C \subseteq B\})$ is a cubical face of Q_M of dimension $|B| - |A|$. In particular, if $A \subseteq B \in \mathscr{I}$, then $conv(\chi_A, \chi_B)$ is an edge if and only if $B = A \cup \{e\}$ for some $e \in E \setminus A$.*

Proof. The linear functional $T(v) = (\chi_A - \chi_{B^c}) \cdot v$ is maximized on F, so F is a face of Q_M. It is clear that F is a cube of dimension $|B| - |A|$. □

Lemma 2.7. *Let $A, B \in \mathscr{I}$ with $A \nsubseteq B$ and $B \nsubseteq A$. If $conv(\chi_A, \chi_B)$ is an edge of Q_M then $|A| = |B|$ and $\overline{A} = \overline{B}$.*

Proof. Assume that $\mathrm{conv}(\chi_A, \chi_B)$ is an edge of Q_M. Suppose, for the sake of contradiction that $|A| < |B|$. Then there exists $e \in B \setminus A$ such that $A \cup \{e\} \in \mathscr{I}$. Then $\chi_A, \chi_B, \chi_{A \cup \{e\}}, \chi_{B \setminus \{e\}}$ are all vertices of Q_M. Furthermore, their convex hull is a parallelogram.

To see this, note that $A \setminus (B \setminus \{e\}) = A \setminus B = (A \cup \{e\}) \setminus B$. Therefore $\mathrm{conv}(\chi_A, \chi_{B \setminus \{e\}})$ and $\mathrm{conv}(\chi_{A \cup \{e\}}, \chi_B)$ are parallel and of equal length. Similarly for $\mathrm{conv}(\chi_A, \chi_{A \cup \{e\}})$ and $\mathrm{conv}(\chi_{B \setminus \{e\}}, \chi_B)$.

This contradicts the fact that $\operatorname{conv}(\chi_A, \chi_B)$ is an edge of Q_M, since $\operatorname{conv}(\chi_A, \chi_B)$ is interior to the parallelogram. Therefore $|A| = |B|$.

Now suppose $\overline{A} \neq \overline{B}$. If $B \subseteq \overline{A}$, then $\overline{B} \subseteq \overline{\overline{A}} = \overline{A}$. But $\operatorname{rk}(A) = |A| = |B| = \operatorname{rk}B$, so $\overline{B} \subseteq \overline{A}$ implies $\overline{B} = \overline{A}$. This is a contradiction, so assume that B is not containted in $\overline{A}$. Then there exists $e \in B$ such that $e \notin \overline{A}$. Since $e \notin \overline{A}$, it must be that $A \cup \{e\} \in \mathscr{I}$. So $\chi_A, \chi_B, \chi_{A\cup\{e\}}, \chi_{B\setminus\{e\}}$ are all vertices of Q_M. As before, the convex hull of these points is a parallelogram containing the edge $\operatorname{conv}(\chi_A, \chi_B)$ in its interior. This is a contradiction, therefore $\overline{A} = \overline{B}$. □

Lemma 2.8. *For any $F \subseteq E$, $P_{M|_F}$ is a face of Q_M.*

Proof. The linear functional $T(v) = (\chi_F - \chi_{F^c}) \cdot v$ will be maximized precisely on

$$\operatorname{conv}\{\chi_B \mid B \text{ is a basis of } M|_F\}.$$

□

Theorem 2.9. *Let $A, B \in \mathscr{I}$ with $|A| \leq |B|$. Then conv(χ_A, χ_B) is an edge of Q_M if and only if one of the following holds:*

1. *$B = A \cup \{e\}$ for some $e \in B \backslash A$.*
2. *$\overline{A} = \overline{B}$ and $B = (A \backslash \{f\}) \cup \{e\}$ for some $f \in A \backslash B$ and $e \in B \backslash A$.*

Proof. ($\Rightarrow$) Assume $\operatorname{conv}(\chi_A, \chi_B)$ is an edge of Q_M. Suppose that $A \subseteq B$. By 2.6, $B = A \cup \{e\}$ for some $e \in B \backslash A$. Now, suppose that $A \nsubseteq B$. By Lemma 2.7, $|A| = |B|$ and $\overline{A} = \overline{B}$. By considering A and B as bases in the restricted matroid $M|_{\overline{A}}$, we can use basis exchange to see that $B = (A \backslash \{f\}) \cup \{e\}$ for some $f \in A \backslash B$ and $e \in B \backslash A$.

($\Leftarrow$) If $B = A \cup \{e\}$ for some $e \in B \backslash A$, then again by Lemma 2.6, $\operatorname{conv}(\chi_A, \chi_B)$ is an edge of Q_M. Now, assume that $\overline{A} = \overline{B}$ and $B = (A \backslash \{f\}) \cup \{e\}$ for some $f \in A \backslash B$ and $e \in B \backslash A$. By Lemma 2.8, $P_{M|_{\overline{A}}}$ is a face of Q_M. Since edges of $P_{M|_{\overline{A}}}$ are given by basis exchange [8], $\operatorname{conv}(\chi_A, \chi_B)$ is an edge of $P_{M|_{\overline{A}}}$. □

Note that the second type of edge in the above theorem comes from basis exchange *within* flats. That is, these edges correspond to a basis exchange in the restriction $M|_F$ where F is a flat of M.

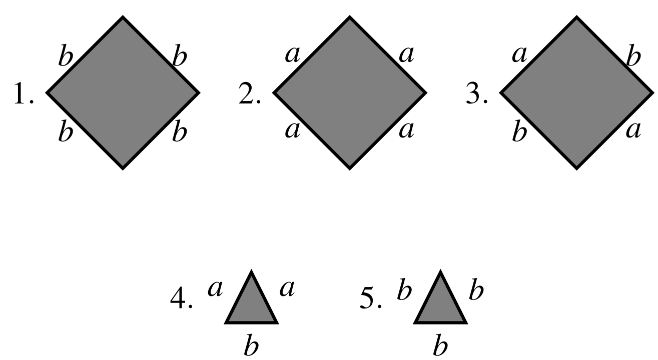
Generalized permutohedra are characterized by every edge lying in a direction $\mathbf{e}_i - \mathbf{e}_j$. Matroid base polytopes have this property as edges of P_M correspond to basis exchange [8]. Viewing base polytopes as generalized permutohedra provides useful tools for their study. While Q_M is not a generalized permutohedron, it is not far off.

Proposition 2.10. *Define* $\tilde{Q}_M := conv\{((rank(M) - |I|), \chi_I) \in \mathbb{R} \times \mathbb{R}^{|E|} \big| I \in \mathscr{I}\}$. *Then* $\tilde{Q}_M$ *is a generalized permutohedron.*

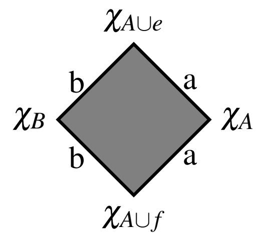
$\tilde{Q}_M$ is the result of embedding Q_M into $\mathbb{R}^{|E|+1}$ by lifting each vertex to height equal to its co-rank. Q_M contains an edge from χ_A to $\chi_{A \cup \{x\}}$ for all $A, A \cup \{x\} \in \mathscr{I}$. Lifting rectifies the direction for such edges. Lifting does not affect the edge direction of the edges of type 2 in Theorem 2.9.

2.2 The 2-Skeleton of Q_M

Theorem 2.11. *Label each edge of* Q_M *with an a or b depending on whether the edge corresponds to adding an element of E or corresponds to basis exchange within a flat of M. The two-dimensional faces of* Q_M *must be of one of the following forms:*



Proof. The only two-dimensional 0/1 polytopes are triangles and quadrilaterals. Let F be a two-dimensional face of P_M. Since the faces of a 0/1 polytope are again 0/1 polytopes, we must have that F is a triangle or quadrilateral. Note that F must contain an even number of 'a' edges. For if not, starting at a vertex $v \in F$ and walking around the boundary of F would result in a disparity in the value of $\chi_v \cdot \mathbb{1}$. Therefore the only possible missing figure in the list above is



We will demonstrate that no 2-face has this form in due course. Suppose F has the above form. Note that B differs from both $A \cup e$ and $A \cup f$ by a basis exchange. That is,

$$\begin{aligned} B &= (A \cup e)\backslash g_1 \cup h_1 \\ &= (A \cup f)\backslash g_2 \cup h_2 \end{aligned}$$

If $e = g_1$, then $B = A \cup h_1$. Therefore Q_M has an edge from χ_A to χ_B, contradicting the fact that F is a face. Thus $e \neq g_1$ (and similarly, $f \neq g_2$). Now

$$\begin{aligned} B &= (A\backslash g_1) \cup \{e, h_1\} \\ &= (A\backslash g_2) \cup \{f, h_2\} \end{aligned}$$

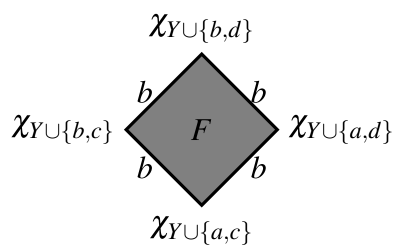
So $e, f \in B$ and hence $h_1 = f$, and $h_2 = e$. Therefore $g_1 = g_2$ and $B = (A\backslash g) \cup \{e, f\}$. This is a contradiction, for the following reason. $\chi_A, \chi_{A\cup e}$, and $\chi_{A\cup f}$ all live in the hyperplane $\chi_{\{g\}} = 1$, therefore so should any two-dimensional face containing them. However, χ_B does not lie in this hyperplane. $\square$

Observation: The quadrilaterals of Type 1 and 2 are squares and the quadrilaterals of type 3 are rectangles.

Theorem 2.11 only says that two-dimensional faces must be of one of five forms (Fig. 2.11), but tt does not guarantee that every polygon of one of these forms is a two-dimensional face. We will investigate this now.

Theorem 2.12. *The following statements characterize the* 2*-skeleton of* Q_M*:*

1. *Every* $2-$*dimensional face of* Q_M *is of type* $1,2,3,4,$ *or* 5 *(Fig. 2.11)*
2. *Let* F *a parallelogram of type 1 with vertices* $\chi_A, \chi_B, \chi_C, \chi_D$ *where* $A = Y \cup \{a,c\}$*,* $B = Y \cup \{b,c\}$*,* $C = Y \cup \{b,d\}$*, and* $D = Y \cup \{a,d\}$*.*



Then F *is a face of* Q_M *iff* $M|_{Y\cup\{a,b,c,d\}}/Y \ncong U(2,4)$*.*

3. *Every polygon of type* $2,3,4,$ *or* 5 *is a face of* Q_M*.*

Proof. **Assertion 1:** This is the content of Theorem 2.11.

Assertion 2: Parallelograms of type 1 sometimes are not faces as is evidenced in $Q_{U(2,4)}$. The convex hull of the characteristic vectors of any four bases of this matroid lies in the interior of the octahedron that is the base polytope of $U(2,4)$. In general, the existence of a $U(2,4)$ minor is the only obstruction to a parallelogram of type 1 being a face of Q_M.

Let $V = Y \cup \{a,b,c,d\}$ and ε be a sufficiently small positive number. If $Y \cup \{a,b\}$ is not independent, then the linear functional φ obtained by taking the dot product with $\chi_V - \chi_{V^c} + \varepsilon\chi_{\{a,b\}}$ is maximized on F and so F is a face. To see this, note that all subsets of V of cardinality greater than $|Y|+2$ are dependent since A,B,C,D are all bases of the same flat. Therefore the vertices of Q_M on which φ might be maximized must be among the 6 subsets of V that contain Y and are of cardinality exactly $|Y|+2$. We can compute that $\varphi(\chi_A) = \varphi(\chi_B) = \varphi(\chi_C) = \varphi(\chi_D) = |Y|+2+\varepsilon$, while $\varphi(\chi_{Y\cup\{c,d\}}) = |Y|+2$.

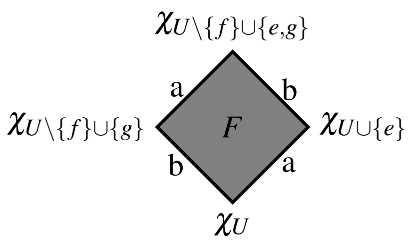
Similarly if $Y\cup\{c,d\}$ is not independent, then the linear functional φ obtained by taking the dot product with $\chi_V - \chi_{V^c} + \varepsilon\chi_{\{c,d\}}$ is maximized on F and so F is a face.

If however, $Y\cup\{a,b\}$ and $U\cup\{c,d\}$ are both independent, then $M|_V/Y \cong U(2,4)$ and $\text{conv}(\chi_A,\chi_B,\chi_C,\chi_D,\chi_{Y\cup\{a,b\}},\chi_{Y\cup\{c,d\}})$ is an octahedral face of Q_M with F in its interior.

Assertion 3:

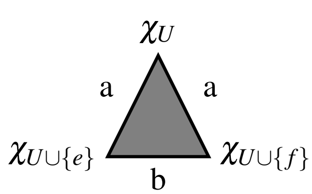
Type 2: Lemma 2.6 guarantees that every square of type 2 is a face of Q_M.

Type 3: Let F be a parallelogram of type 3 in Theorem 2.11. Then we have the following picture:



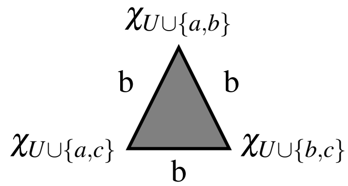
Define the linear functional $\varphi(v) = (\chi_{U\cup\{g\}} - \chi_{(U\cup\{e,f,g\})^c})\cdot v$. Since U and $U\setminus\{f\}\cup\{g\}$ are bases of the same flat, and this flat contains $U\cup\{g\}$, it follows that $U\cup\{g\}$ must be dependent. So $\chi_{U\cup\{g\}}$ is not a vertex of Q_M. Therefore the largest value φ could take at a vertex of Q_M is $|U|$. Note that $\varphi(\chi_U) = \varphi_{(}\chi_{U\cup\{e\}}) = \varphi(U\setminus\{f\}\cup\{g\}) = \varphi(U\setminus\{f\}\cup\{e,g\}) = |U|$. Also for any other subset $V\subseteq U\cup\{e,f,g\}$, $\varphi(\chi_V) < |U|$. Therefore φ is maximized on F.

Type 4: Let F be a triangle of type 4 in Theorem 2.11. Then we have the following picture



Define the linear functional $\varphi(v) = (\chi_U - \chi_{(U\cup\{e,f\})^c})\cdot v$. Note that $\varphi(U) = \varphi(U\cup\{e\}) = \varphi(U\cup\{f\}) = \varphi(U\cup\{e,f\}) = |U|$. However, $U\cup\{e\}$ and $U\cup\{f\}$ are bases of the same flat, and this flat contains $U \cup \{e, f\}$. Therefore, $U \cup \{e, f\}$ is not independent. Hence φ is maximized on F.

Type 5: Let F be a triangle of type 5 in Theorem 2.11. Then we have the following picture



Define the linear functional $\varphi(v) = (\chi_{U\cup\{a,b,c\}} - \chi_{(U\cup\{a,b,c\})^c}) \cdot v$.

Note that $\varphi(\chi_{U\cup\{a,b,c\}}) = |U| + 3$ and $\varphi(\chi_{U\cup\{a,b\}}) = \varphi(\chi_{U\cup\{a,c\}}) = \varphi(\chi_{U\cup\{b,c\}}) = |U| + 2$.

However, $U \cup \{a, b\}, U \cup \{a, c\}, U \cup \{b, c\}$ are bases of the same flat, and that flat contains $U \cup \{a, b, c\}$. So $U \cup \{a, b, c\}$ must be dependent and $\chi_{U\cup\{a,b,c\}}$ is not a vertex of Q_M. Therefore φ is maximized on F. $\square$

At this point, we have fully characterized the 1-skeleton and 2-skeleton of independence polytopes. One could use the theorems in this section to create an (admittedly inefficient) algorithm for computing the 2-skeleton of Q_M. After computing all edges of Q_M via the criterion in Theorem 2.9, one then searches for all triangles and parallelograms of types presented in Theorem 2.11. For parallelograms of type 2, one must additionally check for the existence of the specified $U(2,4)$ minor.

One could pursue a characterization of the 3-skeleton of independence polytopes. Up to combinatorial equivalence, there are 8 three-dimensional $0/1$ polytopes [28]. While possible, such an effort would involve extensive case analysis.

Chapter 3

Shifted Matroids and their Independence Polytopes

This chapter is dedicated to the study of shifted matroids as well as their Independence Polytopes. Here we characterize circuits (Theorem 3.6), characterize inseparable flats (Corollary 3.10), and characterize flacets (Theorem 3.11). Together, these results allow us to state a concise hyperplane description of the base and independence polytopes of shifted matroids (Theorem 3.12). After that we enter into a brief discussion of shifted matroids and their connection to positroids.

3.1 Shifted Matroids

Shifted matroids were characterized in [16] as shifted complexes with a single facet that is maximal with respect to the shifted ordering $\preceq$.

Observation: For the shifted matroid $M = ([n], \langle\langle a_1 \dots a_r \rangle\rangle)$,

1. M is loopless if and only if $a_r = n$
2. M is coloopless if and only if $a_1 \neq 1$.

This is because if $a_n \neq n$, then n is not contained in any basis and is therefore a loop. Similarly if $a_1 = 1$, the 1 is contained in every basis and is therefore a coloop.

In general, M^* is loopless if and only if M is coloopless. Similarly M^* is coloopless if and only if M is loopless. This can be seen very explicitly for shifted matroids by combining the above observation with Proposition 1.12.

To represent shifted matroids, we will choose "sufficiently random" vectors of a certain form. The notion of sufficiently random can be made precise as follows.

Definition 3.1. [1] Let $x_1, \dots, x_k \in \mathbb{R}$. For $S \subseteq [k]$, define $x_S := \prod_{i \in S} x_i$. Consider the 2^{2^k} possible sums of x_S's. If these sums are all distinct, then the collection $x_1, \dots, x_k$ is called *generic*.

The following theorem states that shifted matroids are generically representable.

Theorem 3.2. *Theorem 5.2 [1] Let $a_1 < a_2 < \cdots < a_r$ be arbitrary positive integers. Let $X = (x_{ij})_{1 \le i \le r, 1 \le j \le a_r}$ where $\{x_{ij} \mid 1 \le i \le r$ and $1 \le j \le a_i\}$ is generic and the rest of the x_{ij}'s are equal to 0. Then the vector matroid $M(X)$ is isomorphic to the shifted matroid $\langle\langle a_1 \dots a_r \rangle\rangle$.*

We will use $*$'s to denote generic entries in a matrix.

Example 3.3. $U(r,n)$ is the shifted matroid $M = ([n], \langle\langle (n-r+1)(n-r+2)\dots n \rangle\rangle)$. Below is a generic representation of $U(3,5)$

$$\begin{bmatrix} * & * & * & 0 & 0 \\ * & * & * & * & 0 \\ * & * & * & * & * \end{bmatrix}$$

Example 3.4. Let M be the shifted matroid $\langle\langle 245 \rangle\rangle$. The following generic matrix represents M. The i^{th} column corresponds to the element i of the ground set of M.

$$\begin{bmatrix} * & * & 0 & 0 & 0 \\ * & * & * & * & 0 \\ * & * & * & * & * \end{bmatrix}$$

The bases of M are: $123, 124, 134, 234, 125, 135, 235, 145, 245$

The flats of M are: $\varnothing, 1, 2, 3, 4, 5, 12, 13, 14, 15, 23, 24, 25, 345, 12345$.

The circuits of M are: $1234, 345, 1245, 1235$.

The bases of M^* are: $45, 35, 25, 15, 34, 24, 14, 23, 13$.

3.2 Structure of Shifted Matroids

Definition 3.5. Let M be a shifted matroid. Consider a generic matrix representing M (as in Theorem 3.2). The *height* of k, denoted $\mathrm{ht}(k)$, with respect to M is the number of stars in the k^{th} column of the matrix. The k^{th} *block*, B_k, is the collection of elements of height k. Finally, the k^{th} *terminal segment*, T_k, is the union of $B_1, \ldots, B_k$.

Theorem 3.6. *Let M be a shifted matroid with independence complex $\langle\langle a_1 \ldots a_r \rangle\rangle$. Let C be a set of cardinality $k+1$. Then, C is a circuit of M if and only if $|C \cap T_i| \leq i$ for all $i < k$ and $|C \cap T_i| = k+1$ for all $i \geq k$.*

Proof. Let C be a circuit of M of cardinality $k+1$ and x be the smallest element of C. Then $\tilde{C} = C \setminus \{x\}$ is a rank k independent set. Since $\tilde{C}$ is an independent set rank k, adding any element of height greater than k to $\tilde{C}$ would result in an independent set. Therefore $\mathrm{ht}(x) \leq k$. However, x is the smallest element of C and so $C \subseteq T_k$. Note that $|\tilde{C} \cap T_i| \leq i$ for all $i \in [k]$ (otherwise there would be a dependence). Since $\tilde{C} \subseteq T_k$, we get that $|\tilde{C} \cap T_k| = k$ and since x is smaller than all elements of $\tilde{C}$, we get that $\mathrm{ht}(x) = k$. Consequently, $|\tilde{C} \cap T_i| \leq i$ for all $i < k$ and $|\tilde{C} \cap T_i| = k+1$ for all $i \geq k$.

Now, assume $|C \cap T_i| \leq i$ for all $i < k$ and $|C \cap T_i| = k+1$ for all $i \geq k$. Since C contains $k+1$ elements of height at most k, C is dependent. Let x be the smallest element of C and $\tilde{C} = C \setminus \{x\}$. Then $|\tilde{C} \cap T_i| \leq i$ for all $i \in [k]$. So the restricted matroid $M|_{\tilde{C}}$ is represented by a lower-triangular generic matrix, which is non-singular. Therefore $\tilde{C}$ is independent. Note that $C \setminus \{y\} \preceq \tilde{C}$ for all $y \neq x$. Therefore $C \setminus \{y\}$ is independent for any $y \neq x$. Thus C is a circuit. $\square$

Theorem 3.7. *Let F be an inseparable flat with $|F| \geq 2$. Let $x \in F$ be of greatest height, k. Then $F = T_k$ and $|B_k| > 1$.*

Proof. Note that $F \subseteq T_k$ since the elements of F have height at most k.

Since F is inseparable, $M|_F$ is coloopless. Therefore x is contained in some circuit, C, of $M|_F$. Note that C is also a circuit of M. Suppose $|C| = l+1$. By Theorem 3.6, $|C \cap T_{l-1}| \leq l-1$ and

$|C \cap T_l| = l+1$. So $|C \cap B_l| \geq 2$ and $C \cap B_{l+1} = \varnothing$ since x is of greatest height, $l = k$. Therefore $|B_k| \geq |C \cap B_k| > 1$. Note that C is a circuit of cardinality $k+1$ so $\mathrm{rk}(C) = k$. Since $C \subset T_k$ and $\mathrm{rk}(C) = k = \mathrm{rk}(T_k)$, $\overline{C}$ contains T_k. Since F is a flat and $C \subseteq F$, it follows that $T_k \subseteq \overline{C} \subseteq F$.

So $F = T_k$ and $|B_k| > 1$. $\square$

Lemma 3.8. *Let $r > 1$ and $M = ([n], \langle\langle a_1 \dots a_r \rangle\rangle)$ be a loopless ($a_r = n$) shifted matroid. Then E is separable if and only if $|B_r| = 1$.*

Proof. If $x \in E$ is a coloop, then $E = \{x\} \cup E \backslash \{x\}$ is a separation. Note that $|B_r| = 1$ means exactly that 1 is a coloop of M.

Now assume $E = R \cup S$ is a separation of E. Let $c = \mathrm{rk}(R)$ and $d = \mathrm{rk}(S)$. I.e. $c + d = r$. If R intersected more than c blocks, then $\mathrm{rk}(R)$ would exceed c. Therefore R intersects at most c blocks. Similarly S intersects at most d blocks. But there are $r = c + d$ blocks in total, so R must intersect exactly c blocks and S must intersect exactly d blocks. Thus $R = B_{i_1} \cup \dots \cup B_{i_c}$ and $S = B_{j_1} \cup \dots \cup B_{j_d}$. So without loss of generality, $B_r \subseteq R$. Choosing one element from each B_{i_l} results in an independent set, I, of cardinality c. If $|B_r| > 1$, then one could augment this set with a generic vector of height r. Since $c < r$, the resulting set would be a rank $c+1$ independent set contained in R. This contradicts the fact that $\mathrm{rk}(R) = c$. Therefore $|B_r| = 1$. $\square$

Theorem 3.9. *Let $M = ([n], \langle\langle a_1 \dots a_r \rangle\rangle)$ be a loopless and coloopless shifted matroid and $k \in [r]$.*

1. *T_k is a flat of M.*

2. *T_1 is inseparable.*

3. *If $k > 1$: T_k is inseparable if and only if $|B_k| > 1$.*

Proof. 1. The first fact is easy to see from the generic matrix representation of M (see Theorem 3.2). T_k is the set of elements of height at most k. Any element outside of T_k has height at least $k+1$ and so by genericity will increase the rank when added to T_k.

2. Since M is loopless and $\mathrm{rk}(T_1) = 1$, we have that $\mathrm{rk}(R) = 1$ for any non-empty subset $R \subseteq T_1$. Therefore $\mathrm{rk}(R) + \mathrm{rk}(S) = 2 > \mathrm{rk}(T_1)$ for any non-empty R, S partitioning T_1.

3. Assume $k > 1$. Consider $M' = M|_{T_k}$. M' is a rank k shifted matroid with independence complex $\langle\langle a_{r-k+1} \dots a_r \rangle\rangle$ on ground set T_k. The k^{th} block of this shifted matroid is B_k. By Lemma 3.8, M' is inseparable if and only if $|B_k| > 1$. But M' is inseparable if and only if T_k is inseparable in M.

□

Corollary 3.10. *Let $M = ([n], \langle\langle a_1 \dots a_r \rangle\rangle)$ be a loopless and coloopless shifted matroid. $\varnothing \neq F \subseteq E$ is an inseparable flat of M if and only if one of the following hold:*

1. $F = \{x\}$ where $x \notin T_1$

2. $F = T_1$

3. $F = T_k$ for some k with $|B_k| > 1$.

Proof. ($\Longrightarrow$) Assume F is an inseparable flat.

If $|F| > 1$, then by Theorem 3.7, for some k, we have $F = T_k$ with $|B_k| > 1$.

If $|F| = 1$, then $F = \{x\}$ for some $x \in E$. If $x \notin T_1$, then we are done. Suppose $x \in T_1$. Since $\mathrm{rk}(T_1) = 1 = \mathrm{rk}(F)$, we have that F spans T_1. But F is a flat, so F must contain T_1. Therefore $F = T_1$.

($\Longleftarrow$) If $F = T_1$ or $F = T_k$ with $|B_k| > 1$, then by Theorem 3.9, F is an inseparable flat. If $F = \{x\}$ for $x \notin T_1$, then clearly F is inseparable. Since $x \notin T_1$, the height of x is at least 2. Therefore $\mathrm{rk}(F \cup \{y\}) = 2$ for all $y \neq x$. So F is a flat. □

Recall that a flat F is a *flacet* if $\varnothing \neq F \neq E$, F is inseparable, and F^c is inseparable in M^*. Using the previous results leads to a characterization of the flacets of a shifted matroid.

Theorem 3.11. *Let $M = ([n], \langle\langle a_1 \dots a_r \rangle\rangle)$ be a loopless and coloopless shifted matroid.*

1. *Let $F \neq E$ be an inseparable flat with $|F| \geq 2$. Then F is a flacet.*

2. (a) *If $a_{r-1} = n-1$, then $[n] \setminus \{x\}$ is inseparable in M^* for all $x \in [n]$.*

   (b) *If $a_{r-1} \neq n-1$, then $[n] \setminus \{x\}$ is inseparable in M^* if and only if $x \neq n-1, n$.*

   Consequently, any singleton flat is a flacet.

Proof. Note that $M^* = \langle\!\langle b_1 \dots b_{n-r} \rangle\!\rangle^*$ where $\{b_1, \dots, b_{n-r}\} = [n] \setminus \{a_1, \dots, a_r\}$. Since M is loopless and coloopless, so is M^*. For ease of notation, define $a_0 := 0$ and $b_{(n-r)+1} := n+1$.

Statement 1: Let $F \neq E$ be an inseparable flat with $|F| \geq 2$. By Theorem 3.7, $F = T_k$ where $|B_k| > 1$. Since $F \neq E$, we have $k < r$. Because $|B_k| > 1$, it must be that $a_k > a_{k-1} + 1$. Consequently, $a_{k-1} + 1 = b_i$ for some $i \in [n-r]$. The blocks of M are of the form $B_j = \{a_{j-1}+1, \dots, a_j\}$ for $j = 1, \dots, r$. The blocks of M^* are of the form $B^*_{(n-r)-j+1} = \{b_j, \dots, b_{j+1} - 1\}$ for $j = 1, \dots, n-r$. So

$$T_k = \bigcup_{j=i}^{n-r} B^*_{(n-r)-j+1} \quad \text{and} \quad T_k^c = \bigcup_{j=1}^{i-1} B^*_{(n-r)-j+1} = \bigcup_{j=(n-r)-(i-1)+1}^{n-r} B^*_j.$$

So T_k^c is a terminal segment in M^*. By Theorem 3.9, T_k^c is inseparable in M^* if $|B^*_{(n-r)-(i-1)+1}| > 1$. This is indeed the case since $B^*_{(n-r)-(i-1)+1} = \{b_{i-1}, \dots, b_i - 1\}$ and $b_i - 1 = a_{k-1} \neq b_{i-1}$.

Statement 2: Let $x \in [n]$. Consider $B^*_1 = \{b_{n-r}, \dots, n\}$, the first block of M^*. The first block of $M^*|_{[n]\setminus\{x\}}$ is then $B^*_1 \setminus \{x\}$. Therefore $[n] \setminus \{x\}$ is inseparable in the dual if and only if $|B^*_1 \setminus \{x\}| > 1$. If $a_{r-1} = n-1$, then $b_{n-r} < n-1$ and so $|B^*_1| \geq 3$ and so this holds for all x. Otherwise $B^*_1 = \{n-1, n\}$ and so $|B^*_1 \setminus \{x\}| > 1$ if and only if $x \notin \{n-1, n\}$. $\square$

We are now prepared to state the pièce de résistance of this section.

Theorem 3.12. *Let $M = ([n], \langle\!\langle a_1 \dots a_r \rangle\!\rangle)$ be a loopless and coloopless shifted matroid. Let B_k and T_k be as in Definition 3.5.*

The following is a system of inequalities for P_M:

- $x_1 + \cdots + x_n = rk(M)$

- $x_i \geq 0$ *for all* $i \in [n]$
- $x_i \leq 1$ *for all* $i \notin T_1$
- $\sum_{i \in T_1} x_i \leq 1$
- $\sum_{i \in T_k} x_i \leq k$ *where* $|B_k| > 1$.

The following is a minimal system of inequalities for Q_M*:*

- $x_i \geq 0$ *for all* $i \in [n]$
- $x_i \leq 1$ *for all* $i \notin T_1$
- $\sum_{i \in T_1} x_i \leq 1$
- $\sum_{i \in T_k} x_i \leq k$ *where* $|B_k| > 1$.

Proof. Regarding P_M, the first bullet point comes from 1. in Theorem 2.4, the second bullet point comes from 2., and the final three bullet points come from 3.

Regarding Q_M, the first bullet point comes from 1. while the final three come from 2. in Theorem 2.5. □

3.3 Shifted Matroids are Positroids

Definition 3.13. [2] Let X be an $r \times n$ matrix with real entries such that all maximal minors are non-negative. Such a matrix is called *totally non-negative* and the vector matroid $M(X)$ is called a *positroid*.

Positroids were first introduced by Alexander Postnikov in [22] (though he did not use the term positroid) where he linked them to the study of planar directed networks. Postnikov gave bijections among positroids, grassmann necklaces, and decorated permutations. We will employ the following technical lemma to prove that shifted matroids are positroids.

Before stating the lemma, we define the $t-cyclic$ ordering on $[n]$ by $t <_t t+1 <_t \cdots <_t n <_t 1 <_t \cdots <_t t-1$. You may think of $<_t$ as the ordering obtained from the usual ordering on $[n]$ by "rotating" $[n]$ until t is the smallest element.

Lemma 3.14. *[19] Let M be a matroid of rank k on ground set $[n]$. Let $\mathscr{B}(M)$ denote the set of bases of M. M is a positroid if and only if it satisfies the following condition:*

Let W be any $k-2$ element independent set of $[n]$. For each $a,b,c,d \in [n] \setminus W$ such that $a <_t b <_t <_c <_d$ for some $t \in [n]$, the following relation holds. $W \cup \{a,c\}, W \cup \{b,d\} \in \mathscr{B}(M)$ if and only if $W \cup \{a,b\}, W \cup \{c,d\} \in \mathscr{B}(M)$ or $W \cup \{a,d\}, W \cup \{b,c\} \in \mathscr{B}(M)$.

Theorem 3.15. *Shifted matroids are positroids.*

Proof. Let M be a shifted matroid of rank k on ground set $[n]$. Let $W \subseteq [n]$ be an independent set of size $k-2$. Suppose $a,b,c,d \in [n] \setminus W$ are such that $a <_t b <_t c <_t d$ for some $t \in [n]$. By Lemma 3.14 it is enough to show that

$$W \cup \{a,c\}, W \cup \{b,d\} \in \mathscr{B}(M) \iff W \cup \{a,b\}, W \cup \{c,d\} \in \mathscr{B}(M) \text{ or } W \cup \{a,d\}, W \cup \{b,c\} \in \mathscr{B}(M).$$

Notice that the conditions on both sides are invariant under cyclically shifting a,b,c,d. Therefore WLOG assume $a < b < c < d$.

$(\Rightarrow)$ If $W \cup \{a,c\}, W \cup \{b,d\} \in \mathscr{B}(M)$ then by applying the definition of shiftedness to $W \cup \{b,d\}$ we see that $W \cup \{a,d\}$ and $W \cup \{b,c\}$ are in $\mathscr{B}(M)$.

$(\Leftarrow)$ Suppose $W \cup \{a,b\}, W \cup \{c,d\} \in \mathscr{B}(M)$. Then by applying the definition of shiftedness to $W \cup \{c,d\}$ we see that $W \cup \{a,c\}$ and $W \cup \{b,d\}$ are in $\mathscr{B}(M)$.

Suppose $W \cup \{a,d\}, W \cup \{b,c\} \in \mathscr{B}(M)$. Applying the shifted property to $W \cup \{b,c\}$ gives that $W \cup \{a,c\} \in \mathscr{B}(M)$. Since M is a shifted matroid, $\mathscr{B}(M)$ is a principal order ideal in the poset $\binom{[n]}{k}$ under $\preceq$ [16]. Therefore the join of $W \cup \{a,d\}$ and $W \cup \{b,c\}$, $W \cup \{b,d\}$, must be in $\mathscr{B}(M)$. $\square$

Initially, I had hopes of using the structure of positroids to help study shifted matroids. In [20] hyperplane descriptions are presented for the independence and base polytopes of positroids. These hyperplane descriptions come from certain structures called *counter-clockwise arrows* on the *decorated permutation* corresponding to the positroid in question. However, this point of view was unhelpful in the study of the structure of shifted matroids.

Chapter 4

Truncated Cubes

Let $C(n)$ denote the $0/1$ cube in $\mathbb{R}^n$. That is, $C(n) = [0,1]^n = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : \ 0 \le x_i \le 1\}$.

Definition 4.1. Let r be a non-negative integer. By slicing $C(n)$ with the hyperplane $\sum_{i=1}^{n} x_i = r$, we split the cube into a lower piece

$$TC(r,n) = \{(x_1,\ldots,x_n) \in C(n) \mid \sum_{i=1}^{n} x_i \le r\}.$$

We will call $TC(r,n)$ a *truncated cube*.

Example 4.2. $TC(1,n)$ is an n-simplex. $TC(n,n)$ is the $0/1$ cube in $\mathbb{R}^n$.

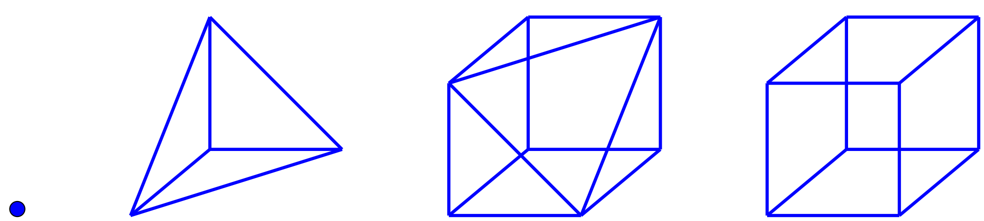


Figure 4.1: $TC(r,3)$ for $r = 0,1,2,3$

Because the vertices of $Q_{U(r,n)}$ satisfy the required inequalities defining $TC(r,n)$, the matroid independence polytope of $U(r,n)$ is contained in $TC(r,n)$. The reverse inclusion also holds and is proven in the following theorem.

Theorem 4.3. *The truncated cube $TC(r,n)$ is the independence polytope of $U(r,n)$.*

Proof. Note that $U(r,n) = \langle\!\langle (n-r+1)(n-r+2)\ldots n \rangle\!\rangle$. By Corollary 3.10, the inseparable flats of $U(r,n)$ are: $\varnothing,[n]$ and $\{i\}$ for each $i\in[n]$. Therefore by Theorem 3.12, a minimal system of hyperplanes describing $Q_{U(r,n)}$ are $0\leq x_i\leq 1$ for each $i\in[n]$ and $\sum\limits_{i\in[n]} x_i\leq r$. These inequalities precisely describe $TC(r,n)$. $\square$

Using the description of $TC(r,n)$ established in the previous theorem, we can compute the Ehrhart polynomial of $TC(2,n)$ using the known formula for the Ehrhart polynomials of simplices, together with Inclusion-Exclusion.

Proposition 4.4. *The Ehrhart polynomial of $TC(2,n)$ is $\binom{2t+n}{n}-n\binom{t+n-1}{n}$.*

Proof. Let $T_k^n=\mathrm{conv}(0,ke_1,\ldots,ke_n)$. That is, T_k^n is the k^{th} dilate of the full dimensional standard simplex in $\mathbb{R}^n$. Let $B=\{(x_1,\ldots,x_n)\in T_2^n\mid x_i>1 \text{ for some } i\in[n]\}$ and $\overline{B}$ be its closure. Here is a picture in the case of $n=3$:

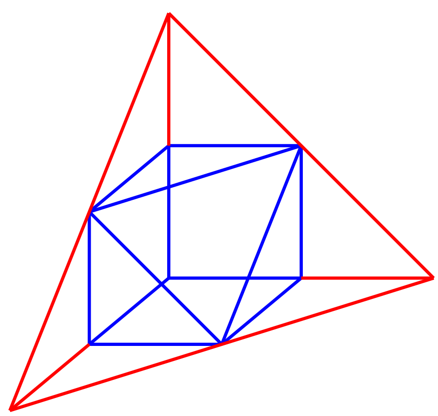


Figure 4.2: T_2^3 (red) with $TC(2,3)$ sitting inside (blue).

B is the union of the three red, half-open tetrahedra.

By Theorem 4.3, $TC(2,n)=T_2^n-B$. Therefore $tTC(2,n)=tT_2^n-tB$. So

$$i(TC(2,n),t)=i(T_2^n,t)-i(B,t)=i(T_1^n,2t)-i(B,t)=\binom{2t+n}{n}-i(B,t).$$

Now, B is the disjoint union of n half-open regions B_i where $\overline{B_i} = e_i + T_1^n$. Note that

$$\overline{B_i} \cap TC(2,n) = \{(x_1,\dots,x_n) \mid x_i = 1,\ \sum_{j\neq i} x_j \leq 1,\ \text{and } 0 \leq x_j \leq 1 \text{ for all } j \neq i\}.$$

By 4.3, this is affinely isomorphic to $Q_{U(1,n-1)}$ which is a full dimensional $n-1$ simplex. Therefore $|t(\overline{B_i} \cap TC(2,n)) \cap \mathbb{Z}^n| = \binom{t+n-1}{n-1}$. Hence

$$\begin{aligned}
|tB \cap \mathbb{Z}| &= \sum_{i=1}^{n} |tB_i \cap \mathbb{Z}| \\
&= \sum_{i=1}^{n} \left(|t\overline{B_i} \cap \mathbb{Z}^n| - |t(\overline{B_i} \cap TC(2,n)) \cap \mathbb{Z}|\right) \\
&= \sum_{i=1}^{n} \left(\binom{t+n}{n} - \binom{t+n-1}{n-1}\right) \\
&= \sum_{i=1}^{n} \binom{t+n-1}{n} \\
&= n\binom{t+n-1}{n}.
\end{aligned}$$

Finally combining this with Equation 4 we see that

$$i(TC(2,n),t) = \binom{2t+n}{n} - n\binom{t+n-1}{n}.$$

□

The leading coefficient of the Ehrhart polynomial is the normalized volume of the polytope. The leading coefficient of $i(TC(2,n),t)$ is $\frac{2^n-n}{n!}$ and so we obtain the following corollary.

Corollary 4.5. *The volume of $TC(2,n)$ is $\frac{2^n-n}{n!}$.*

The previous argument can be made more general by viewing $TC(r,n)$ as sitting inside a properly scaled simplex. This provides a recursive formula for the Ehrhart polynomial of $TC(r,n)$. While this formula is unwieldy, it led to the development of a non-recursive formula (Theorem 4.9).

Theorem 4.6. *The Ehrhart polynomial of $TC(r,n)$ is*

$$\binom{rt+n}{n} + \sum_{k=1}^{r} (-1)^k \binom{n}{k} \left[\binom{(r-k)t+n}{n} - i(TC(r-k,n-k),t) \right].$$

Proof. Given $S \subseteq [n]$, let

$$B_S = \{(x_1,\ldots,x_n) \in T_r^n \mid x_i > 1,\ \forall i \in S\} \text{ and}$$

$$C_S = \{(x_1,\ldots,x_n) - \chi_S \mid (x_1,\ldots,x_n) \in B_S\}.$$

Note that $t(T_r^n \setminus TC(r,n)) = \bigcup_{\emptyset \neq S \subset [n]} tB_S$. Therefore

$$i(T_r^n \setminus TC(r,n),t) = \sum_{\substack{\emptyset \neq S \subseteq [n] \\ |S|<r}} (-1)^{|S|-1} i(B_S,t).$$

Let $R_S = \overline{B_S} \cap TC(r,n)$. Then $i(B_S,t) = i(\overline{B_S},t) - i(R_S,t) = i(\overline{C_S},t) - i(R_S,t)$.

Claim: $\overline{C_S} = T^n_{r-|S|}$.

Suppose $(x_1,\ldots,x_n) \in \overline{B_S}$. Since $\sum_{i\in[n]} x_i \leq r$ and $x_i \geq 1$ for $i \in S$, we have that $1 \leq x_i \leq r-|S|+1$ for $i \in S$, and $0 \leq x_j \leq r-|S|$ for $j \notin S$. Now consider $(y_1,\ldots,y_n) = (x_1,\ldots,x_n) - \chi_S$. By the above observation, $0 \leq y_i \leq r-|S|$ for all $i \in [n]$ and $\sum_{i\in[n]} y_i \leq r-|S|$. Hence,

$$\overline{C_S} = \{(y_1,\ldots,y_n) \mid \sum_{i\in[n]} y_i \leq r-|S|, \text{ and } 0 \leq y_i \leq r-|S|\} = T^n_{r-|S|}.$$

With this in hand, we see that

$$i(\overline{C_S},t) = \binom{(r-|S|)t+n}{n}.$$

We now turn our attention to computing $i(R_S,t)$:

$$\begin{aligned}R_S &= \{(x_1,\ldots,x_n) \in TC(r,n) \mid x_i = 1, \forall i \in S\}\\ &\cong_{\text{aff}} T^{n-|S|}_{r-|S|} \cap [0,1]^{n-|S|}\\ &= Q_{U(r-|S|,n-|S|)}\end{aligned}$$

Where $\cong_{\text{aff}}$ means affinely isomorphic.

This observation yields $i(R_S,t) = i(Q_{U(r-|S|,n-|S|)},t)$.

Finally we have that:

$$\begin{aligned}i(TC(r,n),t) &= i(T^n_r,t) - i(T^n_r \setminus TC(r,n),t)\\ &= \binom{rt+n}{n} - \sum_{\substack{\emptyset\neq S\subseteq[n]\\ |S|<r}} (-1)^{|S|-1} i(B_S,t)\\ &= \binom{rt+n}{n} - \sum_{\substack{\emptyset\neq S\subseteq[n]\\ |S|<r}} (-1)^{|S|-1} [i(\overline{C_S},t) - i(R_S,t)]\\ &= \binom{rt+n}{n} + \sum_{\substack{\emptyset\neq S\subseteq[n]\\ |S|<r}} (-1)^{|S|} \left[\binom{(r-|S|)t+n}{n} - i(Q_{U(r-|S|,n-|S|)},t)\right]\\ &= \binom{rt+n}{n} + \sum_{k=1}^{r} (-1)^k \binom{n}{k} \left[\binom{(r-k)t+n}{n} - i(Q_{U(r-k,n-k)},t)\right].\end{aligned}$$

$\square$

Again, we can pick off the leading coefficient of $i(TC(r,n),t)$ which yields the following corollary.

Corollary 4.7. *The volume of $TC(r,n)$ is*

$$\frac{\sum_{k=0}^{r-1} (-1)^k \binom{n}{k} (r-k)^n}{n!}.$$

The recursive nature of the formula presented in Theorem 4.6 leaves much to be desired. Using SAGE and this recursion, the following was conjectured:

$$i(TC(r,n),t) = \sum_{k=0}^{r-1} (-1)^k \binom{n}{k} \binom{(r-k)t-k+n}{n}.$$

This formula does, in fact, hold. Before presenting the proof, we present a useful lemma.

Lemma 4.8. *Let $A,B \in \mathbb{N}$. Then for any $m \in \mathbb{N}$,*

$$\sum_{i=0}^{m} \binom{i-A+B-1}{B-1} = \binom{m-A+B}{B}$$

Proof. Note that for $i < A$, $\binom{i-A+B-1}{B-1} = 0$, so the result holds if m < A. In the case of $m = A$, the equality holds since $\binom{B-1}{B-1} = 1 = \binom{B}{B}$. Suppose that the equation holds for some $m \geq A$ Then

$$\begin{aligned}
\sum_{i=0}^{m+1} \binom{i-A+B-1}{B-1} &= \sum_{i=0}^{m} \binom{i-A+B-1}{B-1} + \binom{m-A+B}{B-1} \\
&= \binom{m-A+B}{B} + \binom{m-A+B}{B-1} \\
&= \binom{m-A+B+1}{B} \\
&= \binom{m+1-A+B}{B}
\end{aligned}$$

By induction, the formula holds for all $m \in \mathbb{N}$.

□

Theorem 4.9. *The Ehrhart polynomial of $TC(r,n)$ is*

$$i(TC(r,n),t) = \sum_{k=0}^{r-1} (-1)^k \binom{n}{k} \binom{(r-k)t-k+n}{n}.$$

Proof. Integer points in the t^{th} dilate of $TC(r,n)$ correspond to monomials $x_1^{a_1} \dots x_n^{a_n}$ of degree at most tr with $0 \leq a_i \leq t$. Let $[x^i]f(x)$ denote the coefficient of x^i in the formal power series $f(x)$.

Then

$$\begin{aligned}
i(TC(r,n),t) &= \sum_{i=0}^{tr}[x^i](1+x+\cdots+x^t)^n \\
&= \sum_{i=0}^{tr}[x^i]\left(\frac{1-x^{t+1}}{1-x}\right)^n \\
&= \sum_{i=0}^{tr}[x^i]\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{(t+1)k}\right)\left(\sum_{j=0}^{\infty}x^j\binom{j+n-1}{n-1}\right)
\end{aligned}$$

By setting $j = i-(t+1)k$,

$$\begin{aligned}
&= \sum_{i=0}^{tr}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{i-tk-k+n-1}{n-1} \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{i=0}^{tr}\binom{i-tk-k+n-1}{n-1}
\end{aligned}$$

By Lemma 4.8,

$$\begin{aligned}
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{tr-tk-k+n}{n} \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{(r-k)t-k+n}{n}.
\end{aligned}$$

□

Definition 4.10. The hypersimplex $HS(r,n)$ is defined as

$$HS(r,n) = \{(x_1,\ldots,x_n)\in\mathbb{R}^n \mid \sum_{i=1}^{n}x_i = r \text{ and } 0\le x_i\le 1\ \forall i\in[n]\}.$$

$HS(r,n)$ is the base polytope of $U(r,n)$. A formula for the Ehrhart polynomial was previously known and bears a striking similarity to the Ehrhart polynomial of $TC(r,n)$.

Theorem 4.11. *[13] The Ehrhart polynomial of the hypersimplex $HS(r,n)$ is*

$$\sum_{k=0}^{r-1}(-1)^k\binom{n}{k}\binom{(r-k)t-k+n-1}{n-1}.$$

The following result gives a curious connection between the normalized volumes of the two polytopes.

Theorem 4.12. *Let $V(r,n)$ denote the normalized volume of $TC(r,n)$. Then $\frac{\partial V}{\partial r}$ is the normalized volume of $HS(r,n)$.*

Proof. The normalized volume of a polytope is the leading coefficient of its Ehrhart polynomial. Thus,

$$V(r,n) = \frac{1}{n!}\sum_{k=0}^{r-1}(-1)^k\binom{n}{k}(r-k)^n.$$

Then,

$$\begin{aligned}\frac{\partial V}{\partial r} &= \frac{n}{n!}\sum_{k=0}^{r-1}(-1)^k\binom{n}{k}(r-k)^{n-1}\\ &= \frac{1}{(n-1)!}\sum_{k=0}^{r-1}(-1)^k\binom{n}{k}(r-k)^{n-1}.\end{aligned}$$

The above is the leading coefficient of $i(HS(r,n),t)$. □

This relationship is somewhat intuitive because hypersimplices are the "outer-most" face of truncated cubes so slightly perturbing r should change the volume roughly in proportion with the volume of the hypersimplex.

We will now give a formula for the h^* polynomial of $TC(r,n)$.

Definition 4.13. The *hyperslab* $S(r,n)$ is defined as

$$S(r,n) = \{(x_1,\ldots,x_n) \in [0,1]^n \mid r-1 \leq \sum_{i=1}^{n} x_i \leq r\}$$

The *half-open hyperslab* $S'(r,n)$ is defined as

$$S'(r,n) = \{(x_1,\ldots,x_n) \in [0,1]^n \mid r-1 < \sum_{i=1}^{n} x_i \leq r\}.$$

Theorem 4.14. *[18] The h^*-polynomial of $S'(r,n)$ is*

$$\sum_{\substack{w\in\mathfrak{S}_n\\ exc(w)=r-1}} t^{des(w)}.$$

Theorem 4.15. *The h^*-polynomial of $TC(r,n)$ is*

$$\sum_{\substack{w\in\mathfrak{S}_n\\ exc(w)\leq r-1}} t^{des(w)}.$$

Proof. Note that $TC(r,n)$ is the disjoint union of half-open hyperslabs, $TC(r,n) = \bigcup_{k=1}^{r} S'(k,n)$. Therefore the h^*-polynomial of $TC(r,n)$ is the sum of the h^*-polynomials of the $S'(k,n)$. These are computed in [18] as

$$\sum_{\substack{w\in\mathfrak{S}_n\\ \mathrm{exc}(w)=k-1}} t^{\mathrm{des}(w)}.$$

□

Note that this theorem implies that the degree of the h^* polynomial of $TC(r,n)$ is $n-1$ for $r > \lfloor \frac{n}{2} \rfloor$ since $\mathrm{des}([n\; n-1\; \ldots\; 2\; 1]) = n-1$ and $\mathrm{exc}([n\; n-1\; \ldots\; 2\; 1]) = \lfloor \frac{n}{2} \rfloor$.

4.1 Roots of the Ehrhart Polynomials of Truncated Cubes

We will now turn our attention to studying the roots of the Ehrhart polynomials of truncated cubes. Later in this section we will present a conjecture about the location of roots of the Ehrhart polynomials of truncated cubes. Together with the following elementary lemma, the conjecture would imply that the coefficients of these polynomials are positive.

Lemma 4.16. *Let $f(x)$ be a polynomial with real coefficients. If $Re(z) < 0$ for all roots of $f(x)$, then $f(x)$ has positive coefficients.*

Proof. Suppose $Re(z) < 0$ for all roots of $f(x)$. If f has degree 1, then $f(x) = x - a$ for some $a \in \mathbb{R}$. Then a the root of f so $a = Re(a) < 0$. So the coefficients of f are positive.

Suppose f has degree 2. If f has two real roots, a and b. Then $a = Re(a) < 0$ and $b = Re(b) < 0$. So $f(x) = (x-a)(x-b) = x^2 + (-a-b)x + ab$ and we see that f has positive coefficients. If $f(x)$ is irreducible, then the roots of f are $a+bi$ and $a-bi$ for some $a, b \in mathbbR$. So $f(x) = (x-(a+bi))(x-(a-bi)) = x^2 - 2ax + |a+bi|^2$ and we see that f has positive coefficients.

If f has degree greater than 2, we may factor f into a product of linear and irreducible quadratics. The roots of each factor are roots of f and by the previous verifications, each factor will have positive coefficients. Therefore f will have positive coefficients. $\square$

The first main result of this section concerns the integer roots of Ehrhart polynomials of any independence polytope.

Theorem 4.17. *Let M be a loopless and co-loopless matroid. Then the integer roots of $i(Q_M, t)$ are $-1, \ldots, -q$, where $q = \max\{\lfloor \frac{|F|}{r(F)} \rfloor \mid F$ is a non-empty inseparable flat of $M\}$.*

Proof. Let $k \in \mathbb{Z}_{\geq 0}$. It is clear that $i(Q_M, k) > 0$. By Ehrhart reciprocity, $i(Q_M, -k)$ is the number of lattice points in the interior of the k^{th} dilate. As $kQ_M \subseteq \ell Q_M$ for $k \leq \ell$, and Q_M contains no interior lattice point, the integer roots of $i(Q_M, t)$ must form an interval $-1, \ldots, -q$ where $q+1$ is the smallest integer such that $(q+1)Q_M$ contains an interior lattice point. Note that the all ones point, $\mathbb{1}$, must be an interior point of $(q+1)Q_M$ (since Q_M and its dilates are geometrically shifted). The facets of Q_M that are not of the form $x_i = 0$ must be of the form $\sum_{i \in F} x_i \leq r(F)$ where F is a non-empty inseparable flat of M. Thus $|F| = \mathbb{1} \cdot \chi_F < (q+1)r(F)$ for each non-empty inseparable flat F. As well, $qr(F) \leq \mathbb{1} \cdot \chi_F = |F|$ for each non-empty inseparable flat F. So

$$q \leq \frac{|F|}{r(F)} < q+1 \quad \text{for all non-empty inseparable flats } F.$$

Thus $q = \max\{\lfloor \frac{|F|}{r(F)} \rfloor \mid F$ is a non-empty inseparable flat of $M\}$. $\square$

Corollary 4.18. *The integer roots of $i(Q_{\langle\langle a_1 \ldots a_r \rangle\rangle}, t)$ are located at $-j$ for $j = 1, \ldots, q$, where $q = \max\{\frac{\lfloor |T_k|}{k} \rfloor \mid k = 1, \ldots, r\}$.*

This corollary may seem out of left field since we have not mentioned the Ehrhart polynomials of shifted matroids beforehand. I have thought about these polynomials extensively, and have computed many examples. However, I was unable to come up with a unified theory describing these polynomials. I hope in the future that someone might find a way to do so. These polytopes are beautiful and elusive. Perhaps when viewed under a different lens than my own, someone might discover the pattern.

Corollary 4.19. *The integer roots of $i(TC(r,n),t)$ are located at $-j$ for $j = 1, \ldots, \lfloor n/r \rfloor$.*

Conjecture 4.20. *Let $z \in \mathbb{C}$ be a root of the Ehrhart polynomial of $TC(r,n)$. Then*

$$Re(z) < 0.$$

A consequence of this conjecture would be that the coefficients of $i(TC(r,n),t)$ are positive, a fact that is not at all obvious from the formula given in Theorem 4.9. The following theorem establishes the conjecture in the case of $r = 2$.

Theorem 4.21. *Let z be a root of the Ehrhart polynomial of $TC(2,n)$. Then $Re(z) < 0$.*

Proof. Define

$$f_n(z) = n!\binom{2z+n}{n} = (2z+n)(2z+n-1)\ldots(2z+1)$$

and

$$g_n(z) = n!n\binom{z-1+n}{n} = n(z+n-1)(z+n-2)\ldots(z).$$

Then $i(TC(2,n),z) = (\frac{1}{n!}(f_n(z) - g_n(z))$. So $i(TC(2,n),z) = 0$ precisely when $f_n(z) = g_n(z)$. Observe that $z = 0$ is not a root of $i(TC(2,n),t)$ since zero is never a root of an Ehrhart polynomial. Assume $Re(z) \geq 0$ and $z \neq 0$; we will show that $|f_n(z)| > |g_n(z)|$.

Define $\varphi_j = 2z + j$ and $\gamma_j = z + j - 1$. Then $f_n(z) = \prod_{j=1}^{n} \varphi_j$ and $g_n(z) = n \prod_{j=1}^{n} \gamma_j$. Observe that

$$\begin{aligned}
|\varphi_j|^2 &= (2z + j)(2\bar{z} + j) \\
&= 4|z|^2 + 4\operatorname{Re}(z)j + j^2, \\
|\gamma_j|^2 &= (z + j - 1)(\bar{z} + j - 1) \\
&= |z|^2 + 2\operatorname{Re}(z)(j-1) + (j-1)^2.
\end{aligned}$$

We will now show that

$$\frac{|\varphi_j|}{|\gamma_j|} \geq \frac{j^2}{(j-1)^2}$$

for $j \geq 2$.

Let $A = 4|z|^2, B = 4\operatorname{Re}(z)j, C = j^2, D = |z|^2, E = 2\operatorname{Re}(z)(j-1),$ and $F = (j-1)^2$.

Then

$$
\begin{aligned}
F\frac{j^2}{(j-1)^2} &= (j-1)^2\frac{j^2}{(j-1)^2} \\
&= C, \\
E\frac{j^2}{(j-1)^2} &= 2\,\mathrm{Re}(z)(j-1)\frac{j^2}{(j-1)^2} \\
&= 4\,\mathrm{Re}(z)\,j\left(\frac{j}{2(j-1)}\right) \\
&\leq 4\,\mathrm{Re}(z)\,j \\
&= B, \\
D\frac{j^2}{(j-1)^2} &= |z|^2\frac{j^2}{(j-1)^2} \\
&\leq 4|z|^2 \\
&= A.
\end{aligned}
$$

Therefore for $j \geq 2$

$$
\frac{|\varphi_j|^2}{|\gamma_j|^2} = \frac{A+B+C}{D+E+F} \geq \frac{\frac{j^2}{(j-1)^2}(D+E+F)}{D+E+F} = \frac{j^2}{(j-1)^2}.
$$

Note that $|\varphi_1|^2 \geq 4|z|^2 = 4|\gamma_1|^2$. Therefore

$$
\frac{|f_n(z)|^2}{|g_n(z)|^2} = \frac{1}{n}\prod_{j=1}^{n}\frac{|\varphi_j|^2}{|\gamma_j|^2} \geq \frac{1}{4n}\prod_{j=2}^{n}\frac{j^2}{(j-1)^2} = \frac{1}{4n}\frac{n!^2}{(n-1)!^2} = \frac{n^2}{4n} = \frac{n}{4} > 1 \quad \text{if } n \geq 5.
$$

This proves the result for $n \geq 5$. We now verify the result for $n = 1, 2, 3, 4$.

$n = 1$:

$$\begin{aligned} |f_1(z)|^2 &= 4|z|^2 + 4\,\mathrm{Re}(z) + 1 \\ &> |z|^2 \\ &= |g_1(z)|^2. \end{aligned}$$

$n = 2$:

$$\begin{aligned} |f_2(z)|^2 &= (4|z|^2 + 4\,\mathrm{Re}(z) + 1)(4|z|^2 + 8\,\mathrm{Re}(z) + 4) \\ &= 16|z|^4 + 48\,\mathrm{Re}(z)|z|^2 + 20|z|^2 + 32\,\mathrm{Re}(z)^2 + 24\,\mathrm{Re}(z) + 4 \\ &> 2|z|^4 + 4\,\mathrm{Re}(z)|z|^2 + 2|z|^2 \\ &= |g_2(z)|^2. \end{aligned}$$

$n = 3$:

$$\begin{aligned} |f_3(z)|^2 = &\ 64|z|^6 + 384\,\mathrm{Re}(z)|z|^4 + 224|z|^4 + 704\,\mathrm{Re}(z)^2|z|^2 + 768\,\mathrm{Re}(z)|z|^2 + 196|z|^2 + 384\,\mathrm{Re}(z)^3 \\ &+ 576\,\mathrm{Re}(z)^2 + 264\,\mathrm{Re}(z) + 36 \\ &> 3|z|^6 + 18\,\mathrm{Re}(z)|z|^4 + 15|z|^4 + 24\,\mathrm{Re}(z)^2|z|^2 + 36\,\mathrm{Re}(z)|z|^2 + 12|z|^2 \\ &= |g_3(z)|^2 \end{aligned}$$

$n = 4$:

$$
\begin{aligned}
|f_4(z)|^2 &= 256|z|^8 + 2560\,\mathrm{Re}(z)|z|^6 + 1920|z|^6 + 8960\,\mathrm{Re}(z)^2|z|^4 + 12800\,\mathrm{Re}(z)|z|^4 + 4368|z|^4 \\
&\quad + 12800\,\mathrm{Re}(z)^3|z|^2 + 25856\,\mathrm{Re}(z)^2|z|^2 + 16480\,\mathrm{Re}(z)|z|^2 + 3280|z|^2 + 6144\,\mathrm{Re}(z)^4 \\
&\quad + 15360\,\mathrm{Re}(z)^3 + 13440\,\mathrm{Re}(z)^2 + 4800\,\mathrm{Re}(z) + 576 \\
&> 4|z|^8 + 48\,\mathrm{Re}(z)|z|^6 + 56|z|^6 + 176\,\mathrm{Re}(z)^2|z|^4 + 384\,\mathrm{Re}(z)|z|^4 + 196|z|^4 + 192\,\mathrm{Re}(z)^3|z|^2 \\
&\quad + 576\,\mathrm{Re}(z)^2|z|^2 + 528\,\mathrm{Re}(z)|z|^2 + 144|z|^2 \\
&= |g_4(z)|^2
\end{aligned}
$$

Hence $f_n(z) \neq g_n(z)$. So z is not a root of $i(TC(2,n),t)$.

$\square$

Corollary 4.22. *The coefficients of $i(TC(2,n),t)$ are non-negative.*

One could try to employ the previous method to prove Conjecture 4.20 for other values of r. However, this argument is very ad hoc in nature, and it is even difficult to make it work for $r = 3$. So there seems to be little hope for this method generalizing to arbitrary r.

Conjecture 4.20 was made based upon computing roots of hundreds of Ehrhart polynomials of various truncated cubes. Plotting the roots in the complex plane yield rather beautiful and intriguing pictures. I will include some of these pictures below.

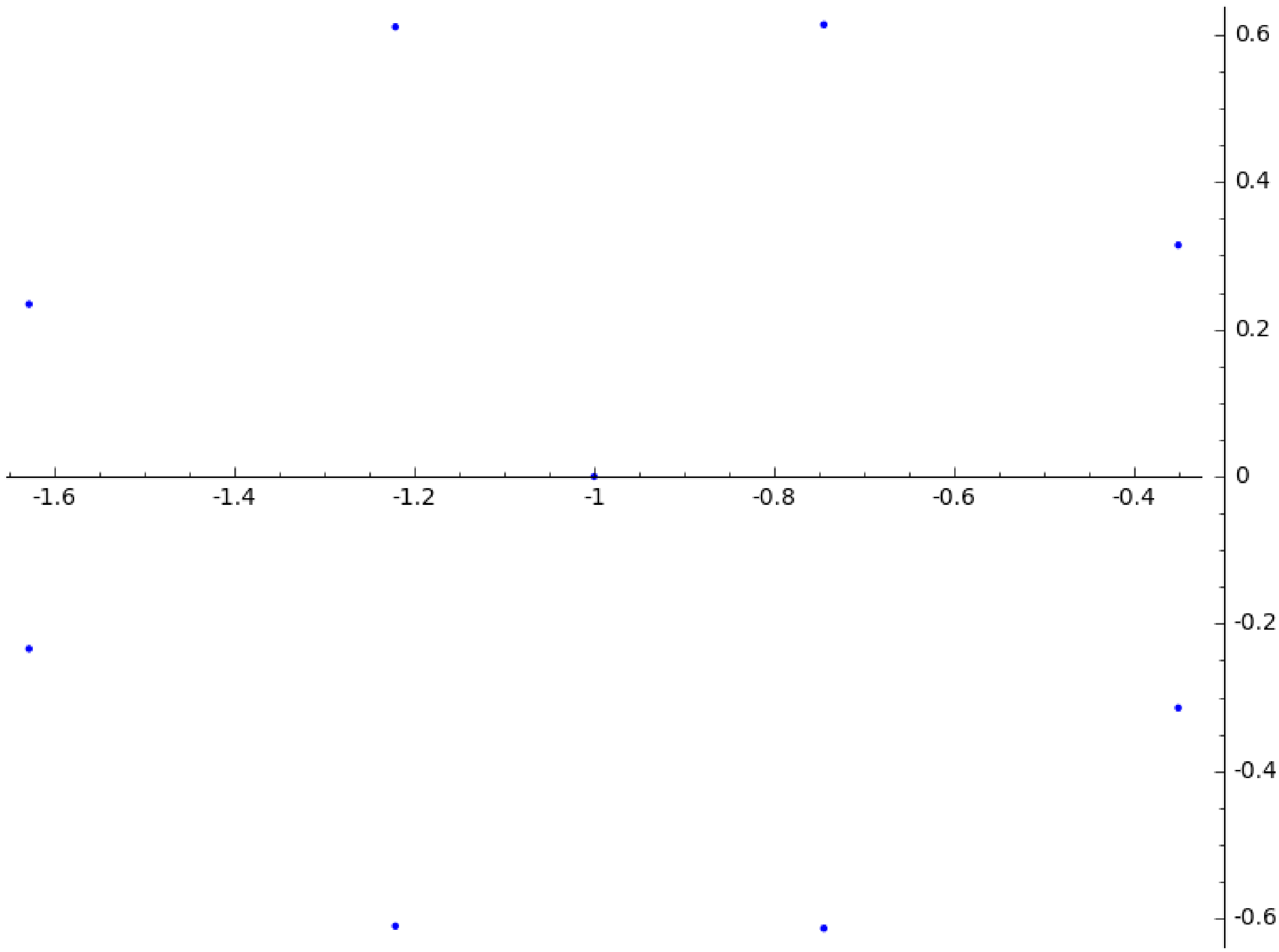


Figure 4.3: The roots of $i(TC(6,9),t)$.

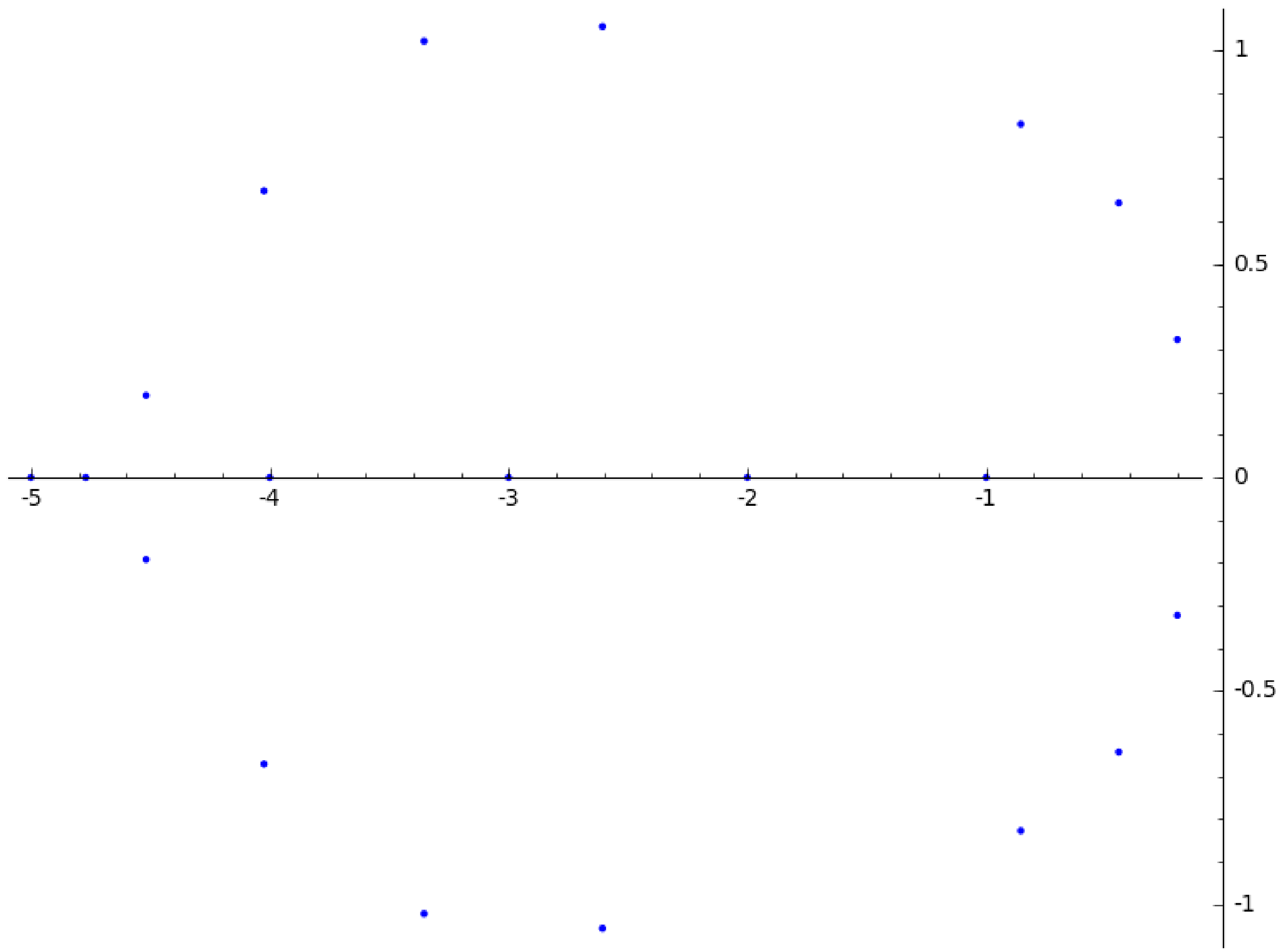


Figure 4.4: The roots of $i(TC(4,20))$.

In the following figures, the roots of $TC(2,n)$ are plotted in the same figure for various n. The red/orange points correspond low values of n while dark blue corresponds to the higher values of n. Notice that as n grows, the non-real roots tend to lie in an ovular shape, with an exceptional few near the origin. A description of this ovular shape still proves elusive, but would be interesting to know. The data suggests that increasing n results in roots with real parts becoming increasingly negative, while increasing r has the opposite effect. For the code used to generate these figures, see SAGE appendix near the end of the document.

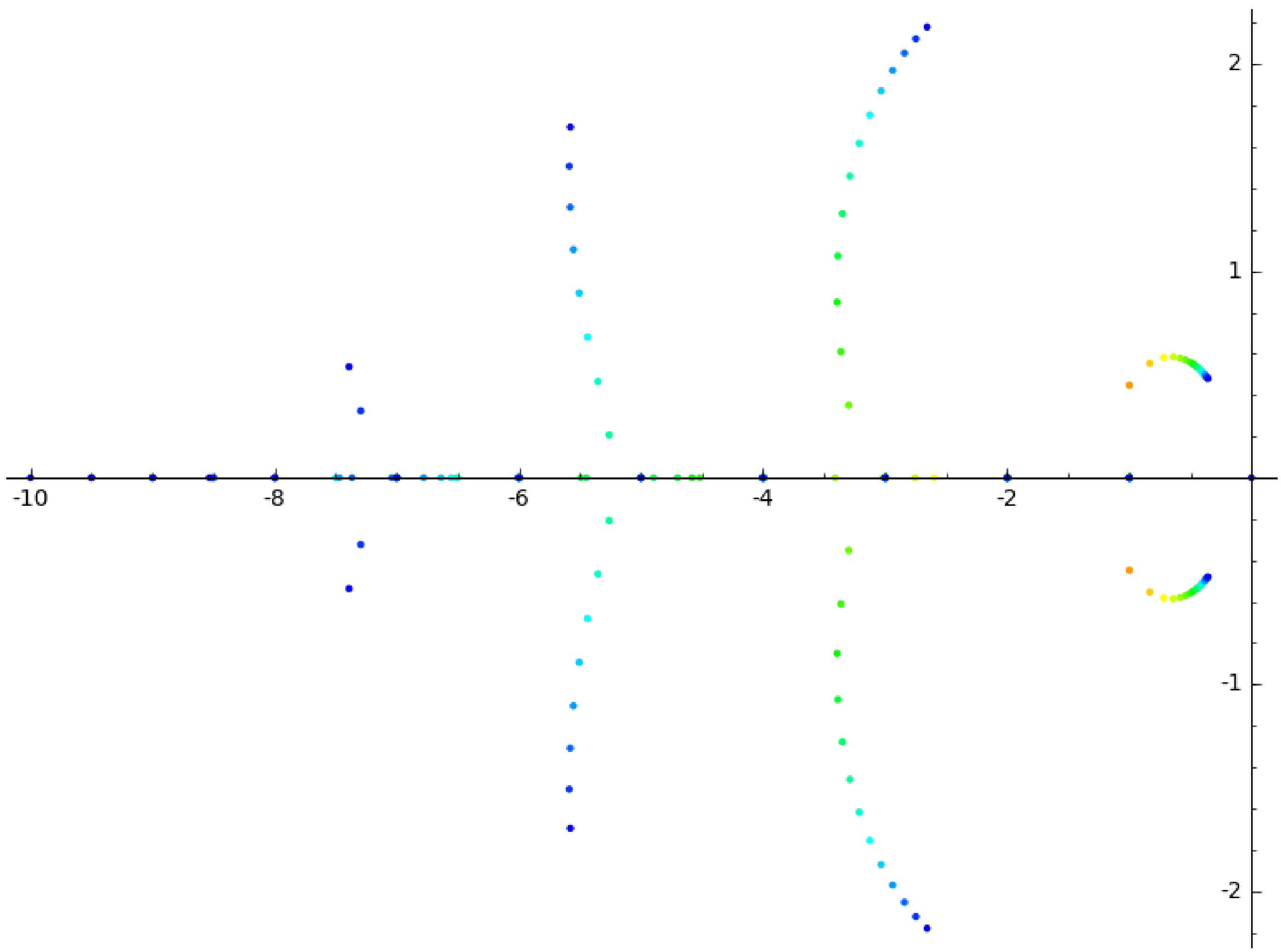


Figure 4.5: The Roots of $TC(2,n)$ for $n \leq 20$.

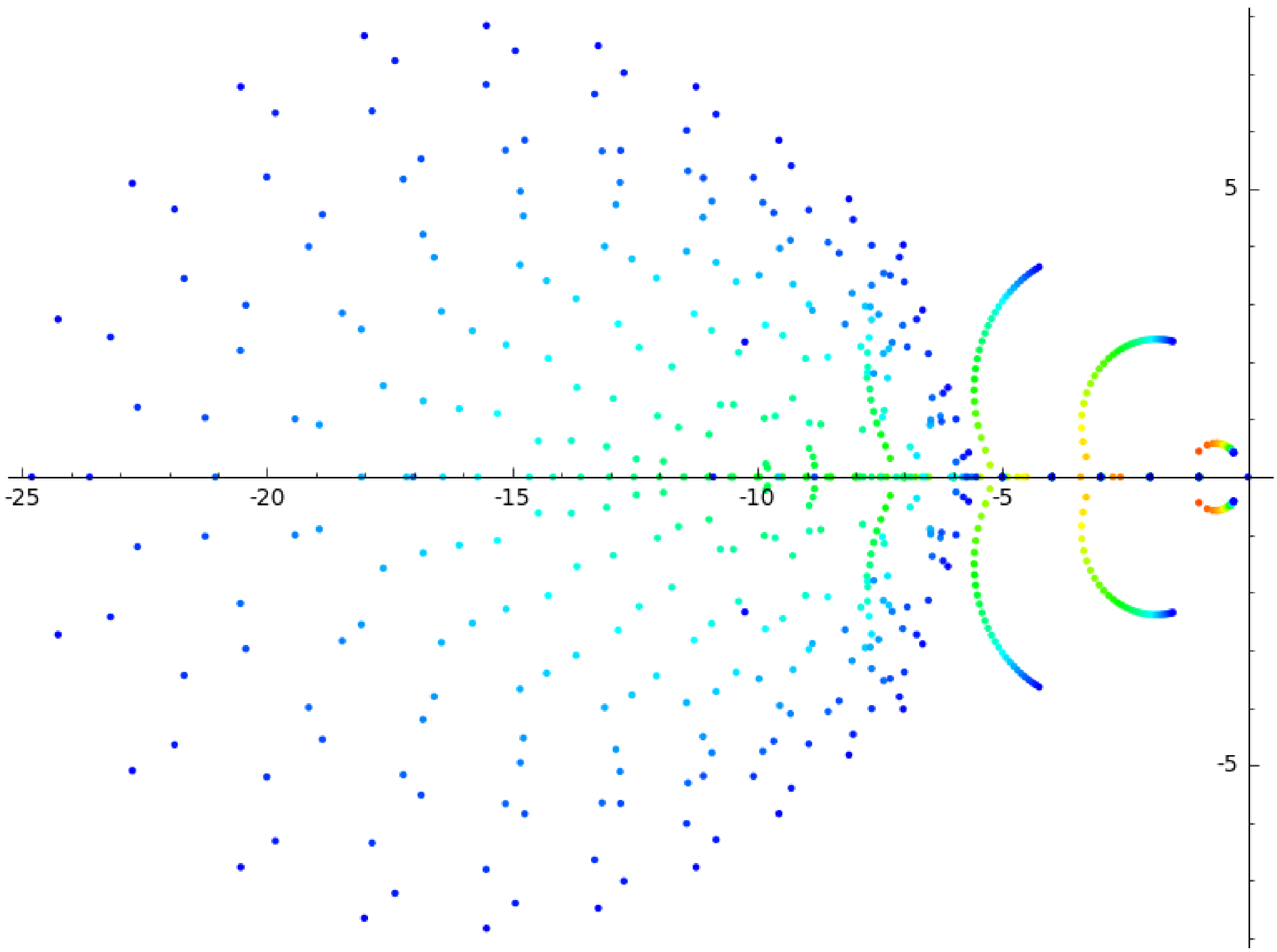


Figure 4.6: The Roots of $TC(2,n)$ for $n \leq 40$.

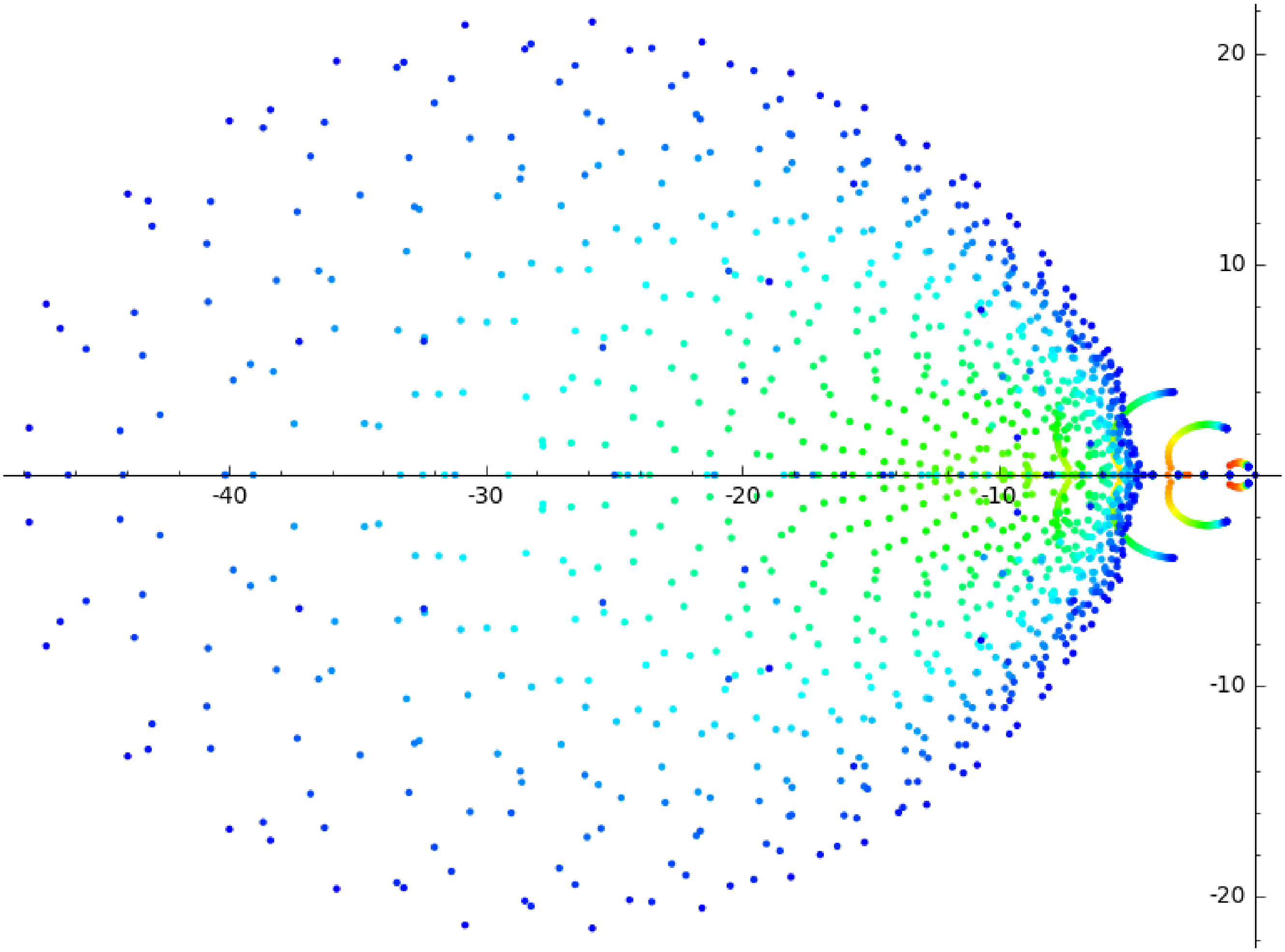


Figure 4.7: The Roots of $TC(2,n)$ for $n \leq 60$.

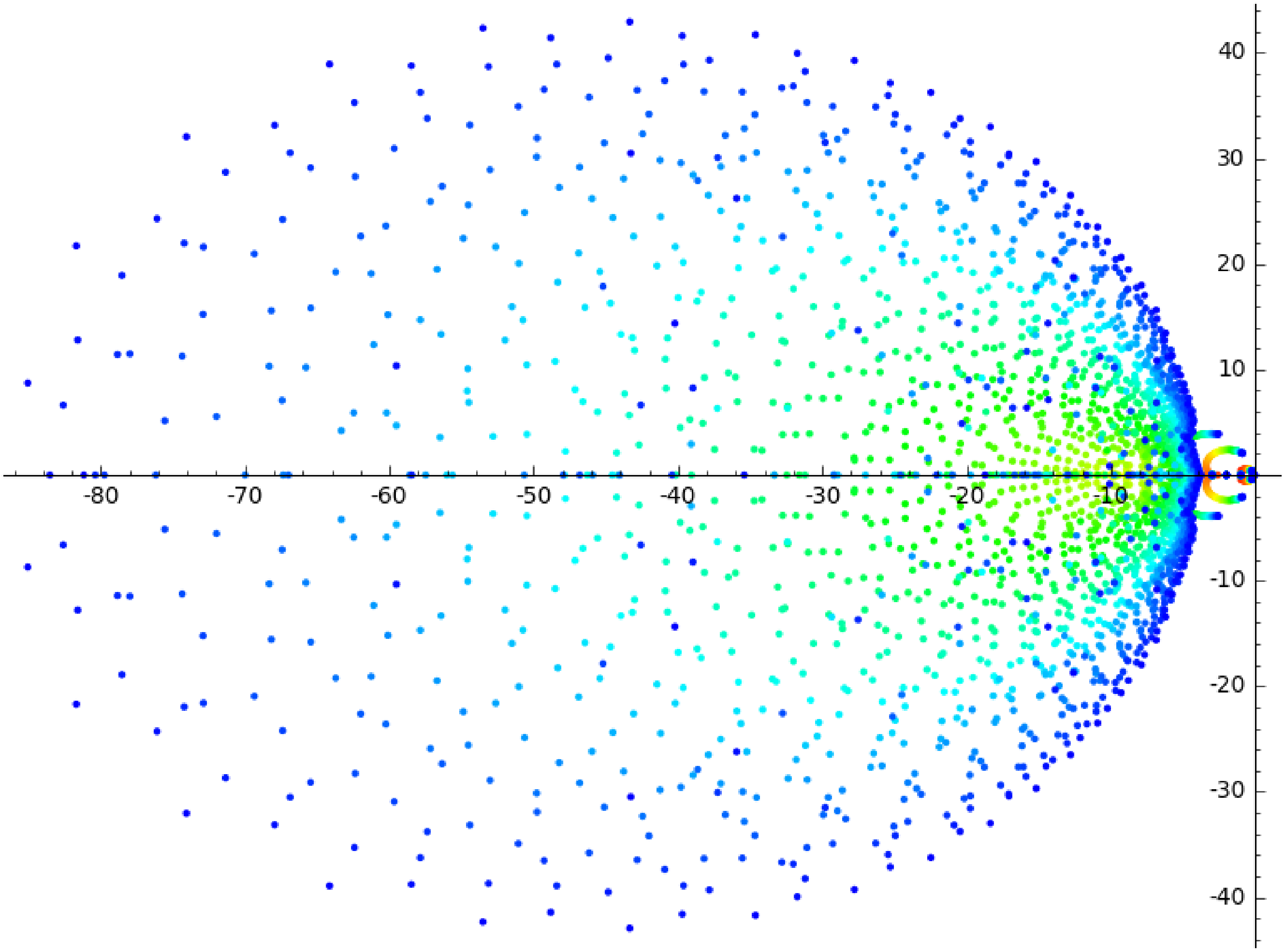


Figure 4.8: The Roots of $TC(2,n)$ for $n \leq 80$.

4.2 Concluding Remarks

The research conducted in this thesis leaves a number of interesting avenues for continued study. We will conclude this thesis with a brief discussion of these possibilities. One possibility is to extend the results on the skeletons of independence polytopes. I believe that the methods used to characterize the 1 and 2-skeletons can be used for higher skeletons. This should be a straightforward, but highly cumbersome task. Therefore it may be well suited for REU students or early grad students.

Another possible topic to explore is the polytope $\tilde{Q}_M$ which is a generalized permutohedron. Perhaps the Hopf structure could be useful in studying Q_M.

One very exciting idea is to try to extend the base polytope constuction into other Coxeter types. Hypersimplices can be viewed as the convex hull of fundamental weights in the Type A root lattice. I explored the idea of taking the convex hull of fundamental weights in the Type B root lattice, and there does seem to be interesting connections between the h^* vectors and Type B descent and excedence statistics. The hope would be for some theorem akin to Theorem 4.15. However, I haven't quite made this work yet.

A very obvious source of future work is the study of the roots of the Ehrhart polynomials of truncated cubes. Proving Conjecture 4.20 would be a lovely result. Additionally describing the equation of the "oval" formed by many of the roots would be interesting as well. I also attempted to develop results on the Ehrhart polynomials of shifted matroids. I think that something can be done here, and it is worth searching for a similar formula to 4.9.

The last idea I have is a relatively recent thought. In discussion with Kevin Marshall, I learned of an object similar to a matroid called a *greedoid*. The analogue of independent sets in a greedoid are called *feasables*. Define the *greedoid feasable polytope* as the convex hull of the χ_F for feasables F of the greedoid.I think that would be extremely interesting to study these polytopes and if Kevin doesn't work on it, then I probably will.

SAGE Code Appendix

```
def Ehr_Poly(r,n):
    i, k = var('i,k')
    return (1/factorial(n))*sum(((-1)^k)*binomial(n,k)* \
    product((r-k)*t+i, i, -k+1, n-k) ,k,0,r-1)


t=var('t') #### Plot roots of Ehrhart Polynomials for TC(2,n), n=2,..., 80
plotty = point([(0,0)])
r = 2
max = 80
for n in range(r,max+1):
    if n%10 ==0:
        print n
    f = Ehr_Poly(r,n)
    rooty = f.roots(ring = CC, multiplicities = False)
    plotty += sum( point([(foo.real_part(),foo.imag_part())], hue=(n/(1.5*max))) \
    for foo in rooty)


plotty
```

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