NORMAL AND PARACOMPACT SPACES AND THEIR PRODUCTS

by

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INTRODUCTION

In 1944, J. Dieudonné, in his paper "Une Généralisation des Espaces Compacts," introduced the concept of paracompactness as a generalization of the topological property of compactness. In this paper, J. Dieudonné proved that in the result "a compact Hausdorff space is normal" one can replace compactness with his generalized property, paracompactness, and while he left open the question as to whether the topological product of two paracompact spaces is paracompact, he did prove that the product of a paracompact space and a compact space is paracompact. Since that time a substantial amount of work has been done integrating this comparatively new topological property, and its subsequent generalization "countable paracompactness," with other already established properties; in particular, normality.

It is the purpose of this paper to set forth a number of theorems connecting normality, paracompactness, and countable paracompactness, and to present the known theorems concerning the topological product of a space enjoying a generalized compact property and a compact space. The main theme, obtained by utilizing the theorems described above, is the following result: The topological product of a normal space X and the closed unit interval is normal if and only if X is countably paracompact. Chapter I is devoted to the necessary definitions and theorems involving normality, paracompactness, and countable paracompactness; we show, for example, that a space is fully normal if and only if it is normal and paracompact. In Chapter II we prove that a number of topological properties are productive and, on the other hand, that the product of two normal spaces need not be normal; and in Chapter III we present the central theorem and also discuss the topological product of a compact space with a space that possesses a generalized compact property.

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Albert F. Joseph

CHAPTER I

SOME DEFINITIONS AND THEOREMS INVOLVING NORMAL SPACES

DEFINITION 1.1: A family T of subsets of a set X is said to be a topology for X, and the pair (X,T) is said to be a topological space (or X is simply referred to as a space), if and only if

(i) X and $\emptyset$ (the empty set) belong to T,

(ii) the union of any subfamily of T belongs to T, and

(iii) the intersection of any two members of T is again a member of T.

The members of T are referred to as open sets and their complements are called closed sets. If N is any subset of X containing some member x in X and if $x \in U \subset N$ for some U in T then N is called a neighborhood of x. Similarly, a neighborhood of a set A is a set that is a neighborhood of every point of A.

DEFINITION 1.2: A subfamily B of T is called

(i) a base for the topology T if and only if for each $x \in X$ and each neighborhood N of x there is a member U of B such that $x \in U \subset N$, and

(ii) a subbase for the topology T if and only if the family of finite intersections of members of B is a base for T.

DEFINITION 1.3: A topological space X is called

(i) a T_1-space if and only if for each $x \in X$, $\{x\}$ is closed,

(ii) a T_2-space (Hausdorff space) if and only if distinct points of X have disjoint neighborhoods,

(iii) regular if and only if for each $x \in X$ and each neighborhood N of x there is a closed neighborhood M of x such that $M \subset N$,

(iv) a T_3-space if and only if it is regular and T_1,

(v) normal if and only if disjoint closed sets have disjoint neighborhoods, and

(vi) a T_4-space if and only if it is normal and T_1.

DEFINITION 1.4: A family $\mathcal{U}$ of subsets of a space X is called a covering of X if $X = \bigcup\{U : U \in \mathcal{U}\}$. A covering $\mathcal{U}$ of a space X is called

(i) open if every member of $\mathcal{U}$ is open,

(ii) countable if $\mathcal{U}$ is countable,

(iii) locally finite if for each $x \in X$ there exists a neighborhood N of x such that N meets (intersects) only finitely many members of $\mathcal{U}$, and

(iv) point finite if for each $x \in X$, x belongs to only finitely many members of $\mathcal{U}$.

A covering $\mathcal{V}$ of X is called a refinement of $\mathcal{U}$ if and only if every member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$; a star refinement if and only if for each $x \in X$ the union of the members of $\mathcal{V}$ containing x is contained in some member of $\mathcal{U}$.

DEFINITION 1.5: A space X is called

(i) compact if and only if every open covering has a finite subfamily that covers X,

(ii) Lindelöf if and only if every open covering has a countable subfamily that covers X,

(iii) countably compact if and only if every countable open covering has a finite subfamily that covers X,

(iv) paracompact if and only if every open covering has a locally finite open refinement,

(v) countably paracompact if and only if every countable open covering has a locally finite open refinement, and

(vi) fully normal if and only if every open covering has an open star refinement.

DEFINITION 1.6: A metric space is a pair (X,d) where d is a metric for the set X. That is, d is a function on the Cartesian product $X \times X$ to the non-negative reals such that for points x,y, and z of X

(i) $d(x,y) = d(y,x)$,

(ii) $d(x,z) \leq d(x,y) + d(y,z)$, and

(iii) $d(x,y) = 0$ if and only if $x = y$.

A base for the metric topology is the family of all open r-spheres ($r > 0$) in X. (See [3]* for a discussion of metric spaces.)

LEMMA 1.1: Let X be a normal space and let $\{U_a : a \in A\}$ be a point finite open covering of X. Then there exists an open refinement $\{V_a : a \in A\}$ such that $\{V_a\}$ is a point finite open covering of X and for all $a \in A$, $V_a \subset \overline{V_a} \subset U_a$ [4].

* Numbers in square brackets refer to the bibliography at the end of this paper.

Proof: If $U_a=X$ for any a, the theorem is trivial. So assume $U_a \neq X$ for all $a \in A$. Now let φ be a function on A such that

(i) $\varphi(a) = U_a$, or $\varphi(a) = V_a$ where $\overline{V_a} \subset U_a$, and

(ii) $\{\varphi(a) : a \in A\}$ is an open covering of X.

Let η be the family of all such φ and order η as follows: $\varphi \leq \varphi'$ whenever $\varphi'(a) = \varphi(a)$ if $\varphi(a) = V_a$ Clearly the pair $(\eta, \leq)$ is a partially ordered system. For let $\varphi, \varphi', \varphi'' \in \eta$ be such that $\varphi \leq \varphi'$ and $\varphi' \leq \varphi''$. Then if $\varphi(a) = V_a$ we have $\varphi'(a) = \varphi(a)$. Therefore $\varphi''(a) = \varphi'(a) = V_a$ and consequently $\varphi \leq \varphi''$. Thus $\leq$ partially orders η. Now let η' be a chain in η and let $\varphi^*(a) = \bigcap \{\varphi'(a) : \varphi' \in \eta'\}$ for $a \in A$. We now show that $\varphi^* \in \eta$. First, $\varphi^*(a)$ is an open set for $a \in A$. For suppose $\varphi'(a) = U_a$ for all $\varphi' \in \eta'$. Then $\varphi^*(a) = U_a$ for all $\varphi' \in \eta'$. Then $\varphi^*(a) = U_a$ which is open. If for some $\varphi'' \in \eta'$, $\varphi''(a) = V_a$, then for $\varphi' \in \eta'$ such that $\varphi' \leq \varphi''$, $\varphi''(a) \subset \varphi'(a)$ and for $\varphi' \in \eta'$ such that $\varphi'' \leq \varphi'$, $\varphi'(a) = \varphi''(a)$. Consequently $\varphi^*(a) = \varphi''(a)$ which is open. Secondly, $\{\varphi^*(a) : a \in A\}$ covers X. For let $x \in X$. Since $\{U_a : a \in A\}$ is a point finite covering of X there exist only finitely many $a \in A$, say $a_1, \cdots, a_n$, such that $x \in U_{a_i}$ for $1 \leq i \leq n$. Also for some $\varphi'' \in \eta'$, $\varphi''(a_i) = \varphi'(a_i)$ for all i and all $\varphi' \in \eta'$ such that $\varphi'' \leq \varphi'$. Thus $\varphi''(a_i) = \varphi^*(a_i)$ for $1 \leq i \leq n$ and since $\{\varphi''(a) : a \in A\}$ covers X we have $x \notin \varphi''(a)$ for $a \neq a_i$ and x in $\varphi''(a_i)$ for at least one i $(1 \leq i \leq n)$. Consequently $\{\varphi^*(a)\}$ covers X and therefore $\varphi^* \in \eta$.

Now let $\varphi' \in \eta'$. If $\varphi'(a) = V_a$ for some $a \in A$ and $\varphi'' \in \eta$

is such that $\varphi''\geq\varphi'$ then $\varphi''(a) = \varphi'(a)$. Consequently $\varphi^*(a)=\varphi'(a)$ and therefore $\varphi'\leq\varphi^*$. Thus φ^* is an upper bound in η for the chain η' and by Zorn's lemma there exists a maximal member φ_1 in η.

We now complete the proof of this lemma by showing that $\varphi_1(a) = V_a$ for all $a\in A$. To show this we assume that for some $b\in A$, $\varphi_1(b) = U_b$. Set $F = X-\bigcup\{\varphi_1(a):a\in A \text{ and } a\neq b\}$. Clearly F is a closed set. If F is empty then $\{\varphi_1(a):a\in A \text{ and } a\neq b\}$ covers X. Therefore define φ_2 as follows: for $a\in A$ such that $a\neq b$ set $\varphi_2(a) = \varphi_1(a)$ and $\varphi_2(b) = \emptyset$ (the empty set). Then $\{\varphi_2(a):a\in A\}$ covers X, $\varphi_1\leq\varphi_2$, and $\varphi_1\neq\varphi_2$ which contradicts the maximality of φ_1. Thus F must be non-empty. But if F is non-empty then F and $X-U_b$ are non-empty, closed and disjoint subsets of the normal space X. Hence there exists disjoint open sets V_b and V_b' such that $F\subset V_b$ and $X-U_b\subset V_b'$. We have therefore $V_b\subset\overline{V_b}\subset X-V_b'\subset U_b$. Now define φ_2 as follows: for $a\in A$ such that $a\neq b$ set $\varphi_2(a) = \varphi_1(a)$ and set $\varphi_2(b)= V_b$. Since $F = X-\bigcup\{\varphi_1(a):a\ A \text{ and } a\neq b\}\subset V_b$, $\{\varphi_2(a):a\in A\}$ covers X and since $\varphi_2(a) = \varphi_1(a)$ for $a\neq b$ and $\varphi_2(b) = V_b$ while $\varphi_1(b) = U_b$ we have $\varphi_2\in\eta$, $\varphi_1\leq\varphi_2$, and $\varphi_1\neq\varphi_2$. Again we have contradicted the maximality of φ_1. Thus $\varphi_1(a) = V_a$ for all $a\in A$ and this completes the proof. We will also have need of the following result:

LEMMA 1.2: Let $\{U_i:i=1,2,\cdots\}$ be a countable open covering of a topological space X. Then if $\{U_i\}$ has an open locally finite refinement there exists an open locally finite refinement $\{V_i\}$ of $\{U_i\}$ with $V_i \subset U_i$.

Proof: Let $\mathcal{W}$ be an open locally finite refinement of a countable open covering $\{U_i\}$ of a space X. For each W in $\mathcal{W}$ let i(W) be the first integer such that $W \subset U_{i(W)}$ and let $V_k = \bigcup \{W : i(W)=k\}$. Then $\{V_k\}$ is an open covering of X with $V_k \subset U_k$. We shall now show that $\{V_k\}$ is locally finite.

Since $\mathcal{W}$ is locally finite, for each x in X there is an open neighborhood N of x that meets but finitely many members $W_1, \cdots, W_n$ of $\mathcal{W}$. Moreover, N meets $V_k = \bigcup \{W : i(W)=k\}$ if and only if there exists W such that i(W)=k and N meets W. This means that $W=W_j$ and hence that $k=i(W_j)$ for some $j=1,2,\cdots,n$. Since there are at most n such integers k, $\{V_k\}$ is locally finite.

Our first theorem gives us a large class of topological spaces that are fully normal.

THEOREM 1.1: Every metric space is fully normal [10].

Proof: Let X be a metric space and let $\mathcal{U} = \{U_a : a \in A\}$ be an open covering of X. For each $x \in X$ there is some $a(x) \in A$ such that $x \in U_{a(x)}$ and therefore there is a real number e(x) such that $0 < e(x) < 1$ and $N(x, 4e(x)) \subset U_{a(x)}$ where $N(x, e(x)) = \{y : d(x,y) < e(x)\}$. Let $\mathcal{U}' = \{N(x, e(x)) : x \in X\}$. Obviously $\mathcal{U}'$ covers X, refines $\mathcal{U}$ and we need only show that $\mathcal{U}'$ is a star refinement of $\mathcal{U}$.

Let $x' \in X$ and consider the set $H = \{x : x' \in N(x, e(x))\}$. Now $x' \in H$ so H is not void. Choose $x^* \in H$ such that $e(x^*) > 2/3$ sup. $\{e(x) : x \in H\}$. Thus if $x \in H$ we have $N(x, e(x)) \subset N(x', 2e(x))$.

For let $y \in N(x,e(x))$ then $d(y,x) < e(x)$ and consequently $d(y,x') < d(y,x) + d(x,x') < 2e(x)$. So $y \in N(x',2e(x))$ and therefore $N(x,e(x)) \subset N(x',2e(x))$. Also by the choice of x^*, $N(x',2e(x)) \subset N(x',3e(x^*))$. Finally $N(x',3e(x^*)) \subset N(x^*,4e(x^*))$. For let $y \in N(x',3e(x^*))$. Then $d(y,x') < 3e(x^*)$ and since $x^* \in H$ $d(x',x^*) < e(x^*)$. Hence $d(y,x^*) < 4e(x^*)$ and therefore $N(x',3e(x^*)) \subset N(x^*,4e(x^*))$. Thus $N(x,e(x)) \subset N(x',2e(x)) \subset N(x',3e(x^*)) \subset N(x^*,4e(x^*))$ for any x such that $x' \in N(x,e(x))$; hence the $\mathcal{U}'$-star of x' is contained in $N(x^*,4e(x^*)) \subset U_{a(x^*)}$. Since x' was arbitrary we have shown that $\mathcal{U}'$ is a star refinement of $\mathcal{U}$ and the proof is complete.

THEOREM 1.2: Every fully normal space is normal and paracompact [9].

Proof: Let X be a fully normal space. We will first show that X is normal. Let A and B be two disjoint closed subsets of X and let $\mathcal{U}$ be an open star refinement of the open covering $\{X-A, X-B\}$. Let U be the union of the members of $\mathcal{U}$ that meet A and V be the union of the members of $\mathcal{U}$ that meet B. Clearly the open sets U and V cover A and B respectively. Moreover they are disjoint. For suppose $x \in U \cap V$; then the star of x of the covering $\mathcal{U}$ meets both A and B, and therefore $\mathcal{U}$ is not a star refinement of $\{X-A, X-B\}$. Thus U and V are disjoint and therefore X is normal.

The space X is also paracompact, for let $\mathcal{U} = \{U_a : a \in A\}$ be an open covering of X. Then there are open coverings $\mathcal{U}^1 = \{U^1\}$, $\mathcal{U}^2 = \{U^2\}, \cdots, \mathcal{U}^n = \{U^n\}, \cdots$ such that $\mathcal{U}^1$ is an open star refinement of $\mathcal{U}$ and $\mathcal{U}^{n+1}$ is an open star refinement of $\mathcal{U}^n$ $(n=1,2,\cdots)$. For any subset Q of X and for any positive integer n let

(i) (Q,n) be the union of all sets U^n of $\mathcal{U}^n$ that meet Q, and

(ii) $(Q,-n) = X-(X-Q,n)$.

Since the set $(X-Q,n)$ is obviously open, the set $(Q,-n)$ is closed. Moreover, we have

(1) $(Q,-n) = \{x:(\{x\},n)\subset Q\}$.

For let $x\in(Q,-n)$. Then $x\notin(X-Q,n)$ and consequently every member of $\mathcal{U}^n$ containing x is contained in Q. Conversely, if every member of $\mathcal{U}^n$ containing x is contained in Q then $x\notin(X-Q,n)$ and therefore $x\in(Q,-n)$. An immediate consequence of (1) is

(2) $((Q,-n),n)\subset Q$.

Now let $y\in((Q,n+1),n+1)$. Then there exists U^{n+1} in $\mathcal{U}^{n+1}$ such that $y\in U^{n+1}$ and U^{n+1} meets $(Q,n+1)$. Let $x\in U^{n+1}\cap(Q,n+1)$ Then $y\in(\{x\},n+1)$ and since $\mathcal{U}^{n+1}$ is an open star refinement of $\mathcal{U}^n$ there exists U^n in $\mathcal{U}^n$ such that $(\{x\},n+1)\subset U^n$. Hence $y\in U^n$ and since U^n meets Q, $U^n\subset(Q,n)$. Thus we have

(3) $((Q,n+1),n+1)\subset(Q,n)$.

The following will also be useful:

(4) $Q\subset P$ implies $(Q,n)\subset(P,n)$,

(5) $m\geq n$ implies $(Q,m)\subset(Q,n)$,

(6) $\overline{Q}\subset(Q,n)$, and

(7) $y\in(\{x\},n)$ if and only if $x\in(\{y\},n)$.

The obvious proofs are omitted.

We now define, for each $a\in A$, $V_a^1=(U_a,-1)$ and $V_a^n=(V_a^{n-1},n)$ for $n\geq 2$. Clearly $V_a^1\subset V_a^2\subset\cdots\subset V_a^n\subset\cdots$, and V_a^n is open if $n\geq 2$. Notice that

(i) $(V_a^1,1)=((U_a,-1),1)\subset U_a$ by (2), and that

(ii) if $(V_a^{k-1},k-1)\subset U_a$ then $(V_a^k,k) = ((V_a^{k-1},k),k)\subset (V_a^{k-1},k-1)\subset U_a$ by (3).

Thus $(V_a^k,k)\subset U_a$ for all k and therefore for any $k\geq 2$

$V_a^k=(V_a^{k-1},k)\subset(V_a^k,k)\subset U_a$ by (4). Thus

(8) $$V_a=\bigcup_{k=1}^{\infty} V_a^k\subset U_a.$$

Futhermore, since $\mathcal{U}^1$ is a star refinement of $\mathcal{U}$, if $x\in X$ then $(\{x\},1)\subset U_a$ for some a and therefore $x\in(U_a,-1)=V_a^1\subset V_a$ by (1). Thus

(9) $X=\bigcup\{V_a:a\in A\}$.

Also for any $x\in V_a$, there exists $n\geq 2$ such that $x\in V_a^{n-1}$ and therefore

(10) $(\{x\},n)\subset V_a^n\subset V_a$.

We now well order the set A and define a transfinite sequence of closed sets H_{na} by setting $H_{n1}=(V_1,-n)$ and $H_{na}=(V_a-\bigcup_{b<a}H_{nb},-n)$ for each n (the sets are closed by the remark preceding (1)). We now have:

(11) If $a\neq b$, no U^n in $\mathcal{U}^n$ can meet both H_{na} and H_{nb}.

For we can suppose $a<b$. Then if U^n meets H_{nb} let $x\in X$ be

such that $x \in U^n \cap H_{nb}$. Then $x \in (V_b - \bigcup_{a<b} H_{na}, -n)$ and by (1) $x \in \{x : (\{x\}, n) \subset (V_b - \bigcup_{a<b} H_{na})\}$. Thus $U^n \subset V_b - \bigcup_{a<b} H_{na}$ and consequently $U^n \cap H_{na}$ is void. Moreover

(12) $\bigcup_{n,a} H_{na} = X$.

For let $x \in X$. By (9) there exists a first $a \in A$ such that $x \in V_a$, and from (10) there exists $n > 0$ such that $(\{x\}, n) \subset V_a$. We assert that $x \in H_{na}$. For suppose not. From (1) $H_{na} = \{x : (\{x\}, n) \subset (V_a - \bigcup_{b<a} H_{nb})$ and therefore $(\{x\}, n)$ contains a point y in H_{nb} for some $b<a$. But then $x \in (H_{nb}, n) \subset ((V_b, -n), n) \subset V_b$ (from (4) and (2)) and this contradicts our choice of a. Thus $x \in H_{na}$ and therefore $\bigcup_{n,a} H_{na} = X$.

Now write $E_{na} = (H_{na}, n+3)$ and $G_{na} = (H_{na}, n+2)$. Then $\overline{E_{na}} = \overline{(H_{na}, n+3)} \subset ((H_{na}, n+3), n+3) \subset (H_{na}, n+2) = G_{na}$ by (6) and (3).

Thus

(13) $H_{na} \subset E_{na} \subset \overline{E_{na}} \subset G_{na}$.

Also

(14) if $a \neq b$, no U^{n+2} in $\mathcal{U}^{n+2}$ can meet both G_{na} and G_{nb}. For suppose U^{n+2} meets both G_{na} and G_{nb}. Then for some $x \in X$, $x \in U^{n+2} \cap G_{na}$. Hence for some U_*^{n+2} in $\mathcal{U}^{n+2}$ containing x, U_*^{n+2} meets H_{na}. Thus $U^{n+2} \cap U_*^{n+2}$ and consequently $(\{x\}, n+2)$ meets both H_{na} and G_{nb}. Since $\mathcal{U}^{n+2}$ is a star refinement of

$\mathcal{U}^{n+1}$ for some U^{n+1} in $\mathcal{U}^{n+1}$, $(\{x\},n+2)\subset U^{n+1}$ and consequently U^{n+1} meets both H_{na} and G_{nb}. But $G_{nb}=(H_{nb},n+2)\subset(H_{nb},n+1)$ by (5). Thus U^{n+1} meets H_{na} and $(H_{nb},n+1)$. Let $y\in X$ be such that $y\in U^{n+1}\cap(H_{nb},n+1)$. Then for some U_{*}^{n+1} in $\mathcal{U}^{n+1}$ containing y, U_{*}^{n+1} meets H_{nb}. Thus $(\{y\},n+1)$ meets both H_{na} and H_{nb} and since $\mathcal{U}^{n+1}$ is a star refinement of $\mathcal{U}^{n}$ for some U^{n} in $\mathcal{U}^{n}$ we have that U^{n} meets both H_{na} and H_{nb}. By (11) this is not possible. Hence (14) follows.

Now write $F_n=\bigcup_a \overline{E_{na}}$. Then F_n is closed. For let $x\in\overline{F_n}$, then every neighborhood N(x) of x meets some $\overline{E_{na}}$, hence some E_{na}. In particular some U^{n+2} in $\mathcal{U}^{n+2}$ containing x meets some E_{na} and by (14) U^{n+2} can meet only one E_{na}. Thus for some $a\in A$, $x\in\overline{E_{na}}\subset F_n$.

Finally we set $W_{1a}=G_{1a}$ and $W_{na}=G_{na}-(F_1\cup F_2\cup\ldots\cup F_{n-1})$ for $n>1$; thus the sets W_{na} are open. We shall show that they form the desired refinement. In the first place $\bigcup_{n,a} W_{na}=X$. For let $x\in X$, then we have x in some H_{na} by (12) and therefore in some $\overline{E_{na}}$. Let m be the smallest integer for which x is in some $\overline{E_{ma}}$. Then for some a, $x\in G_{ma}$ and $x\notin F_1,\cdots,F_m-1$; hence $x\in W_{ma}$. Thus $\{W_{na}\}$ is an open covering of X. Next, using (5), (2), and (8) we have $W_{na}\subset G_{na}=(H_{na},n+2)\subset$

$(H_{na},n) = ((V_a - \bigcup_{b<a} H_{nb},-n),n) \subset V_a - \bigcup_{b<a} H_{nb} \subset V_a \subset U_a$. Thus $\{W_{na}\}$ refines $\mathcal{U}$. We now show that $\{W_{na}\}$ is locally finite. Let $x \in X$; as before, $x \in H_{na}$ for some n and some a so $(\{x\},n+3) \subset E_{na} \subset F_n$. Thus $(\{x\},n+3)$ does not meet any W_{kb} if $k>n$. Further, for a given $k \leq n$, we have $(\{x\},n+3) \subset U^{n+2}$ for some U^{n+2} in $\mathcal{U}^{n+2}$ and U^{n+2} is contained in some U^{k+2} in $\mathcal{U}^{k+2}$. By (14) U^{k+2} can meet G_{kb} for at most one value of b. Thus the neighborhood $(\{x\},n+3)$ of x meets at most n of the sets W_{kb}; hence X is paracompact.

THEOREM 1.3: Every metric space is paracompact and normal.

Proof: Theorems 1.1 and 1.2.

In Theorem 1.2, we showed that a fully normal space is normal. The following theorem gives a necessary and sufficient condition for a normal space to be fully normal.

THEOREM 1.4: A space X is fully normal if and only if it is normal and paracompact[9].

Proof: By Theorem 1.2 if X is fully normal then X is normal and paracompact. On the other hand, suppose X is a paracompact normal space and let $\mathcal{U} = \{U_a\}$ be a locally finite open covering of X. Since a locally finite covering is a point finite covering (definition 1.4) and since X is normal, by Lemma 1.1 there exists open sets $\{V_a\}$ such that $V_a \subset \overline{V}_a \subset U_a$ for all a and $\bigcup \{V_a\} = X$. By hypothesis, each $x \in X$ has an open neighborhood $G(x)$ that meets U_a for only finitely many a's, say for $a \in A(x)$. Let $B(x)$ be those members of $A(x)$ for which $x \in U_a$, and let $C(x)$ be those members of $A(x)$ for which $x \notin \overline{V_a}$. Clearly $B(x) \cup C(x) = A(x)$. Let $W(x) = G(x) \cap (\bigcap\{U_a : a \in B(x)\}) \cap (\bigcap\{X-\overline{V_a} : a \in C(x)\})$, and let $\mathcal{W} = \{W(x); x \in X\}$. Clearly $x \in W(x)$ and by the finiteness of the sets $B(x)$ and $C(x)$, $W(x)$ is open. Thus $\mathcal{W}$ is an open covering of X. We shall show that $\mathcal{W}$ star-refines $\mathcal{U}$. Let $y \in X$ and let b be such that $y \in V_b$. Then if y belongs to $W(x)$, $W(x)$ meets $\overline{V_b}$ and so $b \in A(x)$ and $b \notin C(x)$. Thus $b \in B(x)$, which implies that $W(x) \subset U_b$ by construction. Thus the union of the members of $\mathcal{W}$ containing y is contained in U_b and this completes the proof.

The following theorem, which is due to J. Dieudonné, is a generalization of the result: A compact Hausdorff space is normal.

THEOREM 1.5: Every paracompact Hausdorff space is normal [1].

Proof: Let X be a paracompact Hausdorff space. First we shall show that X is regular. Let F be a closed subset of X and let $a \in X$ be such that $a \notin F$. Then for each $x \in F$ there are disjoint open neighborhoods $W(x)$ and $V(x)$ of a and x respectively. Let $\mathcal{U}$ be the open covering of X consisting of all such $V(x)$ for each $x \in F$, and the complement of F; and let $\mathcal{U}'$ be a locally finite refinement of $\mathcal{U}$. Then there exists an open neighborhood W of a that does not meet F and that meets only finitely many members $U_1', \cdots, U_n'$, of

$\mathcal{U}'$. Let V be the union of the members of $\mathcal{U}'$ that meet F and let $x_i \in F$ be such that $U_i' \subset V(x_i)$ for $1 \le i \le n$. Then

$$U = W \cap W(x_1) \cap W(x_2) \cap \cdots \cap W(x_n)$$

is an open neighborhood of a that does not meet V. Hence U and V are disjoint open neighborhoods of a and F respectively and therefore X is regular.

Now let A and B be disjoint closed subsets of X. Since X is regular, for each $x \in A$ there are open neighborhoods V(x) of x and W(x) of B that are disjoint. Consider the open covering $\mathcal{U}$ of X consisting of the sets V(x) for each $x \in A$ and the complement of A, and let $\mathcal{U}'$ be a locally finite refinement of $\mathcal{U}$ that covers X. The union V of the members of $\mathcal{U}'$ that meet A is clearly an open neighborhood of A and for each $y \in B$ there is an open neighborhood N(y) of y that does not meet A and that meets only finitely many members $U_1', U_2', \cdots, U_n'$ of $\mathcal{U}'$. Let $x_i \in A$ be such that $U_i' \subset V(x_i)$ for $1 \le i \le n$, and set $W(y) = N(y) \cap W(x_1) \cap \cdots \cap W(x_n)$. Clearly W(y) does not meet V, hence $U = \bigcup \{W(y) : y \in B\}$ does not meet V. Hence V and U are open neighborhoods of A and B respectively that are disjoint and the proof is complete.

COROLLARY: For Hausdorff spaces, paracompactness and full normality are equivalent.

Proof: Theorems 1.4 and 1.5.

Using the fact that every normal Hausdorff space with a countable base is metrizable (see [3]) and theorem 1.3 (every metric space is paracompact) we have the following: Every normal Hausdorff space with a countable base is paracompact. We refer you to the appendix for an example, due to M. E. Rudin, of a separable* normal Hausdorff space that is not paracompact.

Theorem 1.4, which is due to A. H. Stone, is, in a certain sense, a justification of the concept of paracompactness. In like manner, C. H. Dowker, has justified the concept of countable paracompactness which is presented as Theorem 3.6 in Chapter III of this paper. The following theorem, which is also due to Dowker, exhibits a number of conditions on a normal space that are equivalent to countable paracompactness.

THEOREM 1.6: The following properties of a normal space X are equivalent:

(i) The space X is countably paracompact.

(ii) Every countable open covering of X has a point-finite open refinement.

(iii) Every countable open covering $\{U_i\}$ has an open refinement $\{V_i\}$ with $\overline{V_i} \subset U_i$.

(iv) Given a decreasing sequence $\{F_i\}$ of closed sets with vacuous intersection, there is a sequence $\{G_i\}$ of open sets with vacuous intersection such that $F_i \subset G_i$.

(v) Given a decreasing sequence $\{F_i\}$ of closed sets with vacuous intersection, there is a sequence

* A space X is separable if there exists a countable subset A of X such that $\overline{A} = X$.

$\{A_i\}$ of closed G_δ-sets* with vacuous intersection such that $F_i \subset A_i$.

In the proof of the above theorem we will need the following lemma:

LEMMA 1.3: Any open F_σ-set in a normal space can be written as the countable union of closed sets F_i $(i=1,2,\cdots)$ such that for all i F_i is contained in the interior of F_{i+1}.

Proof: Let B be an open F_σ-set in a normal space X and let $B=\bigcup_{i=1}^{\infty} B_i$ where each B_i is closed. Set $B_1=F_1$. If $B_1=B$ then the proof is complete. So suppose $B_1 \neq B$ and we have constructed closed sets $F_1,\cdots,F_n$ such that $F_n \neq B$, F_i is contained in the interior of F_{i+1} for $1 \leq i \leq n$, and $B_i \subset F_i \subset B$ for $1 \leq i \leq n$. Then there is a least integer $j \geq n+1$ such that $B_j \not\subset F_n$. Therefore since X is normal and $B_j \cup F_n$ is a closed set contained in the open set B there is an open set V_{n+1} such that $B_j \cup F_n \subset V_{n+1} \subset \overline{V}_{n+1} \subset B$. Set $F_{n+1}=\overline{V}_{n+1}$. Then $B_{n+1} \subset F_{n+1} \subset B$ and F_n is clearly contained in the interior of F_{n+1}. Thus if $F_{n+1}=B$ the proof is complete. If $F_{n+1} \neq B$ for any n then by the above induction we can construct the desired closed sets and since $B_n \subset F_n \subset B$ we have $\bigcup_n F_n=B$.

Proof of Theorem 1.6: (i) $\longrightarrow$ (ii). This is clear since a locally finite covering is obviously a point finite covering.

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* A set A is called a G_δ-set if it is the intersection of some countable collection of open sets, and is called an F_σ-set if it is the union of some countable collection of closed sets.

(ii) $\longrightarrow$ (iii). Let $\{U_i\}$ be any countable open covering of X. Then by (ii) $\{U_i\}$ has a point finite open refinement $\mathcal{W}$. For each W in $\mathcal{W}$ let i(W) be the first integer such that $W \subset U_{i(W)}$ and let G_i be the union of those W in $\mathcal{W}$ for which $i(W)=i$. Clearly $\{G_i\}$ is a point finite refinement of $\{U_i\}$ such that, for each i, $G_i \subset U_i$ (see proof of Lemma 1.2). Since X is normal, there is a point-finite open refinement $\{V_i\}$ of $\{G_i\}$ such that, for each i, $V_i \subset \overline{V_i} \subset G_i$ (by Lemma 1.1). Hence $\overline{V_i} \subset U_i$ and (iii) follows.

(iii) $\longrightarrow$ (iv). Let $\{F_i\}$ be a decreasing sequence of closed sets such that $\bigcap_{i=1}^{\infty} F_i$ is void. Set $U_i = X - F_i$ for all i. Then U_i is open and $\bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} (X-F_i) = X - \bigcap_{i=1}^{\infty} F_i = X$. Therefore $\{U_i\}$ is a countable open covering of X. By (iii) there is an open refinement $\{V_i\}$ of $\{U_i\}$ that covers X with the property that $\overline{V_i} \subset U_i$ for all i. Set $G_i = X - \overline{V_i}$. Then G_i is open and since $U_i \supset \overline{V_i}$ we have

$$G_i = X - \overline{V_i} \supset X - U_i = X - (X - F_i) = F_i .$$

Moreover $\bigcap_{i=1}^{\infty} G_i = \bigcap_{i=1}^{\infty} (X - \overline{V_i}) = X - \bigcup_{i=1}^{\infty} \overline{V_i} = X - X$. Thus $F_i \subset G_i$ for all i and $\bigcap_{i=1}^{\infty} G_i$ is void.

(iv) $\longrightarrow$ (v). Let $\{F_i\}$ be a decreasing sequence of closed sets such that $\bigcap_{i=1}^{\infty} F_i$ is void. Then, by (iv), there is a sequence of open sets $\{G_i\}$ with vacuous intersection such that $F_i \subset G_i$ for each i. Hence, for each i, $X - G_i$ and F_i are disjoint closed sets of the normal space X and by Urysohn's Lemma (see [3]) there is a continuous function f_i from X to the closed unit interval $[0,1]$ such that $f_i(x)=0$ if $x \in F_i$ and $f_i(x)=1$ if $x \in X-G_i$. For all integers i and j set

$$G_{ij}=\{x \in X : f_i(x)<1/j\};$$

and for all integers i set

$$A_i=\bigcap_{j=1}^{\infty} G_{ij}=\{x \in X : f_i(x)=0\}.$$

Since f_i is continuous G_{ij} is open and A_i is a closed G_δ-set. Moreover, by the definition of f_i, $F_i \subset A_i \subset G_i$, and since $\bigcap_{i=1}^{\infty} G_i$ is empty, $\bigcap_{i=1}^{\infty} A_i$ is empty.

(v) $\longrightarrow$ (i). Let $\{U_i\}$ be a countable open covering of X and let $F_i = X-\bigcup_{k<i} U_k$. Then $\{F_i\}$ is a decreasing sequence of closed sets and since $\bigcup_{i=1}^{\infty} U_i = X$, $\bigcap_{i=1}^{\infty} F_i$ is empty. Then by (v), there is a sequence $\{A_i\}$ of closed G_δ-sets with $F_i \subset A_i$ and $\bigcap_{i=1}^{\infty} A_i$ void. Set $B_j=X-A_j$. Then B_j is an open F_σ-set and by Lemma 1.3 we may assume $B_j = \bigcup_{i=1}^{\infty} B_j^i$ where each B_j^i is closed and each B_j^i is contained in the interior H_j^{i+1} of B_j^{i+1}. Since for all i, $B_j^i \subset H_j^{i+1} \subset B_j^{i+1}$, we have $B_j= \bigcup_{i=1}^{\infty} H_j^i= X-A_j$ and $B_j^i \subset B_j= X-A_j \subset X-F_j= \bigcup_{k<j} U_k$.

Now let $V_i = U_i-\bigcup_{j<i} B_j^i$; then V_i is open. If $j<i$, then $B_j^i \subset \bigcup_{k<j} U_k \subset \bigcup_{k<i} U_k$. Hence $\bigcup_{j<i} B_j^i \subset \bigcup_{k<i} U_k$. Therefore

$U_i - \bigcup_{k<i} U_k \subset U_i - \bigcup_{j<i} B_j^i = V_i$. So let $x \in X$. Then there exists i such that $x \in U_i$ and $x \notin U_k$ for $k<i$: hence $x \in V_i$. Consequently $\{V_i\}$ covers X and clearly $\{V_i\}$ is an open refinement of $\{U_i\}$. Moreover, for each $x \in X$ there is some A_j such that $x \notin A_j$ ($\bigcap_{i=1}^{\infty} A_i$ is void). But if $x \notin A_j$, then $x \in X - A_j = B_j$ and consequently for some k, $x \in H_j^k$. Thus if $i>j$ and $i>k$ $H_j^k \subset B_j^i$ and $H_j^k \cap (U_i - \bigcup_{j<i} B_j^i)$ is void. But $V_i = U_i - \bigcup_{j<i} B_j^i$. Thus $x \in H_j^k$, an open set, that can intersect V_i only if $i \leq \max. \{j,k\}$. Hence $\{V_i\}$ is a locally finite open refinement of $\{U_i\}$ and consequently X is countably paracompact. This completes the proof of the theorem.

COROLLARY: Every perfectly normal space is countably paracompact.

Proof: A perfectly normal space is a normal space X in which every closed set is a G_δ-set. Hence if $\{F_i\}$ is a decreasing sequence of closed sets with vacuous intersection then it is also a decreasing sequence of closed G_δ-sets with vacuous intersection and by (v) of the foregoing theorem X is countably paracompact.

It is not true that every normal space is countably paracompact as the following example shows. Let X be the set of real numbers and let the empty set, X, and sets of the form $\{x \in X : x < a\}$ (where $a \in X$) be open. Then it is clear that we have defined a topology for X and since

X has no pairwise disjoint closed sets, X must be trivially normal. But $\{G_i\}$, where $G_i = \{x \in X : x < i\}$, is a countable cover for X which has no locally finite refinement. Consequently X is not countably paracompact.

Note that the above space X is not Hausdorff. To the best of the author's knowledge it is not known whether there exists a normal Hausdorff space that is not countably para-compact. The question of the existence of such a space is known as Dowker's Problem. M. E. Rudin [7], however, has shown that if a Souslin space* exists then Dowker's Question is answered in the affirmative, i.e., there exists a normal Hausdorff space which is not countably paracompact. We omit the construction of Rudin's example.

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* A space S is a Souslin space if S is a gapless linearly ordered space (with the order topology) such that
1.) S is not separable, and
2.) Every collection of disjoint segments (each containing at least two elements) in S is countable.

CHAPTER II

PRODUCTS OF SPACES ENJOYING A COMMON TOPOLOGICAL PROPERTY

Let "p" be a given topological property. One naturally asks if "p" is productive, i.e., if given a family of spaces having "p", does the product of these same spaces also have "p". This subject is the substance of this chapter and we begin with a few definitions.

DEFINITION 2.1: Let $\{X_a : a \in A\}$ be a family of topological spaces. The Cartesian product $\prod_{a \in A} X_a$ of $\{X_a\}$ is defined to be the set of all functions f on A such that $f(a) \in X_a$ for each $a \in A$. The function $P_a : \prod_{a \in A} X_a \longrightarrow X_a$ defined by $P_a(x) = x_a$, where x_a is the "a-th" coordinate of x, is called the projection map of $\prod_{a \in A} X_a$ onto X_a. The topology on the set $X = \prod_{a \in A} X_a$, called the product topology, is motivated by the requirement that the projection maps be continuous and consequently the defining subbase for this topology is the family of all sets of the form $P_a^{-1}[U]$ where U is open in X_a.

We now present a short sequence of theorems exhibiting a number of topological properties that are productive.

THEOREM 2.1: The product of an arbitrary family of T_1-spaces is a T_1-space.

Proof: Let $\{X_a\}$ be an arbitrary family of T_1-spaces and let $x \in X = \prod_a X_a$. We shall show that $X - \{x\}$ is open. Suppose $y \in X - \{x\}$.

Then for some a, $x_a \neq y_a$. Hence y_a belongs to the open set $X_a - \{x_a\} = U$ and $P_a^{-1}[U]$ is a neighborhood of y which is contained in $X - \{x\}$. Thus $X - \{x\}$ is a neighborhood of each of its points and is, therefore, an open set.

THEOREM 2.2: The product of an arbitrary family of T_2-spaces is a T_2-space.

Proof: Let $\{X_a\}$ be an arbitrary family of T_2-spaces and let x and y be distince points in the product. Then for some a, $x_a \neq y_a$ and so there are open neighborhoods U and V of x_a and y_a respectively that are disjoint. Consequently $P_a^{-1}[U]$ and $P_a^{-1}[V]$ are disjoint neighborhoods of x and y in the product.

THEOREM 2.3: The product of an arbitrary family of regular spaces is regular.

Proof: Let $\{X_a : a \in A\}$ be an arbitrary family of regular spaces and let $X = \prod_{a \in A} X_a$ be the product space. Suppose $x \in X$ and W is an open neighborhood of x. Let U be a member of the defining base such that $x \in U \subset W$. Then

$$U = \bigcap \{P_a^{-1}[U_a] : a \in \gamma\}$$

where γ is a finite subset of A and U_a is an open set containing x_a in X_a. Since each X_a is regular there exists for each a an open set V_a with $x_a \in V_a \subset \overline{V_a} \subset U_a$. Therefore $x \in V = \bigcap \{P_a^{-1}[\overline{V_a}] : a \in \gamma\} \subset U \subset W$ and V is a closed neighborhood of x. Thus X is regular.

DEFINITION 2.2: A space X is called completely regular if and only if for each member x of X and each neighborhood

U of x there is a continuous function f on X to the closed unit interval $[0,1]$ such that $f(x)=0$ and $f(y)=1$ for all y in X-U. A completely regular T_1-space is called a Tychonoff space.

THEROEM 2.4: The product of an arbitrary family of completely regular spaces is completely regular [3].

Proof: Let $X = \prod_{a\in A} X_a$ be the product of an arbitrary family of completely regular spaces, let $x\in X$, and let U be a neighborhood of x. Then there exists a finite subset γ of A and a family of open sets $\{U_a : a\in\gamma\}$ such that $x\in \bigcap_{a\in\gamma} P_a^{-1}[U_a] \subset U$. But each X_a is completely regular; hence for each $a\in\gamma$, there exists a continuous function $f_a : X_a \longrightarrow [0,1]$ such that $f_a(x_a)=0$ and $f_a[X_a - U_a]=\{1\}$. Then $f_a \circ P_a$ is a continuous function from X to $[0,1]$ such that $f_a\circ P_a(x)=0$ and $f_a\circ P_a[X-P_a^{-1}[U_a]] = \{1\}$. Define $f:X\longrightarrow[0,1]$ by $f(t)=\max.\{f_a\circ P_a(t) : a\in\gamma\}$, for $t\in X$. The function f is the maximum of finitely many continuous functions on X and hence is a continuous function. Moreover $f(x)=0$ and $f[X-U]\subset f[X-\bigcap_{a\in\gamma} P_a^{-1}[U_a]]\subset f[\bigcup_{a\in\gamma}(X-P_a^{-1}[U_a])]=\{1\}$. Thus X is completely regular.

COROLLARY: The product of an arbitrary family of Tychonoff spaces is a Tychonoff space.

Proof: Theorems 2.1 and 2.4.

A natural candidate, as a productive topological property, for the next theorem in this sequence is normality. But the product of an arbitrary family of normal spaces is not necessarily normal, and, as mentioned in the introduction, it is the purpose of this paper to show, to some extent, how drastically normality fails to be productive. We shall content ourselves, in this chapter, to show that the product of two normal spaces is not necessarily normal. However, the example we shall use to show this, which is due to R. H. Sorgenfrey, also shows that a number of other topological properties are not productive. Before we present the example, we now give the following definitions and a useful theorem.

DEFINITION 2.3: A sequence $x_1, x_2, \cdots, x_n \cdots$ of points of a metric space is called a Cauchy sequence if for any real number $e>0$ there exists an integer $n(e)$ such that for all $n,m > n(e)$, $d(x_n, x_m) < e$. A metric space X is said to be complete if every Cauchy sequence in X converges to a point of X.

DEFINITION 2.4: A subset A of a space X is said to be nowhere dense if the interior of the closure ($\mathring{\bar{A}}$) of A is void. A space X is said to be of the first category if X is a countable union of nowhere dense sets. If X is not of the first category, X is said to be of the second category.

LEMMA 2.1: The finite union of nowhere dense sets is nowhere dense.

Proof: It will suffice to show that if A and B are two closed sets such that both $\mathring{A}$ and $\mathring{B}$ are void, then $\overset{\circ}{\overline{A \cup B}}$ is void. Suppose A and B are two closed sets with vacuous interior. Then B-A is open in $A \cup B$ and contained in B;

hence $(B-A) \cap \widehat{A \cup B}^{\circ}$ is open in $\widehat{A \cup B}^{\circ}$ and therefore absolutely open. Thus $(B-A) \cap \widehat{A \cup B}^{\circ} \subset \mathring{B} = \emptyset$. Therefore $\widehat{A \cup B}^{\circ} \subset A$ but $\mathring{A} = \emptyset$ and consequently $\widehat{A \cup B}^{\circ} = \emptyset$.

THEOREM 2.5: If X is a complete metric space then X is of the second category [5] .

Proof: Let $A = \bigcup_{i=1}^{\infty} A_i$, where each A_i is nowhere dense, be a subset of a complete metric space X. We shall show $A \neq X$. We proceed inductively, first noting that since the union of a finite number of nowhere dense sets is nowhere dense we may suppose $A_i \subset A_{i+1}$.

Now for some x_1 in X we have x_1 in $X-\overline{A_1}$, or else $\overline{A_1} = X$, contradicting the fact that A_1 is nowhere dense. Moreover, there is a neighborhood $N(x_1,e_1) = \{x : d(x,x_1) < e_1\}$ of x_1 such that $\overline{N(x_1,e_1)}$ does not meet A_1. Now suppose we have found, for $1 \leq i \leq n$, points x_i and neighborhoods $N(x_i,e_i)$ of x_i such that

(i) $0 < e_i \leq e_{i-1}/2$,

(ii) $\overline{N(x_i,e_i)}$ does not meet A_i, and

(iii) $\overline{N(x_i,e_i)} \subset \overline{N(x_{i-1},e_{i-1})}$.

Then $N(x_n,e_n) \not\subset \overline{A_{n+1}}$ since A_{n+1} is nowhere dense and consequently there is a point x_{n+1} in $N(x_n,e_n)-A_{n+1}$ and a real number $e_{n+1} \leq e_n/2$ such that $N(x_{n+1},e_{n+1}) \subset N(x_n,e_n)$ and $\overline{N(x_{n+1},e_{n+1})}$ does not meet A_{n+1}. Therefore the above conditions are satisfied for k=n+1 and by induction we have obtained a decreasing sequence $\{\overline{N(x_i,e_i)}\}$ of non-empty closed sets whose diameter approaches zero. It is clear that $\{x_i\}$

is a Cauchy sequence eventually in each $N(x_i, e_i)$. Thus, since X is a complete metric space, $\{x_i\}$ converges to a point x in X belonging to all sets $\overline{N(x_i, e_i)}$. Therefore $x \notin A$ and $A \neq X$.

We now give the previously mentioned example as

THEOREM 2.6: There is a paracompact Hausdorff space X with the Lindelöf property such that $X \times X$ is not normal and does not have the Lindelöf property [8] .

Proof: Let X be the set of all non-negative real numbers with the half-open interval topology, i.e., a base for the topology is the family of all half-open intervals $[a,b)$ for a and b in X.

(1) The space X is Hausdorff. For let $x,y \in X$, say $x<y$. Then there is a point $z \in X$ such that $x<z<y$ and therefore $[0,z)$ and $[z,y+1)$ are disjoint open neighborhoods of x and y respectively.

(2) The space X is paracompact. Let $\mathcal{U}$ be an open covering of X and let $\mathcal{U}'$ be a refinement of $\mathcal{U}$ whose members are sets of the form $[a,b)$ where $a,b \in X$. Let E be the set of all points q of X which satisfy the relation $a<q<b$ for no neighborhood $[a,b) \in \mathcal{U}'$. We shall now show that E is closed relative to the natural topology of the real numbers. Let $z \in X-E$, then there is a set $[a,b) \in \mathcal{U}'$ such that $a<z<b$, and since the open interval (a,b) contains no points of E, z does not belong to the closure of E. Thus E is closed relative to the natural topology of the real numbers and consequently the

complement of E is the union of a countable number of disjoint open intervals $I_1, I_2, \cdots$. Clearly for each n the left endpoint of I_n is a point of E and conversely every point of E is a left end point of some interval I_n. Hence if I_n is the interval (q_n, f_n) it follows that $E = \bigcup_n \{q_n\}$ and hence that $X = \bigcup_n (\{q_n\} \cup I_n)$. For each n there is a member $[q_n, w_n) = D_n$ of $\mathcal{U}'$ (where $w_n < f_n$), so let γ_n be the collection of all open intervals (a,b) such that $[a,b)$ belongs to $\mathcal{U}'$ and $[a,b) \subset [q_n, f_n)$. Now suppose $z \in [w_n, f_n)$. Then, since z does not belong to E, there exists $[t,h)$ in $\mathcal{U}'$ such that $z \in [t,h)$ and since $f_n \in E$, $f_n \notin [t,h)$. Thus (t,h) belongs to γ_n and so γ_n is an open covering of $[w_n, f_n)$. Now under the natural topology of the real numbers $[w_n, f_n)$ is a metric space and therefore paracompact by Theorems 1.1 and 1.2. It follows that there is a locally finite open refinement δ_n of γ_n that covers $[w_n, f_n)$. Let $\mathcal{E}_n$ be the collection of intersections of members of δ_n with $[w_n, f_n)$. The members of $\mathcal{E}_n$ are open in the topology for X since each member of $\mathcal{E}_n$ is a union of sets of the form $[a,b)$ or (a,b). Moreover, each member of $\mathcal{E}_n$ is a subset of a member of $\mathcal{U}'$ and therefore a subset of a member of $\mathcal{U}$. Thus if η_n is the collection obtained by adding the open set D_n to $\mathcal{E}_n$, η_n is a locally finite open covering of $\{q_n\} \cup I_n$ whose members are subsets of $\mathcal{U}$. Therefore $\eta = \bigcup_n \eta_n$ is an open refinement of $\mathcal{U}$ that covers X. η is a locally finite covering of X, for let $x \in X$. Then there exists exactly one n such that x belongs to $[q_n,f_n)=D_n \cup [w_n,f_n)$. If $x\in D_n$ then, by the construction, D_n is an open neighborhood of x and no other member of η meets D_n; and if $x\in[w_n,f_n)$ then some neighborhood U of x meets only finitely many sets of η_n, and hence the neighborhood $U \cap [w_n,f_n)$ meets only finitely many members of η. Thus X is paracompact.

(3) The space X is also Lindelöf. First, notice that X is fully normal. For by (1) and (2) X is a Hausdorff and paracompact space and therefore normal by Theorem 1.5 and consequently fully normal by Theorem 1.4. Now let $\mathcal{U}$ be an open covering of X, $\mathcal{U}'$ be an open star refinement of $\mathcal{U}$ (we may suppose the sets of $\mathcal{U}'$ are of the form $[a,b)$), γ be a family of disjoint sets of $\mathcal{U}'$, and γ^* be the family of all such γ. Partially ordering γ^* by set inclusion, it is easy to see that every chain in γ^* has an upper bound and consequently, by Zorn's Lemma, γ^* has a maximal member γ. γ is certainly countable, so let $\gamma = \{[a_i,b_i)\}$ and for each i let U_i and V_i be some members of $\mathcal{U}$ containing the stars of a_i and b_i respectively. Then the family of all such U_i and V_i is countable and, moreover, covers X. For let $x\in X$. If x belongs to $[a_i,b_i)$, for some i, then $x\in U_i$. So suppose x does not belong to $[a_i,b_i)$ for any integer i. Then for some $[a,b)$ in $\mathcal{U}'$, $x\in[a,b)$, and since γ is maximal, $[a,b)$ meets $[a_i,b_i)$ for some integer i. Therefore x belongs to U_i or V_i. Thus X is Lindelöf.

(4) The product space $X \times X$ is not normal. Let Y=

$\{(x,y) \in X \times X : x+y=1\}$, A be the set of all points of Y whose first coordinate is irrational, and let B= Y-A. Clearly A and B are disjoint. They are also closed. For suppose (x_0, y_0) does not belong to A. If $x_0 + y_0 \geq 1$, then for any positive number e, the neighborhood $[x_0, x_0+e) \times [y_0, y_0+e)$ of (x_0, y_0) does not meet A; and if $x_0 + y_0 < 1$, the neighborhood $[x_0, x_0+e) \times [y_0, y_0+e)$ of (x_0, y_0) does not meet A if $e = \sqrt{2}\, d$, where 2d is the distance from the point (x_0, y_0) to the segment x+y=1. Thus A, and similarly B, is closed.

Suppose U and V are open sets of $X \times X$ that cover A and B respectively. Let $Q(x) = \{e : [x, x+e) \times [1-x, 1-x+e) \subset U\}$ for x such that $(x, 1-x) \in A$. Now define a function f on A as follows: f(x)=1 if $1 \in Q(x)$ and f(x)= sup. Q(x) if $1 \notin Q(x)$. Then f is defined on the set of irrational numbers T in the closed unit interval I= $[0,1]$ and f is never zero. Moreover, $T = \bigcup_{n=1}^{\infty} T_n$, where $T_n = \{x : f(x) \geq 1/n\}$ and $I = T \cup I'$ where I' is the rational numbers of I. We assert that: for some rational number r in I, and for some integer n, r belongs to the Euclidean closure of T_n. Suppose not. Then, for all integers n, $\overset{o}{\overline{T_n}}$ does not contain any rational numbers of I, and consequently $\overset{o}{\overline{T_n}}$ is void, i.e., T_n is nowhere dense for all n. But I' is countable and consequently I=$[0,1]$ is of the first category. Since I= $[0,1]$ is a complete metric space, we have contradicted Theorem 2.5 and therefore our assertion is true. Thus there is a rational number r belonging to the closure of T_n, for some n. Now let $V' = [r, r+e) \times [1-r, 1-r+e)$ be a neighborhood of (r,1-r) that is contained in V and let $q>0$ be such that $2q<\min.\{e,1/n\}$. By the above, there is an irrational number x such that $|r-x|<q$ and $x \in T_n = \{x : f(x) \geq 1/n\}$. We shall now show that $(r+q, 1-r)$ belongs to $U \cap V$. Clearly $(r+q, 1-r)$ belongs to V. If $x>r$, then

$$0<(r+q)-x = r-x+q<2q<1/n, \text{ so}$$

$$x<r+q<x+1/n \text{ and}$$

$$(1-r)-(1-x) = x-r<q<1/n, \text{ so}$$

$$1-x<1-r<1-x+1/n.$$

Hence $(r+q, 1-r) \in [x, x+1/n) \times [1-x, 1-x+1/n) \subset U$. If $x<r$, a similar demonstration will achieve the same result. Thus U and V cannot be disjoint and therefore $X \times X$ is not normal.

(5) The product space $X \times X$ is not Lindelöf. Define the set Y as before and let $\mathcal{U}$ be the open covering consisting of $X \times X - Y$ and open sets of the form $[x, x-1) \times [1-x, 1-x+1)$ for $(x, 1-x)$ in Y. Then the covering $\mathcal{U}$ is not countable and no countable subfamily of $\mathcal{U}$ can cover $X \times X$. Thus $X \times X$ is not Lindelöf.

We have, therefore, obtained (by Theorems 1.4, 1.5, 2.2, and 2.6) the following:

(i) The topological product of two Lindelöf spaces need not be Lindelöf .

(ii) The topological product of two normal spaces need not be normal.

(iii) The topological product of two paracompact (and Hausdorff) spaces need not be paracompact (or even normal).

(iv) The topological product of two fully normal (and Hausdorff) spaces need not be fully normal (or even normal).

CHAPTER III

PRODUCTS OF A GIVEN SPACE WITH A COMPACT SPACE

The first part of this chapter will be devoted to theorems involving the topological product of a compact space and a space enjoying a generalized compact property. The second part will deal with normality of product spaces.

The first theorem will be the classical theorem of Tychonoff on the product of compact spaces. We will need the following definition.

DEFINITION 3.1 : A family $\mathcal{U}$ of sets has the finite intersection property if and only if the intersection of the members of each finite subfamily of $\mathcal{U}$ is non-void.

THEOREM 3.1 : The topological product of an arbitrary family of compact spaces is compact [3].

Proof: Let $X = \prod_{a \in A} X_a$ be a product of compact spaces and let F be a family of closed subsets of X having the finite intersection property. Set

$$F^* = \{\mathcal{B} : \mathcal{B} \text{ is a family of subsets of X having the finite intersection property and such that } F \subset \mathcal{B}\}.$$

Partially ordering F^* by set inclusion, it is easily seen that every chain in F^* has an upper bound in F^* and therefore by Zorn's lemma F^* has a maximal member $\mathcal{B}$. Thus $\mathcal{B}$ is the largest family with the finite intersection property containing F. Since a space is compact if and only if every family of closed sets having the finite intersection property has a non-void intersection, it suffices to show that $\bigcap\{\overline{B}:B\in\mathcal{B}\}\neq\emptyset$.

It is easy to see that the family $\mathcal{B}$ has the following properties:

(i) $B\in\mathcal{B}$ and $C\supset B \longrightarrow C\in\mathcal{B}$.

(ii) $B_1,B_2\in\mathcal{B}\longrightarrow B_1\cap B_2\in\mathcal{B}$.

(iii) $C\cap B\neq\varphi$ for all $B\in\mathcal{B}\longrightarrow C\in\mathcal{B}$, and

(iv) $P_a[\mathcal{B}] = \{P_a[B] : B\in\mathcal{B}\}$ has the finite intersection property. (Since $\varphi\neq P_a[B_1\cap B_2]\subset P_a[B_1]\cap P_a[B_2]$).

Since X_a is compact, X_a has the finite intersection property and therefore, by (iv) for each $a\in A$ there exists x_a in X_a such that $x_a\in\{P_a[B]:B\in\mathcal{B}\}$. Hence $U\cap P_a[B]\neq\varphi$ for each neighborhood U of x_a and consequently $P_a^{-1}[U]\cap B\neq\varphi$ for each neighborhood U of x_a and each $B\in\mathcal{B}$. Therefore every basic neighborhood of x is in $\mathcal{B}$ and so $x\in\overline{B}$ for all $B\in\mathcal{B}$. Thus X is compact.

In Chapter II we saw that the product of two paracompact spaces need not be paracompact. However, we do have the following:

THEOREM 3.2 : The topological product of a compact space and a paracompact space is paracompact [1].

Proof: Let X be a compact space, Y a paracompact space, and let $\mathcal{U}$ be an open covering of $X\times Y$. For each (x,y) in $X\times Y$ there exist open sets V_{xy},W_{xy} in X and Y respectively such that (x,y) is in $V_{xy}\times W_{xy}$ and $V_{xy}\times W_{xy}$ is contained in some set of $\mathcal{U}$. The resulting family $\{V_{xy}\times W_{xy}:(x,y)\in X\times Y\}$ is an open refinement of $\mathcal{U}$ covering $X\times Y$. Now since X is compact, there exists, for each $y \in Y$, a finite subset A_y of X such that $\{V_{xy} : x \in A_y\}$ covers X. Let $W_y = \bigcap\{W_{xy} : x \in A_y\}$. The family $\{W_y : y \in Y\}$ covers Y and therefore has an open locally finite refinement $\{N_\beta : \beta \in B\}$ that covers Y. For each $\beta \in B$, choose $y_\beta \in Y$ such that $N_\beta \subset W_{y_\beta}$. Finally, let $\mathcal{U}' = \{V_{xy_\beta} \times N_\beta : \beta \in B \text{ and } x \in A_{y_\beta}\}$. $\mathcal{U}'$ refines $\mathcal{U}$ since $V_{xy_\beta} \times N_\beta \subset V_{xy_\beta} \times W_{y_\beta} \subset V_{xy_\beta} \times W_{xy_\beta}$, if $\beta \in B$, and $x \in A_{y_\beta}$. $\mathcal{U}'$ is a locally finite cover of $X \times Y$, for let (x,y) belong to $X \times Y$. Then there exists $\beta \in B$ such that $y \in N_\beta$ and there exists $z \in A_{y_\beta}$ such that $x \in V_{zy_\beta}$. Hence (x,y) belongs $V_{zy_\beta} \times N_\beta$, i.e., $\mathcal{U}'$ covers $X \times Y$. On the other hand, there is an open set W containing y such that W meets only finitely many N_β, and for each such β, X meets only finitely many V_{zy_β} with $z \in A_{y_\beta}$. Hence $X \times W$ meets only finitely many sets of $\mathcal{U}'$. Thus $X \times Y$ is paracompact.

Replacing paracompactness by countable paracompactness, we now have the following analogous result.

THEOREM 3.3: The topological product of a countably paracompact space and a compact space is countably paracompact [2].

Proof: Let X be a countably paracompact space, Y be a compact space, and let $\{U_i : i=1,2,\cdots\}$ be a countable covering of $X \times Y$. For each positive integer i, let V_i be the set of all $x \in X$ such that $\{x\} \times Y \subset \bigcup_{j \leq i} U_j$. If $x \in V_i$, then for every point (x,y) of $\{x\} \times Y$ there are sets M and N such that M is open in X, N is open in Y, and $(x,y) \in M \times N \subset \bigcup_{j \leq i} U_j$. Now since Y is compact a finite number of these sets N cover Y; let M_x be the intersection of the corresponding finite number of the sets M. Then $x \in M_x$, M_x is open and $M_x \times Y \subset \bigcup_{j \leq i} U_j$; hence

$M_x \subset V_i$. Therefore V_i is open for all integers i. Also, for any $x \in X$, $\{x\} \times Y$ is compact and therefore (since $\{U_i\}$ covers $\{x\} \times Y$) $\{x\} \times Y$ is contained in some finite subfamily of $\{U_i\}$. It follows that $x \in V_i$, for some i, and therefore $\{V_i\}$ is a covering of X. Since $\{V_i\}$ is countable and X is countably paracompact, $\{V_i\}$ has an open locally finite refinement $\{G_i\}$ such that $G_i \subset V_i$ (by lemma 1.2).

For each i and for each j such that $j \leq i$, let $G_{ij} = (G_i \times Y) \cap U_j$. Then G_{ij} is an open covering of $X \times Y$. For let (x,y) belong to $X \times Y$, then for some i, $x \in G_i$, hence (x,y) belongs to $G_i \times Y$ and since $G_i \subset V_i$, $(x,y) \in \{x\} \times Y \subset \bigcup_{j \leq i} U_i$. Therefore, for some $j \leq i$, $(x,y) \in U_j$ and it follows that $(x,y) \in G_{ij}$. Moreover, since $G_{ij} \subset U_j$, $\{G_{ij}\}$ is a refinement of $\{U_i\}$. Also, if (x,y) belongs to $X \times Y$, x belongs to some open set H which meets only a finite number of the sets of $\{G_i\}$. Then $H \times Y$ is an open set containing (x,y) which can meet G_{ij} only if H meets G_i. But, for each i, there are only a finite number of the sets G_{ij}. Hence $H \times Y$ meets only a finite number of sets of $\{G_{ij}\}$; hence $\{G_{ij}\}$ is locally finite. Thus $X \times Y$ is countably paracompact.

The techniques of the above proof are useful in proving the corresponding theorem for countable compactness.

THEOREM 3.4: The topological product of a countably compact space and a compact space is countably compact.

Proof: Let X be a countably compact space, Y be a compact space, and let $\{U_i : i=1,2,\cdots\}$ be a countable open covering of $X \times Y$. For each positive integer i let V_i be the set of all $x \in X$ such that $\{x\} \times Y \subset \bigcup_{j \leq i} U_j$. Proceeding exactly as in the preceeding theorem, $\{V_i\}$ is a countable open covering of X. Since X is countably compact, finitely many members $V_{i_1}, \cdots, V_{i_n}$ of $\{V_i\}$ cover X. Let

$$\mathcal{U} = \{U_j : j \leq N = \max. \{i_1, \cdots, i_n\}\}.$$

$\mathcal{U}$ is a finite subfamily of $\{U_i\}$ that covers $X \times Y$, for let $(x,y) \in X \times Y$. Then x belongs to one of the sets $V_{i_1}, \cdots, V_{i_n}$, say to V_{i_1}, and therefore $(x,y) \in \{x\} \times Y \subset \bigcup_{j \leq i_1} U_j \subset \bigcup_{j \leq N} U_j$.

We now turn directly to the question of the normality of product spaces. As we saw in Chapter II the product of two normal Hausdorff spaces need not be normal. We shall subsequently show that the product of a normal space and the closed unit interval need not be normal. First, we have the following lemma:

Lemma 3.1: There is a countable base for the open sets of a compact metric space.

Proof: For each positive integer n, $\mathcal{U}^n = \{N(x,1/n) : x \in X\}$ is an open covering of a compact metric space X and for each n, there exists a finite subfamily of $\mathcal{U}^n$ that covers X. The union $\mathcal{U}$ of these finite subfamilies is a countable covering of X and we shall show that $\mathcal{U}$ is the desired base. Let U be an open set containing some point x in X. Then $d(x, X-U) = \inf. \{d(x,z) : z \in X-U\} = q > 0$, and there exists a positive integer n and a member $N(z,1/n)$ of $\mathcal{U}$ such that $x \in N(z,1/n)$ and $2/n < q$. Hence $d(w,x) \leq d(w,z) + d(z,x) < 1/n + 1/n < q$ if $w \in N(z,1/n)$ and therefore $x \in N(z,1/n) \subset U$.

We now give a sufficient condition for a product space to be normal.

THEOREM 3.5: The topological product of a countably paracompact normal space and a compact metric space is normal [2].

Proof: Let X be a countably paracompact normal space, let Y be a compact metric space, and let A and B be two disjoint closed sets in the product space $X \times Y$. By the above lemma there is a countable base $\{G_i\}$ for the open sets of Y. Let $H_\gamma = \bigcup_{i \in \gamma} G_i$, where γ is any finite set of positive integers and, for each $x \in X$, let A_x and B_x be the sets defined by: $\{x\} \times A_x = (\{x\} \times Y) \cap A$ and $\{x\} \times B_x = (\{x\} \times Y) \cap B$. Since $\{x\} \times Y$ is homeormorphic to Y, A_x and B_x are closed (and disjoint).

Let $U_\gamma = \{x : A_x \subset H_\gamma \subset \overline{H_\gamma} \subset Y - B_x\}$. U_γ is open, for let $x_0 \in X$ be such that $A_{x_0} \subset H_\gamma$. Then, for each $y \in Y - H_\gamma$, $(x_0, y) \notin A$ and, since A is closed, there are open sets M in X and N in Y such that (x_0, y) belongs to $M \times N$ and $M \times N$ does not meet A. Since $Y - H_\gamma$ is closed in Y, it is compact and therefore a finite number of the sets N cover $Y - H_\gamma$. If M_{x_0} is the intersection of the corresponding finite number of the sets M, then M_{x_0} is open and $M_{x_0} \times (Y - H_\gamma)$ does not meet A. Hence if $x \in M_{x_0}$ and $y \in A_x$ then $(x,y) \in A$. Therefore $(x,y) \notin M_{x_0} \times (Y - H_\gamma)$ and consequently $A_x \subset H_\gamma$. Thus $A_x \subset H_\gamma$ for all $x \in M_{x_0}$ and therefore the set $\{x : A_x \subset H_\gamma\}$ is open. Now let x_0 be a point of X for which $\overline{H_\gamma} \subset Y - B_{x_0}$. Then for each $y \in \overline{H_\gamma}$, $(x_0, y) \notin B$ and, since B is closed, there are open sets M in X and N in Y such that (x_0, y)

belongs to $M \times N$ and $M \times N$ does not meet B. Since $\overline{H_\gamma}$ is closed in Y, it is compact, and so a finite number of the sets M covers $\overline{H_\gamma}$. If M_{x_0} is the intersection of the corresponding finite number of the sets M, then M_{x_0} is open and $M_{x_0} \times \overline{H_\gamma}$ does not meet B. Now let $x \in M_{x_0}$ and $y \in B_x$. Then $(x,y) \in B$; hence (x,y) does not belong to $M_{x_0} \times \overline{H_\gamma}$, and $y \notin \overline{H_\gamma}$, so that $\overline{H_\gamma} \subset Y - B_x$. Thus $\overline{H_\gamma}$ is contained in $Y-B_x$ for all $x \in M_{x_0}$ and therefore the set $\{x : \overline{H_\gamma} \subset Y - B_x\}$ is open. Since U_γ is the intersection of the latter set and the set $\{x : A_x \subset H_\gamma\}$, U_γ must be open.

Now let $x \in X$; then for each y in A_x, $y \notin B_x$ since A and B are disjoint. Moreover, since Y is a metric space, Y is normal (Theorems 1.1 and 1.4) and since B_x is closed there is a G_i such that $y \in G_i \subset \overline{G_i} \subset Y - B_x$. Since Y is compact, a finite number of these sets G_i covers the closed set A_x, i.e., for some finite set γ of positive integers $A_x \subset \bigcup_{i \in \gamma} G_i = H_\gamma$ and $\overline{H_\gamma} = \bigcup_{i \in \gamma} \overline{G_i} \subset Y - B_x$. Hence x belongs to U_γ. Thus the open sets U_γ cover X. Since there are only a countable number of finite subsets γ of the positive integers, the covering $\{U_\gamma\}$ of X is countable. Since X is countably paracompact, by Lemma 1.2, $\{U_\gamma\}$ has a locally finite refinement $\{W_\gamma\}$ that covers X with $W_\gamma \subset U_\gamma$. Moreover, by (iii) of Theorem 1.6, $\{W_\gamma\}$ has a refinement $\{V_\gamma\}$ (still locally finite) that covers X with $V_\gamma \subset \overline{V_\gamma} \subset W_\gamma \subset U_\gamma$. Let U be the open set $\bigcup_\gamma (V_\gamma \times H_\gamma)$. For any point (x,y) of A, and for some V_γ, $x \in V_\gamma \subset U_\gamma$. Then $y \in A_x \subset H_\gamma$ and therefore $(x,y) \in V_\gamma \times H_\gamma$. Thus $A \subset U$. Moreover, since $\{V_\gamma\}$ is a locally finite covering of X, every member x of X is contained in an open set G(x) which meets only finitely many of the sets V_γ; and hence the neighborhood $G(x)\times Y$ of (x,y) meets only finitely many of the sets $V_\gamma \times H_\gamma$. Hence (x,y) belongs to the closure of some $V_\gamma \times H_\gamma$, i.e., $\overline{U} = \bigcup_\gamma \overline{V_\gamma \times H_\gamma}$. But $\overline{V_\gamma \times H_\gamma} = \overline{V_\gamma}\times\overline{H_\gamma}$. Therefore $\overline{U} = \bigcup_\gamma (\overline{V_\gamma}\times\overline{H_\gamma}) \subset \bigcup_\gamma (U_\gamma \times \overline{H_\gamma})$. But if $(x,y)\in \bigcup_\gamma (U_\gamma \times \overline{H_\gamma})$ then $(x,y)\in U_\gamma \times \overline{H_\gamma}$ for some γ and consequently $\overline{H_\gamma} \subset Y-B_x$. Thus $(x,y)\notin B$. Hence the open set U contains A and its closure does not meet B. Therefore $X\times Y$ is normal.

In conclusion, we now give a necessary and sufficient condition on a normal space X for the topological product of X and the closed unit interval to be normal. The following theorem, which is also due to C. H. Dowker, justifies the concept of countable paracompactness.

THEOREM 3.6: The following three properties of a topological space X are equivalent:

(i) The space X is countably paracompact and normal,

(ii) If g is a lower semicontinuous real function on X and h is an upper semicontinuous real function on X and if $h(x) < g(x)$ for all $x \in X$, then there exists a continuous real function f such that $h(x) < f(x) < g(x)$ for all $x \in X$.

(iii) The topological product $X \times I$ of X with the closed unit interval $I = [0,1]$ is normal.

Proof: (i) $\longrightarrow$ (ii). Let X be a countable paracompact normal space and let g and h be lower and upper semicontinuous real functions with $h(x) < g(x)$ for all $x \in X$. For each rational number r let $G_r = \{x : h(x) < r < g(x)\}$. Since g is lower semi-continuous, $\{x : g(x) > r\}$ is open, and, since h is upper semi-continuous, $\{x : h(x) < r\}$ is open. Thus G_r is open for all r.

For each $x \in X$ we have $h(x) < g(x)$ and so there is a rational number r such that $h(x) < r < g(x)$; hence $x \in G_r$. Thus $\{G_r\}$ is a countable open covering of X. Hence, since X is countably paracompact and normal, there is, by Lemma 1.2 and Theorem 1.6 a locally finite open covering $\{U_r\}$ of X with $U_r \subset G_r$ and also a locally finite open covering $\{V_r\}$ of X such that $\overline{V_r} \subset U_r$.

By Urysohn's Lemma, there is, for each r, a continuous function f_r from X to $[-\infty, r]$ where $[-\infty, r]$ has the usual topology and $f_r(x) = -\infty$ if $x \notin U_r$, $f_r(x) = r$ if $x \in \overline{V_r}$. Let $f(x)$ be the least upper bound of the extended real numbers $f_r(x)$. Each point x_0 of X is contained in an open set $N(x_0)$ which meets only a finite number of the sets U_r. Hence, in $N(x_0)$ for all but a finite number of values of r, $f_r(x) = -\infty$. Thus in the neighborhood $N(x_0)$, $f(x)$ is the least upper bound of a finite number of continuous functions, hence f is continuous on $N(x_0)$. But x_0 is arbitrary. Hence f is continuous on X. In U_r, $f_r(x) \leq r < g(x)$ and in $X - U_r$, $f_r(x) = -\infty < g(x)$. Thus $f_r(x) < g(x)$ for each r and, for each x, $f(x)$ is the least upper bound of a finite number of $f_r(x)$, each less than $g(x)$. Thus $f(x) < g(x)$. Each x is in some V_r, and for this r, $f_r(x) = r$; hence $f(x) \geq f_r(x) = r > h(x)$. Thus $f(x) > h(x)$. Therefore $h(x) < f(x) < g(x)$.

(ii) $\longrightarrow$ (i) Let X be a space satisfying condition (ii) and let A and B be two disjoint closed subsets of X. Define functions h and g on X as follows:

$$h(x) = 1 \text{ if } x \in A,\ h(x) = 0 \text{ if } x \notin A, \text{ and}$$

$$g(x) = 1 \text{ if } x \in B,\ g(x) = 2 \text{ if } x \notin B.$$

Clearly g is lower semicontinuous, h is upper semicontinuous and $h(x) < g(x)$ for all $x \in X$. Hence there is a continuous real function f on X with $h(x) < f(x) < g(x)$. Let $U = \{x : f(x) > 1\}$ and $V = \{x : f(x) < 1\}$. Then U and V are disjoint open sets; if $x \in A$ we have $1 = h(x) < f(x)$, hence $x \in U$, and if $x \in B$ we have $1 = g(x) > f(x)$, hence $x \in V$. Thus $A \subset U$ and $B \subset V$ and therefore X is normal.

Now let $\{F_i : i = 1, 2, \cdots\}$ be a decreasing sequence of closed sets with vacuous intersection. Define functions h and g on X as follows, $h(x) = 0$ for all $x \in X$ and $g(x) = 1/i+1$ for $x \in F_i - F_{i+1}$ for $(i = 0, 1, 2, \cdots)$ where F_0 means the whole space X. The function h is continuous, hence upper semicontinuous. To show that g is lower semicontinuous let the real number q be given. For $q \geq 1$, $\{x : g(x) \leq q\} = X$; for $q \leq 0$, $\{x : g(x) \leq q\}$ is void; and if $0 < q < 1$ then for some positive integer i, $1/i+1 \leq q < 1/i$ hence $\{x : g(x) \leq q\} = F_i$. Thus, in any case, $\{x : g(x) \leq q\}$ is closed and therefore g is lower semicontinuous. Hence there is a continuous real function f on X with $0 < f(x) < g(x)$ for $x \in X$. Let $G_i = \{x : f(x) < 1/i+1\}$. Then G_i is open, $F_i \subset G_i$ and, since $f(x) > 0$ for all x, $\bigcap_{i=1}^{\infty} G_i$ is void. Thus by Theorem 1.6, X is countably paracompact.

(i) $\longrightarrow$ (iii). This follows immediately from Theorem 3.5 and the fact that the closed unit interval is a compact metric space.

(iii) $\longrightarrow$ (i). Let X be a space for which $X \times I$ is normal. Then X is homeomorphic to the closed subset $X \times \{0\}$ of the normal space $X \times I$ and therefore X is normal. Let $\{F_i : i=1,2,\cdots\}$ be a decreasing sequence of closed sets in X with vacuous intersection. Then, since the half open interval $[0,1/i)$ is open in $I=[0,1]$, $W_i = (X-F_i) \times [0,1/i)$ is open in $X \times I$. Let A be the closed set $X \times I - \bigcup_{i=1}^{\infty} W_i$. If $x \in X$, then, for some i, $x \in X-F_i$ and $(x,0) \in W_i$ and hence $(x,0) \notin A$. Hence if $B = X \times \{0\}$, A and B are closed disjoint subsets of the normal space $X \times I$. Therefore there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Set $G_i = \{x : (x,1/i) \in U\}$; then G_i is open. For each $x \in X$, $(x,0) \in B$ and hence for sufficiently large i, $(x,1/i)$ belongs to V. For V is an open neighborhood of $(x,0)$ and therefore the sequence $\{(x,1/i) : i=1,2,\cdots\}$ is eventually in V. Thus for some i, $(x,1/i) \in V$ and so $(x,1/i) \notin U$. Thus $\bigcap_{i=1}^{\infty} G_i$ is void. Moreover for $x \in F_i$, if $j \leq i$, $F_i \subset F_j$ and $x \notin X-F_j$; and if $j > i$, $1/i \notin [0,1/j)$. Hence, in any case, $(x,1/i) \notin W_j$ for $j=1,2,\cdots$. Hence $(x,1/i) \in A \subset U$ and $x \in G_i$. Therefore $F_i \subset G_i$ and by Theorem 1.6, X is countably paracompact. This completes the proof of the theorem.

REMARK: Since a normal space need not be countably paracompact (see the example given at the end of Chapter I) the topological product of a normal space and the closed unit interval need not be normal. It is not known, to this author's knowledge, whether there exists a normal Hausdorff space X such that $X \times I$ is not normal. But by the above theorem and the example due to M. E. Rudin mentioned at the end of Chapter I, there exists such a normal Hausdorff space if there exists a Souslin space.

THEOREM A: There is a separable normal Hausdorff space X which is not paracompact and does not have the Lindelöf property [6] .

Proof: To construct the desired space we shall need the following lemma:

For each ordinal α in the set ω_1 of all countable ordinals there is a function f_α defined on the set ω_0 of positive integers, with values in ω_0, such that: if $\alpha<\beta$, there exists an integer $m(\alpha,\beta)= m(\beta,\alpha)$ such that $f_\alpha(i)< f_\beta(i)$ whenever $i>m(\alpha,\beta)$.

Proof of the lemma is by transfinite induction. Define $f_o:\omega_o\longrightarrow\omega_o$ by $f_o(i)= 1$ for all $i\in\omega_o$. Let $\beta\in\omega_1$ and suppose for each $\alpha<\beta$ that f_α is defined in such a way that, whenever $\alpha_1<\alpha_2<\beta$, there exists $m(\alpha_1,\alpha_2)$ for which

$$i>m(\alpha_1,\alpha_2)\longrightarrow f_{\alpha_1}(i)<f_{\alpha_2}(i).$$

The set $\{\alpha:\alpha<\beta\}$ is countable and hence can be reordered as a sequence $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$. Define $f_\beta:\omega_0\rightarrow\omega_0$ by

$$f_\beta(i)= 1+\sum_{j=1}^{i} f_{\alpha_j}(i) \text{ for } i\in\omega_0.$$

Then for each $\alpha_n<\beta$, let $m(\alpha_n,\beta)= n$ and observe that

$$i>n\longrightarrow f_\beta(i)=1+\sum_{j=1}^{i} f_{\alpha_j}(i)>f_{\alpha_n}(i).$$

The lemma now follows by induction.

In the proof of this theorem Greek letters will always denote countable ordinals; and the letters i,j,k, and n will stand for positive integers.

(1) Construction of the space X. Let A denote the set of all ordered pairs (m,n) and B the set of all countable ordinals. Set $X = A \cup B$. We shall now use the set $\{f_\alpha : \alpha \in \omega_1\}$ of functions constructed in the above lemma to define a topology for X.

The set N will be a neighborhood if and only if it belongs to one of the following classes.

(i) Every point of A is a neighborhood of itself.

(ii) If α is not a limit ordinal, then corresponding to each n there is a neighborhood of α which consists of α itself, and all pairs $(k, f_\alpha(k))$ with $k > n$.

(iii) If α is a limit ordinal, choose an ordinal $\beta < \alpha$, and a positive integer $n(\gamma)$ for each γ such that $\beta < \gamma \leq \alpha$. For each such collection of choices there is a neighborhood of α which consists of (a) all γ such $\beta < \gamma \leq \alpha$, and (b) all pairs $(k, f_\gamma(k))$ with $k > n(\gamma)$ and $\beta < \gamma \leq \alpha$.

Now, for $x \in A$, let $\mathcal{U}(x)$ be the family of all sets containing x; for $\alpha \in B$ such that α is not a limit ordinal let $\mathcal{U}(\alpha)$ be the family of all sets containing a neighborhood of α as defined in (ii); and for $\alpha \in B$ such that α is a limit ordinal let $\mathcal{U}(\alpha)$ be the family of all sets containing a neighborhood of α as defined in (iii). It is easily seen that $\mathcal{U}(x)$ for $x \in X$ is a neighborhood system of x relative to a topology for X. Moreover the neighborhoods as defined in (i), (ii), and (iii) are open with respect to the generated topology.

This completes the construction of the space X. We shall now show that X has the desired properties.

(2) The space X is separable. Since every neighborhood of any point of B meets the countable set A, $\bar{A}=X$. Thus X is separable.

(3) The space X is Hausdorff. Let x and y be distinct points of X. If x and y both belong to A the $\{x\}$ and $\{y\}$ are disjoint neighborhoods of x and y respectively. If $x=(m,n)$ and $y=\alpha$, where α is not a limit ordinal, choose the integer $n(\alpha)$ such that $n(\alpha)>m$. By the constructions (i) and (ii) of (1) x and y have disjoint neighborhoods. If $x=(m,n)$ and $y=\alpha$, where α is a limit ordinal choose $\beta<\alpha$ and integers $n(\gamma)>m$ for all γ such that $\beta<\gamma\leq\alpha$. By the constructions (i) and (iii) of (1), x and y have disjoint neighborhoods. If $x=\alpha$ and $y=\beta$, where α and β are not limit ordinals choose $n=k=m(\alpha,\beta)$. By the construction (ii) of (1) the neighborhoods of x and y corresponding to the choices of n and k, respectively, are disjoint. If $x=\alpha$ and $y=\beta$, where α is not a limit ordinal and β is, choose $\tau<\beta$ if $\beta<\alpha$ and $\alpha<\tau<\beta$ if $\beta>\alpha$. Then choose $n(\gamma)=m(\alpha,\gamma)$ for all γ such that $\tau<\gamma\leq\beta$. By the constructions (ii) and (iii) of (1) the intersection of the basic neighborhood of $y=\beta$ (corresponding to the choice of τ and the choice of $n(\gamma)$ for all $\tau<\gamma\leq\beta$) and any basic neighborhood of $x=\alpha$ is void. Finally, if $x=\alpha$ and $y=\beta$, where α and β are both limit ordinals (say $\alpha<\beta$) choose $\tau<\alpha$ and choose δ such that $\alpha<\delta<\beta$. For all γ such that $\tau<\gamma<\alpha$

choose $n(\gamma)=m(\gamma,\alpha)$; choose $n(\alpha)=m(\alpha,\gamma)$ for some γ such that $\tau<\gamma<\alpha$; and for all λ such that $\delta<\lambda\leq\beta$ choose $n(\lambda)=m(\lambda,\alpha)$. Now let N be the neighborhood of α corresponding to the choices of τ and $n(\gamma)$ for all γ such that $\tau<\gamma\leq\alpha$, and let N' be the neighborhood of β corresponding to the choices of δ and $n(\lambda)$ for all λ such that $\delta<\lambda\leq\beta$. Then N and N' are disjoint neighborhoods of α and β respectively. For by the choice of δ, N and N' have no ordinals in common and by the choices of τ, δ, $n(\gamma)$, and $n(\lambda)$, if γ is such that $\tau<\gamma\leq\alpha$ and λ is such that $\delta<\lambda\leq\beta$, then $f_\gamma(k)\leq f_\alpha(k)<f_\lambda(k)$ whenever $(k,f_\gamma(k))$ belongs to N and $(k,f_\lambda(k))$ belongs to N'. Thus N and N' have no members of A in common. Hence N and N' are disjoint and we have now proved that $\ddot{X}$ is Hausdorff.

(4) The space X is normal. Let H and K be closed disjoint subsets of X. If both H and K are uncountable, then $H\cap B$ and $K\cap B$ are both uncountable and there exists sequences $\beta_1, \beta_2, \cdots; \beta_n, \cdots$ and $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ such that for each n, β_n belongs to H, α_n belongs to K, and $\beta_n<\alpha_n<\beta_{n+1}$. Let τ be the common limit ordinal of these two sequences. By the construction (iii) of (1) every neighborhood of τ intersects both H and K and since these sets are closed $\tau\in H\cap K$. Thus $H\cap K$ is not void. This contradiction shows that either H or K, say H, is countable. Since H is countable there exists a countable ordinal α_0 such that if $\alpha\in H$, $\alpha<\alpha_0$.*

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* The existence of such a countable ordinal follows from a well known theorem of ordinal numbers. See [3] .

The construction of disjoint open sets covering H and K will be carried out with the aid of the integers $n(\alpha)$ now to be defined for each α.

(i) For $\alpha > \alpha_0$, choose $n(\alpha) > m(\alpha, \alpha_0)$.

(ii) Order the ordinals which do not exceed α_0 in a simple countable sequence $\alpha_0, \alpha_1, \cdots \alpha_n, \cdots$. Take $n(\alpha_0) = 1$. Then having chosen $n(\alpha_0), \cdots, n(\alpha_{i-1})$, choose $n(\alpha_i) > m(\alpha_i, \alpha_j)$ where $0 \leq j < i$.

Suppose that $\alpha \in K$. Since α is a countable ordinal and since H is a closed set there exists an ordinal β which is maximal with respect to the two properties: β is in H, and $\beta < \alpha$. (It is assumed here that H contains ordinals less than α; in the contrary case take $\beta = 1$.) If α is a limit ordinal construct the neighborhood $U'(\alpha)$ of α in accordance with (iii) of (1) using the β above and the integers $n(\gamma)$ described above in (i) and (ii). If α is not a limit ordinal construct the neighborhood $U'(\alpha)$ of α in accordance with (ii) of (1), taking $n = n(\alpha)$. In either case $U'(\alpha)$ does not intersect $H \cap B$. By noting that the neighborhoods described in (1) are open and that X-H is open it is easily seen that there exists an open neighborhood $U(\alpha)$ of α contained in $U'(\alpha)$ that does not intersect $H \cap A$ and therefore that does not intersect H.

If $\alpha \in H$, carry out the same construction interchanging H and K to obtain an open neighborhood $V(\alpha)$ of α that does not intersect K. Now let U and V be the sum of all the neighborhoods $U(\alpha)$ and $V(\alpha)$ for α in K and H, respectively. Clearly both U and V are open, and U and V contain no ordinals in common (by the choices of β). Moreover, U and V contain no points of A in common. For suppose $(k, f_\alpha(k)) \in U$ and $(k, f_\gamma(k) \in V$. Then $\gamma \neq \alpha < \alpha_0$ and $f_\alpha(k) < f_\gamma(k)$ by (i) and (ii), if $\gamma > \alpha_0$ and by (ii) if $\gamma < \alpha_0$.

Finally, set $U' = U \cup (K \cap A)$ and $V' = V \cup (H \cap A)$. Since all sets are open and since $K \cap B \subset U$ and $H \cap B \subset V$, U' and V' are open disjoint subsets of X that cover K and H respectively. Thus X is normal.

(5) The space X is not paracompact. Let $\mathcal{U}$ be the covering of X consisting of all neighborhoods described in (1). Then $\mathcal{U}$ is an open covering and every member of $\mathcal{U}$ is a countable set. If $\mathcal{U}'$ is any open refinement of $\mathcal{U}$, then the members of $\mathcal{U}'$ are also countable. Suppose every point of A has a neighborhood that intersects only finitely many members of $\mathcal{U}'$. This is equivalent to the statement: Every member of A is contained in but finitely many members of $\mathcal{U}'$. However, since A is countable and since all open sets meet A, the union of the members of $\mathcal{U}'$ is countable. But the set B of all countable ordinals is not countable. Thus $\mathcal{U}'$ cannot cover X. This contradiction shows that some member of A must be contained in infinitely many members of $\mathcal{U}'$. Consequently no refinement of $\mathcal{U}$ can be locally finite. Thus X is not paracompact.

(6) The space X is not Lindelöf. For let $\mathcal{U}$ be the open

covering of X described in the proof of (5). It is clear from the proof of (5) that no countable subcover of $\mathcal{U}$ can cover X. Thus X is not Lindelöf and the proof of the theorem is complete.

BIBLIOGRAPHY

1. Dieudonné, J., Une généralisation des espaces compacts, Journal de Mathématiques Pures et Appliquées, Vol. 23 (1944), 65-76.

2. Dowker, C. H., On countably paracompact spaces, Canadian Journal of Mathematics, Vol. 3 (1951), 219-224.

3. Kelley, J. L., General Topology, D. Van Nostrand Co., Inc., 1955.

4. Lefschetz, S., Algebraic Topology, American Mathematical Society Colloquium Publications, Vol. 27 (1942).

5. Munroe, M. E., Introduction to Measure and Integration, Addison-Wesley Publishing Company, Inc., 1953.

6. Rudin, M. E., A separable normal nonparacompact space, Proceedings American Mathematical Society, Vol. 7 (1956), 940,041.

7. Rudin, M. E., Countable paracompactness and Souslin's problem, Canadian Journal of Mathematics, Vol. 7 (1955), 543-547.

8. Sorgenfrey, R. H., On the topological product of paracompact spaces, Bulletin American Mathematical Society, Vol. 53 (1947), 631,632.

9. Stone, A. H., Paracompactness and product spaces, Bulletin American Mathematical Society, Vol. 54 (1948), 977-982.

10. Tukey, J. W., Convergence and uniformity in general topology, Annals of Mathematical Studies, No. 2, Princeton, 1940.