Serre's Condition and Depth of Stanley-Reisner Rings

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Abstract

The aim of this work is to garner a deeper understanding of the relationship between depth of a ring and connectivity properties of the spectrum of that ring. We examine with particular interest the case where our ring is a Stanley-Reisner ring. In this circumstance, we consider the simplicial complex that corresponds to the spectrum of R. We examine properties of simplicial complexes whose Stanley-Reisner rings satisfy depth conditions such as Cohen-Macaulay and Serre's condition (S_ℓ). We leverage these properties to use algebraic tools to examine combinatorial problems. For example, the gluing lemma in (Hol18) allows us to construct bounds on the diameter of a class of graphs acting as a generalization of the 1-skeleton of polytopes.

Throughout, we give special consideration to Serre's condition (S_ℓ). We create a generalized Serre's condition (S_ℓ^j) and prove equivalent homological, topological, and combinatorial properties for this condition. We generalize many well-known results pertaining to (S_ℓ) to apply to (S_ℓ^j). This work also explores a generalization of the nerve complex and considers the correlation between the homologies of the nerve complex of a Stanley-Reisner ring and depth properties of that ring. Finally we explore rank selection theorems for simplicial complexes. We prove many results on depth properties of simplicial complexes. In particular, we prove that rank selected subcomplexes of balanced (S_ℓ) simplicial complexes retain (S_ℓ). The primary focus of this work is on Stanley-Reisner rings, however, other commutative, Noetherian rings are also considered.

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Contents

Chapter 1

Introduction

The fields of Combinatorics and Commutative Algebra are closely related by the correspondence between simplicial complexes and square-free monomial ideals. This correspondence allows classification of simplicial complexes to be accomplished through the classification of Stanley-Reisner rings. This thesis explores the relationship between depth properties of Stanley-Reisner rings and the connectedness properties of their associated simplicial complexes. In particular, this document examines the properties of complexes with Stanley-Reisner rings that satisfy Cohen-Macaulay, Serre's condition (S_ℓ), or a new generalization of Serre's condition (S_ℓ^j).

1.1 Overview

In this section, we will explore the main topics of this thesis and introduce the main results. For a more detailed account of these topics, see each chapter.

In Chapter 3, I use Serre's condition and Alexander duality to explore a new perspective on the polynomial Hirsch conjecture. The polynomial Hirsch conjecture states that a d-dimensional polyhedron with n facets has diameter bounded above by a polynomial expression in $n-d$. This conjecture is a natural weakening of the now disproven Hirsch conjecture. A common approach to this problem is to work with generalizations of polyhedra. We consider generalizations of polyhedra whose 1-skeletons $\mathcal{G}$ have vertices that are subsets of size d of $\{1, 2, ..., n\}$, such that $\mathcal{G}$ has the following properties (see Section 1 of (EHRR10)):

1. For each $u, v \in V(G)$ $\exists$ a path connecting u and v whose intermediate vertices all contain $u \cap v$.

2. The edge (u, v) is present if and only if $|u \cap v| = d - 1$.

A graph having properties (1) and (2) is equivalent to that graph being the dual graph of a Stanley-Reisner ring satisfying Serre's condition (S_2). Using this relationship, we are able to build graphs with properties (1) and (2) of maximal diameter for small values of n and d. We are also able to construct graphs for higher n and d that serve as lower bounds for the diameter. The following table provides a lower bound on the maximal diameter of graphs with properties (1) and (2) with small values of n and d.

Table 1.1: Bounds for small n and d

	$d=2$	$d=3$	$d=4$
$n=6$	4	4	2
$n=7$	5	5	3
$n=8$	6	6	6
$n=9$	7	7	7
$n \geq 10$	$n-2$	$n-1$	$n-2$

The first four rows of the table contain sharp bounds.

Using local duality and cohomology, we are able to prove an algebraic analogue of a gluing lemma from (HK98). This gluing lemma allows us to glue (S_2) complexes together to make larger diametered (S_2) complexes. With these new complexes, we create lower bounds for larger n and d.

Theorem 1.1.1. *Let Δ and Δ' be $(d-1)$-dimensional complexes on n vertices whose Stanley-Reisner rings each satisfy (S_ℓ). The Stanley-Reisner ring of $\Delta \cup \Delta'$ satisfies (S_ℓ) if the two complexes are glued along a pure complex of dimension at least $d-2$ whose Stanley-Reisner ring satisfies $(S_{\ell-1})$.*

We are also able to demonstrate upper bounds on the diameter for quite general n and d. This is achieved by using blocks and layers as delineated in (EHRR10). We take advantage of the properties a dual graph must have if its Stanley-Reisner ring satisfies (S_2).

In Chapter 4, I present a condition (S_ℓ^j). The class of rings satisfying (S_ℓ^j) is larger than the class of rings that satisfy (S_ℓ) but retains many of the good properties possessed by (S_ℓ) rings. A ring R satisfies (S_ℓ^j) if for all $P \in \operatorname{Spec} R$, $\operatorname{depth} R_P \geq \min\{\ell, \dim R_P - j\}$. A multitude is known about Serre's condition; in this chapter, we examine many of these properties and how they change as (S_ℓ) is relaxed to (S_ℓ^j).

We begin by proving a functorial condition equivalent to (S_ℓ^j). This equivalence unlocks the use of homological methods in examining the (S_ℓ^j) condition and is used in many of the proofs of this Chapter's theorems. To introduce this equivalence, we define the following notation. Given a ring $R = S/I$, let Q_I be a minimal prime of I of smallest height. Given a prime $\mathfrak{p}$ which contains I, let $Q_\mathfrak{p}$ be a minimal prime of I contained in $\mathfrak{p}$ with smallest height. Let $\alpha_\mathfrak{p} = \operatorname{ht} Q_\mathfrak{p} - \operatorname{ht} Q_I$. We note that $\alpha_\mathfrak{p} = 0$ for all $\mathfrak{p}$ when R is equidimensional.

Theorem 1.1.2. *Let S be an n-dimensional polynomial ring with maximal homogeneous ideal m and let I be a homogeneous ideal or let S be an n-dimensional complete regular local ring with maximal ideal m and let I be an ideal of S. Let $R = S/I$. Then R satisfies (S_ℓ^j) if and only if for all $P \in \operatorname{Spec} S$ containing I with $\operatorname{ht} P < n - i + \ell$, we have that $P \notin \operatorname{Supp} \operatorname{Ext}_S^{n-i}(R, S)$ for all $i = 0, ..., d - j - 1 - \alpha_P$.*

In case that the ring is pure, but all other assumptions hold, this simplifies to:

Theorem 1.1.3. *R satisfies (S_ℓ^j) if and only if* $\dim \operatorname{Ext}_S^{n-i}(R,S) \leq i-\ell$ *for all* $i = 0, ..., d-j-1$.

We prove a bound on cohomological dimension, which also provides information about regularity of the Alexander dual and about projective dimension of the quotient ring.

Theorem 1.1.4. *Let S be an n-dimensional regular local ring containing a field and let $a \subset S$ be a pure ideal of height c.*

(1) *If S/a satisfies Serre's condition (S_2^j) and* $\dim S/a \geq 1+j$, *then*

$$\operatorname{cd}(S,a) \leq n-1-\lfloor \tfrac{n-2-j}{c} \rfloor$$

(2) *Suppose that S is essentially of finite type over a field. If S/a satisfies Serre's condition (S_3^j) and* $\dim S/a \geq 2+j$, *then*

$$\operatorname{cd}(S,a) \leq n-2-\lfloor \tfrac{n-3-j}{c} \rfloor$$

We generalize the result presented in Chapter 3 about dual graphs. We consider a generalization of the dual graph described independently in (NBSW17).

Definition 1.1.5. Let $G^j(R)$ be the graph with $V(G^j(R)) = \{v_i = P_i\}$ where the P_i are the minimal primes of R, $E(G^j(R)) = \{(v_j, v_k) | 1 \leq \operatorname{ht}(P_j + P_k) \leq j\}$. We note $G^1(R)$ is the dual graph of R.

We say a ring is *j-locally connected* if for any $P \in \operatorname{Spec} R$, $G^j(R_P)$ is connected. We prove that a Stanley-Reisner ring satisfies (S_2^j) if and only if it is $j+1$-locally connected.

Yanagawa related (S_2) to resolutions of the Alexander dual in (Yan00). We further generalize his result.

Definition 1.1.6. Let $R = S/I$ be an equidimensional Stanley-Reisner ring. We say that $I^\vee$ satisfies $(N_{c,\ell}^j)$ if

$$[\operatorname{Tor}_\gamma(I^\vee, \mathbb{K})]_\beta = 0 \text{ for all } \gamma < \ell \text{ and for all } c+j+\gamma < \beta \leq n.$$

Theorem 1.1.7. *Let $R = S/I$ be a d-dimensional, equidimensional Stanley-Reisner ring with codimension c. Then the following are equivalent for $\ell \geq 2$:*

(i) R satisfies (S_ℓ^j).

(ii) $I^\vee$ satisfies $(N_{c,\ell}^j)$.

We generalize Reisner's criterion which relates the Cohen-Macaulayness of a Stanley-Reisner ring to its homologies of links.

Theorem 1.1.8. *Let $\mathbb{K}$ be a field and Δ be a pure simplicial complex of dimension $d-1$. Then Δ satisfies (S_ℓ^j) over $\mathbb{K}$ if and only if for every $F \in \Delta$ (including $F = \emptyset$) with $|F| \leq d - i - j - 2$ and for every $-1 \leq i \leq \ell - 2$, $\tilde{H}_i(\mathrm{lk}_\Delta(F); \mathbb{K}) = 0$ holds true.*

Finally, we explore monomial ideals. We show that (S_ℓ^j) preserves localization and polarization. We show that S/I satisfying (S_ℓ^j) implies that $S/\sqrt(I)$ satisfies (S_ℓ^j).

In Chapter 5, we define a generalization of the nerve complex. These "higher nerves" retain connectivity information from the original family of sets. We explore in particular, the higher nerves of simplicial complexes and discover how the higher nerves can be used to recover properties of the initial simplicial complex. We demonstrate how to recover the the depth of a simplicial complex and its f-vector from the homologies of these higher nerves. We then define the j^{th} LCM complex and relate the homologies of these complexes to the Castelnuovo-Mumford regularity of a monomial ideal. These constructions and theorems came from joint work with Hailong Dao, Joseph Doolittle, Ken Duna, Bennet Goekner, and Justin Lyle.

Definition 1.1.9. Let $A = \{A_1, A_2, ..., A_n\}$ be a family of sets. Consider

$$\mathrm{N}(A) := \{F \subseteq [r] \colon \cap_{i \in F} A_i \neq \emptyset\}.$$

This simplicial complex is the *nerve complex* of A.

We restrict our consideration to the case where A is the set of facets of a simplicial complex. We note that the definition we are about to introduce can be crafted for much more general A.

Definition 1.1.10.

$$\mathrm{N}_i(\Delta) := \{F \subseteq [r] \colon |\cap_{j\in F} A_j| \geq i\}.$$

We call this simplicial complex the i^{th} *nerve complex* of Δ and we refer to the $N_i(\Delta)$ as the *higher nerve complexes* of Δ.

With these higher nerves we prove a generalization of Borsuk's nerve theorem (Bor48). To see this result, we must first introduce some notation.

Definition 1.1.11. Let $\mathcal{F}_{>k}(\Delta)$ denote the face poset of Δ restricted to faces of Δ with cardinality strictly greater than k.

Definition 1.1.12. The *order complex* of a poset P, denoted $\mathcal{O}(P)$, is the simplicial complex whose faces are all chains in P.

We will often use the following shorthand:$[\Delta]_{>k} = \mathcal{O}(\mathcal{F}_{>k}(\Delta))$. We prove the following theorem:

Proposition 1.1.13 (Generalized Nerve Theorem)**.** *$[\Delta]_{>j}$ is homotopy equivalent to $\mathrm{N}_{j+1}(\Delta)$.*

Using this Generalized Nerve Theorem and some technical lemmas we are able to prove the following theorem, which relates both depth and the f-vector of a simplicial complex to the homologies of its higher nerves.

Theorem 1.1.14. *Let Δ be a simplicial complex and $k[\Delta]$ be its Stanley-Reisner ring. Then:*

1. $\tilde{H}_i(\mathrm{N}_j(\Delta)) = 0$ *for* $i + j > d$ *and* $1 \leq j \leq d$.

2. $\mathrm{depth}(k[\Delta]) = \inf\{i + j \colon \tilde{H}_i(\mathrm{N}_j(\Delta)) \neq 0\}$.

3. *For* $i \geq 0$, $f_i(\Delta) = \sum_{j=i+1}^{d} \binom{j-1}{i} \chi(\mathrm{N}_j(\Delta))$.

In Chapter 6, we explore rank selection theorems for balanced simplicial complexes.

Definition 1.1.15. A *balanced simplicial complex* is a pair (Δ, π) satisfying:

1. Δ is $d-1$ dimensional simplicial complex on a vertex set V.

2. $\pi = (V_1, \ldots, V_d)$ is an ordered partition of V.

3. For every facet $F \in \Delta$ and every $i \in [d]$, $|F \cap V_i| \leq 1$.

Definition 1.1.16. The *S-rank selected subcomplex* of Δ is the subcomplex of Δ induced on $\bigcup_{i \in S} V_i$.

In (Sta79), Stanley introduced the above definition and proved:

Theorem 1.1.17. *Let (Δ, π) be a balanced simplicial complex of dimension $d-1$, where $\pi = (V_1, \ldots, V_d)$ and, for any $S \subseteq [d]$, let Δ_S denote the S-rank selected subcomplex of Δ.*

If $k[\Delta]$ satisfies Cohen-Macaulay, then $k[\Delta_S]$ satisfies Cohen-Macaulay for any $S \subseteq [d]$.

Using the same hypothesis, we prove:

Theorem 1.1.18. *If $k[\Delta]$ satisfies (S_ℓ), then $k[\Delta_S]$ satisfies (S_ℓ) for any $S \subseteq [d]$.*

In answering this question, we take a broader approach remniscient of (Hib91) in which Hibi introduces excellent sets of faces.

We say that $J \subseteq V(\Delta)$ is an *independent set* for Δ if $\{a, b\} \notin \Delta$ for any $a, b \in J$ with $a \neq b$. We say that $J \subseteq V(\Delta)$ is an *excellent set* for Δ if J is an independent set for Δ and $J \cap F \neq \varnothing$ for every facet $F \in \Delta$.

All balanced complexes are balanced between excellent sets of vertices, and therefore if we prove a fact true for excellent sets of vertices, the fact follows as a corollary for balanced complexes.

Definition 1.1.19. Let T be a face of the simplicial complex Δ, then $\operatorname{astar}_\Delta T := \{G \in \Delta \mid T \cap G = \varnothing\} = \Delta|_{V-T}$

For a simplicial complex Δ and an excellent set of vertices J, we define $\tilde{\Delta}$ to be the antistar of J over Δ. We prove the following from which Theorem 1.1.18 follows as a corollary.

Lemma 1.1.20. *Suppose $k[\Delta]$ satisfies (S_ℓ). Then $k[\tilde{\Delta}]$ satisfies (S_ℓ).*

Using similar techniques as used to prove the above results, we are able to prove the pair of theorems:

Theorem 1.1.21. *If P is a finite poset satisfying (S_ℓ), then $\tilde{H}_{i-1}(\mathcal{O}(P_{>j})) = 0$ whenever $i + j < d$ and $0 \leq i < \ell$. In particular, if Δ is a simplicial complex satisfying (S_ℓ), then $\tilde{H}_{i-1}([\Delta]_{>j}) = 0$ whenever $i + j < d$ and $0 \leq i < \ell$.*

Theorem 1.1.22. *If $\tilde{H}_{i-1}([\Delta]_{>j}) = 0$ whenever $i + j < d$ and $0 \leq i \leq \ell$, then Δ satisfies (S_ℓ).*

Since, by the generalized nerve theorem 5.3.7, $\tilde{H}_{i-1}([\Delta]_{>j}) \cong \tilde{H}_{i-1}(N_{j+1}(\Delta))$ for any i and j, Theorems 6.3.3 and 6.3.4 also serve as a version of Theorem 1.1.14 (2) for (S_ℓ).

We extend a result of Hibi and Munkres (Hib91; Mun84) to prove the following depth formula for balanced simplicial complexes:

Theorem 1.1.23. *If (Δ, π) is a balanced simplicial complex with $\pi = (V_1, \ldots, V_d)$, then*

$$\operatorname{depth} \Delta = \min\{i + |S| \mid \tilde{H}_{i-1}(\Delta_{[d]-S}) \neq 0\}.$$

Finally, we prove an analogous result for complexes satisfying Gorenstein* by first proving an extension to Theorem 1.1.14(3).

Definition 1.1.24. $\operatorname{core} \Delta := \Delta|_{\operatorname{core} V(\Delta)}$.

Definition 1.1.25. We say that Δ is Gorenstein if the ring $k[\Delta]$ is Gorenstein; if, in addition, $\operatorname{core}\Delta = \Delta$, we say that Δ is Gorenstein*.

Definition 1.1.26. Let T be a face of the simplicial complex Δ, then $\operatorname{lk}_\Delta T := \{G \in \Delta \mid T \cup G \in \Delta \text{ and } T \cap G = \varnothing\}$

Theorem 1.1.27. *Suppose Δ is pure. Let χ denote the Euler characteristic and $\tilde{\chi}$ denote the reduced Euler characteristic. Then*

$$\sum_{\substack{T \in \Delta \\ |T| = k}} \tilde{\chi}(\operatorname{lk}_\Delta(T)) = \chi(N_{k+1}(\Delta)) - \chi(N_k(\Delta))$$

This theorem leads to:

Corollary 1.1.28. *Suppose Δ is Gorenstein*. Then*

$$\dim_k \tilde{H}_{i-1}([\Delta]_{>j}) = \begin{cases} \dim_k \tilde{H}_{j-1}(\Delta^{(j-1)}) & \text{if } i = d - j \\ 0 & \text{if } i \neq d - j. \end{cases}$$

The converse holds if $\operatorname{lk}_\Delta(T)$ is non-acyclic for each $T \in \Delta$.

Chapter 2

Background

2.1 Historical Note

The study of Commutative Algebra began in the late nineteenth century as a means to better understand algebraic geometry. Algebraic geometry is deeply concerned with the natural endeavor of finding the vanishing loci of polynomials. Given k a field, and an ideal $I \subseteq k[x_1, ..., x_n]$, let $V(I)$ denote the vanishing locus of I. In 1893, Hilbert proved his Nullstellensatz, which showed that for k an algebraically closed field, the points of $V(I)$ are in bijection with the maximal ideals containing I. Thus the study of $V(I)$ could be done by examining $k[x_1, .., x_n]/I$. This study of quotient rings broadened to the study of commutative Noetherian rings and began to develop into a field itself.

In 1928, Wolfgang Krull defined what would come to be called the Krull dimension of a commutative Noetherian ring (Sch08). For polynomial rings, this concept operated as an analogue of geometric dimension, while also being well-defined for more general commutative rings. Krull proved his famous principal ideal theorem using this construction of dimension. Through the rest of this document, when speaking of dimension, we shall be speaking of this Krull dimension. Commutative Algebra flourished as Krull introduced the tools of localization and completion.

In (Ree57), Rees introduces the grade of an ideal $I \subset R$ over an R-module M. When the I is the unique maximal ideal of R and M is R itself, we refer to the grade of the I over M as the depth of R. Depth and dimension are foundational attributes of rings, and much of the work in the field of commutative algebra has centered on the study of depth and dimension. Depth is always bounded above by dimension, and it is an area of great interest to study rings with the property $\operatorname{depth} R = \dim R$. Rings possessing this property are called Cohen-Macaulay rings after the work of Francis Sowerby Macaulay and Irvin Cohen.

Serre crafted what is now known as Serre condition (S_ℓ) as a method for checking the normality of a ring. A Cohen-Macaulay ring satisfies (S_ℓ) for all $\ell \leq d$. Serre condition (S_ℓ) is a natural weakening of Cohen-Macaulay and rings satisfying Serre condition have become an area of fervent study (DHV16; HTYZN11; MT09; PSFTY14; Ter07; Yan00).

Although commutative algebra has historically been intimately tied to algebraic geometry, the field developed a strong relationship with the field of combinatorics beginning in the 1970s. This relationship was born of a correspondence between simplicial complexes and square-free monomial ideals. This correspondence led to the association of rings and simplicial complexes. These rings were named Stanley-Reisner rings after Richard Stanley and Gerald Reisner and provided the opportunity to examine combinatorial problems from an algebraic vantage point. The intersection of the two fields blossomed as the techniques of homological algebra and combinatorics were now applicable to solve problems in both fields. With these developments, classifying complexes by the algebraic properties of their Stanley-Reisner rings became a priority for many researchers. For example, a simplicial complex is said to be Cohen-Macaulay if its Stanley-Reisner ring is Cohen-Macaulay. In (Rei76), Gerald Reisner presented the following criterion for a simplicial complex to be Cohen-Macaulay.

Theorem 2.1.1 (Reisner's Criterion)**.** *The following are equivalent:*

1. $k[x_1, ..., x_n]/I_\Delta$ is Cohen Macaulay

2. For each $F \in \Delta$ the reduced homology of $\operatorname{lk} F$ with coefficients in k vanishes in all dimensions except possibly the dimension of $\operatorname{lk} F$.

Though proved by different methods, Reisner's result can be derived from the later released but unpublished Hochster formula, which established a relationship between multigraded Betti numbers of square-free monomial ideals and simplicial homology.

Theorem 2.1.2. *(Hochster) (BH98)[Hochster's Formula (unpublished)] Let Δ be a simplicial complex. Then the Hilbert series of the local cohomology modules of S/I_Δ with respect to the fine grading is given by:*

$$Hilb_{H^i_{\mathfrak{m}}(S/I_\Delta)}(t) = \sum_{T \in \Delta} \dim_k \tilde{H}_{i-|T|-1}(\operatorname{lk}_\Delta T) \prod_{v_j \in T} \frac{t_j^{-1}}{1 - t_j^{-1}}.$$

Reisner's result can be presented more generally so that it relates the homology of links of a simplicial complex to the depth of the Stanley-Reisner ring.

Theorem 2.1.3. *Let Δ be a simplicial complex. Then $\operatorname{depth} S/I_\Delta \geq t$ if and only if $\tilde{H}_{i-1}(\operatorname{lk}_\Delta T) = 0$ for all $T \in \Delta$ with $i + |T| < t$.*

In 1975, Richard Stanley proved the upper bound conjecture using Reisner's criterion (Sta75). The results of Stanley, Hochster, and Reisner have proved instrumental in the development of algebraic combinatorics and will be used regularlry throughout this document.

2.2 Algebraic Combinatorics

Combinatorics is the study of finite sets of elements and their relationship with each other. This wide field spans counting, graphs, polytopes, simplicial complexes and many other topics. This thesis focuses on algebraic combinatorics, in particular the study of simplicial complexes and their associated Stanley-Reisner rings.

Fix $n \geq 0$ and consider the set $X = \{x_1, ..., x_n\}$. A simplicial complex on the vertex set X is a collection of subsets S of X with the property: if $A \in S$ and $B \subseteq A$ then $B \in S$.

We call these subsets *faces* of Δ. Faces which are maximal with respect to containment are called *facets*. The dimension of the complex is one smaller than the cardinality of the largest facet.

A major focus of Algebraic Combinatorics is the study of the reduced homologies of simplicial complexes and their subcomplexes. These reduced homologies correspond to topological connectivity conditions. For a complex Δ, $\tilde{H}_{-1}(\Delta) = 0$ if and only if the complex is non-empty. The zeroth homology counts connected components; $\dim \tilde{H}_0(\Delta)$ is one fewer than the number of connected components of Δ. As will be seen, these reduced homologies are instrumental in understanding depth properties of the complex's associated ring.

2.3 Commutative Algebra

Commutative Algebra is the study of commutative rings, ideals of commutative rings, and modules over commutative rings. We will be particularly interested in quotients of the ring $S = k[x_1, ..., x_n]$, where k is a field. A monomial of S is an element which factors as a product of the x_i's. A monomial is square-free if no variable is repeated in the factorization. An ideal I is a square-free monomial ideal if its minimal generators are square-free monomials. A quotient ring of the form S/I where I is a square-free monomial ideal is a Stanley-Reisner ring.

By the Stanley-Reisner correspondence, Stanley-Reisner rings have a bijective relationship with simplicial complexes. To generate a Stanley-Reisner ring from a simplicial complex, we begin by taking a polynomial ring over n variables, where n is the number of vertices in the simplicial complex. We then consider the facets of the simplicial complex. For each facet, we construct a prime ideal which is generated by all the variables, which are not represented by vertices of the facet. Intersecting these prime ideals generates the quotient ideal for our Stanley-Reisner ring. The polynomial ring modulo this quotient ring produces the simplicial complex's associated Stanley-Reisner ring.

A major area of interest in Commutative Algebra is depth and depth conditions. For R a commutative Noetherian local ring with maximal ideal m, the depth is the length of all maximal regular sequences of R. Depth is naturally bounded above by the dimension of the ring. A not necessarily local ring R is said to satisfy the Cohen-Macaulay property if for each prime ideal P in the spectrum of R, $\operatorname{depth} R_P = \dim R_P$. Serre's condition operates as a measure of how close a ring is to satisfying Cohen-Macaulay. A ring R satisfies Serre's condition (S_ℓ) if for all prime ideals P in the spectrum of R, $\operatorname{depth} R_P \geq \min\{\ell, \dim R_P\}$.

Chapter 3

On the Diameter of Dual Graphs of (S_2) Stanley-Reisner Rings

3.1 Introduction

In this section, we use the (S_2) property to aid in constructing bounds on the diameter of polyhedra. The (S_2) property simplifies the process of constructing large diameter examples for small dimensional polyhedra with small number of facets and allows us to prove a gluing theorem, which grants us the ability to make examples of arbitrarily large diametered polyhedra with large numbers of facets. We also construct upper bounds in a manner inspired by (EHRR10).

The polynomial Hirsch conjecture states that a d-dimensional polyhedron with n facets has diameter bounded above by a polynomial expression in $n-d$. The diameter of a polyhedron is the diameter of its 1-skeleton. The polynomial Hirsch conjecture is a weakening of the Hirsch conjecture, which was disproved by Klee and Walkup (KW67) in the general case and Santos (San11) in the bounded case. For a history of the Hirsch and polynomial Hirsch conjectures, see (San13). In this paper we construct bounds which improve on bounds from existing literature (Lar70; Bar74; EHRR10), but are not polynomial. Our bounds are sharp for small n and d. Many authors have examined the diameters of generalizations of polyhedra (e.g.(AD74; CS16; EHRR10; Kal92)). We consider generalizations of polyhedra whose 1-skeletons $\mathcal{G}$ have vertices that are subsets of size d of $\{1, 2, ..., n\}$, such that $\mathcal{G}$ has the following properties (see Section 1 of (EHRR10)):

1. For each $u, v \in V(\mathcal{G})$ there exists a path connecting u and v whose intermediate vertices all contain $u \cap v$.

2. The edge (u, v) is present if and only if $|u \cap v| = d - 1$.

Condition (2) is equivalent to the statement: $\mathcal{G}$ is the adjacency graph of a pure simplicial complex. Satisfying condition (1) in addition to condition (2) says $\mathcal{G}$ is the adjacency graph of a pure simplicial complex satisfying the "normal" condition as defined in Definition 3.1 in (San13). (See Remark 3.2.13 for more details). Generalized polyhedra of this type have been considered in section 4.1 of (Kal92).

Dual graphs are an object of wide interest in commutative algebra and algebraic geometry (e.g. (Har62; BBV17; BV15; BMS18; NBSW17)). From our setting, we shall consider the dual graph to have vertices corresponding to the minimal primes of a ring, however, the dual graph can be constructed in a more general setting with vertices corresponding to the irreducible components of a scheme. It is a famous result of Hartshorne (Har62) that if X is a connected projective scheme such that $\mathcal{O}_{X,x}$ satisfies Serre's condition (S_2) for all $x \in X$, then the dual graph of X is connected. This result is commonly known in its less general form to say arithmetically Cohen–Macaulay projective schemes have connected dual graphs.

Stanley-Reisner rings satisfying (S_2) have recently attracted much attention (MT09; HTYZN11; PSFTY14; DHV16). It is known that a graph having properties (1) and (2) is equivalent to that graph being the dual graph of a Stanley-Reisner ring satisfying Serre's condition (S_2). We shall combine techniques from commutative algebra and combinatorics to prove bounds on the diameter of these graphs.

We define $\mu(d, n)$ to be the largest diameter of a dual graph of an (S_2) Stanley-Reisner ring of dimension d and codimension $n - d$. One of the main results of this paper is the determination of the precise values of $\mu(d, n)$ for small n and d (see Table 1).

To that end we first construct upper bounds for quite general n and d. For instance, Theorem 3.3.2 shows $\mu(3, n) \leq \max(2n - 10, n - 2)$. In Theorem 3.3.5, we prove $\mu(d, n) \leq 2^{d-2}(n - d)$, which improves on the bound of (EHRR10) (See Remark 3.3.6). In Theorem 3.3.9, we prove $\mu(d, n) \leq 3 \cdot 2^{\frac{n-d-5}{2}}(n - d)$. This result is derived using our bound from Theorem 3.3.5. Combining these results with manually generated constructions (Section 3.6), we can produce the following table of exact values of $\mu(d, n)$:

Table 3.1: $\mu(d, n)$ for small n and d

	d=2	d=3	d=4
n=6	4	4	2
n=7	5	5	3
n=8	6	6	6
n=9	7	7	7
n=10+	$n-2$	$\geq n-1$	$\geq n-2$

In Section 6 of (San11), Santos builds arbitrarily large complexes whose diameters exceed the Hirsch bound by a fixed fraction. This is achieved by using a *gluing lemma* from (HK98), which states that two d-dimensional polytopes, P_1, P_2, can be glued together yielding a new polytope P with $\operatorname{diam} P \geq \operatorname{diam} P_1 + \operatorname{diam} P_2 - 1$. We present Theorem 3.4.2, an algebraic analogue to the gluing lemma. This theorem tells us that two $d-1$-dimensional complexes with (S_ℓ) rings (we shall call these (S_ℓ) complexes) glued together along a pure, ($S_{\ell-1}$) subcomplex of dimension at least $d - 2$ yield an (S_ℓ) complex. The proof of Theorem 3.4.2 is achieved using local cohomology and local duality.

Applying Theorem 3.4.2, we are able to construct complexes whose Stanley-Reisner rings have dual graphs with arbitrarily large diameter which (with proper labeling) have properties (1) and (2). For appropriate complexes Δ and Δ',

$$\operatorname{diam} G(k[\Delta]) + \operatorname{diam} G(k[\Delta']) = \operatorname{diam} G(k[\Delta \cup \Delta']).$$

Gluing multiple copies of examples from the small n and d cases together, we construct graphs with properties (1) and (2) in dimensions 3 and 4 with diameters $\frac{5}{4}(n-d)$ and $\frac{3}{2}(n-d)$ respectively (see Theorem 3.5.3, Theorem 3.5.1). We show graphs with properties (1) and (2) and diameter $\frac{3}{2}(n-d)$ can be constructed for all $d \geq 4$ (see Remark 3.5.6).

Introduction of terms is covered in Section 3.2.1. In Section 3.2.2, we demonstrate that a graph having properties (1) and (2) is equivalent to that graph being the dual graph of a Stanley-Reisner ring satisfying Serre's condition (S_2). In Section 3.3, we prove the upper bounds introduced earlier in this section. Details of the process of gluing to preserve (S_2) are discussed in Section 3.4, and constructions of glued complexes are discussed in Section 3.5 with examples displayed in Figures 3.1 and 3.2. In Section 3.6, we show the constructions needed which justify Table 3.1 and investigate the relationship between (S_2) and Buchsbaum.

3.2 Background and Notation

3.2.1 Introduction of Terms

Definition 3.2.1. A d-dimensional *polyhedron* is a non-empty intersection of finitely many closed half spaces of $\mathbb{R}^d$.

Definition 3.2.2. A *facet* of a d-dimensional polytope is a $d-1$-dimensional face of the polytope.

Definition 3.2.3. The 1-*skeleton of a polyhedron* is the set of vertices and edges of the polyhedron.

A polyhedron is called *non-degenerate* if each vertex is the intersection of d facets. Any polyhedron can be transformed into a non-degenerate polyhedron by perturbation without decreasing its diameter (EHRR10). Therefore, we may restrict our attention to non-degenerate polyhedra.

Definition 3.2.4. A *facet* of a simplicial complex Δ is a simplex of Δ which is not properly contained in another simplex of Δ.

Definition 3.2.5. A *pure* simplicial complex is a simplicial complex whose facets all have the same dimension.

We remind the reader that a $(d-1)$-dimensional simplicial complex has a d-dimensional Stanley-Reisner ring. We will use as notation Δ_R to be the simplicial complex with Stanley-Reisner ring R. We shall use Δ when the ring is either unspecified or clear from context.

Let k be a field and $S = k[x_1, ..., x_n]$. Let Δ be a pure, $(d-1)$-dimensional simplicial complex with Stanley-Reisner ring $R = S/I$, where I is the intersection of the minimal prime ideals P_i of R. For each P_i there exists a facet of Δ, call it F_i such that P_i is generated by $\{x_j | x_j \notin F_i\}$ (see e.g., the survey (FMS14) for proof). Thus purity of Δ is equivalent to each P_i being generated by $n-d$ distinct variables.

Definition 3.2.6. Let $\mathcal{G}(R)$ be the graph with $V(\mathcal{G}(R)) = \{v_i = \Pi x_j\}$ where the x_j's generate P_i, $E(\mathcal{G}(R)) = \{(v_j, v_k) | \operatorname{ht}(P_j + P_k) = 1\}$. Then $\mathcal{G}(R)$ is the *dual graph* of R.

This type of graph is often constructed in a more general setting applying to schemes (e.g. (BBV17)). This definition follows the definition of Hochster and Huneke (HH94) and is equivalent to other definitions, (e.g. (BBV17)). In this paper we consider dual graphs of Stanley-Reisner rings. It should be noted that not every graph is a dual graph of a Stanley-Reisner ring (BV15)).

Definition 3.2.7. Let Δ be a simplicial complex on $\{1, 2, \cdots, n\}$. The *Alexander dual* of Δ is

$$\Delta^{\vee} = \{F \subseteq \{1, 2, \cdots, n\} | \{1, 2, \cdots, n\} \backslash F \notin \Delta\}.$$

Let S/I be the Stanley-Reisner ring of Δ, $S/I^{\vee}$ be the Stanley-Reisner ring of $\Delta^{\vee}$. We refer to $I^{\vee}$ as the Alexander dual of I. The Alexander dual of I is generated by the product of the generators of each minimal prime ideal of I (see e.g. the survey (FMS14) for proof).

Example 3.2.8. Let $S = k[x_1, x_2, x_3, x_4, x_5, x_6]$ and

$$I = \langle x_1x_3x_5, x_1x_3x_6, x_1x_4x_5, x_1x_4x_6, x_2x_3x_5, x_2x_3x_6, x_2x_4x_5, x_2x_4x_6 \rangle =$$

$$\langle x_1, x_2 \rangle \cap \langle x_3, x_4 \rangle \cap \langle x_5, x_6 \rangle.$$

Then $I^{\vee} = \langle x_1x_2, x_3x_4, x_5x_6 \rangle$.

Remark 3.2.9. We notice that the vertices of $G(S/I)$ are in one to one correspondence with the generators of $I^{\vee}$. Also, when S/I is an equidimensional d-dimensional ring, each vertex of $G(S/I)$ is comprised of $n - d$ variables. Finally, notice that by definition every dual graph of an equidimensional ring has property (2).

Definition 3.2.10. Define $\bar{\mathcal{G}}(R)$, to be isomorphic to $\mathcal{G}(R)$ but each vertex labeled by a square-free monomial μ is now labeled by the complementary square-free monomial.

The graph $\bar{\mathcal{G}}(R)$ is a relabeling of $\mathcal{G}(R)$. We construct this labeling so that our graphs fit the setting of (EHRR10), and so that we can determine if our graphs have properties (1) and (2).

Remark 3.2.11. We note $\bar{\mathcal{G}}(R)$ is the facet-ridge graph of the complex with Stanley-Reisner ring R.

Definition 3.2.12. A ring R is *locally connected* if for any two vertices $\bar{v}_i, \bar{v}_j \in \bar{\mathcal{G}}(R)$, there exists a path between them such that each vertex in the path contains $\bar{v}_i \cap \bar{v}_j$.

Locally connected is also referred to as ultra connected in (Kal92).

Remark 3.2.13. Locally connected graphs are referred to as *normal* graphs in certain combinatorial circles (San13). Normality of a graph is a notion unrelated to normality of a ring. To avoid this confusion, we use the name "locally connected." This name has been motivated by Theorem 3.2.17, which connects this property of $\bar{\mathcal{G}}(R)$ to localization of the ring R.

3.2.2 Serre's Condition

We have shown graphs with properties (1) and (2) from (EHRR10) are an abstraction of 1-skeletons of non-degenerate polyhedra. Ensuring a graph satisfies property (2) is not difficult. Using Serre's condition and syzygy matrices, we demonstrate a simple method to ensure a dual graph of an (S_2) Stanley-Reisner ring satisfies property (1).

Definition 3.2.14. A ring satisfies *Serre's condition* (S_ℓ) if for all P in $\operatorname{Spec} R$,

$$\operatorname{depth} R_P \geq \min\{\ell, \dim R_P\}.$$

Definition 3.2.15. Let M be an R module with minimal generating set $\{z_1, z_2, ..., z_k\}$. A *first syzygy* of M is a non-zero vector $(a_1, ..., a_k) \in R^k$ such that $a_1 z_1 + ... + a_k z_k = 0$. A *first syzygy matrix* of a module is a matrix whose columns span all the first syzygies of that module.

Theorem 3.2.16 (Yanagawa)**.** *The Stanley-Reisner ring $R = S/I$ satisfies (S_2) if and only if R is equidimensional and $I^\vee$ has a first syzygy matrix with only linear entries.*

This theorem is true by Corollary 3.7 in (Yan00). Since checking the linearity of a first syzygy matrix can be easily done computationally, this theorem provides us with a simple way to demonstrate if a Stanley-Reisner ring satisfies (S_2). Now we connect (S_2) to our locally connected condition. We will use local duality, the Ext module, and local cohomology. For background on these topics, see a Homological Algebra text (e.g. (Wei94)).

Theorem 3.2.17. *Let R be an equidimensional Stanley-Reisner ring. The following are equivalent:*

1. *R satisfies (S_2).*
2. *For any prime ideal P generated by variables, $\mathcal{G}(R_P)$ is connected.*
3. *$\bar{\mathcal{G}}(R)$ is locally connected.*

Proof. (2) $\Leftrightarrow$ (3): Let S be a subset of $\{x_1, ...x_n\}$. Let $\bar{\mathcal{G}}(R)_S$ be the induced subgraph of $\bar{\mathcal{G}}(R)$ with

$$V(\bar{\mathcal{G}}(R)_S) = \{v \in V(\bar{\mathcal{G}}(R)) | x_i \in v \text{ for all } x_i \in S\}.$$

$\bar{\mathcal{G}}(R)$ is locally connected if and only if $\bar{\mathcal{G}}(R)_S$ is connected for any choice of S. By the definition of $\bar{\mathcal{G}}(R)$, $\bar{\mathcal{G}}(R)_S$ is a relabeling of a subgraph of $\mathcal{G}(R)$ whose vertices are the minimal prime ideals of R contained in P, the prime generated by $\{x_j | x_j \notin S\}$. This subgraph is $\mathcal{G}(R_P)$. Therefore $\mathcal{G}(R_P)$ connected for all primes generated by variables is equivalent to $\bar{\mathcal{G}}(R)$ being locally connected.

(1) $\Rightarrow$ (2): Suppose $\mathcal{G}(R_P)$ is not connected for some nonempty set of primes generated by variables in $\operatorname{Spec} R$. Note that every prime in this set must have height at least 2. Let us choose any prime P maximal in this set. If $\mathcal{G}(R_P)$ is not connected then $\bar{\mathcal{G}}(R_P)$ is not connected. Suppose $\bar{\mathcal{G}}(R_P)$ contains two distinct components, each of which contain $x_i \in \operatorname{Supp} R_P$. Then $\bar{\mathcal{G}}(R_P)$ can be localized at $S = x_i$ to yield a disconnected graph, contradicting the maximality of P. Therefore, each connected component of $\bar{\mathcal{G}}(R_P)$ is composed of disjoint

sets of variables. The vertices of $\bar{\mathcal{G}}(R_P)$, however, represent the facets of the simplicial complex Δ_{R_P}. Therefore, we have that Δ_{R_P} is not connected, which implies $H^1_{PR_P}(R_P) \neq 0$. Thus $\operatorname{depth} R_P \leq 1$; however, $\operatorname{ht} P \geq 2$, and thus $\dim R_P \geq 2$. Therefore, R is not (S_2).

(1) $\Leftarrow$ (2): Using local duality, we have a ring R satisfies (S_2) if and only if $\dim \operatorname{Ext}^{n-i}_S(R, \omega_S) \leq i - 2$ for all $i < d$ (see Lemma 3.4.1). From (Yan00), since R is a Stanley-Reisner ring, $\operatorname{Ext}^{n-i}_S(R, \omega_S)$ is a square-free module. Thus $\operatorname{Ext}^{n-i}_S(R, \omega_S)$ is uniquely determined by its primes generated by variables. The dimension of $\operatorname{Ext}^{n-i}_S(R, \omega_S)$ determines if R satisfies (S_2). Therefore, we only need consider primes generated by variables when showing that R satisfies (S_2).

If $\mathcal{G}(R_P)$ is connected for all primes generated by variables in $\operatorname{Spec} R$, then Δ_{R_P} is connected for all primes generated by variables with height at least 2 in $\operatorname{Spec} R$. When $\operatorname{ht} P \geq 2$, Δ_{R_P} is connected if and only if $H^1_{PR_P}(R_P) = 0$. Further, $H^0_{PR_P}(R_P) = 0$ for all $P \in \operatorname{Spec} R$, since R is a Stanley-Reisner ring. Thus $\operatorname{depth} R_P \geq 2$ for all primes generated by variables with height at least 2, and $\operatorname{depth} R_P \geq 1$ for all primes generated by variables with height 1. Thus R satisfies (S_2). $\square$

Definition 3.2.18. Given a graph $\mathcal{G}$ with each vertex labeled with the same number of variables the following are equivalent:

1. $\mathcal{G}$ has properties (1) and (2) from (EHRR10).

2. $\mathcal{G}$ is $\bar{\mathcal{G}}(R)$ with R an equidimensional Stanley-Reisner ring, and $\mathcal{G}$ has local connectedness.

3. $\mathcal{G}$ is $\bar{\mathcal{G}}(R)$ with R an (S_2) Stanley-Reisner ring.

4. $\mathcal{G}$ is $\bar{\mathcal{G}}(R)$ with $R = S/I$ an equidimensional Stanley-Reisner ring such that the Alexander dual of I, $I^\vee$, has first syzygy matrix with all linear entries.

Proof. (1) $\Rightarrow$ (2): Let each vertex label of $\mathcal{G}$ be a facet of a pure simplicial complex Δ. Let the Stanley-Reisner ring of Δ be R. Then $\mathcal{G}$ has the same vertex set as $\bar{\mathcal{G}}(R)$. Property (2) implies that $\mathcal{G}$ has the same edge set as the dual graph of R. Property (1) is equivalent to locally connected.

(1) $\Leftarrow$ (2): Property (2) is required by the definition of a dual graph. Property (1) is precisely the same as local connectedness.

(2) $\Leftrightarrow$ (3): A ring with property (S_2) is equidimensional. Thus, by Theorem 3.2.17, $\bar{\mathcal{G}}(R)$ is locally connected if and only if R satisfies (S_2).

(3) $\Leftrightarrow$ (4): See Theorem 3.2.16. $\square$

3.3 Upper Bounds

In this section we prove upper bounds for the diameters of dual graphs of (S_2) Stanley-Reisner rings. This is achieved by working with $\bar{\mathcal{G}}(R)$ (see Section 3.2.1).

Definition 3.3.1. A *strictly increasing path* is a path such that the k^{th} vertex has distance k from the starting vertex.

Theorem 3.3.2. *For $n \geq 3$, $\mu(3, n) \leq \max(2n - 10, n - 2)$.*

Proof. We construct this upper bound in a manner inspired by (EHRR10). Let R be an (S_2) Stanley-Reisner ring, such that $\bar{\mathcal{G}}(R)$ has vertices named ABC, DEF with maximum distance in the graph. Let us assign to each vertex v the integer $\text{dist}(ABC, v)$, where $\text{dist}(ABC, v)$ denotes the length of the shortest path from ABC to v. We define layers $L_i = \{v \in V(\bar{\mathcal{G}}(R)) | \text{dist}(ABC, v) = i\}$.

Define a block of layers to be a set of layers $\{L_i | a \leq i \leq b\}$ for some integers a, b. Let c be the largest integer such that L_c contains A, B, or C. Let $B_1 = \{L_i | 0 \leq i \leq c\}$. Let n_0 be the number of variables in B_1. Without loss of generality, A is contained in L_c. By local connectedness, there exists a path consisting of vertices which all contain A from L_0 to L_c. This path can have maximum length $n_0 - 3$ (see the $d = 2$, $n = n_0 - 1$ case in Section 3.6).

Next, we construct a second block. Let d be the largest integer such that L_d contains a variable of L_c; call this variable a_1. Let $B_2 = \{L_i | c < i \leq d\}$. Let n_1 be the number of variables in B_2 but not in B_1. The diameter of this layer will be bounded by the maximum length of a path in which each vertex contains a_1. This path will have maximum length of $n_1 + n_0 - 3 - 3$ (the second -3 is to account for the fact that A, B, C cannot be in the layers of this block).

Construct the third block B_3 in the same way. Its longest path will have maximal length $n_2 + n_1 - 3 - 3$. By construction, B_3 cannot have any elements in common with B_1. Also, B_3 does not contain any variables in the $c + 1$ layer (there are at least 3 such variables).

Continue in this manner.

We sum the lengths of the blocks and add 1 for each path between blocks to obtain:

$2n_0 + 2n_1 + \cdots + 2n_{k-2} + n_{k-1} - 5(k - 2) - 8$ where k is the number of blocks.

$k \geq 3$ implies $\operatorname{diam} \bar{\mathcal{G}}(R) \leq 2n - 13$.

$k = 1$ implies $n_0 = n$ and $\operatorname{diam} \bar{\mathcal{G}}(R) \leq n - 3$.

Let $k = 2$. If ADE, ADF, or AEF is a vertex, then our path containing A has length at most $n-3$, and some vertex in that path is adjacent to DEF. Therefore, $\operatorname{diam} \bar{\mathcal{G}}(R) \leq n-2$.

Let L_j be the largest layer containing the vertex x_i. Define $L_{x_i} = j$. Let $L_A \geq L_B \geq L_C$. Let $i^* = \min\{i \mid AD \subseteq v \in L_i\} \leq \min\{i \mid AE \subseteq v \in L_i\} \leq \min\{i \mid AF \subseteq v \in L_i\}$. Denote the vertex containing AD in L_{i^*} to be v^* (if more than one exists choose any such one).

Consider the case $L_A = n - 3$. To maximize i^*, we must have a path with tail:

$$Ax_iD, Ax_jD, Ax_jE, Ax_kE, Ax_kF.$$

Thus the path from ABC to v^* must have length at most $n - 7$. The maximum length from

v^* to DEF is $n-3$. Thus, $L_D \leq 2n-10$.

Consider the case where $L_A = n-4$. Take $v \in L_{n-4}$ such that $A \in v$ and construct a path from ABC to v. If the path contains AD, AE, and AF, then the path from ABC to v^* must have length at most $n-7$. If this path does not contain one of those, say AF, then $i^* \leq n-6$. If $i^* = n-6$, then $L_A \geq L_B \geq L_C$ implies any strictly increasing path from v^* to DEF of vertices all containing D cannot have both a vertex containing B and a vertex containing C. Thus $L_D \leq i^* + n - 4 = 2n-10$. Consider $i^* < n-6$. Any path from v^* to DEF will be at most length $n-3$. Thus $L_D \leq 2n-10$.

Now suppose $L_A \leq n-5$. Then $i^* \leq n-5$. Further, $B, C \notin L_j$ for all $j > n-5$. Thus any path from L_A to DEF in which each vertex contains D, will have maximal length $n-3-2$. Thus $L_D \leq 2n-10$.

$\square$

Definition 3.3.3. $\mu(4,8) \leq 6$.

Proof. Let us consider $v_1, v_2 \in \bar{\mathcal{G}}(R)$. If $v_1 \cap v_2 \neq \emptyset$ then we can reduce to the $n=7$, $d=3$ case, and thus $\operatorname{dist}(v_1, v_2) \leq 5$. Thus assume $v_1 \cap v_2 = \emptyset$. Connectivity implies there exists v_3 adjacent to v_1. The vertex v_3 must have a non-trivial intersection with v_2, thus $\operatorname{dist}(v_2, v_3) \leq 5$. Therefore, $\operatorname{dist}(v_1, v_2) \leq 6$. $\square$

Definition 3.3.4. For $d \geq 4$, $\mu(d, d+4) = 6$.

Proof. Let R be a codimension-four, (S_2) Stanley-Reisner ring of dimension at least 4. Take $v_1, v_2 \in V(\bar{\mathcal{G}}(R))$. Then, v_1, v_2 will each contain $n-4$ variables, and $v_1 \cap v_2$ will contain at least $n-8$ variables. Thus there must be a path from v_1 to v_2 in which each vertex in that path contains those $n-8$ shared variables. Thus $\mu(d, d+4) \leq \mu(4,8)$. Furthermore, we may take the graph in Figure 3.8 (see Section 3.6) and add the same $d-4$ variables to each vertex to show $\mu(d, d+4) \geq 6$. $\square$

Let us now construct bounds for more general values of n and d.

Theorem 3.3.5. *$\mu(d,n) \leq 2^{d-2}(n-d)$, for all $d \geq 2$ and all $n \geq d$.*

Proof. Applying Theorem 3.3.2, $\mu(3,n) \leq 2n-6$ for all $n \geq d$. Thus the $d=3$ case holds. We begin induction on d. Let us partition the vertices of $\bar{\mathcal{G}}(R)$ into layers and blocks, as in the proof of Theorem 3.3.2.

If $\bar{\mathcal{G}}(R)$ has 1 block, then there is a variable which is contained in each layer of the graph. Thus $\operatorname{diam} \bar{\mathcal{G}}(R) \leq \mu(d-1,n-1)$. But by induction $\mu(d-1,n-1) \leq 2^{d-3}(n-d) \leq 2^{d-2}(n-d)$.

Now suppose we have multiple blocks. Then the first block will be bounded in diameter by $\mu(d-1,n_0-1)$, where n_0 is the number of variables in the block. The second block will be bounded in diameter by $\mu(d-1,n_0+n_1-d-1)$. The third block will be bounded by $\mu(d-1,n_1+n_2-d-1)$, and so on. Using k for the number of blocks and using the induction hypothesis, we get:

$$\mu(d,n) \leq 2^{d-3}(n_0-d)+1+2^{d-3}(n_1+n_0-d-1-(d-1))+1+2^{d-3}(n_2+n_1-d-1-(d-1))+1$$

$$+\cdots+2^{d-3}(n_{k-1}+n_{k-2}-d-1-(d-1)) \leq 2^{d-2}n-2^{d-3}(d+2d(k-1))+k-1 \leq$$

$$2^{d-2}n-2^{d-3}(3d)+1 \leq 2^{d-2}(n-d).$$

□

Remark 3.3.6. In (EHRR10), Eisenbrand et al. proved Larman's (Lar70) bound $2^{d-1}n$ holds for graphs with property (1). Theorem 3.3.5 shows a stronger bound holds for graphs with properties (1) and (2). In (Lar70), Larman showed that $2^{d-3}n$ is an upper bound for the diameter of polytopes of dimension at least 3. In (Bar74), Barnette strengthened Larman's bound to $\frac{1}{3}2^{d-3}(n-d+\frac{5}{2})$. Our bound is slightly weaker than the bounds of Barnette and

Larman; however, in Section 3.5 we will show by construction that the bounds of Barnette and Larman do not hold in our generality (see Theorem 3.5.1 and 3.5.3).

From Proposition 2.10 of (KW67), it follows that $\mu(d,n) \leq \mu(n-d, 2(n-d))$. This fact gives rise to *the d-step conjecture*, which states $\mu(d, 2d) \leq d$ for all d. We may rewrite this conjecture as $\mu(n-d, 2(n-d)) \leq n-d$. Thus, the d-step conjecture is equivalent to the Hirsch conjecture. A natural generalization of the d-step conjecture is $\mu(d, 2d) \leq p(d)$ where $p(d)$ is a polynomial in d. Again $d = n-d$, and thus we have that this conjecture is equivalent to the polynomial Hirsch conjecture.

We examine upper bounds on $\mu(d, d+k)$. We note $\mu(d, d+k) = \mu(k, 2k)$ by (KW67).

Theorem 3.3.7. $\mu(d, d+5) \leq 8$.

Proof. Choose any $v_1, v_2 \in \bar{\mathcal{G}}(R)$. From (KW67), we have $\mu(d,n) \leq \mu(n-d, 2n-2d)$. Thus we may reduce to the $d = 5$ case. Let $v_1 = ABCDE$. We will consider cases based on v_2 to deduce the bound on diameter.

If $v_2 = FGHIJ$, then without loss of generality, we will have $BCDEF$ and $CDEFG$ in $V(\bar{\mathcal{G}}(R))$. The vertices $CDEFG$ and $FGHIJ$ will have distance bounded by $\mu(3,8)$ (see Table 3.1). Thus $\text{dist}(CDEFG, FGHIJ) \leq 6$. Thus $\text{dist}(v_1, v_2) \leq 8$.

If $v_2 = EFGHI$, then either $\bar{\mathcal{G}}(R)$ contains $ABCEF$ or $\bar{\mathcal{G}}(R)$ contains $ABCEJ$ and $ABEFJ$. For both scenarios, the graph will be bounded by two more than the $d = 3$, $n = 8$ case. Thus $\text{dist}(v_1, v_2) \leq 8$.

If $\text{degree}(v_1 \cap v_2) \geq 2$, then $\text{dist}(v_1, v_2)$ is bounded by the $d = 3$, $n = 8$ case and is at most 6.

□

Theorem 3.3.8. $\mu(d, d+6) \leq 14$.

Proof. Choose any $v_1, v_2 \in \bar{\mathcal{G}}(R)$. As before, we can reduce to the $d = n-d$ case. Let $v_1 = ABCDEF$.

If degree($v_1 \cap v_2$) ≥ 3, then diam $\bar{\mathcal{G}}(R)$ is bounded by 7 (the $d = 3$ $n = 9$ case, see Table 3.1).

Now suppose degree($v_1 \cap v_2$) ≤ 2. There must exist a vertex v_3 such that degree($v_3 \cap v_1$) ≥ 3 and degree($v_3 \cap v_2$) ≥ 3.

But then dist$(v_1, v_3) \leq 7$, and dist$(v_3, v_2) \leq 7$. Thus dist$(v_1, v_2) \leq 14$.

□

Theorem 3.3.9. *For $d \geq 2$:*

$$\mu(d, d+k) \leq 3 \cdot 2^{\frac{n-d-5}{2}}(n-d).$$

Proof. We only need to consider the case $(n-d, 2(n-d))$.

We will first consider the case $n-d$ is even. In this case,

$$\mu(n-d, 2(n-d)) \leq 2\mu\left(\frac{1}{2}(n-d), \frac{3}{2}(n-d)\right) \leq 2(2^{\frac{n-d}{2}-2}(n-d))$$

$$= 2^{\frac{n-d-2}{2}}(n-d) \leq 3 \cdot 2^{\frac{n-d-5}{2}}(n-d).$$

Next we consider the case $n-d$ is odd. Then

$$\mu(n-d, 2(n-d))$$

$$\leq \mu\left(\left\lfloor\frac{n-d}{2}\right\rfloor, 2(n-d) - \left\lceil\frac{n-d}{2}\right\rceil\right) + \mu\left(\left\lceil\frac{n-d}{2}\right\rceil, 2(n-d) - \left\lfloor\frac{n-d}{2}\right\rfloor\right)$$

$$\leq 2^{\lfloor\frac{n-d}{2}\rfloor-2}(n-d) + 2^{\lceil\frac{n-d}{2}\rceil-2}(n-d) = 3 \cdot 2^{\frac{n-d-5}{2}}(n-d).$$

□

3.4 Gluing

In (San11), Santos uses a gluing lemma to construct polyhedra with arbitrarily many facets whose diameters exceed $n-d$ by a fixed fraction. We will construct an algebraic analogue to this gluing lemma, which will allow us to construct (S_ℓ) complexes with arbitrarily many facets whose diameters exceed $n-d$ by a fixed fraction.

The facet-ridge graph of an (S_ℓ) complex is $\bar{\mathcal{G}}(R)$, where $R = S/I$ is that complex's Stanley-Reisner ring. Thus by making these (S_2) complexes, we are making dual graphs of Stanley-Reisner rings satisfying (S_2) with arbitrarily large n whose diameters exceed $n-d$ by a fixed fraction.

In this section, we will make use of the ext module $\operatorname{Ext}^i_S(R,S)$ and the cohomology module $H^i_{PR_P}(R_P)$. For background on these modules, see a text on Homological Algebra (e.g. (Wei94)). We can describe (S_ℓ) in terms of the dimension of the Ext module. This fact is key to the proof that proper gluing will maintain (S_ℓ). The following lemma appears without proof in (Vas05, Proposition 3.1).

Lemma 3.4.1. *Let Δ be a pure complex with Stanley-Reisner ring R. R satisfies (S_ℓ) if and only if*

$$\dim \operatorname{Ext}^{n-i}_S(R,\omega_S) \leq i-\ell \quad \textit{for all } i = 0, \cdots d-1.$$

Proof. We reconstruct the proof from (DHV16).

Suppose R satisfies (S_ℓ). If $i < \ell \leq \operatorname{depth} R$ then $\operatorname{Ext}^{n-i}_S(R,S) = 0$. Thus,

$$\dim \operatorname{Ext}^{n-i}_S(R,S) = -\infty,$$

and we are finished. Otherwise, let us take P a prime ideal of S containing I. Let h be the height of P. If $h < n-i+\ell$, then $h-n+i < \ell$. Further $\dim R_P = h-n+d > h-n+i$.

Thus depth $R_P > h - n + i$. Therefore,

$$0 = H^{(h-n+i)}_{PS_P}(R_P) \cong \operatorname{Ext}^{n-i}_{S_P}(R_P, S_P) \cong \operatorname{Ext}^{n-i}_S(R, S)_P \quad \text{ for all } i = 0, ..., d-1.$$

So $P \notin \operatorname{Supp} \operatorname{Ext}^{n-i}_S(R, S)$ whenever $\operatorname{ht} P < n - (i - \ell)$. Thus $\dim \operatorname{Ext}^{n-i}_S(R, S) \le i - \ell$.

Suppose $\dim \operatorname{Ext}^{n-i}_S(R, S) \le i - \ell$ for all $i = 0, ..., d-1$. Let $\operatorname{ht} I = c$. Let V_h be the set of prime ideals of S of height h containing I for $h = c, ..., n$. Then we have the following equivalences:

$$R \text{ satisfies } (S_\ell) \quad \Leftrightarrow$$

$$\operatorname{depth} R_P \ge \min\{\ell, h - c\} =: b \quad \text{ for all } h = c, ..., n, \text{ for all } P \in V_h \quad \Leftrightarrow$$

$$H^i_{PS_P}(R_P) = 0 \quad \text{ for all } h = c, ..., n, \text{ for all } P \in V_h, \text{ for all } i < b \quad \Leftrightarrow$$

$$\operatorname{Ext}^{h-i}_{S_P}(R_P, S_P) = 0 \text{ for all } h = c, ..., n, \text{ for all } P \in V_h, \text{ for all } i < b \quad \Leftrightarrow$$

$$\dim \operatorname{Ext}^{h-i}_{S_P}(R_P, S_P) < n - h \text{ for all } h = c, ..., n, \text{ for all } P \in V_h, \text{ for all } i < b.$$

When $i < b \le h - c$, $n - h + i < n - c = d$. For all $i < d$, $\dim \operatorname{Ext}^{n-i}_S(R, S) \le i - \ell$. Thus

$$\dim \operatorname{Ext}^{h-i}_S(R, S) = \dim \operatorname{Ext}^{n-(n-h+i)}_S(R, S) \le n - h + i - \ell < n - h$$

$$\text{for all } h = c, \cdots, n, \text{ for all } i < b.$$

$\square$

Theorem 3.4.2. *Let Δ and Δ' be $(d-1)$-dimensional complexes on n vertices whose Stanley-Reisner rings each satisfy (S_ℓ). The Stanley-Reisner ring of $\Delta \cup \Delta'$ satisfies (S_ℓ) if the two complexes are glued along a pure complex of dimension at least $d-2$ whose Stanley-Reisner* *ring satisfies* $(S_{\ell-1})$.

Proof. Let us use the notation R_Δ to refer to the Stanley-Reisner ring of Δ.

If $\ell = 1$, any gluing will preserve the (S_1) property, since every simplicial complex satisfies (S_1).

Thus let us consider $\ell \geq 2$, noting every (S_2) complex is pure (Yan00).

Take the short exact sequence

$$0 \to R_{\Delta \cup \Delta'} \to R_\Delta \oplus R_{\Delta'} \to R_{\Delta \cap \Delta'} \to 0.$$

Then take the long exact sequence in Ext:

$$\cdots \to \operatorname{Ext}_S^{n-i}(R_\Delta \oplus R_{\Delta'}, \omega_S) \to \operatorname{Ext}_S^{n-i}(R_{\Delta \cup \Delta'}, \omega_S) \to \operatorname{Ext}_S^{n-(i-1)}(R_{\Delta \cap \Delta'}, \omega_S) \to \cdots .$$

Since Ext is an additive functor:

$$\operatorname{Ext}_S^{n-i}(R_\Delta \oplus R_{\Delta'}, \omega_S) \cong \operatorname{Ext}_S^{n-i}(R_\Delta, \omega_S) \oplus \operatorname{Ext}_S^{n-i}(R_{\Delta'}, \omega_S).$$

Since R_Δ and $R_{\Delta'}$ are (S_ℓ), Lemma 3.4.1 gives

$$\dim \operatorname{Ext}_S^{n-i}(R_\Delta, \omega_S) \leq i - \ell$$

and

$$\dim \operatorname{Ext}_S^{n-i}(R_{\Delta'}, \omega_S) \leq i - \ell.$$

Therefore

$$\dim \operatorname{Ext}_S^{n-i}(R_\Delta, \omega_S) \oplus \operatorname{Ext}_S^{n-i}(R_{\Delta'}, \omega_S) \leq i - \ell \quad \text{for all } i < d.$$

Suppose $R_{\Delta\cap\Delta'}$ is an equidimensional, $(S_{\ell-1})$ ring of dimension at least $d-1$. Then we also have

$$\dim \operatorname{Ext}_S^{n-(i-1)}(R_{\Delta\cap\Delta'}, \omega_S) \leq (i-1)-(\ell-1) = i-\ell \quad \text{for all } i < d.$$

Therefore,

$$\dim \operatorname{Ext}_S^{n-i}(R_{\Delta\cup\Delta'}, \omega_S) \leq i-\ell \quad \text{for all } i < d.$$

□

Thus, gluing any two (S_2) complexes along a pure subcomplex of dimension at least $d-2$ yields an (S_2) complex. In particular, gluing two (S_2) complexes along a facet yields an (S_2) complex.

3.5 Complexes Built by Gluing

In this section, we create lower bounds for large n and d by taking copies of the graphs in Section 3.6 and gluing them along a shared vertex. The graphs from Section 3.6 are facet-ridge graphs of (S_2) complexes. Thus, gluing along a vertex is equivalent to gluing the complexes along a facet. Therefore, by Theorem 3.4.2, gluing along a vertex yields a facet-ridge graph of an (S_2) complex.

Theorem 3.5.1. $\mu(4, 4k+4) \geq 6k$.

Proof. We construct a graph composed of k copies of the graph in Figure 3.8 by gluing the vertex $ABCD$ to the vertex $EFGH$. Each copy adds 6 to the diameter, since all adjacent vertices lie in the same copy of Figure 3.8. The new graph retains local connectedness by Theorem 3.4.2. □

The complex in Figure 3.1 is an example when $k = 2$.

Thus, we have a lower bound of $\frac{3}{2}(n - d)$ when $n = 4k + 4$ and $d = 4$.

Definition 3.5.2. $\mu(4, 4k + 4 + j) \geq 6k + j$.

Proof. Start with the graph made of k copies of Figure 3.8. Then append the vertex $x_{6k-3}x_{6k-2}x_{6k-1}x_{6k+1}$. If $j \geq 2$ then append $x_{6k-2}x_{6k-1}x_{6k+1}x_{6k+2}$. If $j = 3$ then append $x_{6k-1}x_{6k+1}x_{6k+2}x_{6k+3}$. □

For the $d = 3$ case, we will consider three graphs: The graph G_0 from Figure 3.6, the graph G_1 from Figure 3.7, and the graph G_2, which is the graph G_1 with $\{I, J, K\}$ appended to the end.

Theorem 3.5.3. $\mu(3, 8k + 2) \geq 10k - 1$.

Proof. Construct a graph composed of $k-1$ copies of G_2 by gluing ABC to IJK. Then glue a copy of G_1 to the end. Each copy of G_2 adds 10 to diameter, since all adjacent vertices lie in the same copy of G_2. Gluing G_1 adds 9 to the diameter. The new graph retains local connectedness by Theorem 3.4.2. □

This yields a lower bound of $\frac{5}{4}(n - d)$ when $n = 8k + 2$, $d = 3$.

Definition 3.5.4. $\mu(3, 8k + 3 + j) \geq 10k + j$ when $j \geq 0$.

Proof. Take the graph constructed in Theorem 3.5.3. Append the vertices $\{n - 2 + i, n - 1 + i, n + i\}$ for $i = 1...j + 1$. This will be a locally connected graph of diameter $10k + j$. □

Theorem 3.5.5. $\mu(3, 8k + 3 + j) \geq 10k + j + 1$ *when* $j \geq 4$.

Proof. Glue G_0 to $k - 1$ copies of G_2. Then glue one copy of G_1. If $j \geq 5$ then append $\{n - 6 + i, n - 5 + i, n - 4 + i\}$ for $i = 5...j$. This graph has diameter $10k + j + 1$ and is locally connected by Theorem 3.4.2. □

Figure 3.2 depicts the case where $d = 3$, $n - d = 12$.

Remark 3.5.6. To construct a graph with $d \geq 4$, codim $= 4k$, and diameter $\frac{3}{2}(4k)$, begin with the construction for $d = 4$, $n = 4k + 4$ given above. This construction has the desired diameter $\frac{3}{2}(4k)$. Add the new variables $x_{n+1}, \ldots, x_{n+(d-4)}$ to each vertex of this graph to generate the desired graph.

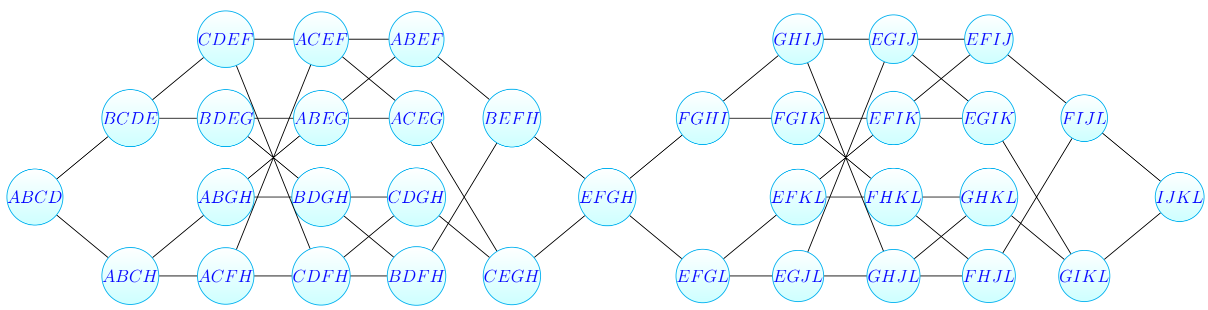

Figure 3.1

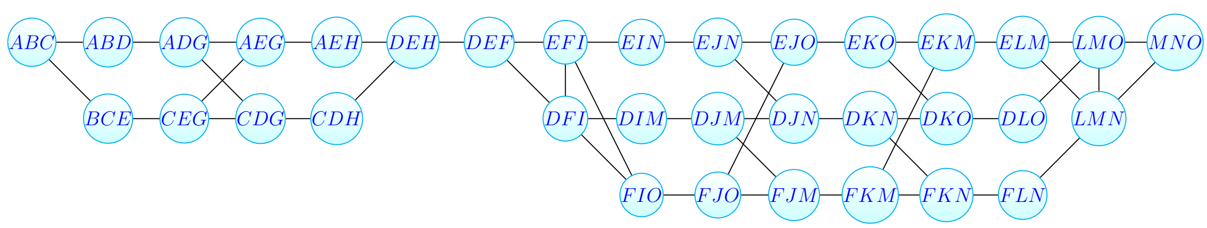

Figure 3.2

3.6 Constructing Graphs to Identify Lower Bounds

In this section, for d and n fixed, we construct lower bounds for the maximum diameters of dual graphs of equidimensional Stanley-Reisner rings satisfying (S_2). We achieve this by constructing graphs with properties (1) and (2). Recall, $\mu(d, n)$ is the largest diameter of a dual graph of an equidimensional, (S_2) Stanley-Reisner ring of dimension d and codimension $n - d$.

Theorem 3.6.1. *Table 1 (see Section 3.1) presents $\mu(d,n)$ for small values of d and n.*

This theorem is proved by the propositions of this section.

Proposition 3.6.2. *For $n \geq 2$, $\mu(2,n) = n-2$.*

Proof. R satisfies (S_2) if and only if $\bar{\mathcal{G}}(R)$ is locally connected. In the $d=2$ case, locally connected is equivalent to connected. Thus we construct a connected graph. Create a vertex $v_1 = x_1x_2$. We wish to create another vertex v_2 adjacent to v_1. Without loss of generality, $v_2 = x_1x_3$. Any vertex not adjacent to v_1 but adjacent to v_2 must be of the form x_3x_i $(i \neq 1,2,3)$. Thus we may choose $v_3 = x_3x_4$. Continuing this process, we see that $\mu(2,n)$ is bounded above by the number of variables in R that are not contained in v_1. We also see that this construction yields a graph of diameter $n-2$. Thus $\mu(2,n) = n-2$. Figure 3.3 shows a graph with properties (1) and (2) of diameter 3 when $n=5, d=2$. $\square$

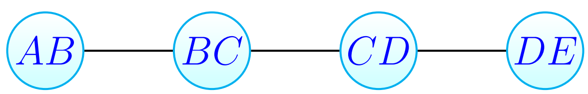

Figure 3.3

Proposition 3.6.3. $\mu(3,6) = 3$.

Proof. Let us consider any distinct pair of vertices $v_1, v_2 \in \bar{\mathcal{G}}(R)$. If $\deg(v_1 \cap v_2) = 2$ then v_1, v_2 are adjacent. If $\deg(v_1 \cap v_2) = 1$, then local connectedness of the graph requires that there exists a path from v_1 to v_2 such that each vertex in the path contains $v_1 \cap v_2$. Thus applying Proposition 3.6.2, the distance between these two vertices is bounded above by $\mu(2,5) = 3$. If $\deg(v_1 \cap v_2) = 0$, then every vertex is either adjacent to v_1 or adjacent to v_2. Therefore $\mu(3,6) \leq 3$. Adding F to every vertex label in Figure 3.3 produces a diameter-3 graph with properties (1) and (2) with $d=3$, $n=6$. Therefore $\mu(3,6) = 3$. $\square$

Definition 3.6.4. For $d \geq 2$, $\mu(d, d+3) = 3$.

Proof. Let R be a codimension-3, (S_2) Stanley-Reisner ring. Take $v_1, v_2 \in V(\bar{\mathcal{G}}(R))$. Then, v_1, v_2 will each contain $n-3$ variables, and $v_1 \cap v_2$ will contain at least $n-6$ variables. Thus there must be a path from v_1 to v_2 in which each vertex in that path contains those $n-6$ shared variables. Thus $\mu(d, d+3) \leq \mu(3, 6)$. Furthermore, we may take the graph with vertex set $\{ABC, BCD, CDE, DEF\}$ and add the same $d-3$ variables to each vertex to show $\mu(d, d+3) \geq 3$. $\square$

Proposition 3.6.5. $\mu(3, 7) = 5$.

Proof. Figure 3.4 is an example of a diameter-5 graph with properties (1) and (2) with $d = 3$, $n = 7$. Thus $\mu(3, 7) \geq 5$. In Theorem 3.3.2, we proved $\mu(3, 7) \leq 5$. $\square$

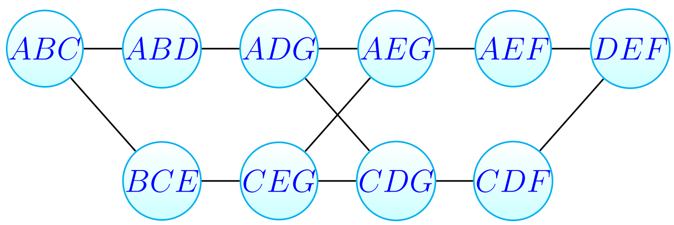

Figure 3.4

To see that $\bar{\mathcal{G}}(R)$ is locally connected, we take any subset of variables S and check that the vertices containing S form a connected subgraph. Below, we color the variables containing E red.

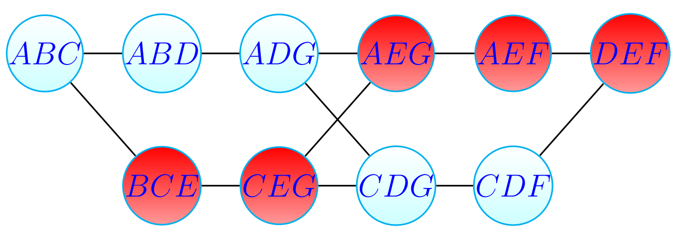

Figure 3.5

Proposition 3.6.6. $\mu(3, 8) = 6$.

Proof. Figure 3.6 is an example of a diameter-6 graph with properties (1) and (2) with $d = 3$, $n = 8$. Thus $\mu(3, 8) \geq 6$. In Theorem 3.3.2, we proved $\mu(3, 8) \leq 6$. $\square$

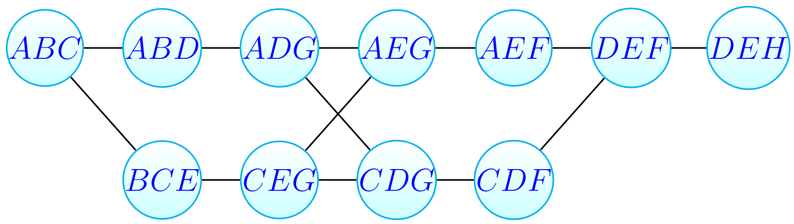

Figure 3.6

Proposition 3.6.7. $\mu(3, 9) = 7$.

Proof. By 3.3.2, $\mu(3, 9) \leq 8$. We will exhaustively prove that a graph with properties (1) and (2) and with diameter 8 cannot be constructed when $n = 9$, $d = 3$. We will use the names $A, B, C, D, E, F, G, H, I$ to represent our variables.

Suppose there exists a diameter-8 graph $\mathcal{G}$ with properties (1) and (2). Then $\mathcal{G}$ contains at least two vertices which are connected by paths of length no less than 8. Without loss of generality, let one of these vertices be ABC, name the other v_2. If $v_2 \cup ABC \neq \emptyset$ then the shortest path from v_2 to ABC is at most $\mu(8, 2) = 6$. Thus without loss of generality $v_2 = GHI$.

The graph $\mathcal{G}$ being connected implies $V(\mathcal{G})$ contains either ABD or ABG (without loss of generality). If $ABG \in V(\mathcal{G})$, the shortest path from ABG to GHI will be at most $\mu(8, 2) = 6$, and thus the diameter of the graph will be at most 7. Thus, we cannot have a vertex containing G, H, or I adjacent to ABC. By symmetry, we note we cannot have a vertex containing A, B, or C adjacent to GHI.

Thus, $ABD \in V(\mathcal{G})$.

Suppose a vertex containing G, H, or I has shortest path 2 to ABC in $V(\mathcal{G})$. Then without loss of generality, $ADG \in V(\mathcal{G})$. Then $\mathcal{G}$ must have a path of length 6 from ADG

to GHI containing $\mathcal{G}$ in each vertex. BC cannot be contained in any vertex of this path or else the graph will not have diameter 8, since any vertex containing BC is adjacent to ABC. But a path of length 6 with each vertex containing $\mathcal{G}$ must contain all 8 variables, since $\mu(7,2) < 6$. Furthermore, we have that B and C cannot be in a vertex adjacent to GHI. Therefore, we have that the vertex following ADG in the path must contain B or C. Either way this vertex may not contain A since ABC would then be adjacent to this vertex, and the graph would no longer have diameter 8. If this vertex contains BD we also would have an adjacnecy which would shoten the diameter of the graph. Thus our next vertex in the path must be CDG. Without loss of generality, our next vertex must be CEG. Our path must finish BEG, BFG, FGH, GHI.

Thus, $ABD \in V(\mathcal{G})$ implies $V(\mathcal{G})$ contains

$$ABC, ABD, ADG, CDG, CEG, BEG, BFG, FGH, GHI.$$

The locally connected property of $\mathcal{G}$ implies the vertices containing B must form connected subgraph. To not shorten the diameter of the graph, we need a path connecting ABC to BFG of length 6 and every vertex must contain B. Since $\mu(7,3) < 6$ we must use every variable in this path. There must be a vertex in the path adjacent to ABD that does not contain AB, BC, BE, BF, BG, or HI. Thus we must have BDH or BDI. There also must be a vertex in the path adjacent to BDH/BDI that does not contain AB, BC, BE, BF, BG, or HI. This however is impossible.

Therefore, we cannot have a vertex containing G, H, or I with shortest path to ABC less than 3. By symmetry, a vertex containing A, B, or C cannot have shortest path to GHI less than 3.

Without loss of generality, our graph contains the vertices ABC, ABD, ADE, and GHI. Suppose we have a vertex containing G, H, or I that has shortest path 3 to ABC. Without loss of generality, $V(\mathcal{G})$ contains AEG or DEG.

First suppose $DEG \in V(\mathcal{G})$. The vertices DEG and GHI must be connected by a path with length at least 5 in which each vertex contains G. As noted above, we cannot have a vertex containing A, B, or C with shortest path 2 or fewer to GHI. There must be a vertex adjacent to GHI containing G. This vertex, however, cannot contain A, B, C, D, E or else it will shorten the diameter of the graph to less than 8. The desired vertex must be FGH (without loss of generality). There also must be a vertex adjacent to FGH which contains G and does not contain H, I, A, B, C, D, E. This is impossible. Thus $DEG \notin V(\mathcal{G})$.

Suppose $AEG \in V(\mathcal{G})$. The path connecting AEG and GHI must have a length of at least 5 in which each vertex contains G. There must be a vertex adjacent to GHI containing G and not containing A, B, C, E. Thus the desired vertex must be FGH or DGH. There also must be a vertex adjacent to FGH/DGH which contains G and does not contain H, I, A, B, C, E. The desired vertex must be DFG.

Continuing, we get $V(\mathcal{G})$ must contain one of:

$$GHI, DGH, DFG, BFG, BEG, AEG, ADE, ABD, ABC,$$

$$GHI, DGH, DFG, CFG, CEG, AEG, ADE, ABD, ABC,$$

$$GHI, FGH, DFG, CDG, CEG, AEG, ADE, ABD, ABC.$$

Let us consider these cases individually. If $V(\mathcal{G})$ contains

$$GHI, DGH, DFG, BFG, BEG, AEG, ADE, ABD, ABC$$

then there must be a connected subgraph of vertices containing D. This subgraph must have diameter at least 6 or $\mathcal{G}$ does not have diameter 8. The subgraph must either be

$$ABD, ADE, CDE, CDI, DFI, DFG, DGH$$

or

$$ABD, ADE, DEI, CDI, CDF, DFG, DGH.$$

Suppose $V(\mathcal{G})$ contains $ABD, ADE, CDE, CDI, DFI, DFG, DGH$. Then $V(\mathcal{G})$ contains

$$ABC, ABD, ADE, AEG, BEG, BFG, DFG, DGH, GHI, CDE, CDI, DFI.$$

This graph must also have the property that the vertices containing C form a connected subgraph. Let us construct a path connecting ABC to CDE. The vertex adjacent to ABC must not contain G, H, I, D, E, or BF, so it must be ACF. The vertex adjacent to ACF must not contain G, H, I, D, B, A so it must be CEF. This graph must also have the property that the vertices containing I form a connected subgraph. The vertex adjacent to GHI must not contain A, B, C, D, F or GE, so it must be EHI. The vertex adjacent to EHI, must not contain A, B, C, G, H or DF, DE, EF. This is impossible. Thus $V(\mathcal{G})$ does not contain

$$ABD, ADE, CDE, CDI, DFI, DFG, DGH$$

.

Suppose $V(\mathcal{G})$ contains $ABD, ADE, DEI, CDI, CDF, DFG, DGH$. Then $V(\mathcal{G})$ contains

$$ABC, ABD, ADE, AEG, BEG, BFG, DFG, DGH, GHI, DEI, CDI, CDF.$$

Let us attempt to construct a path to connect the vertices containing C. The vertex adjacent to ABC cannot contain D, F, G, H, I, so it must be BCE or ACE. The vertex adjacent to ACE/BCE must not contain A, B, D, F, G, H, I, thus no such vertex exists. Thus $V(\mathcal{G})$ does not contain

$$GHI, DGH, DFG, BFG, BEG, AEG, ADE, ABD, ABC.$$

Suppose $V(\mathcal{G})$ contains

$$GHI, DGH, DFG, CFG, CEG, AEG, ADE, ABD, ABC.$$

We attempt to construct a path to connect the vertices containing C. The vertex adjacent to ABC cannot contain E, F, G, H, I, so it must be ACD or BCD. The vertex adjacent to ACD/BCD must not contain A, B, E, F, G, H, I. No such vertex exists. Thus $V(\mathcal{G})$ does not contain

$$GHI, DGH, DFG, CFG, CEG, AEG, ADE, ABD, ABC.$$

Suppose $V(\mathcal{G})$ contains

$$GHI, FGH, DFG, CDG, CEG, AEG, ADE, ABD, ABC.$$

We attempt to construct a path to connect the vertices containing C. The vertex adjacent to ABC cannot contain D, E, G, H, I, so it must be ACF or BCF. The vertex adjacent to ACF/BCF must not contain A, B, D, E, G, H, I. No such vertex exists. Thus $V(\mathcal{G})$ does not contain

$$GHI, FGH, DFG, CDG, CEG, AEG, ADE, ABD, ABC.$$

Thus, $V(\mathcal{G})$ does not contain a vertex containing G, H, or I with shortest path to ABC

of length three or fewer.

Suppose $\mathcal{G}$ has a vertex containing G whose shortest path to ABC is length four. Without loss of generality, the graph contains ABC, ABD, ADE. Let us consider vertices adjacent to ADE, which do not contain G, H, I and are not adjacent to ABC or ABD. Our possibilities are CDE, DEF, and AEF.

No vertex of $\mathcal{G}$ with path length to ABC smaller than 4 contains G, H, I, and $\mu(6,3) = 3$. These two facts imply any vertex of $\mathcal{G}$ with path length to ABC equal to 4 contains G, H, or I.

If our graph contains ABC, ABD, ADE, CDE, then it must contain CDG or CEG (or it must contain more vertices with shortest path length to ABC less than 4. Those cases will either be considered ahead or are symmetric to this case).

Suppose $CDG \in V(\mathcal{G})$. Let us construct a path between CDG and GHI in which each vertex contains G. The vertex adjacent to GHI must not contain A, B, C, D so it must be EGH or FGH. The next vertex in the path cannot contain A, B, C, D, H, I, so it must be EFG. The next vertex cannot contain A, B, C, H, I, and it cannot contain DE, so it must be DFG. This implies the vertex adjacent to GHI is not FGH.

Thus $V(\mathcal{G})$ contains: $ABC, ABD, ADE, CDE, CDG, DFG, EFG, EGH, GHI$. A path connects the vertices containing C. Thus either ACF or BCF is in $V(\mathcal{G})$. The graph must contain a vertex adjacent to ACF/BCF that does not contain A, B, D, G, H, I or EF, but this is impossible. Thus $CDG \notin V(\mathcal{G})$.

Suppose $CEG \in V(\mathcal{G})$. Let us construct a path between CEG and GHI in which each vertex contains G. The vertex adjacent to GHI must not contain A, B, C, E so it must be DGH or FGH. The next vertex in the path cannot contain A, B, C, E, H, I, so it must be DFG. The next vertex cannot contain A, B, C, H, I, and it cannot contain DE, so it must be EFG. This implies the vertex adjacent to GHI is not FGH.

Thus we have the vertices: $ABC, ABD, ADE, CDE, CEG, GEF, GDF, GHD, GHI$. The vertices containing C must be connected. The vertex set $V(\mathcal{G})$ contains either ACF or BCF, and $V(\mathcal{G})$ must also contain a vertex adjacent to ACF/BCF that does not contain A, B, E, G, H, I or DF. This is impossible. Thus $CEG \notin V(\mathcal{G})$. Thus $CDE \notin V(\mathcal{G})$.

If $V(\mathcal{G})$ contains ABC, ABD, ADE, DEF then it must contain DFG or EFG. Let us construct a path between GHI and DFG/EFG in which each vertex contains G. First suppose $DFG \in V(\mathcal{G})$. The vertex adjacent to GHI cannot contain A, B, C, D, F, so it must be EGH. The vertex adjacent to EGH cannot contain A, B, C, D, F, H, I, but no such vertex exists. Thus $DFG \notin V(\mathcal{G})$.

Now suppose $EFG \in V(\mathcal{G})$. The vertex adjacent to GHI cannot contain A, B, C, E, F, so it must be DGH. The vertex adjacent to DGH cannot contain A, B, C, E, F, H, I, but no such vertex exists. Thus $EFG \notin V(\mathcal{G})$. Thus $DEF \notin V(\mathcal{G})$.

If our graph contains ABC, ABD, ADE, AEF then it must contain AFG or EFG. Let us constrcut a path between GHI and AFG/EFG in which each vertex contains G. First suppose $AFG \in V(\mathcal{G})$. The vertex adjacent to GHI cannot contain A, B, C, F, so it must be EGH or DGH. The vertex adjacent to DGH/EGH cannot contain A, B, C, F, H, I or DE, but no such vertex exists. Thus $AFG \notin V(\mathcal{G})$.

Now suppose $EFG \in V(\mathcal{G})$. The vertex adjacent to GHI cannot contain A, B, C, E, F, so it must be DGH. The vertex adjacent to DGH cannot contain A, B, C, E, F, H, I, but no such vertex exists. Thus $EFG \notin V(\mathcal{G})$. Thus $AEF \notin V(\mathcal{G})$.

Thus, we have shown $\mathcal{G}$ does not contain a vertex containing G with shortest path length to ABC less than four. However, $\mu(6, 3) = 3$ tells us that every graph will either have such a vertex or will have diameter at most 3. Therefore, we have proven that $\mathcal{G}$ does not exist, and thus $\mu(9, 3) \leq 7$. We have $\mu(9, 3) = 7$, since

$$ABC, ABD, ADG, AEG, AEF, DEF, DEH, EHI, BCE, CEG, CDG, CDF$$

is an example of a diameter 7 graph with properties (1) and (2) when $n = 9, d = 3$.

$\square$

Proposition 3.6.7 gives a bound only one better than the general upper bound given in Theorem 3.3.2.

Proposition 3.6.8. $\mu(3, 10) \geq 9$.

Proof. Figure 3.7 is an example of a diameter-9 graph with properties (1) and (2) with $d = 3$, $n = 10$. $\square$

Proposition 3.6.8 gives a bound only one better than the bound of Theorem 3.3.2.

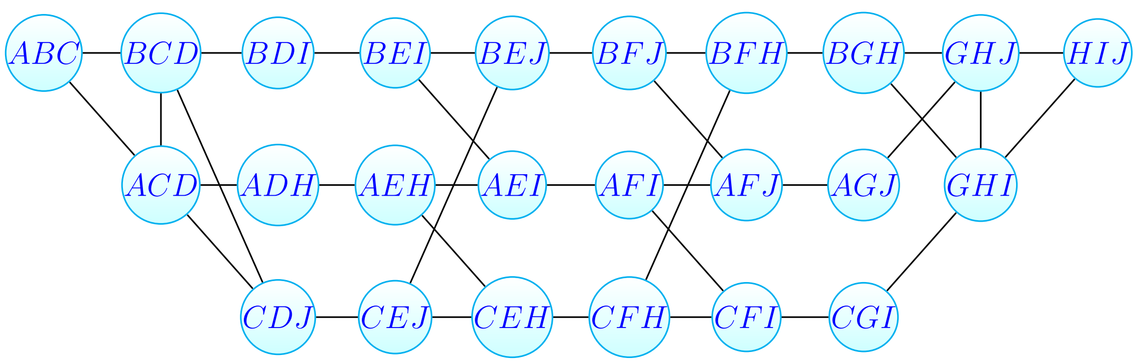

Figure 3.7

Proposition 3.6.9. $\mu(3, n) \geq n - d + 2$ *for all* $n \geq 10$.

Proof. We can construct an example of a diameter-$(n-d+2)$ graph with properties (1) and (2) with $d = 3$, $n = 10 + j$ by taking the graph in Figure 3.7 and appending the vertices:

$$IJx_1, Jx_1x_2, x_1x_2x_3, \cdots, x_{j-2}x_{j-1}x_j.$$

$\square$

Buchsbaum complexes have long been studied in combinatorial algebra (Hib96; Ter96; Han01; TY06). It is of interest that all of the complexes we have examined thus far are Buchsbaum. The following is likely known to experts, but we include a proof here.

Proposition 3.6.10. *Let $R = S/I$ be an equidimensional Stanley-Reisner ring of dimension 3. Then R is connected and Buchsbaum if and only if R satisfies (S_2).*

Proof. Let Δ be a complex with Stanley-Reisner ring $R = S/I$. Let m be the unique maximal homogeneous ideal of S. Using local duality (see Lemma 3.4.1), we have that a 3-dimensional equidimensional Stanley-Reisner ring R satisfies (S_2) if and only if

$$\dim \operatorname{Ext}_S^{n-i}(R, \omega_S) \leq i - 2 \quad \text{for all } i < 3.$$

Thus, R satisfies (S_2) if and only if $H^0_m(R) = H^1_m(R) = 0$ and $H^2_m(R)$ is finitely generated.

Suppose R is Buchsbaum and connected. Connectivity implies $H^0_m(R) = H^1_m(R) = 0$. If R is Buchsbaum, then R is Generalized Cohen-Macaulay, which implies that $H^i_m(R)$ is finitely generated for all $i < d$. Thus for R an equidimensional Stanley-Reisner ring of dimension 3, Buchsbaum and connected imply (S_2).

Suppose now that R satisfies (S_2). We will use the combinatorial definition of Buchsbaum, which says a complex is Buchsbaum if it is pure and has the property that the link of any non-empty face has zero reduced homology except possibly in top dimension.

We first note the Stanley-Reisner ring of Δ satisfying (S_2) implies Δ is pure and connected. Next we note that every link of a non-empty face of Δ has Stanley-Reisner ring R_P, where P is a prime ideal and $\dim R_P < \dim R$. Thus R being a 3-dimensional, (S_2) ring implies R_P is Cohen-Macaulay for all P such that $\dim R_P < \dim R$. Thus R_P has zero reduced homology except possibly in top dimension. Thus R is Buchsbaum and connected. $\square$

Note that this theorem does not apply in the higher dimension cases. In fact, most of our examples in higher dimension, including Figure 3.8 below, are not Buchsbaum.

Proposition 3.6.11. $\mu(4, 8) = 6$.

Proof. Figure 3.8 is an example of a diameter-6 graph with propeties (1) and (2) with $d = 4$, $n = 8$. In Corollary 3.3.3, we will prove $\mu(4, 8) \leq 6$. $\square$

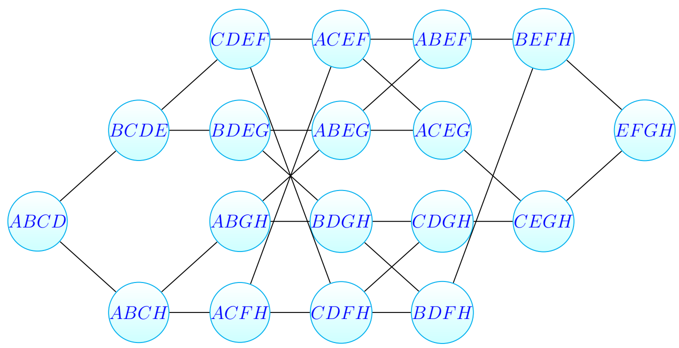

Figure 3.8

Proposition 3.6.12. $\mu(4, 9) = 7$.

Proof. Let $\mathcal{G}$ be a graph with properties (1) and (2) and diameter at least 8. First let us consider two vertices with maximum distance in $\mathcal{G}$. If the intersection of these vertices is non-trivial, their distance is bounded above by $\mu(3, 8) = 6$. Thus, the vertices of maximal distance in $\mathcal{G}$ must have trivial intersection. Call them $ABCD$ and $FGHI$. Suppose there exists a $v \in V(\mathcal{G})$ such that v is adjacent to $ABCD$, and v contains F, G, H or I. The shortest path from v to $FGHI$ will be bounded above by $\mu(3, 8) = 6$, and thus $\mathcal{G}$ will have diameter at most 7. Thus no such vertex is contained in $\mathcal{G}$. Simillarly no vertex containing A, B, C or D adjacent to $FGHI$ is contained in $\mathcal{G}$.

Connectivity of $\mathcal{G}$ requires $\mathcal{G}$ have at least one vertex adjacent to $ABCD$ and at least one vertex adjacent to $FGHI$. These vertices must both contain the only variable which is not in $ABCD$ or $FGHI$, call this variable E. Since $\mathcal{G}$ is locally connected, $\mathcal{G}$ must contain a connected subgraph composed only of the vertices containing E. Every vertex containing E will also contain two variables from either $ABCD$ or $FGHI$. Take such a vertex, $ABEF$. Then $\mathcal{G}$ must have a connected subgraph made up of only the vertices containing AB. However, we already have that any vertex adjacent to $ABCD$ must contain E. Thus $ABEF$ must have shortest path length 2 to $ABCD$. Any vertex containing E is distance at most 2 from $ABCD$ or $FGHI$. Thus $\mathcal{G}$ is at most diameter 5. Thus we have a contradiction. Thus no diameter-8 graph with properties (1) and (2) exists.

To construct a diameter-7 graph with properties (1) and (2), take the graph in Figure 3.8 and append the vertex $EFGI$. □

Proposition 3.6.13. *$\mu(4, n) \geq n - d + 2$ for all $n \geq 8$.*

Proof. We can construct an example of a diameter-$(n - d + 2)$ graph with propeties (1) and (2) with $d = 4$, $n = 8 + j$ by taking the graph in Figure 3.8 and appending the vertices

$$EFGx_1, FGx_1x_2, \cdots x_{j-3}x_{j-2}x_{j-1}x_j.$$

□

3.7 Final Remarks

These results raise several questions for further study. Primarily, is $\mu(d, n)$ bounded above by a polynomial in the codimension? A positive answer to this question would affirm the polynomial Hirsch conjecture. Our work still leaves the possibility that $\mu(d, n)$ is bounded by a linear function.

Another interesting question is the following: Do the bounds of Larman (Lar70) and Barnette (Bar74) hold for larger values of d? We have seen that the bounds of Larman and Barnette do not hold for our $d = 3$ case. Figure 3.1 shows the bound of Barnette does not hold for $d = 4$, and we can cone over this complex to show the bound of Barnette does not hold for $d = 5$; however, we do not have counterexamples in any higher dimensions.

We have found graphs of maximal diameter for (S_2) Stanley-Reisner rings with $d = 3, 4$ and small n. We then used those graphs in conjunction with gluing to make graphs with large diameters with respect to codimension. It would be valuable to know what the largest diameter would be for graphs of (S_2) Stanley-Reisner rings with $d = 5, 6$ and small n, specifically, $d = 5, n = 10$ and $d = 6, n = 12$. Answers to these questions could lead to new asymptotic lower bounds and could give insight on how these bounds would grow with respect to d.

Chapter 4

A Generalized Serre Condition

4.1 Introduction

In this chapter, we produce a generalization of Serre's condition (S_ℓ), which we call (S_ℓ^j). We will prove (S_ℓ^j) analogues to many (S_ℓ) theorems, and we shall prove that a variety of criterion are equivalent to satisfying (S_ℓ^j)

Let R be a commutative Noetherian ring. Recall Serre's condition (S_ℓ).

Definition 4.1.1. A ring R satisfies Serre's condition (S_ℓ) if for all $\mathfrak{p} \in \operatorname{Spec} R$,

$$\operatorname{depth} R_\mathfrak{p} \geq \min\{\ell, \dim R_\mathfrak{p}\}.$$

For a d-dimensional ring, being Cohen-Macaulay is equivalent to satisfying (S_d). A multitude of authors have examined Serre's condition, and the popularity of the topic has continued to grow (cf. (DHV16; HTYZN11; MT09; PSFTY14; Ter07; Yan00)). Serre's condition, like the Cohen-Macaulay condition, ties homological properties to geometric properties. We define a generalization of Serre's condition, which also links homological properties to geometric properties.

Definition 4.1.2. A ring R satisfies (S_ℓ^j) property if for all $\mathfrak{p} \in \operatorname{Spec} R$,

$$\operatorname{depth} R_{\mathfrak{p}} \geq \min\{\ell, \dim R_{\mathfrak{p}} - j\}.$$

In this chapter, we examine results about (S_ℓ) from a variety of sources in the literature. We prove generalizations for the (S_ℓ^j) property. The following will be used implicitly in the chapter when appropriate.

Proposition 4.1.3. *Let $\phi : R \to S$ be a faithfully flat homomorphism of Noetherian rings. Then if S satisfies (S_ℓ^j) then so does R.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$. Since ϕ is faithfully flat, there exists $\mathfrak{q} \in \operatorname{Spec} S$ such that $\mathfrak{p} = \mathfrak{q} \cap R$. From (BH98), $\dim R_{\mathfrak{p}} = \dim S_{\mathfrak{q}}$ and $\operatorname{depth} R_{\mathfrak{p}} = \operatorname{depth} S_{\mathfrak{q}}$. Thus $\operatorname{depth} R_{\mathfrak{p}} = \operatorname{depth} S_{\mathfrak{q}} \geq \min\{n, \dim S_{\mathfrak{q}} - j\} = \min\{n, \dim R_{\mathfrak{p}} - j\}$. Thus R satisfies (S_ℓ^j). $\square$

Remark 4.1.4. We shall also make use of the fact that for a Stanley-Reisner ring R and its associated simplicial complex Δ, localizing R at a prime $\mathfrak{p}$ generated by variables yields the same ring as the following process. Localize Δ at a face F generated by the variables which are not generators of $\mathfrak{p}$. Then localize the Stanley-Reisner ring of $\operatorname{lk}_\Delta F$ at its unique maximal homogeneous ideal. Throughout the chapter, we shall use $R_{\mathfrak{p}}$ and the Stanley-Reisner ring of $\operatorname{lk}_\Delta F$ interchangeably for convenience and brevity.

We now describe the organization of the chapter.

In Section 2, we prove an equivalence between rings satisfying (S_ℓ^j) and the support of Ext functors. In this section we consider rings of the form $R = S/I$ where S is an n-dimensional polynomial ring and I is a homogeneous ideal or S is an n-dimensional complete regular local ring and I is an ideal of S.

In Section 3, we examine theorems from the literature which bound cohomological dimension when R satisfies (S_2) or (S_3). We prove generalizations of these bounds that apply when

R is equidimensional and satisfies (S_2^j) or (S_3^j). Using these bounds, we bound projective dimension of R when $\mathfrak{a}$ is a pure square-free monomial ideal and R satisfies (S_2^j) or (S_3^j). In this section, we will consider S to be an n-dimensional regular local ring containing a field, $\mathfrak{a}$ to be an ideal of S of pure height and our ring to be $R = S/\mathfrak{a}$.

In Section 4, we consider the Hochster-Huneke graph of R (denoted by $G(R)$) where R is a local ring or a quotient of a polynomial ring and a homogeneous ideal. It is known that a Stanley-Reisner ring satisfies (S_2) if and only if every localization of R at a prime has a connected Hochster-Huneke graph (Kum08; Hol18). We create a generalization of the Hochster-Huneke graph. We show that every localization of R at a prime has a connected generalized Hochster-Huneke graph if and only if R satisfies (S_2^j).

In Section 5, we expand upon a result of Yanagawa (Yan00) which states that a Stanley-Reisner ring S/I satisfies (S_ℓ) if and only if the Alexander dual of I satisfies a specific homological condition. We combine this result with Theorem 3.4 to prove a bound on regularity of pure square-free monomial ideals.

Given a Stanley-Reisner ring R with simplicial complex Δ, Reisner's criterion provides a method to check the Cohen-Macaulayness of R by examining the reduced homology groups of Δ (Rei76). Terai made an analogous theorem for describing whether R satisfies (S_ℓ) (Ter07). In Section 6, we generalize Terai's result by giving an equivalent condition for (S_ℓ^j) using homology of links in the equidimensional case.

In Section 7, we examine monomial ideals of polynomial rings over a field. Herzog, Takayama, and Terai (HTT05) proved for a monomial ideal I and a polynomial ring S, S/I being Cohen-Macaulay implies $S/\sqrt{I}$ is Cohen-Macaulay. We present an analogous theorem showing S/I satisfying (S_ℓ^j) implies $S/\sqrt{I}$ satisfies (S_ℓ^j).

In section 8, we prove a theorem relating i-skeletons of simplicial complexes and the (S_ℓ^j) property. We discuss i-skeletons' importance with respect to depth.

Unless otherwise stated, k will be a field, $S = k[x_1, x_2, ..., x_n]$, and I will be an ideal of

S.

4.2 An Equivalent Functorial Condition

In this section, we characterize the generalized Serre's condition as a homological condition. This is an extension of Lemma 3.4.1.

A catenary noetherian local ring satisfying (S_2) is equidimensional by (Har62, Remark 2.4.1). This is not true for (S_ℓ^j) even for large values of ℓ and small, positive values of j.

Example 4.2.1. Consider the ring:

$$k[x_1, x_2, ..., x_{11}]/\langle x_1 x_{11}, x_2 x_{11}\rangle$$

This ideal has two minimal primes $\langle x_1, x_2\rangle$ and $\langle x_{11}\rangle$. This ideal is clearly non-equidimensional but satisfies (S_ℓ^1) for any choice of ℓ.

Thus we will need the following notation.

Given a ring $R = S/I$, let Q_I be a minimal prime of I of smallest height. Given a prime $\mathfrak{p}$ which contains I, let $Q_\mathfrak{p}$ be a minimal prime of I contained in $\mathfrak{p}$ with smallest height. Let $\alpha_\mathfrak{p} = \operatorname{ht} Q_\mathfrak{p} - \operatorname{ht} Q_I$. We note that $\alpha_\mathfrak{p} = 0$ for all $\mathfrak{p}$ when R is equidimensional.

Theorem 4.2.2. *Let S be an n-dimensional polynomial ring with maximal homogeneous ideal $\mathfrak{m}$ and let I be a homogeneous ideal or let S be an n-dimensional complete regular local ring with maximal ideal $\mathfrak{m}$ and let I be an ideal of S. Let $R = S/I$ and let $d = \dim R$. Then R satisfies (S_ℓ^j) if and only if for all $\mathfrak{p} \in \operatorname{Spec} S$ containing I with $\operatorname{ht}\mathfrak{p} < n - i + \ell$, we have that $\mathfrak{p} \notin \operatorname{Supp}\operatorname{Ext}_S^{n-i}(R, S)$ for all $i = 0, ..., d - j - 1 - \alpha_\mathfrak{p}$.*

Proof. Suppose R satisfies (S_ℓ^j). If $i < \ell \leq \operatorname{depth} R$ then $H_\mathfrak{m}^i(R) = 0 = \operatorname{Ext}_S^{n-i}(R, S)$, and we are finished. Otherwise, let us take $\mathfrak{p} \in \operatorname{Spec} S$ such that $I \subseteq \mathfrak{p}$. Let h be the height of

$\mathfrak{p}$. If $h < n - i + \ell$, then

$$\operatorname{Ext}^{n-i}_S(R,S)_{\mathfrak{p}} \cong \operatorname{Ext}^{n-i}_{S_{\mathfrak{p}}}(R_{\mathfrak{p}}, S_{\mathfrak{p}}) = 0 \quad \text{ for all } i = 0, ..., d - j - 1 - \alpha_{\mathfrak{p}}.$$

This is true because $n - i = h - (h - n + i)$ and $\dim R_{\mathfrak{p}} - j = h - n + d - j - \alpha_{\mathfrak{p}} > h - n + i$. So $\mathfrak{p} \notin \operatorname{Supp} \operatorname{Ext}^{n-i}_S(R,S)$ for $i = 0, ..., d - j - 1 - \alpha_{\mathfrak{p}}$ whenever $\operatorname{ht} \mathfrak{p} < n - (i - \ell)$.

Suppose $\mathfrak{p} \notin \operatorname{Supp} \operatorname{Ext}^{n-i}_S(R,S)$ for $i = 0, ..., d-j-1-\alpha_{\mathfrak{p}}$ whenever $\operatorname{ht} \mathfrak{p} < n-(i-\ell)$. Let V_h be the set of prime ideals of S of height h containing I for $h = n - d, ..., n$. Henceforth, let us use c to denote $n - d$.

Then we have the following equivalences:

$$R \text{ satisfies } (S^j_\ell) \quad \Leftrightarrow$$

$$\operatorname{depth} R_{\mathfrak{p}} \geq \min\{\ell, h - c - j - \alpha_{\mathfrak{p}}\} =: b \quad \text{ for all } h = c, ..., n \text{ for all } \mathfrak{p} \in V_h \quad \Leftrightarrow$$

$$H^i_{\mathfrak{p}S_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0 \quad \text{ for all } h = c, ..., n \text{ for all } \mathfrak{p} \in V_h, \text{ for all } i < b \quad \Leftrightarrow$$

$$\operatorname{Ext}^{h-i}_{S_{\mathfrak{p}}}(R_{\mathfrak{p}}, S_{\mathfrak{p}}) = 0 \text{ for all } h = c, ..., n \text{ for all } \mathfrak{p} \in V_h, \text{ for all } i < b \quad \Leftrightarrow$$

$$\mathfrak{p} \notin \operatorname{Supp} \operatorname{Ext}^{h-i}_S(R,S) \text{ for all } h = c, ..., n \text{ for all } \mathfrak{p} \in V_h, \text{ for all } i < b.$$

When $i < b \leq h - c - j - \alpha_{\mathfrak{p}}$, $n - h + i < n - c - j - \alpha_{\mathfrak{p}} = d - j - \alpha_{\mathfrak{p}}$.

From the assumption, for all $n-h+i < d-j-\alpha_{\mathfrak{p}}$, $\mathfrak{p} \notin \operatorname{Supp} \operatorname{Ext}^{n-(n-h+i)}_S(R,S)$ whenever $h \leq n - (n - h + i - \ell)$; thus,

$$\mathfrak{p} \notin \operatorname{Supp} \operatorname{Ext}^{h-i}_S(R,S) = \operatorname{Supp} \operatorname{Ext}^{n-(n-h+i)}_S(R,S) \text{ for all } i < b.$$

□

The result can be stated in a cleaner way when the ring in question is equidimensional.

In the equidimensional case, the above proof can be adjusted to give:

Corollary 4.2.3. *Let S be an n-dimensional polynomial ring with maximal homogenous ideal $\mathfrak{m}$ and let I be a homogeneous ideal or let S be an n-dimensional complete regular local ring with maximal ideal $\mathfrak{m}$ and let I be an ideal of S. Let $R = S/I$ be an equidimensional ring. Then R satisfies (S_ℓ^j) if and only if $\dim \operatorname{Ext}_S^{n-i}(R, S) \leq i - \ell$ for all $i = 0, ..., d - j - 1$.*

4.3 Bounds on Cohomological Dimension

Local cohomology is widely important in the fields of commutative algebra and algebraic geometry. One measure of this vanishing is the cohomological dimension. Let S be a Noetherian local ring and $\mathfrak{a}$ be an ideal of S. The cohomological dimension of $\mathfrak{a}$ in S is:

$$\operatorname{cd}(S, \mathfrak{a}) = \sup\{i \in \mathbb{Z}_{\geq 0} | H^i_{\mathfrak{a}}(M) \neq 0 \text{ for some } S\text{-module } M\}.$$

The cohomological dimension has been a topic garnering wide interest (cf. (Har68; Ogu73; PS73; HL90; Lyu93; Var13; DT16)). It is known that $\operatorname{cd}(S, \mathfrak{a})$ is less than or equal to $\dim S$. We desire to improve this bound for special cases. We will take advantage of various existing results to aid in this endeavor. If $\operatorname{depth} S/\mathfrak{a} \geq 1$, then $\operatorname{cd}(S, \mathfrak{a}) \leq n - 1$, which is an immediate consequence of the Hartshorne-Lichtenbaum vanishing theorem ((Har68, Theorem 3.1)). From the work of Ogus, we have $\operatorname{depth} S/\mathfrak{a} \geq 2$, then $\operatorname{cd}(S, \mathfrak{a}) \leq n - 2$ (Remark below Corollary 2.11 (Ogu73)).

In (HL90), Huneke and Lyubeznik prodce a theorem which helps construct more of these bounds. In (DT16), the theorem is reinterpreted for convenience of application.

Theorem 4.3.1 (Huneke-Lyubeznik, see (DT16, Theorem 3.7))**.** *Let (S,m) be a d-dimensional regular local ring containing a field and let $\mathfrak{a} \subset S$ be an ideal of pure height c. Let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. Assume there exist integers $\ell' \geq \ell \geq c$ such that:*

(1) $f(\ell) \geq c$,

(2) $\operatorname{cd}(S_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}}) \leq f(\ell' + 1) - c + 1$ *for all prime ideals* $\mathfrak{p} \supset \mathfrak{a}$ *with* $\operatorname{ht} \mathfrak{p} \leq \ell - 1$,

(3) $\operatorname{cd}(S_{\mathfrak{p}}, \mathfrak{a}_{\mathfrak{p}}) \leq f(\operatorname{ht} \mathfrak{p})$ *for all prime ideals* $\mathfrak{p} \supset \mathfrak{a}$ *with* $\ell \leq \operatorname{ht} \mathfrak{p} \leq \ell'$,

(4) $f(r - s - 1) \leq f(r) - s$ *for every* $r \geq \ell' + 1$ *and every* $c - 1 \geq s \geq 1$.

Then $\operatorname{cd}(S, \mathfrak{a}) \leq f(d)$ *if* $d \geq \ell$.

In (DT16), Dao and Takagi proved the following corollary:

Corollary 4.3.2 (Dao-Takagi). *Let* $(S, \mathfrak{m})$ *be an* n*-dimensional regular local ring essentially of finite type over a field. If* $\mathfrak{a}$ *is an ideal of* S *such that* $\operatorname{depth} S/\mathfrak{a} \geq 3$, *then* $\operatorname{cd}(S, \mathfrak{a}) \leq n - 3$.

Dao and Takagi then used this corollary in conjuction with Theorem 3.1 to prove:

Theorem 4.3.3 (Dao-Takagi). *Let* S *be an* n*-dimensional regular local ring containing a field and let* $\mathfrak{a} \subset S$ *be an ideal of height* c.

(1) *If* $S/\mathfrak{a}$ *satisfies Serre's condition* (S_2) *and* $\dim S/\mathfrak{a} \geq 1$, *then*

$$\operatorname{cd}(S, \mathfrak{a}) \leq n - 1 - \lfloor \tfrac{n-2}{c} \rfloor$$

(2) *Suppose that* S *is essentially of finite type over a field. If* $S/\mathfrak{a}$ *satisfies Serre's condition* (S_3) *and* $\dim S/\mathfrak{a} \geq 2$, *then*

$$\operatorname{cd}(S, \mathfrak{a}) \leq n - 2 - \lfloor \tfrac{n-3}{c} \rfloor$$

In this chapter, we generalize this theorem to consider (S_2^j) and (S_3^j) for any j.

Theorem 4.3.4. *Let* S *be an* n*-dimensional regular local ring containing a field and let* $\mathfrak{a} \subset S$ *be a pure ideal of height* c.

(1) *If* $S/\mathfrak{a}$ *satisfies Serre's condition* (S_2^j) *and* $\dim S/\mathfrak{a} \geq 1 + j$, *then*

$$\operatorname{cd}(S, \mathfrak{a}) \leq n - 1 - \lfloor \tfrac{n-2-j}{c} \rfloor$$

(2) *Suppose that S is essentially of finite type over a field. If $S/\mathfrak{a}$ satisfies Serre's condition (S_3^j) and* $\dim S/\mathfrak{a} \geq 2 + j$*, then*

$$\operatorname{cd}(S, \mathfrak{a}) \leq n - 2 - \lfloor \tfrac{n-3-j}{c} \rfloor$$

Proof. For (1) we apply Theorem 4.3.1 with $f(m) = m - 1 - \lfloor \frac{m-2-j}{c} \rfloor$, $\ell = c + 1 + j$, and $\ell' = 2c + 1 + j$. Condition (2) of Theorem 4.3.1 becomes $\operatorname{cd}(S_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}) \leq c + j$ for all primes of height at most $c + j$. This follows since $\operatorname{cd}(S_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}) \leq \operatorname{ht} \mathfrak{p} \leq c + j$. Condition (3) of Theorem 4.3.1 breaks into two cases. If $\operatorname{ht} \mathfrak{p} = \ell$ then we get $\operatorname{cd}(S_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}) \leq \operatorname{ht} \mathfrak{p} - 1$. This is true, since $\operatorname{depth} S_\mathfrak{p}/\mathfrak{a}S_\mathfrak{p} \geq \min\{2, c + j + 1 - c - j\} = 1$. If $\operatorname{ht} \mathfrak{p} > \ell$ then we get $\operatorname{cd}(S_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}) \leq \operatorname{ht} \mathfrak{p} - 2$. This is true, since $\operatorname{depth} S_\mathfrak{p}/\mathfrak{a}S_\mathfrak{p} \geq 2$.

For (2) we apply Theorem 4.3.1 with $f(m) = m - 2 - \lfloor \frac{m-3-j}{c} \rfloor$, $\ell = c + 2 + j$, and $\ell' = 2c + 2 + j$. Condition (2) of Theorem 4.3.1 breaks into two cases. If $\operatorname{ht} \mathfrak{p} = \ell - 1$, then $\operatorname{cd}(S_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}) \leq \operatorname{ht} \mathfrak{p} - 1$. This is true since $\operatorname{depth} S_\mathfrak{p}/\mathfrak{a}S_\mathfrak{p} \geq 1$. Otherwise, (2) becomes $\operatorname{cd}(S_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}) \leq \operatorname{ht} \mathfrak{p}$, which is true. Demonstrating condition (3) of Theorem 4.3.1 has two parts. If $\operatorname{ht} \mathfrak{p} = \ell = c + 2 + j$, then $\operatorname{depth} S/\mathfrak{a} \geq 2$, and thus we have $cd(S, \mathfrak{a}) \leq \operatorname{ht} \mathfrak{p} - 2 = f(\operatorname{ht} \mathfrak{p})$. If $c + 3 + j \leq \operatorname{ht} \mathfrak{p} \leq 2c + 2 + j$, then $f(\operatorname{ht} \mathfrak{p}) = \operatorname{ht} \mathfrak{p} - 3$. However, $\operatorname{depth} S_\mathfrak{p}/\mathfrak{a}S_\mathfrak{p} \geq 3$. Therefore, Corollary 4.3.2 implies $\operatorname{cd}(S_\mathfrak{p}, \mathfrak{a}_\mathfrak{p}) \leq \operatorname{ht} \mathfrak{p} - 3$. □

Corollary 4.3.5. *Let S be an n-dimensional regular local ring containing a field and let $\mathfrak{a} \subset S$ be a pure, square-free monomial ideal of height c.*

(1) If $S/\mathfrak{a}$ satisfies Serre's condition (S_2^j) and $\dim S/\mathfrak{a} \geq 1 + j$*, then*

$$\operatorname{pd} S/\mathfrak{a} \leq n - 1 - \lfloor \tfrac{n-2-j}{c} \rfloor$$

(2) Suppose that S is essentially of finite type over a field. If $S/\mathfrak{a}$ satisfies Serre's condition (S_3^j) and $\dim S/\mathfrak{a} \geq 2 + j$*, then*

$$\operatorname{pd} S/\mathfrak{a} \leq n - 2 - \lfloor \tfrac{n-3-j}{c} \rfloor$$

Proof. When $\mathfrak{a}$ is a square-free monomial ideal, $\operatorname{cd}(S, \mathfrak{a}) = \operatorname{pd} S/\mathfrak{a}$ (Lyu84). Therefore, applying Theorem 4.3.4 we get the desired bounds. $\square$

4.4 A Generalized Hochster-Huneke Graph

For this section we consider two kinds of rings, local rings and rings which are a quotient of a polynomial ring and a homogeneous ideal. When we say ring, we shall mean these types of rings unless otherwise noted.

The Hochster-Huneke graph is a graphical representation of the minimal prime ideals of a ring R. This graph is sometimes called the dual graph of $\operatorname{Spec} R$ and has been examined from many perspectives throughout the literature (cf (HH94; BBV17; BV15; Hol18)). The connectivity of the Hochster-Huneke graph and its localizations provides information about algebraic properties of R. We will consider a generalization of the Hochster-Huneke graph.

Definition 4.4.1 ((HH94))**.** Let $G(R)$ be the graph with $V(G(R)) = \{v_i = P_i\}$ where the P_i are the minimal primes of R, $E(G(R)) = \{(v_k, v_\ell) | \operatorname{ht}(\mathfrak{p}_k + \mathfrak{p}_\ell) = 1\}$. Then $G(R)$ is the *Hochster-Huneke graph* of R.

We note the following definition is equivalent to Definition 3.2.12.

Definition 4.4.2. A ring is *locally connected* if for any $\mathfrak{p} \in \operatorname{Spec} R$, $G(R_\mathfrak{p})$ is connected.

Theorem 4.4.3 ((Kum08, Theorem 6.1.5), Theorem 3.2.17)**.** *Let R be a Stanley-Reisner ring. Then R satisfies (S_2) if and only if R is locally connected.*

We desire to make a generalized Hochster-Huneke graph in order to make an analogue of the previous theorem for rings satisfying (S_2^j). We note that our generalization is the same generalization given in (NBSW17).

Definition 4.4.4. Let $G^j(R)$ be the graph with $V(G^j(R)) = \{v_i = P_i\}$ where the $\mathfrak{p}_i$ are the minimal primes of R, $E(G^j(R)) = \{(v_k, v_\ell) | 1 \leq \mathrm{ht}(\mathfrak{p}_k + \mathfrak{p}_\ell) \leq j\}$. We note $G^1(R)$ is the Hochster-Huneke graph of R.

Definition 4.4.5. A ring is *j-locally connected* if for any $\mathfrak{p} \in \operatorname{Spec} R$, $G^j(R_\mathfrak{p})$ is connected.

Theorem 4.4.6. *Let R be a Stanley-Reisner ring. Then, R satisfies (S_2^j) if and only if R is $j+1$-locally connected.*

Proof. By Theorem 4.2.2, we have R satisfying (S_2^j) can be viewed as a condition on $\operatorname{Supp} \operatorname{Ext}_S^{n-i}(R, S)$. From [Ya00], we have that R is a Stanley-Reisner ring implies that $\operatorname{Ext}_S^{n-i}(R, S)$ is a square-free module. Thus, $\operatorname{Ext}_S^{n-i}(R, S)$ is uniquely determined by its prime ideals generated by variables. Therefore, we only need to consider primes generated by variables when showing R satisfies (S_2^j).

Assume R is $(j+1)$-locally connected and $\dim R = d$. If $j \geq d-1$, then $1 \geq d-j$. All Stanley-Reisner rings are (S_1), thus this case is trivially true.

Now we suppose $j < d-1$. When $d > j+1$, $G^{j+1}(R)$ being connected implies Δ_R is connected. Thus, for all localizations of R such that $\dim R_\mathfrak{p} > j+1$ we have that $\Delta_{R_\mathfrak{p}}$ is connected. Therefore, we have that $\operatorname{depth} R_\mathfrak{p} \geq 2$, for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R_\mathfrak{p} \geq 2 + j$. This combined with the fact that all Stanley-Reisner rings satisfy (S_1) gives that R satisfies (S_2^j).

Now suppose R satisfies (S_2^j). By Corollary 2.3 of (Har62), we have that R is locally connected in codimension $1+j$. Thus $G^{j+1}(R_\mathfrak{p})$ is connected for all $\mathfrak{p} \in \operatorname{Spec} R$.

$\square$

4.5 Resolution of the Alexander Dual

Let us use notation from (FMS14) to introduce Alexander Duality. Let k be a field and $S = k[x_1, ..., x_n]$. Let m be a monomial in S, and let $\mathfrak{p}_m = (x_i : x_i | m)$.

A prime ideal $\mathfrak{p}$ is an associated prime of a square free monomial ideal I if $\mathfrak{p} \supseteq I$ and $\mathfrak{p}$ is minimal among primes ideals that contain I. The set of all associated primes of I is written $\mathrm{Ass}(I)$.

Definition 4.5.1. Let I be a square-free monomial ideal. The *Alexander dual* of I is

$$I^\vee = (m : \mathfrak{p}_m \in \mathrm{Ass}(I)).$$

In (ER98), Eagon and Reiner proved that a Stanley-Reisner ring R/I is Cohen-Macaulay if and only if $I^\vee$ has linear resolution. In (Yan00), Yanagawa generalized their result to say a Stanley-Reisner ring $R = S/I$ satisfies (S_ℓ) if and only if $I^\vee$ has syzygy matrices with linear entries up to homological degree $\ell - 1$ (Yan00). We further generalize these statements for equidimensional Stanley-Reisner rings satisfying (S_ℓ^j).

Definition 4.5.2. Let $R = S/I$ be an equidimensional Stanley-Reisner ring. We say that $I^\vee$ satisfies $(N_{c,\ell}^j)$ if

$$[\mathrm{Tor}_\gamma(I^\vee, k)]_\beta = 0 \text{ for all } \gamma < \ell \text{ and for all } c + j + \gamma < \beta \leq n.$$

Theorem 4.5.3. *Let $R = S/I$ be a d-dimensional, equidimensional Stanley-Reisner ring with codimension c. Then the following are equivalent for $\ell \geq 2$:*

(i) R satisfies (S_ℓ^j).

(ii) $I^\vee$ satisfies $(N_{c,\ell}^j)$.

Proof. By Corollary 4.2.3, R satisfies (S_ℓ^j) if and only if

$$\dim \operatorname{Ext}_S^{n-i}(R,S) \leq i-\ell \text{ for all } i=0,...,d-j-1.$$

We rewrite for convenience:

$$\dim \operatorname{Ext}_S^{\alpha}(R,S) \leq n-\alpha-\ell \text{ for all } c+j<\alpha\leq n.$$

The above is true if and only if for any prime $\mathfrak{p}$ with height $h<\alpha+\ell$ and support F we have:

$$(\operatorname{Ext}_S^{\alpha}(R,S))_F = 0 \text{ for } c+j<\alpha\leq n.$$

By Theorem 3.4 of [Ya00], this is true if and only if

$$[\operatorname{Tor}_{h-\alpha}(I^\vee,k)]_{F^c} = 0 \text{ for } c+j<\alpha\leq n.$$

By rewriting, we recover our definition of $(N_{c,\ell}^j)$.

□

Remark 4.5.4. From this result, we have R satisfies (S_2^j) if and only if all entries of the first syzygy matrix of R have degree at most $j+1$. Furthermore, R satisfies (S_ℓ^j) if the sum of the highest degrees of the first $\ell-1$ syzygy matrices is less than $j+\ell-1$.

Corollary 4.5.5. *Let $\mathfrak{a}$ be a pure, square-free monomial ideal of an n-dimensional polynomial ring over a field.*

(1) If $\mathfrak{a}$ satisfies $(N_{c,2}^j)$ and $c\leq n-1$, then

$$\operatorname{reg}\mathfrak{a} \leq n-1-\lfloor \tfrac{n-2-j}{c} \rfloor$$

(2) If $\mathfrak{a}$ satisfies $(N_{c,3}^j)$ and $c\leq n-2$, then

$$\operatorname{reg}\mathfrak{a} \leq n - 2 - \lfloor \tfrac{n-3-j}{c} \rfloor$$

Proof. From Theorem 4.5.3 we get that $S/\mathfrak{a}^\vee$ satisfies (S_2^j), and $\operatorname{cd}(S, \mathfrak{a}^\vee) = \operatorname{reg}\mathfrak{a}$. Therefore, applying Theorem 4.3.4 we get the desired bounds. □

4.6 A Generalization of Reisner's criterion

Reisner's criterion is a method for checking the Cohen-Macaulayness of a Stanley-Reisner ring by considering the reduced homology groups of the ring's associated complex. The i^{th} reduced homology group of Δ is $\tilde{H}_i(\Delta; k)$. The following theorem is from (Rei76).

Theorem 4.6.1 (Reisner's Criterion)**.** *Let k be a field and Δ be a simplicial complex of dimension $d-1$. Then Δ is Cohen-Macaulay over k if and only if for every $F \in \Delta$ and for every $i < \dim(\mathrm{lk}_\Delta(F))$, $\tilde{H}_i(\mathrm{lk}_\Delta(F); k) = 0$ holds true.*

Terai has formulated an analogue of this theorem for Stanley-Reisner rings satisfying (S_ℓ) $(\ell \geq 2)$.

Theorem 4.6.2 ((Ter07) described after Theorem 1.4)**.** *Let k be a field and Δ be a simplicial complex of dimension $d-1$. Then Δ satisfies (S_ℓ) over k if and only if for every $F \in \Delta$ (including $F = \emptyset$) with $|F| \leq d - i - 2$ and for every $-1 \leq i \leq \ell - 2$, $\tilde{H}_i(\mathrm{lk}_\Delta(F); k) = 0$ holds true.*

Now we present a lemma, which serves as an analogue to one direction of Reisner's criterion for rings satisfying (S_ℓ^j). Following this Lemma, we prove a full analogue of Reisner's criterion for equidimensional rings satisfying (S_ℓ^j). For the rest of this section, R shall be the Stanley-Reisner ring of Δ and $\mathfrak{m}$ shall be the unique maximal homogeneous ideal of R.

Lemma 4.6.3. *Let k be a field and Δ be a simplicial complex of dimension $d-1$. For a face F let d_F denote the cardinality of the largest facet containing F. Then Δ satisfying (S_ℓ^j)*

($\ell \geq 2$) over k implies for every $F \in \Delta$ (including $F = \emptyset$) with $|F| \leq d_F - i - j - 2$ and for every $-1 \leq i \leq \ell - 2$, $\tilde{H}_i(\mathrm{lk}_\Delta(F); k) = 0$.

Proof. Consider $\tilde{H}_i(\mathrm{lk}_\Delta(F); k)$ with $-1 \leq i \leq \ell - 2$ and $|F| \leq d_F - i - j - 2$. Let us localize R at a prime $\mathfrak{p}$ generated by the variables contained in F. Let $\dim R_\mathfrak{p} = d_\mathfrak{p}$. Then (S_ℓ^j) condition implies $H^\beta_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) = 0$ for all $\beta < \min(\ell, d_\mathfrak{p} - j) := b_\mathfrak{p}$. Hochster's formula, then, implies $\tilde{H}_{\beta-|F'|-1}(\mathrm{lk}_{\mathrm{lk}\,F}(F'); k) = 0$ for all F' and all $\beta < b_\mathfrak{p}$. In particular, $\tilde{H}_{\beta-1}(\mathrm{lk}_{\mathrm{lk}\,F}(\emptyset); k) = \tilde{H}_{\beta-1}(\mathrm{lk}_\Delta(F); k) = 0$ for all $\beta < b_\mathfrak{p}$. If we can show $i + 1 < b_\mathfrak{p}$, then we will have $\tilde{H}_i(\mathrm{lk}_\Delta(F); k) = 0$ as desired. We have $i \leq \ell - 2$ which implies $i + 1 < \ell$. We also have $|F| \leq d_F - j - i - 2$, which implies that $i < d_F - j - 1 - |F|$. Thus, $i + 1 < d_F - |F| - j$, and $d_F - |F| = d_\mathfrak{p}$. Thus, $i + 1 < b_\mathfrak{p}$. □

Theorem 4.6.4. *Let k be a field and Δ be a pure simplicial complex of dimension $d-1$. Then Δ satisfies (S_ℓ^j) over k if and only if for every $F \in \Delta$ (including $F = \emptyset$) with $|F| \leq d-i-j-2$ and for every $-1 \leq i \leq \ell - 2$, $\tilde{H}_i(\mathrm{lk}_\Delta(F); k) = 0$ holds true.*

Proof. Suppose Δ satisfies (S_ℓ^j). Then the result follows from Lemma 4.6.3.

Suppose for every $F \in \Delta$ with $|F| \leq d - i - j - 2$ and for every $-1 \leq i \leq \ell - 2$, $\tilde{H}_i(\mathrm{lk}_\Delta(F); k) = 0$.

Then let us consider $\tilde{H}_{\alpha-|F|-1}(\mathrm{lk}_\Delta(F); k)$. Let us examine the case where $\alpha < b := \min\{\ell, d-j\}$. We have $|F| > d-i-j-2 = d-\alpha+|F|+1-j-2 = d-\alpha-1+|F|-j$ if and only if $\alpha + 1 > d - j$ which implies $\alpha \geq d - j \geq b$. Thus, $\tilde{H}_{\alpha-|F|-1}(\mathrm{lk}_\Delta(F); k) = 0$ for any F so long as $\alpha < b$. We also note that $\alpha < b$ implies $\alpha < \ell$ which implies $\alpha - |F| - 1 \leq \ell - 2$. Thus, if we have $\alpha < b$, we have $\tilde{H}_{\alpha-|F|-1}(\mathrm{lk}_\Delta(F); k) = 0$ for all F. Applying Hochster's formula, we get that $H^i_\mathfrak{m}(R) = 0$ for all $i < b$. Therefore, we have $\operatorname{depth} R \geq b$.

We must prove that for any localization $R_\mathfrak{p}$ with dimension $d_\mathfrak{p}$, we have $\operatorname{depth} R_\mathfrak{p} \geq b_\mathfrak{p} := \min\{\ell, d_\mathfrak{p} - j\}$. From (Yan00), we have that R is a Stanley-Reisner ring implies that $\operatorname{Ext}^{n-i}_S(R, S)$ is a square-free module. Thus, $\operatorname{Ext}^{n-i}_S(R, S)$ is uniquely determined by its primes generated by variables. Support of $\operatorname{Ext}_S^{n-i}(R,S)$ determines if R satisfies (S_ℓ^j). Therefore, we only need consider primes generated by variables when showing R satisfies (S_ℓ^j). To examine $R_\mathfrak{p}$ we consider the simplicial complex $\operatorname{lk}_\Delta F$, where F is generated by the variables which are not generators of R (see Remark 4.1.4).

We then have $\tilde{H}_i(\operatorname{lk}_{\operatorname{lk} F}(F');k) = \tilde{H}_i(\operatorname{lk}_\Delta(F \cup F');k)$ which is 0 when $|F \cup F'| \leq d-i-j-2$ and when $i \leq \ell - 2$. Thus, we have $\tilde{H}_i(\operatorname{lk}_{\operatorname{lk} F}(F');k) = 0$ when $|F| + |F'| \leq d-i-j-2$ and $i \leq \ell - 2$, and thus $\tilde{H}_i(\operatorname{lk}_{\operatorname{lk} F}(F');k) = 0$ when $|F'| \leq d - |F| - i - j - 2$. We have $d - |F| \geq d_\mathfrak{p}$. Thus, $\tilde{H}_i(\operatorname{lk}_{\operatorname{lk} F}(F');k) = 0$ when $|F'| \leq d_\mathfrak{p} - i - j - 2$ and $i \leq \ell - 2$. Thus, by the above argument, we have $\operatorname{depth} R_\mathfrak{p} \geq \min\{\ell, d_\mathfrak{p} - j\}$. Thus, we have R satisfies (S_ℓ^j). $\square$

Corollary 4.6.5. *Let k be a field and Δ be a pure simplicial complex. Then the Stanley-Reisner ring of Δ, R, satisfies (S_2^j) over k if and only if for every face $F \in \Delta$ with $\dim(\operatorname{lk}_\Delta(F)) \geq 1+j$, $\operatorname{lk}_\Delta(F)$ is connected. Note (S_2^j) is independent of the base field.*

Proof. Suppose Δ satisfies (S_2^j). Then for any prime ideal $\mathfrak{p}$ with $\dim R_\mathfrak{p} \geq 2+j$ we have $\operatorname{depth} R_\mathfrak{p} \geq \min\{2, \dim R_\mathfrak{p} - j\} = 2$. Now we take any face F of our complex such that $\dim \operatorname{lk}_\Delta F \geq 1+j$. The Stanley-Reisner ring of $\operatorname{lk}_\Delta F$ is $R_{\tilde{\mathfrak{p}}}$ where $\tilde{\mathfrak{p}}$ is the prime ideal generated by the variables not contained in F. Thus, $\dim R_{\tilde{\mathfrak{p}}} \geq 2+j$. Thus, $H_m^1(R_{\tilde{\mathfrak{p}}}) = 0$. Thus, by Hochster's formula, we have $\tilde{H}_{1-|F'|-1}(\operatorname{lk}_{\operatorname{lk} F} F') = 0$ for all F'. In particular, $\tilde{H}_0(\operatorname{lk}_{\operatorname{lk} F} \emptyset) = 0$, which implies $\operatorname{lk}_{\operatorname{lk} F} \emptyset = \operatorname{lk}_\Delta F$ is connected.

Now suppose $\dim \operatorname{lk} F \geq 1+j$ implies $\operatorname{lk} F$ is connected. Thus, we have $\tilde{H}_0(\operatorname{lk}_{\operatorname{lk} F} \emptyset) = 0$. The link of F has dimension at least 1. Thus, if we take a face of $\operatorname{lk} F$ F' such that $|F'| = 1$, then $\operatorname{lk}_{\operatorname{lk} F} F'$ is not empty. Thus, $\tilde{H}_{-1}(\operatorname{lk}_{\operatorname{lk} F} F') = 0$ for all $|F'| = 1$. Thus, we get that $H_\mathfrak{m}^1(R_\mathfrak{p}) = 0$ where $\mathfrak{p}$ is the prime generated by the variables contained in F. Thus, we have $\operatorname{depth} R_\mathfrak{p} \geq 2$. This argument holds for all primes generated by variables such that $\dim R_\mathfrak{p} \geq 2+j$. If we have a prime $\mathfrak{p}$ such that $\dim R_\mathfrak{p} < 2+j$, we still have $\operatorname{depth} R_\mathfrak{p} \geq 1$, since $R_\mathfrak{p}$ is a Stanley-Reisner ring. Therefore, R satisfies (S_2^j). $\square$

4.7 Monomial Ideals

In this section, we consider monomial ideals that are not necessarily square-free. For convenience, we will establish a few conventions. When we consider the depth of a Stanley-Reisner ring, we shall be considering its depth with respect to the unique maximal homogeneous ideal. When we speak of the localized Stanley-Reisner ring of a link, we shall mean the Stanley-Reisner ring of the link localized at its unique maximal homogeneous ideal. Throughout this section, let k be a field.

One of the most powerful techniques for working with such ideals is polarization. We reproduce the definition of (MT09).

Definition 4.7.1. Let I be a monomial ideal in $k[x_1, ..., x_n]$. Let the minimal generators of I be $u_1, ..., u_m$ where $u_i = \prod_{j=1}^{n} x_j^{a_{ij}}$. For $1 \leq j \leq n$, let $a_j = \max\{a_{ij}\}$, and let $N = \max\{a_j\}$. Then let

$$T = k[x_{1,1}x_{1,2}, ..., x_{1,N}, x_{2,1}, x_{2,2}, ..., x_{2,N}, ..., x_{n,1}, x_{n,2}, ..., x_{n,N}].$$

Definition 4.7.2. The polarization of a monomial $u = x_1^{a_1}x_2^{a_2}...x_n^{a_n}$ is

$$\mathrm{pol}(u) = \prod_{1 \leq k \leq n, a_k \neq 0} (x_{k,1}x_{k,2}...x_{k,a_k})$$

Let J be the square-free monomial ideal of T generated by $\{\mathrm{pol}(u_i)\}$. We call J the *polarization* of I.

In (HTT05), Herzog, Takayama, and Terai proved that for a monomial ideal I, S/I being Cohen-Macaulay implies $S/\sqrt{I}$ is Cohen-Macaulay. In (PSFTY14), Pournaki et al. generalize this result to say S/I satisfying (S_ℓ) implies $S/\sqrt{I}$ satisfies (S_ℓ). We generalize this result further to hold for (S_ℓ^j).

Lemma 4.7.3. *Localization preserves* (S_ℓ^j).

Proof. Let R satisfy (S_ℓ^j). Let S be a multiplicatively closed set in R. Let $\mathfrak{q}$ be a prime ideal of R_S. The ideal $\mathfrak{q}$ is the extension of a prime ideal $\mathfrak{p}$ in R and so $(R_S)_\mathfrak{q} \cong R_\mathfrak{p}$. Since R satisfies (S_ℓ^j),

$$\operatorname{depth}(R_S)_\mathfrak{q} = \operatorname{depth} R_\mathfrak{p} \geq \min\{\ell, \dim R_\mathfrak{p} - j\} = \min\{\ell, \dim(R_S)_\mathfrak{q} - j\}.$$

Therefore, R_S satisfies (S_ℓ^j). $\square$

In (MT09), Murai and Terai prove that polarization preserves (S_ℓ). We will generalize their argument in the following lemmas.

Lemma 4.7.4. *If* $\operatorname{depth} R \geq \dim R - j$ *then* $\operatorname{depth} R_\mathfrak{p} \geq \dim R_\mathfrak{p} - j$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$ *where* $\mathfrak{p}$ *is generated by variables.*

Proof. The ring $R_\mathfrak{p}$ where $\mathfrak{p}$ is a prime generated by variables is the 0 ring, a field, or the localized Stanley-Reisner ring of the link of F, which is generated by the variables not in the generating set of $\mathfrak{p}$. From (MT09), we have $\operatorname{depth}(k[\operatorname{lk}_\Delta(v)]) \geq \operatorname{depth}(k[\Delta]) - 1$. Therefore, $\operatorname{depth} R_\mathfrak{p} \geq \operatorname{depth} R - |F| \geq \dim R - j - |F| \geq \dim R_\mathfrak{p} - j$. $\square$

Lemma 4.7.5. *Polarization preserves (S_ℓ^j).*

Proof. We mimic the proof of (MT09).

Let us consider an (S_ℓ^j) ring $k[x_1, ..., x_n]/I$. Let $I^{pol} \subseteq T$ be the polarization of I. Let Δ be the complex whose Stanley-Reisner ring is T/I^{pol}. Let F be an arbitrary face of Δ. We will be finished when we prove $\operatorname{depth}(k[\operatorname{lk}_\Delta(F)]) \geq \min\{r, \dim \operatorname{lk}_\Delta(F) + 1 - j\}$.

Let us write $V_k = \{x_{k,1}, x_{k,2}, ..., x_{k,N}\}$. We may write $F = F_1 \cup F_2 \cup \cdots \cup F_s \cup V_{s+1} \cup \cdots \cup V_n$, where F_k is a proper subset of V_k for $1 \leq k \leq s$.

Set $S' = k[x_1, ..., x_s]$, $I' = (I : x_{s+1}^N, ..., x_n^N) \cap S'$. Let $\mathfrak{p} = \langle x_1, ..., x_s \rangle$. Then:

$$\operatorname{depth}(S'/I') = \operatorname{depth}(S/I)_{\mathfrak{p}} \geq \min\{r, \dim(S/I)_{\mathfrak{p}} - j\} = \min\{r, \dim(S'/I') - j\}$$

Let $B = k[\cup_{k=1}^{s} V_k]$. Let $J \subseteq B$ be the monomial ideal generated by the polarization of the minimal generators of I'. Let Γ be the set of all square-free monomials in B which are not in J. We note Γ is a simplicial complex if we identify square-free monomials with sets of variables. Since J is a square-free monomial ideal, J is generated by the monomials that generate I_Γ and the variables of B which are not in Γ. Therefore, we have $\operatorname{depth}(k[\Gamma]) = \operatorname{depth}(B/J)$. Because J is the polarization of I' we get (from (BH98)):

$$\operatorname{depth}(k[\Gamma]) = \operatorname{depth}(B/J) = \operatorname{depth}(S'/I') + s(N-1).$$

By construction of Δ and Γ, we get $\operatorname{lk}_\Delta(V_{s+1} \cup ... \cup V_n) = \Gamma$. Thus, $\operatorname{lk}_\Delta(F) = \operatorname{lk}_\Gamma(F_1 \cup .... \cup F_s)$.

Thus, it will be sufficient to show

$$\operatorname{depth}(k[\operatorname{lk}_\Gamma(F_1 \cup \cdots \cup F_s)]) \geq \min\{r, \dim \operatorname{lk}_\Gamma(F_1 \cup \cdots \cup F_s) + 1 - j\}.$$

Suppose $\operatorname{depth}(S'/I') \geq \dim(S'/I') - j$.

From the process of polarization we can see that

$$\dim(B/J) = \dim(S'/I') + s(N-1).$$

Thus, we have $\operatorname{depth} B/J \geq \dim B/J - j$. By lemma 7.4, we have

$$\operatorname{depth}(k[\operatorname{lk}_\Gamma(F_1 \cup \cdots \cup F_s)]) \geq \dim \operatorname{lk}_\Gamma(F_1 \cup \cdots \cup F_s) + 1 - j.$$

Now suppose $\operatorname{depth}(S'/I') \geq \ell$. From (MT09), we have

$$\operatorname{depth}(k[\operatorname{lk}_\Gamma(F_1 \cup \cdots \cup F_s)]) \geq \ell.$$

$\square$

Theorem 4.7.6. *Let $A = k[x_1, \cdots, x_n]$ be the polynomial ring in n variables over a field k. Let I be a monomial ideal of A. Suppose that A/I satisfies (S_ℓ^j). Then $A/\sqrt{I}$ satisfies (S_ℓ^j).*

Proof. Using the lemmas and theorem in this section, we follow the argument of Theorem 8.3 of Pournaki et al (PSFTY14).

Let T/J be the polarization of A/I, where $T = k[x_1, ..., x_n, Y]$ is the new polynomial ring over k with Y as the set of new variables. By Theorem 4.7.5, T/J satisfies (S_ℓ^j). Let W be the multiplicatively closed subset $k[Y]\backslash 0$ in T and consider $F = k(Y)$. Then the localization of T/J at W is isomorphic to:

$$F[x_1, ..., x_n]/\sqrt{I} \cong (k[x_1, ..., x_n]/\sqrt{I}) \otimes_k F.$$

Since localization preserves generalized Serre's condition, $(k[x_1, ..., x_n]/\sqrt{I}) \otimes_k F$ satisfies (S_ℓ^j). Now to finish the proof, we will mirror the argument found in Section 2.1 of (BH98). We wish to show $(k[x_1, ..., x_n]/\sqrt{I}) \otimes_k F$ satisfying (S_ℓ^j) implies $(k[x_1, ..., x_n]/\sqrt{I})$ satisfies (S_ℓ^j). Let us consider $\mathfrak{p} \in \operatorname{Spec}(k[x_1, ..., x_n]/\sqrt{I})$. Since $(k[x_1, ..., x_n]/\sqrt{I}) \otimes_k F$ is faithfully flat, there exists a $\mathfrak{q} \in \operatorname{Spec}(k[x_1, ..., x_n]/\sqrt{I}) \otimes_k F$ such that $\mathfrak{p} = (k[x_1, ..., x_n]/\sqrt{I}) \cap \mathfrak{q}$ and $\mathfrak{q}$ is minimal over $\mathfrak{p}((k[x_1, ..., x_n]/\sqrt{I}) \otimes_k F)$. Therefore, $\operatorname{ht} \mathfrak{p} = \operatorname{ht} \mathfrak{q}$.

We will take $M = R = (k[x_1, ..., x_n]/\sqrt{I})_\mathfrak{p}$, $N = S = ((k[x_1, ..., x_n]/\sqrt{I}) \otimes_k F)_\mathfrak{q}$. Since $(k[x_1, ..., x_n]/\sqrt{I}) \otimes_k F$ is flat over $(k[x_1, ..., x_n]/\sqrt{I})$, N is flat over R. Thus, applying Theorem 1.2.16 and A.11 from (BH98) we get:

$$\operatorname{depth}_S M \otimes_R N = \operatorname{depth}_R M + \operatorname{depth}_S N/mN$$

$$\dim_S M \otimes_R N = \dim_R M + \dim_S N/mN$$

The fiber of the extension $R \to S$ is a localization of $k(\mathfrak{p}) \otimes F$. Thus, the fiber is Cohen-Macaulay by Proposition 2.11 of (BH98). The fiber of that map is also $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q} = N_\mathfrak{q}/\mathfrak{p}N_\mathfrak{q}$, because $\mathfrak{p}$ is the maximal ideal of $R_\mathfrak{p}$. Thus, $N_\mathfrak{q}/\mathfrak{p}N_\mathfrak{q}$ is Cohen-Macaulay. Thus, subtracting the above equations gives:

$$\dim M \otimes_R N - \operatorname{depth}_S M \otimes_R N = \dim M - \operatorname{depth}_R M.$$

Since $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{q}$, we get $\dim R_\mathfrak{p} = \dim S_\mathfrak{q}$. Thus, $\operatorname{depth} R_\mathfrak{p} = \operatorname{depth} S_\mathfrak{q} \geq \min\{r, \dim S_\mathfrak{q} - j\} = \min\{r, \dim R_\mathfrak{p} - j\}$.

□

4.8 Skeletons

In this section, we examine the i-skeletons of simplicial complexes. We begin with the definition.

Definition 4.8.1. The simplicial complex $\Delta^{(i)} := \{F \in \Delta | \dim F \leq i\}$ is the *i-skeleton* of Δ.

These i-skeletons are subcomplexes that retain many important properties of the original complex. In particular, these complexes are a powerful tool for understanding depth. A complex whose Stanley-Reisner ring has depth b has a Cohen-Macaulay $b-1$-skeleton.

Proposition 4.8.2 ((HTYZN11, Proposition 2.3))**.** *Let k be a field and Δ a simplicial complex. If Δ satisfies (S_ℓ) over k, then $\Delta^{(i)}$ also satisfies (S_ℓ) over k for $(2 \leq \ell \leq i+1)$.*

Theorem 4.8.3. *Let k be a field and Δ a simplicial complex with Stanley-Reisner ring R. If Δ is pure and R satisfies (S_ℓ^j) over k, then $\Delta^{(i)}$ is also pure and its Stanley-Reisner ring satisfies $(S_\ell^{\max\{0,j+i+1-d\}})$ over k for $(2 \leq \ell \leq i+1)$.*

Proof. Let us consider R that satisfies (S_ℓ^j). Then $\operatorname{depth} R_\mathfrak{p} \geq \min\{\ell, \dim R_\mathfrak{p} - j\}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. As noted earlier, we may restrict our consideration to primes generated by variables. Let $\Delta_\mathfrak{p}$ be the simplicial complex whose localized Stanley-Reisner ring is $R_\mathfrak{p}$. We note $(\Delta_\mathfrak{p})^{(i-d+\dim R_\mathfrak{p})} \cong (\Delta^{(i)})_\mathfrak{p}$ when $d - \dim R_\mathfrak{p} \leq i$. We also note that if $d - \dim R_\mathfrak{p} > i$ then $(\Delta^{(i)})_\mathfrak{p}$ has Stanley-Reisner ring k or 0, and thus is Cohen-Macaulay. It is known that the b-skeleton of a simplicial complex is Cohen-Macaulay if and only if the depth of that complex is greater than or equal to $b+1$.

We consider two cases. First, $i+j+1 \leq d$. This is equivalent to $|(\Delta_\mathfrak{p})^{(i-d+\dim R_\mathfrak{p})}| \leq \dim R_\mathfrak{p} - j$. If $\operatorname{depth} R_\mathfrak{p} \geq i - d + \dim R_\mathfrak{p} + 1$ then $(\Delta_\mathfrak{p})^{(i-d+\dim R_\mathfrak{p})}$ is Cohen-Macaulay. Otherwise, $\operatorname{depth} R_\mathfrak{p} < |(\Delta_\mathfrak{p})^{(i-d+\dim R_\mathfrak{p})}| \leq \dim R_\mathfrak{p} - j$. Thus,
$\operatorname{depth} R_\mathfrak{p} \geq \ell$. Thus, $\operatorname{depth}(\Delta_\mathfrak{p})^{(i-d+\dim R_\mathfrak{p})} \geq \ell$. Combining these, we get:

$\operatorname{depth}(\Delta_\mathfrak{p})^{(i-d+\dim R_\mathfrak{p})} \geq \min\{\ell, |(\Delta_\mathfrak{p})^{(i-d+\dim R_\mathfrak{p})}|\}$.

Therefore, the Stanley-Reisner ring of $(\Delta)^i$ satisfies (S_ℓ).

Now consider the case $i+j+1 > d$. Again if $\operatorname{depth} R_\mathfrak{p} \geq i - d + \dim R_\mathfrak{p} + 1$ then $(\Delta_\mathfrak{p})^{(i-d+dimR_\mathfrak{p})}$ is Cohen-Macaulay. Otherwise, $\operatorname{depth}(\Delta_\mathfrak{p})^{(i-d+\dim R_\mathfrak{p})} = \operatorname{depth} R_\mathfrak{p}$
$\geq \min\{\ell, \dim R_\mathfrak{p} - j\} = \min\{\ell, i+1-d+\dim R_\mathfrak{p} - (i+j+1-d)\}$. Thus, we have the Stanley-Reisner ring of $(\Delta^{(i)})_\mathfrak{p}$ satisfies $(S_\ell^{i+j+1-d})$. Combining these two cases gives the proof of the theorem.

□

Theorem 4.8.4. *Let k be a field and Δ a simplicial complex with Stanley-Reisner ring R.*

If R satisfies (S_ℓ^j) over k, then the Stanley-Reisner ring of $\Delta^{(i)}$ satisfies (S_ℓ^j) over k for $(2 \leq \ell \leq i+1)$.

Proof. This proof will be similar to the previous proof.

Let us consider R that satisfies (S_ℓ^j).

We consider two cases. First, $i+1-n+\dim S_\mathfrak{p} \leq \dim R_\mathfrak{p} - j$. This is equivalent to

$|(\Delta_\mathfrak{p})^{(i-n+\dim S_\mathfrak{p})}| \leq \dim R_\mathfrak{p} - j$. If $\operatorname{depth} R_\mathfrak{p} \geq i-n+\dim S_\mathfrak{p}+1$ then $(\Delta_\mathfrak{p})^{(i-n+\dim S_\mathfrak{p})}$ is Cohen-Macaulay. Otherwise, $\operatorname{depth} R_\mathfrak{p} < |(\Delta_\mathfrak{p})^{(i-n+\dim S_\mathfrak{p})}| \leq \dim R_\mathfrak{p} - j$. Thus, $\operatorname{depth} R_\mathfrak{p} \geq \ell$. Thus, $\operatorname{depth}(\Delta_\mathfrak{p})^{(i-n+\dim S_\mathfrak{p})} \geq \ell$. Combining these, we get

$\operatorname{depth}(\Delta_\mathfrak{p})^{(i-n+\dim S_\mathfrak{p})} \geq \min\{\ell, |(\Delta_\mathfrak{p})^{(i-n+\dim S_\mathfrak{p})}|\}$.

Therefore, the Stanley-Reisner ring of $(\Delta)^i$ satisfies (S_ℓ).

Now consider the case $i+1-n+\dim S_\mathfrak{p} > \dim R_\mathfrak{p} - j$. Again if $\operatorname{depth} R_\mathfrak{p} \geq i-n+\dim S_\mathfrak{p}+1$ then $(\Delta_\mathfrak{p})^{(i-n+dimS_\mathfrak{p})}$ is Cohen-Macaulay. Otherwise, $\operatorname{depth}(\Delta_\mathfrak{p})^{(i-n+\dim S_\mathfrak{p})} = \operatorname{depth} R_\mathfrak{p} \geq \min\{\ell, \dim R_\mathfrak{p} - j\} = \min\{\ell, i+1-n+\dim S_\mathfrak{p} - (i+j+1-n-\dim R_\mathfrak{p} + \dim S_\mathfrak{p})\}$. We note if $i+j+1-n-\dim R_\mathfrak{p}+\dim S_\mathfrak{p} \geq j$ then $i+1-\dim R_\mathfrak{p} \geq (n-\dim S_\mathfrak{p})$. We note $n-\dim S_\mathfrak{p}$ is positive and thus we get $i+1 \geq \dim R_\mathfrak{p}$. In this case, we note we would merely be examining $\Delta_\mathfrak{p}$. Thus, we have the Stanley-Reisner ring of $(\Delta^{(i)})_\mathfrak{p}$ satisfies (S_ℓ^j). Combining these two cases gives the proof of the theorem.

□

Chapter 5

Higher Nerves of Simplicial Complexes

5.1 Introduction

The work in this chapter is joint work. Sections 1 through 7 are based on joint work with Hailong Dao, Joseph Doolittle, Ken Duna, Bennet Goeckner, and Justin Lyle. Section 8 is based on joint work with Bennet Goeckner.

The nerve complex has been an important object of study in algebraic combinatorics (Bjö03; Grü70; Bor48; Bas03; KM05; LSVJ11; CJS15; PUV16). We remind the reader of its definition:

Let $A = \{A_1, A_2, \ldots, A_r\}$ be a family of sets.

Definition 5.1.1. Consider

$$\mathrm{N}(A) := \{F \subseteq [r] \colon \cap_{i \in F} A_i \neq \emptyset\}.$$

This simplicial complex is the *Nerve Complex* of A.

Of special interest is the case where A is the set of facets of a simplicial complex Δ; in this case, one sets $N(\Delta) := N(A)$. We propose a natural extension of this notion.

Definition 5.1.2. Let $A = \{A_1, A_2, \ldots, A_r\}$ be the set of facets of a simplicial complex Δ. Define

$$\mathrm{N}_i(\Delta) := \{F \subseteq [r] \colon |\cap_{j \in F} A_j| \geq i\}.$$

We call this simplicial complex the i^{th} *nerve Complex* of Δ and we refer to the $N_i(\Delta)$ as the *higher nerve complexes* of Δ.

When $i = 1$, this definition recovers $N(\Delta)$.

The Nerve Theorem of Borsuk (Bor48) gives that $\mathrm{N}(\Delta)$ and Δ have the same homologies. We now explain how the higher nerves relate to the original complex in a more subtle manner. Namely, their homologies determine important algebraic and combinatorial properties of Δ. We summarize our main quantitative results below.

Theorem 5.1.3 (Main Theorem). *Let k be a field, let Δ be a simplicial complex, and let $k[\Delta]$ be the associated Stanley-Reisner ring. Let $\tilde{H}_i$ denote ith reduced simplicial homology with coefficients in k, and let χ denote Euler characteristic. Then:*

1. $\tilde{H}_i(\mathrm{N}_j(\Delta)) = 0$ *for* $i + j > d$ *and* $1 \leq j \leq d$ *(see Corollary 5.3.8).*

2. $\operatorname{depth}(k[\Delta]) = \inf\{i + j \colon \tilde{H}_i(\mathrm{N}_j(\Delta)) \neq 0\}$ *(see Theorem 5.5.2).*

3. *For* $i \geq 0$, $f_i(\Delta) = \displaystyle\sum_{j=i+1}^{d} \binom{j-1}{i} \chi(\mathrm{N}_j(\Delta))$ *(see Theorem 5.6.1).*

In short, the numbers $b_{ij} = \dim \tilde{H}_i(\mathrm{N}_j(\Delta))$ for $0 \leq i \leq d - j$ and $1 \leq j \leq d$ can be presented in a table which determine both the depth and the f-vector (and thus also the h-vector) of Δ. We provide an explicit example below.

Example 5.1.4. Consider the simplicial complex Δ with facets

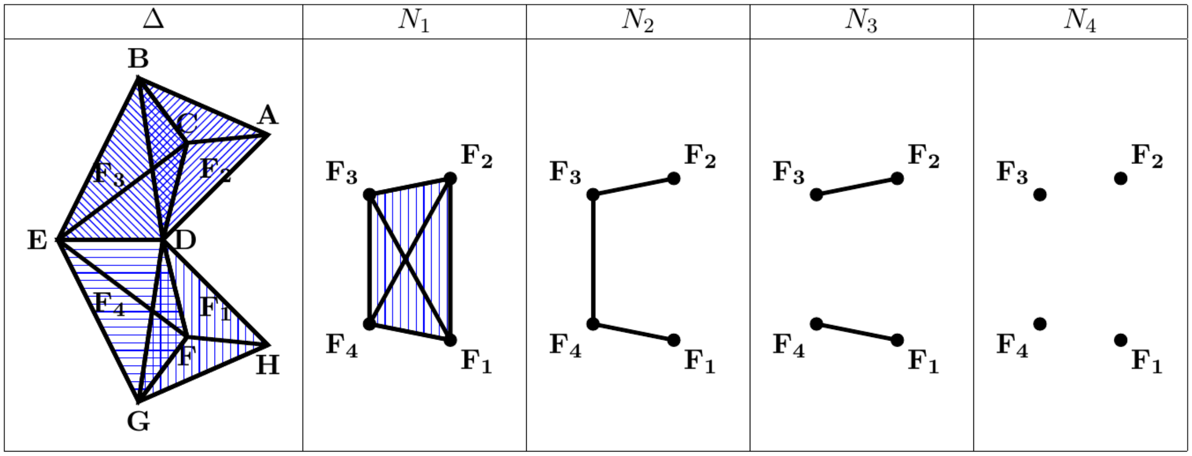

Table 5.1: Nerves of Δ

$\{ABCD, BCDE, DEFG, DFGH\}$.

The following are geometric realizations of the complex and its higher nerves:

	$\tilde{H}_0$	$\tilde{H}_1$	$\tilde{H}_2$	χ
N_1	0	0	0	1
N_2	0	0	0	1
N_3	1	0	0	2
N_4	3	0	0	4

Table 5.2: Nerve Homologies

Using our main theorem and the Table 2, depth $k[\Delta] = 3$ and $f(\Delta) = (1, 8, 17, 14, 4)$.

Using our main result, we provide a formula to compute the regularity of any monomial ideal, not necessarily square-free, in Theorem 5.7.2. Other algebraic properties such as Serre's condition (S_r) can also be detected from the nerve table: see Chapter 6.

One way to gather the data stored by the higher nerves is to consider the nerve complex of a simplicial complex Δ along with an integer labeling of the faces of the nerve complex. For the labeling, we assign each face with the highest nerve in which it appears. We designate this construction the *nervous system* of Δ. This object retains enough information about the original simplicial complex to determine it completely up to isomorphism. We also define the k-*nervous system* to be the nerve complex along with an integer labeling of each face, such that each face is labeled with the smaller of k and the highest nerve in which the face appears. We note the k-nervous system is the nervous system when k is at least d.

Remark 5.1.5. Though we will not consider it in this thesis, one can also define higher nerves in a more general setting. Let A be a collection of subsets of a topological space X. Define $N_i(A) := \{F \subseteq [r] \colon \dim \cap_{j \in F} A_j \geq i\}$, where dim represents Krull dimension. In this setting, special interest is given to the case where X is a Noetherian algebraic scheme; in this case, one sets $N_i(X) := N_i(A)$, where A is the collection of irreducible components of X. In particular, if $X = \operatorname{Spec} R$ for a local ring R, then the $N_i(X)$ provide a natural generalization of the Lyubeznik complex of R (see (Lyu07, Theorem 1.1) for the definition). If, instead, $X = \operatorname{Spec} R$ for R a Stanley-Reisner ring of a simplicial complex Δ, then the complex defined in this remark coincides with that of Definition 5.1.2, via the Stanley-Reisner correspondence.

We now briefly describe the structure of this chapter. In Section 5.2, we cover combinatorial background and fix the notation we will use throughout the chapter. In Section 5.3, we recall and prove certain basic facts about depth and connectivity of a complex, which motivate our results and will be used in our proofs. We provide a strengthened version of the classical Nerve Theorem that suits our purpose in Proposition 5.3.7. This proposition is a critical component of parts (1) and (2) of our main theorem. We conclude this section by proving part (1) of our main theorem. In Section 5.4, we provide several lemmata, the main technical tools of most of our proofs. Section 5.5 is devoted to the proof of the second part of our main theorem. Section 5.6 gives the proof of the third part of our main theorem and provides a formula for the h-vector in terms of homologies of higher nerves in Corollary 5.6.2. Section 5.7 applies our main theorem to give a formula for computing the Castelnuovo-Mumford regularity of any monomial ideal. In Section 5.8, we explore unpublished SAGE code created to facilitate use of the nerve complex. We also introduce the nervous system and k-nervous system along with code for implementation of these objects.

5.2 Notation and Definitions

In this section we introduce the notation we will use throughout this chapter. Unless otherwise stated, we fix the field k and let $\tilde{H}_i$ denote i^{th} reduced simplicial or singular homology, whichever is appropriate, always with coefficients in k.

We will use $V(\Delta)$ to represent the vertex set of a simplicial complex Δ; we will use V instead of $V(\Delta)$ when the choice of Δ is clear; we also set $n := |V(\Delta)|$ and $S := k[x_1, \dots, x_n]$. We denote the subcomplex of Δ induced by the vertex set W by $\Delta|_W := \{F \in \Delta \colon F \subseteq W\}$.

Given a subset $T \subseteq V(\Delta)$, we may define the star, the anti-star, and the link of T, denoted $\mathrm{st}_\Delta(T)$, $\mathrm{astar}_\Delta(T)$, and $\mathrm{lk}_\Delta(T)$, respectively, as follows:

$$\begin{aligned} \mathrm{st}_\Delta T &:= \{G \in \Delta \colon T \cup G \in \Delta\} \\ \mathrm{astar}_\Delta T &:= \{G \in \Delta \colon T \cap G = \varnothing\} = \Delta|_{V \setminus T} \\ \mathrm{lk}_\Delta T &:= \{G \in \Delta \colon T \cup G \in \Delta \text{ and } T \cap G = \varnothing\} = \mathrm{st}_\Delta T \cap \mathrm{astar}_\Delta T \end{aligned}$$

The star and link of T are the void complex exactly when $T \notin \Delta$, and the link of T is the irrelevant complex $\{\varnothing\}$ exactly when T is a facet. On the other hand, the anti-star of any $T \subsetneq V(\Delta)$ is nonempty.

We call $\Delta^{(k)} := \{\sigma \in \Delta : |\sigma| \leq k+1\}$ the *k-skeleton of* Δ.

Definition 5.2.1. Let $\mathcal{F}_{>k}(\Delta)$ denote the face poset of Δ restricted to faces of Δ with cardinality strictly greater than k.

We note the face poset of Δ is $\mathcal{F}_{>-1}(\Delta)$. Furthermore $\mathcal{F}_{>d}(\Delta)$ is the empty poset.

Definition 5.2.2. The *order complex* of a poset P, denoted $\mathcal{O}(P)$, is the simplicial complex whose faces are all the chains in P.

We will denote the geometric realization of Δ as $||\Delta||$.

Given a complex Δ, its *barycentric subdivision* may be defined as $\operatorname{sd}\Delta := \mathcal{O}(\mathcal{F}_{>0}(\Delta))$. The following is well-known (see Corollary 5.7 of (Gib77) for example).

Lemma 5.2.3. *The realization* $||\Delta||$ *is homeomorphic to* $||\operatorname{sd}\Delta||$. *In particular,* $\tilde{H}_i(\Delta) = \tilde{H}_i(\operatorname{sd}\Delta)$ *for all* i.

We let $\rho : \mathcal{F}_{>0}(\Delta) \to V(\operatorname{sd}\Delta)$ be the map which sends an element of $\mathcal{F}_{>0}(\Delta)$ to itself viewed as a vertex of $\operatorname{sd}\Delta$.

We will often use the following shorthand:

$$\begin{aligned}[]
[\Delta]_{>k} &= \mathcal{O}(\mathcal{F}_{>k}(\Delta)) \\
&= \operatorname{sd}\Delta\big|_{V(\operatorname{sd}\Delta)\setminus V(\operatorname{sd}(\Delta^{(k-1)}))}
\end{aligned}$$

Notice that the image of ρ may be restricted to $V([\Delta]_{>k})$ by restricting its domain to $\mathcal{F}_{>k}(\Delta)$. A *simplicial map* $f : \Delta_1 \to \Delta_2$ is a function $f : V(\Delta_1) \to V(\Delta_2)$ so that for all $\sigma \in \Delta_1$, $f(\sigma) \in \Delta_2$. We say a simplicial map f is a *simplicial isomorphism* if f has an inverse that is a simplicial map. Note that if $f : Q \to P$ is an order-reversing or order-preserving poset map, then $f : \mathcal{O}(Q) \to \mathcal{O}(P)$ is a simplicial map.

Given a simplicial complex Δ, we also consider algebraic properties of its Stanley-Reisner ring $k[\Delta]$. Unless otherwise stated, we write d for $\dim k[\Delta]$, the Krull dimension of the ring $k[\Delta]$. We also use $s(\Delta)$ to mean the size of the smallest facet of Δ.

We further note that

$$\dim k[\Delta] = \max\{|F| \colon F \text{ is a facet of } \Delta\}$$
$$\operatorname{depth} k[\Delta] = \max\{i : \Delta^{(i-1)} \text{ is Cohen-Macaulay}\} \leq s(\Delta).$$

5.3 Preparatory Results

In this section, we begin by exploring what is known in the literature and use our construction to prove some immediate results. Many of these results follow as a consequence of our main theorem, but their immediacy shows that our construction is a natural one. We then prove a generalization of Borsuk's Nerve Theorem for simplicial complexes.

We now present Hochster's formula, which will be used throughout the chapter. It relates the i^{th} local cohomology module of $k[\Delta]$ supported on $\mathfrak{m}$, denoted $H^i_{\mathfrak{m}}(k[\Delta])$, to the reduced homology of links of certain faces of Δ. Here $\mathfrak{m}$ is the ideal of $k[\Delta]$ generated by the residue classes of all variables in S.

Theorem 5.3.1 (Hochster (BH98)). *Let Δ be a simplicial complex. Then the Hilbert series of the local cohomology modules of $k[\Delta]$ with respect to the fine grading is given by:*

$$\operatorname{Hilb}_{H^i_{\mathfrak{m}}(k[\Delta])}(t) = \sum_{T\in\Delta} \dim_k \tilde{H}_{i-|T|-1}(\operatorname{lk}_\Delta T) \prod_{v_j\in T} \frac{t_j^{-1}}{1-t_j^{-1}}.$$

One has $\operatorname{depth} k[\Delta] = \min\{i \colon H^i_{\mathfrak{m}}(k[\Delta]) \neq 0\}$ and $\dim k[\Delta] = \max\{i \colon H^i_{\mathfrak{m}}(k[\Delta]) \neq 0\}$, so Hochster's formula allows us to characterize depth and dimension of $k[\Delta]$ in terms of homologies of links of faces. The following is a generalization of Reisner's well known criterion for Cohen-Macaulayness.

Corollary 5.3.2. *Let Δ be a simplicial complex. Then* $\operatorname{depth} k[\Delta] \geq t$ *if and only if* $\tilde{H}_{i-1}(\operatorname{lk}_\Delta T) = 0$ *for all $T \in \Delta$ with $i + |T| < t$.*

The following theorem, known as the Borsuk Nerve Theorem, is one of the main tools for working with the classical nerve complex.

Theorem 5.3.3 ((Bor48, Section 9, Corollary 2))**.** *Δ and $\mathrm{N}_1(\Delta)$ have same homotopy type. In particular, $\tilde{H}_i(\Delta) \cong \tilde{H}_i(\mathrm{N}_1(\Delta))$ for all i.*

Note if $\operatorname{depth} k[\Delta] \geq t$, then $\tilde{H}_{i-1}(\Delta) = \tilde{H}_{i-1}(\mathrm{N}_1(\Delta)) = 0$ for $i < t$ by Corollary 5.3.2 and Corollary 5.3.3.

Following from the definition of higher nerves, we are able to quickly derive the following results. We will prove much stronger results in subsequent sections.

Lemma 5.3.4. *If $i \leq s(\Delta)$ and $\mathrm{N}_i(\Delta)$ is connected, then $\Delta^{(1)}$ is an i-connected graph.*

Proof. Since $\mathrm{N}_i(\Delta)$ is connected, there is a spanning tree of $\mathrm{N}_i(\Delta)^{(1)}$. Let S be a set of all vertices of Δ except for at most $i-1$ of them. We have that $\mathrm{N}_1(\Delta|_S)$ is connected, since the facets of $\Delta|_S$ are a subset of the facets of Δ, and the induced spanning tree is preserved. Since connectedness is equivalent to trivial 0^{th} reduced homology and $\mathrm{N}_1(-)$ preserves reduced homology, $\Delta|_S$ is connected. Therefore $\Delta^{(1)}$ is i-connected. $\square$

Corollary 5.3.5. *Let $t = \operatorname{depth} k[\Delta]$. Then $\Delta^{(1)}$ is a $(t-1)$-connected graph.*

Proof. Since $\Delta^{(t-1)}$ is Cohen-Macaulay, the facet-ridge graph of $\Delta^{(t-1)}$ is connected by (Har62); that is, between any pair of $(t-1)$-faces of Δ, there is a sequence of $(t-1)$-faces, so that each consecutive pair intersects in a $(t-2)$-face. Then for any pair of facets of Δ, by choosing a $(t-1)$-face for each, and finding such a sequence between them, we construct from this a sequence of facets so that each consecutive pair intersects in a $(t-2)$-face. Therefore $N_{t-1}(\Delta)$ is connected, and the result then follows from Lemma 5.3.4. $\square$

An easy proof of Borsuk's Nerve Theorem (Theorem 5.3.3) uses the following result.

Theorem 5.3.6 ((Qui78), Proposition 1.6)**.** *Let $f : \Delta \to \mathcal{O}(P)$ be a simplicial map. If for all $x \in P$ we have that $f^{-1}(P_{\geq x})$ is contractible, then f induces a homotopy equivalence between Δ and $\mathcal{O}(P)$.*

Using the above theorem, we are able to provide a generalization of the classical Nerve Theorem.

Proposition 5.3.7 (Generalized Nerve Theorem)**.** *$[\Delta]_{>j}$ is homotopy equivalent to $\mathrm{N}_{j+1}(\Delta)$.*

Proof. We use a similar approach as that of Theorem 10.6 in (Bjö95).

Let $P = \mathcal{F}_{>0}(\mathrm{N}_{j+1}(\Delta))$ and define $f : \mathcal{F}_{>j}(\Delta) \to P$ by

$$f(\sigma) = \{F_i : \sigma \subseteq F_i \text{ facet of } \Delta\}.$$

This map is order-reversing, and it is well-defined, since $|\sigma| \geq j + 1$. Therefore, $f : \mathcal{O}(\mathcal{F}_{>j}(\Delta)) \to \mathcal{O}(P)$ is a simplicial map. For any $\tau \in P$, we have that

$$f^{-1}(P_{\geq \tau}) = \bigcap_{F_i \in \tau} F_i,$$

which is a face of Δ and is thus contractible. Therefore, by Theorem 5.3.6, f induces a homotopy equivalence between $\mathcal{O}(\mathcal{F}_{>j}(\Delta))$ and $\mathcal{O}(P)$. Since $\mathcal{O}(P)$ is the barycentric subdivision of $\mathrm{N}_{j+1}(\Delta)$, Lemma 5.2.3 says that $||\mathcal{O}(P)|| \cong ||\,\mathrm{N}_{j+1}(\Delta)||$, and therefore, $\mathcal{O}(\mathcal{F}_{>j}(\Delta)) = [\Delta]_{>j}$ is homotopy equivalent to $\mathrm{N}_{j+1}(\Delta)$. □

Notice when $j = 0$, we recover the classical Nerve Theorem.

We may now prove part (1) of our main theorem as a corollary.

Corollary 5.3.8. *For a simplicial complex Δ, $\tilde{H}_i(N_j(\Delta)) = 0$ for $i + j > d$ and $1 \leq j \leq d$.*

Proof. By Proposition 5.3.7, we get

$$\tilde{H}_i(N_j(\Delta)) = \tilde{H}_i([\Delta]_{>j-1}).$$

But $[\Delta]_{>j-1}$ has dimension at most $d-j$ and the result follows. $\square$

5.4 Lemmata

In this section, we introduce several lemmata that will be integral to proving our main theorem.

Lemma 5.4.1. *Let T be a face of Δ and $|T| = k > 0$. Then, $\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T)) \cong [\mathrm{lk}_\Delta(T)]_{>0}$ as simplicial complexes. In particular, $\tilde{H}_i(\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T))) \cong \tilde{H}_i(\mathrm{lk}_\Delta(T))$ for every i.*

Proof. First note that if T is a facet then $\mathrm{lk}_\Delta(T) = \{\varnothing\} = [\mathrm{lk}_\Delta(T)]_{>0}$. But, since T is a facet, $\{\rho(T)\}$ must be a facet of $[\Delta]_{>k-1}$, since this is a chain of maximal length containing $\rho(T)$. Thus $\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T)) = \{\varnothing\} = [\mathrm{lk}_\Delta(T)]_{>0}$, and thus we have the result if T is a facet.

Now, suppose $T \in \Delta$ is not a facet and define $f : V([\mathrm{lk}(T)]_{>0}) \to V(\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T)))$ by $f(\rho(\tau)) = \rho(\tau \cup T)$. One can check that f is a simplicial isomorphism.

Then f induces a homeomorphism between the geometric realizations of $[\mathrm{lk}_\Delta(T)]_{>0}$ and $\mathrm{lk}_{[\Delta]_{k-1}}(\rho(T))$, and the result follows from Lemma 5.2.3.

$\square$

Lemma 5.4.2. *Let T be a non-trivial, non-facet face of Δ with $|T| = k$. Let i be such that $\tilde{H}_{i-1}(\Delta) = \tilde{H}_{i-1}([\Delta]_{>k-1}) = \tilde{H}_i(\Delta) = \tilde{H}_i([\Delta]_{>k-1}) = 0$. Then*

$$\tilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) \cong \tilde{H}_{i-1}(\mathrm{astar}_{[\Delta]_{>k-1}}(\rho(T))).$$

Proof. Notice that $\mathrm{st}(T) \cup \mathrm{astar}_\Delta(T) = \Delta$ and $\mathrm{st}(T) \cap \mathrm{astar}_\Delta(T) = \mathrm{lk}_\Delta(T) \neq \varnothing$, thus we have a Mayer-Vietoris exact sequence in reduced homology:

$$\cdots \to \tilde{H}_i(\Delta) \to \tilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) \to \tilde{H}_{i-1}(\mathrm{st}(T)) \oplus \tilde{H}_{i-1}(\mathrm{astar}_\Delta(T)) \to \tilde{H}_{i-1}(\Delta) \to \cdots$$

Since $\mathrm{st}(T)$ is a cone, it is acyclic. Since $\tilde{H}_{i-1}(\Delta) = \tilde{H}_i(\Delta) = 0$, this sequence gives that $\tilde{H}_{i-1}(\mathrm{astar}_\Delta(T)) \cong \tilde{H}_{i-1}(\mathrm{lk}_\Delta(T))$.

By the same reasoning, we have $\tilde{H}_{i-1}(\mathrm{astar}_{[\Delta]_{>k-1}}(\rho(T))) \cong \tilde{H}_{i-1}(\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T)))$.

By Lemma 5.4.1, we have $\tilde{H}_{i-1}(\mathrm{lk}_{[\Delta]_{>k-1}}(\rho(T))) \cong \tilde{H}_{i-1}([\mathrm{lk}_\Delta(T)]_{>0})$. □

Lemma 5.4.3. *Let Δ be a simplicial complex and $J \subsetneq V = V(\Delta)$ such that $\dim(\Delta|_J) = 0$. Assume that $\tilde{H}_{i-1}(\Delta) = \tilde{H}_i(\Delta) = 0$. Then*

$$\tilde{H}_{i-1}(\Delta|_{V \setminus J}) \cong \bigoplus_{x \in J} \tilde{H}_{i-1}(\Delta|_{V \setminus \{x\}}).$$

Proof. We will proceed by induction on $|J|$. When $|J| = 1$, the result is immediate. Suppose the result holds for any J of cardinality k for some $k \geq 1$, and suppose now that $|J| = k+1$. Let $x \in J$ and $J' = J \setminus \{x\}$. Suppose $\sigma \in \Delta$. If $x \in \sigma$, then $\sigma \in \Delta|_{V \setminus J'}$; otherwise if σ contained some $y \in J'$, then $\{x, y\} \in \Delta$, contradicting the fact that $\dim(\Delta|_J) = 0$. If $x \notin \sigma$, then $\sigma \in \Delta|_{V \setminus \{x\}}$. Therefore, $\Delta = \Delta|_{V \setminus J'} \cup \Delta|_{V \setminus \{x\}}$. Note that $\Delta|_{V \setminus J'} \cap \Delta|_{V \setminus \{x\}} = \Delta|_{V \setminus J} \neq \varnothing$.

We have the following Mayer-Vietoris sequence in reduced homology:

$$\cdots \to \tilde{H}_i(\Delta) \to \tilde{H}_{i-1}(\Delta|_{V \setminus J}) \to \tilde{H}_{i-1}(\Delta|_{V \setminus J'}) \oplus \tilde{H}_{i-1}(\Delta|_{V \setminus \{x\}}) \to \tilde{H}_{i-1}(\Delta) \to \cdots$$

Because $\tilde{H}_{i-1}(\Delta) = \tilde{H}_i(\Delta) = 0$, we have that

$$\tilde{H}_{i-1}(\Delta|_{V \setminus J}) \cong \tilde{H}_{i-1}(\Delta|_{V \setminus J'}) \oplus \tilde{H}_{i-1}(\Delta|_{V \setminus \{x\}}).$$

By induction, $\tilde{H}_{i-1}(\Delta|_{V\setminus J'}) \cong \bigoplus_{y\in J'} \tilde{H}_{i-1}(\Delta|_{V\setminus\{y\}})$. Therefore

$$\tilde{H}_{i-1}(\Delta|_{V\setminus J}) \cong \bigoplus_{x\in J} \tilde{H}_{i-1}(\Delta|_{V\setminus\{x\}}).$$

□

5.5 Depth and Higher Nerves

Theorem 5.5.1. *The following are equivalent:*

1. $\tilde{H}_{i-1}(\mathrm{N}_{j+1}(\Delta)) = 0$ *for all* $i, j \geq 0$ *such that* $i + j < m$.

2. $\tilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) = 0$ *for all* $i, j \geq 0$, $|T| = j$, *and* $i + j < m$.

Proof. We begin the proof by showing that each condition implies $m \leq s(\Delta)$ and thus we will never need to consider the case when T is a facet. Consider the first condition: if $m > s(\Delta)$, then we may take $j = s(\Delta) - 1, i = 1$. This nerve will have an isolated vertex corresponding to the facet of smallest size. The nerve will not be connected unless that facet is the only facet. However, if this facet is the only facet, then we contradict the first condition for $j = s(\Delta), i = 0$. Now consider the second condition: suppose $m > s(\Delta)$. Then take $j = s(\Delta)$, $i = 0$. Then we have a contradiction when T is a facet.

To prove equivalence, we will induct on j. Thus, let us begin by considering the case $j = 0$. The first set of equations is then $\tilde{H}_{i-1}(\mathrm{N}_1(\Delta)) = 0$ for all $i < m$. Using Theorem 5.3.3 (1), we get that this statement is equivalent to $\tilde{H}_{i-1}(\Delta) = 0$ for all $i < m$. When $j = 0$, the second set of equations is in fact $\tilde{H}_{i-1}(\Delta) = 0$ for all $i < m$, since $|T| = 0$ implies T is the empty set. Thus we have equivalence when $j = 0$.

Now, let us take as our induction hypothesis that our theorem holds for $j = k - 1$. Consider $j = k < m$. Assuming either set of equations holds, the $j = 0$ case again says that $\tilde{H}_{i-1}(\Delta) = 0$ for all $i < m$. By Proposition 5.3.7 and the $j = k-1$ case, either set of equations yields $\tilde{H}_{i-1}([\Delta]_{>k}) = 0$ for all $i < m-(k-1)$. Therefore, we may apply Lemma 5.4.2 for all $i < m-(k-1)-1 = m-k$. Thus, we have

$$\bigoplus_{\substack{T\in\Delta \\ |T|=k}} \tilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) \cong \bigoplus_{\substack{T\in\Delta \\ |T|=k}} \tilde{H}_{i-1}(\mathrm{astar}_{[\Delta]_{>k-1}}(\rho(T))).$$

Applying Lemma 5.4.3, we get:

$$\tilde{H}_{i-1}([\Delta]_{>k}) \cong \bigoplus_{\substack{T\in\Delta \\ |T|=k}} \tilde{H}_{i-1}(\mathrm{astar}_{[\Delta]_{>k-1}}(\rho(T))).$$

And by Proposition 5.3.7:

$$\tilde{H}_{i-1}([\Delta]_{>k}) \cong \tilde{H}_{i-1}(\mathrm{N}_{k+1}(\Delta)).$$

Thus, we have completed the proof by induction. $\square$

Combining Corollary 5.3.2 and Theorem 5.5.1, we obtain the second part of our main theorem, Theorem 5.1.3, restated here:

Theorem 5.5.2. *For a simplicial complex Δ,* $\mathrm{depth}(k[\Delta]) = \inf\{i+j \mid \tilde{H}_i(\mathrm{N}_j(\Delta)) \neq 0\}$.

Remark 5.5.3. Since depth is a topological property ((Mun84, Theorem 3.1)), we always have $\mathrm{depth}\, k[\Delta] = \mathrm{depth}\, k[\mathrm{sd}\, \Delta]$ by Lemma 5.2.3. One can apply (Hib91, Proposition 2.8) repeatedly to show that $\mathrm{depth}[\Delta]_{>j} \geq \mathrm{depth}\, k[\Delta] - j$ for every $j \leq d$. In particular, this implies $\tilde{H}_i(N_j(\Delta)) = 0$ for $i + j < \mathrm{depth}\, k[\Delta]$. Therefore, one immediately obtains $\mathrm{depth}\, k[\Delta] \leq \inf\{i+j \mid \tilde{H}_i(\mathrm{N}_j(\Delta)) \neq 0\}$. However, the converse to (Hib91, Proposition 2.8) does not hold, even with additional hypotheses on vanishing of homology, and therefore, these methods are incapable of establishing the reverse inequality.

5.6 The f-vector and the h-vector

In this section, we prove part 3 of Theorem 5.1.3. We set $\chi(\mathrm{N}_j(\Delta))$ to be the Euler characteristic of $N_j(\Delta)$ and $\tilde{\chi}(\mathrm{N}_j(\Delta))$ to be the reduced Euler characteristic of $N_j(\Delta)$. We use $f_i(\Delta)$ to indicate the i^{th} entry in the f-vector of Δ.

Theorem 5.6.1. *Let $i \geq 0$,*

$$f_i(\Delta) = \sum_{j=i+1}^{d} \binom{j-1}{i} \chi(\mathrm{N}_j(\Delta))$$

We note that f_{-1} is always 1.

Proof. Before we proceed, we introduce some additional notation:

Let $f_{h,k}$ be the number of h-faces in $\mathrm{N}_k(\Delta)$. We note that for any complex Δ, $f_{h,k}$ is 0 for large enough h and for large enough k.

If a face appears in $\mathrm{N}_{k+1}(\Delta)$, then that face also appears in $\mathrm{N}_k(\Delta)$. We wish to count the h-faces of $\mathrm{N}_k(\Delta)$ which first appear in $\mathrm{N}_k(\Delta)$. This number is given by

$$f_{h,k} - f_{h,k+1}.$$

For a collection of facets ρ let $\varphi(\rho) = \cap_{F\in\rho} F$. Note that for a given $\alpha \in \Delta$, the set of ρ such that $\alpha \subseteq \varphi(\rho)$ is Boolean. Let $y, x_1, \ldots, x_n$ be indeterminates, and let $x_\alpha = \prod_{i\in\alpha} x_i$. Then,

$$\sum_{\rho} \sum_{\alpha\subseteq\varphi(\rho)} (-1)^{|\rho|} x_\alpha y^{|\alpha|} = \sum_{\alpha\in\Delta} x_\alpha y^{|\alpha|} \sum_{\substack{\rho \\ \alpha\subseteq\varphi(\rho)}} (-1)^{|\rho|} = 0.$$

This is because for each α, the set of such ρ is Boolean, and therefore, $\sum_{\substack{\rho \\ \alpha\subseteq\varphi(\rho)}} (-1)^{|\rho|} = 0.$

Now, setting $x_i = 1$ for all i and solving for the $\rho = \emptyset$ term yields:

$$\sum_{\alpha\in\Delta} y^{|\alpha|} \;=\; -\sum_{\rho\neq\emptyset}(-1)^{|\rho|}\sum_{\alpha\subseteq\varphi(\rho)} y^{|\alpha|} \;=\; \sum_{\rho\neq\emptyset}(-1)^{|\rho|-1}\sum_{j=0}^{|\varphi(\rho)|}\binom{|\varphi(\rho)|}{j}y^j.$$

Taking the $(i+1)^{st}$ coefficient of each side yields:

$$\begin{aligned}
f_i(\Delta) &= \sum_{\substack{\rho\neq\emptyset \\ |\varphi(\rho)|\geq i+1}} (-1)^{|\rho|-1}\binom{|\varphi(\rho)|}{i+1} \\
&= \sum_{h=0}^{\infty}\sum_{k=i+1}^{\infty}(-1)^h\binom{k}{i+1}\#\{\rho \mid |\rho|-1=h,\ \rho\in N_k(\Delta)\backslash N_{k+1}(\Delta)\} \\
&= \sum_{h=0}^{\infty}(-1)^h\sum_{k=i+1}^{\infty}\binom{k}{i+i}(f_{h,k}-f_{h,k+1}) \\
&= \sum_{h=0}^{\infty}(-1)^h\sum_{k=i+1}^{\infty}(f_{h,k}-f_{h,k+1})\sum_{j=i+1}^{k}\binom{j-1}{i} \\
&= \sum_{h=0}^{\infty}(-1)^h\sum_{j=i+1}^{d}\sum_{k=j}^{\infty}\binom{j-1}{i}(f_{h,k}-f_{h,k+1}) \\
&= \sum_{j=i+1}^{d}\binom{j-1}{i}\sum_{h=0}^{\infty}(-1)^h f_{h,j} \\
&= \sum_{j=i+1}^{d}\binom{j-1}{i}\chi(\mathrm{N}_j(\Delta)).
\end{aligned}$$

□

For the convenience of the reader, we have worked out the corresponding formula for the h-vector $(h_0=1, h_1, \ldots, h_d)$ of Δ.

Corollary 5.6.2. *For $k\geq 1$ we have:*

$$h_k(\Delta) = (-1)^{k-1}\sum_{j\geq 1}\binom{d-j}{k-1}\tilde{\chi}(N_j(\Delta)).$$

We also record the following:

Corollary 5.6.3. *If Δ_1 and Δ_2 are simplicial complexes with $\tilde{H}_{i-1}(N_j(\Delta_1)) \cong \tilde{H}_{i-1}(N_j(\Delta_2))$ for all i, j, then Δ_1 and Δ_2 have identical f-vectors and h-vectors.*

5.7 LCM-lattice and Regularity of Monomial Ideals

In this section, we use our main theorem, Theorem 5.1.3, to deduce a formula for the Castelnuovo-Mumford regularity of any monomial ideal I, denoted by $\operatorname{reg}(I)$. We first fix some notation motivated by (Wel99). Suppose $f_1, \ldots, f_r$ are the minimal monomial generators of I.

Definition 5.7.1. We define the j-th LCM complex of I to be:

$$L_j(I) := \{F \subseteq [r] \colon |lcm_{i\in F}(f_i)| \leq j\}.$$

Theorem 5.7.2. *Let I be a monomial ideal. Then:*

$$\operatorname{reg}(I) = \sup\{j - i \; | \tilde{H}_i(L_j(I)) \neq 0\}.$$

Proof. Let $I^{pol} = (g_1, \ldots, g_r)$ be the polarization of I. Then it is well-known that $\operatorname{reg}(I) = \operatorname{reg}(I^{pol})$ (see for instance (Pee11, Theorem 21.10)). From the construction of the g_i's from the f_i's, it is obvious that for any subset $F \subseteq [r]$, $lcm_{i\in F}(f_i)$ and $lcm_{i\in F}(g_i)$ have the same size. Thus, the problem reduces to the case when I is a square-free monomial ideal.

Now let $I^\vee$ be the Alexander dual of I. It is the Stanley-Reisner ideal of some complex Δ. We have that

$$\operatorname{reg}(I) = \operatorname{pd} S/I^\vee = n - \operatorname{depth} S/I^\vee$$

by the Eagon-Reiner theorem ((MS05, Theorem 5.59)) and the Auslander-Buchsbaum formula. We now note that each g_i is precisely the product of variables in the complement of the corresponding facet F_i of Δ. Thus $L_j(I) = N_{n-j}(\Delta)$. Putting all of these together, we have:

$$\mathrm{reg}(I) = n - \inf\{i + j \mid \tilde{H}_i(L_{n-j}(I)) \neq 0\} = \sup\{j - i \mid \tilde{H}_i(L_j(I)) \neq 0\}$$

as desired.

□

Remark 5.7.3. Our formula above should be compared with Theorem 2.1 in (Wel99).

5.8 Nerve Code, Nervous Systems, and k-Nervous Systems

In this section, we introduce unpublished code written in SAGE (TheYY) to facilitate the use of the nerve, the nervous system, and the k-nervous system. This code was written with Goeckner.

Though the collection of higher nerves retain enough information about the complex to determine the depth and f-vector of the complex, the collection of higher nerves do not retain all of the information about the nerve. This is made apparent in Chapter 6, where we will show that the higher nerves do not retain enough information to determine if the original complex satisfied Serre's condition.

The nervous system is the nerve of a complex along with an integer labelling of each face. Each face is labelled with the highest nerve in which it appears. This object retains enough information about the original simplicial complex to determine it completely up to isomorphism. The k-nervous system is the nerve complex along with an integer labelling of each face, such that each face is labelled with the smaller of k and the highest nerve in which the face appears.

The following code is commented for ease of understanding and implementation.

```
#a program to construct a higher nerve from the simplicial complex
def nerve(X,j=1): ## Defaults to "first" nerve
    Pow = powerset(X.facets())
    F = []
    for p in Pow:
        if len(p) > 0:
            inter = set(p[0])
            for i in range(len(p)):
                inter = inter.intersection(p[i])
            if len(inter) >= j:
                F.append(p)
    return SimplicialComplex(F)

#We wish to make this nerve program faster
#We provide two options to handle very different types of complexes.
#Below, we implement a method that starts from the bottom and builds up
    nerves until the intersection is no longer of size at least j.
#This method will be much faster for sparse nerves.
#nerve_bottom_up will call a recursive program with the empty face
    (SimplicialComplex()).
#nerve_bottom_up will receive a facet list from the recursive program. From
    this nerve_bottom_up will make a simplicial complex

def nerve_bottom_up(X, nerve_number=1):
    ordered_facet_list = X.facets()
    last_facet_in_list = -1 #keeps track of which facets we should be
        considering moving forward. If we only consider facets after this
        integer we will avoid double counting.
    temp = nerve_bottom_up_recursive(
        ordered_facet_list,last_facet_in_list,nerve_number,[],Set([]))
```

```
    return SimplicialComplex(temp)


#the recursive program adds the input facet_list to answer
#The recursive program will then add a facet to our facet list which was not
    previously included.
#If inter and the new facet have an intersection of size at least j we call
    the recursive program on the new facet list and new intersection.
#we append anythign that returns to our answer in the current iteration of
    the recursion; finally we return our answer


#X will be a simplicial complex, facet_list gets built up until the
    intersection becomes too small.  We can avoid bigger intersections by
    doing this.
def nerve_bottom_up_recursive(
        ordered_facet_list, last_facet_in_list, j, facet_list, inter):
    answer = [] #initializing a list of lists
    if last_facet_in_list != -1:
        copy_list = list(facet_list)
        answer.append(copy_list)
    #now we will look at facets which are not in facet_list and whose
        addition will make combinations which will not be checked elsewhere.
    for i in range(last_facet_in_list+1, len(ordered_facet_list)):
        if inter == Set([]):
            tempinter = [(ordered_facet_list[i])]
        else:
            tempinter = [(list(set(inter[0]) & set(ordered_facet_list[i])))]
        if len(list(tempinter[0])) >= j:
            temp_facet_list = list(facet_list)
            temp_facet_list.append(ordered_facet_list[i])
            add_to_answer = nerve_bottom_up_recursive(
                ordered_facet_list, i, j, temp_facet_list, tempinter)
```

```
            answer = answer + add_to_answer
    return answer

def nerve_top_down(X, nerve_number=1):
    answer = []
    facet_list = X.facets()
    inter = intersect_list(facet_list)
    if len(inter) >= nerve_number:
        answer = [facet_list]
    else:
        answer = nerve_top_down_recursive(facet_list,nerve_number)
    return SimplicialComplex(answer)

#we know that the intersection of everything in facet list will not be size
    nerve_number. We try one smaller
def nerve_top_down_recursive(facet_list, nerve_number):
    #we'll do the i=0 case separately
    answer = []
    for i in range(0,len(facet_list)):
        temp_list = list(facet_list) #makes a copy that doesn't point to the
            same location
        del temp_list[i]
        inter = intersect_list(temp_list)
        if len(inter) >= nerve_number:
            answer = answer + [temp_list]
        else:
            add_to_answer = nerve_top_down_recursive(temp_list,nerve_number)
            answer = answer + add_to_answer
    return answer

#The Nervous System is a way to keep track of all of the nerves of a complex
```

```
    at once.  It is the original
#nerve complex with each face labelled by the largest nerve in which that
    face appears.
#Nervous Systems describe complexes up to isomorphism.
def nervousSystem(X):
    Pow = powerset(X.facets())
    F = []
    maxCard = 0
    for p in Pow:
        if len(p) > 0:
            inter = set(p[0])
            for i in range(1,len(p)):
                inter = inter.intersection(p[i])
            if len(inter) >= 1:
                F.append([p,len(inter)])
                if maxCard < len(p):
                    maxCard = len(p)
    return [F,maxCard]


class Nervous_System:

    def __init__(self,list,int):
        X = [list, int]
        self.gamma = X[0]
        self.maxCard = X[1]

    #This method takes a Nervous System and returns the associated Simplicial
        Complex

    def unnerve(self):
```

```
facet_list = [] #this will be a list of facets, from this we will
    create the simplicial complex we intend to return
vertex_array = [] #an array of arrays. we will add
    (sigma,new_vertices,number of new vertices)
vertex_counter = 1 #a counter of how many vertices have been added
    minus 1. used to add new vertices which will be uniquely named
    integers.
d = self.maxCard
if d != 1: #if d=1 then we go straight to constructing the facets
    for sigma in self.gamma:
        if sigma[1] < 0:
            return false
        if len(sigma[0]) == d:
            new_vertices = [] #an array where we will add new
                vertices.
            for count in range(sigma[1]):
                new_vertices.append(vertex_counter)
                vertex_counter = vertex_counter+1
            vertex_array.append(
                [sigma[0],new_vertices,sigma[1]])
    #we have made vertices from the d-dimensional faces of the
        nervous system
    #we repeat this process for d-1,d-2,...2. After that we must
        actually build the facets outside this if statement
    for j in range(d-2):
        for sigma in self.gamma:
            if len(sigma[0]) == d-1-j:
                s = 0 #we shall subtract this from sigma[1] to
                    determine how many vertices need to be added.
                for delta in vertex_array:
                    subsets = Set(delta[0]).subsets()
```

```
                    if Set(sigma[0]) in subsets:
                        s = s+delta[2]
                t = sigma[1]-s
                if t < 0:
                    return false
                new_vertices = [] #an array where we will add new
                    vertices.
                for count2 in range(t):
                    new_vertices.append(vertex_counter)
                    vertex_counter = vertex_counter+1
                vertex_array.append(
                        [sigma[0], new_vertices, t])
#now we make the facets by looking at the vertices of the nervous
    system
for sigma in self.gamma: #this loop finds sigma[0] = 1, these sigmas
    then represent the facets we wish to build
    if len(sigma[0]) == 1:
        temp = [] #a temporary array to help us build the facet
            corresponding to sigma
        s=0
        for delta in vertex_array:
            subsets = Set(delta[0]).subsets()
            if Set(sigma[0]) in subsets:
                s = s+delta[2]
                #append the vertices created for delta
                for alpha in range(delta[2]):
                    temp.append(delta[1][delta[2]-alpha-1])
        t = sigma[1]-s
        if t < 0:
            return false
        for count3 in range(t):
```

```
                    temp.append(vertex_counter)
                    vertex_counter = vertex_counter+1
                tempsubsets = Set(temp).subsets()
                for omega in facet_list:   #this for loop will check to make
                    sure we don't have redundant facets
                    subsets2 = Set(omega).subsets()
                    if Set(temp) in subsets2:
                        return false
                    if Set(omega) in tempsubsets:
                        return false
                facet_list.append(temp)
        answer = SimplicialComplex(facet_list)
        return answer

#The k-Nervous System is the nerve complex such that each face of the complex
    is labelled with the smaller of
#the number k and the largest nerve in which the face appears.  If k =
    dimension of the Stanley-Reisner ring
#then we have the Nervous System.

def knervousSystem(X,k):
    Pow = powerset(X.facets())
    F = []
    Label = []
    maxCard = 0
    for p in Pow:
        if len(p) > 0:
            inter = set(p[0])
            i = 1
            while i < len(p):
                inter = inter.intersection(p[i])
```

```
                i = i + 1
            if len(inter) >= 1:
                if len(inter) >= k:
                    F.append([p,k])
                    if maxCard < len(p):
                        maxCard = len(p)
                elif len(inter) < k:
                    F.append([p,len(inter)])
                    if maxCard < len(p):
                        maxCard = len(p)
    return [F,maxCard]

class kNervous_System:

    def __init__(self,list,int):
        X = [list, int]
        self.gamma = X[0]
        self.maxCard = X[1]

    def printknervous(self):
        print self.gamma
        print self.maxCard

    #we can return a simplicial complex associated to a k-Nervous System, but
        it is not the only such complex

    def kunnerve(self,k):
        facet_list = [] #this will be a list of facets, from this we will
            create the simplicial complex we intend to return
        vertex_array = [] #an array of arrays. we will add
            (sigma,new_vertices,number of new vertices)
```

```
vertex_counter = 1 #a counter of how many vertices have been added
    minus 1.  used to add new vertices which will be uniquely named
    integers.
d = self.maxCard
i = d-1
if d != 1:
    for sigma in self.gamma:
        if sigma[1] < 0:
            return false
        if sigma[1] > k:
            return false
        if len(sigma[0]) == d:
            new_vertices = [] #an array where we will add new
                vertices.
            count = 0
            while count < sigma[1]:
                new_vertices.append(vertex_counter)
                vertex_counter = vertex_counter+1
                count = count+1
            vertex_array.append([sigma[0], new_vertices, count])
    while i > 1:
        for sigma in self.gamma:
            if len(sigma[0]) == i:
                s = 0 #we shall subtract this from sigma[1] to
                    determine how many vertices need to be added.
                for delta in vertex_array:
                    subsets = Set(delta[0]).subsets()
                    if Set(sigma[0]) in subsets:
                        s = s+delta[2]
                t = sigma[1]-s
                if t < 0:
```

```
                    if t != k:
                        return false
                new_vertices = [] #an array where we will add new
                    vertices.
                counter2 = 0
                while counter2 < t:
                    new_vertices.append(vertex_counter)
                    vertex_counter = vertex_counter+1
                    counter2 = counter2+1
                vertex_array.append(
                        [sigma[0],new_vertices,counter2])
        i = i-1
for sigma in self.gamma:
    if len(sigma[0]) == 1:
        if sigma[1] > k:
            return false
        temp = [] #a temporary array to help us build the facet
            corresponding to sigma
        s=0
        for delta in vertex_array:
            subsets = Set(delta[0]).subsets()
            if Set(sigma[0]) in subsets:
                s = s+delta[2]
                #append the vertices created for delta
                tempcount=delta[2]
                while tempcount > 0:
                    temp.append(delta[1][tempcount-1])
                    tempcount = tempcount - 1
        t = sigma[1]-s
        if t < 0:
            if t != k:
```

```
                    return false
                count3 = 0
                while count3 < t:
                    temp.append(vertex_counter)
                    vertex_counter = vertex_counter+1
                    count3 = count3+1
                tempsubsets = Set(temp).subsets()
                for omega in facet_list:   #this for loop will check to make
                    sure we don't have redundant facets
                    subsets2 = Set(omega).subsets()
                    if Set(temp) in subsets2:
                        temp.append(vertex_counter)
                        vertex_counter = vertex_counter+1
                    if Set(omega) in tempsubsets:
                        omega.append(vertex_counter)
                        vertex_counter = vertex_counter+1
                facet_list.append(temp)
                print facet_list
        answer = SimplicialComplex(facet_list)
        return answer

#returns the list of homologies of the higher nerves.
def homology_list(X):
    d = dim(X)+1
    rows = []
    temp = []
    Homs = X.homology()
    for j in range(d):
        temp.append(Homs[j])
    rows.append(temp)
    for i in range (2,d+1): ## Start w/ second nerve
```

```
        temp = []
        N = nerve(X, i)
        Homs = N.homology()
        for j in range (d):
            if j > dim(N):
                temp.append(SimplicialComplex([0]).homology()[0]) #appends
                    the 0 homology group
            else:
                temp.append(Homs[j])
        rows.append(temp)
    return rows

#takes a list of the dimensions of the homologies of the nerves and
    calculates the depth
def depth_by_nerve_list(L):
    #A variable counting depth. We continue to decrease it as we examine our
        information.
    depth = len(L[0])
    i = 1 #keeps track of which nerve we are considering.
    while i < depth:
        j=0
        while i+j < depth:
            if L[i-1][j] != 0:
                depth = i+j
            j=j+1
        i = i + 1
    return depth


#takes a simplicial complex and calculates it's depth using the nerve
    homologies and the theorem of DDDGHL
```

```
def depth_by_nerve(X):
    Y = SimplicialComplex([0])
    Test = Y.homology()[0]              #I am unaware of a cleaner way to
        get this

    #A variable counting depth. We continue to decrease it as we examine our
        information.
    depth = dim(X)+1
    j=0
    while 1+j < depth:
        if X.homology()[j] != Test:
                depth = 1+j
        j = j+1

    i=2 #Start with 2nd Nerve b/c first nerve has same homologies as the
        complex

    #Given a bound on depth, k, the kth nerve and above cannot impose
        stronger bounds on depth.
    while i < depth:
        N = nerve(X, i)
        j=0
        while i+j < depth:
            if N.homology()[j] != Test:
                depth = i+j
            j=j+1
        i = i + 1
    return depth

#Makes a table that shows the reduced homologies of the higher nerves of the
    simplicial complex X
```

```
def nerveTable(X):
    d = dim(X)+1
    rows = []
    temp = []
    temp.append("")
    for j in range(d): #makes the labels
        temp.append("H"+str(j))
    rows.append(temp)
    temp = []
    Homs = X.homology()
    temp.append("N1")
    for j in range(d): #we do the first nerve separately because it has the
        same homologies as X and is thus less computationally intensive
        temp.append(Homs[j])
    rows.append(temp)
    for i in range (2,d+1):
        temp = []
        temp.append("N"+str(i))
        N = nerve(X,i)
        Homs = N.homology()
        for j in range (d):
            if j > dim(N):
                temp.append(0)
            else:
                temp.append(Homs[j])
        rows.append(temp)
    print table(rows, header_column = True, header_row = True)


#Takes a list of length d, where d is the dimension of original complex. The
    ith element of this list is a list of the first d reduced homologies of
```

```
    the ith nerve.
#This program takes this information and returns the f-vector of the complex.
def f_vector_nerve(L):
    answer = []
    answer.append(1) #negative first element of f-vector is always 1
    d = len(L[0])     #dimension of the complex
    #below we calculate the reduced Euler Charactersistics for the higher
        Nerves
    euler = []
    for j in range(1,d+1):
        temp = 0
        for n in range(d):
            temp = temp + (-1)^(n)*(L[j-1][n])  #calculate the reduced euler
                coefficients
        euler.append(temp)
    #now we calculate the f-vector using the euler characteristics
    for i in range(d):
        A = 0
        for j in range(i+1,d+1):
            A = A+(binomial(j-1,i)*(euler[j-1]+1)) #calculate a single
                element of the f-vector
        answer.append(A)
    return answer

#Takes a list of length d, where d is the dimension of original complex. The
    ith element of this list is a list of the first d reduced homologies of
    the ith nerve.
#This program takes this information and returns the h-vector of the complex.
def h_vector_nerve(L):
    answer = []
    answer.append(1) #h_0 = 1
```

```
d = len(L[0])     #dimension of complex
#below we calculate the reduced Euler Charactersistics for the higher
    Nerves
euler = []
for j in range(1,d+1):
    temp = 0
    for n in range(d):
        temp = temp + (-1)^(n)*(L[j-1][n])  #calculate the reduced euler
            coefficients
    euler.append(temp)
#below we calculate the h-vector with the Euler characteristics
for k in range(1,d+1):
    A = 0
    for j in range(1,d+1):
        A = A + (euler[j-1])*binomial(d-j,k-1) #euler[j-1] is the Euler
            char of the jth Nerve
    A = A*(-1)^(k-1)
    answer.append(A)
return answer
```

Chapter 6

Rank Selection Theorems for Balanced Simplicial Complexes

6.1 Introduction

The work in this chapter is based on joint work with Justin Lyle.

Two objects which have attracted attention in recent years are balanced simplicial complexes and simplicial complexes satisfying Serre's condition (S_ℓ) (see (DHV16; PSFTY14; HTYZN11; MT09; TY08; Ter07; Bjö03; Yan00; KN16; BVT12; Het06; JV18)).

Let (Δ, π) be a balanced simplicial complex of dimension $d-1$ with ordered partition $\pi = (V_1, \ldots, V_d)$, and let $k[\Delta]$ denote its Stanley-Reisner ring over the field k. If $S \subseteq [d]$, we let Δ_S denote the S-rank selected subcomplex of Δ; that is, Δ_S is the subcomplex of Δ induced on $\bigcup_{i \in S} V_i$. For convenience, we also set $\tilde{\Delta}_S := \Delta_{[d]-S}$. The so-called rank selection theorems of Stanley ((Sta79)) and Munkres ((Mun84)) show that Δ_S often inherits nice properties, in a homological sense, from Δ. Specifically, we have the following:

Theorem 6.1.1 ((Sta79)). *Let (Δ, π) be a balanced simplicial complex. If $k[\Delta]$ is Cohen-Macaulay, then $k[\Delta_S]$ is Cohen-Macaulay for any $S \subseteq [d]$.*

This can be extended, from a more general statement of Munkres, to the following:

Theorem 6.1.2 ((Mun84))**.** *Let* (Δ, π) *be a balanced simplicial complex. Then, for any* $S \subseteq [d]$*,* $\operatorname{depth} k[\tilde{\Delta}_S] \geq \operatorname{depth} k[\Delta] - |S|$*.*

As Serre's condition (S_ℓ), like depth, extends the Cohen-Macaulay property, it is natural to consider if there is any extension of these results to (S_ℓ). We prove this is indeed the case:

Theorem 6.1.3. *Let* (Δ, π) *be a balanced simplicial complex of dimension* $d-1$*, with ordered partition* $\pi = (V_1, \dots, V_d)$*. If* $k[\Delta]$ *satisfies Serre's condition* (S_ℓ)*, then* $k[\Delta_S]$ *satisfies* (S_ℓ) *for any* $S \subseteq [d]$*.*

A motivating example of a balanced simplicial complex is the order complex $\mathcal{O}(P)$ of a poset P, whose elements are partitioned by height. In this case, $\mathcal{O}(P_{>j})$ is the subcomplex of $\mathcal{O}(P)$ with the bottom j ranks removed. For this case, we prove the following:

Theorem 6.1.4. *Let* P *be a finite poset. We use* $\mathcal{O}(P)$ *for the order complex of* P*.*

1. *If* $k[\mathcal{O}(P)]$ *satisfies* (S_ℓ)*, then* $\tilde{H}_{i-1}(\mathcal{O}(P_{>j}); k) = 0$ *whenever* $i + j < d$ *and* $0 \leq i < \ell$*.*

2. *If* P *is the face poset of a simplicial complex and* $\tilde{H}_{i-1}(\mathcal{O}(P_{>j}); k) = 0$ *whenever* $i+j < d$ *and* $0 \leq i \leq \ell$*, then* $k[\mathcal{O}(P)]$ *satisfies* (S_ℓ)*.*

We also provide a more direct extension of Theorem 6.1.2, and a formula for $\operatorname{depth} k[\Delta]$ in terms of reduced homologies of rank selected subcomplexes.

Proposition 6.1.5. *Let* (Δ, π) *be a balanced simplicial complex of dimension* $d-1$*, with ordered partition* $\pi = (V_1, \dots, V_d)$*. If* $\tilde{H}_{\operatorname{depth} k[\Delta]-1}(\Delta) = 0$*, then there is an* $i \in [d]$ *such that* $\operatorname{depth} k[\tilde{\Delta}_{\{i\}}] = \operatorname{depth} k[\Delta] - 1$*.*

Theorem 6.1.6. *Let* (Δ, π) *be a balanced simplicial complex of dimension* $d-1$*, with ordered partition* $\pi = (V_1, \dots, V_d)$*. Then*

$$\operatorname{depth} k[\Delta] = \min\{i + |S| \mid \tilde{H}_{i-1}(\tilde{\Delta}_S; k) \neq 0\}.$$

Finally, we use higher nerve complexes (see Definition 5.1.2 to provide a formula for sums of reduced Euler characteristics of links.

Theorem 6.1.7. *Suppose Δ is pure and let P be the face poset of Δ. Write χ for Euler characteristic and $\tilde{\chi}$ for reduced Euler characteristic. Then*

$$\sum_{\substack{T \in \Delta \\ |T| = k}} \tilde{\chi}(\operatorname{lk}_\Delta(T)) = \chi(\mathcal{O}(P_{>k})) - \chi(\mathcal{O}(P_{>k-1})).$$

We now briefly describe the structure of our paper. In Section 2, we set notation and provide the algebraic and combinatorial background we appeal to throughout the paper. In Section 3 we prove Theorems 6.1.3 and 6.1.4, and in section 4, we prove Proposition 6.1.5 and Theorem 6.1.6. In section 5, we prove Theorem 6.1.7 and provide an application to Gorenstein* complexes. The last section discusses open problems related to this work and provides examples indicating the sharpness of our results.

6.2 Background and Notation

In this section we set notation and provide algebraic and combinatorial background. Once and for all, fix the base field k. We let $\tilde{H}_i$ denote ith simplicial or singular homology, whichever is appropriate, always taken with respect to the field k. We use χ for Euler characteristic and $\tilde{\chi}$ for reduced Euler characteristic.

Given a simplicial complex Δ we write $k[\Delta]$ for its Stanley-Reisner ring over k. We write $V(\Delta)$ for the vertex set of Δ, but, if Δ is clear from context, we generally write V for $V(\Delta)$ and n for $|V|$; we set $A := k[x_1, \dots, x_n]$. We write $f_i(\Delta)$ for the number of i-dimensional faces of Δ, and $h_i(\Delta)$ for the ith entry of the h-vector of Δ; so $h_i(\Delta) = \sum_{i=0}^{k} \binom{d-i}{k-i}(-1)^{k-i} f_{i-1}(\Delta)$. We let $||\Delta||$ denote the geometric realization of Δ. We call $\Delta^{(k)} := \{\sigma \in \Delta : \dim \sigma \leq k\}$ the k-skeleton of Δ.

Given a subset $T \subseteq V(\Delta)$, we use $\Delta|_T := \{\sigma \in \Delta \mid T \subseteq \sigma\}$ for the induced subcomplex of Δ on T. We may then define the star, the anti-star, and the link of T, respectively, as follows:

$$\begin{aligned}
\operatorname{st}_\Delta T &:= \{G \in \Delta \mid T \cup G \in \Delta\} \\
\operatorname{astar}_\Delta T &:= \{G \in \Delta \mid T \cap G = \varnothing\} = \Delta|_{V-T} \\
\operatorname{lk}_\Delta T &:= \{G \in \Delta \mid T \cup G \in \Delta \text{ and } T \cap G = \varnothing\} = \operatorname{st}_\Delta T \cap \operatorname{astar}_\Delta T
\end{aligned}$$

Of import, $\operatorname{st}_\Delta(T)$ is a cone over $\operatorname{lk}_\Delta(T)$ for any $T \in \Delta$, in particular is acyclic. When $T = \{v\}$, we abuse notation and write $\operatorname{st}_\Delta(v)$, $\operatorname{astar}_\Delta(v)$, and $\operatorname{lk}_\Delta(v)$.

We say that $J \subseteq V(\Delta)$ is an independent set for Δ if $\{a, b\} \notin \Delta$ for any $a, b \in J$ with $a \neq b$. Motivated by (Hib91), we say that $J \subseteq V(\Delta)$ is an excellent set for Δ if J is an independent set for Δ and $J \cap F \neq \varnothing$ for every facet $F \in \Delta$. When Δ is clear from context, we simply say that J is an independent set or that J is an excellent set, as appropriate.

The main computational tools of this paper are two exact sequences recorded in the following propositions:

Proposition 6.2.1. *Suppose $T \in \Delta$ is not a facet. Then there is a Mayer-Vietoris exact sequence of the form*

$$\cdots \to \tilde{H}_i(\Delta) \to \tilde{H}_{i-1}(\operatorname{lk}_\Delta(T)) \to \tilde{H}_{i-1}(\operatorname{st}(T)) \oplus \tilde{H}_{i-1}(\operatorname{astar}_\Delta(T)) \to \tilde{H}_{i-1}(\Delta) \to \cdots$$

Proposition 6.2.2. *Suppose $\{x\} \subsetneq J \subsetneq V$ is an independent set. Set $J' = J - \{x\}$. Then there is a Mayer-Vietoris exact sequence of the form*

$$\cdots \to \tilde{H}_i(\Delta) \to \tilde{H}_{i-1}(\mathrm{astar}_\Delta(J)) \to \tilde{H}_{i-1}(\mathrm{astar}_\Delta(J')) \oplus \tilde{H}_{i-1}(\mathrm{astar}_\Delta(x)) \to \tilde{H}_{i-1}(\Delta) \to \cdots$$

We will say Δ satisfies (S_ℓ) if $k[\Delta]$ does. Every simplicial complex satisfies (S_1), and a simplicial complex satisfies (S_d) if and only if it is Cohen-Macaulay. The following is an immediate consequence of Hochster's formula (BH98, Theorem 5.3.8) and gives a useful characterization of depth for Stanley-Reisner rings in terms of reduced homologies of links:

Proposition 6.2.3. *Let Δ be a simplicial complex. Then* $\operatorname{depth} k[\Delta] \geq t$ *if and only if* $\tilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) = 0$ *for all* $T \in \Delta$ *with* $i + |T| < t$.

The corresponding result for (S_ℓ) can be found in (Ter07):

Proposition 6.2.4 ((Ter07))**.** *Let Δ be a simplicial complex. Then Δ satisfies (S_ℓ) for $\ell \geq 2$ if and only if $\tilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) = 0$ whenever $i + |T| < d$ and $0 \leq i < \ell$. In particular, (S_ℓ) complexes are pure if $\ell \geq 2$.*

Define $\operatorname{core} V(\Delta) := \{v \in V(\Delta) \mid \mathrm{st}_\Delta(v) \neq \Delta\}$ and set $\operatorname{core} \Delta := \Delta|_{\operatorname{core} V(\Delta)}$. We say that Δ is Gorenstein if the ring $k[\Delta]$ is Gorenstein; if, in addition, $\operatorname{core} \Delta = \Delta$, we say that Δ is Gorenstein*. One has the following, see (BH98, Theorem 5.6.1):

Theorem 6.2.5. *A simplicial complex Δ is Gorenstein* if and only if*

$$\tilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) \cong \begin{cases} k & \textit{if } i = d - |T| \\ 0 & \textit{if } i \neq d - |T| \end{cases}$$

Now, suppose P is a poset. If $p \in P$, we let $\mathrm{ht}(p)$ denote the length of a longest chain $p_1 \prec p_2 \prec \cdots \prec p_i = p$ and let $\operatorname{ht} P := \max\{\operatorname{ht} p \mid p \in P\}$. We denote by $P_{>j}$ the poset obtained by restricting to elements $p \in P$ so that $\operatorname{ht} p > j$. The *order complex* of P, denoted

$\mathcal{O}(P)$, is the simplicial complex on P consisting of all chains of elements in P. Let $\mathcal{F}(\Delta)$ denote the face poset of Δ. We set $[\Delta]_{>j} := \mathcal{O}(\mathcal{F}(\Delta)_{>j})$. We note that when $j = 0$, $[\Delta]_{>0}$ is the barycentric subdivision of Δ. The following is well known (see (Gib77, Corollary 5.7), for example):

Lemma 6.2.6. *The realization $||\Delta||$ is homeomorphic to $||[\Delta]_{>0}||$. In particular, $\tilde{H}_i(\Delta) \cong \tilde{H}_i([\Delta]_{>0})$ for all i.*

We let $\rho : \Delta - \{\varnothing\} \to V([\Delta]_{>0})$ be the map which sends T to itself viewed as a vertex of $[\Delta]_{>0}$.

There are several advantages of working with $[\Delta]_{>k}$. For instance, Lemma 5.4.1

Definition 6.2.7. A balanced simplicial complex is a pair (Δ, π) satisfying:

1. Δ is $d - 1$ dimensional simplicial complex on a vertex set V.
2. $\pi = (V_1, \ldots, V_d)$ is an ordered partition of V.
3. For every facet $F \in \Delta$ and every $i \in [d]$, $|F \cap V_i| \leq 1$.

Balanced simplicial complexes were introduced by Stanley in (Sta79). One can find more information on them in (BSF87; BGS82; Gar80); (Sta96) gives a more modern treatment of the subject. An important property of balanced simplicial complexes is that each V_i is an independent set for Δ, and, if Δ is pure, the V_i are excellent sets for Δ. If (Δ, π) is a balanced simplicial complex with $\pi = (V_1, \ldots, V_d)$, and if $S \subseteq [d]$, we define the *S-rank selected subcomplex* of Δ to be the complex $\Delta_S := \Delta|_{\bigcup_{i \in S} V_i}$; for notational convenience, we also set $\tilde{\Delta}_S = \Delta_{[d]-S}$. If (Δ, π) is a balanced simplicial complex, we often suppress the ordered partition π and simply refer to Δ as a balanced simplicial complex; in this case we always write $\pi = (V_1, \ldots, V_d)$ for the corresponding ordered partition.

Now, let P be a finite poset. If we set $V_i := \{p \mid \mathrm{ht}(p) = i\}$ and $\pi = (V_1, \ldots, V_{\mathrm{ht}\,P})$, then $(\mathcal{O}(P), \pi)$ is a balanced simplicial complex. In particular, this means $[\Delta]_{>j}$ is always a balanced simplicial complex for any j.

Finally, we will need the higher nerves defined in 5.1.2.

6.3 Rank Selection Theorems for Serre's Condition

In this section we prove some general statements and use them to derive Theorems 6.1.3 and 6.1.4.

Lemma 6.3.1. *Suppose $J \subseteq V$ is excellent and Δ satisfies (S_ℓ). Set $\tilde{\Delta} := \mathrm{astar}_\Delta(J)$. Then $\tilde{\Delta}$ satisfies (S_ℓ).*

Proof. We proceed by induction on ℓ. The claim is clear when $\ell = 1$, since every simplicial complex satisfies (S_1). So, suppose we know the result for all $1 \leq j \leq \ell$ and suppose Δ satisfies $(S_{\ell+1})$. Inductive hypothesis gives us that $\tilde{\Delta}$ satisfies (S_ℓ).

By Proposition 6.2.4, we have that $\tilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) = 0$ whenever $i + |T| < d$ and $0 \leq i \leq \ell$, $\tilde{H}_{i-1}(\mathrm{lk}_{\tilde{\Delta}}(T)) = 0$ whenever $i + |T| < d - 1$ and $0 \leq i < \ell$, and we need only show that $\tilde{H}_{\ell-1}(\mathrm{lk}_{\tilde{\Delta}}(T)) = 0$ for all $T \in \tilde{\Delta}$ with $\ell + |T| < d - 1$.

Pick $T \in \tilde{\Delta}$ such that $\ell + |T| < d - 1$. Let $\sigma \supseteq T$ be a facet of Δ. Since J is excellent, there is a $b \in J \cap \sigma$, and thus $b \cup T \in \Delta$. Since $b \notin T$, this means $b \in \mathrm{lk}_\Delta(T)$. Note $T \cup \{b\}$ cannot be a facet of Δ, since this would mean $|T| + 1 = d$, whilst $\ell + |T| < d - 1$. Set $S = J \cap V(\mathrm{lk}_\Delta(T))$. Then evidently we have $\mathrm{lk}_{\tilde{\Delta}}(T) = \mathrm{astar}_{\mathrm{lk}_\Delta(T)}(S)$. By Proposition 6.2.1, we have, for any $b \in S$, the exact sequence:

$$\tilde{H}_\ell(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(b)) \xrightarrow{i_b^*} \tilde{H}_\ell(\mathrm{lk}_\Delta(T)) \to \tilde{H}_{\ell-1}(\mathrm{lk}_{\mathrm{lk}_\Delta(T)}(b)) \to \tilde{H}_{\ell-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(b)) \to \tilde{H}_{\ell-1}(\mathrm{lk}_\Delta(T))$$

where i_b^* is the induced map coming from the inclusion $i_b : \mathrm{astar}_{\mathrm{lk}_\Delta(T)}(b) \hookrightarrow \mathrm{lk}_\Delta(T)$. Since

$\mathrm{lk}_{\mathrm{lk}_\Delta(T)}(b) = \mathrm{lk}_\Delta(T \cup \{b\})$ and since $\ell + |T| < d - 1$, we have $\tilde{H}_{\ell-1}(\mathrm{lk}_{\mathrm{lk}_\Delta(T)}(b)) = 0$. Since $\tilde{H}_{\ell-1}(\mathrm{lk}_\Delta(T)) = 0$, we obtain $\tilde{H}_{\ell-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(b)) = 0$ and that i_b^* is surjective, from exactness.

Now, since J is an independent set in Δ, S is an independent set in $\mathrm{lk}_\Delta(T)$. We claim that $\tilde{H}_{\ell-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(I)) = 0$ for any $\varnothing \subsetneq I \subseteq S$. To see this, we induct on $|I|$. Note that the claim is true when $|I| = 1$, from above. Now suppose the claim is true for every I with $|I| = k$, and suppose we are given an I with $|I| = k + 1$. Write $I = L \cup \{b\}$ so that $|L| = k$. By Proposition 6.2.2 we have the exact sequence

$$\begin{tikzcd}
\tilde{H}_\ell(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(b)) \oplus \tilde{H}_\ell(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(L)) \arrow[r, "i_b^* - k^*"] & \tilde{H}_\ell(\mathrm{lk}_\Delta(T)) \arrow[dll, rounded corners] & \\
\tilde{H}_{\ell-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(I)) \arrow[r] & \tilde{H}_{\ell-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(b)) \oplus \tilde{H}_{\ell-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(L))
\end{tikzcd}$$

where k^* is the induced map coming from the inclusion $k : \mathrm{astar}_{\mathrm{lk}_\Delta(T)}(L) \hookrightarrow \mathrm{lk}_\Delta(T)$.

By inductive hypothesis, we have that $\tilde{H}_{\ell-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(b)) \oplus \tilde{H}_{\ell-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(L)) = 0$. As we saw previously, i_b^* is surjective so that $i_b^* - k^*$ is as well. Thus we obtain $\tilde{H}_{\ell-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(I)) = 0$ from exactness. Therefore, induction gives us that
$\tilde{H}_{\ell-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T)}(S)) = \tilde{H}_{\ell-1}(\mathrm{lk}_{\tilde{\Delta}}(T)) = 0$, and thus, $\tilde{\Delta}$ satisfies $(S_{\ell+1})$.

□

Theorems 6.1.3 and 6.1.4 (1) now follow as quick consequences of Lemma 6.3.1:

Theorem 6.3.2. *Let Δ be a balanced simplicial complex. If Δ satisfies (S_ℓ), then Δ_S satisfies (S_ℓ) for any $S \subseteq [d]$.*

Proof. The claim is clear when $\ell = 1$. When $\ell \geq 2$, Δ is pure, and the result follows by applying Lemma 6.3.1 inductively on each $i \in [d] - S$.

□

Theorem 6.3.3. *If P is a finite poset satisfying (S_ℓ), then $\tilde{H}_{i-1}(\mathcal{O}(P_{>j})) = 0$ whenever $i + j < d$ and $0 \leq i < \ell$. In particular, if Δ is a simplicial complex satisfying (S_ℓ), then $\tilde{H}_{i-1}([\Delta]_{>j}) = 0$ whenever $i + j < d$ and $0 \leq i < \ell$.*

Proof. Suppose P is (S_ℓ). By Theorem 6.3.2, $\mathcal{O}(P_{>j})$ satisfies (S_ℓ) for each $0 \leq j \leq d-1$. In particular, $\tilde{H}_{i-1}(\mathcal{O}(P_{>j})) = 0$ for $i < d - j$ and $0 \leq i < \ell$. It only remains to remark that if Δ is a simplicial complex satisfying (S_ℓ), then, since $||\Delta|| \cong ||[\Delta]_{>0}||$ and since (S_ℓ) is a topological property ((Yan11, Theorem 4.4 (d))), $[\Delta]_{>0}$ satisfies (S_ℓ).

□

Remarkably, Theorem 6.3.3 admits a partial converse (Theorem 6.1.4 (2)) when P is the face poset of a simplicial complex.

Theorem 6.3.4. *If $\tilde{H}_{i-1}([\Delta]_{>j}) = 0$ whenever $i + j < d$ and $0 \leq i \leq \ell$, then Δ satisfies (S_ℓ).*

Proof. We follow a similar approach to that of Lemma 6.3.1; we induct on ℓ. The result is clear when $\ell = 1$. Suppose we know the result for ℓ and suppose $\tilde{H}_{i-1}([\Delta]_{>j}) = 0$ whenever $i + j < d$ and $0 \leq i \leq \ell + 1$. From induction hypothesis, we have that Δ satisfies (S_ℓ). Note that we assumed, in particular, that $\tilde{H}_0([\Delta]_{>j}) = 0$ whenever $j < d - 1$. Thus, no facet of Δ can have cardinality less than or equal to $d - 1$; that is, Δ is pure. Since Δ is (S_ℓ), we have $\tilde{H}_{i-1}(\mathrm{lk}_\Delta(T)) = 0$ whenever $i + |T| < d$ and $0 \leq i < \ell$, and we need only show that $\tilde{H}_{\ell-1}(\mathrm{lk}_\Delta(T)) = 0$ whenever $|T| < d - \ell$. To see this, we proceed by induction on $|T|$. When $|T| = 0$, we have $\tilde{H}_{\ell-1}(\mathrm{lk}(T)) = \tilde{H}_{\ell-1}(\Delta) = \tilde{H}_{\ell-1}([\Delta]_{>0}) = 0$. Suppose $\tilde{H}_{\ell-1}(\mathrm{lk}(T)) = 0$ whenever $j = |T| < d - \ell$, and consider $T \in \Delta$ with $j + 1 = |T| < d - \ell$.

Letting $S = \{\rho(T) \mid T \in \Delta, |T| = j + 1\}$ and writing $S = I \cup \{\rho(T)\}$, we have, by Proposition 6.2.2, the exact sequence

$$\tilde{H}_{\ell-1}([\Delta]_{>j+1}) \longrightarrow \tilde{H}_{\ell-1}(\mathrm{astar}_{[\Delta]_{>j}}(\rho(T))) \oplus \tilde{H}_{\ell-1}(\mathrm{astar}_{[\Delta]_{>j}}(I)) \longrightarrow \tilde{H}_{\ell-1}([\Delta]_{>j})$$

Since $\tilde{H}_{\ell-1}([\Delta]_{>j+1}) = 0 = \tilde{H}_{\ell-1}([\Delta]_{>j})$, we have $\tilde{H}_{\ell-1}(\mathrm{astar}_{[\Delta]_{>j}}(\rho(T))) \oplus \tilde{H}_{\ell-1}(\mathrm{astar}_{[\Delta]_{>j}}(I)) = 0$. In particular, $\tilde{H}_{\ell-1}(\mathrm{astar}_{[\Delta]_{>j}}(\rho(T))) = 0$.

As Δ is pure, T is not a facet, and so $\mathrm{lk}_\Delta(T) \neq \{\varnothing\}$ whence $\mathrm{lk}_{[\Delta]_{>j}}(\rho(T)) \neq \{\varnothing\}$. By Proposition 6.2.1, we have the exact sequence

$$\tilde{H}_{\ell}(\mathrm{astar}_{[\Delta]_{>j}}(\rho(T))) \to \tilde{H}_{\ell}([\Delta]_{>j}) \to \tilde{H}_{\ell-1}(\mathrm{lk}_{[\Delta]_{>j}}(\rho(T))) \to$$

$$\tilde{H}_{\ell-1}(\mathrm{astar}_{[\Delta]_{>j}}(\rho(T))) \to \tilde{H}_{\ell-1}([\Delta]_{>j})$$

Since $\tilde{H}_{\ell-1}(\mathrm{astar}_{[\Delta]_{>j}}(\rho(T))) = 0 = \tilde{H}_{\ell}([\Delta]_{>j})$, it follows that $\tilde{H}_{\ell-1}(\mathrm{lk}_{[\Delta]_{>j}}(\rho(T))) = 0 = \tilde{H}_{\ell-1}(\mathrm{lk}(T))$. Thus, Δ satisfies $(S_{\ell+1})$, and the result follows from induction. □

Remark 6.3.5. When $\ell = 2$, the conclusion of Theorem 6.3.3 is equivalent to $\tilde{H}_0([\Delta]_{>d-2}) = 0$, since, for a pure complex, connectivity of $[\Delta]_{>j}$ implies connectivity of $[\Delta]_{>j-1}$.

Remark 6.3.6. Since, by Theorem 1.1.14 (4), $\tilde{H}_{i-1}([\Delta]_{>j}) \cong \tilde{H}_{i-1}(N_{j+1}(\Delta))$ for any i and j, Theorems 6.3.3 and 6.3.4 also serve as a version of Theorem 1.1.14 (2) for (S_ℓ).

6.4 Depth of Rank Selected Subcomplexes

The following lemma follows from (Hib91, Proposition 2.8) and a slightly weaker version can be found in (Mun84, Theorem 6.4):

Lemma 6.4.1. *Suppose* $\operatorname{depth} k[\Delta] \geq \ell$ *and that* J *is an independent set. Set* $\tilde{\Delta} = \mathrm{astar}_\Delta(J)$. *Then* $\operatorname{depth} k[\tilde{\Delta}] \geq \ell - 1$.

We first provide a refinement of this lemma:

Lemma 6.4.2. *Let* $\operatorname{depth}\Delta = \ell$ *and suppose* $\tilde{H}_{\ell-1}(\Delta) = 0$*. Choose* $T \in \Delta$ *of minimal cardinality such that* $\tilde{H}_{\ell-|T|-1}(\mathrm{lk}_\Delta(T)) \neq 0$ *(that such a* T *exists follows from Proposition 6.2.3). Let* J *be an independent set and suppose* $T = T' \cup \{b\}$ *with* $b \in J$*. Set* $\tilde{\Delta} = \mathrm{astar}_\Delta(J)$*. Then* $\tilde{H}_{\ell-|T'|-2}(\mathrm{lk}_{\tilde{\Delta}}(T')) \neq 0$*. In particular,* $\operatorname{depth}\tilde{\Delta} = \ell - 1$*.*

Proof. If T is a facet of Δ, then we have that $|T| = \ell$, and, as $\mathrm{lk}_\Delta(T) = \mathrm{lk}_{\mathrm{lk}_\Delta(T')}(b)$, that $\{b\}$ is a facet of $\mathrm{lk}_\Delta(T')$. By our minimality hypothesis, $\tilde{H}_0(\mathrm{lk}_\Delta(T')) = 0$. It follows that $\mathrm{lk}_\Delta(T')$ is a simplex with facet $\{b\}$, and so $\mathrm{lk}_{\tilde{\Delta}}(T') = \mathrm{astar}_{\mathrm{lk}_\Delta(T')}(b) = \{\varnothing\}$. Thus T' is a facet of $\tilde{\Delta}$, and so $\tilde{H}_{\ell-1-|T'|-1}(\mathrm{lk}_{\tilde{\Delta}}(T')) = \tilde{H}_{-1}(\mathrm{lk}_{\tilde{\Delta}}(T')) \neq 0$.

Otherwise, set $S = J \cap V(\mathrm{lk}_\Delta(T'))$ and note that $\mathrm{lk}_{\tilde{\Delta}}(T') = \mathrm{astar}_{\mathrm{lk}_\Delta(T')}(S)$. Lemma 6.2.1 gives the following exact sequence

$$\tilde{H}_{\ell-|T|}(\mathrm{lk}_\Delta(T')) \to \tilde{H}_{\ell-|T|-1}(\mathrm{lk}_{\mathrm{lk}_\Delta(T')}(b)) \to \tilde{H}_{\ell-|T|-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T')}(b)) \to \tilde{H}_{\ell-|T|-1}(\mathrm{lk}_\Delta(T'))$$

By minimality of $|T|$ and Lemma 6.4.1, we have $\tilde{H}_{\ell-|T|}(\mathrm{lk}_\Delta(T')) = \tilde{H}_{\ell-|T|-1}(\mathrm{lk}_\Delta(T')) = 0$. Thus, $\tilde{H}_{\ell-|T|-1}(\mathrm{lk}_{\mathrm{lk}_\Delta(T')}(b)) \cong \tilde{H}_{\ell-|T|-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T')}(b))$. But, $\mathrm{lk}_{\mathrm{lk}_\Delta(T')}(b) = \mathrm{lk}_\Delta(T' \cup \{b\}) = \mathrm{lk}_\Delta(T)$, and so, in particular, $\tilde{H}_{\ell-|T|-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T')}(b)) \neq 0$.

But now, Lemma 5.4.3 gives that

$$\tilde{H}_{i-|T|-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T')}(S)) \cong \bigoplus_{x \in S} \tilde{H}_{i-|T|-1}(\mathrm{astar}_{\mathrm{lk}_\Delta(T')}(x)),$$

in particular, is nonzero. That $\operatorname{depth}\tilde{\Delta} = \ell - 1$ now follows from Lemma 6.4.1 and Proposition 6.2.3.

□

Proposition 6.4.3. *Let* Δ *be a balanced simplicial complex. Suppose* $\tilde{H}_{\ell-1}(\Delta) = 0$*. Then there exists an* i *such that* $\operatorname{depth}\mathrm{astar}_\Delta(V_i) = \ell - 1$*.*

Proposition 6.1.5 now follows immediately.

With these results in hand, we now prove Theorem 6.1.6:

Theorem 6.4.4. *If Δ is a balanced simplicial complex, then*

$$\operatorname{depth}\Delta = \min\{i + |S| \mid \tilde{H}_{i-1}(\tilde{\Delta}_S) \neq 0\}.$$

Proof. That

$$\operatorname{depth}\Delta \leq \min\{i + |S| \mid \tilde{H}_{i-1}(\tilde{\Delta}_S) \neq 0\}$$

follows at once from Lemma 6.4.1, so we need only concern ourselves with the reverse inequality. We proceed by induction on $\operatorname{depth}\Delta$, noting that the claim is clear when $\operatorname{depth}\Delta = 0$, that is, when $\Delta = \{\varnothing\}$. Suppose $\operatorname{depth}\Delta = \ell$. The claim is clear if $\tilde{H}_{\ell-1}(\Delta) \neq 0$, so we may suppose this is not the case. By Proposition 6.4.3, there is an i with $\operatorname{depth}\operatorname{astar}_\Delta(V_i) = \ell - 1$. From inductive hypothesis, we have $\ell - 1 = \min\{i + |S| \mid \tilde{H}_{i-1}(\operatorname{astar}_\Delta(V_i)_{[d]-S})\}$. In particular, there is an $S \subseteq [d-1]$ with $\tilde{H}_{\ell-|S|-2}(\operatorname{astar}_\Delta(V_i)) = \tilde{H}_{\ell-|S\cup\{i\}|-1}(\tilde{\Delta}_{|S\cup\{i\}|}) \neq 0$, and the result follows.

∎

Corollary 6.4.5. *Let P be a finite poset. For any $S \subseteq \{1, \ldots, \operatorname{ht} P\}$, let $\tilde{P}_S$ denote the poset obtained by restricting P to elements whose height is not in S. Then*

$$\operatorname{depth}\mathcal{O}(P) = \min\{i + |S| \mid \tilde{H}_{i-1}(\mathcal{O}(\tilde{P}_S)) \neq 0\}.$$

6.5 Euler Characteristics of Links and Truncated Posets

We now shift our attention to Theorem 6.1.7. Similar to the proof of (HN02, Lemma 1 (ii)), a simple counting argument shows

$$\sum_{T \in F_k} f_{i-1}(\mathrm{lk}_\Delta(T)) = \binom{i+k}{k} f_{i+k-1}(\Delta).$$

As in (Swa05, Proposition 2.3) one can combine this with Theorem 1.3 (3) to obtain a formula for $\sum_{\substack{T\in\Delta \\ |T|=k}} h_i(\mathrm{lk}_\Delta(T))$ in terms of Euler characteristics of higher nerves. We follow a similar approach to obtain a particularly simple formula for $\sum_{\substack{T\in\Delta \\ |T|=k}} \tilde{\chi}(\mathrm{lk}_\Delta(T))$:

Theorem 6.5.1. *Suppose Δ is pure. Then*

$$\sum_{\substack{T\in\Delta \\ |T|=k}} \tilde{\chi}(\mathrm{lk}_\Delta(T)) = \chi([\Delta]_{>k}) - \chi([\Delta]_{>k-1})$$

Proof. We make use of the following identity:

$$\sum_{i=0}^{j} (-1)^{i+1} \binom{i+k}{k} \binom{j}{i+k-1} = \begin{cases} -1 & j = k-1 \\ 1 & j = k \\ 0 & j \neq k, k-1 \end{cases}$$

Set $F_k = \{T \in \Delta, |T| = k\}$. Now,

$$\begin{aligned}
\sum_{T\in F_k} \tilde{\chi}(\mathrm{lk}_\Delta(T)) &= \sum_{i=0}^{d-k} \sum_{T\in F_k} (-1)^{i+1} f_{i-1}(\mathrm{lk}_\Delta(T)) \\
&= \sum_{i=0}^{d-k} (-1)^{i+1} \binom{i+k}{k} f_{i+k-1}(\Delta) \\
&= \sum_{i=0}^{d-k} \sum_{j=i+k-1}^{d-1} (-1)^{i+1} \binom{i+k}{k} \binom{j}{i+k-1} \chi(N_{j+1}(\Delta)) \\
&= \sum_{j=0}^{d-1} \sum_{i=0}^{j} (-1)^{i+1} \binom{i+k}{k} \binom{j}{i+k-1} \chi(N_{j+1}(\Delta)) \\
&= \chi(N_{k+1}(\Delta)) - \chi(N_k(\Delta))
\end{aligned}$$

Th result then follows from Theorem 1.1.14 (4).

□

Note that, as long as $k \neq d$, $\sum_{T \in F_k} \tilde{\chi}(\mathrm{lk}_\Delta(T)) = \chi([\Delta]_{>k}) - \chi([\Delta]_{>k-1}) = \tilde{\chi}([\Delta]_{>k}) - \tilde{\chi}([\Delta]_{>k-1})$.

Corollary 6.5.2. *Suppose* Δ *is pure. Then*

$$\sum_{k=j}^{i} \sum_{T \in F_k} \tilde{\chi}(\mathrm{lk}_\Delta(T)) = \chi([\Delta]_{>i}) - \chi([\Delta]_{>j-1}).$$

In particular,

$$\sum_{k=0}^{i} \sum_{T \in F_k} \tilde{\chi}(\mathrm{lk}_\Delta(T)) = \chi([\Delta]_{>i}).$$

As an application, we provide a result analogous to those of sections 6.3 and 6.4 for Gorenstein* complexes.

Corollary 6.5.3. *Suppose* Δ *is Gorenstein*. Then*

$$\dim_k \tilde{H}_{i-1}([\Delta]_{>j}) = \begin{cases} \dim_k \tilde{H}_{j-1}(\Delta^{(j-1)}) & \text{if } i = d - j \\ 0 & \text{if } i \neq d - j. \end{cases}$$

The converse holds if $\mathrm{lk}_\Delta(T)$ *is non-acyclic for each* $T \in \Delta$.

Proof. By Theorem 1.1.14 (4), $\tilde{H}_{i-1}([\Delta]_{>j}) \cong \tilde{H}_{i-1}(N_{j+1}(\Delta))$ for any i and j. Thus, by Theorems 1.1.14 (1) and 6.2.5, both conditions imply Δ is Cohen-Macaulay, in particular, that $\Delta^{(j-1)}$ is Cohen-Macaulay for every j ((Frö90, Theorem 8)). In this case, we have

$$\dim_k \tilde{H}_{j-1}(\Delta^{(j-1)}) = (-1)^j \tilde{\chi}(\Delta^{(j-1)}) = \sum_{k=0}^{j} (-1)^{j-k} f_{k-1}(\Delta).$$

Suppose Δ is Gorenstein*. Then, by Theorem 6.2.5

$$\tilde{H}_{i-1}(\mathrm{lk}_{\Delta}(T)) \cong \begin{cases} k & \text{if } i = d - j \\ 0 & \text{if } i \neq d - j \end{cases}$$

Likewise, since Δ is Cohen-Macaulay, we have $\tilde{H}_{i-1}(N_{j+1}(\Delta)) = 0$ unless $i = d - j$ by Theorem 1.1.14. By Corollary 6.5.2 we have

$$\sum_{k=0}^{j} \sum_{T \in F_k} \tilde{\chi}(\mathrm{lk}_{\Delta}(T)) = \sum_{k=0}^{j} \sum_{T \in F_k} (-1)^{d-k-1} = \sum_{k=0}^{j} (-1)^{d-k-1} f_{k-1}(\Delta) =$$

$$(-1)^{d-j-1} \dim_k \tilde{H}_{d-j-1}([\Delta]_{>j})$$

and the result follows.

Now suppose $\mathrm{lk}_{\Delta}(T)$ is non-acyclic for each $T \in \Delta$ and that

$$\dim_k \tilde{H}_{i-1}([\Delta]_{>j}) = \begin{cases} \sum_{k=0}^{j} (-1)^{j-k} f_{k-1}(\Delta) & \text{if } i = d - j \\ 0 & \text{if } i \neq d - j. \end{cases}$$

.

Since Δ is Cohen-Macaulay, $\tilde{H}_{i-1}(\mathrm{lk}_{\Delta}(T)) = 0$ unless $i = d - |T|$. Now we induct on $|T|$ to show that $\tilde{H}_{d-|T|-1}(\mathrm{lk}_{\Delta}(T)) \cong k$ for each T. When $T = \varnothing$ we have $\dim \tilde{H}_{d-1}(\mathrm{lk}_{\Delta} T) = \dim \tilde{H}_{d-1}(\Delta) \cong \dim \tilde{H}_{d-1}([\Delta]_{>0}) = f_{-1}(\Delta) = 1$. Now suppose $\tilde{H}_{d-|T|-1}(\mathrm{lk}_{\Delta}(T)) \cong k$ whenever $|T| < j$. Then

$$\sum_{k=0}^{j} \sum_{T \in F_k} \tilde{\chi}(\mathrm{lk}_{\Delta}(T)) = \tilde{\chi}([\Delta]_{>j}) = (-1)^{d-j-1} \dim_k \tilde{H}_{d-j-1}(N_{j+1}(\Delta)) = \sum_{k=0}^{j} (-1)^{d-k-1} f_{k-1}(\Delta)$$

Similarly,

$$\sum_{k=0}^{j-1}\sum_{T\in F_k}\tilde{\chi}(\mathrm{lk}_\Delta(T)) = \sum_{k=0}^{j-1}(-1)^{d-k-1}f_{k-1}(\Delta),$$

and thus

$$\sum_{T\in F_j}\tilde{\chi}(\mathrm{lk}_\Delta(T)) = \sum_{T\in F_j}(-1)^{d-j-1}\dim_k \tilde{H}_{d-j-1}(\mathrm{lk}_\Delta(T)) = (-1)^{d-j-1}f_{j-1}(\Delta).$$

Then

$$\sum_{T\in F_j}\dim_k \tilde{H}_{d-j-1}(\mathrm{lk}_\Delta(T)) = f_{j-1}(\Delta),$$

but, since $\mathrm{lk}_\Delta(T)$ is non-acyclic for each T, we must have $\dim_k \tilde{H}_{d-j-1}(\mathrm{lk}_\Delta(T)) = 1$ for each $T \in F_j$, by pigeonhole. The result now follows from induction.

□

Remark 6.5.4. We claim the result of Corollary 6.5.3 is analogous to those of Sections 6.3 and 6.4, but this is perhaps not obvious. To see this, note that $\dim_k \tilde{H}_{i-1}(\Delta^{(j-1)}) = \dim_k \tilde{H}_{i-1}(P^{-1}_{>d-j})$ where P is the face poset of Δ (excluding $\varnothing$). In essence, our result says that, when Δ is Gorenstein*, removing j ranks from the bottom of P gives the same homologies as removing $d-j$ ranks from the top, though they are in different degrees.

6.6 Open Problems and Examples

We

say that $A \subseteq \Delta$ is independent if $\sigma\cup\tau \notin \Delta$ for all $\sigma,\tau \in A$ with $\sigma \neq \tau$. We say that A is excellent if, additionally, for every facet F of Δ, $F \supseteq \sigma$ for some (necessarily unique) $\sigma \in A$. Note that $J = \{v_1,\ldots,v_m\} \subseteq V$ is independent (resp. excellent) if and only if $\{\{v_1\},\ldots,\{v_m\}\}$ is an independent (resp. excellent) subset of Δ. If $A \subseteq \Delta$ is independent, we set

$$\Delta_A := \Delta - \{\sigma \in \Delta \mid \sigma \supseteq \tau \text{ for some } \tau \in A\}.$$

If $A = \{\{v_1\}, \ldots, \{v_m\}\}$ where $J = \{v_1, \ldots, v_m\} \subseteq V$ is independent, then $\Delta_A = \operatorname{astar}_\Delta(J)$. Essentially the same argument as (Hib91, Proposition 2.8) shows the following extension of Lemma 6.4.2:

Proposition 6.6.1. *Suppose $A \subseteq \Delta$ is independent. Then* $\operatorname{depth} k[\Delta] \geq \ell$ *implies* $\operatorname{depth} k[\Delta_A] \geq \ell - 1$.

We conjecture a similar extension of Lemma 6.3.1.

Conjecture 6.6.2. *Suppose $A \subseteq \Delta$ is excellent. If Δ satisfies (S_ℓ), then Δ_A satisfies (S_ℓ).*

Remark 6.6.3. If A is independent and $\ell \geq 2$ the conclusion can only hold if A is excellent, since (S_2) complexes are pure. Similar to Proposition 6.6.1, one can modify the argument of (Hib91, Proposition 2.8) to show that Δ_A satisfies $(S_{\ell-1})$ whenever Δ satisfies (S_ℓ) and A is excellent. However, as in the proof of Theorem 6.3.3, one often needs to cut away excellent subsets inductively, and, for this purpose, $(S_{\ell-1})$ is not generally good enough. A positive answer to this conjecture would allow one to extend Theorem 6.1.3 to balanced complexes of a more general type, along the lines of (Hib91, Section 3).

The following examples show that converses of Theorems 6.3.3 and 6.3.4 do not hold, even for face posets of simplicial complexes:

Example 6.6.4. Consider the complex with facets:

$$\{4,5,6\}, \{1,5,6\}, \{1,3,5\}, \{2,3,6\}, \{2,5,6\}, \{2,4,6\}.$$

This complex is not (S_2) and has $\tilde{H}_{i-1}([\Delta]_{>j}) = 0$ for all i, j with $i + j < d$ and $0 \leq i < 2$.

Example 6.6.5. Consider the complex with facets:

$$\{4,5,6\},\{3,5,6\},\{2,3,5\},\{2,3,4\},\{1,3,4\},\{2,4,6\}.$$

This complex is (S_2) but has non-trivial $\tilde{H}_1([\Delta]_{>0})$.

In fact, $\tilde{H}_{i-1}([\Delta_1]_{>j}) \cong \tilde{H}_{i-1}([\Delta_2]_{>j})$ for every i and every j. Since Example 6.6.5 is (S_2) and Example 6.6.4 is not, this shows that (S_2) cannot be determined in general by reduced homologies of the $[\Delta]_{>j}$. Further, Example 6.6.5 is Buchsbaum while Example 6.6.4 is not, so Buchsbaum cannot be determined either. In a similar fashion, the following example shows that Gorenstein cannot be detected in general.

Example 6.6.6. Let Δ_1 be the complex with facets

$$\{2,3,4\},\{1,3,4\},\{1,2,5\},\{2,3,5\},\{1,2,4\},\{1,3,5\}$$

and Δ_2 the complex with facets

$$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\}.$$

Then $[\Delta_1]_{>j}$ and $[\Delta_2]_{>j}$ have isomorphic homologies for each j, but Δ_1 is Gorenstein whilst Δ_2 is not (it is not even 2-Cohen-Macaulay).

The above discussion leads us to ask the following general question:

Question 1. *In addition to the reduced homologies of the $[\Delta]_{>j}$, what information does one need to determine if a simplicial complex satisfies homological conditions such as (S_ℓ), Buchsbaum, or Gorenstein?*

Appendix A

Appendix of Notation

We list some notation which is consistently used throughout this document.

1. R is a commutative Noetherian ring
2. $\mu(d, n)$ is the largest diameter of a Hochster-Huneke graph of an (S_2) Stanley-Reisner ring of dimension d and codimension $n - d$.
3. $\mathcal{G}(R)$ is the Hochster-Huneke graph of R.
4. $\mathcal{G}^j(R)$ is the generalized Hochster-Huneke graph of R.
5. $V(G)$ is the vertex set of the graph G.
6. $E(G)$ is the edge set of the graph G.
7. $\Delta^\vee$ is the Alexander dual of Δ.
8. $I^\vee$ is the Alexander dual of I.
9. $\bar{\mathcal{G}}(R)$ is the Hochster-Huneke graph of R relabeled so that its vertices are the complements of $V(\mathcal{G})$.

10. $\operatorname{st}_\Delta T$ is the star of the face T over the complex Δ. It is the set $\{G \in \Delta \mid T \cup G \in \Delta\}$

11. $\operatorname{astar}_\Delta T$ is the antistar of the face T over the complex Δ. It is the set $\{G \in \Delta \mid T \cap G = \varnothing\} = \Delta|_{V-T}$

12. $\operatorname{lk}_\Delta T$ is the link of the face T over the complex Δ. It is the set $\{G \in \Delta \mid T \cup G \in \Delta \text{ and } T \cap G = \varnothing\} = \operatorname{st}_\Delta T \cap \operatorname{astar}_\Delta T$

13. $\mathcal{O}(P)$ is the order complex of the poset P.

14. $\mathrm{N}_i(\Delta)$ is the i^{th} nerve of Δ.

15. $\chi(\Delta)$ represents the Euler Characteristic of Δ.

16. $\mathcal{F}_{>k}(\Delta)$ is the face poset of Δ restricted to faces of Δ with cardinality strictly greater than k.

17. $\operatorname{sd}\Delta := \mathcal{O}(\mathcal{F}_{>0}(\Delta))$ is the barycentric subdivision of Δ.

18. $[\Delta]_{>k} = \mathcal{O}(\mathcal{F}_{>k}(\Delta))$

19. $\tilde{\chi}(\Delta)$ represents the reduced Euler Characteristic of Δ.

20. Δ_S is the subcomplex of Δ induced on $\bigcup_{i\in S} V_i$ where the V_i represent the ordered partition of a balanced simplicial complex.

21. $\tilde{\Delta}_S := \Delta_{[d]-S}$.

Unless otherwise stated, all rings are assumed to be commutative and Noetherian, and all modules will be finitely generated.

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