Decompositions of Simplicial Complexes

By

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Abstract

In this thesis we study the interplay between various combinatorial, algebraic, and topological properties of simplicial complexes. We focus on when these properties imply the existence of decompositions of the face poset. In Chapter 2, we present the counterexample to Stanley's partitionability conjecture that appeared in [DGKM16], we give a characterization of the h-vectors of Cohen–Macaulay relative complexes, and we construct a family of disconnected partitionable complexes.

In Chapter 3, we introduce colorated cohomology, which aims to combine the theories of color shifting and iterated homology. Colorated cohomology gives rise to certain decompositions of balanced complexes that preserve the balanced structure. We give conditions that would guarantee the existence of a weaker form of Stanley's partitionability conjecture for balanced Cohen–Macaulay complexes.

We consider Stanley's conjecture on k-fold acyclic complexes in Chapter 4, and we show that a relaxation of this conjecture holds in general. We also show that the conjecture holds in the case when k is the dimension of a given complex, and we present a framework that may lead to a counterexample to the original version of this conjecture.

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This thesis is the culmination of my time as student and represents years of work. Though my name is the only one appearing on the byline of this document, a seemingly uncountable number of people have contributed to this work in some way or another. I am unable to adequately thank everyone who has helped me throughout this whole ordeal, but I will try to do so anyway.

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There is one name that has been conspicuously absent from this list so far, and that is my advisor, Jeremy Martin. Jeremy fits in to all of the above categories (please go back and reread the above, occasionally adding his name), and I am unable to fully express my gratitude for all he's done for me.

Lastly, I want to thank my fiancée Hilary and our cat Napoleon. Every day I look forward to spending as much time as possible with you guys. If I knew how to formally dedicate a thesis, I'd definitely dedicate it to you two.

Contents

Introduction

Simplicial complexes arise naturally in many fields of mathematics. In topology, when considering a topological space X, it is often easier to find a simplicial complex Δ whose geometric realization is homeomorphic to X and perform computations with it, rather than working with X directly. In algebra, simplicial complexes correspond precisely to square-free monomial ideals of polynomial rings, which have proved useful in studying homogeneous ideals in general. In combinatorics, simplicial complexes occupy a central place in the literature—arising as higher dimensional analogues of simple graphs and as order complexes in the study of posets—and have proved worthy of study in their own right.

The central theme of this thesis is the interplay between topological, algebraic, and combinatorial aspects of simplicial complexes. Work in this direction traces back to at least the 1970s, when Richard Stanley, Melvin Hochster, and Gerald Reisner laid the groundwork of Stanley-Reisner theory and combinatorial commutative algebra; see, for example, [Hoc72], [Rei76], [Hoc77], and [Sta77]. In 1975, Stanley used the theory of Cohen–Macaulay rings to prove the Upper Bound Conjecture for simplicial spheres in [Sta75], solidifying the idea that complexes with Cohen–Macaulay face rings were a particularly important class of simplicial complexes. In the years since, combinatorial commutative algebra has developed into a major modern field; see, for example, [BH93], [Sta96], [MS05], and [FMS14].

We will be particularly interested in how various combinatorial, algebraic, and topological properties imply the existence of certain decompositions of the face poset of a simplicial complex. In 1979, Stanley conjectured in [Sta79] that any simplicial complex with a Cohen–Macaulay face ring could be written as the disjoint union of boolean intervals whose maximal elements were themselves maximal faces of the simplicial complex. Such a decomposition is known as a *partitioning*, and complexes that admit partitionings are said to be *partitionable*. This conjecture was recently shown to be false by Art Duval, Caroline Klivans, Jeremy Martin, and the author in [DGKM16], but related decomposition conjectures remain open. In 1980, Garsia conjectured in [Gar80] that Cohen–Macaulay complexes arising as order complexes of posets were partitionable. In 1993, Stanley conjectured in [Sta93] that complexes with well-behaved links have decompositions into boolean intervals of appropriate sizes. The majority of this thesis is work motivated by these conjectures.

Chapter 1 presents the fundamentals of simplicial complexes and Stanley-Reisner theory. We introduce many well-studied properties of simplicial complexes and survey known results relating to these properties. We also summarize several common techniques used in the study of simplicial complexes, such as Hochster's Theorem, algebraic shifting, and iterated homology.

Chapter 2 covers work arising from the aforementioned conjectures and includes overviews of joint projects with connections to the rest of the thesis. In particular, we present brief sections on the counterexample to Stanley's partitionability conjecture (based on joint work with Art Duval, Caroline Klivans, and Jeremy Martin) and partition extenders (based on joint work with Joseph Doolittle and Alexander Lazar). We also include some additional work on partitionability and relative simplicial complexes.

Chapter 3 focuses on decompositions of balanced simplicial complexes. We introduce the notion of *colorated cohomology*, which aims to combine the theories of color shifting, introduced by Eric Babson and Isabella Novik in [BN06], and iterated homology, introduced by Art Duval and Lauren Rose in [DR00]. Art Duval and Ping Zhang showed in [DZ01] that iterated homology provides a way to decompose simplicial complexes into boolean trees, a weaker analogue of boolean intervals. We show that colorated cohomology gives rise to boolean trees that preserve a balanced structure, and we present open questions on decompositions of balanced Cohen–Macaulay complexes.

In Chapter 4, we investigate Stanley's conjecture on decompositions of k-fold acyclic complexes. We show that k-fold acyclicity imposes additional structure on the face ring and show that the conjecture holds in the case when $k = \dim \Delta$. We also show that a relaxation of this conjecture holds for all k, and we present a general construction that may provide a counterexample to Stanley's original conjecture.

Chapter 1

Background

1.1 Basic definitions

Simplicial Complexes

The main objects we will study in this thesis are simplicial complexes, which can be thought of as higher dimensional analogues of simple graphs from graph theory. The following chapter presents background material; references throughout are [BH93], [Mar17], [Sta96], and [Sta12].

Definition 1.1.1. Given a set $[n] = \{1, 2, \ldots, n\}$, an **abstract simplicial complex** on $[n]$ is $\Delta \subseteq 2^{[n]}$ such that $\sigma \in \Delta$ and $\tau \subseteq \sigma \implies \tau \in \Delta$. In other words, Δ is closed under taking subsets.

If $\sigma \in \Delta$, we call σ a **face** of Δ. Maximal faces are called **facets**. If $F_1, \ldots, F_k$ are the facets of Δ, we often write $\Delta = \langle F_1, \ldots, F_k \rangle$, that is, Δ is the complex generated by the facets $F_1, \ldots, F_k$. The dimension of a face σ is defined as $\dim \sigma = |\sigma| - 1$, and the dimension of Δ is $\dim \Delta = \max\{\dim \sigma \;:\; \sigma \in \Delta\}$. A simplicial complex is said to be **pure** if all its facets have the same dimension. We call 0-dimensional faces **vertices**, 1-dimensional faces **edges**, 2-dimensional faces **triangles**, and so on. If there exists a facet F such that $\sigma \subseteq F$ and $\dim \sigma = \dim F - 1$, then σ is a **ridge**; if Δ is pure then a ridge is just a face of codimension 1. If $\dim \Delta \leq 1$, then Δ is a simple graph in the graph-theoretic sense.

Note in particular that $\varnothing \in \Delta$ for all simplicial complexes except for the complex $\Delta = \varnothing$, which is called the **void complex**. We will often want to distinguish between the void complex and the complex whose only face is the empty set, that is $\Delta = \{\varnothing\}$, which is known as the **irrelevant complex**. Note that $\dim \{\varnothing\} = -1$; we will set $\dim \varnothing = -\infty$. Unless otherwise specified, we will adopt the convention throughout that $\dim \Delta = d - 1$, thus each facet has at most d vertices.

The f**-vector** of a simplicial complex Δ is $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$ where $f_i = |\{\sigma \in \Delta \,:\, \dim \sigma = i\}|$, the number of i-dimensional faces of Δ. The h**-vector** is $h(\Delta) = (h_0, h_1, \ldots, h_d)$, where the entries are defined by

$$h_k = \sum_{i=0}^{k} (-1)^{i-k} \binom{d-i}{k-i} f_{i-1}. \tag{1.1.1}$$

In other words, the h-vector can be obtained from an invertible linear transformation of the f-vector. Thus the f-vector of a complex completely determines its h-vector and vice versa.

We may also consider polynomials that encode the same information. We define the f**-polynomial** of Δ to be

$$f(\Delta, t) = \sum_{\sigma \in \Delta} t^{|\sigma|} = f_{-1} + f_0 t + f_1 t^2 + \cdots + f_{d-1} t^d \tag{1.1.2}$$

We may also define the h**-polynomial** as

$$h(\Delta, t) = (1-t)^d f\left(\frac{t}{1-t}\right) = h_0 + h_1 t + \cdots + h_d t^d \tag{1.1.3}$$

with the second equality given by (1.1.1). We note that f_{i-1} is the coefficient of t^i in the f-polynomial, but h_i is the coefficient of t^i in the h-polynomial.

If Γ is a simplicial complex and every face of Γ is a face of Δ, then we say that Γ is a **subcomplex** of Δ, denoted $\Gamma \subseteq \Delta$. A subcomplex Γ is **proper** if there exists some $\sigma \in \Delta$ such that $\sigma \notin \Gamma$. Given $W \subseteq [n]$, we can define the **induced subcomplex** of Δ on W as $\Delta_W = \{\sigma \in \Delta \,:\, \sigma \subseteq W\}$.

Given a face $\sigma \in \Delta$, its **link** (with respect to Δ) is

$$\operatorname{link}_\Delta \sigma = \{\tau \in \Delta \,:\, \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \varnothing\}$$

which we will often denote simply as $\operatorname{link} \sigma$. The **star** of σ is $\operatorname{star}_\Delta \sigma = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta\}$. It is easy to see that $\operatorname{link}_\Delta \sigma \subseteq \operatorname{star}_\Delta \sigma \subseteq \Delta$ as subcomplexes. Another subcomplex of note is the i**-skeleton** of Δ, defined as $\operatorname{skel}_i \Delta = \{\sigma \in \Delta \,:\, \dim \sigma \leq i\}$.

If Δ and Γ are disjoint simplicial complexes, their **join** is the complex $\Delta \star \Gamma = \{\sigma \cup \tau \,:\, \sigma \in \Delta \text{ and } \tau \in \Gamma\}$. If Δ has a single facet σ (i.e. Δ is a simplex), then we often write the join as $\sigma \star \Gamma$. If σ is a vertex, we say $\sigma \star \Gamma$ is the **cone** of Γ; if $|\sigma| = k$ we say that $\sigma \star \Gamma$ is the k**-fold cone** of Γ. We note that $\operatorname{star} \sigma = \sigma \star \operatorname{link} \sigma$.

We now turn our attention to the geometric realizations of simplicial complexes. A set $C \subseteq \mathbb{R}^n$ is **convex** if $a, b \in C$ implies that $at + (1-t)b \in C$ for all $t \in [0,1]$, i.e. the line segment from a to b is contained entirely within C. Given a set $S \subseteq \mathbb{R}^n$, the **convex hull** of S, denoted $\operatorname{conv}(S)$, is defined to be the smallest convex set containing S. Given a simplicial complex Δ, we denote its **geometric realization** as $|\Delta|$. If $\Delta \subseteq 2^{[n]}$, then $|\Delta|$ lives naturally in $\mathbb{R}^{n-1}$: To create $|\Delta|$, place the vertices of Δ in general position. Then we may define $|\Delta| = \bigcup_{\sigma \in \Delta} \operatorname{conv}(\sigma)$. If X is a topological space, then we say that Δ is a **triangulation** of X if $|\Delta|$ is homeomorphic to X. Where there is no possibility of confusion, we will often refer to an abstract simplicial complex Δ and its geometric realization $|\Delta|$ as the same object.

Posets

Definition 1.1.2. A **poset** $(P, \leq)$ is a set P together with a relation $\leq$ that satisfies the following three properties for all $x, y, z \in P$:

1. $x \leq x$.

2. If $x \leq y$ and $y \leq x$, then $x = y$.

3. If $x \leq y$ and $y \leq z$, then $x \leq z$.

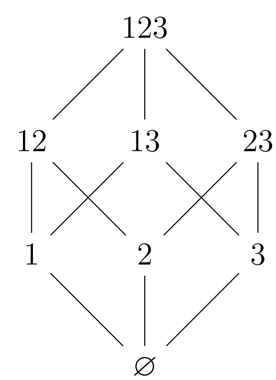
We will occasionally use the notation $x < y$ to mean that $x \leq y$ and $x \neq y$, and we will often shorten the notation of a poset to P when the relation is unambiguous. If $x < y$ and there is no $z \in P$ such that $x < z < y$, then we say that y **covers** x, often denoted $x \lessdot y$. We say that x and y are **incomparable** if neither $x \leq y$ nor $y \leq x$. Perhaps the most natural example of a poset is the **Boolean poset** $(2^{[n]}, \subseteq)$. For more information on posets, see for example [Sta12] for a general introduction; we note here only a few important properties of these objects. A poset is **bounded** if it has a unique minimal element $\hat{\mathbf{0}}$ and a unique maximal element $\hat{\mathbf{1}}$, i.e. for all $x \in P$, $\hat{\mathbf{0}} \leq x \leq \hat{\mathbf{1}}$. In our example $(2^{[n]}, \subseteq)$, we see that $\hat{\mathbf{0}} = \varnothing$ and $\hat{\mathbf{1}} = [n]$.

A finite poset is said to be **ranked** if there is a rank function $r : P \to \mathbb{N}$ such that if $x \lessdot y$, then $r(x) = r(y) - 1$. In our running example, if $S \in 2^{[n]}$ then $r(S) = |S|$, and $S \lessdot T$ if $S \subseteq T$ and $|T \setminus S| = 1$. Some sources use the term **graded** instead of ranked, but for our purposes these terms will be interchangeable.

Given a poset $(P, \leq_P)$, we say that $(Q, \leq_Q)$ is a **subposet** of P if $Q \subseteq P$ and Q inherits its order relation from P. An **order ideal** of P is a subposet Q with the property that $x \in Q$ implies $y \in Q$ for all $y \leq x$. Similarly, an **order filter** is a subposet Q with the property that $x \in Q$ implies $y \in Q$ for all $x \leq y$. A **chain** C is a subposet of P such that for all $x, y \in C$, either $x \leq y$ or $y \leq x$ (i.e., C is totally ordered). An **antichain** is a subset of P with the property that all of its elements are incomparable. Given $a, b \in P$, the (closed) **interval** $[a,b]$ is defined as $[a,b] = \{c \in P \,:\, a \leq c \leq b\}$. If $[a,b]$ is isomorphic to $(2^{[k]}, \subseteq)$ for some $k \in \mathbb{Z}_{\geq 0}$, then $[a,b]$ is a **boolean interval** of rank k.

Returning to simplicial complexes, we can alternatively think of a simplicial complex on the vertex set $[n]$ as an order ideal in the boolean lattice $(2^{[n]}, \subseteq)$. In particular, given a simplicial complex Δ, we can consider the poset $P(\Delta) = (\Delta, \subseteq)$, known as the **face poset** of Δ. In this thesis, we will primarily be considering decompositions of the face poset of a simplicial complex into boolean intervals.

We can represent a poset P using its **Hasse diagram**, the graph with a vertex for each element of P and an edge between x and y if $x \lessdot y$, oriented so that x is below y on the page. For example, here is the Hasse diagram for $(2^{[3]}, \subseteq)$, which is also the face poset of the simplicial complex $\Delta = \langle 123 \rangle$ (abbreviating $\{1,2,3\}$ as 123, etc.).



Given a poset P, we define its **order complex** $\mathcal{O}(P)$ to be

$$\mathcal{O}(P) = \{\sigma \,:\, \sigma = \{x_1 \leq \cdots \leq x_\ell\} \text{ where } x_i \in P\}$$

In other words, $\mathcal{O}(P)$ is the simplicial complex whose faces are chains of P. Given a simplicial complex, the **barycentric subdivision** of Δ is defined as $\mathrm{sd}(\Delta) = \mathcal{O}(P(\Delta \setminus \{\varnothing\}))$. It is known that $\mathrm{sd}(\Delta)$ is homeomorphic to Δ.

1.2 Simplicial homology

For most topological notions, we direct the reader to [Hat02] for general reference, but we will briefly overview the idea of *simplicial homology* and highlight a few particularly useful topological tools that will be used throughout this thesis.

As throughout the rest of this document, we will fix a field $\Bbbk$ over which we will perform all calculations. Let Δ be a $(d-1)$-dimensional simplicial complex Δ. For $-1 \leq k \leq d-1$, define $C_k(\Delta)$ to be the k^{th} **simplicial chain group**, the formal sum of k-simplices (i.e. k-faces) of Δ with coefficients in $\Bbbk$. Given a k-simplex $\sigma = [v_0, \dots, v_k]$ with $v_0 \leq \cdots \leq v_k$ (for some fixed order on the n vertices of Δ), then we define the **simplicial boundary map** $\partial_k : C_k \to C_{k-1}$ as

$$\partial_k[\sigma] = \sum_{i \in [n]} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k]$$

where the hat denotes removal, and extend this linearly to C_k. We then form the **simplicial chain complex**

$$0 \to C_{d-1}(\Delta) \xrightarrow{\partial_{d-1}} C_{d-2}(\Delta) \xrightarrow{\partial_{d-2}} \dots \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) \to 0. \tag{1.2.1}$$

It is a standard exercise that $\partial_k \circ \partial_{k+1} = 0$, i.e. $\operatorname{im} \partial_{k+1} \subseteq \ker \partial_k$ and thus these groups and the maps ∂ together form what is known as a *chain complex*.

We then define the k^{th} **reduced simplicial homology group** of Δ over $\Bbbk$ as

$$\tilde{H}_k(\Delta) = \tilde{H}_k(\Delta;\Bbbk) = \ker \partial_k / \operatorname{im} \partial_{k+1}.$$

These homology groups are important topological invariants that only depend on the homotopy type of the geometric realization $|\Delta|$. The k^{th} homology group can be thought of as counting the k-dimensional "holes" of $|\Delta|$. Considering these groups as vector spaces over $\Bbbk$, we say that the k^{th} **(reduced) Betti number** of Δ is $\tilde{\beta}_k = \dim_{\Bbbk} \tilde{H}_k(\Delta)$, the vector space dimension of $\tilde{H}_k(\Delta)$. For example, $\tilde{\beta}_0$ is one less than the number of connected components of Δ. We say that Δ is **acyclic (over** $\Bbbk$**)** if all of its reduced homology groups vanish.

The following is a well-known fact that follows from basic linear algebra.

Proposition 1.2.1 (Euler-Poincaré Formula)**.** Let Δ be a simplicial complex, $f(\Delta)$ be its f-vector, and $\tilde{\beta}_k$ be the reduced Betti numbers of Δ. Then

$$\sum_{i \geq -1} (-1)^i f_i = \sum_{k \geq 0} \tilde{\beta}_k$$

regardless of the field $\Bbbk$ chosen for homology calculations.

One corollary of Proposition 1.2.1 is that if Δ is acyclic over *some* field, then its f-polynomial factors as $f(\Delta,t) = (1+t)g(t)$ where $g(t)$ is some other polynomial. In [Sta93], Stanley showed, via an explicit construction, that this $g(t)$ is in fact the f-polynomial of some subcomplex $\Delta' \subseteq \Delta$. Chapter 4 is focused on extending this result to a more general version of acyclicity.

One topological tool that will appear often in this thesis is the Mayer-Vietoris sequence, which relates the homology of a complex to the homology of two subcomplexes and their intersection. While this holds for spaces in more generality, we will state it only in the form that we need for simplicial complexes. We first define a key term: If a chain complex such as (1.2.1) above has the property that $\operatorname{im}\partial_{k+1} = \ker\partial_k$ for each k, then it is said to be an **exact sequence**.

Theorem 1.2.2 (Mayer-Vietoris Sequence)**.** [Hat02, p. 149]

Let Δ, Γ be nonempty simplicial complexes. Then the chain complex

$$\cdots \to \tilde{H}_{i+1}(\Delta \cup \Gamma) \to \tilde{H}_i(\Delta \cap \Gamma) \to \tilde{H}_i(\Delta) \oplus \tilde{H}_i(\Gamma) \to \tilde{H}_i(\Delta \cup \Gamma) \to \dots$$

is an exact sequence.

We will often be able to decompose a complex Δ into subcomplexes Δ_1 and Δ_2 such that the homology groups of Δ_1, Δ_2, and $\Delta_1 \cap \Delta_2$ are easy to describe, and the Mayer-Vietoris sequence will then allow us to compute the homology of Δ.

Given simplicial complexes $\Gamma \subseteq \Delta$, we can form the chain complexes for Γ and Δ as in 1.2.1, and then define the **relative chain groups** $C_k(\Delta, \Gamma)$ as

$$C_k(\Delta, \Gamma) = \frac{C_k(\Delta)}{C_k(\Gamma)}.$$

The chain maps $\partial : C_k(\Delta) \to C_{k-1}(\Delta)$ take k-chains in Γ to $(k-1)$-chains in Γ, so the quotient map $\overline{\partial} : C_k(\Delta, \Gamma) \to C_{k-1}(\Delta, \Gamma)$ is well-defined and a chain map. Thus we get the chain complex

$$0 \to C_{d-1}(\Delta, \Gamma) \xrightarrow{\overline{\partial}_{d-1}} C_{d-2}(\Delta, \Gamma) \xrightarrow{\overline{\partial}_{d-2}} \dots \xrightarrow{\overline{\partial}_1} C_0(\Delta, \Gamma) \xrightarrow{\overline{\partial}_0} C_{-1}(\Delta, \Gamma) \to 0 \quad (1.2.2)$$

from which we define the **(reduced) relative simplicial homology groups** $\tilde{H}_k(\Delta, \Gamma) = \ker\overline{\partial}_k / \operatorname{im}\overline{\partial}_{k+1}$.

A frequent tool in relative homology will be the following long exact sequence in relative homology.

Proposition 1.2.3 (Long Exact Sequence in Relative Homology)**.** [Hat02, p. 115]

Let $\Gamma \subseteq \Delta$ be simplicial complexes. Then

$$\cdots \to \tilde{H}_{i+1}(\Delta,\Gamma) \to \tilde{H}_i(\Gamma) \to \tilde{H}_i(\Delta) \to \tilde{H}_i(\Delta,\Gamma) \to \dots$$

is an exact sequence.

It is worth noting the similarity between Proposition 1.2.3 and the Mayer-Vietoris sequence in Theorem 1.2.2. We will often strive to reduce questions about simplicial complexes to questions about the pair (Δ,Γ), which is known as a **relative complex** and will be discussed in more detail in Section 1.5.

There is a dual notion of simplicial homology called **simplicial cohomology**. For our purposes, it is formed using the same simplicial chain groups as for simplicial homology. Thinking of the original simplicial boundary maps $\partial_k : C_k \to C_{k-1}$ as matrices, we take the dual maps $\partial_k^* : C_{k-1} \to C_k$ to be the transpose of these matrices. Then the k^{th} **simplicial cohomology group of** Δ is

$$\tilde{H}^k(\Delta) = \ker \partial_{k+1}^* / \partial_k^*.$$

Due to the *universal coefficient theorem*, since we are computing these groups over a field $\Bbbk$, it turns out that

$$\tilde{H}_k(\Delta) \cong \tilde{H}^k(\Delta)$$

for all k. However, it will sometimes be more useful to use a cohomological approach; see, for example, Sections 1.6, 1.7, and 3.3.

1.3 Shellability, constructibility, and partitionability

Shellability, constructibility, and partitionability are three properties of simplicial complexes that can be defined entirely combinatorially. Shellability has been studied extensively, while the other two are less well known. In this thesis we will focus primarily on partitionability and various related decompositions of the face poset of a simplicial complex, though shellability in particular will appear frequently, since any shellable complex is also partitionable.

Definition 1.3.1. A simplicial complex Δ is said to be **shellable** if its facets can be ordered $F_1, \ldots, F_k$ such that either of the following equivalent statements is true:

1. $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ is pure of dimension $\dim(F_i) - 1$ for $2 \leq i \leq k$.

2. $\langle F_i \rangle \setminus \langle F_1, \ldots, F_{i-1} \rangle$ has a unique minimal face (often denoted R_i and called a **restriction face**) for $2 \leq i \leq k$.

This order of the facets is known as a **shelling order** (or simply **shelling**) of Δ. It is an easy exercise to show that the two above criteria are equivalent. Shellability was originally only considered for pure complexes but was extended to nonpure complexes by Björner and Wachs in [BW96] and [BW97]. The definitions given above do not require that Δ be pure, and we assume throughout that shellability does not assume purity unless otherwise noted.

If Δ has k facets and $j < k$, we say that a **partial shelling** of Δ is an order $F_1, \ldots, F_j$ on j of the facets that meets the criteria of Definition 1.3.1. If *every* partial shelling of Δ can be extended to a (complete) shelling of Δ, then Δ is said to be **extendably shellable**. In this case, for each facet F of Δ, there is some shelling in which F is the first facet in the shelling order. It is computationally difficult to check in general whether a complex Δ is shellable, but if Δ is extendably shellable, then a greedy algorithm will suffice.

Proposition 1.3.2. [BW96, Theorem 4.1] If Δ is shellable, then Δ is homotopy equivalent to a wedge of spheres which are indexed by the h-triangle of Δ (a nonpure analogue of the h-vector). In particular, if Δ is pure and shellable of dimension $d-1$, then Δ is homotopy equivalent to a wedge of $(d-1)$-dimensional spheres.

Furthermore, the restriction faces of a shellable complex Δ that are themselves facets give bases of the homology groups of Δ [BW96, Corollary 4.4]. Proposition 1.3.2 also implies that if Δ is pure and shellable, then Δ only has top-dimensional homology.

Constructibility is a generalization of pure shellability and was introduced by Hochster in [Hoc72]. While in a shelling, we only add one facet at the time, constructibility extends this notion to adding a subcomplex that itself is constructible.

Definition 1.3.3. A pure $(d-1)$-dimensional complex Δ is **constructible** if one of the following is true:

1. Δ is a simplex.

2. $\Delta = \Delta_1 \cup \Delta_2$ where Δ_1, Δ_2 are $(d-1)$-dimensional constructible complexes and $\Delta_1 \cap \Delta_2$ is a $(d-2)$-dimensional constructible complex.

Shellable complexes are easily seen to be constructible (let Δ_2 be a single facet each time in the definition above to recover a shelling), but the converse is not true in general. However, in low enough dimension, these notions are equivalent. If $\dim \Delta = 0$, then Δ is vacuously both shellable and constructible.

Proposition 1.3.4. If Δ is pure and $\dim \Delta = 1$, then the following are equivalent:

1. Δ is shellable.

2. Δ is constructible.

3. Δ is connected.

However, when $\dim \Delta = 2$ there are already many known examples of complexes that are constructible but not shellable. A particularly good reference on constructible complexes—which contains several such examples—is Masahiro Hachimori's PhD thesis [Hac00a]. Another resource that lists several interesting examples is Hachimori's "Simplicial Complex Library" [Hac01].

The decomposition property that we will focus on in this thesis is **partitionability**, which is even less well-studied than constructibility. Partitionability was first introduced and initially studied in the context of probability and operations research in [Bal77], [Pro77], [BN79], and [BP82].

Definition 1.3.5. A simplicial complex Δ with facets $F_1, \ldots, F_k$ is **partitionable** if it can be written as

$$\Delta = \bigsqcup_{i \in [k]} [R_i, F_i]$$

the disjoint union of boolean intervals whose maximal elements are facets of Δ. Such a decomposition is known as a **partitioning**, and the R_i are the **restriction faces** of the partitioning.

It is immediate that shellability implies partitionability; a shelling in the sense of Definition 1.3.1 (2) is a partitioning. However, there is no order imposed on the facets in a partitioning, and shellability is in fact a much stronger condition, even in dimension 1. For example, consider the following:

Example 1.3.6. The complex $\Delta = \langle 12, 23, 13, 45 \rangle$ is a disconnected graph, but it can be partitioned $\Delta = [1, 12] \sqcup [2, 23] \sqcup [3, 13] \sqcup [\varnothing, 45]$.

In Section 2.4, we will provide some additional examples of partitionable complexes that are not shellable.

For many years it was not known whether constructibility implied partitionability. However, due to recent work of Art Duval, Caroline Klivans, Jeremy Martin, and the author [DGKM16], it is now known that there exist constructible complexes that are not partitionable. An explicit example is discussed in Section 2.1.

A strong motivation to study partitionability of simplicial complexes is the following result, which interprets the h-vector of a *pure* simplicial complex in terms of a partitioning (if one exists).

Proposition 1.3.7. [Sta96, Proposition III.2.3] Let Δ be a pure simplicial complex with a partitioning $\bigsqcup[R_i, F_i]$, and let $h(\Delta) = (h_0, h_1, \ldots, h_d)$ be the h-vector of Δ. Then

$$h_j = |\{R_i \ : \ |R_i| = j\}|\,.$$

In other words, the h_j count the number of restriction faces of size j in *any* partitioning of Δ.

As for many results that we will state for pure complexes, Björner and Wachs have proved a similar statement in [BW96] for nonpure complexes and the h-triangle.

As we will note in Section 2.1, many classes of simplicial complexes (including shellable and constructible complexes) have particularly well-behaved h-vectors, raising the question of whether these complexes are partitionable. In Section 1.7, we will also see that Proposition 1.3.7 has an extension to a weaker type of decomposition, so we may ask similar questions even for non-partitionable complexes.

If the h-vector of a simplicial complex Δ has any negative entries, then Proposition 1.3.7 immediately implies that Δ is not partitionable. However, the converse is not true (see, for example, Proposition 2.1.8). To the best of our knowledge, it is unknown whether partitionability places any additional restrictions on the h-vector other than non-negativity.

For non-partitionable complexes, a combinatorial interpretation of the h-vector is less apparent. One possible approach uses *partition extenders*, introduced in the author's joint work with Joseph Doolittle and Alexander Lazar [DGL18]. We show that the h-vector of any pure simplicial complex can be written as the difference of an h-vector of a partitionable simplicial complex and the h-vector of a partitionable *relative complex* in a natural way. More details will be presented in Section 2.5.

1.4 Stanley-Reisner rings

An important algebraic object associated with a simplicial complex is its *Stanley-Reisner ring* (or *face ring*).

Definition 1.4.1. Let $\Bbbk$ be a field. If Δ is a simplicial complex on $[n]$, then its **Stanley-Reisner ideal** is $I_\Delta = \left\langle \prod_{i\in\sigma} x_i \;:\; \sigma \notin \Delta \right\rangle \subseteq \Bbbk[x_1,\ldots,x_n]$, the ideal generated by non-faces of Δ. Its **Stanley-Reisner ring** $\Bbbk[\Delta]$ is defined as

$$\Bbbk[\Delta] = \frac{\Bbbk[x_1,\ldots,x_n]}{I_\Delta}.$$

If $\sigma \notin \Delta$ and $\tau \in \Delta$ for all $\tau \subsetneq \sigma$, then we call σ a **missing face** (or **minimal non-face**) of Δ. We note that I_Δ is generated by monomials corresponding to the missing faces of Δ. If all missing faces of Δ have cardinality 2, then Δ is said to be **flag**. Order complexes are easily shown to be flag, so in particular if $\Delta = \mathrm{sd}(\Gamma)$ for some simplicial complex Γ, then Δ is flag.

A $\Bbbk$-algebra R is **graded** if it admits a decomposition $R = \bigoplus_{i\geq 0} R_i$ as $\Bbbk$-vector spaces and $R_iR_j \subseteq R_{i+j}$. It is $\mathbb{Z}^\ell$**-graded** if $R = \bigoplus_{\alpha\in\mathbb{Z}^\ell} R_\alpha$ and $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$.

Stanley-Reisner rings are graded, with $\Bbbk[\Delta]_i$ being the $\Bbbk$-span of monomials of degree i. The **Hilbert series** of a $\mathbb{Z}^\ell$-graded $\Bbbk$-algebra R is defined as

$$\operatorname{Hilb}(R,t) = \sum_{\alpha \in \mathbb{Z}^\ell} \dim_\Bbbk (R_\alpha)\, t^\alpha$$

where $t^\alpha = t_1^{\alpha_1} \dots t_\ell^{\alpha_\ell}$. The following fact along with Proposition 1.3.7 motivates many of the decomposition conjectures and theorems considered in this thesis.

Proposition 1.4.2. [Sta96, Theorem II.1.4] Let Δ be a $(d-1)$-dimensional simplicial complex. Then the Hilbert series of $\Bbbk[\Delta]$ can be written

$$\operatorname{Hilb}(\Bbbk[\Delta],t) = \frac{h(\Delta,t)}{(1-t)^d}$$

where $h(\Delta,t)$ is the h-polynomial, defined in (1.1.3).

Proposition 1.4.2 naturally leads one to ask: If the Stanley-Reisner ring of a complex has well-behaved structure (in particular, something that controls its Hilbert series), does the complex have a partitioning? The most famous question in this form is due to Stanley, who asked whether complexes with Cohen–Macaulay face rings are always partitionable. This question was answered in the negative in the author's joint work with Art Duval, Caroline Klivans, and Jeremy Martin in [DGKM16], and the explicit counterexample is described in Section 2.1 of this thesis. Before discussing this further, we will present more algebraic background.

The **(Krull) dimension** of a ring R is the maximum length of a chain of prime ideals in R. We call $\theta_1,\dots,\theta_m \in R$ a **regular sequence** if θ_i is a non-zerodivisor of $R/(\theta_1,\dots,\theta_{i-1})R$ for all $i = 2,\dots,m$. The **depth** of R is the length of the longest regular sequence in R. It is always the case that $\operatorname{depth} R \le \dim R$, and R is said to be **Cohen–Macaulay** (or simply **CM**) if $\operatorname{depth} R = \dim R$. A simplicial complex whose face ring is CM is called simply a **Cohen–Macaulay complex** (this can depend on the choice of $\Bbbk$). There are many characterizations of depth, dimension, and Cohen–Macaulayness for Stanley-Reisner rings, and we will survey a few below.

Proposition 1.4.3. [Sta96, Theorem II.1.3] If Δ is a simplicial complex, then $\dim \Bbbk[\Delta] = \dim \Delta + 1$. In other words, the dimension of the Stanley-Reisner ring is the cardinality of the largest facet.

Returning to Proposition 1.4.2, we note that the power of $(1-t)$ in the denominator of the Hilbert series is in fact the dimension of the ring. One particularly succinct way of encoding the depth and dimension of a ring is by using *local cohomology modules*. While we will not define them here, a worthwhile introduction to local cohomology modules is [ILL+07]. We will often use the following central result on local cohomology.

Proposition 1.4.4. [Sta96, Theorem I.6.3] Let R be a graded commutative ring and $H_m^i(R)$ its i^{th} local cohomology module where $m = \bigoplus_{i>0} R_i = R_+$ the irrelevant ideal. Then $\operatorname{depth} R = \min\left\{i \ : \ H_m^i(R) \neq 0\right\}$ and $\dim R = \max\left\{i \ : \ H_m^i(R) \neq 0\right\}$.

Proposition 1.4.4 implies that R is Cohen–Macaulay exactly when all but one of its local cohomology modules vanish.

There is an especially useful result due to Hochster (see, for example, [Sta96, Theorem II.4.1]) that relates the Hilbert series of local cohomology modules of a Stanley-Reisner ring $\Bbbk[\Delta]$ to the simplicial homology of the links of faces of Δ.

Theorem 1.4.5. [Sta96, Theorem II.4.1, due to Hochster (unpublished)] Let Δ be a complex on $[n]$ and let $H_m^i(\Bbbk[\Delta])$ be the i^{th} local cohomology module of $\Bbbk[\Delta]$. Then the $\mathbb{Z}^n$-graded Hilbert series of $H_m^i(\Bbbk[\Delta])$ can be written

$$\operatorname{Hilb}\left(H_m^i\left(\Bbbk[\Delta]\right), t_1, \ldots, t_n\right) = \sum_{\sigma \in \Delta} \dim_{\Bbbk} \tilde{H}_{i-|\sigma|-1}\left(\operatorname{link}_\Delta \sigma; \Bbbk\right) \prod_{j \in \sigma} \frac{t_j^{-1}}{1 - t_j^{-1}}$$

where $\tilde{H}_i\,(X;\Bbbk)$ denotes the i^{th} reduced simplicial homology of X computed over $\Bbbk$.

Hochster's theorem provides a powerful way to compute the depth of a Stanley-Reisner ring in terms of links of faces of Δ. An especially well-known consequence is the following characterization of Cohen–Macaulay complexes due to Reisner.

Theorem 1.4.6 (Reisner's Criterion, [Rei76])**.** The Stanley-Reisner ring $\Bbbk[\Delta]$ is Cohen–Macaulay if and only if

$$\tilde{H}_i(\operatorname{link}_\Delta \sigma;\Bbbk) = 0$$

for all $\sigma \in \Delta$ and for all $i < \dim(\operatorname{link}_\Delta \sigma)$.

One immediate consequence of Reisner's Criterion is a relationship between several of the decomposition properties discussed in Section 1.3 and Cohen–Macaulayness. A quick proof of this result relies on a Mayer-Vietoris sequence, which is discussed briefly in Section 1.2.

Theorem 1.4.7. Let Δ be a simplicial complex. The following implications are always true.

$$\Delta \text{ is pure shellable} \implies \Delta \text{ is constructible} \implies \Delta \text{ is CM over } \textit{any} \text{ field}$$

Each of these implications is strict in dimensions ≥ 2.

In dimension one being CM is equivalent to being connected, so Proposition 1.3.4 can be extended to include CM-ness in this list of equivalences. Furthermore, Munkres showed in [Mun84] that Cohen–Macaulayness is topological, i.e., it depends only on the geometric realization (and the field $\Bbbk$) and not any particular triangulation. While we often focus on the CM case, we are also interested in measuring the depth of a complex (i.e., the depth of its face ring) in general. This is a more nuanced question than that of the dimension of the face ring, and there are several ways of approaching it. The following formula is an extension of Reisner's criterion.

Proposition 1.4.8. [Sta96, Exercise 34] Let Δ be a simplicial complex and $\Bbbk[\Delta]$ its Stanley-Reisner ring. Then

$$\operatorname{depth}\Bbbk[\Delta] = \max\{i \ : \ \operatorname{skel}_i \Delta \text{ is CM}\} + 1$$

where $\operatorname{skel}_i \Delta$ is the i-skeleton of Δ.

Another combinatorial way of studying depth was recently introduced in the author's joint work with Hailong Dao, Joseph Doolittle, Ken Duna, Brent Holmes, and Justin Lyle in [DDD$^+$17]. In this preprint, we define a generalization of the well-known nerve complex as follows.

Definition 1.4.9. Given a simplicial complex Δ with facets $F_1, \ldots, F_k$, the j^{th} **nerve** of Δ is

$$N_j(\Delta) = \{\{F_i\} \ : \ |\cap F_i| \geq j\}$$

a simplicial complex whose vertices correspond to facets of Δ with cardinality greater than or equal to j. When $n = 1$, this recovers the classical **nerve complex**; see, for example [Bor48], [Grü70], and [Bjö95].

The nerve complex has been studied extensively, and in particular is known to be homotopy equivalent to Δ. This is not true for $N_j(\Delta)$ in general; rather, these complexes are homotopy equivalent to certain order complexes coming from subposets of the face poset of Δ. One result on higher nerves that will be useful in Chapter 4 is the following theorem from [DDD$^+$17].

Theorem 1.4.10. [DDD+17, Theorem 5.2] Let Δ be a simplicial complex and $\Bbbk[\Delta]$ its Stanley-Reisner ring. Then

$$\operatorname{depth} \Bbbk[\Delta] = \min \left\{ i + j \ : \ \tilde{H}_i(N_j(\Delta)) \neq 0 \right\}.$$

1.5 Relative simplicial complexes

Given simplicial complexes $\Gamma \subseteq \Delta$, we define the **relative (simplicial) complex** as $\Theta = (\Delta, \Gamma)$, a relative pair in the sense mentioned in Section 1.2. The face poset of a relative complex is $\Theta = (\Delta, \Gamma) = \Delta \setminus \Gamma$, i.e. all faces of Δ that are not in Γ. We will be interested in relative complexes primarily because many questions about simplicial complexes can be reduced to questions on relative complexes, but also because relative complexes have proved worthy of study in their own right. Furthermore, relative complexes are a more general class of objects, since any simplicial complex can be written as $\Delta = (\Delta, \varnothing)$. We will refer to these relative complexes as **absolute**. (We note that this is distinct from the relative complex $\Theta = (\Delta, \{\varnothing\})$, the set of all faces of Δ except for the empty face.)

Many properties of simplicial complexes can be defined similarly for relative complexes. For example, the definitions of f- and h-vectors, dimension, purity, and partitionability are the same for relative complexes as for simplicial complexes. Similarly, there are analogous definitions for relative shellability and relative Cohen–Macaulayness. The analogue of the Stanley-Reisner ring in the relative setting is the **Stanley-Reisner module** $I_\Theta \subseteq \Bbbk[\Delta]$, defined as

$$I_\Theta = \left\langle \prod_{i \in \sigma} x_i \ : \ \sigma \in \Delta \setminus \Gamma \right\rangle$$

which can be regarded as a module over $\Bbbk[\Delta]$ or over $\Bbbk[x_1,\dots,x_n]$. A relative complex Θ is **Cohen–Macaulay** if I_Θ is a Cohen–Macaulay module over $\Bbbk[\Delta]$. Versions of Hochster's Theorem and Reisner's Criterion hold for relative complexes as well:

Theorem 1.5.1. [AS16, Theorem 1.8] Let $\Theta = (\Delta, \Gamma)$ be a relative complex on $[n]$, and let I_Θ be its Stanley-Reisner module in $\Bbbk[\Delta]$, and let $H^i_m(I_\Theta)$ be the i^{th} local cohomology module of I_Θ. Then the $\mathbb{Z}^n$-graded Hilbert series of $H^i_m(I_\Theta)$ can be written as

$$\mathrm{Hilb}\left(H^i_m\left(I_\Theta\right), t_1, \dots, t_n\right) = \sum_{\sigma\in\Delta} \dim_{\Bbbk} \tilde{H}_{i-|\sigma|-1}\left(\mathrm{link}_\Theta\,\sigma; \Bbbk\right) \prod_{j\in\sigma} \frac{t_j^{-1}}{1-t_j^{-1}}$$

where $\mathrm{link}_\Theta\,\sigma = (\mathrm{link}_\Delta\,\sigma, \mathrm{link}_\Gamma\,\sigma)$ is the relative link of σ in Θ.

Theorem 1.5.2. [Sta87, Theorem 5.3] Let $\Theta = (\Delta, \Gamma)$. The Stanley-Reisner module I_Θ is Cohen–Macaulay over $\Bbbk$ if and only if

$$\tilde{H}_i(\mathrm{link}_\Theta\,\sigma; \Bbbk) = 0$$

for all $\sigma \in \Delta$ and for all $i < \dim(\mathrm{link}_\Delta\,\sigma)$.

More will be said about Cohen–Macaulay relative complexes in Chapter 2, particularly in Section 2.3.

1.6 Algebraic shifting

In [Kal01], Kalai introduced *algebraic shifting*, a way to transform a given simplicial complex Δ into a related complex $S(\Delta)$ that is combinatorially much easier to describe and study but that preserves much of the same properties of the original complex.

Definition 1.6.1. A simplicial complex Δ on $[n]$ is **shifted** if, for every face $\sigma = \{v_1,\dots,v_k\}$ with $v_1 < \cdots < v_k$, then $(\sigma \setminus \{v_i\}) \cup \{v\} \in \Delta$ for all $v < v_i$.

The definition easily extends to any simplicial complex equipped with a total order in its vertex set. One useful feature of shifted complexes is the following proposition.

Proposition 1.6.2. If Δ is shifted, then Δ is shellable.

The shelling order on Δ in this case is given by the lexicographic order of the facets of Δ. Together with Proposition 1.3.2, this observation implies that all shifted complexes are homotopy wedges of spheres.

There are two types of algebraic shifting, *exterior* and *symmetric*; we will focus exclusively on exterior shifting, which is defined in terms of the *exterior Stanley-Reisner ring* of a complex.

To define exterior shifting, we must use the **exterior Stanley-Reisner ring** $\Lambda[\Delta]$, which is the exterior analogue of the Stanley-Reisner ring (again taken over a field $\Bbbk$). The only difference is that multiplication in the exterior ring is given by the wedge product, that is for variables $x_i, x_j \in \Lambda[\Delta]$, we have that $x_i \wedge x_j = -x_j \wedge x_i$. This shows that $\mathrm{Hilb}(\Lambda[\Delta], t) = f_{-1} + f_0 t + f_1 t^2 + \cdots + f_{d-1} t^d = f(\Delta, t)$. The exterior Stanley-Reisner ring is a natural place in which we can compute cohomology, as multiplying by the form $x_1 + \cdots + x_n$ (or any generic linear form) corresponding to all of the vertices of Δ is the same as the coboundary operator.

Given vertices $v_1, \ldots, v_n$ of Δ and $i \in [n]$, define the i^{th} **generic linear form** as

$$f_i = \sum_{j=1}^{n} \alpha_{ij} x_j \tag{1.6.1}$$

where α_{ij} are n^2 algebraically independent elements (which can be adjoined to $\Bbbk$ if necessary). Similarly, if $T = \{i_1, \ldots, i_k\}$ with $i_1 < \cdots < i_k$, then define $f_T = f_{i_1} \wedge \cdots \wedge f_{i_k}$.

Definition 1.6.3. Let Δ be a simplicial complex on $[n]$ with exterior Stanley-Reisner ring $\Lambda[\Delta]$. Then the **(exterior) algebraic shifting** of Δ is

$$S(\Delta) = \{T \subseteq [n] \ : \ f_T \notin \text{span}\{f_R \ : \ |R| = |T| \text{ and } R <_{\text{lex}} T\}\}$$

where $<_{\text{lex}}$ denotes lexicographic order.

This definition produces a complex $S(\Delta)$ that is itself shifted. Algebraic shifting has proved to be a powerful tool in studying simplicial complexes and their f-vectors, due in part to the following result.

Theorem 1.6.4. [BK88], [Kal01] Let Δ be a simplicial complex and $S(\Delta)$ its algebraic shifting. Then:

1. If Δ is shifted, then $S(\Delta) = \Delta$.

2. Shifting preserves Hilbert series, i.e., $\mathrm{Hilb}(\Bbbk[S(\Delta)]) = \mathrm{Hilb}(\Bbbk[\Delta])$. Equivalently, $f(S(\Delta)) = f(\Delta)$ and $h(S(\Delta)) = h(\Delta)$.

3. Shifting preserves Betti numbers, i.e., $\tilde{\beta}_i(S(\Delta)) = \tilde{\beta}_i(\Delta)$ for all i.

4. Shifting preserves depth, i.e. $\operatorname{depth} \Bbbk[S(\Delta)] = \operatorname{depth} \Bbbk[\Delta]$.

A consequence of (4) is the following.

Theorem 1.6.5. Δ is CM if and only if $S(\Delta)$ is pure.

1.7 Iterated homology

One offshoot of algebraic shifting is the theory of *iterated homology* developed by Duval and Rose in [DR00] and extended by Duval and Zhang in [DZ01]. It has found applications in nonpure shellability and decompositions of Cohen–Macaulay complexes (see Theorem 2.1.12). The following groups, defined within the exterior Stanley Reisner ring $\Lambda[\Delta]$, are needed to define iterated homology.

Definition 1.7.1. [DR00, p. 286] Let Δ be a simplicial complex. If $0 \leq r \leq k+1 \leq d$, then

$$\begin{aligned}
\Lambda^k[r](\Delta) &= f_1 \wedge \ldots f_r \wedge \Lambda^{k-r}[\Delta] \\
Z^k[r](\Delta) &= \left\{ x \in \Lambda^k[r](\Delta) \ : \ f_{r+1} \wedge x = 0 \right\} \\
B^k[r](\Delta) &= \begin{cases} f_{r+1} \wedge \Lambda^{k-1}[r](\Delta) & \text{if } r < k+1 \\ 0 & \text{if } r = k+1 \end{cases} \\
H^k[r](\Delta) &= Z^k[r](\Delta)/B^k[r](\Delta)
\end{aligned}$$

where the f_i are generic linear forms as defined in (1.6.1). The groups $H^k[r](\Delta)$ are the r^{th} **iterated cohomology groups**, and the r^{th} **iterated Betti numbers** are $\beta^k[r](\Delta) = \dim_{\Bbbk} H^k[r](\Delta)$.

We survey below a few results of iterated homology that will be used throughout this thesis. Producing balanced analogues of these results is the main focus of Sections 3.3 and 3.4.

Proposition 1.7.2. [DR00, Theorem 4.1] Let Δ be a $(d-1)$-dimensional simplicial complex, $S(\Delta)$ its algebraic shifting, and $\beta^k[r](\Delta)$ an iterated Betti number (where $0 \leq r \leq k+1 \leq d$). Then

$$\beta^k[r](\Delta) = |\{\text{facets } T \in S(\Delta) \ : \ |T| = k+1 \text{ and } \operatorname{init}(T) = r\}| .$$

where $\operatorname{init}(T) = \max\{i \geq 0 \ : \ [i] \subseteq T\}$ if $1 \in T$ and $\operatorname{init}(T) = 0$ otherwise.

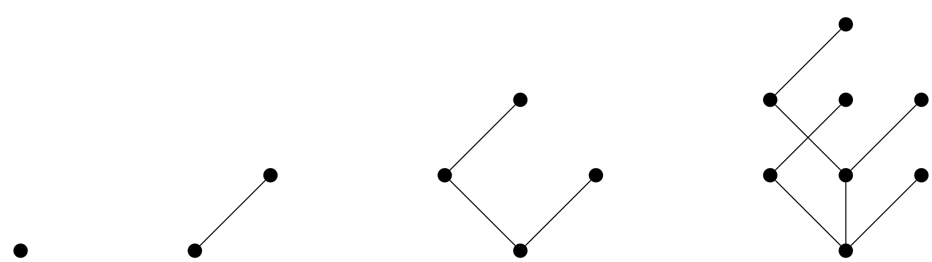
In particular, if $r = 0$, then iterated homology is precisely the ordinary cohomology of Δ, since multiplication by a generic linear form is a coboundary operator. Proposition 3.3.2 is a balanced analogue of Proposition 1.7.2.

In [DZ01], the authors introduce the following definition.

Definition 1.7.3. Given a poset P, a **boolean tree** is a subposet of P with the following recursive definition:

- A rank-0 boolean tree is a single element of P.
- A rank-k boolean tree is defined recursively as follows: Let T_1 and T_2 be two disjoint rank-$(k-1)$ boolean trees with minimal elements r_1 and r_2 such that $r_1 \lessdot r_2$. Then $T_1 \cup T_2$ is a rank k boolean tree with minimal element r_1.

Here are the rank-k boolean trees for $0 \leq k \leq 3$.



Duval and Zhang prove a decomposition theorem [DZ01, Theorem 3.2] which gives rise to boolean tree decompositions of some complexes. We note that Theorem 3.3.4 is a balanced analogue of the Duval-Zhang result.

A set B of faces of a simplicial complex Δ is an r**-Betti set** if $f_{k-r}(B) = \beta_k[r](\Delta)$ for all k.

Theorem 1.7.4. [DZ01, Theorem 3.2] Let Δ be a $(d-1)$-dimensional simplicial complex. Then there exists a chain of subcomplexes

$$\varnothing = \Delta^{(d+1)} \subseteq \cdots \subseteq \Delta^{(r)} \subseteq \Delta^{(r-1)} \subseteq \cdots \subseteq \Delta^{(1)} \subseteq \Delta^{(0)} = \Delta,$$

where

$$\Delta^{(r)} = \Delta^{(r+1)} \sqcup B^{(r)} \sqcup \Omega^{(r+1)} \;\; (0 \leq r \leq d),$$

and bijections

$$\eta^{(r)} : \Delta^{(r)} \to \Omega^{(r)} \;\; (1 \leq r \leq d),$$

such that, for each r.

1. $\Delta^{(r+1)}$ and $\Delta^{(r+1)} \sqcup B^{(r)}$ are subcomplexes of $\Delta^{(r)}$;

2. $B^{(r)}$ is an r-Betti set; and

3. for any $\sigma \in \Delta^{(r)}$, we have $\sigma \subsetneq \eta^{(r)}(\sigma)$ and $\left|\eta^{(r)}(\sigma) \setminus \sigma\right| = 1$.

The following is an important corollary of Theorem 1.7.4.

Theorem 1.7.5. [DZ01, Corollary 3.5] Let Δ be a simplicial complex. Then there is a decomposition of Δ into disjoint boolean trees such that the number of boolean trees of rank r with a $(k-r)$-dimensional minimal element is $\beta^k[r](\Delta)$, a iterated Betti number of Δ.

Proposition 3.4.4 is a balanced analogue of Theorem 1.7.5.

Boolean trees have the same rank generating functions as boolean intervals, though they are much less restrictive in structure. However, if Δ is pure and has a decomposition into disjoint boolean trees whose tops are the facets of Δ (i.e., replace “boolean interval” with “boolean tree” in the definition of a partitioning, Definition 1.3.5), then Proposition 1.3.7 provides a combinatorial interpretation for the minimal elements of the boolean trees.

Chapter 2

Partitionability

2.1 Stanley's partitionability conjecture

A central goal in the study of simplicial complexes is to characterize the f- and h-vectors of complexes that have various properties. The celebrated *Kruskal-Katona Theorem* (proved independently by Kruskal [Kru63], Katona [Kat68], and Schützenberger [Sch59] in response to a conjecture by Schützenberger) completely classifies which integer vectors are the f-vectors of simplicial complexes, hence also answers this question for h-vectors.

In [Sta77], Stanley provided a similar characterization of the h-vectors of shellable and Cohen–Macaulay (CM) complexes. Given two positive integers $\ell, i \in \mathbb{Z}_{>0}$, there exists a unique expansion (called the i^{th} Macaulay expansion of ℓ)

$$\ell = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}$$

such that $n_i > n_{i-1} > \cdots > n_j > j \geq 1$. Using this decomposition and the notation of [Sta96], we define $\ell^{\langle i \rangle}$ as

$$\ell^{\langle i \rangle} = \binom{n_i + 1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j + 1}{j+1}$$

for positive ℓ and define $0^{\langle i \rangle} = 0$. Then we say that $h = (h_0, h_1, \ldots, h_d)$ is an M**-vector** (or O**-sequence**) if $h_0 = 1$ and $0 \leq h_{i+1} \leq h_i^{\langle i \rangle}$ for $i \geq 1$. In [Sta77], Stanley proved the following result, which he has referred to as "a 'Kruskal-Katona Theorem' for Cohen–Macaulay complexes" [Sta96, p. 58].

Theorem 2.1.1. [Sta77, Theorem 6] Let $h = (h_0, h_1, \ldots, h_d)$ be an integer vector. Then the following are equivalent.

1. h is an M-vector.
2. $h = h(\Delta)$ where Δ is a pure shellable simplicial complex.
3. $h = h(\Delta)$ where Δ is a constructible simplicial complex.
4. $h = h(\Delta)$ where Δ is a Cohen–Macaulay simplicial complex.

The implication (2) $\implies$ (3) $\implies$ (4) is immediate by Theorem 1.4.7. Similarly, (4) $\implies$ (2) is due to Theorems 1.6.5 and 1.6.4. Two immediate consequences of Theorem 2.1.1 are that h-vectors of these complexes are non-negative (i.e., $h_i \geq 0$) and gap-free (i.e., $h_i = 0$ implies that $h_{i+1} = 0$). Neither of these are true for simplicial complexes in general—see Example 2.5.1 and Example 2.4.4 respectively.

We remind the reader of Proposition 1.3.7, which states that if Δ is pure and partitionable, then $h(\Delta)$ enumerates the minimal faces in any partitioning of Δ. This proposition—along with the structure that Theorem 2.1.1 guarantees for the h-vectors of

Cohen–Macaulay complexes—motivated the following conjecture, which Stanley called "a central combinatorial conjecture on Cohen–Macaulay complexes" [Sta96, p. 85].

Conjecture 2.1.2 (Partitionability Conjecture)**.** [Sta79, p. 149] If Δ is Cohen–Macaulay, then Δ is partitionable.

This conjecture is false. In [DGKM16], Art Duval, Caroline Klivans, Jeremy Martin, and the author constructed an explicit counterexample to Conjecture 2.1.2. We will outline this result in the remainder of this section. The main ingredient in the construction of the counterexample is a non-partitionable relative complex that has the CM property. We first note the following lemmas.

Lemma 2.1.3. Let Δ_1, Δ_2 be simplicial complexes and $\sigma \in \Delta_1 \cup \Delta_2$. Then $\operatorname{link}_{\Delta_1 \cup \Delta_2} \sigma = \operatorname{link}_{\Delta_1} \sigma \cup \operatorname{link}_{\Delta_2} \sigma$ and $\operatorname{link}_{\Delta_1 \cap \Delta_2} \sigma = \operatorname{link}_{\Delta_1} \sigma \cap \operatorname{link}_{\Delta_2} \sigma$.

Proof. We prove the first equality, and the other holds similarly.

$$\begin{aligned}
\operatorname{link}_{\Delta_1 \cup \Delta_2} \sigma &= \{\tau \in \Delta_1 \cup \Delta_2 \ : \ \tau \cup \sigma \in \Delta_1 \cup \Delta_2, \tau \cap \sigma = \varnothing\} \\
&= \{\tau \in \Delta_1 \ : \ \tau \cup \sigma \in \Delta_1 \cup \Delta_2, \ \tau \cap \sigma = \varnothing\} \\
&\quad \cup \{\tau \in \Delta_2 \ : \ \tau \cup \sigma \in \Delta_1 \cup \Delta_2, \ \tau \cap \sigma = \varnothing\} \\
&= \{\tau \in \Delta_1 \ : \ \tau \cup \sigma \in \Delta_1, \ \tau \cap \sigma = \varnothing\} \\
&\quad \cup \{\tau \in \Delta_2 \ : \ \tau \cup \sigma \in \Delta_2, \ \tau \cap \sigma = \varnothing\} \\
&= \operatorname{link}_{\Delta_1} \sigma \cup \operatorname{link}_{\Delta_2} \sigma. \qquad \square
\end{aligned}$$

The following result, allows us to glue Cohen–Macaulay complexes along Cohen–Macaulay induced subcomplexes of sufficiently high dimension to produce a new complex that is itself Cohen–Macaulay.

Lemma 2.1.4. [DGKM16, Lemma 2.2] Let Δ_1 and Δ_2 be $(d-1)$-dimensional Cohen–Macaulay complexes. If $\Delta_1 \cap \Delta_2$ is Cohen–Macaulay and $\dim(\Delta_1 \cap \Delta_2) \geq d-2$, then $\Omega = \Delta_1 \cup \Delta_2$ is Cohen–Macaulay.

Proof. We will show that Reisner's Criterion (Theorem 1.4.6) holds for each face $\sigma \in \Omega$. If $\sigma \in \Delta_1 \setminus \Delta_2$, then $\operatorname{link}_\Omega \sigma = \operatorname{link}_{\Delta_1} \sigma$ and Reisner's Criterion holds by assumption. The argument is similar if $\sigma \in \Delta_2 \setminus \Delta_1$.

If instead $\sigma \in \Delta_1 \cap \Delta_2$, then Lemma 2.1.3 and Theorem 1.2.2 gives the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_i(\operatorname{link}_{\Delta_1} \sigma) \oplus \tilde{H}_i(\operatorname{link}_{\Delta_2} \sigma) \to \tilde{H}_i(\operatorname{link}_\Omega \sigma) \to \tilde{H}_{i-1}(\operatorname{link}_{\Delta_1 \cap \Delta_2} \sigma) \to \ldots.$$

Observe that

$$\dim(\operatorname{link}_{\Delta_1} \sigma) = \dim(\operatorname{link}_{\Delta_2} \sigma) = \dim\left(\operatorname{link}_{(\Delta_1 \cup \Delta_2)} \sigma\right) = d - |\sigma| + 1$$

and similarly that

$$d - |\sigma| - 2 \leq \dim\left(\operatorname{link}_{(\Delta_1 \cap \Delta_2)} \sigma\right) \leq d - |\sigma| + 1.$$

Since Δ_1, Δ_2, and $\Delta_1 \cap \Delta_2$ are Cohen–Macaulay, each of these links have trivial homology in non-top dimension. Thus $\tilde{H}_i(\operatorname{link}_{(\Delta_1 \cup \Delta_2)} \sigma) = 0$ for $i < \dim(\operatorname{link}_{(\Delta_1 \cup \Delta_2)} \sigma)$. Therefore $\Delta_1 \cup \Delta_2$ is Cohen–Macaulay. $\square$

Remark 2.1.5. If we replace each instance of "Cohen–Macaulay" in Lemma 2.1.4 with "constructible," the result also holds.

We may repeat the process in Lemma 2.1.4 arbitrarily many times while preserving Cohen–Macaulayness, which leads to the following theorem.

Theorem 2.1.6. [DGKM16, Theorem 3.1] Let $\Theta = (\Delta, \Gamma)$ be a relative complex such that

1. Δ and Γ are Cohen–Macaulay;

2. Γ is an induced subcomplex of Δ such that $\dim \Gamma \geq \dim \Delta - 1$; and

3. Θ is not partitionable.

Let k be the total number of faces of Γ and let $N > k$. Then gluing N copies of Δ together along Γ will produce a non-partitionable Cohen–Macaulay simplicial complex.

Proof. Call the result of this construction Υ.

A complex of simplices is simplicial if and only if distinct faces have distinct vertex sets. If we glue two simplicial complexes along non-induced subcomplexes, then there will be two distinct faces that have the same vertex set. However, if the gluing is done along an induced subcomplex, then the resulting complex Υ is still simplicial. Similarly, by Proposition 2.1.4, Υ is Cohen–Macaulay.

Finally, assume that there is a partitioning

$$\Upsilon = \bigsqcup [R_i, F_i]. \tag{2.1.1}$$

Since there are more copies of (Δ, Γ) in Υ than there are faces of Γ, there must be some copy of (Δ, Γ) such that no face in (Δ, Γ) is in an interval in this partitioning with a face of Γ. But then this means that this partitioning (2.1.1) contains a partitioning of (Δ, Γ), which is a contradiction. Therefore Υ cannot be partitionable. $\square$

Remark 2.1.7. It is worth noting that Γ is an induced subcomplex of Δ if and only if all minimal faces of the relative complex (Δ, Γ) are vertices.

Theorem 2.1.6 does not guarantee that such a pair (Δ, Γ) exists. However, in [DGKM16], we constructed an example that meets the criteria of Theorem 2.1.6. It arises as a subcomplex of Ziegler's Ball [Zie98], a particularly small 3-ball that is constructible (and hence CM) but not shellable.

Proposition 2.1.8. Let Δ be the complex with the following facets (where the set $\{1,2,4,9\}$ is abbreviated 1249, etc.)

$$\begin{gathered} 1249, 1269, 1569, 1589, 1489, 1458, 1457, \\ 4578, 1256, 0125, 0256, 0123, 1234, 1347. \end{gathered} \tag{2.1.2}$$

Let Γ be the induced subcomplex $\Gamma = \Delta_{\{0,2,3,4,6,7,8\}}$, which has facets

$$026, 023, 234, 347, 478. \tag{2.1.3}$$

Then $\Theta = (\Delta, \Gamma)$ meets the necessary properties of Theorem 2.1.6.

Define C_n to be the complex created by gluing n copies of Δ together along Γ. Since $f(\Gamma, 1) = 24$, gluing 25 copies of Δ together will produce a non-partitionable Cohen–Macaulay complex. Thus C_{25} is a counterexample to Conjecture 2.1.2.

Remark 2.1.9. It is easy to see that Δ and Γ in Example 2.1.8 are Cohen–Macaulay: In fact they are shellable, and the orders given in (2.1.2) and (2.1.3) respectively are shelling orders. Similarly, it is clear that Γ is induced and $\dim \Gamma = \dim \Delta - 1$. It is a straightforward argument to show that Θ is not partitionable; see [DGKM16, Theorem 3.3] for details.

Remark 2.1.10. While Theorem 2.1.6 guarantees that gluing 25 copies of Δ together along Γ will produce a non-partitionable complex, it turns out that fewer copies are needed. In fact, C_3, the complex constructed by gluing 3 copies of Δ along Γ, is also non-partitionable [DGKM16, Theorem 3.5]. To the best of my knowledge, it is currently the smallest known counterexample to Conjecture 2.1.2.

Remark 2.1.11. Since Δ and Γ are both pure shellable, they are also constructible. Lemma 2.1.4 and Theorem 2.1.6 also hold for constructible complexes, so C_3 is also an example of a non-partitionable constructible complex. This answers Hachimori's question [Hac00b] of whether all constructible complexes were partitionable.

In Section 4.3, we will present similar gluing arguments that could be used in disproving a related conjecture.

While a Cohen–Macaulay complex need not be partitionable, it does admit a reasonably well-behaved decomposition, thanks to a result of Duval and Zhang. If Δ can be decomposed into disjoint boolean trees such that the top elements of all of the boolean trees are facets of Δ, we will call such a decomposition an **honest boolean tree decomposition**. (This is the same as a partitioning, but with "boolean intervals" replaced with "boolean trees.")

Theorem 2.1.12. [DZ01, Theorem 5.4] If Δ is Cohen–Macaulay, then Δ has an honest boolean tree decomposition.

Since Δ is Cohen–Macaulay if and only if $S(\Delta)$ is pure, Theorem 2.1.12 follows from Theorem 1.7.5 and Proposition 1.7.2.

2.2 Stanley depth

A combinatorial version of depth—now referred to as *Stanley depth*—has recently attracted a great deal of attention; see [PSFTY09] for an introduction and overview. Given a polynomial ring $S = \Bbbk[x_1, \ldots, x_n]$ and a M a $\mathbb{Z}^n$-graded S module, a **Stanley**

decomposition D of M is a vector space decomposition

$$M = \bigoplus_{i=1}^{r} \Bbbk[X_i] \cdot m_i$$

where each X_i is a subset of $\{x_1, \ldots, x_n\}$ and each m_i is a homogeneous element of M. The **Stanley depth** of M is defined as

$$\operatorname{sdepth} M = \max_D \{\min(|X_1|, \ldots, |X_r|)\}$$

where D ranges over all Stanley decompositions of M. This definition gave rise to the following conjecture.

Conjecture 2.2.1 (Depth Conjecture)**.** [Sta82, Conjecture 5.1] For all $\mathbb{Z}^n$-graded $\Bbbk[x_1, \ldots, x_n]$-modules M,

$$\operatorname{sdepth} M \geq \operatorname{depth} M.$$

In [HJY08, Corollary 4.5], Herzog, Jahan, and Yassemi showed that Conjecture 2.2.1 would have implied Conjecture 2.1.2. Therefore the counterexample in Proposition 2.1.8 is also the first known counterexample to Conjecture 2.2.1.

2.3 h-vectors of Cohen–Macaulay relative complexes

With the M-vector characterization of h-vectors of shellable, constructible, and Cohen–Macaulay complexes known (see Theorem 2.1.1) the following is a natural question.

Question 2.3.1. Is there a characterization of the h-vectors of shellable or Cohen–Macaulay relative complexes? Do the same constraints hold for the h-vectors for these classes of relative complexes?

Stanley provided a partial answer to this question in [Sta87, Proposition 5.2], showing that the h-vector of a Cohen–Macaulay relative complex Θ is the sum of m-vectors that have been shifted in dimension corresponding to the cardinalities of the minimal faces of Θ. Question 2.3.1 was also recently studied by Codenotti, Khattän, and Sanyal [CKS17], who also extended this question to complexes which they called *fully* Cohen–Macaulay complexes. Their work [CKS17, Theorem 10] implies Proposition 2.3.2, which we prove independently in this section.

A relative complex $\Theta = (\Delta, \Gamma)$ is **shellable** if its facets can be ordered $F_1, \ldots, F_k$ such that

$$(\langle F_1, \ldots, F_{i+1} \rangle \cap \Theta) \setminus (\langle F_1, \ldots, F_i \rangle \cap \Theta)$$

has a unique minimal element for each $i \in [k-1]$.

If Θ is pure shellable, then Θ is Cohen–Macaulay, just as in the simplicial complex case [Sta96, p. 118]. The following proposition gives an answer to the second part of Question 2.3.1.

Proposition 2.3.2. Let $h = (h_0, h_1, \ldots, h_d) \in \mathbb{Z}^{d+1}$. The following are equivalent, assuming that the relative complexes Θ in question are not *absolute* complexes, i.e., $\varnothing \notin \Theta$.

1. $h_0 = 0$ and $h_i \geq 0$ for $i \in [d]$.

2. $h = h(\Theta)$ for a pure shellable relative complex Θ.

3. $h = h(\Theta)$ for a Cohen–Macaulay relative complex Θ.

Note that if $\varnothing \in \Theta$, then $\Theta = (\Delta, \varnothing) = \Delta$ is a simplicial complex, and Theorem 2.1.1 applies instead of Proposition 2.3.2. We will prove Proposition 2.3.2 by constructing a pure shellable relative complex Θ such that $h(\Theta) = h$ for any given vector h of the form in (1). Furthermore, for $\Theta = (\Delta, \Gamma)$, the complexes Δ and Γ are connected, pure, and shifted (both with respect to the same order).

Proof. Recall that if a relative complex is pure and shellable, then it is automatically Cohen–Macaulay, so (2) $\implies$ (3).

We note that (3) $\implies$ (1) is implied by [Sta87, Proposition 5.2].

Thus the only implication left to show is (1) $\implies$ (2), which we will do by construction. Given $h = (0, h_1, \ldots, h_d)$ with $h_i \geq 0$ (assume without loss of generality that $h_d \neq 0$), we will consider each h_i separately and construct h_i many disjoint rank-$(d-i)$ boolean intervals in a certain way to ensure shellability of Θ. In particular, we will construct $\Theta = (\Delta, \Gamma)$ and show that Δ and Γ are pure and shifted and that Θ is shellable.

Let $k = \sum_{i=1}^{d} h_i$. Define

$$\mathcal{F} = \{12\ldots(d-1)(d+\ell-1) \ : \ \ell \in [k]\}$$

and let Δ be the complex generated by the elements of $\mathcal{F}$. Thus Δ has k facets which are all $(d-1)$-simplices glued together along the ridge $12\ldots(d-1)$. This complex is pure, shifted, and shellable. Call the facet $F_j = 12\ldots(d-1)j$ the j^{th} facet of Δ.

We will define Γ as a collection of ridges within each facet of Δ, and these ridges will correspond to a particular entry h_i. We will start with h_d and work back to h_1. The first h_d facets of Δ will correspond to h_d, the next h_{d-1} facets will correspond to h_{d-1}, etc. Let F_j be the j^{th} facet of Δ and assume that it corresponds to h_i. We will define

$$\Gamma_j = \langle \text{The lexicographically first } i \text{ many ridges of } F_j \rangle$$

Define $\Theta_j = (\langle F_j \rangle, \Gamma_j)$, which is a boolean interval of rank-$(d-i)$ with minimal element M_j. Then $|M_j| = i$ and $M_j = \{\text{The last } i \text{ vertices of } F_j\}$. Certainly each of these Γ_j are pure, shifted, and shellable, and $\Gamma = \bigcup_{j\in[k]} \Gamma_j$ is also pure, shifted, and shellable.

If $\Theta = (\Delta, \Gamma)$, then Θ is the union of all of the Θ_j, which are disjoint boolean intervals, and $h(\Theta) = (0, h_1, \ldots, h_d)$. □

2.4 An algebraic notion of partitionability? / Partitionable complexes with low depth

Dress [Dre93] and Simon [Sim94] studied the *cleanness* of modules, an idea that originated in [DDM96] by Dahmen, Dress, and Micchelli. [1]

Given M a module over a commutative ring R, a **filtration** $\mathcal{F}$ is a finite set $\{M_i\}$ of R-modules such that

$$\langle 0 \rangle = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_k = M.$$

A filtration is **clean** if, for all $i \in [k]$, $M_i/M_{i-1} \cong R/P_i$ where P_i is a minimal prime of R over $\operatorname{Ann}(M)$, the annihilator of M. A module is **clean** if it has a clean filtration.

Theorem 2.4.1. [Dre93] [Sim94, Theorem 2.1.1] A simplicial complex Δ is shellable if and only if $\Bbbk[\Delta]$ is clean when regarded as a module over itself.

Remark 2.4.2. In Theorem 2.4.1, Dress and Simon did not assume that Δ was pure; this is notable because this predates Björner and Wachs's introduction of nonpure shellability in [BW96].

[1] The article [DDM96] was originally written in 1990 but was not officially published until 1996.

In fact, a clean filtration of $\Bbbk[\Delta]$ can be used to construct a shelling of Δ. This leads naturally to the following question, which was originally posed to the author by John Shareshian.

Question 2.4.3. Is there an algebraic criterion similar to cleanness for partitionability?

Both a filtration and a shelling rely heavily on an order (respectively on some submodules or on facets), but there is no sense of an order on the facets in a partitioning. In general, there is no apparent way to systematically assemble a subcollection of the intervals of a partitioning. Consider the following example.

Example 2.4.4. [Sta96, p. 85, due to Björner] Let $\Delta = \langle 123, 124, 134, 234, 456 \rangle$. The decomposition

$$\Delta = [\varnothing, 456] \cup [1, 124] \cup [2, 234] \cup [3, 134] \cup [123, 123]$$

is one of the two possible partitionings of Δ.

While it is easy in this example to provide an algebraic structure for the interval $[\varnothing, 456]$ (namely $\Bbbk[x_4, x_5, x_6]$), it is not so for other proper subcollections of these intervals. This is because, for example, $\Gamma = [\varnothing, 456] \cup [1, 124]$ is not itself a simplicial complex or relative complex (because $124 \in \Gamma$ but $2 \notin \Gamma$), so Γ does not have an obvious algebraic structure to assign to it. Contrast this to the shellable case—at each step in a shelling, the object that has been constructed is a simplicial complex.

While the complex in Example 2.4.4 is pure and partitionable, it is not Cohen–Macaulay (in particular, $\mathrm{link}(4)$ fails Reisner's Criterion, Theorem 1.4.6). This is not surprising; it has long been known that partitionability does not imply Cohen–Macaulayness.

Example 2.4.4 also shows that the h-vectors of partitionable complexes have fewer restrictions than those of CM or shellable complexes. In particular, $h(\Delta) = (1,3,0,1)$, which is not an M-vector as it is not gap-free (see Theorem 2.1.1). We can see that $\dim \Bbbk[\Delta] = 3$ and $\operatorname{depth} \Bbbk[\Delta] = 2$, so this complex is relatively close to being CM. Our goal for the remainder of this section is to construct pure partitionable complexes that have arbitrarily large differences between their depth and dimension, as this would show that partitionability imposes no restriction on the depth of a complex's face ring.

We know that $\operatorname{depth} \Bbbk[\Delta] \geq 1$ for any non-trivial simplicial complex and $\operatorname{depth} \Bbbk[\Delta] = 1$ exactly when Δ has more than one connected component. Similarly, $\dim \Bbbk[\Delta] = \dim \Delta + 1$. For each dimension, we will construct a disconnected (hence depth-1) pure partitionable complex of that dimension.

To motivate the construction, consider the following example:

Example 2.4.5. The complex $\langle 12, 13, 23, 45 \rangle$ is partitionable and disconnected.

In fact, we can completely characterize which graphs are partitionable. In the proposition below, we say that a connected component is **nontrivial** if it contains at least one edge.

Proposition 2.4.6. A graph is partitionable if and only if it contains at most one non-trivial acyclic connected component.

Proof. Assume that Δ has at least two non-trivial acyclic connected components, X and Y. One edge of Δ can be paired with the empty face in a partitioning of Δ; assume this edge is not in Y. Thus all of the intervals partitioning Y must be of the form $[v_i, v_i v_j]$ for vertices $v_i \in Y$ and edges $v_i v_j \in Y$. But, since Y is acyclic, it has exactly one more vertex than edge. Therefore Y cannot be partitioned, so Δ is not partitionable.

Instead assume that Δ has at most one acyclic connected component. Shell this component as usual (if all components contain a cycle, pick an arbitrary component and shell it). For each other component, find a cycle with vertices $v_1, v_2, \ldots, v_k, v_1$. Partition the elements of this cycle as

$$[v_1, v_1v_2] \sqcup [v_2, v_2v_3] \sqcup \cdots \sqcup [v_{k-1}, v_{k-1}v_k] \sqcup [v_k, v_1v_k].$$

For the remaining edges in this component, add them on one-by-one, as if in a shelling. Thus Δ is partitionable. $\square$

Full criteria for partitionability of higher-dimensional complexes are not known.

Theorem 2.4.7. Define

$$\begin{aligned} \alpha_i &= i(i+1)\ldots(d+i-1) \\ \beta_i &= 12\ldots i(d+i)\ldots(2d-1) \\ \gamma &= (2d)(2d+1)\ldots(3d-1). \end{aligned}$$

Let Δ be generated by the facets:

$$\{\alpha_i \,:\, i \in [d]\} \cup \{\beta_i \,:\, i \in [d-1]\} \cup \{\gamma\}$$

Then Δ is pure and partitionable with $h(\Delta) = (1, 2d-1, 0, \ldots, 0)$, $\dim \Bbbk[\Delta] = d$, and $\operatorname{depth} \Bbbk[\Delta] = 1$.

Proof. All facets of Δ have cardinality d, so $\dim \Bbbk[\Delta] = d$. Since Δ is disconnected, $\operatorname{depth} \Bbbk[\Delta] = 1$.

We claim that

$$\Delta = \left(\bigsqcup_{i \in [d]} [i, \alpha_i] \right) \sqcup \left(\bigsqcup_{i \in [d-1]} [d+i, \beta_i] \right) \sqcup [\varnothing, \gamma] \tag{2.4.1}$$

is a partitioning. Let I_τ be the interval from the above decomposition whose unique minimal face is τ. Except for the interval $I_\varnothing = [\varnothing, (2d)\dots(3d-1)]$, all these minimal faces τ are vertices in $[2d-1]$.

Let $\sigma \in \Delta$. If $\sigma \subseteq (2d)\dots(3d-1)$, then σ is only contained in $I_\varnothing$.

Let $\sigma = \sigma_1 \dots \sigma_m \notin I_\varnothing$, where σ_i are vertices and $\sigma_1 < \cdots < \sigma_m$. If $\sigma_m - \sigma_1 < d$, then $\sigma \in I_{\sigma_1}$. If instead $\sigma_m - \sigma_1 \geq d$, then $\sigma \in I_{\sigma_i}$, where $i = \min\{j \ : \ j > d \text{ and } \sigma_j \in \sigma\}$. Thus the above is a partitioning for Δ. $\square$

Example 2.4.5 is the $d = 2$ case of this construction. Here is the construction when $d = 3$: Let $\Delta = \langle 123, 234, 345, 145, 125, 678 \rangle$, so $\dim \Bbbk[\Delta] = 3$ and $\operatorname{depth} \Bbbk[\Delta] = 1$. Then

$$\Delta = [1, 123] \sqcup [2, 234] \sqcup [3, 345] \sqcup [4, 145] \sqcup [5, 125] \sqcup [\varnothing, 678]$$

is a partitioning of Δ.

2.5 Partition extenders

If Δ is pure and partitionable, then Proposition 1.3.7 gives a specific combinatorial interpretation for the h-vector of Δ. However, if Δ is not partitionable, then no such interpretation exists, so the question remains: What, if anything, does the h-vector of a simplicial complex count in general?

Recall that the h-vector can contain negative entries.

Example 2.5.1. Consider the bowtie complex $B = \langle 123, 345 \rangle$.

Then $f(B) = (1, 5, 6, 2)$ and $h(B) = (1, 2, -1, 0)$, so Δ cannot be partitionable. However, we can extend B to the complex $B' = \langle 123, 234, 345 \rangle$.

B $\qquad\qquad$ B'

There are two pertinent facts about B'. First, it is shellable, hence partitionable. Second, the relative complex $(B',B) = \{24,234\}$ is also partitionable. As a consequence, we can now write the h-vector of B as

$$h(B) = h(B') - h(B',B).$$

In other words, we can write the h-vector as the difference of the h-vectors of two related complexes (one of which is a relative complex). This idea gives rise to *partition extenders*, which are introduced in [DGL18].

Definition 2.5.2. Let Δ be a pure simplicial complex. Then a pure complex $\Gamma \supseteq \Delta$ is a **partition extender** of Δ if

1. $\dim\Gamma = \dim\Delta$,

2. Γ is partitionable, and

3. (Γ,Δ) is partitionable.

The following observation motivates the search for partition extenders: If Γ is a partition extender for Δ, then

$$h(\Delta) = h(\Gamma) - h(\Gamma,\Delta)$$

and thus $h(\Delta)$ can be written as the difference of two h-vectors of partitionable complexes. Note that for any complexes $A \subseteq B$ of the same dimension, we can write $h(B,A) = h(B) - h(A)$ (but this is not true if $\dim A < \dim B$).

Proposition 2.5.3. [DGL18] If Δ is a pure simplicial complex, then Δ has a partition extender.

To create a partition extender for Δ, we first greedily construct a maximal partitionable subcomplex $\Upsilon \subseteq \Delta$. For every face $\sigma \in \Delta \setminus \Upsilon$, we add a new complex Γ_σ in such a way that the relative complexes $(\Gamma_\sigma, \langle \sigma \rangle)$ and $(\Gamma_\sigma, \langle \partial\sigma \rangle)$ are both partitionable. Then $\Gamma = \Upsilon \cup \bigsqcup_{\sigma \in \Delta \setminus \Upsilon} \Gamma_\sigma$ is a partition extender for Δ.

We can ask if similar objects exist if we replace "partitionable" in Definition 2.5.3 with other properties that pertain to both simplicial and relative complexes. For example, we have been able to say the following about *Cohen–Macaulay extenders*.

Proposition 2.5.4. [DGL18] A simplicial complex Δ has a Cohen–Macaulay extender if and only if $\operatorname{depth} \Bbbk[\Delta] \geq \dim \Bbbk[\Delta] - 1$.

Chapter 3

Balanced complexes

3.1 Balanced complexes

In [Sta79], Stanley introduced the notion of *balanced complexes*, which are simplicial complexes equipped with certain additional combinatorial structure. In the following years, balanced complexes have generated a considerable amount of attention; see, for example [BFS87], [BN06], [Fro08], [BVT13], [JKV17]. While Stanley originally gave a more general definition, we will focus here on what he originally called *completely* balanced complexes in [Sta79].

Definition 3.1.1. A $(d-1)$-dimensional simplicial complex is **balanced** if its vertices can be colored using d colors so that every face $\sigma \in \Delta$ contains at most one vertex of each color.

Stanley also assumed purity in his definition [Sta79, p. 143] since he was primarily interested in balanced Cohen–Macaulay complexes. We will not assume balanced complexes are pure unless otherwise noted. Furthermore, we will often assume not only that the vertices of Δ can be colored as in Definition 3.1.1 but also that a balanced complex comes equipped with a specific coloring. We can think of this coloring as an ordered partition on the vertices of Δ as $V = V_1 \sqcup \cdots \sqcup V_d$ where

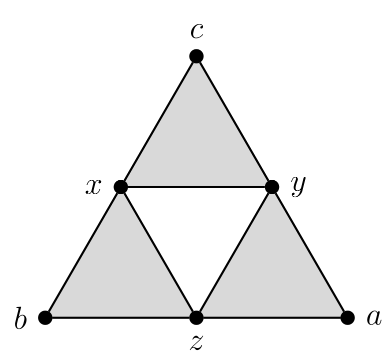
$V_i = \{\text{vertices of color } i\} = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$ such that for all $\sigma \in \Delta$ and for all $i \in [d]$, $|\sigma \cap V_i| \leq 1$.

A $(d-1)$-dimensional complex is balanced if and only if its 1-skeleton $\mathrm{skel}_1 \Delta$ can be properly colored (in the graph theoretical sense) using d colors. Balancedness is not a topological property. For example, a 3-cycle is not balanced because it is not two-colorable, but a 4-cycle is balanced (a graph is balanced precisely if it is bipartite). In fact any complex Δ is homeomorphic to a balanced complex, its barycentric subdivision. This is part of a larger phenomenon.

Proposition 3.1.2. Let Δ be an order complex of a finite poset, i.e., $\Delta = \mathcal{O}(P)$ for a finite poset P. Then Δ is balanced.

To see that Proposition 3.1.2 is true, for each element $x \in P$, consider the longest chain in P such that $x_1 < x_2 < \cdots < x_{j-1} < x$. Assign x the color j. Then no chain in P will contain two elements of the same color and thus $\Delta = \mathcal{O}(P)$ is balanced. We note that if P is ranked, then this is equivalent to coloring x by $r(x)$, the rank of x, but even non-ranked posets have balanced (albeit nonpure) order complexes.

There are also many balanced complexes that do not arise as order complexes of posets. Consider the following example.



The above example is balanced, but it is not an order complex. Recall that order complexes are flag, i.e., the minimal non-faces of Δ all have cardinality 2. However, we can see that xyz is a minimal non-face.

Let Δ be a balanced complex with a given coloring. We can consider refinements of the original f- and h-vectors. Given a face $\sigma \in \Delta$, we will define

$$\operatorname{color}(\sigma) = \{i \,:\, v_{i,j} \in \sigma \text{ for some } j \in [n_i]\} = \{i \,:\, \sigma \text{ contains a vertex of color } i\}.$$

The **flag f-vector** of a balanced complex is

$$(\alpha_\Delta(C))_{C \subseteq [d]}$$

where $\alpha_\Delta(C) = |\{\sigma \,:\, \operatorname{color}(\sigma) = C\}|$. We similarly define the **flag h-vector** as

$$(\beta_\Delta(C))_{C \subseteq [d]}$$

where $\beta_\Delta(C) = \sum_{D \subseteq C} (-1)^{|C \setminus D|} \alpha_\Delta(D)$.

These flag vectors are refinements of the original f- and h-vectors; in other words, $f_i(\Delta) = \sum_{|C|=i+1} \alpha_\Delta(C)$ and $h_i(\Delta) = \sum_{|C|=i} \beta_\Delta(C)$. The balanced structure of Δ produces a $\mathbb{Z}^d$-grading on the face ring $\Bbbk[\Delta]$, since no face contains more than one vertex of any of the d colors. Similarly the flag h-vector gives the coefficients of the $\mathbb{Z}^d$-graded Hilbert series of $\Bbbk[\Delta]$.

Proposition 3.1.3. [Sta79, Proposition 3.2] If Δ is a $(d-1)$-dimensional balanced complex with a given coloring, then

$$\operatorname{Hilb}(\Bbbk[\Delta]; t_1, \ldots, t_d) = \frac{\sum_C \beta_\Delta(C) \prod_{i \in C} t_i}{(1-t_1) \ldots (1-t_d)}$$

is the $\mathbb{Z}^d$-graded Hilbert series of $\Bbbk[\Delta]$.

Compare the above to Proposition 1.4.2; in particular, if we set $t_i = t$ for all t_i, then this recovers Proposition 1.4.2.

Furthermore, there is an analogue of Proposition 1.3.7 for balanced complexes.

Proposition 3.1.4. [Sta79, p. 149] If Δ is a pure, balanced, partitionable complex, then

$$\beta_\Delta(C) = |\{R_i \ : \ \mathrm{color}(R_i) = C\}|$$

where R_i are the restriction faces of a partitioning of Δ.

We will often focus our attention on balanced complexes that are also Cohen–Macaulay (in fact, this is the context in which Stanley originally defined these complexes). If Δ is balanced and $C \subseteq [d]$, its C**-color-selected subcomplex** (often referred to as the C**-rank-selected subcomplex**) is

$$\Delta_C = \{\sigma \in \Delta \ : \ \mathrm{color}(\sigma) \subseteq C\}.$$

In other words, we restrict the complex to vertices with colors in C. There is a connection between the flag h-vector and the reduced Euler characteristics of these color-selected subcomplexes; in particular, $\beta_\Delta(C) = (-1)^{|C|-1}\chi(\tilde{\Delta}_C)$ [Sta79, Proposition 3.5]. We will be more interested in the following.

Theorem 3.1.5. Let Δ be a $(d-1)$-dimensional balanced Cohen–Macaulay complex and let $C \subseteq [d]$.

1. [Sta79, follows from Theorem 4.4] $\beta_\Delta(C) \geq 0$.

2. [Sta79, Theorem 4.3] Δ_C is also Cohen–Macaulay.

Part 1 follows from Proposition 3.1.3. Part 2 follows from the following argument: A balanced CM complex has a system of parameters that is homogeneous in the finely graded sense [Sta96, Proposition III.4.3], and restricting to any set of colors C gives a homogeneous system of parameters of the face ring of Δ_C [Sta96, Theorem III.4.5].

Even with the example from Proposition 2.1.8, it was still unknown whether *balanced* CM complexes were partitionable. However, Juhnke-Kubitzke and Venturello recently showed the following.

Theorem 3.1.6. [JKV17] There exist balanced Cohen–Macaulay complexes that are not partitionable.

If Δ is assumed to be balanced, then the construction in Theorem 2.1.6 produces a non-partitionable balanced simplicial complex as well. Juhnke-Kubitzke and Venturello were able to show that an appropriate subdivision of the relative complex in Proposition 2.1.8 is balanced, and thus produced a non-partitionable balanced Cohen–Macaulay complex.

In [Gar80], Garsia made the following conjecture, which remains open.

Conjecture 3.1.7. [Gar80, Remark 5.2] If Δ is a Cohen–Macaulay order complex, then Δ is partitionable.

It is unclear whether a similar type of argument as in Theorem 2.1.6 would apply to provide a counterexample to Conjecture 3.1.7. If the conjecture is true, it would imply that the barycentric subdivision of a CM complex is partitionable, since the barycentric subdivision of a complex is the order complex of its face poset not including the empty set. If the conjecture is false, we may ask the following question:

Question 3.1.8. If Δ is CM, is there a number of barycentric subdivisions we can take to guarantee that the resulting complex is partitionable? Is this number constant for all

CM complexes; for example, is one barycentric subdivision always enough to guarantee partitionability?

3.2 Color shifting

Algebraic shifting has proved very powerful in the study of Cohen–Macaulay complexes. However, if Δ is balanced, then its shifting $S(\Delta)$ is no longer guaranteed to be balanced (in fact, it generally isn't). To extend the idea of algebraic shifting to balanced complexes while preserving the balanced structure, Babson and Novik introduced *color shifting* in [BN06]. Rather than creating a generic initial ideal as in algebraic shifting, the idea is to construct an initial ideal that is only generic within each color class of vertices.

Definition 3.2.1. Let Δ be a balanced complex such that each of its color classes $V_i = \{v_{i,1}, \dots, v_{i,n_i}\}$ is totally ordered. We say that Δ is **color-shifted** if $v_{i,j} \in \sigma \in \Delta$ implies that $(\sigma \setminus \{v_{i,j}\}) \cup \{v_{i,k}\} \in \Delta$ for all $k < j$.

Note that the order on the vertices within each color class is needed. This definition is analogous to Definition 1.6.1 but with the added condition that we only decrease the variable index within each color class.

We will define a slightly different version of color shifting from what is discussed in [BN06]; Babson and Novik define color shifting based on symmetric shifting whereas our definition comes from exterior shifting. Thus our definition will use the exterior Stanley-Reisner ring $\Lambda[\Delta]$.

Instead of defining generic linear forms of all of the vertices together as in Definition 1.6.3, we will only make generic linear forms supported on each color class. This will allow us to preserve the balancedness/color structure of Δ.

Set $i \in [d]$. Working in $\Lambda[\Delta]$, define the **color-generic linear forms of color** i to be

$$f_{i,j} = \sum_{k=1}^{n_i} \alpha_{i,j,k} v_{i,k} \tag{3.2.1}$$

for $j \in [n_i]$ and where $\alpha_{i,j,k}$ are all algebraically independent (adjoin indeterminates to $\Bbbk$ as needed). We will often refer to the forms in 3.2.1 as **generic vertices**, as these turn out to be the vertices of the color shifting of Δ, which will be defined below. Similarly, sets of these $f_{i,j}$ will correspond to faces of the color shifting when they satisfy an appropriate criterion.

Given a set $T = \{f_{i_1,j_1}, \dots, f_{i_\ell,j_\ell}\}$ of these generic linear forms, define

$$f_T = f_{i_1,j_1} \wedge \cdots \wedge f_{i_\ell,j_\ell}. \tag{3.2.2}$$

Recall that rearranging the order of these terms can change these by a negative sign, so we will in general assume that these sets are presented with the forms in lexicographic order. Since Δ is assumed to be balanced, we note that if $i_m = i_n$ for any $f_{i_m,j_m}, f_{i_n,j_n} \in T$, then $f_T = 0$ in $\Lambda[\Delta]$.

With the above generic color forms, we can define the following.

Definition 3.2.2. Let Δ be a balanced simplicial complex on $[n]$ with exterior Stanley-Reisner ring $\Lambda[\Delta]$. Then the **(exterior) color shifting** of Δ is

$$CS(\Delta) = \{T \subseteq [n] \ : \ f_T \notin \text{span}\{f_R \ : \ |R| = |T| \text{ and } R <_{\text{lex}} T\}\}$$

where $<_{\text{lex}}$ denotes lexicographic order.

Notice that this definition is identical to Definition 1.6.3 except for the generic forms in use. In Definition 1.6.3, *fully* generic forms are used while in Definition 3.2.2 we restrict the generic forms to each color class to preserve the balanced structure.

3.3 Colorated cohomology

We now will define a colored version of iterated homology, which was discussed in Section 1.7. The following definitions and results are balanced analogues of similar ones appearing in [DR00] and [DZ01].

Given a set of colors $C = \{i_1, i_2, \dots, i_\ell\} \subseteq [d]$, define its **initial vertices** to be $\mathrm{init}(C) = \{f_{i_1,1}, f_{i_2,1}, \dots, f_{i_\ell,1}\}$ and define

$$f_C := f_{\mathrm{init}(C)} = f_{i_1,1} \wedge f_{i_2,1} \wedge \cdots \wedge f_{i_\ell,1}$$

where these $f_{i,j}$ are defined as in (3.2.1).

Working in the exterior Stanley-Reisner ring $\Lambda[\Delta]$, we define the following groups. Let $C \subseteq [d]$ be a set of colors, and let $j \in [d]$ be a specific color.

$$\begin{aligned}
\Lambda^k_C(\Delta) &= f_C \wedge \Lambda^{k-|C|}[\Delta] \\
Z^k_{C,j}(\Delta) &= \left\{ x \in \Lambda^k_C(\Delta) \ : \ f_{j,1} \wedge x = 0 \right\} \\
B^k_{C,j}(\Delta) &= f_{j,1} \wedge \Lambda^{k-1}_C(\Delta) \\
H^k_{C,j}(\Delta) &= Z^k_{C,j}(\Delta) / B^k_{C,j}(\Delta)
\end{aligned} \tag{3.3.1}$$

The groups $H^k_{C,j}$ are the **colorated cohomology groups** of Δ. Notice that if $j \in C$, then these simplify to

$$Z^k_{C,j}(\Delta) = \Lambda^k_C(\Delta)$$
$$B^k_{C,j}(\Delta) = 0$$

and if $j \notin C$, then

$$B^k_{C,j}(\Delta) = \Lambda^k_{C \cup \{j\}}(\Delta).$$

We want to consider the **colorated Betti numbers** $\beta^k_{C,j}(\Delta)$, which are defined as

$$\beta^k_{C,j} = \dim_{\Bbbk}\left(H^k_{C,j}\right)$$

the dimensions of $H^k_{C,j}(\Delta)$ as a vector space over $\Bbbk$. First, we need a few more preliminary definitions.

Let $\Gamma = CS(\Delta)$, the color shifting of Δ. If $T \in \Gamma$, then $\operatorname{ics}(T) = \{j \,:\, f_{j,1} \in T\}$ is the **initial color segment** of T. Define

$$\Gamma^k_C = \{T \in \Gamma = CS(\Delta) \,:\, |T| = k+1 \text{ and } C \subseteq \operatorname{ics}(T)\}.$$

Lemma 3.3.1. $\Lambda^k_C(\Delta) = \operatorname{span}\left\{f_T \,:\, T \in \Gamma^k_C\right\}$. In fact, these f_T form a basis of $\Lambda^k_C(\Delta)$.

Proof. Let $y \in \Lambda^k_C(\Delta)$. Then $y = f_C \wedge x$ where $x \in \Lambda^{k-|C|}[\Delta]$ by the definition of $\Lambda^k_C(\Delta)$. We can write $x = \sum \gamma_R f_R$ where these f_R are color generic forms as defined in Equations 3.2.1 and 3.2.2. Since Δ is balanced, $f_C \wedge f_R = 0$ in $\Lambda[\Delta]$ whenever $\operatorname{color}(R) \cap C \neq \varnothing$. Therefore

$$y = \sum_{\operatorname{color}(R) \cap C = \varnothing} \gamma_R f_{R \cup \operatorname{init}(C)}$$

and so $y \in \operatorname{span}\left\{f_T \,:\, T \in \Gamma_C^k\right\}$.

Say instead that $T \in \Gamma_C^k$. Then

$$f_T = f_C \wedge f_{\text{color}(T)\setminus C} \in \Lambda_C^k(\Delta).$$

Lastly, notice that these f_T are independent by the definition of the color shifting of Δ, so they form a basis of their span. □

We will use Lemma 3.3.1 in the proof of the following proposition, which relates the colorated Betti numbers of Δ to $CS(\Delta)$, the color shifting of Δ.

We first note if $j \in C$, then

$$\beta_{C,j}^k = \dim_{\Bbbk} H_{C,j}^k = \dim_{\Bbbk} Z_{C,j}^k = \dim \Lambda_C^k = \left|\Gamma_C^k\right|$$

So if $j \in C$, these colorated Betti numbers do not provide us with any new information relating to color j. However, they are interesting in the case when $j \notin C$.

Proposition 3.3.2. Let Δ be a balanced simplicial complex of dimension $(d-1)$ and $CS(\Delta)$ be the color-shifting of Δ. Then the colorated Betti numbers of Δ are

$$\beta_{C,j}^k = |\{T \in CS(\Delta) \,:\, |T| = k+1,\ \operatorname{init}(C) \subseteq T,\ f_{j,1} \notin T, \text{ and } T \cup \{f_{j,1}\} \notin CS(\Delta)\}|$$

if $j \notin C$.

Proof. Let C be a set of colors and T be a face of $CS(\Delta)$. Recall that $\operatorname{init}(C) = \{f_{j,1} \,:\, j \in C\}$ is the set of initial vertices of C and that $\operatorname{ics}(T) = \{j \,:\, f_{j,1} \in T\}$ is the initial color segment of T. We can see that $\operatorname{init}(C) \subseteq T$ if and only if $C \subseteq \operatorname{ics}(T)$.

We will be performing all calculations over the field $\Bbbk$, so in particular the groups defined in (3.3.1) are all finite-dimensional vector spaces. By rank-nullity,

$$\dim Z^k_{C,j}(\Delta) = \dim \Lambda^k_C(\Delta) - \dim B^{k+1}_{C,j}(\Delta).$$

Thus we have

$$\begin{aligned}
\beta^k_{C,j} = \dim H^k_{C,j} &= \dim Z^k_{C,j}(\Delta) - \dim B^k_{C,j}(\Delta) \\
&= \dim \Lambda^k_C(\Delta) - \dim B^{k+1}_{C,j}(\Delta) - \dim B^k_{C,j}(\Delta) \\
&= \dim \Lambda^k_C(\Delta) - \dim \Lambda^{k+1}_{C\cup\{j\}}(\Delta) - \dim \Lambda^k_{C\cup\{j\}}(\Delta)
\end{aligned}$$

We now calculate these dimensions. Recall that $\Gamma^k_C = \{T \in CS(\Delta) \ : \ |T| = k+1 \text{ and } C \subseteq \operatorname{ics}(T)\}$. By Lemma 3.3.1,

$$\begin{aligned}
\dim \Lambda^k_C(\Delta) = \left|\Gamma^k_C\right| &= |\{T \in CS(\Delta) \ : \ |T| = k+1 \text{ and } C \subseteq \operatorname{ics}(T)\}| \\
&= |\{T \in CS(\Delta) \ : \ |T| = k+1 \text{ and } \operatorname{init}(C) \subseteq T\}|.
\end{aligned}$$

Similarly, since $j \notin C$, thus

$$\dim \Lambda^k_{C\cup\{j\}}(\Delta) = \left|\Gamma^k_{C\cup\{j\}}\right| = \left|\left\{T \in \Gamma^k_C \ : \ f_{j,1} \in T\right\}\right|.$$

Finally, considering the bijection $T \mapsto T' = T \cup \{f_{j,1}\}$,

$$\begin{aligned}
\dim \Lambda^{k+1}_{C\cup\{j\}}(\Delta) &= \left|\left\{T' \in \Gamma^{k+1}_C \ : \ \operatorname{init}(C\cup\{j\}) \subseteq T'\right\}\right| \\
&= \left|\left\{T \in \Gamma^k_C \ : \ f_{j,1} \notin T,\ T \cup \{f_{j,1}\} \in CS(\Delta)\right\}\right|
\end{aligned}$$

Therefore

$$\begin{aligned}\beta^k_{C,j} = & |\{T \in CS(\Delta) \ : \ |T| = k+1 \text{ and } \operatorname{init}(C) \subseteq T\}| \\ & - \left|\left\{T \in \Gamma^k_C \ : \ f_{j,1} \notin T,\, T \cup \{f_{j,1}\} \in CS(\Delta)\right\}\right| \\ & - \left|\left\{T \in \Gamma^k_C \ : \ f_{j,1} \in T\right\}\right| \\ = & |\{T \in CS(\Delta) \ : \ |T| = k+1,\ \operatorname{init}(C) \subseteq T,\ f_{j,1} \notin T, \text{ and } T \cup \{f_{j,1}\} \notin CS(\Delta)\}|\end{aligned}$$

when $j \notin C$. □

Proposition 3.3.2 is a colored analogue of Proposition 1.7.2. However, the key distinction between these two results is that Proposition 1.7.2 counts specific *facets* of the shifting $S(\Delta)$ whereas Proposition 3.3.2 counts faces that are not necessarily maximal in the color shifting $CS(\Delta)$. It is this seemingly small detail that makes searching for *balanced* boolean tree decompositions of balanced Cohen–Macaulay complexes noticeably different from the version presented in [DZ01]. We will discuss this issue in more depth in Section 3.4.

Definition 3.3.3. Let $C \subseteq [d]$ and $j \in [d] \setminus C$. We call a collection of faces B of a simplicial complex Δ a C, j-**colorated Betti set** if $\operatorname{color}(\sigma) \cap (C \cup \{j\}) = \{j\}$ for all $\sigma \in B$ and $f_{k-|C|}(B) = \beta^k_{C,j}(\Delta)$ for all k.

The following is our main result on colorated cohomology. It allows us to decompose a balanced complex in a way that preserves its balanced structure. It is a balanced analogue of Theorem 1.7.4.

Theorem 3.3.4. Let Δ be a pure, balanced, $(d-1)$-dimensional simplicial complex. Specify the coloring of Δ and order the colors 1 through d. Then there exist sets $\Delta^{(i)}$ (for $0 \leq i \leq d$), $B^{(i)}$ (for $0 \leq i \leq d$), and $\Omega^{(i)}$ (for $1 \leq i \leq d$) with the following properties:

1. $\Delta^{(i)}$ is the complex formed by removing all vertices of colors 1 through i.

2. $\eta^{(i)} : \Delta^{(i)} \to \Omega^{(i)}$ is a bijection, defined $\eta^{(i)}(\sigma) = \sigma \cup \{v_{i,j}\}$ where $j = \max\left\{k \ : \ \sigma \cup \left\{v_{i,k}\right\} \in \Delta^{(i-1)}\right\}$.

3. $\Delta^{(i)} = \Delta^{(i+1)} \sqcup B^{(i)} \sqcup \Omega^{(i+1)}$.

4. $\Delta^{(i+1)}$ and $\Delta^{(i+1)} \sqcup B^{(i)}$ are subcomplexes of $\Delta^{(i)}$.

5. $B^{(i)}$ is enumerated by $\beta_{[i],i+1}(\Delta)$; i.e., $B^{(i)}$ is a colorated Betti set.

Proof. Recall that if $C \subseteq [d]$, then the C-color-selected complex of Δ is defined as $\Delta_C = \{\sigma \in \Delta \ : \ \text{color}(\sigma) \subseteq C\}$. For a color $i \in [d]$, recall that $V_i = \{v_{i,1}, \ldots, v_{i,n_i}\}$ is the set of vertices of color i.

For any subset $\Delta' \subseteq \Delta$, we define $\Bbbk\Delta'$ to be the $\Bbbk$-span within $\Lambda[\Delta]$ of monomials corresponding to faces of Δ'. In other words, if $\sigma = \{v_{i_1,j_1}, \ldots, v_{i_\ell,j_\ell}\} \in \Delta'$ then we consider $v_{i_1,j_1} \wedge \cdots \wedge v_{i_\ell,j_\ell}$ in $\Bbbk\Delta'$.

For $i \in [d]$ and $x \in \Lambda[\Delta]$, define the map $\delta_i(x) = (v_{i,1} + \cdots + v_{i,n_i}) \wedge x$. Let δ_0 be the identity map and $\delta_{(i)} = \delta_i \ldots \delta_1$.

STEP 1 – <u>Defining $\Delta^{(i)}$</u> : Define $\Delta^{(i)} = \Delta_{[d]\setminus[i]}$, the color selected subcomplex on the final $d-i$ colors. Notice that $\Delta^{(0)} = \Delta$. For $i > 0$, define

$$I_i = \delta_{(i)}\Bbbk\Delta$$

where $\Bbbk\Delta$ is the $\Bbbk$-span of faces of Δ in $\Lambda[\Delta]$. Notice that if $\text{color}(\sigma) \cap [i] \neq \varnothing$, then $\delta_{(i)}\sigma = 0$ since Δ is balanced. Therefore

$$I_i = \delta_{(i)}\Bbbk\Delta_{[d]\setminus[i]} = \delta_{(i)}\Bbbk\Delta^{(i)}.$$

If $\sigma, \tau \in \Delta^{(i)}$ and $\sigma \neq \tau$, then the support of the images of σ and τ under $\delta_{(i+1)}$ are disjoint, and thus $\delta_{(i+1)}$ is injective on the restricted domain of $\Bbbk\Delta^{(i)}$. Since $\left\{\sigma \ : \ \sigma \in \Delta^{(i)}\right\}$ is a basis for $\Bbbk\Delta^{(i)}$, therefore $\left\{\delta_{(i)}\sigma \ : \ \sigma \in \Delta^{(i)}\right\}$ is also a basis for $\delta_{(i)}\Bbbk\Delta^{(i)} = I_i$.

STEP 2 – Defining $B^{(i)}$: Let L_i be the lexicographically least basis of

$$K_i = \Bbbk\delta_{(i)}\Delta^{(i)}/\Bbbk\delta_{(i+1)}\Delta^{(i)}.$$

Then define

$$B^{(i)} = \left\{\sigma \in \Delta^{(i)} \ : \ \delta_{(i)}\sigma \in L_i\right\} \setminus \Delta^{(i+1)}$$

We can rewrite this given our description of $\Delta^{(i)}$ above:

$$\begin{aligned} B^{(i)} &= \left\{\sigma \in \Delta^{(i)} \setminus \Delta^{(i+1)} \ : \ \delta_{(i)}\sigma \in L_i\right\} \\ &= \left\{\sigma \in \Delta \ : \ \mathrm{color}(\sigma) \cap [i+1] = \{i+1\} \text{ and } \delta_{(i)}\sigma \in L_i\right\} \end{aligned}$$

Returning to K_i, we see that

$$\begin{aligned} K_i &= \Bbbk\delta_{(i)}\Delta^{(i)}/\Bbbk\delta_{(i+1)}\Delta^{(i)} \\ &= \Bbbk\delta_{(i)}\Delta^{(i)}/\Bbbk\delta_{(i+1)}\Delta^{(i+1)} \\ &= \delta_{(i)}\left[\Bbbk\Delta^{(i)}/\Bbbk\delta_{i+1}\Delta^{(i+1)}\right] \\ &= \delta_{(i)}\left[\Bbbk\Delta^{(i+1)} \oplus \bigoplus_{\sigma \in \Delta^{(i+1)}} \Bbbk\left\{\sigma \cup \{v\} \ : \ v \in V_{i+1}\right\}/\delta_{i+1}\sigma\right] \end{aligned}$$

with the last equality because the faces of $\Delta^{(i+1)}$ are not affected by imposing the new relation. Modding out by $\delta_{i+1}\sigma$ gives us precisely one new relation among the faces

$\sigma \cup \{v\}$ for $v \in V_{i+1}$ in K_i. If k is the largest index such that $\sigma \cup \left\{v_{i+1,k}\right\} \in \Delta^{(i)}$, then

$$v_{i+1,k} \wedge \sigma = \sum_{j<k} v_{i+1,j} \wedge \sigma$$

in K_i. (Note that it is possible that this sum on the right is zero, if $v_{i+1,k}$ is the only vertex of color $i+1$ that can be added to σ.) Therefore

$$\begin{gathered} B^{(i)} = \{\sigma \in \Delta \ : \ \text{color}(\sigma) \cap [i+1] = \{i+1\}\, , \ v_{i+1,j} \in \sigma \ \text{ for some } j, \\ \text{and } (\sigma \setminus \{v_{i+1,j}\}) \cup \left\{v_{i+1,k}\right\} \in \Delta \text{ for some } k > j\}. \end{gathered}$$

By the definition of $B^{(i)}$ above, we see that $\Delta^{(i+1)} \sqcup B^{(i)}$ is a (lex least) basis for $\delta_{(i)} \Bbbk \Delta^{(i)} / \delta_{(i+1)} \Bbbk \Delta^{(i)}$.

STEP 3 – Defining $\Omega^{(i+1)}$: For each face $\sigma \in \Delta^{(i)}$, define the map η by

$$\eta(\sigma) = \sigma \cup \{v_{i,j}\} \text{ where } j = \max\left\{k \ : \ \sigma \cup \left\{v_{i,k}\right\} \in \Delta^{(i-1)}\right\}$$

and then define $\Omega^{(i)}$ to be the image of $\Delta^{(i)}$ under this map. Since Δ is pure, there always exists such a vertex $v_{i,j}$ for each σ. Therefore η and Ω are well-defined, and $\Omega^{(i+1)} \subseteq \Delta^{(i)}$. We also note that $B^{(i)} \cap \Omega^{(i+1)} = \varnothing$. Thus

$$\Delta^{(i)} = \Delta^{(i+1)} \sqcup B^{(i)} \sqcup \Omega^{(i+1)} \tag{3.3.2}$$

for each i.

STEP 4 – Showing that $\Delta^{(i+1)}$ and $\Delta^{(i+1)} \sqcup B^{(i)}$ are subcomplexes of $\Delta^{(i)}$: By definition, $\Delta^{(i+1)}$ is a subcomplex of $\Delta^{(i)}$. Consider instead $\Gamma^{(i+1)} = \Delta^{(i+1)} \sqcup B^{(i)}$. By definition, $\Gamma^{(i+1)} \subseteq \Delta^{(i)}$ as sets; we must show that $\Gamma^{(i+1)}$ is itself a simplicial complex.

Since $\Delta^{(i+1)}$ is a simplicial complex, this reduces to showing that if $\sigma \in B^{(i)}$, then $\tau \in \Gamma^{(i+1)}$ for each $\tau \subseteq \sigma$.

Let $\sigma \in B^{(i)}$ and say $\sigma = \sigma' \cup \{v_{i+1,j}\}$. Then $\sigma' \in \Delta^{(i+1)}$, so we need only show that the face $\tau = \tau' \cup \{v_{i+1,j}\}$ belongs to $\Gamma^{(i+1)}$ for each proper face $\tau' \subsetneq \sigma'$. Note that $\tau \notin \Delta^{(i+1)}$, so by the decomposition (3.3.2), it follows that $\tau \in B^{(i)} \sqcup \Omega^{(i+1)}$. If $\tau \in \Omega^{(i+1)}$ then $\tau' \cup \left\{v_{i+1,k}\right\} \notin \Delta^{(i)}$ for all $k > j$, by the construction of $\Omega^{(i+1)}$. But then also $\sigma' \cup \left\{v_{i+1,k}\right\} \notin \Delta^{(i)}$ for all $k > j$, which is precisely the statement that $\sigma \in \Omega^{(i+1)}$, a contradiction since $B^{(i)} \cap \Omega^{(i+1)} = \varnothing$. Therefore $\Gamma^{(i+1)}$ is a simplicial complex.

STEP 5 – <u>Showing that the $B^{(i)}$ are colorated Betti sets</u> : We will first show that $\left|B^{(i)}\right| = \beta_{[i],i+1}$. Consider the map

$$\delta_{i+1} : I_i \to I_{i+1}$$

where $I_i = \delta_{(i)}\Bbbk\Delta$ as in Step 1. By the definition of $\Delta^{(i)}$, we know that $\left|\Delta^{(i+1)}\right| = \dim I_{i+1} = \dim\left(\operatorname{im}\delta_{i+1}\right)$. By rank-nullity, thus

$$\left|\Delta^{(i+1)}\right| = \dim I_{i+1} = \dim\left(\operatorname{im}\delta_{i+1}\right) = \dim\left(I_i\right) - \dim\left(\ker\delta_{i+1}\right).$$

Similarly, using the definition of $B^{(i)}$, we have

$$\begin{aligned}\left|\Delta^{(i+1)} \sqcup B^{(i)}\right| &= \dim\left(\delta_{(i)}\Bbbk\Delta^{(i)}/\delta_{(i+1)}\Bbbk\Delta^{(i)}\right)\\ &= \dim\left(\delta_{(i)}\Bbbk\Delta/\delta_{(i+1)}\Bbbk\Delta\right)\\ &= \dim\left(I_i\right) - \dim\left(\operatorname{im}\delta_{i+1}\right).\end{aligned}$$

Combining the above, we see that

$$\left|B^{(i)}\right| = \left|\Delta^{(i+1)} \sqcup B^{(i)}\right| - \left|\Delta^{(i+1)}\right| = \dim(\ker \delta_{i+1}) - \dim(\operatorname{im} \delta_{i+1}).$$

Therefore $\left|B^{(i)}\right| = \beta_{[i],i+1}$. We now note that this equality holds if we extend this to including k, the cardinalities of the faces involved, as cohomology acts on each graded piece separately. $\square$

3.4 Decompositions of balanced complexes

In [JKV17], Juhnke-Kubitzke and Venturello showed that there exist balanced Cohen–Macaulay complexes that are not partitionable. This prompts the question of whether balanced Cohen–Macaulay complexes admit some weaker decomposition that provides a combinatorial explanation for the positivity of their flag h-vectors. Duval and Zhang showed in [DZ01, Theorem 5.4] (reproduced here as Theorem 2.1.12) that if Δ is Cohen–Macaulay, then it admits an honest boolean tree decomposition. However, in the case that Δ is balanced, the boolean trees arising from the Duval–Zhang construction do not explain positivity of the flag h-vector.

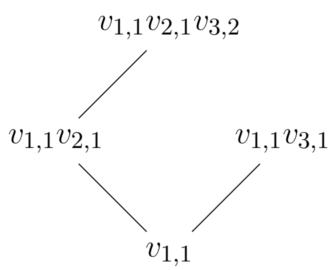
Given $P \subseteq P(\Delta)$, a subposet of the face poset of a balanced complex Δ, we define its **color poset** as $\mathcal{C}(P) = \{\mathrm{color}(x) \,:\, x \in P\}$.

Definition 3.4.1. A rank-k boolean tree Υ is a **balanced boolean tree** if its color poset $\mathcal{C}(\Upsilon)$ is a rank-k boolean interval.

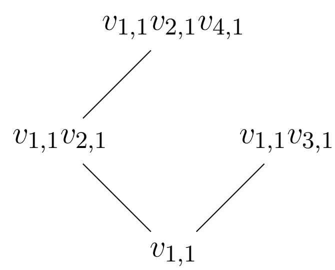
This gives us a precise way to state the above question. Recall that a decomposition of a simplicial complex into disjoint boolean trees an *honest* boolean tree decomposition if the tops of each of the boolean trees in the decomposition are facets of Δ.

Conjecture 3.4.2. (Martin, unpublished) If Δ is a balanced Cohen–Macaulay complex, then there exists an honest balanced boolean tree decomposition of Δ.

Preserving balancedness is not an issue for intervals—if Δ is a balanced simplicial complex, then any interval in the face poset of Δ is automatically balanced and thus any partitioning interprets the flag h-vector. This is not always the case for boolean trees. Consider the following two boolean trees.



Balanced



Not Balanced

The boolean tree on the left is balanced while the boolean tree on the right is not. We can see that the color poset of the boolean tree on the left is a rank-2 boolean interval while the color poset of the tree on the right is not.

Remark 3.4.3. An equivalent rephrasing of Definition 3.4.1 is the following:

- A rank-0 boolean tree is balanced.
- If two rank-$(k-1)$ balanced boolean trees Υ_1 and Υ_2 have minimal elements r_1 and r_2 such that $r_1 \lessdot r_2$ and $\mathcal{C}(\Upsilon_2) = \{C \cup \{j\} \ : \ C \in \mathcal{C}(\Upsilon_1)\}$ for some color j, then $\Upsilon = \Upsilon_1 \cup \Upsilon_2$ is a rank-k balanced boolean tree.

Proposition 3.4.4. If Δ is pure and balanced, then Δ can be decomposed into disjoint balanced boolean trees such that the number of rank-i balanced boolean trees is the colorated Betti number $\beta_{[i],i+1}(\Delta)$.

Proof. We begin by noting that every face of Δ is itself a rank-0 balanced boolean tree.

Assume that the first i iterations of the theorem have decomposed Δ into a family of balanced boolean trees of rank $\leq i$, and that the minimal elements of these trees are all of the faces in $\Delta^{(i)}$, which is defined in Theorem 3.3.4 as $\Delta^{(i)} = \Delta_{[d]\setminus[i]}$. By Theorem 3.3.4,

$$\Delta^{(i)} = \Delta^{(i+1)} \sqcup B^{(i)} \sqcup \Omega^{(i+1)}$$

and for each $\sigma \in \Delta^{(i+1)}$ there exists $\eta(\sigma) \in \Omega^{(i+1)}$ such that $\eta(\sigma) = \sigma \cup \{v_{i+1,j}\}$ for some vertex $v_{i+1,j} \in V_{i+1}$. For each $\sigma \in \Delta^{(i+1)}$, let Υ_σ be the balanced boolean tree with minimal element σ and $\Upsilon_{\eta(\sigma)}$ the balanced boolean tree with minimal element $\eta(\sigma)$. Then $\Upsilon_\sigma, \Upsilon_{\eta(\sigma)}$ together meet the criterion in Remark 3.4.3 and thus $\Upsilon = \Upsilon_\sigma \cup \Upsilon_{\eta(\sigma)}$ is a rank-$(i+1)$ boolean tree with minimal element $\sigma \in \Delta^{(i+1)}$.

At each step, the only trees that do not get matched to create larger trees are those with minimal elements in $B^{(i)}$. Therefore, at the end of this process, there are $\beta_{[i],i+1}$ balanced boolean trees of rank i, and the minimal element σ of a tree of rank i will have the property that $\operatorname{color}\sigma \cap [i+1] = \{i+1\}$, by the definition of $B^{(i)}$. $\square$

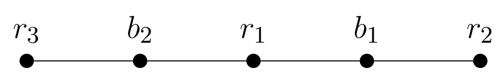
This is a direct analogue of Theorem 1.7.5, which allowed Duval and Zhang to prove Theorem 2.1.12. However, while the above process does create balanced boolean trees, it does not in general match enough faces of Δ to construct *honest* boolean trees in the Cohen–Macaulay case (i.e., it alone cannot prove a balanced version of Theorem 2.1.12). This is due to the fact that the unmatched elements at each step correspond to colorated Betti sets, which in general are much larger than their colorless counterparts.

However, if the complex admits appropriate orderings of colors and color classes, then there is hope of iterating the process outlined in Proposition 3.4.4 to create an *honest* balanced boolean tree decomposition. In particular, after Proposition 3.4.4 is applied to Δ, if we can apply it recursively to each of the $B^{(i)}$, then these applications together will form an honest balanced boolean tree decomposition.

There is at least some hope of doing this when Δ is a balanced Cohen–Macaulay complex, and this is the case we are most interested in. By Proposition 3.1.5 (2), we can see that each of the $\Delta^{(i)}$ themselves are CM in this case. If we could also show that the $B^{(i)}$ are also well behaved, then we could complete this decomposition via induction.

This all hinges on the order placed upon the vertices. It is of note that in the original Duval-Zhang setup, the vertices had an order which affected the matchings, but this order did not affect whether Theorem 1.7.5 would produce an honest boolean tree decomposition for a Cohen–Macaulay complex. However, in the balanced case, it is possible to order the vertices of a Cohen–Macaulay complex so that Proposition 3.4.4 cannot be iterated to produce an honest balanced boolean tree decomposition.

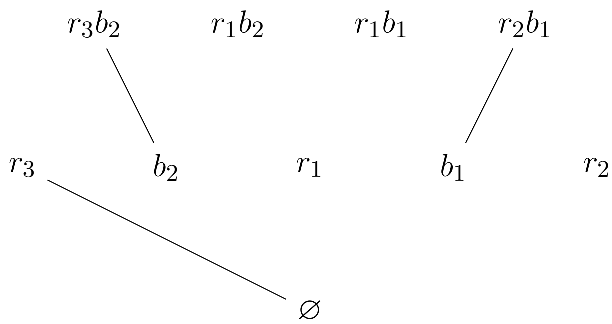
Example 3.4.5. Consider the complex $\Delta = \langle r_1 b_1, r_1 b_2, r_2 b_1, r_3 b_2 \rangle$, which is balanced (the r_i are red vertices and the b_i blue vertices).



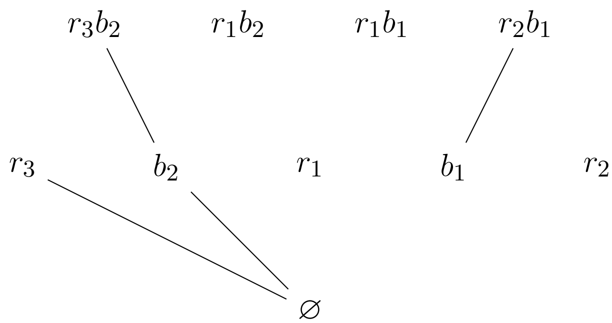
This complex is a connected graph; thus it is partitionable and this partitioning gives an honest balanced boolean tree decomposition of Δ. We order the colors $r < b$ and apply our algorithm for creating balanced boolean trees. Here is the face poset of $\Delta^{(0)} = \Delta$ without any covering relations.

$$\begin{array}{ccccc} & r_3b_2 & r_1b_2 & r_1b_1 & r_2b_1 \\ r_3 & b_2 & r_1 & b_1 & r_2 \\ & & \varnothing & & \end{array}$$

Then $\Delta^{(1)} = \langle b_1, b_2 \rangle$. We match each face $\sigma \in \Delta^{(1)}$ with a covering face that contains the largest red vertex possible. We add this matching in below. The unmatched faces together form $B^{(0)}$.

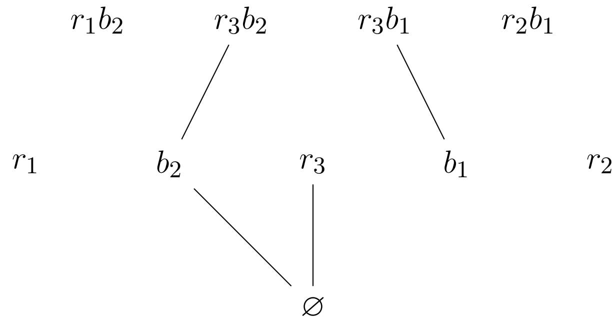


Then $\Delta^{(2)} = \{\varnothing\}$. We match this face to the largest blue vertex.

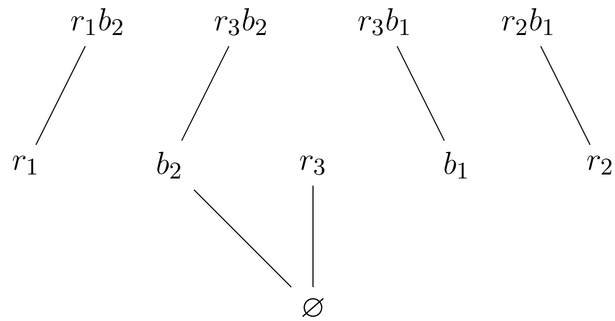


To create an honest decomposition, we would now need to perform further matchings within $B^{(0)}$. However, we can see from inspection that it is impossible to match r_2 to any unmatched face. However, if we relabel the central red vertex as either r_2 or r_3, then this process will produce an honest balanced boolean tree decomposition.

Assume we swap the vertex labels of r_1 and r_3. Then the first two steps of the matching give the following.



Then we can perform this matching algorithm for the faces of $B^{(0)}$, first separating this set into faces containing r_1 and faces containing r_2. This gives the following honest balanced boolean tree decomposition.



Given a balanced Cohen–Macaulay complex, if we want to avoid the situation in Example 3.4.5, we must be able to guarantee that we can order the colors and the vertices within each color class in a particular way. If it were true that

$$B_j^i := \operatorname{link}_{\Delta^{(i-1)}}(v_{i,j}) \cap \left(\bigcup_{k>j} \operatorname{link}_{\Delta^{(i-1)}} v_{i,k} \right) \tag{3.4.1}$$

is Cohen–Macaulay for all $i \in [d]$, $j \in [n_i]$, then B_j^i is always Cohen–Macaulay, so Proposition 3.4.4 could be inductively applied to create balanced boolean trees for each of these B_j^i. Observing that

$$B^{(i)} = \bigcup_{j \in [n_i]} \left\{ \sigma \cup \{v_{i,j}\} \ : \ \sigma \in B_j^i \right\}$$

we see that this would produce an honest balanced boolean tree decomposition for Δ.

Chapter 4

k-fold acyclic complexes

4.1 Stanley's and Duval's decomposition theorems

For this final chapter, we will assume that $\dim \Delta = d$, unless otherwise specified.

Recall that a simplicial complex Δ is said to be **acyclic** (over $\Bbbk$) if its reduced homology groups all vanish, i.e., $\tilde{H}_i(\Delta) = \tilde{H}_i(\Delta; \Bbbk) = 0$ for all i. Acyclicity can depend on $\Bbbk$; for example, any triangulation of the real projective plane $\mathbb{R}P^2$ is acyclic over a field $\Bbbk$ if $\mathrm{char}(\Bbbk) \neq 2$ but $\tilde{H}_2(\mathbb{R}P^2) \neq 0$ if $\mathrm{char}(\Bbbk) = 2$.

If there is some field $\Bbbk$ over which Δ is acyclic, then it is an immediate consequence of Proposition 1.2.1 that the f-polynomial of Δ can be factored as $f(\Delta, t) = (1+t)g(t)$. Stanley showed that this polynomial $g(t)$ is itself the f-polynomial of some other simplicial complex Γ. In fact, he was able to show the following stronger result.

Theorem 4.1.1. [Sta93, Theorem 1.2] Let Δ be a simplicial complex that is acyclic over some field. Then there exist Δ', Ω such that $\Delta = \Delta' \sqcup \Omega$ and a bijection $\eta : \Delta' \to \Omega$ with the following properties:

1. Δ' is a subcomplex of Δ.

2. $\sigma \subseteq \eta(\sigma)$ and $|\eta(\sigma) \setminus \sigma| = 1$ for all $\sigma \in \Delta'$.

B is a **Betti set** of Δ if $f_i(B) = \beta_i(\Delta)$ for all i. Duval [Duv94, Theorem 1.1] generalized Theorem 4.1.1 to all simplicial complexes by introducing a Betti set B of the complex such that $\Delta = \Delta' \sqcup \Omega \sqcup B$ and $\Delta' \sqcup B$ is a subcomplex of Δ. Furthermore, Duval and Zhang extended this generalization to a sequence of decompositions through the use of iterated homology [DZ01, Theorem 3.2] (Theorem 1.7.4). Theorem 3.3.4 is a balanced analogue of the Duval–Zhang result.

Stanley also asked whether a stronger version of Theorem 4.1.1 could be shown for complexes which possess a stronger notion of acyclicity.

Definition 4.1.2. A simplicial complex Δ is said to be k**-fold acyclic** (over a field $\Bbbk$) if $\mathrm{link}_\Delta(\sigma)$ is acyclic for all $\sigma \in \Delta$ such that $|\sigma| < k$.

We can see that Δ is acyclic if and only if Δ is 1-fold acyclic, since $\mathrm{link}_\Delta \varnothing = \Delta$. Similarly, Δ is 2-fold acyclic whenever Δ itself is acyclic and $\mathrm{link}_\Delta v$ is acyclic for all vertices $v \in \Delta$.

With Definition 4.1.2 in mind, we may restate Theorem 4.1.1 as follows: If Δ is 1-fold acyclic (over some field), then Δ may be decomposed as

$$\Delta = \bigsqcup_{i \in I} [R_i, G_i]$$

where each $[R_i, G_i]$ is a rank-1 interval and $\Delta' = \{R_i \ : \ i \in I\}$ is a subcomplex of Δ. Notice here that $G_i = \eta(R_i)$ where η is the map from Theorem 4.1.1.

This observation leads naturally to the conjecture that Stanley made regarding k-fold acyclic complexes.

Conjecture 4.1.3. [Sta93, Conjecture 2.4] Let Δ be k-fold acyclic (over some field). Then Δ may be decomposed as

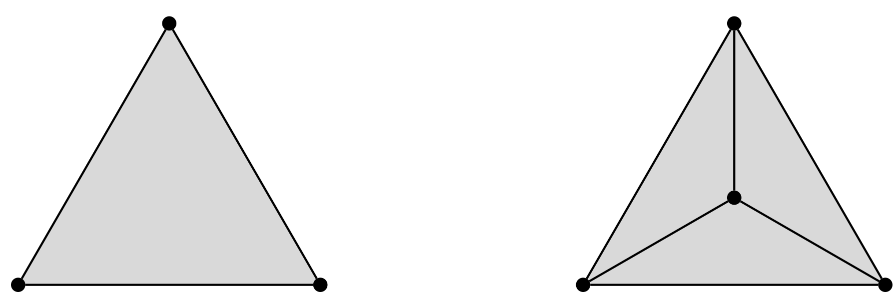
$$\Delta = \bigsqcup_{i \in I} [R_i, G_i] \tag{4.1.1}$$

where each $[R_i, G_i]$ is a rank k interval and $\Delta' = \{R_i \,:\, i \in I\}$ is a subcomplex of Δ.

While Equation 4.1.1 appears very similar to Definition 1.3.5 for partitionability, the key difference is that the faces G_i need not be facets of Δ. However, a partitioning of Δ in which the R_i form a subcomplex (or, more generally, a partitioning in which the intervals can be broken into subintervals whose minimal faces together form a subcomplex) is also a decomposition in the sense of Conjecture 4.1.3.

4.2 Basic facts about k-fold acyclicity

Notice that k-fold acyclicity is not topological for $k > 1$. Consider the below examples.



Both of these complexes are homeomorphic to a 2-ball and thus are acyclic. The left complex is in fact 3-fold acyclic since every non-facet face of the left complex has an acyclic link. However, the right complex is only 1-acyclic, since the link of the center vertex has nontrivial homology.

Recall that the join of two complexes Δ and Γ on disjoint vertex sets is $\Delta \star \Gamma = \{\sigma \cup \tau \ : \ \sigma \in \Delta \text{ and } \tau \in \Gamma\}$. In the case that $\Delta = \langle \sigma \rangle$ is a simplex with $|\sigma| = k$, the join $\langle \sigma \rangle \star \Gamma = \Delta \star \Gamma$ is called a **k-fold cone**.

It is easy to see that k-fold cones are k-fold acyclic. In fact, k-fold cones are the simplest class of complexes for which Conjecture 4.1.3 holds. Say that $\Delta = \sigma \star \Delta'$. Then Δ may be decomposed as

$$\Delta = \bigsqcup_{\tau \in \Delta'} [\tau, \tau \cup \sigma]$$

which fulfills the requirements of Conjecture 4.1.3. Thus Conjecture 4.1.3 identifies a specific way in which k-fold acyclic complexes may behave like k-fold cones.

In general, Proposition 4.2.2 gives a relationship between cones and acyclicity. We first need the following lemma.

Lemma 4.2.1. Let $\sigma = \sigma_1 \cup \sigma_2$ such that that $\sigma_1 \in \Delta$, $\sigma_2 \in \Gamma$. Then

$$\operatorname{link}_{\Delta \star \Gamma} \sigma = \operatorname{link}_{\Delta} \sigma_1 \star \operatorname{link}_{\Gamma} \sigma_2.$$

Proof. This lemma follows from the below equalities.

$$\begin{aligned}
\operatorname{link}_{\Delta \star \Gamma} \sigma &= \{\tau \in \Delta \star \Gamma \ : \ \sigma \cup \tau \in \Delta \star \Gamma,\ \sigma \cap \tau = \varnothing\} \\
&= \{\tau_1 \cup \tau_2 \ : \ \tau_1 \in \Delta,\ \tau_2 \in \Gamma,\ \sigma \cup (\tau_1 \cup \tau_2) \in \Delta \star \Gamma,\ \sigma \cap (\tau_1 \cup \tau_2) = \varnothing\} \\
&= \{\tau_1 \cup \tau_2 \ : \ \tau_1 \in \Delta,\ \tau_2 \in \Gamma,\ \sigma_1 \cup \tau_1 \in \Delta,\ \sigma_2 \cup \tau_2 \in \Gamma,\ \sigma \cap (\tau_1 \cup \tau_2) = \varnothing\} \\
&= \{\tau_1 \in \Delta \ : \ \sigma_1 \cup \tau_1 \in \Delta,\ \sigma_1 \cap \tau_1 = \varnothing\} \star \{\tau_2 \in \Gamma \ : \ \sigma_2 \cup \tau_2 \in \Gamma,\ \sigma_2 \cap \tau_2 = \varnothing\} \\
&= \operatorname{link}_{\Delta} \sigma_1 \star \operatorname{link}_{\Gamma} \sigma_2
\end{aligned}$$

□

Proposition 4.2.2. If Δ is j-fold acyclic and $|\sigma| = k$, then $\sigma \star \Delta$ is $(j+k)$-fold acyclic.

Proof. Let $\tau \in \Gamma = \sigma \star \Delta$. Then $\tau = \sigma' \cup \tau'$ where $\sigma' \subseteq \sigma$ and $\tau' \in \Delta$. Then $\operatorname{link}_\Gamma \tau = (\sigma \setminus \sigma') \star \operatorname{link}_\Delta \tau'$ by Lemma 4.2.1. Note that $|\tau| = |\sigma'| + |\tau'|$. Assume that $|\tau| < j + k$. If $|\sigma'| < k$, then the link of τ is a cone and thus is acyclic. If instead $|\sigma'| = k$ then $\sigma' = \sigma$ and $\operatorname{link}_\Gamma \tau = \operatorname{link}_\Delta \tau'$. But since $|\tau'| < j$, therefore this link is acyclic as well. Thus Γ is $(j+k)$-fold acyclic. $\square$

We are also able to describe the acyclicity of links of faces in k-fold acyclic complexes, but we first need the following lemma.

Lemma 4.2.3. Let $\sigma, \tau \in \Delta$ and $\sigma \cap \tau = \varnothing$. Then

$$\operatorname{link}_{(\operatorname{link}_\Delta \sigma)} \tau = \operatorname{link}_\Delta(\sigma \cup \tau).$$

Proof. The proof is the following straightforward calculation.

$$\begin{aligned}
\operatorname{link}_{(\operatorname{link}_\Delta \sigma)} \tau &= \{\gamma \in \operatorname{link}_\Delta \sigma \ : \ \gamma \cup \tau \in \operatorname{link}_\Delta \sigma \text{ and } \gamma \cap \tau = \varnothing\} \\
&= \{\gamma \in \Delta \ : \ \gamma \cup \tau \in \operatorname{link}_\Delta \sigma,\ \gamma \cup \sigma \in \Delta,\ \gamma \cap \sigma = \varnothing, \text{ and } \gamma \cap \tau = \varnothing\} \\
&= \{\gamma \in \Delta \ : \ \gamma \cup \tau \in \operatorname{link}_\Delta \sigma,\ \gamma \cup \sigma \in \Delta, \text{ and } \gamma \cap (\sigma \cup \tau) = \varnothing\} \\
&= \{\gamma \in \Delta \ : \ \gamma \cup (\sigma \cup \tau) \in \Delta \text{ and } \gamma \cap (\sigma \cup \tau) = \varnothing\} \\
&= \operatorname{link}_\Delta(\sigma \cup \tau)
\end{aligned}$$

This proposition can be modified to apply to faces $\sigma, \tau \in \Delta$ with nonempty intersection by replacing τ with $\tau \setminus \sigma$. $\square$

Proposition 4.2.4. Let Δ be k-fold acyclic. Then $\operatorname{link}_\Delta \sigma$ is $(k - |\sigma|)$-fold acyclic for every $\sigma \in \Delta$.

Proof. Consider a face $\tau \in \operatorname{link}_\Delta \sigma$ such that $|\tau| < k - |\sigma|$. Then by Lemma 4.2.3,

$$\operatorname{link}_{(\operatorname{link}_\Delta \sigma)} \tau = \operatorname{link}_\Delta (\sigma \cup \tau)$$

which must be acyclic, since $|\sigma \cup \tau| = |\sigma| + |\tau| < k$. $\square$

Furthermore, k-fold acyclicity also provides some algebraic structure on a complex's Stanley-Reisner ring.

Proposition 4.2.5. Let Δ be a k-fold acyclic simplicial complex. If Δ has more than one facet, then $\operatorname{depth} \Bbbk[\Delta] \geq k+1$.

Proof. By Proposition 1.4.4, recall that $\operatorname{depth} \Bbbk[\Delta] = \min\left\{i \,:\, H^i_m(\Bbbk[\Delta]) \neq 0\right\}$ where $H^i_m(\Bbbk[\Delta])$ is the i^{th} local cohomology module of $\Bbbk[\Delta]$ (with m taken to be the irrelevant ideal).

Assume $k \geq 1$. Hochster's Theorem together with Proposition 1.4.4 implies that $\operatorname{depth} \Bbbk[\Delta] \leq k$ if and only if there is a face $\sigma \in \Delta$ and a number $e \leq k$ such that

$$\tilde{H}_{e-|\sigma|-1}(\operatorname{link}_\Delta \sigma) \neq 0.$$

Notice that if $|\sigma| > e$, then $e - |\sigma| - 1 < -1$ and this homology is trivial by definition. If $|\sigma| < e$, then $\operatorname{link}_\Delta \sigma$ is acyclic since $e \leq k$ and thus $\tilde{H}_{e-|\sigma|-1}(\operatorname{link}_\Delta \sigma) = 0$.

Therefore the only case we need to check is when $|\sigma| = e$, when we are considering $\tilde{H}_{e-|\sigma|-1}(\operatorname{link}_\Delta \sigma) = \tilde{H}_{-1}(\operatorname{link}_\Delta \sigma)$. This homology group is only nontrivial when $\operatorname{link}_\Delta \sigma = \{\varnothing\}$, that is, when σ is a facet of Δ. We may assume there is a ridge $\tau \subseteq \sigma$ that is contained in another facet σ' of Δ. (If this is not the case, then Δ is disconnected and thus $\tilde{H}_0(\operatorname{link}_\Delta \varnothing) = \tilde{H}_0(\Delta) \neq 0$, which contradicts k-fold acyclicity.)

But τ is a ridge of σ, so $|\tau| = |\sigma| - 1 < k$ and $\operatorname{link}_\Delta \tau$ is disconnected. This contradicts k-fold acyclicity. Therefore $\operatorname{depth} \Bbbk[\Delta] \geq k+1$. $\square$

In fact, the bound of Proposition 4.2.5 is tight in any dimension. Consider Δ to be two d-simplices glued along a k-simplex, with $k < d$. This complex is k-fold acyclic (by, for example, Proposition 4.3.2 in the following section), and so $\operatorname{depth} \Bbbk[\Delta] \geq k+1$. However, it is an easy computation (by, say, Proposition 1.4.10) to see that $\operatorname{depth} \Bbbk[\Delta] = k+1$.

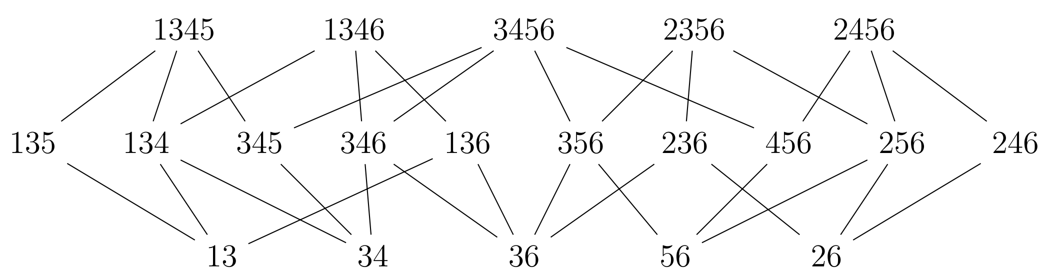
4.3 Gluing and a relative counterexample

If $\Theta = (\Delta, \Gamma)$ is a relative complex, recall that for $\sigma \in \Delta$, the relative link of σ in Θ is $\operatorname{link}_\Theta \sigma = (\operatorname{link}_\Delta \sigma, \operatorname{link}_\Gamma \sigma)$. We define a relative complex $\Theta = (\Delta, \Gamma)$ to be a **k-fold acyclic relative complex** if $\operatorname{link}_\Theta \sigma$ is acyclic for all $\sigma \in \Delta$ such that $|\sigma| < k$.

It turns out that Conjecture 4.1.3 does not hold for relative complexes in general. Consider the following example.

Example 4.3.1. [DGKM16, Remark 3.6 (vertices relabeled)]

Let $\Delta = \langle 1345, 1346, 3456, 2356, 2456 \rangle$ and $\Gamma = \langle 145, 146, 235, 245 \rangle$. Then the relative complex $\Theta = (\Delta, \Gamma)$ is 2-fold acyclic. The following is the face poset of Θ.



One can see from inspection of this face poset that Θ cannot be written as the disjoint union of rank-2 boolean intervals.

It is natural to look for a counterexample to Conjecture 4.1.3 by gluing together copies of a relative counterexample, similarly to the construction in Proposition 2.1.8.

We first need to note that gluing together complexes preserves k-fold acyclicity in a natural way.

Proposition 4.3.2. Let Δ_1 and Δ_2 be simplicial complexes such that Δ_1 is j-fold acyclic, Δ_2 is k-fold acyclic, and $\Delta_1 \cap \Delta_2$ is ℓ-fold acyclic. Then $\Delta_1 \cup \Delta_2$ is m-fold acyclic, where $m = \min\{j, k, \ell\}$.

Proof. Let $\sigma \in \Delta_1 \cup \Delta_2$ and assume $|\sigma| < m$. If $\sigma \in \Delta_1 \setminus \Delta_2$, then $\operatorname{link}_{(\Delta_1 \cup \Delta_2)} \sigma = \operatorname{link}_{\Delta_1} \sigma$ and thus $\operatorname{link}_{(\Delta_1 \cup \Delta_2)} \sigma$ is acyclic. The same holds if $\sigma \in \Delta_2 \setminus \Delta_1$.

If instead $\sigma \in \Delta_1 \cap \Delta_2$, then we will apply Lemma 2.1.3 and Theorem 1.2.2. This gives the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_i(\operatorname{link}_{\Delta_1} \sigma) \oplus \tilde{H}_i(\operatorname{link}_{\Delta_2} \sigma) \to \tilde{H}_i(\operatorname{link}_{\Delta_1 \cup \Delta_2} \sigma) \to \tilde{H}_{i-1}(\operatorname{link}_{\Delta_1 \cap \Delta_2} \sigma) \to \dots.$$

Since Δ_1, Δ_2, and $\Delta_1 \cap \Delta_2$ are m-fold acyclic, the homology groups of the links of σ in each of these complexes vanish since $|\sigma| < m$. This implies that $\tilde{H}_i(\operatorname{link}_{\Delta_1 \cup \Delta_2} \sigma) = 0$ for all i. Therefore $\Delta_1 \cup \Delta_2$ is m-fold acyclic. $\square$

With Proposition 4.3.2 in hand, we can prove a k-fold acyclic version of Theorem 2.1.6.

Theorem 4.3.3. Let $\Theta = (\Delta, \Gamma)$ be a relative complex such that

1. Δ and Γ are k-fold acyclic;

2. Γ is an induced subcomplex of Δ; and

3. Θ cannot be written as a disjoint union of rank k boolean intervals.

Let k be the total number of faces of Γ and let $N > k$. Then gluing N copies of Δ together along Γ will produce a k-fold acyclic complex that cannot be written as a disjoint union of rank k boolean intervals.

Proof. The proof follows the same reasoning as the proof for Theorem 2.1.6. $\square$

However, we note that Example 4.3.1 is not of the form (Δ, Γ) where Γ is an *induced* complex of Δ, so it cannot be used with Theorem 4.3.3 to produce a counterexample to Conjecture 4.1.3. An extensive search has been made using well-known complexes with interesting decomposition properties (for example, the non-shellable balls constructed by Rudin [Rud58], Ziegler [Zie98], and Benedetti and Lutz [BL13], along with related complexes constructed from these examples) but so far no relative complexes meeting all of the criteria of Theorem 4.3.3 have been constructed, and it is unknown whether such a relative complex exists.

4.4 Decompositions into boolean trees

While Conjecture 4.1.3 remains open, we can prove a weakened version of it by replacing boolean intervals with boolean trees.

Proposition 4.4.1. [Sta93, due to Kalai, noted in the proof of Proposition 2.3] If Δ is k-fold acyclic, then its algebraic shifting $S(\Delta)$ is also k-fold acyclic.

Proposition 4.4.2. If Δ is shifted and k-fold acyclic, then Δ is a k-fold cone. In other words, $\Delta = \langle 12\dots k\rangle \star \Delta'$ for some subcomplex Δ'.

Proof. Since Δ is shifted, $\Delta = S(\Delta)$. By [BK88, Theorem 4.3],

$$\beta_i(\Delta) = |\{\text{facets } T \in \Delta \ : \ |T| = i+1 \text{ and } T \cup \{1\} \notin \Delta\}|$$

(Notice that this is the $r = 0$ case of Proposition 1.7.2). Since Δ is assumed to be k-fold acyclic, it is in particular acyclic. Thus $\beta_i(\Delta) = 0$ for all i, which implies that $\Delta = \langle 1\rangle \star \Gamma_1$ for some complex Γ_1. By [DR00, Proposition 2.3], Γ_1 is shifted on the remaining vertices, and we also know that Γ_1 is $(k-1)$-fold acyclic. Repeating this argument, we see that $\Delta = \langle 1\rangle \star \langle 2\rangle \star \cdots \star \langle k\rangle \star \Delta' = \langle 12\ldots k\rangle \star \Delta'$ for some subcomplex Δ', i.e., Δ is a k-fold cone. $\square$

An immediate corollary of the above is the following.

Corollary 4.4.3. [Sta93, essentially Proposition 2.3] If Δ is k-fold acyclic, then its f-polynomial can be factored as $f(\Delta,t) = (1+t)^k f(\Delta',t)$, where $f(\Delta',t)$ is the f-polynomial of another complex Δ'.

We are now able to prove the following relaxation of Conjecture 4.1.3.

Theorem 4.4.4. Let Δ be k-fold acyclic. Then Δ can be written as the disjoint union of boolean trees of rank k. Furthermore, the minimal faces of these boolean trees together form a subcomplex Δ'.

Proof. The proof is similar to the proof of Theorem 1.7.5. We will make use of Theorem 1.7.4, and we will use the notation of that theorem.

By Proposition 4.4.2, $S(\Delta) = \langle 1\ldots k\rangle \star \Delta'$ for some complex Δ'. Proposition 1.7.2 relates the iterated Betti numbers of Δ to the algebraic shifting $S(\Delta)$. In particular, it says that

$$\beta^i[r](\Delta) = |\{\text{facets } F \in S(\Delta) \ : \ |F| = i+1 \text{ and } \mathrm{init}(F) = r\}|.$$

Since $S(\Delta)$ is a k-fold cone, $\mathrm{init}(F) \geq k$ for all facets $F \in S(\Delta)$, and thus $\beta^i[r](\Delta) = 0$ for $r < k$.

Step 0: Note that all faces of $\Delta = \Delta^{(0)}$ form rank-0 boolean trees.

We will perform the following step k times: Assume this step has been completed $i < k$ times, so the minimal elements of boolean trees of rank i are all of the faces of

$\Delta^{(i)}$. By Theorem 1.7.4,

$$\begin{aligned}\Delta^{(i)} &= \Delta^{(i+1)} \sqcup B^{(i)} \sqcup \Omega^{(i+1)} \\ &= \Delta^{(i+1)} \sqcup \Omega^{(i+1)}\end{aligned}$$

with the second equality by 1.7.4 (2) since $i < k$. For each face $\sigma \in \Delta^{(i+1)}$, we combine the rank-$i$ boolean trees with minimal elements σ and $\eta^{(i+1)}(\sigma)$ to form rank-$(i+1)$ boolean trees. Since $B^{(i)} = \varnothing$, there are no rank-i boolean trees remaining after this step.

Furthermore, if we stop this process after k iterations, we see that the minimal elements of the resulting boolean trees are precisely the faces of $\Delta^{(k+1)} \sqcup B^{(k)}$. We know that

$$\Delta^{(k+1)} \sqcup B^{(k)} \subseteq \Delta^{(k)} \subseteq \Delta^{(k-1)} \subseteq \cdots \subseteq \Delta^{(0)} = \Delta$$

as subcomplexes, therefore the minimal elements of these boolean trees together form a subcomplex $\Delta' = \Delta^{(k+1)} \sqcup B^{(k)}$. $\square$

4.5 $\dim \Delta$-fold acyclic complexes

Recall that in this chapter we adopt the convention $\dim \Delta = d$. We will use this section to show that Conjecture 4.1.3 holds when $k = d = \dim \Delta$. Thus the conjecture is known to be true for $k = 1$ (due to Stanley's original theorem) and $k = \dim \Delta$, which is as high as possible (unless Δ is a simplex, which is $(\dim \Delta + 1)$-fold acyclic). It is unknown whether the conjecture holds for $1 < k < \dim \Delta$, except that it is known to be true when $\dim \Delta \leq 2$ due to a result of Duval, Klivans, and Martin (unpublished). Much of this section is based on elements of the Duval–Klivans–Martin proof of the $\dim \Delta = 2$ case.

Assume Δ is d-fold acyclic. Then by Proposition 4.2.5, $\operatorname{depth}\Bbbk[\Delta] \geq d+1$. Since $\operatorname{depth}\Bbbk[\Delta] \leq \dim\Bbbk[\Delta] = d+1$, it follows that $\operatorname{depth}\Bbbk[\Delta] = \dim\Bbbk[\Delta]$ and therefore Δ is Cohen–Macaulay. However, we will show that Δ possesses an even stronger structure defined below, and this will provide a decomposition of Δ into rank-d boolean intervals.

Definition 4.5.1. A simplicial complex Δ is a **stacked simplicial complex** if Δ is pure with a shelling order $F_1,\dots,F_j$ such that $\langle F_1,\dots,F_i\rangle \cap \langle F_{i+1}\rangle$ has a single facet for all $i \in [j-1]$.

The term "stacked" is intended to suggest the notion of a *stacked polytope*. A polytope is **stacked** if it is a simplex or if it is constructed by taking the cone over a single facet of a stacked polytope (see, for example, [Grü03]). If $\Gamma = \partial P$ is the boundary complex of a stacked polytope P, then it is necessarily a sphere; however, such a complex Γ is also the boundary of a stacked simplicial complex Δ. However, Definition 4.5.1 is slightly more lenient than that of a stacked polytope. Note that $\Delta = \langle 123, 124, 125\rangle$ is a stacked simplicial complex, but if $\Gamma = \partial\Delta$ there is no polytope P (stacked or not) such that $\Gamma = \partial P$.

Remark 4.5.2. Definition 4.5.1 is equivalent to the term "facet constructible," which appears in [DS17, Section 4].

If Δ is a stacked simplicial complex, then Δ is extendably shellable (see p. 13). Once we have the following lemma, we can say even more about stacked complexes.

Lemma 4.5.3. If Δ is (pure) shellable, then $\operatorname{link}_\Delta \sigma$ is (pure) shellable for all $\sigma \in \Delta$.

Proof. Each facet of $\operatorname{link}_\Delta \sigma$ is of the form G where $G \cup \sigma = F$, a facet of Δ. Given a shelling order on the facets of Δ, we can order the facets of $\operatorname{link}_\Delta \sigma$ as $G_1,\dots,G_k$ where $G_i = F_{j_i} \cap \operatorname{link}_\Delta \sigma$ and $j_1 < j_2 < \cdots < j_k$ in the shelling order of Δ.

Then, for each $1 < i \le k$, we have that

$$\langle G_i \rangle \cap \langle G_1, \ldots, G_{i-1} \rangle = (\langle F_{j_i} \rangle \cap \langle F_1, \ldots, F_{j_i - 1} \rangle) \cap \operatorname{link}_\Delta \sigma$$

is pure of dimension $d - \dim \sigma$. Therefore this is a shelling order of G. □

Proposition 4.5.4. Let Δ be a d-dimensional stacked simplicial complex. Then Δ is d-fold acyclic and Δ can be written as the disjoint union of rank-d boolean intervals, the minimal elements of which form a subcomplex $\Delta' \subseteq \Delta$. In other words, Conjecture 4.1.3 holds for stacked complexes.

Proof. We will prove that stacked complexes are d-fold acyclic by induction on $d = \dim \Delta$. Notice that if $\dim \Delta = 1$, then Δ is stacked if and only if Δ is a connected acyclic graph (i.e., a tree), and thus is 1-fold acyclic.

Assume the result holds for $k < d$ and let $\dim \Delta = d$. Let $\sigma \in \Delta$ such that $|\sigma| < d$. Then $\operatorname{link}_\Delta \sigma$ is shellable by Lemma 4.5.3. Recall that $\operatorname{star}_\Delta \sigma = \langle \sigma \rangle \star \operatorname{link}_\Delta \sigma$. Therefore $\operatorname{link}_\Delta \sigma$ must also be stacked and $\dim \operatorname{link}_\Delta \sigma < \dim \Delta$ as long as $\sigma \neq \varnothing$. If, however, $\sigma = \varnothing$, then $\operatorname{link}_\Delta \sigma = \Delta$, which is acyclic by assumption. Thus Δ is d-fold acyclic, and we are done by induction.

Given a stacked complex Δ, a shelling $F_1, \ldots F_j$ gives rise to the following partitioning:

$$\Delta = [\varnothing, F_1] \sqcup [v_2, F_2] \sqcup [v_3, F_3] \sqcup \cdots \sqcup [v_j, F_j]$$

For any vertex $v_1 \in F_1$, we can write $[\varnothing, F_1] = [\varnothing, F_1 \setminus \{v_1\}] \sqcup [v_1, F_1]$. Therefore Δ can be decomposed as

$$\Delta = [\varnothing, F_1 \setminus \{v_1\}] \sqcup [v_1, F_1] \sqcup [v_2, F_2] \sqcup [v_3, F_3] \sqcup \cdots \sqcup [v_j, F_j]$$

and $\Delta' = \{\varnothing, v_1, v_2, \ldots, v_j\}$ is a subcomplex of Δ. Thus Conjecture 4.1.3 holds for stacked complexes. □

Therefore the goal of the remainder of this section will be to show that if a d-dimensional complex Δ is d-fold acyclic, then Δ is stacked and thus Conjecture 4.1.3 holds when $k = d = \dim \Delta$. We also note that Hailong Dao (personal communication) has been able to show that Conjecture 4.1.3 holds for $k = \dim \Delta$ using results from [DS17] and techniques from homological algebra.

We first need several technical lemmas before we can prove our main result.

Lemma 4.5.5. Let $\Gamma \subseteq \Delta$ be simplicial complexes such that $\dim \Gamma = \dim \Delta = d$. If $\tilde{H}_d(\Gamma) \neq 0$ then $\tilde{H}_d(\Delta) \neq 0$.

Proof. (Sketch) If $\Gamma \subseteq \Delta$, then the maps $C_i(\Gamma) \to C_i(\Delta)$ are injective. Therefore the induced map $\tilde{H}_d(\Gamma) \to \tilde{H}_d(\Delta)$ is injective. □

Lemma 4.5.6. Let Δ be d-dimensional and d-fold acyclic. Then Δ is pure.

Proof. Certainly Δ is connected; otherwise $\tilde{H}_0(\Delta) \neq 0$. Assume that Δ is not pure. Thus there is some face contained in facets of different dimensions. Take σ to be a face that is maximal with respect to this property. We must have both $|\sigma| < d$ and $\operatorname{link} \sigma$ is disconnected. This is a contradiction to d-fold acyclicity. □

Lemma 4.5.7. Let Δ be d-fold acyclic and d-dimensional. If X is generated by a partial shelling of Δ, then X is also d-fold acyclic.

Proof. We note that Δ is pure by Lemma 4.5.6.

X is pure and is shellable by assumption, thus it only has top dimensional homology. However, Δ has no top dimensional homology and $X \subseteq \Delta$ of the same dimension, thus $\tilde{H}_d(X) = 0$ by Lemma 4.5.5.

Similarly, let $\sigma \in X$ and say $|\sigma| < d$. The shelling of X induces a shelling of $\operatorname{link}_X \sigma$, thus $\operatorname{link}_X \sigma$ is pure shellable because Δ is pure. Therefore $\operatorname{link}_X \sigma$ may have homology only in its top dimension. But again, since $\operatorname{link}_X \sigma \subseteq \operatorname{link}_\Delta \sigma$ of the same dimension and $\operatorname{link}_\Delta \sigma$ is acyclic by assumption, thus $\operatorname{link}_X \sigma$ is acyclic by Lemma 4.5.5. Therefore X is d-fold acyclic. $\square$

Lemma 4.5.8. Let Δ be d-fold acyclic and d-dimensional. If X is generated by a partial shelling of Δ and Y is generated by the remaining facets of Δ, then $X \cap Y$ is pure of dimension $d-1$.

Proof. Since Δ is d-fold acyclic, $X \cap Y$ is non-empty, otherwise $\tilde{H}_0(\Delta) \neq 0$. Let σ be a maximal face of $X \cap Y$. Recall that $\operatorname{link}_\Delta \sigma = \operatorname{link}_X \sigma \cup \operatorname{link}_Y \sigma$ by Proposition 2.1.3, and thus is disconnected and therefore *not* acyclic. If $\dim \sigma < d-1$, i.e. $|\sigma| < d$, then this would contradict d-fold acyclicity of Δ. Thus $X \cap Y$ is pure of dimension $d-1$. $\square$

Lemma 4.5.9. Let Δ be a d-dimensional stacked simplicial complex. If X is generated by a partial shelling and F is a facet of Δ such that $\dim X \cap \langle F \rangle = d-1$, then $X \cap \langle F \rangle$ has a single facet.

Proof. This follows from the fact that stacked simplicial complexes are extendably shellable.

Assume $X \cap \langle F \rangle$ has more than one facet. Since X is generated by a partial shelling, then there must be a way to extend it to a full shelling. At some point this extended shelling contains F. Then this implies that when F is added in the full shelling of Δ that it is attached along more than one ridge of Δ. But this contradicts the assumption that Δ is stacked. $\square$

We now reword the Duval–Klivans–Martin result using our terminology.

Theorem 4.5.10. [Duval–Klivans–Martin, unpublished] If Δ is 2-dimensional and 2-fold acyclic, then Δ is stacked.

We will now show the general case by induction on dimension.

Theorem 4.5.11. If Δ is d-dimensional and d-fold acyclic, then Δ is stacked.

Proof. The statement holds for $d = 2$ by Theorem 4.5.10 and trivially for $d < 2$.

Let $\dim\Delta = d > 2$ and assume that the statement of the theorem holds for dimensions lower than d. Let $X \subseteq \Delta$ be generated by a partial (stacked) shelling of Δ, and let F be a facet of $Y = \langle$remaining facets of $\Delta\rangle$ such that $\dim X \cap \langle F\rangle = d-1$.

Assume that $X \cap \langle F\rangle$ has at least two facets, σ_1 and σ_2, and assume without loss of generality that $\dim\sigma_1 = d-1$. Define $\sigma = \sigma_1 \cap \sigma_2$. Note that $\mathrm{link}_\Delta\,\sigma$ is $(d-|\sigma|)$-fold acyclic and $\dim \mathrm{link}_\Delta\,\sigma = d-|\sigma|$. If $\sigma \neq \varnothing$, then $\mathrm{link}_\Delta\,\sigma$ is stacked by our induction hypothesis. Furthermore, $\mathrm{link}_\Delta\,\sigma$ inherits the induced (stacked) shelling of X, and $F \setminus \sigma$ cannot be part of any stacked shelling of $\mathrm{link}_\Delta\,\sigma$ by Lemma 4.5.9. Thus we reach a contradiction.

If $\sigma = \varnothing$, then $\mathrm{link}_\Delta\,\sigma = \Delta$, so we cannot apply induction on dimension in this case. Instead, we want to consider the homology of $\mathrm{link}_{X\cup\langle F\rangle}\,\sigma = X \cup \langle F\rangle$. Note that in this case $X \cap \langle F\rangle$ is two disjoint simplices, σ_1 a ridge of Δ and σ_2 a vertex. So $\tilde{H}_0\,(X \cap \langle F\rangle) \neq 0$, and a Mayer-Vietoris sequence (Proposition 1.2.2) implies that $\tilde{H}_1\,(X \cup \langle F\rangle) \neq 0$. But $\tilde{H}_1(\Delta) = 0$ by d-fold acyclicity.

Thus for some 1-chain Q supported in $X \cup \langle F\rangle$, there exists a collection of remaining facets Z of Δ that support a 2-chain in Δ such that the boundary of this 2-chain is the 1-chain Q. Furthermore, the support of this 1-chain must include an edge with endpoints σ_2 and some $v \in \sigma_1$; otherwise this would contradict d-fold acyclicity of X.

We know that $\mathrm{link}_\Delta\,v$ is $(d-1)$-dimensional and $(d-1)$-fold acyclic, so it is stacked by our induction hypothesis. Thus there is some shelling of $\mathrm{link}_\Delta\,v$ that begins with all of the facets of $(X \cup \langle F \rangle) \cap \operatorname{link}_\Delta v$ which eventually adds all of the facets of $\langle Z \rangle \cap \operatorname{link}_\Delta v$. However, this contradicts the assumption that $\operatorname{link}_\Delta v$ is stacked by Lemma 4.5.9, since some facet of $\langle Z \rangle \cap \operatorname{link}_\Delta v$ must intersect the partial shelling in more than one ridge. Therefore $\operatorname{link}_\Delta v$ is not $(d-1)$-fold acyclic and so Δ is not d-fold acyclic, which is a contradiction. This completes the proof. $\square$

Combining Theorem 4.5.11 and Proposition 4.5.4, we see that a d-dimensional complex Δ is stacked if and only if it is d-fold acyclic. This leads immediately to our main result of this section.

Corollary 4.5.12. Conjecture 4.1.3 holds when $k = \dim \Delta$.

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