Parameter estimation for stochastic differential equations driven by fractional Brownian motion

By

Hongjuan Zhou

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Committee members

David Nualart, Chairperson

Yaozhong Hu, Co-chair

Zhipeng Liu

Terry Soo

Jianbo Zhang

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The Dissertation Committee for Hongjuan Zhou certifies
that this is the approved version of the following dissertation:

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Abstract

This dissertation systematically considers the inference problem for stochastic differential equations (SDE) driven by fractional Brownian motion. For the volatility parameter and Hurst parameter, the estimators are constructed using iterated power variations. To prove the strong consistency and the central limit thoerems of the estimators, the asymptotics of the power variatons are studied, which include the strong consistency, central limit theorem, and the convergence rate for the iterated power variations of the Skorohod integrals with respect to fractional Brownian motion. The iterated logarithm law of the power variations of fractional Brownian motion is proved. The joint convergence along different subsequence of power variations of Skorohod integrals is also studied in order to derive the central limit theorem for the estimators.

Another important topic considered in this dissertation is the estimation of drift parameters of the SDEs. A least squares estimator (LSE) is proposed and the strong consistency is proved for the fractional Ornstein-Uhlenbeck process that is the solution to the linear SDE. The fourth moment theorem is applied to obtain the central limit theorems. Then the LSE is considered for the drift parameter of the multi-dimensional nonlinear SDE. While proving the strong consistency of LSE, the regularity structure of the SDE's solution is explored and the maximal inequality for the Skorohod integrals is derived. The main tools used in this research are Malliavin calculus and some Gaussian analysis elements.

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Contents

Chapter 1

Introduction

Stochastic models have been widely used to describe various phenomena in many research areas, such as physics, economics and finance etc. The important examples include the semimartingale models that demonstrate the Markovian property. However, as the Hurst phenomena and the fractal property of financial market were discovered, researchers began to widely investigate the non-Markovian models. The fractional Brownian motion (fBm) and the stochastic processes driven by fBm are the essential representatives. A typical model of great interest is

$$dX_t = f(\theta, t, X_t)dt + \sigma_t dB_t^H \,, \tag{0.1}$$

with initial condition $X_0 \in \mathbb{R}$, where $B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian (fBm) motion of Hurst parameter $H \in (0,1)$. The volatility σ_t is a stochastic process with β-Hölder continuous trajectories, where $\beta > 1 - H$. Under this condition on σ_t, the stochastic integral $\int_0^t \sigma_s dB_s^H$ is well defined as a pathwise Riemann-Stieljes integral (see, for instance, [41]).

There are different assumptions that are imposed on the drift funciton f such that the above stochastic differential equation has a unique solution (see [32, 16] and the reference therein). In the papaer [32], the drift function f is Lipschitz continuous and satisfies a boundness condition, i.e.,

$$|f(\theta,t,x)-f(\theta,t,y)| \le C|x-y|,\ \forall\, x,\ y \in \mathbb{R},\ \forall\, t\ \in [0,T]$$

and

$$f(\theta,t,x) \le C|x| + f_0(t), \forall x \in \mathbb{R}, \forall\, t\ \in [0,T].$$

In the paper [16], they consider the drift funciton f in the form of $f(X_t)$, and require that f is one-sided dissipative Lipschitz and it has polynomial growth together with its derivative, i.e.,

$$\langle x-y, f(x)-f(y)\rangle \le -L|x-y|^2,\ \forall\, x,\ y,$$

$$|f(x)| + |Df(x)| \le K(1+|x|^q),$$

for some $q \ge 1$ and a constant $K > 0$. In each chapter, we may clearly state the conditions of f such that the stochastic differential equation has a unique solution.

Now we assume that one trajectory of the stochastic process X_t has been obtained. We are interested in the estimation of the parameters H, σ_t and θ. It is worth mentioning that the inference problem under multiple trajectories has been well established where the law of large numbers could be applied. However, in real world usually there is only one trajectory available, and this challenging problem is discussed in this thesis.

The statistical estimation of the integrated volatility has already been studied in the recent decades. Barndorff-Nielsen *et al* ([4] - [5]) studied estimation of volatility for Brownian semimartingale and Brownian semi-stationary processes by using power, bipower, or multipower variations. However, those results cannot be applied to the fractional Ornstein-Uhlenbeck process due to its lack of the semimartingale property. To tackle this difficulty, Berzin and León use the regression models in the paper [7] to estimate the volatility and the Hurst parameter. Some other researchers used quadratic variations to estimate the Hurst parameter of the fBm. Interested readers are referred to the papers [25, 23]. In this research work, we will apply general power variations to estimate volatility and Hurst parameter. This will involve some research work on the asymptotic behavior of power variations, which has been discussed by Nualart, Corcuera and Woerner in the paper [15]. They studied the asymptotic behavior of the power variation of the stochastic integral $Z_t = \int_0^t u_s dB_s^H$, which is defined as

$$V_p^n(Z)_t = \sum_{i=1}^{[nt]} |Z_{i/n} - Z_{(i-1)/n}|^p$$

for any $p > 0$. They proved that if the process $u = \{u_t, t \geq 0\}$ has finite q-variation on any finite interval, for some $q < 1/(1-H)$, then, as $n \to \infty$,

$$n^{-1+pH} V_p^n(Z)_t \to c_{1,p} \int_0^t |u_s|^p ds$$

uniformly in probability in any compact sets of t, where $c_{1,p} = \mathbb{E}|B_1^H|^p$. The corresponding central limit theorem was also obtained for $H \in (0, \frac{3}{4}]$. These results can be applied to construct an estimator based on the power variation of $\int_0^t \sigma_s dB_s^H$ to estimate the integrated volatility $\int_0^t |\sigma_s|^p ds$ when $H \in (0, \frac{3}{4}]$. However, the condition $H \in (0, \frac{3}{4}]$ is critical in [15]. The first objective of this research is to remove this restriction. To this end, we shall use higher order (or iterated) power variations defined as

$$V_{k,p}^n(Z)_t = \sum_{i=1}^{[nt]-k+1} \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} Z_{(i+j-1)/n} \right|^p ,$$

for any integer $k \geq 1$. In Section 3.1, we study the asymptotic behavior of these higher order power variations of the general stochastic integral $Z_t = \int_0^t u_s dB_s^H$. The application of these results to estimate the integrated volatility are presented in Section 3.4. In particular, when $\sigma_t = \sigma$ we can use

$$|\hat{\sigma}_T|^p = \frac{n^{-1+pH} V^n_{k,p}(X)_T}{c_{k,p} T}$$

to estimate σ, where $c_{k,p}$ is a constant. The almost sure convergence and the central limit theorems of the estimators for both the integrated volatility and the volatility itself are established.

Another related problem of power variations is the convergence rate. We know that as an immediate consequence of ergodic theory,

$$n^{-1+pH} V^n_p(B)_t = n^{-1+pH} \sum_{i=1}^{[nt]} |B_{\frac{i}{n}} - B_{\frac{i-1}{n}}|^p \to \mathbb{E}|B_1|^p$$

almost surely as $n \to \infty$. The central limit theorem can be obtained by the Wiener Chaos projection and Fourth moment theorem. However, it is unknown whether the law of iterated logarithm exists, and the total variation distance between the power variation sequence and its limit distribution is not known either. In this research, we will investigate these problems thoroughly for the first time. We have discovered that, if $p \geq 3$, the law of iterated logarithm is valid for all $H \in (0,1)$ if iterated power variations are used.

Based on the results on power variations, we develop the estimation method for the Hurst parameter using change-of-frequency method. The estimator is proposed as

$$\hat{H}_{\lambda,n} = \frac{1}{p}\left(1 - \frac{\log V^{\lambda n}_{k,p}(X)_t - \log V^n_{k,p}(X)_t}{\log \lambda}\right), \quad t \in [0,T],$$

where $\lambda > 1$ is the scaling constant. We will prove the consistency and the central limit theorem.

To estimate the drift parameter θ, We may assume that the Hurst parameter H and the volatility σ are known or have been estimated by the above methods. There have been two popular types of estimators for this drift parameter. One is the maximum likelihood estimator and the other one is the least square estimator. In the Brownian motion case, they coincide, but for the fractional Ornstein-Uhlenbeck processes they are different (see [20] and [24]).

For the linear SDE, also known as fractional Ornstein-Uhlenbeck processes, there have been many results on this topic when $H \in [\frac{1}{2}, 1)$. Interested readers are referred to the papers [10, 20, 36, 21, 26, 22, 40, 9]. In the case of continuous observations, Kleptsyna and Le Breton ([24]) studied the maximum likelihood estimator (MLE) and proved the almost sure convergence. It is worth noting that Tudor and Viens ([38]) have also obtained the almost sure convergence of both the MLE and a version of the MLE using discrete observations for all $H \in (0,1)$. Bercu, Courtin and Savy proved in [6] the central limit theorem for the MLE in the case of $H > \frac{1}{2}$. They claimed without proof that the above convergence is also valid for $H \in (0, \frac{1}{2})$.

On the other hand, Hu and Nualart ([20]) proposed the least square estimator and another ergodic type estimator. They obtained almost sure convergence and the central limit theorem for $H \in [\frac{1}{2}, \frac{3}{4})$. Sottinen and Viitasaari derived a central limit theorem and a Berry-Esseen bound for the ergodic type estimator when $H \in (0,1)$ in a recently published paper [36]. However, they did not give an explicit expression for the limiting variance.

Moreover, when $H \in (0, \frac{1}{2}) \cup [\frac{3}{4}, 1)$, the central limit theorems for the least square estimator have not been known yet. One of the objectives in this thesis is to prove the asymptotic consistency by using a new method, different from that in [20], which is valid for all $H \in (0,1)$. This method involves the relationship between the divergence and Stratonovich integrals and the integration by parts technique and it is based on the pathwise properties of the fractional Ornstein-Uhlenbeck process established in a paper [12] by Cheridito, Kawaguchi and Maejima. A central limit theorem for the least square estimator will be established. We will make a comparison of the asymptotic variance for these three estimators. We will use the ergodic-type estimator to construct a consistent estimator for high frequency data (if only discrete observations are available). The asymptotic behavior of this estimator in the discrete case is also studied.

For a general nonlinear SDE, let us first mention the paper [38] in which the maximum likelihood estimator is analyzed. The paper [2] is more related to our work, where Neuenkirch and Tindel studied the discrete observation case and proved the strong consistency of the following estimator

$$\bar{\theta}_n = \operatorname{argmin}_\theta \left| \frac{1}{n\alpha_n^2} \sum_{k=0}^{n-1} \left(|X_{t_{k+1}} - X_{t_k} - f(X_{t_k};\theta)\alpha_n|^2 - \sum_{j=1}^{d} |\sigma_j|^2 \alpha_n^{2H} \right) \right|$$

when $H > \frac{1}{2}$, where $\alpha_n = t_k - t_{k-1}$ satisfies that $\alpha_n n^\alpha$ converges to a constant as $n \to \infty$ for some small $\alpha > 0$. Their approach relies on Young's inequality from the rough path theory to handle Skorohod integrals, which cannot be applied for the case $H \in (0, \frac{1}{2}]$.

In this research, motivated by the parameter estimation, we contribute some stochastic analysis results on the Skorohod integrals. Through Malliavin calculus and factorization method, a maximum inequality for Skorohod integrals is developed. Moreover, some useful results on the solution of stochastic differential equations with drift function in the form of $-\theta f(X_t)$ are also derived, for example, the moment estimation and the regularity analysis of the solution. These are important ingredients to prove the strong consistency of the least squares estimator.

Chapter 2

Preliminary

2.1 Fractional Brownian motion

The fractional Brownian motion (fBm) $B^H = \{B_t^H, t \in \mathbb{R}\}$ with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process, defined on a complete probability space $(\Omega, \mathscr{F}, P)$, with the following covariance function

$$\mathbb{E}(B_t^H B_s^H) = R_H(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \tag{1.1}$$

This process is self-similar of order $H > 0$, that is, for any $a > 0$ the processes $\{B_{at}^H, t \in \mathbb{R}\}$ and $\{a^H B_t^H, t \in \mathbb{R}\}$ are the same in law. From (1.1), it is easy to see that

$$\mathbb{E}|B_t^H - B_s^H|^2 = |t-s|^{2H}.$$

Then it follows from Kolmogorov's continuity criterion that on any finite interval, almost surely all paths of fBm are α-Hölder continuous with $\alpha < H$. Denote by η_T the

α-Hölder coefficient of fBm on the interval $[0,T]$, i.e.,

$$\eta_T = \sup_{t\neq s\in[0,T]} \frac{\left|B_t^H - B_s^H\right|}{|t-s|^\alpha}\,. \tag{1.2}$$

Clearly, $\mathbb{E}|\eta_T|^q = T^{q(H-\alpha)}\mathbb{E}|\eta_1|^q$ for any $q>1$, by the self-similarity property of fBm.

Let $\mathscr{F}$ denote the σ-field obtained from the completion of the σ-field generated by B^H. Let $\mathscr{E}$ denote the space of all real valued step functions on $\mathbb{R}$. The Hilbert space $\mathfrak{H}$ is defined as the closure of $\mathscr{E}$ endowed with the inner product

$$\langle \mathbb{1}_{[a,b]}, \mathbb{1}_{[c,d]}\rangle_{\mathfrak{H}} = \mathbb{E}\left((B_b^H - B_a^H)(B_d^H - B_c^H)\right)\,.$$

Under the convention that $\mathbb{1}_{[0,t]} = -\mathbb{1}_{[t,0]}$ if $t<0$, the mapping $\mathbb{1}_{[0,t]} \mapsto B_t^H$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian space $\mathscr{H}_1$ spanned by B^H. We denote this isometry by $\mathfrak{H} \ni \varphi \mapsto B^H(\varphi)$.

If $f,g\in\mathfrak{H}$ and g is a continuously differentiable function with compact support, we can use step functions in $\mathscr{E}$ to approximate f and g and by a limiting argument we deduce

$$\langle f,g\rangle_{\mathfrak{H}} = \int_{\mathbb{R}^2} f(t)g'(s)\frac{\partial R_H(t,s)}{\partial t}dtds \tag{1.3}$$

(see [19]). We can also use Fourier transform to compute $\langle f,g\rangle_{\mathfrak{H}}$, namely,

$$\langle f,g\rangle_{\mathfrak{H}} = \frac{1}{c_H^2}\int_{\mathbb{R}} \mathscr{F}f(\xi)\overline{\mathscr{F}g(\xi)}|\xi|^{1-2H}d\xi, \tag{1.4}$$

where $c_H = \left(\frac{2\pi}{\Gamma(2H+1)\sin(\pi H)}\right)^{\frac{1}{2}}$ (see [35]). When $H>1/2$, for any $f,g\in L^{1/H}([0,T])$, if we extend f and g to be zero on $\mathbb{R}\cap[0,T]^c$, then $f,g\in\mathfrak{H}$ and we have the following simple identity

$$\langle f,g\rangle_{\mathfrak{H}} = \alpha_H \int_{[0,T]^2} f(u)g(v)|u-v|^{2H-2}dudv\,, \tag{1.5}$$

where $\alpha_H = H(2H-1)$.

Next we introduce the d-dimensional fBm $B=\{(B_t^1,\dots,B_t^d), t\geq 0\}$ with Hurst parameter $H\in(0,1)$, which is a zero mean Gaussian process whose components are independent and have the covariance function

$$\mathbb{E}(B_t^i B_s^i) = R_H(t,s) := \frac{1}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H}), \tag{1.6}$$

for $i=1,\dots,d$.

Let $\mathscr{E}^d$ denote the set of $\mathbb{R}^d$-valued step functions on $[0,\infty)$ with compact support. The Hilbert space $\mathfrak{H}^d$ is defined as the closure of $\mathscr{E}^d$ endowed with the inner product

$$\langle(\mathbb{1}_{[0,s_1]},\dots,\mathbb{1}_{[0,s_d]}),(\mathbb{1}_{[0,t_1]},\dots,\mathbb{1}_{[0,t_d]})\rangle_{\mathfrak{H}^d} = \mathbb{E}\left[\left(\sum_{j=1}^d B_{s_j}^j\right)\left(\sum_{j=1}^d B_{t_j}^j\right)\right] = \sum_{i=1}^d R_H(s_i,t_i)\,.$$

Then the mapping $(\mathbb{1}_{[0,t_1]},\dots,\mathbb{1}_{[0,t_d]})\mapsto \sum_{j=1}^d B_{s_j}^j$ can be extended to a linear isometry between $\mathfrak{H}^d$ and the Gaussian space $\mathscr{H}_1$ spanned by B. We denote this isometry by $\varphi\in\mathfrak{H}^d\mapsto B(\varphi)$.

When $H=\frac{1}{2}$, B is just a d-dimensional Brownian motion and $\mathfrak{H}^d = L^2([0,\infty);\mathbb{R}^d)$. When $H\in(\frac{1}{2},1)$, let $|\mathfrak{H}|^d$ be the linear space of $\mathbb{R}^d$-valued measurable functions φ on $[0,\infty)$ such that

$$\|\varphi\|_{|\mathfrak{H}|^d}^2 = \alpha_H \sum_{j=1}^d \int_{[0,\infty)^2} |\varphi_r^j||\varphi_s^j||r-s|^{2H-2}drds < \infty\,,$$

where $\alpha_H = H(2H-1)$. Then $|\mathfrak{H}|^d$ is a Banach space with the norm $\|\cdot\|_{|\mathfrak{H}|^d}$ and $\mathscr{E}^d$ is dense in $|\mathfrak{H}|^d$. Furthermore, for any $\varphi \in L^{\frac{1}{H}}([0,\infty);\mathbb{R}^d)$, we have

$$\|\varphi\|_{|\mathfrak{H}|^d} \leq b_{H,d}\|\varphi\|_{L^{\frac{1}{H}}([0,\infty);\mathbb{R}^d)}, \tag{1.7}$$

for some constant $b_{H,d} > 0$ (See [29]). Thus, we have continuous embeddings

$$L^{\frac{1}{H}}([0,\infty);\mathbb{R}^d) \subset |\mathfrak{H}|^d \subset \mathfrak{H}^d$$

for $H > \frac{1}{2}$.

When $H \in (0, \frac{1}{2})$, the covariance of the fBm B^j can be expressed as

$$R_H(t,s) = \int_0^{s\wedge t} K_H(s,u)K_H(t,u)du,$$

where $K_H(t,s)$ is a square integrable kernel defined as

$$K_H(t,s) = d_H\left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t v^{H-\frac{3}{2}}(v-s)^{H-\frac{1}{2}}dv\right),$$

for $0 < s < t$, with d_H being a constant depending on H (see [29]). The kernel K_H satisfies the following estimates

$$|K_H(t,s)| \leq c_H\left((t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}}\right), \tag{1.8}$$

and

$$\left|\frac{\partial K_H}{\partial t}(t,s)\right| \leq c'_H(t-s)^{H-\frac{3}{2}}, \tag{1.9}$$

for all $s < t$ and for some constants c_H, c'_H. Now we define a linear operator K_H from $\mathscr{E}^d$ to $L^2([0,\infty);\mathbb{R}^d)$ as

$$K_H(\phi)(s) = \left(K_H(T,s)\phi(s) + \int_s^T (\phi(t) - \phi(s)) \frac{\partial K_H}{\partial t}(t,s) dt \right) \mathbb{1}_{[0,T]}(s) , \qquad (1.10)$$

where the support of ϕ is included in $[0,T]$. One can show that this definition does not depend on T. Then the operator K_H can be extended to an isometry between the Hilbert space $\mathfrak{H}^d$ and $L^2([0,\infty);\mathbb{R}^d)$ (see [29]), and if $\phi \in \mathfrak{H}^d$ has support in $[0,T]$, then (1.10) holds. For $\phi \in \mathfrak{H}^d$ with support in $[0,T]$, we define

$$\begin{aligned} \|\phi\|^2_{K_T^d} \;:=\; & \int_0^T |\phi(t)|^2 \left((T-t)^{2H-1} + t^{2H-1} \right) dt \\ & + \int_0^T \left(\int_s^T |\phi(t) - \phi(s)| (t-s)^{H-\frac{3}{2}} dt \right)^2 ds . \end{aligned}$$

By the estimates (1.8) and (1.9), there exists a constant C depending on H such that for any $\phi \in \mathfrak{H}^d$ with support in $[0,T]$,

$$\|\phi\|^2_{\mathfrak{H}^d} = \|K_H(\phi)\|^2_{L^2([0,\infty);\mathbb{R}^d)} \le C \|\phi\|^2_{K_T^d} . \qquad (1.11)$$

2.2 Malliavin Calculus

We define two types of stochastic integrals: Stratonovich integral and divergence integral (Skorohod integral). Given a stochastic process $\{v(t), t \ge 0\}$ such that $\int_0^t |v(s)| ds < \infty$ a.s. for all $t > 0$, the Stratonovich integral $\int_0^t v(s) \circ dB_s^H$ is defined as the following limit in probability if it exists

$$\lim_{\varepsilon \to 0} \int_0^t v(s) \dot{B}_s^{H,\varepsilon} ds ,$$

where $\dot{B}_s^{H,\varepsilon}$ is a symmetric approximation of $\dot{B}_s^H$:

$$\dot{B}_s^{H,\varepsilon} = \frac{1}{2\varepsilon}(B_{s+\varepsilon}^H - B_{s-\varepsilon}^H)\,.$$

Before we define the divergence integral, we present some background of Malliavin calculus. For a smooth and cylindrical random variable $F = f(B^H(\varphi_1), \ldots, B^H(\varphi_n))$, with $\varphi_i \in \mathfrak{H}$ and $f \in C_b^\infty(\mathbb{R}^n)$ (f and all of its partial derivatives are bounded), we define its Malliavin derivative as the $\mathfrak{H}$-valued random variable given by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^H(\varphi_1), \ldots, B^H(\varphi_n))\varphi_i.$$

By iteration, one can define the k-th derivative $D^k F$ as an element of $L^2(\Omega; \mathfrak{H}^{\otimes k})$. For any natural number k and any real number $p \geq 1$, we define the Sobolev space $\mathbb{D}^{k,p}$ as the closure of the space of smooth and cylindrical random variables with respect to the norm $||\cdot||_{k,p}$ defined by

$$||F||_{k,p}^p = \mathbb{E}(|F|^p) + \sum_{i=1}^k \mathbb{E}(||D^i F||_{\mathfrak{H}^{\otimes i}}^p).$$

The divergence operator δ is defined as the adjoint of the derivative operator D in the following manner. An element $u \in L^2(\Omega; \mathfrak{H})$ belongs to the domain of δ, denoted by $\operatorname{Dom}\delta$, if there is a constant c_u depending on u such that

$$|\mathbb{E}(\langle DF, u\rangle_{\mathfrak{H}})| \leq c_u ||F||_{L^2(\Omega)}$$

for any $F \in \mathbb{D}^{1,2}$. If $u \in \operatorname{Dom}\delta$, then the random variable $\delta(u)$ is defined by the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(\langle DF, u\rangle_{\mathfrak{H}})\,,$$

which holds for any $F \in \mathbb{D}^{1,2}$. If $u = \{u_t, t \in [0,T]\}$ is a stochastic process, whose trajectories belong to $\mathfrak{H}$ almost surely (with the convention $u_t = 0$ if $t \notin [0,T]$) and $u \in \operatorname{Dom} \delta$, we make use of the notation $\int_0^T u_t dB_t^H = \delta(u)$ and call $\delta(u)$ the divergence integral of u with respect to the fractional Brownian motion B^H on $[0,T]$. It is worth noting that the divergence integral of fBm with respect to itself does not exist if $H \in (0, \frac{1}{4})$ because the paths of the fBm are too irregular (see [13]). For this reason, in [13] the authors introduce an extended divergence integral δ^* such that $\operatorname{Dom} \delta^* \cap L^2(\Omega; \mathfrak{H}) = \operatorname{Dom} \delta$ and the extended divergence operator δ^* restricted to $\operatorname{Dom} \delta$ coincides with the divergence operator. In a similar way we can introduce the iterated divergence operator δ^k for each integer $k \geq 2$, defined by the duality relationship

$$\mathbb{E}(F\delta^k(u)) = \mathbb{E}\left(\langle D^k F, u\rangle_{\mathfrak{H}^{\otimes k}}\right),$$

for any $F \in \mathbb{D}^{k,2}$, where $u \in \operatorname{Dom} \delta^k \subset L^2(\Omega; \mathfrak{H}^{\otimes k})$.

For any integer $m \geq 1$, we use $\mathfrak{H}^{\otimes m}$ and $\mathfrak{H}^{\odot m}$ to denote the m-th tensor product and the m-th symmetric tensor product of the Hilbert space $\mathfrak{H}$, respectively. We denote by $\mathscr{H}_m$ the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_m(B^H(\varphi)) : \varphi \in \mathfrak{H}, ||\varphi||_{\mathfrak{H}} = 1\}$, where H_m is the m-th Hermite polynomial defined by

$$H_m(x) = \frac{(-1)^m}{m!} e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-\frac{x^2}{2}}, \quad m \geq 1,$$

and $H_0(x) = 1$. The space $\mathscr{H}_m$ is called the Wiener chaos of order m. The m-th multiple integral of $\varphi \in \mathfrak{H}^{\odot m}$ is defined by the identity $I_m(\varphi) = \delta^m(\varphi)$, and in particular, $I_m(\phi^{\otimes m}) = H_m(B^H(\phi))$ for any $\phi \in \mathfrak{H}$. The map I_m provides a linear isometry between $\mathfrak{H}^{\odot m}$(equipped with the norm $\frac{1}{\sqrt{m!}}||\cdot||_{\mathfrak{H}^{\otimes m}}$) and $\mathscr{H}_m$ (equipped with $L^2(\Omega)$ norm) (see [28], Theorem 2.7.7). By convention, $\mathscr{H}_0 = \mathbb{R}$ and $I_0(x) = x$.

The space $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathscr{H}_m$, which is known as the Wiener chaos expansion. Thus, any square integrable random variable $F \in L^2(\Omega)$ has the following expansion,

$$F = \sum_{m=0}^{\infty} I_m(f_m),$$

where $f_0 = \mathbb{E}(F)$, and $f_m \in \mathfrak{H}^{\odot m}$ are uniquely determined by F. We denote by J_m the orthogonal projection onto the m-th Wiener chaos $\mathscr{H}_m$. This means that $I_m(f_m) = J_m(F)$ for every $m \geq 0$.

For all $t \geq 0$ and $F \in L^2(\Omega)$, we define the Ornstein-Uhlenbeck semigroup $(P_t)_{t\geq 0}$ as

$$P_t(F) = \sum_{m=0}^{\infty} e^{-mt} J_m(F) \in L^2(\Omega)\,.$$

Denote $L = \frac{d}{dt}|_{t=0} P_t$ the infinitesimal generator of $(P_t)_{t\geq 0}$ on $L^2(\mu)$. Then we have

$$LF = -\sum_{m=1}^{\infty} mJ_m(F)$$

for any $F \in \mathrm{Dom}L$ (see [28]). We define the pseudo-inverse of L as

$$L^{-1}F = -\sum_{m=1}^{\infty} \frac{1}{m} J_m F.$$

The following lemma establishes the relationship among P_t, D, L^{-1} (see [28]).

Lemma 2.2.1. *Suppose $F \in \mathbb{D}^{1,2}$ and $\mathbb{E}(F) = 0$. Then we have the identity*

$$-DL^{-1}F = \int_0^{\infty} e^{-t} P_t DF\, dt\,.$$

Next, we define the derivative operator and its adjoint, the divergence with respect to d-dimensional fractional Brownian motion. Consider a smooth and cylindrical random variable of the form $F = f(B_{t_1},\dots,B_{t_n})$, where $f \in C_b^\infty(\mathbb{R}^{d\times n})$ (f and its partial derivatives are all bounded). We define its Malliavin derivative as the $\mathfrak{H}^d$-valued random variable given by $DF = (D^1F,\dots,D^dF)$ whose jth component is given by

$$D_s^jF = \sum_{i=1}^n \frac{\partial f}{\partial x_i^j}(B_{t_1},\dots,B_{t_n})\mathbb{1}_{[0,t_j]}(s).$$

By iteration, one can define higher order derivatives $D^{j_1,\dots,j_i}F$ that take values on $(\mathfrak{H}^d)^{\otimes i}$. For any natural number p and any real number $q \geq 1$, we define the Sobolev space $\mathbb{D}^{p,q}$ as the closure of the space of smooth and cylindrical random variables with respect to the norm $\|\cdot\|_{p,q}$ given by

$$\|F\|_{p,q}^q = \mathbb{E}(|F|^q) + \sum_{i=1}^p \mathbb{E}\left[\left(\sum_{j_1,\dots,j_i=1}^d \|D^{j_1,\dots,j_i}F\|^2_{(\mathfrak{H}^d)^{\otimes i}}\right)^{\frac{q}{2}}\right].$$

Similarly, if $\mathbb{W}$ is a general Hilbert space, we can define the Sobolev space of $\mathbb{W}$-valued random variables $\mathbb{D}^{p,q}(\mathbb{W})$.

For $j = 1,\dots,d$, the adjoint of the Malliavin derivative operator D^j, denoted as δ^j, is called the divergence operator or Skorohod integral (see [29]). A random element u belongs to the domain of δ^j, denoted as $\mathrm{Dom}(\delta^j)$, if there exists a positive constant c_u depending only on u such that

$$|\mathbb{E}(\langle D^jF, u\rangle_{\mathfrak{H}})| \leq c_u\|F\|_{L^2(\Omega)}$$

for any $F \in \mathbb{D}^{1,2}$. If $u \in \mathrm{Dom}(\delta^j)$, then the random variable $\delta^j(u)$ is defined by the duality relationship

$$\mathbb{E}\left(F\delta^j(u)\right) = \mathbb{E}(\langle D^j F, u\rangle_{\mathfrak{H}}),$$

for any $F \in \mathbb{D}^{1,2}$. In a similar way, we can define the divergence operator on $\mathfrak{H}^d$ and we have $\delta(u) = \sum_{j=1}^d \delta^j(u_j)$ for $u = (u_1, \dots, u_d) \in \cap_{j=1}^d \mathrm{Dom}(\delta^j)$. We make use of the notation $\delta(u) = \int_0^\infty u_t dB_t$ and call $\delta(u)$ the divergence integral of u with respect to the fBm B.

For $p > 1$, as a consequence of Meyer's inequality, the divergence operator δ is continuous from $\mathbb{D}^{1,p}(\mathfrak{H}^d)$ into $L^p(\Omega)$, which means

$$\mathbb{E}(|\delta(u)|^p) \le C_p \left(\mathbb{E}(\|u\|_{\mathfrak{H}^d}^p) + \mathbb{E}(\|Du\|_{\mathfrak{H}^d \otimes \mathfrak{H}^d}^p)\right), \tag{2.1}$$

for some constant C_p depending on p.

2.3 Convergence results

Let $\{e_k, k \ge 1\}$ be a complete orthonormal system in the Hilbert space $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot n}, g \in \mathfrak{H}^{\odot m}$, and $p = 1, \dots, n \wedge m$, the p-th contraction between f and g is the element of $\mathfrak{H}^{\otimes(m+n-2p)}$ defined by

$$f \otimes_p g = \sum_{i_1, \dots, i_p = 1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_p}\rangle_{\mathfrak{H}^{\otimes p}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_p}\rangle_{\mathfrak{H}^{\otimes p}}.$$

The following result (known as the fourth moment theorem) provides necessary and sufficient conditions for the convergence of some random variables to a normal distribution (see [30, 31, 28]).

Theorem 2.3.1. *Let $n \geq 2$ be a fixed integer. Consider a collection of elements $\{f_T, T > 0\}$ such that $f_T \in \mathfrak{H}^{\odot n}$ for every $T > 0$. Assume further that*

$$\lim_{T\to\infty} \mathbb{E}[I_n(f_T)^2] = \lim_{T\to\infty} n!\|f_T\|^2_{\mathfrak{H}^{\otimes n}} = \sigma^2.$$

Then the following conditions are equivalent:

1. $\lim_{T\to\infty} \mathbb{E}[I_n(f_T)^4] = 3\sigma^2$.

2. *For every $p = 1, \ldots, n-1$, $\lim_{T\to\infty} ||f_T \otimes_p f_T||_{\mathfrak{H}^{\otimes 2(n-p)}} = 0$.*

3. *As T tends to infinity, the n-th multiple integrals $\{I_n(f_T), T \geq 0\}$ converge in distribution to a standard Gaussian random variable $N(0, \sigma^2)$.*

4. $\|D(I_n(f_T))\|^2_{\mathfrak{H}} \xrightarrow[T\to\infty]{L^2(\Omega)} n\sigma^2$.

Remark 2.3.2. The multidimensional version of the above theorem is also stated and proved in [30, 28, 33].

For the two real-valued random variables F and G, the total variation distance between the laws of F and G is defined by the quantity

$$d_{\mathrm{TV}}(F,G) = \sup_{B\in\mathscr{B}(\mathbb{R})} |P(F \in B) - P(G \in B)| \,,$$

where the supremum is taken over Borel sets B of $\mathbb{R}$. Then we have the following bounds on normal approximation inside a Wiener chaos (see [28]).

Proposition 2.3.3. Let $n \geq 2$ be an integer, and $F_T = I_n(f)$ be a multiple integral of order n with $\mathbb{E}(F_T^2) = 1$. Let N be a random variable with the standard normal distribution.

Then the total variation distance between F_T and N is bounded as follows.

$$d_{\mathrm{TV}}(F_T,N) \le 2\sqrt{\mathrm{Var}\left(\frac{1}{n}\|DF_T\|_{\mathfrak{H}}^2\right)}\,.$$

In the paper [30], Nualart and Ortiz-Lattore apply the fourth moment theorem to establish the following weak convergence result for an arbitrary sequence of centered square integrable random vectors.

Theorem 2.3.4. *Let $\{F_k, k \in \mathbb{N}\}$ be a sequence of d-dimensional centered square integrable random vectors with the following Wiener chaos expansions:*

$$F_k = \sum_{m=1}^{\infty} J_m F_k\,.$$

Suppose that:

(i) $\lim_{M\to\infty} \limsup_{k\to\infty} \sum_{m=M+1}^{\infty} \mathbb{E}[|J_m F_k|^2] = 0\,.$

(ii) For every $m \ge 1$, $1 \le i,j \le d$, $\lim_{k\to\infty} \mathbb{E}[(J_m F_k^i)(J_m F_k^j)] = C_m^{ij}$.

(iii) For all $v \in \mathbb{R}^d$, $\sum_{m=1}^{\infty} v^T C_m v = v^T C v$, where C is a $d \times d$ symmetric nonnegative definite matrix.

(iv) For all $m \ge 1$, $1 \le i,j \le d$,

$$\langle D(J_m F_k^i), D(J_m F_k^j)\rangle_{\mathfrak{H}} \xrightarrow[k\to\infty]{L^2(\Omega)} m C_m^{ij}\,.$$

Then, F_k converges in distribution to the d-dimensional normal law $N_d(0,C)$ as k tends to infinity.

Next let us recall the definition of the Rosenblatt process that will appear in the limit theorems of the following chapters. Fix $H > 3/4$ and $t \in [0,1]$. Consider the sequence of functions of two variables

$$\xi_{n,t} = 2^n \sum_{i=1}^{[2^n t]} \mathbf{1}^{\otimes 2}_{((i-1)2^{-n}, i2^{-n}]}\,.$$

Through a direct computation using (1.5) one can show that this sequence is Cauchy in $\mathfrak{H}^{\otimes 2}$ and converges to distribution denoted by $\delta_{0,t}$ and defined by

$$\langle \delta_{0,t}, f\rangle = \int_0^t f(s,s)ds, \tag{3.1}$$

for any test function f on $\mathbb{R}^2$. It turns out (see [27] for the proofs) that the sequence $I_2(\xi_{n,t})$ converges in L^2 as n tends to infinity to the Rosenblatt random variable $R_t = I_2(\delta_{0,t})$. For any $f \in L^{1/H}([0,1]^2)$, we have the following formula, letting f equal to zero on $\mathbb{R}^2 \cap [0,1]^c$,

$$\mathbb{E}(R_t I_2(f)) = 2\langle \delta_{0,t}, f\rangle_{\mathfrak{H}^{\otimes 2}} = 2\alpha_H^2 \int_0^t dv \int_{[0,1]^2} f(u_1,u_2)|u_1 - v|^{2H-2}|u_2 - v|^{2H-2} du_1 du_2\,. \tag{3.2}$$

In the paper [3], the authors establish an explicit connection between Stein matrices and the law of iterated logarithm, which is stated as the following proposition.

Proposition 2.3.5. Let $X = \{X_n, n \geq 1\}$ be a sequence of centered random variables. We assume that X satisfies the following conditions.

1. There exists a function $\delta : \mathbb{N} \to \mathbb{R}^+$ such that $\delta(n) = \mathscr{O}(n^\alpha L(n))$ for some $\alpha \in (0,1]$ where L satisfies $\lim_{x\to\infty} L(ax)/L(x) = 1$ for any $a > 0$, and for all $n_1 < n_2$,

   $$\left| \mathbb{E}\left(\frac{X_{n_2} - X_{n_1}}{\delta(n_2 - n_1)} \right)^2 - 1 \right| \leq \frac{C}{1 + \log(n_2 - n_1)},$$

   where $C > 0$ is a constant independent of n_1, n_2.

2. For every $m \in \mathbb{N}$, and every increasing sequence $\vec{n} = \{n_i\}_{1\le i\le 2m} = \{[a^{(b+i)^{1+\alpha}}]\}$ where $a > 1, \alpha > 0$ and $m, b \in \mathbb{N}$, we define the vector $\vec{R} = (R_1, \dots, R_m)$ where

$$R_i = \frac{X_{n_{2i}} - X_{n_{2i-1}}}{\delta(n_{2i} - n_{2i-1})}.$$

Then $\vec{R}$ admits a $m \times m$ Stein matrix $\tau = (\tau_{ij})$.

3. There exists $a > 1$ such that for every $\alpha > 0$, there exists a positive constant C depending on a, α satisfying the following inequalities.

$$\sqrt{\mathrm{Var}(\tau_{ii}(\vec{R}))} \le \frac{C}{1 + \log(n_{2i} - n_{2i-1})}, \quad i = 1, \dots, m,$$

and

$$\sqrt{\mathbb{E}(\tau_{ij}(\vec{R})^2)} \le \frac{C}{1 + \log(n_{2i} - n_{2i-1})}, \quad i \ne j.$$

4. There exist positive constants C, θ such that for all $r \ge 1$, $n_1 < n_2$,

$$\left(\mathbb{E}|\tau\left(\frac{X_{n_2} - X_{n_1}}{\delta(n_2 - n_1)}\right) - 1|^r\right)^{\frac{1}{r}} \le \frac{Cr^\theta}{1 + \log(n_2 - n_1)},$$

where τ is the Stein factor of $\frac{X_{n_2} - X_{n_1}}{\delta(n_2 - n_1)}$.

Then

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\delta^2(n)\log\log n}} = 1, \ a.s.$$

$$\liminf_{n\to\infty} \frac{X_n}{\sqrt{2\delta^2(n)\log\log n}} = -1, \ a.s.$$

We end this section by stating the following theorem proved in the paper [14] on the asymptotic behavior of weighted random sums. It will be used in the next section to prove the central limit theorem of the power variation of stochastic integrals.

Theorem 2.3.6. *Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. Fix a time interval $[0,T]$ and consider a double sequence of random variables $\xi = \{\xi_{i,m}, m \in \mathbb{Z}_+, 1 \leq i \leq [mT]\}$. Assume the double sequence ξ satisfies the following hypotheses.*
(H1) Denote $g_m(t) := \sum_{i=1}^{[mt]} \xi_{i,m}$. The finite dimensional distributions of the sequence of processes $\{g_m(t), t \in [0,T]\}$ converges $\mathscr{F}$-stably to those of $\{B(t), t \in [0,T]\}$ as $m \to \infty$, where $\{B(t), t \in [0,T]\}$ is a standard Brownian motion independent of $\mathscr{F}$.
(H2) ξ satisfies the tightness condition

$$\mathbb{E}\left|\sum_{i=j+1}^{k} \xi_{i,m}\right|^4 \leq C\left(\frac{k-j}{m}\right)^2$$

for any $1 \leq j < k \leq [mT]$.
If $\{f(t), t \in [0,T]\}$ is an $\alpha-$Hölder continuous process with $\alpha > 1/2$ and we set $X_m(t) := \sum_{i=1}^{[mt]} f(\frac{i}{m})\xi_{i,m}$, then we have the $\mathscr{F}$-stable convergence

$$X_m(t) \xrightarrow[m\to\infty]{\mathscr{L}} \int_0^t f(s)dB(s),$$

in the Skorohod space $\mathscr{D}[0,T]$.

Chapter 3

Estimation for the volatility parameter and Hurst parameter

3.1 Asymptotic behavior of power variations

We first recall the definition of p-variation. For any $p>0$, the p-variation of a real-valued function f on an interval $[a,b]$ is defined as

$$\mathrm{var}_p(f;[a,b]) = \sup_\pi \Big(\sum_{i=1}^n |f(t_i)-f(t_{i-1})|^p\Big)^{1/p},$$

where the supremum runs over all partitions $\pi=\{a=t_0<t_1<\cdots<t_n=b\}$. If f is α-Hölder continuous on the interval $[a,b]$, $\alpha\in(0,1]$, then we set

$$\|f\|_\alpha := \sup_{a\le s<t\le b} \frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

It is known that an α -Hölder continuous function f on the interval $[a,b]$ has finite $1/\alpha$-variation on this interval. If f and g have finite p-variation and finite q-variation on the interval $[a,b]$ respectively and $1/p+1/q>1$, the Riemann-Stieltjes integral $\int_a^b f dg$ exists (see Young [41]). By Young's result, the stochastic integral $\int_0^t u_s dB_s^H$ is well defined as a pathwise Riemann-Stieltjes integral provided that the trajectories of the process $\{u_t, t \geq 0\}$ have finite q-variation on any finite interval for some $q < 1/(1-H)$.

Next we introduce high order power variations and prove some asymptotic results for the high order power variations of stochastic integrals with respect to fBm. The high order power variations will be used to construct estimators for Hurst parameter, the volatility and the integrated volatility of some stochastic processes in the next sections.

Consider a sequence of random variables $\{X_{i-1}, i \geq 1\}$. Denote the first order difference $\Delta X_{i-1} = \Delta_1 X_{i-1} = X_i - X_{i-1}$. Define the k-th order difference by induction as follows $\Delta_k X_{i-1} = \Delta_{k-1} X_i - \Delta_{k-1} X_{i-1}$ for $k = 2, 3, \ldots$, namely,

$$\Delta_k X_{i-1} = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} X_{i+j-1}\,.$$

Let $B^H = \{B_t^H, t \geq 0\}$ be a fBm with Hurst parameter $H \in (0,1)$. For any $j \geq 0$, we can write down the covariance function of the k-th order difference of the sequences $\{B_n^H, n \geq 0\}$ and $\{B_{n+j}^H, n \geq 0\}$ as follows

$$\rho_{k,H}(j) := \mathbb{E}[(\Delta_k B_{n+j}^H)(\Delta_k B_n^H)] = \frac{1}{2} \sum_{i=-k}^{k} (-1)^{1-i} \binom{2k}{k-i} |j-i|^{2H}\,.$$

Since all the moments of a mean zero Gaussian can be expressed by its variance, we see that the p-th moment of $\Delta_k B_n^H$ is given by

$$c_{k,p} = \mathbb{E}[|\Delta_k B_n^H|^p] = \frac{2^{p/2}\Gamma((p+1)/2)}{\Gamma(1/2)} [\rho_{k,H}(0)]^{p/2}. \tag{1.1}$$

Notice that the quantities $\rho_{k,H}(j)$ and $c_{k,p}$ are independent of n, due to the fact that the fBm has stationary increments.

From the fact that $\rho_{k,H}(j) = o(j^{2H-2k})$ for j large it follows that

$$\sum_{j=0}^{\infty} \rho_{k,H}^2(j) \begin{cases} = \infty & \text{when } k = 1 \text{ and } \frac{3}{4} \le H < 1, \\ < \infty & \text{when } k = 1 \text{ and when } 0 < H < \frac{3}{4}, \\ < \infty & \text{when } k \ge 2\,. \end{cases}$$

Let $p > 0$ and let $n \ge 1$ be an integer. We define the k-th order p-variation of a stochastic process $Z = \{Z_t, t \ge 0\}$ as

$$V_{k,p}^n(Z)_t = \sum_{i=1}^{[nt]-k+1} |\Delta_k Z_{\frac{i-1}{n}}|^p = \sum_{i=1}^{[nt]-k+1} \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} Z_{\frac{i+j-1}{n}} \right|^p, \tag{1.2}$$

where we use the convention that the sum is zero if $[nt] - k + 1 < 1$.

The following proposition shows the convergence of the k-th order p-variation for stochastic integrals of fractional Brownian motion, extending a result in [15] which is valid when $k = 1$.

Theorem 3.1.1. *Let $k \ge 2$ and let $H \in (0,1)$. Suppose that $\{u_t, t \in [0,T]\}$ is a stochastic process whose sample paths are Hölder continuous with exponent $a \in (1-H, 1]$. Consider the pathwise Riemann-Stieltjes integral*

$$Z_t = \int_0^t u_s dB_s^H\,, \quad t \in [0,T].$$

Then for any $p > 0$, as $n \to \infty$,

$$n^{-1+pH} V_{k,p}^n(Z)_t \to c_{k,p} \int_0^t |u_s|^p ds \tag{1.3}$$

almost surely, uniformly on $[0,T]$, where $c_{k,p}$ is the constant introduced in (1.1).

Proof. Denote by $\|\cdot\|_\infty$ the supremum norm on $[0,T]$. For any $t\in[0,T]$ and any $m\geq n\geq 1$, by the definition of $V^m_{k,p}(Z)_t$, we have

$$\begin{aligned}
& m^{-1+pH}V^m_{k,p}(Z)_t - c_{k,p}\int_0^t |u_s|^p ds \\
= \; & m^{-1+pH}\sum_{i=1}^{[mt]-k+1}\left(\left|\Delta_k Z_{\frac{i-1}{m}}\right|^p - \left|u_{\frac{i}{m}}\Delta_k B^H_{\frac{i-1}{m}}\right|^p\right) \\
& + m^{-1+pH}\left(\sum_{i=1}^{[mt]-k+1}\left|u_{\frac{i}{m}}\Delta_k B^H_{\frac{i-1}{m}}\right|^p - \sum_{i=1}^{[nt]-k+1}\left|u_{\frac{i-1}{n}}\right|^p\sum_{j\in I_n(i)}\left|\Delta_k B^H_{\frac{j-1}{m}}\right|^p\right) \\
& + m^{-1+pH}\sum_{i=1}^{[nt]-k+1}\left|u_{\frac{i-1}{n}}\right|^p\sum_{j\in I_n(i)}\left|\Delta_k B^H_{\frac{j-1}{m}}\right|^p - c_{k,p}n^{-1}\sum_{i=1}^{[nt]-k+1}\left|u_{\frac{i-1}{n}}\right|^p \\
& + c_{k,p}\Big(\frac{1}{n}\sum_{i=1}^{[nt]-k+1}\left|u_{\frac{i-1}{n}}\right|^p - \int_0^t |u_s|^p ds\Big) \\
=: \; & A^{(m)}_t + B^{(n,m)}_t + C^{(n,m)}_t + D^{(n)}_t,
\end{aligned} \tag{1.4}$$

where $I_n(i)=\{j:\frac{j-1}{m}\in(\frac{i-1}{n},\frac{i}{n}]\},\quad 1\leq i\leq [nt]-k+1.$

Because of the stationary property of the increments of B^H, the high order difference sequence $\{\Delta_k B^H_{j-1}, j\geq 1\}$ is stationary as well. Thus, for any fixed $n\in\mathbb{N}$ and $1\leq i\leq [nt]-k+1$, we apply the ergodic theorem to obtain

$$m^{-1+pH}n\sum_{j\in I_n(i)}\left|\Delta_k B^H_{\frac{j-1}{m}}\right|^p - c_{k,p}\to 0, \tag{1.5}$$

almost surely as $m\to\infty$. This implies

$$\lim_{m\to\infty}\|C^{(n,m)}\|_\infty = 0 \tag{1.6}$$

almost surely, for any fixed $n\geq 1$.

In the following arguments, we will use the two elementary inequalities

$$
\begin{aligned}
|x+y+z|^p &\leq 3^{(p-1)^+}[|x|^p+|y|^p+|z|^p], &(1.7)\\
||x|^p-|y|^p| &\leq (p\vee 1)2^{(p-2)^+}[|x-y|^p+|y|^{(p-1)^+}|x-y|^{(p\wedge 1)}] &(1.8)
\end{aligned}
$$

for any $p \geq 0$, and any $x,y,z \in \mathbb{R}$.

For the term $B_t^{(n,m)}$,

$$
\begin{aligned}
\|B^{(n,m)}\|_\infty &\leq m^{-1+pH}\sum_{i=1}^{[nT]}\sum_{j\in I_n(i)}\left||u_{\frac{j}{m}}|^p-|u_{\frac{i-1}{n}}|^p\right|\left|\Delta_k B_{\frac{j-1}{m}}\right|^p\\
&\leq \frac{1}{n}\sum_{i=1}^{[nT]}\sup_{s\in(\frac{i-1}{n},\frac{i+1}{n}]}\left||u_{\frac{i-1}{n}}|^p-|u_s|^p\right|\left(m^{-1+pH}n\sum_{j\in I_n(i)}\left|\Delta_k B_{\frac{j-1}{m}}\right|^p\right)\\
&\leq C(n^{-ap}+\||u|^p\|_\infty n^{-a(p\wedge 1)}),
\end{aligned}
$$

where the last step we have used the result of (1.5) for the second factor for each fixed n, and for the first factor, we have applied the inequlaity (1.8) and the the Hölder continuity of u. Therefore,

$$
\lim_{n\to\infty}\lim_{m\to\infty}\|B^{(n,m)}\|_\infty = 0, \tag{1.9}
$$

almost surely.

The term $D_t^{(n)}$ is the remainder of a Riemann sum approximation. For all $p > 0$, we have

$$
\begin{aligned}
|D_t^{(n)}| &= \left|\sum_{i=1}^{[nt]-k+1}\frac{1}{n}\left|u_{\frac{i-1}{n}}\right|^p-\int_{\frac{i-1}{n}}^{\frac{i}{n}}|u_s|^p ds\right|+\int_{\frac{[nt]-k}{n}}^{t}|u_s|^p ds\\
&\leq \frac{1}{n}\sum_{i=1}^{[nT]}\sup_{s\in(\frac{i-1}{n},\frac{i}{n}]}\left|\left|u_{\frac{i-1}{n}}\right|^p-|u_s|^p\right|+\frac{k+1}{n}\||u|^p\|_\infty\\
&\leq C(n^{-ap}+\||u|^p\|_\infty n^{-a(p\wedge 1)})+Cn^{-1}\||u|^p\|_\infty,
\end{aligned}
$$

where in the last step we have used inequality (1.8) and the Hölder continuity of u. Therefore,

$$\lim_{n\to\infty} \|D^{(n)}\|_\infty = 0\,, \tag{1.10}$$

almost surely.

It remains to deal with the term $A^{(m)}$. Using inequality (1.8), we obtain

$$\begin{aligned}
|A_t^{(m)}| &\le m^{-1+pH} \sum_{i=1}^{[mt]+1-k} \left| |\Delta_k Z_{\frac{i-1}{m}}|^p - |u_{\frac{i}{m}} \Delta_k B^H_{\frac{i-1}{m}}|^p \right| \\
&\le (p\vee 1)2^{(p-2)_+} m^{-1+pH} \left\{ \sum_{i=1}^{[mt]+1-k} \left[\left| \Delta_k Z_{\frac{i-1}{m}} - u_{\frac{i}{m}} \Delta_k B^H_{\frac{i-1}{m}} \right|^p \right.\right. \\
&\quad \left.\left. + \left| u_{\frac{i}{m}} \Delta_k B^H_{\frac{i-1}{m}} \right|^{(p-1)^+} \left| \Delta_k Z_{\frac{i-1}{m}} - u_{\frac{i}{m}} \Delta_k B^H_{\frac{i-1}{m}} \right|^{p\wedge 1} \right] \right\} \\
&=: (p\vee 1)2^{(p-2)_+} [E^{(m)}_{k,p}(t) + F^{(m)}(t)]\,.
\end{aligned} \tag{1.11}$$

First, we use mathematical induction on k to prove $\lim_{m\to\infty} \|E^{(m)}_{k,p}\|_\infty = 0$, almost surely. For $k = 1$, the result is true by the proof of Theorem 1 in [15]. Assume the convergence holds true for $k-1$. We can express $E^{(m)}_{k,p}(t)$ in the following way

$$E^{(m)}_{k,p}(t) = m^{-1+pH} \sum_{i=1}^{[mt]+1-k} \left| \Phi^{(m)}_{i,1} - \Phi^{(m)}_{i,2} + \Phi^{(m)}_{i,3} \right|^p,$$

where

$$\begin{aligned}
\Phi^{(m)}_{i,1} &= \Delta_{k-1} Z_{\frac{i}{m}} - u_{\frac{i+1}{m}} \Delta_{k-1} B^H_{\frac{i}{m}}, \\
\Phi^{(m)}_{i,2} &= \Delta_{k-1} Z_{\frac{i-1}{m}} - u_{\frac{i}{m}} \Delta_{k-1} B^H_{\frac{i-1}{m}},
\end{aligned}$$

and

$$\Phi_{i,3}^{(m)} = \Delta_{k-1} B^H_{\frac{i}{m}} (u_{\frac{i+1}{m}} - u_{\frac{i}{m}}).$$

Then, applying inequality (1.7) yields

$$\begin{aligned} E_{k,p}^{(m)}(t) &\leq 3^{(p-1)^+} m^{-1+pH} \sum_{i=1}^{[mt]+1-k} \left(|\Phi_{i,1}^{(m)}|^p + |\Phi_{i,2}^{(m)}|^p + |\Phi_{i,3}^{(m)}|^p \right) \\ &\leq 3^{(p-1)^+} \left(2E_{k-1,p}^{(m)}(t) + m^{-1+pH} \sum_{i=1}^{[mt]+1-k} |\Phi_{i,3}^{(m)}|^p \right). \end{aligned}$$

Choosing $0 < \varepsilon < a + H - 1$, we can write

$$m^{-1+pH} \sum_{i=1}^{[mt]+1-k} |\Phi_{i,3}^{(m)}|^p \leq C m^{pH-pa-p(H-\varepsilon)} \|u\|_a^p \|B\|_{H-\varepsilon}^p,$$

for some constant C depending on T, p, ε, k and H. Using the induction hypothesis, and taking into account that $-a+\varepsilon < H-1 < 0$, we conclude that $\|E_{k,p}^{(m)}\|_\infty$ converges to zero almost surely, as m tends to infinity.

Finally, the infinity norm of the term $F^{(m)}$ can be bounded by

$$\|F^{(m)}\|_\infty \leq C \|u\|_\infty^{(p-1)^+} \|B\|_{H-\varepsilon}^{(p-1)^+} m^{-(p-1)^+(H-\varepsilon)} \|E_{k,p\wedge 1}^{(m)}\|_\infty,$$

where again C is a constant depending on T, p, ε, k and H. Then, $\|F^{(m)}\|_\infty$ goes to 0 almost surely, as $m \to \infty$.

Thus, by (1.11) we have $\|A^{(m)}\|_\infty \to 0$ almost surely, as $m \to \infty$. The proposition follows then from this convergence and the limits established in (1.6), (1.9) and (1.10).

□

Next we study the central limit theorem of (1.3). We will use the notation

$$v_1^2 = \sum_{m=2}^{\infty} \frac{c_m^2}{m!} \Big[1 + 2 \sum_{j=1}^{\infty} \Big(\frac{\rho_{k,H}(j)}{\rho_{k,H}(0)} \Big)^m \Big] , \tag{1.12}$$

where $c_m = m!(\rho_{k,H}(0))^{\frac{p}{2}} \mathbb{E}[H_m(N)|N|^p]$ and N is a standard Gaussian random variable. We shall first deal with the case of the fractional Brownian motion ($Z_t = B_t^H$) and then consider the general case of stochastic integrals.

Proposition 3.1.2. Fix a positive integer $k \geq 2$. Let $H \in (0,1)$, $T > 0$ and $p > 0$. Then

$$\Big(B_t^H, \sqrt{n} \Big(n^{-1+pH} V_{k,p}^n (B^H)_t - c_{k,p} t \Big) \Big) \to (B_t^H, v_1 W_t) \tag{1.13}$$

in law in the space $\mathscr{D}([0,T])^2$ equipped with the Skorohod topology, where v_1 is defined by (1.12) and $W = \{W_t, t \in [0,T]\}$ is a Brownian motion, independent of the fractional Brownian motion B^H.

Proof. The proof will be completed in two steps.

Step 1: We show the convergence of the finite-dimensional distributions. Let the intervals $(a_l, b_l], l = 1, \dots, \nu$, be pairwise disjoint in $[0,T]$. Define the random vectors $B = (B_{b_1}^H - B_{a_1}^H, \dots, B_{b_\nu}^H - B_{a_\nu}^H)$ and $X^{(n)} = (X_1^{(n)}, \dots, X_\nu^{(n)})$, where

$$X_l^{(n)} = n^{-\frac{1}{2}+pH} \sum_{j \in \mathscr{I}_{nl}} \Big| \Delta_k B^H_{\frac{j-1}{n}} \Big|^p - \sqrt{n} c_{k,p} |b_l - a_l|,$$

and $\mathscr{I}_{nl} = ([na_l] - k + 1, [nb_l] - k + 1]$, for $l = 1, \dots, \nu$. We claim that

$$(B, X^{(n)}) \xrightarrow[n \to \infty]{\mathscr{L}} (B, V) , \tag{1.14}$$

where B, V are independent and V is a centered Gaussian vector, whose components are independent and have variances $v_1^2|b_l - a_l|$. Here v_1^2 is defined in (1.12).

Set $\xi_j = B_j^H - B_{j-1}^H$ and $h(x) = |x|^p - c_{k,p}$. Then $\{\xi_j, j \geq 1\}$ is a stationary Gaussian sequence. Introduce the random vectors $B^{(n)} = (B_1^{(n)}, \dots, B_\nu^{(n)})$ and $Y^{(n)} = (Y_1^{(n)}, \dots, Y_\nu^{(n)})$, where

$$B_l^{(n)} = n^{-H} \sum_{[na_l] < j \leq [nb_l]} \xi_j \,,$$

$$Y_l^{(n)} = \frac{1}{\sqrt{n}} \sum_{j \in \mathscr{I}_{nl}} h(\Delta_{k-1}\xi_j), \quad 1 \leq l \leq \nu \,.$$

By the self-similarity property of fBm, the convergence of (1.14) will follow from the convergence

$$(B^{(n)}, Y^{(n)}) \xrightarrow[n\to\infty]{\mathscr{L}} (B, V) \,. \tag{1.15}$$

We are going to prove (1.15) by Theorem 2.3.4. Consider the normalized sequence

$$N_j = \frac{\Delta_{k-1}\xi_j}{\sqrt{\rho_{k,H}(0)}}, \quad j \geq 1. \tag{1.16}$$

Since the function $h(x)$ has Hermite rank 2, the term $Y_l^{(n)}$ can be decomposed as

$$Y_l^{(n)} = \sum_{m \geq 2} J_m Y_l^{(n)} := \sum_{m \geq 2} \frac{c_m}{\sqrt{n}} \sum_{j \in \mathscr{I}_{nl}} H_m(N_j) \,,$$

where $J_m Y_l^{(n)}$ is the projection of $Y_l^{(n)}$ on the m-th Wiener chaos, and

$$c_m = m!\mathbb{E}[H_m(N) h(\sqrt{\rho_{k,H}(0)} N)] = m!(\rho_{k,H}(0))^{\frac{p}{2}} \mathbb{E}[H_m(N)|N|^p] \,,$$

with N being a standard Gaussian random variable. We have the following five statements.

(i) $\lim_{n\to\infty}\mathbb{E}[B_h^{(n)}B_l^{(n)}] = \mathbb{E}[(B_{b_h}^H - B_{a_h}^H)(B_{b_l}^H - B_{a_l}^H)]$ for all $1 \le h, l \le \nu$.

(ii) $\mathbb{E}(B_h^{(n)} J_m Y_l^{(n)}) = 0$, for all $1 \le h, l \le \nu$. This is clear because $B_h^{(n)} \in \mathscr{H}_1$ and $J_m Y_l^{(n)} \in \mathscr{H}_m$ with $m \ge 2$.

(iii) For all $1 \le l \le \nu$, we have

$$\limsup_{n\to\infty} \sum_{m=M+1}^{\infty} \mathbb{E}[|J_m Y_l^{(n)}|^2] = \limsup_{n\to\infty} \sum_{m=M+1}^{\infty} \frac{c_m^2}{n} \sum_{i,j\in\mathscr{I}_{nl}} \mathbb{E}[H_m(N_i)H_m(N_j)]$$
$$\le \limsup_{n\to\infty} \frac{[nb_l]-[na_l]}{n} \sum_{m=M+1}^{\infty} \frac{c_m^2}{m!}\left[1+2\sum_{i=1}^{[nb_l]-[na_l]}\left|\frac{\rho_{k,H}(i)}{\rho_{k,H}(0)}\right|^m\right],$$

which equals the constant $b_l - a_l$ multiplying the tail of v_1^2, and it converges to 0 as $M \to \infty$.

(iv) For all $1 \le l, h \le \nu$, we have

$$\mathbb{E}(J_m Y_l^{(n)} J_m Y_h^{(n)}) = \frac{c_m^2}{n} \sum_{j\in\mathscr{I}_{nl}} \sum_{i\in\mathscr{I}_{nh}} \mathbb{E}[H_m(N_j)H_m(N_i)].$$

As $n \to \infty$, this quantity converges to

$$\Sigma_{lh} = \delta_{lh} c_m^2 (b_l - a_l) \frac{1}{m!}\left[1+2\sum_{j=1}^{\infty}\left(\frac{\rho_{k,H}(j)}{\rho_{k,H}(0)}\right)^m\right].$$

(v) For all $1 \le l, h \le \nu$, we have

$$\langle DJ_m Y_l^{(n)}, DJ_m Y_h^{(n)}\rangle_{\mathfrak{H}} = \frac{c_m^2}{n} \sum_{j\in\mathscr{I}_{nl}} \sum_{i\in\mathscr{I}_{nh}} H_{m-1}(N_j)H_{m-1}(N_i)\mathbb{E}(N_i N_j),$$

which converges to $m\Sigma_{lh}$ in $L^2(\Omega)$ as n goes to infinity. To show this, we explain the details for $l=h$. The case $l \neq h$ can be treated in a similar way.

$$\begin{aligned}\|DJ_mY_l^{(n)}\|_{\mathfrak{H}}^2 &= \frac{c_m^2}{n}\sum_{i\in\mathscr{I}_{nl}} H_{m-1}^2(N_i) \\ &\quad +2\frac{c_m^2}{n}\sum_{i\in\mathscr{I}_{nl}}\sum_{j=1}^{[nb_l]-[na_l]-1} H_{m-1}(N_i)H_{m-1}(N_{i+j})\frac{\rho_{k,H}(j)}{\rho_{k,H}(0)}.\end{aligned}$$

Denote $\zeta_i = \sum_{j=1}^{\infty} H_{m-1}(N_i)H_{m-1}(N_{i+j})\frac{\rho_{k,H}(j)}{\rho_{k,H}(0)}$. We can show that the sequence ζ_i converges almost surely and in $L^2(\Omega)$ using the fact that

$$\sup_j \mathbb{E}\left[\left|H_{m-1}(N_i)H_{m-1}(N_{i+j})\right|^2\right] < \infty$$

and $\sum_{j=0}^{\infty} |\rho_{k,H}(j)|^2 < \infty$. Meanwhile, since N_i given by (1.16) is stationary and ergodic so is $\{\zeta_i, i \geq 1\}$. By the ergodic theorem, we have thus in $L^2(\Omega)$

$$\begin{aligned}&\lim_{n\to\infty}\|DJ_mY_l^{(n)}\|_{\mathfrak{H}}^2 \\ &= c_m^2(b_l-a_l)\left(\mathbb{E}[H_{m-1}^2(N_1)] + 2\sum_{j=1}^{\infty}\mathbb{E}[H_{m-1}(N_1)H_{m-1}(N_{1+j})]\frac{\rho_{k,H}(j)}{\rho_{k,H}(0)}\right),\end{aligned}$$

which equals $m\Sigma_{lh}$ for $l=h$.

These can be used to verify the conditions in Theorem 2.3.4 to obtain the convergence $(B^{(n)}, Y^{(n)}) \xrightarrow[n\to\infty]{\mathscr{L}} (B, V)$ and correspondingly the convergence (1.14) stands true.

Step 2: Let

$$g_n(t) = n^{-\frac{1}{2}+pH}\sum_{j=1}^{[nt]-k+1} |\Delta_k B^H_{\frac{j-1}{n}}|^p - \sqrt{n}t c_{k,p}\,. \tag{1.17}$$

We need to show that the sequence of processes g_n is tight in $\mathscr{D}([0,T])$. To this end we want to prove $\mathbb{E}(|g_n(r)-g_n(s)|^2|g_n(t)-g_n(r)|^2) \le C(t-s)^2$ for any $s<r<t$. First, let us compute $\mathbb{E}(|g_n(t)-g_n(s)|^4)$ for $s<t$,

$$\mathbb{E}(|g_n(t)-g_n(s)|^4) = \frac{1}{n^2}\mathbb{E}\Big(\sum_{j=[ns]-k+1}^{[nt]-k} h(\Delta_k B_j^H) + c_{k,p}([nt]-nt-[ns]+ns)\Big)^4.$$

Using the elementary inequality $|a+b|^4 \le 8(|a|^4+|b|^4)$, we can bound the right-hand side of the above equation as follows

$$\begin{aligned}
\mathbb{E}(|g_n(t)-g_n(s)|^4) &\le \frac{8}{n^2}\mathbb{E}\Big(\Big|\sum_{j=[ns]-k+1}^{[nt]-k} h(\Delta_k B_j^H)\Big|^4\Big) + 8c_{k,p}^4\frac{([nt]-nt-[ns]+ns)^4}{n^2} \\
&\le K_1\frac{([nt]-[ns])^2}{n^2}\Big(\sum_{j=0}^{\infty}\rho_{k,H}^2(j)\Big)^2 + 8c_{k,p}^4\frac{([nt]-nt-[ns]+ns)^4}{n^2} \\
&\le C\frac{([nt]-[ns])^2}{n^2} + \frac{C}{n^2}, \qquad (1.18)
\end{aligned}$$

where the second inequality follows from Proposition 4.2 in [37]. The constant K_1 is independent of n,t,s, but it may depend on the function h and the distribution of $\Delta_k B_j^H$.

Now for $s<r<t$, if $t-s \ge 1/n$, applying the above inequality (1.18), we have

$$\begin{aligned}
\mathbb{E}(|g_n(r)-g_n(s)|^2|g_n(t)-g_n(r)|^2) &\le \mathbb{E}(|g_n(r)-g_n(s)|^4 + |g_n(t)-g_n(r)|^4) \\
&\le C\frac{([nt]-[ns])^2}{n^2} + \frac{C}{n^2}.
\end{aligned}$$

Clearly, the right-hand side of the above inequality is at most $C(t-s)^2$.

If $t-s<1/n$, then either s and r or t and r lie in the same subinterval $((j-1)/n, j/n]$ for some j. It suffices to look at the former case. By (1.17),

$$g_n(r)-g_n(s) = \sqrt{n}c_{k,p}(s-r).$$

Using this fact and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\mathbb{E}(|g_n(r)-g_n(s)|^2|g_n(t)-g_n(r)|^2) &= nc_{k,p}^2(r-s)^2\mathbb{E}|g_n(t)-g_n(r)|^2\\ &\le nc_{k,p}^2(r-s)^2\sqrt{\mathbb{E}|g_n(t)-g_n(r)|^4}\\ &\le C(t-s)^2,\end{aligned}$$

where in the last step we have used (1.18) for $\mathbb{E}|g_n(t)-g_n(r)|^4$. The desired tightness property follows from Theorem 13.5 in [8]. □

Theorem 3.1.3. *Let $H\in(0,1)$ and $k\ge 2$. Fix $p>0$ and suppose $u=\{u_t,t\in[0,T]\}$ is a stochastic process with Hölder continuous sample paths of order $a>\max(1-H,\frac{1}{2(p\wedge 1)})$ so that the pathwise Riemann-Stieltjes integral $Z_t=\int_0^t u_s dB_s^H$ is well-defined. Then*

$$(B_t^H, n^{-\frac{1}{2}+pH}V_{k,p}^n(Z)_t - c_{k,p}\sqrt{n}\int_0^t |u_s|^p ds)\to(B_t^H, v_1\int_0^t |u_s|^p dW_s),$$

in law in the space $\mathscr{D}([0,T],\mathbb{R}^2)$ equipped with the Skorohod topology, where v_1 is defined by (1.12)*, $W=\{W_t,t\in[0,T]\}$ is a Brownian motion independent of the fractional Brownian motion B^H.*

Proof. We start with the following decomposition of the concerned quantity

$$\begin{aligned}& n^{-\frac{1}{2}+pH}V_{k,p}^n(Z)_t - c_{k,p}\sqrt{n}\int_0^t |u_s|^p ds\\ =\;& n^{-\frac{1}{2}+pH}\sum_{j=1}^{[nt]+1-k}\Big(\Big|\Delta_k Z_{\frac{j-1}{n}}\Big|^p-\Big|u_{\frac{j}{n}}\Delta_k B^H_{\frac{j-1}{n}}\Big|^p\Big)\\ &+\Big(n^{-\frac{1}{2}+pH}\sum_{j=1}^{[nt]+1-k}\Big|u_{\frac{j}{n}}\Delta_k B^H_{\frac{j-1}{n}}\Big|^p-\frac{c_{k,p}}{\sqrt{n}}\sum_{j=1}^{[nt]+1-k}\Big|u_{\frac{j}{n}}\Big|^p\Big)\\ &+c_{k,p}\Big(\frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]+1-k}\Big|u_{\frac{j}{n}}\Big|^p-\sqrt{n}\int_0^t |u_s|^p ds\Big)\end{aligned}$$

$$=: \quad A_t^{(n)} + B_t^{(n)} + c_{k,p} C_t^{(n)}\,.$$

Using the Hölder continuity of u, we can show $\lim_{n\to\infty} \|C^{(n)}\|_\infty = 0$ almost surely. The fact that $\lim_{n\to\infty} \|A^{(n)}\|_\infty = 0$ almost surely can be proved by the same arguments as in the proof of Theorem 3.1.1 under the condition $a > \frac{1}{2(p\wedge 1)}$. It remains to show that

$$B_t^{(n)} \xrightarrow[n\to\infty]{\mathscr{L}} v_1 \int_0^t |u_s|^p dW_s\,, \tag{1.19}$$

in the Skorohod topology of $\mathscr{D}([0,T])$. Denote

$$\begin{aligned} g_n(t) &= n^{-\frac{1}{2}+pH} \sum_{i=1}^{[nt]+1-k} \left|\Delta_k B^H_{\frac{i-1}{n}}\right|^p - \frac{[nt]}{\sqrt{n}} c_{k,p}, \\ \xi_{j,n} &= g_n\left(\frac{j+k-1}{n}\right) - g_n\left(\frac{j+k-2}{n}\right) = n^{-\frac{1}{2}+pH} \left|\Delta_k B^H_{\frac{j-1}{n}}\right|^p - \frac{c_{k,p}}{\sqrt{n}}\,. \end{aligned}$$

Then $B_t^{(n)} = \sum_{j=1}^{[nt]+1-k} |u_{j/n}|^p \xi_{j,n}$. In order to finish the proof of (1.19), we are going to apply Theorem 2.3.6. We shall verify the hypotheses (H1) and (H2). By Proposition 3.1.2 and its proof, $(B_t^H, g_n(t)) \xrightarrow[n\to\infty]{\mathscr{L}} (B_t^H, v_1 W_t)$, so the sequence of processes $\{g_n(t), t\in[0,T]\}$ satisfies the hypothesis (H1). Using a similar argument as that for (1.18), namely by Proposition 4.2 in [37] again, the family of random variables ξ satisfies the tightness condition (H2). This concludes the proof of the theorem. □

Corollary 3.1.4. If a stochastic process $\{Y_t, t\in[0,T]\}$ satisfies $n^{-\frac{1}{2}+pH} V^n_{k,p}(Y)_t \to 0$ almost surely on $[0,T]$ and if $\{Z_t, t\in[0,T]\}$ satisfies the conditions of Theorem 3.1.3, then

$$\left(B_t^H, n^{-\frac{1}{2}+pH} V^n_{k,p}(Y+Z)_t - c_{k,p}\sqrt{n}\int_0^t |u_s|^p ds\right) \xrightarrow[n\to\infty]{\mathscr{L}} \left(B_t^H, v_1 \int_0^t |u_s|^p dW_s\right)$$

in law in $\mathscr{D}([0,T])^2$ equipped with the Skorohod topology, where $W = \{W_t, t \in [0,T]\}$ is a Brownian motion independent of the fractional Brownian motion B^H.

Remark 3.1.5. When k=1, Theorem 3.1.1, Proposition 3.1.2, Theorem 3.1.3 and Corollary 3.1.4 are proved in [14] and [15] for $H \in (0, \frac{3}{4})$. We need to use higher order ($k \geq 2$) power variations to estimate the volatility or integrated volatility for a general Hurst parameter case.

3.2 Convergence rate of power variations

Proposition 3.2.1. Let the intervals $(a_j, b_j]$ be pairwise disjoint in $[0,\infty)$, and denote the intervals $\mathscr{I}_{nj} = ([na_j],[nb_j]]$, for $j = 1,\dots,\nu$ and $\nu \geq 1$. Define the random vector $Y^{(n)} = (Y_1^{(n)},\dots,Y_\nu^{(n)})$, where

$$Y_l^{(n)} = \frac{1}{\sqrt{n}} \sum_{j \in \mathscr{I}_{nl}} h(N_l), \quad 1 \leq l \leq \nu\,, \tag{2.1}$$

where

$$h(x) = \left(\rho_{k,H}(0)\right)^{\frac{p}{2}} |x|^p - c_{k,p}, \tag{2.2}$$

and

$$N_l = \frac{\Delta_k B_{j-1}^H}{\sqrt{\rho_{k,H}(0)}}. \tag{2.3}$$

Then we have the following results.

$$d_{\mathrm{TV}}(Y^{(n)}, Z) \leq C n^{4H-4k+1} \vee n^{-1/2}, \tag{2.4}$$

where Z is multi-dimensional Gaussian with law $N(0,\Sigma)$, and Σ is a $\nu \times \nu$ matrix with components

$$\Sigma_{ij} = v_1^2 |b_i - a_i| \delta_{ij} := \sigma_i^2 \delta_{ij}. \tag{2.5}$$

In the following parts of this chapter, we will use the following notation.

$$\rho(j) = \mathbb{E}(N_1 N_{j+1}) \tag{2.6}$$

for the stationary sequence $(N_j)_{j \geq 0}$. Clearly,

$$\rho(j) = \frac{\rho_{k,H}(j)}{\rho_{k,H}(0)} \sim j^{2H-2k}.$$

Before we prove this proposition, we need the following two auxiliary lemmas.

Lemma 3.2.2. *Let the sequence* $\{Y_j^{(n)}, n \geq 1\}$ *be defined by* (2.1). *Then*

$$\mathbb{E}(Y_i^{(n)} Y_j^{(n)}) \to \delta_{ij} \sigma_i^2$$

as $n \to \infty$ *at the rate of* $n^{4H-4k+1} \vee n^{-1}$*, where* σ_i *is defined in* (2.5).

Proof. First we write

$$\mathbb{E}(Y_i^{(n)} Y_j^{(n)}) = \frac{1}{n} \sum_{l_1 \in \mathscr{I}_{ni}} \sum_{l_2 \in \mathscr{I}_{nj}} \mathbb{E}(h(N_{l_1}) h(N_{l_2})) .$$

For $i \neq j$, we use Gebebein's inequality,

$$\mathbb{E}(Y_i^{(n)} Y_j^{(n)}) \leq \frac{1}{n} \sum_{l_1 \in \mathscr{I}_{ni}} \sum_{l_2 \in \mathscr{I}_{nj}} \rho(l_2 - l_1)^2 = \mathscr{O}(n^{4H-4k+1} \vee n^{-1}) .$$

For $i=j$,

$$\begin{aligned}\mathbb{E}((Y_i^{(n)})^2) &= \frac{1}{n}\left(\sum_{l\in\mathscr{I}_{ni}}\mathbb{E}(h(N_l)^2)+2\sum_{l_1<l_2\in\mathscr{I}_{ni}}\mathbb{E}(h(N_{l_1})h(N_{l_2}))\right)\\ &\to |b_i-a_i|\left(\mathbb{E}(h(N)^2)+2\sum_{l=1}^{\infty}\mathbb{E}(h(N_{1l})h(N_{2l}))\right)=|b_i-a_i|\nu_1^2\,,\end{aligned}$$

at the rate of $n^{-1}\vee n^{4H-4k+1}$, where N is standard Gaussian and (N_{1l},N_{2l}) is centered Gaussian with covariance matrix

$$\begin{pmatrix}1 & \rho(l)\\ \rho(l) & 1\end{pmatrix}.$$

□

Lemma 3.2.3. *Denote*

$$I_1:=\int_0^\infty\int_0^\infty e^{-t-s}h''(N_{l_2})h''(N_{l_4})P_th'(N_{l_1})P_sh'(N_{l_3})dsdt\,,$$

and

$$I_2:=\int_0^\infty\int_0^\infty e^{-2t-2s}h'(N_{l_2})h'(N_{l_4})P_th''(N_{l_1})P_sh''(N_{l_3})dsdt\,.$$

Then $I_1,I_2\in L^r(\Omega)$ for any $r>0$. Moreover,

$$|\mathbb{E}I_1|\le C(|\rho(l_1-l_2)|+|\rho(l_1-l_4)|+|\rho(l_1-l_3)|), \tag{2.7}$$

and

$$|\mathbb{E}I_2|\le C\sum_{i\neq j;i,j=1}^{4}|\rho(l_i-l_j)|. \tag{2.8}$$

Proof. Clearly $I_1,I_2\in L^r(\Omega)$ for any $r>0$, because h'',P_th' have finite moments. Since $P_th'(N_{l_1})$ is centered, using the identity $P_th'(N_{l_1})=-\delta DL^{-1}P_th'(N_{l_1})$ and applying duality, we obtain

$$\begin{aligned}\mathbb{E}I_1 &= -\int_{[0,\infty)^2} e^{-t-s}\mathbb{E}\langle DL^{-1}P_t h'(N_{l_1}), D(h''(N_{l_2})h''(N_{l_4})P_s h'(N_{l_3}))\rangle_{\mathfrak{H}} ds dt \\ &=: I_{11}+I_{12}+I_{13}\,,\end{aligned}$$

where

$$\begin{aligned}I_{11} &= -\int_{[0,\infty)^2} e^{-t-s}\mathbb{E}\left(h''(N_{l_4})P_s h'(N_{l_3})h'''(N_{l_2})\langle DL^{-1}P_t h'(N_{l_1}), DN_{l_2}\rangle_{\mathfrak{H}}\right) ds dt\,, \\ I_{12} &= -\int_{[0,\infty)^2} e^{-t-s}\mathbb{E}\left(h''(N_{l_2})P_s h'(N_{l_3})h'''(N_{l_4})\langle DL^{-1}P_t h'(N_{l_1}), DN_{l_4}\rangle_{\mathfrak{H}}\right) ds dt\,, \\ I_{13} &= -\int_{[0,\infty)^2} e^{-t-s}\mathbb{E}\left(h''(N_{l_2})h''(N_{l_4})\langle DL^{-1}P_t h'(N_{l_1}), DP_s h'(N_{l_3})\rangle_{\mathfrak{H}}\right) ds dt\,.\end{aligned}$$

For the term I_{11}, since $P_t h'(N_{l_1})$ is centered, we apply Lemma 2.2.1 and use the semi-group property of P_t to obtain

$$\begin{aligned}I_{11} &= \int_{[0,\infty)} e^{-s} ds \\ &\quad \times\mathbb{E}\left(h''(N_{l_4})P_s h'(N_{l_3})h'''(N_{l_2})\int_{[0,\infty)^2} e^{-\theta-t}\langle P_{\theta+t}Dh'(N_{l_1}), DN_{l_2}\rangle_{\mathfrak{H}} d\theta dt\right) \\ &= \rho(l_1-l_2)\int_{[0,\infty)^3} e^{-s-\theta-t}\mathbb{E}\left(h''(N_{l_4})P_s h'(N_{l_3})h'''(N_{l_2})P_{\theta+t}h''(N_{l_1})\right) ds d\theta dt\,.\end{aligned}$$

Using Cauchy-Swartz inequality and taking into account the fact that h', h'', h''' have finite moments, we have

$$|I_{11}| \le C|\rho(l_1-l_2)|\,.$$

Similary, we deduce

$$|I_{12}| \le C|\rho(l_1-l_4)|\,.$$

For the term I_{13}, we use the similar argument but also taking into account the relationship $DP_s = e^{-s}P_sD$. In this way,

$$I_{13} = \rho(l_1 - l_3)\int_{[0,\infty)^3} e^{-2s-t-\theta}\mathbb{E}\left(h''(N_{l_2})h''(N_{l_4})P_{t+\theta}h''(N_{l_1})P_sh''(N_{l_3})\right) dsdtd\theta\,.$$

Therefore,

$$|I_{13}| \le C|\rho(l_1 - l_3)|\,.$$

This finishes the proof of (2.7).

The proof of (2.8) is similar after centering the function h''. Namely, denote $M = \mathbb{E}(h''(N))$ where N is standard Gaussian and $\tilde{h}'' = h'' - M$. Then the term I_2 can be written as

$$\mathbb{E}I_2 = \sum_{i=1}^{3} I_{2i},$$

where

$$\begin{aligned}
I_{21} &:= \mathbb{E}\int_0^\infty\int_0^\infty e^{-2t-2s}h'(N_{l_2})h'(N_{l_4})P_t\tilde{h}''(N_{l_1})P_sh''(N_{l_3})dsdt\,,\\
I_{22} &:= M\mathbb{E}\int_0^\infty\int_0^\infty e^{-2t-2s}h'(N_{l_2})h'(N_{l_4})P_s\tilde{h}''(N_{l_3})dsdt\,,\\
I_{23} &:= M^2\int_0^\infty\int_0^\infty e^{-2t-2s}\mathbb{E}(h'(N_{l_2})h'(N_{l_4}))dsdt\,.
\end{aligned}$$

We use the similar arguments for the terms I_{21} and I_{22} as that for (2.7), and obtain

$$|I_{21}| \le C\sum_{i=2}^{4}|\rho(l_1 - l_i)|\,, \qquad |I_{22}| \le C(|\rho(l_3 - l_2)| + |\rho(l_3 - l_4)|)\,.$$

Finally, applying Gebelein's inequality to the term I_{23} yields

$$|I_{23}| \le C|\rho(l_4 - l_2)|\,.$$

□

Proof of Proposition 3.2.1. We first write

$$Y_l^{(n)} = LL^{-1}Y_l^{(n)} = \delta(-DL^{-1}Y_l^{(n)}) = \delta(\int_0^\infty e^{-t}P_tDY_l^{(n)}dt) =: \delta(u_l^{(n)}),$$

where

$$u_l^{(n)} = \frac{1}{\sqrt{n}} \sum_{j\in \mathscr{I}_{nl}} \int_0^\infty e^{-t}P_t h'(N_j)\varphi_j dt\,,$$

with $\varphi_j \in \mathfrak{H}$ satisfies $\|\varphi_j\|_{\mathfrak{H}}^2 = \mathbb{E}|N_j|^2$. To prove the thoerem, it suffices to prove (2.4), for which we use Theorem 6.1.1 in [28] by verifying the following result,

$$\sqrt{\mathbb{E}\left(\langle DY_i^{(n)}, u_j^{(n)}\rangle_{\mathfrak{H}} - \delta_{ij}\sigma_i^2\right)^2} = \mathscr{O}(n^{4H-4k+1} \vee n^{-1/2})$$

for $i, j = 1, \dots, \nu$. Considering

$$\begin{aligned} &\mathbb{E}\left(\langle DY_i^{(n)}, u_j^{(n)}\rangle_{\mathfrak{H}} - \delta_{ij}\sigma_i^2\right)^2 \\ \le\;& 2\mathrm{Var}(\langle DY_i^{(n)}, u_j^{(n)}\rangle_{\mathfrak{H}}) + 2\left(\mathbb{E}(Y_i^{(n)}Y_j^{(n)}) - \delta_{ij}\sigma_i^2\right)^2. \end{aligned} \tag{2.9}$$

and Lemma 3.2.2, we just need to study the first summand on the right-hand side of the the above inequality. Observe that

$$\mathrm{Var}(\langle DY_i^{(n)}, u_j^{(n)}\rangle_{\mathfrak{H}}) = \mathrm{Var}\left(\frac{1}{n}\sum_{l_1\in\mathscr{I}_{nj}}\sum_{l_2\in\mathscr{I}_{ni}} \rho(l_2-l_1)h'(N_{l_2})\int_0^\infty e^{-t}P_t h'(N_{l_1})dt\right). \tag{2.10}$$

By Poincare inequality,

$$\mathrm{Var}(\langle DY_i^{(n)}, u_j^{(n)}\rangle_{\mathfrak{H}}) \le \mathbb{E}\left\|\frac{1}{n}\sum_{l_1\in\mathscr{I}_{nj}}\sum_{l_2\in\mathscr{I}_{ni}} \rho(l_2-l_1)\int_0^\infty e^{-t}D(h'(N_{l_2})P_t h'(N_{l_1}))dt\right\|_{\mathfrak{H}}^2.$$

By Lemma 3.2.3 and the elementary inequality $\|X+Y\|^2 \leq 2\|X\|^2 + 2\|Y\|^2$,

$$\begin{aligned}\mathrm{Var}(\langle DY_i^{(n)}, u_j^{(n)}\rangle_{\mathfrak{H}}) \;\leq\; & \frac{2}{n^2}\sum_{l_1,l_3\in\mathscr{I}_{nj}}\sum_{l_2,l_4\in\mathscr{I}_{ni}}|\rho(l_2-l_1)\rho(l_4-l_3)\rho(l_2-l_4)| \\ & (|\rho(l_1-l_2)|+|\rho(l_1-l_3)|+|\rho(l_1-l_4)|) \\ & +|\rho(l_2-l_1)\rho(l_4-l_3)\rho(l_3-l_1)|\sum_{i\neq j,i,j=1}^{4}|\rho(l_i-l_j)|\,.\end{aligned}$$

We recall that the convolution for two sequences $\{u(l)\}_{l\in\mathbb{Z}}$ and $\{v(l)\}_{l\in\mathbb{Z}}$ is defined as $u*v(i-j)=\sum_{l\in\mathbb{Z}}u(i-l)v(j-l)$ whenever $u(-l)=u(l)$ and $v(-l)=v(l)$. Then we expand the interval $\{l_1,l_3\in\mathscr{I}_{ni}\}\cup\{l_2,l_4\in\mathscr{I}_{nj}\}$ to be $\{l_1,\ldots,l_4\in[0,[nb_\nu]]\}$. In this case, by setting

$$\tilde{\rho}_n(l)=|\rho(l)|\mathbb{1}_{|l|\leq[nb_\nu]},$$

and analyzing the summand in the above inequality one by one, and

$$\begin{aligned}\mathrm{Var}(\langle DY_i^{(n)}, u_j^{(n)}\rangle_{\mathfrak{H}}) \;\leq\; & \frac{8}{n}\sum_{l\in\mathbb{Z}}\tilde{\rho}_n^2*(\tilde{\rho}_n*\tilde{\rho}_n)(l)+\frac{4}{n}\sum_{l\in\mathbb{Z}}(\tilde{\rho}_n*\tilde{\rho}_n)^2(l) \\ & +\frac{6}{n}\sum_{l\in\mathbb{Z}}(((\tilde{\rho}_n*\tilde{\rho}_n)\tilde{\rho}_n)*\tilde{\rho}_n)(l) \\ \leq\; & \frac{8}{n}\|\tilde{\rho}_n^2*(\tilde{\rho}_n*\tilde{\rho}_n)\|_{\ell^1}+\frac{4}{n}\|\tilde{\rho}_n*\tilde{\rho}_n\|_{\ell^2}^2+\frac{6}{n}\|((\tilde{\rho}_n*\tilde{\rho}_n)\tilde{\rho}_n)*\tilde{\rho}_n\|_{\ell^1}\,.\end{aligned}$$

Applying Young's convolution inequality yields

$$\begin{aligned}&\|\tilde{\rho}_n^2*(\tilde{\rho}_n*\tilde{\rho}_n)\|_{\ell^1}\leq\|\tilde{\rho}_n^2\|_{\ell^p}\|\tilde{\rho}_n\|_{\ell^r}\|\tilde{\rho}_n\|_{\ell^r},\ \text{for } 1/p+1/r+1/r=3\,,\\ &\|\tilde{\rho}_n*\tilde{\rho}_n\|_{\ell^2}^2\leq\|\tilde{\rho}_n\|_{\ell^{\frac{4}{3}}}^4\,,\\ &\|((\tilde{\rho}_n*\tilde{\rho}_n)\tilde{\rho}_n)*\tilde{\rho}_n\|_{\ell^1}\leq\|\tilde{\rho}_n\|_{\ell^a}\|\tilde{\rho}_n\|_{\ell^b}\|\tilde{\rho}_n\|_{\ell^r}\|\tilde{\rho}_n\|_{\ell^q}\ \text{for } 1/a+1/b+1/r+1/q=3\,,\end{aligned}$$

where for the third inequality we also apply Hölder inequality to handle the norm of $(\tilde{\rho}_n*\tilde{\rho}_n)\tilde{\rho}_n$. Notice that $\|\tilde{\rho}_n^2\|_{\ell^p}\leq Cn^{(2(2H-2k)+1/p)_+}$ and $\|\tilde{\rho}_n\|_{\ell^r}\leq Cn^{(2H-2k+1/r)_+}$ for any $p,r>0$, because of the fact $\rho(l)\sim l^{2H-2k}$. Thus,

$$\mathrm{Var}(\langle DY_i^{(n)},u_j^{(n)}\rangle_{\mathfrak{H}})\leq Cn^{8H-8k+2}\vee n^{-1}\,.$$

Plugging this inequality into (2.9), we complete the proof.

□

Proposition 3.2.4. [Law of iterated logarithm for power variations of fBm] Let $p\in\{2\}\cup[3,\infty)$ and $k\geq 2$. Define the sequence W_n as

$$W_n=n^{pH-1}\sum_{j=1}^{[nt]-k+1}\left|\Delta_k B_{\frac{j-1}{n}}\right|^p-c_{k,p}t\,.$$

Then

$$\limsup_{n\to\infty}\frac{nW_n}{\sqrt{C_p n\log\log n}}=1\,,$$

$$\liminf_{n\to\infty}\frac{nW_n}{\sqrt{C_p n\log\log n}}=-1\,,$$

almost surely.

Proof. It sufficies to use Proposition 2.3.5 with the function $\delta(n)=\sqrt{n}$ to prove the law of iterated logarithm for the sequence

$$\widetilde{X}_n:=\sum_{j=1}^{n}\left|\Delta_k B_{j-1}\right|^p-c_{k,p}=\sum_{j=1}^{n}h(N_j)\,,$$

where N_j is given by (2.3) and the function $h(x)$ is defined by (2.2). Clearly $\widetilde{X}_n\in L^2(\Omega)$ has Wiener chaos expansion as

$$\widetilde{X}_n:=\sum_{j=1}^{n}\sum_{m\geq 2}c_m H_m(N_j),$$

where $c_m = m!\langle |\cdot|^p, H_m(\cdot)\rangle_{L^2(\mu)}$, with $d\mu = e^{-\frac{x^2}{2}}dx$. Moreover, the coefficients c_m satisfies $\sum\limits_{m\geq 2} \frac{c_m^2}{m!} < \infty$.

Firstly, let ν_1^2 be defined by (1.12). For all $n_1 < n_2 \in \mathbb{N}$,

$$\begin{aligned}
& \mathbb{E}\left(\frac{\widetilde{X}_{n_2} - \widetilde{X}_{n_1}}{\sqrt{(n_2-n_1)\nu_1^2}}\right)^2 - 1 \\
= & \frac{1}{(n_2-n_1)\nu_1^2} \sum_{j,l=n_1+1}^{n_2} \sum_{m,m'\geq 2} c_m c_{m'} \mathbb{E}\left(H_m(N_j)H_{m'}(N_l)\right) - 1 \\
= & \frac{1}{(n_2-n_1)\nu_1^2} \sum_{j,l=n_1+1}^{n_2} \sum_{m\geq 2} (\rho(l-j))^m \frac{c_m^2}{m!} - 1 \\
= & \frac{2}{\nu_1^2} \sum_{j>n_2-n_1} \sum_{m\geq 2} \rho(j)^m \frac{c_m^2}{m!}
\end{aligned}$$

Now we use the estimate $\rho(j) = o(j^{2H-2k})$ to get

$$\begin{aligned}
\left|\mathbb{E}\left(\frac{\widetilde{X}_{n_2} - \widetilde{X}_{n_1}}{\sqrt{(n_2-n_1)\nu_1^2}}\right)^2 - 1\right| & \leq \frac{2}{\nu_1^2} \sum_{m\geq 2} \frac{c_m^2}{m!} \sum_{j>n_2-n_1} \rho(j)^2 \\
& \leq C(n_2-n_1)^{2(2H-2k)+1} \leq \frac{C}{1+\log(n_2-n_1)} \qquad (2.11)
\end{aligned}$$

Secondly, we consider the random vector $Y_{\overline{n}} = (Y_1, \ldots, Y_d)$ where the component is given by

$$Y_i = \frac{\widetilde{X}_{n_{2i}} - \widetilde{X}_{n_{2i-1}}}{\sqrt{(n_{2i}-n_{2i-1})\nu_1^2}}, \quad i = 1, \ldots, d.$$

We write

$$Y_i = LL^{-1}Y_i = \delta(-DL^{-1}Y_i) = \delta(\int_0^\infty e^{-t} P_t DY_i dt) =: \delta(S_i),$$

where

$$S_i = \frac{1}{\sqrt{(n_{2i}-n_{2i-1})\nu_1^2}} \sum_{j\in\mathscr{I}_i} \int_0^\infty e^{-t} P_t h'(N_j)\phi_j dt\,,$$

with $\phi_j \in \mathfrak{H}$ satisfies $\|\phi_j\|_{\mathfrak{H}}^2 = \mathbb{E}|N_j|^2$, and the interval $\mathscr{I}_i = [n_{2i-1}+1, n_{2i}]$.

Clearly, the vector $Y_{\overline{n}}$ admits the stein matrix as $\tau_{i,j} = \mathbb{E}(\langle S_i, DY_j\rangle_{\mathfrak{H}} | Y_{\overline{n}})$. Next, we calculate $\mathrm{Var}(\tau_{i,i}(Y_{\overline{n}}))$ as

$$\begin{aligned}\mathrm{Var}(\tau_{i,i}(Y_{\overline{n}})) &\leq \mathrm{Var}(\langle S_i, DY_i\rangle_{\mathfrak{H}})\\ &= \frac{1}{|\mathscr{I}_i|^2\sigma^4}\mathrm{Var}\left(\sum_{j,l\in\mathscr{I}_i}\int_0^\infty e^{-t}P_t h'(N_j)h'(N_l)\rho(j-l)dt\right).\end{aligned}$$

We use the similar argument as the one for (2.10) to obtain

$$\mathrm{Var}(\tau_{i,i}(Y_{\overline{n}})) = \mathscr{O}((n_{2i}-n_{2i-1})^{4H-4k+1}) \leq \frac{C}{1+\log(n_{2i}-n_{2i-1})}.$$

Finally, denote the stein factor of Y_1 as

$$\tau(Y_1) := \tau_{11}(Y_1) = \langle S_1, DY_1\rangle_{\mathfrak{H}} = \frac{1}{(n_2-n_1)\sigma^2}\sum_{j,l\in\mathscr{I}_1}\int_0^\infty e^{-t}P_t h'(N_j)h'(N_l)\rho(j-l)dt\,.$$

We have for an abitrary even natural number r,

$$(\mathbb{E}|\tau(Y_1)-\mathbb{E}(\tau(Y_1))|^r)^{\frac{1}{r}} \leq (r-1)^{\frac{1}{2}}(\mathbb{E}\|D\tau(Y_1)\|_{\mathfrak{H}}^r)^{\frac{1}{r}}\,.$$

Note that

$$\|D\tau(Y_1)\|_{\mathfrak{H}}^2 \leq \frac{1}{(n_2-n_1)^2\sigma^4}\sum_{j_1,j_2,l_1,l_2\in\mathscr{I}_1}\rho(j_1-l_1)\rho(j_2-l_2)\rho(l_2-l_1)A(j_1,j_2,l_1,l_2)$$

where

$$A(j_1, j_2, l_1, l_2) := \int_{[0,\infty)^2} e^{-t-s} h''(N_{l_1}) h''(N_{l_2}) P_t h'(N_{j_1}) P_t h'(N_{j_2}) ds dt.$$

Therefore,

$$\begin{aligned}
&(\mathbb{E}|\tau(Y_1) - \mathbb{E}(\tau(Y_1))|^r)^{\frac{1}{r}} \\
&\leq \frac{(r-1)^{\frac{1}{2}}}{\sigma^2} \left(\sum_{j_1, j_2, l_1, l_2 \in \mathscr{I}_1} \rho(j_1 - l_1) \rho(j_2 - l_2) \rho(l_2 - l_1) \|A(j_1, j_2, l_1, l_2)\|_{L^{\frac{r}{2}}(\Omega)} \right)^{\frac{1}{2}} \\
&\leq \frac{C(r-1)^{\frac{1}{2}}}{\sigma^2} \left(\frac{1}{n_2 - n_1} \|\tilde{\rho}_n * \tilde{\rho}_n * \tilde{\rho}_n\|_{\ell^1} \right)^{\frac{1}{2}}.
\end{aligned}$$

Applying Young's convolution inequality yields

$$(\mathbb{E}|\tau(Y_1) - \mathbb{E}(\tau(Y_1))|^r)^{\frac{1}{r}} \leq \frac{C(r-1)^{\frac{1}{2}}}{\sigma^2} (n_2 - n_1)^{3H-3k+1}.$$

Taking into account the inequality (2.11) and $\mathbb{E}(\tau(Y_1)) = \mathbb{E}(Y_1^2)$, we have

$$(\mathbb{E}|\tau(Y_1) - 1|^r)^{\frac{1}{r}} \leq \frac{C(r-1)^{\frac{1}{2}}}{\sigma^2(1 + \log(n_2 - n_1))}.$$

□

As a by-product of the above Proposition 3.2.4 and the proof of Theorem 3.1.1, we obtain the convergence rate of the terms defined by (1.4) as follows.

$$\|C^{(n,m)}\|_\infty = \mathscr{O}((m/\log\log m)^{-\frac{1}{2}}), \quad \|B^{(n,m)}\|_\infty = \mathscr{O}(n^{-a(p\wedge 1)}),$$

$$\|D^{(n)}\|_\infty = \mathscr{O}(n^{-a(p\wedge 1)}), \quad \|A^{(m)}\|_\infty = \mathscr{O}(m^{-(p\wedge 1)(a-\varepsilon)}),$$

for any $\varepsilon \in (0, H+a-1)$. Correspondingly, we can get the rate of convergence for (1.3), which is stated as the following proposition.

Proposition 3.2.5. Let the stochastic process Z_t be defined by Theorem 3.1.1, i.e.

$$Z_t = \int_0^t u_s dB_s^H,\ t \in [0,T],$$

and the conditions of Theorem 3.1.1 hold and, in addition, $a(p \wedge 1) > \frac{1}{2}$, and $k \geq 2$. Then

$$n^{-1+pH} V_{k,p}^n(Z)_t - c_{k,p} \int_0^t |u_s|^p ds = \mathscr{O}((n/\log\log n)^{-\frac{1}{2}}), \quad \text{a.s..}$$

3.3 Joint convergence along different subsequences of power variations

For this topic, the paper [11] discussed the signed cubic variation of fBm. Here we consider a general power variation of fBm. Fix a natural number λ. Define $\widetilde{\Delta} X_{i-\lambda} = X_i - X_{i-\lambda}$ and $\widetilde{\Delta}_k X_{i-\lambda} = \widetilde{\Delta}_{k-1} X_i - \widetilde{\Delta}_{k-1} X_{i-\lambda}$. Clearly when $\lambda = 1$, $\widetilde{\Delta}_k = \Delta_k$. Moreover, we have

$$\widetilde{\Delta}_k X_i = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} X_{i+\lambda l},$$

and correspondingly we obtain for every $i, j \in \mathbb{R}$,

$$\Phi(i,j) := \mathbb{E}(\widetilde{\Delta}_k B_i \Delta_k B_j) = -\frac{1}{2} \sum_{l=0}^{k} \sum_{l'=0}^{k} (-1)^{l+l'} \binom{k}{l} \binom{k}{l'} \left| i + \lambda l - j - l' \right|^{2H}. \quad (3.1)$$

The following lemma is a consequence of the properties of fBm.

Lemma 3.3.1. *The function $\Phi(i,j)$ has the following properties.*

(1) $\forall c \in \mathbb{R}$, $\Phi(i,j) = \Phi(i+c, j+c)$.

(2) $\Phi(i,j) = \Phi(j+k, i+\lambda k) = \Phi(j+k-\lambda k, i)$.

Lemma 3.3.2. *The function* $\Phi(i,j)$ *admits the following integral expression.*

$$\begin{aligned}\Phi(i,j) &= \alpha_k(H)\int_i^{i+\lambda}\int_{t_k}^{t_k+\lambda}\cdots\int_{t_2}^{t_2+\lambda} \\ &\qquad \int_{j+k-1}^{j+k}\int_{s_k-1}^{s_k}\cdots\int_{s_2-1}^{s_2}(t_1-s_1)^{2H-2k}ds_1\cdots ds_k dt_1\cdots dt_k\,,\end{aligned}$$

for all $i,j\in\mathbb{R}$*, where* $\alpha_k(H) = \frac{1}{2}((2H-2k+2)\cdot\dots\cdot(2H))^{-1}$.

Proof. It is trivial to see that the statement is valid for $k=1$. Suppose it is true for $k-1$. Then

$$\mathbb{E}(\widetilde{\Delta}_k B_i \Delta_k B_j) = \mathbb{E}\left((\widetilde{\Delta}_{k-1}B_{i+\lambda} - \widetilde{\Delta}_{k-1}B_i)(\Delta_{k-1}B_{j+1} - \Delta_{k-1}B_j)\right). \tag{3.2}$$

Note that

$$\begin{aligned}&\mathbb{E}(\widetilde{\Delta}_{k-1}B_{i+\lambda}\Delta_{k-1}B_{j+1}) \\ =\;& \alpha_{k-1}(H)\int_{i+\lambda}^{i+2\lambda}\int_{t_k}^{t_k+\lambda}\cdots\int_{t_3}^{t_3+\lambda} \\ &\qquad \int_{j+k-1}^{j+k}\int_{s_k-1}^{s_k}\cdots\int_{s_3-1}^{s_3}(t_2-s_2)^{2H-2k+2}ds_2\cdots ds_k dt_2\cdots dt_k \\ =\;& \alpha_{k-1}(H)\int_{i}^{i+\lambda}\int_{t_k}^{t_k+\lambda}\cdots\int_{t_3}^{t_3+\lambda} \\ &\qquad \int_{j+k-1}^{j+k}\int_{s_k-1}^{s_k}\cdots\int_{s_3-1}^{s_3}(t_2+\lambda-s_2)^{2H-2k+2}ds_2\cdots ds_k dt_2\cdots dt_k\,.\end{aligned}$$

We denote the region

$$\begin{aligned}&\mathscr{D} = \{(t_2,\cdots,t_k,s_2,\cdots,s_k : s_3-1<s_2<s_3,\cdots,s_k-1<s_{k-1}<s_k, \\ &j+k-1<s_k<j+k, t_3<t_2<t_3+\lambda,\cdots,t_k<t_{k-1}<t_k+\lambda, i<t_k<i+\lambda\}\,.\end{aligned}$$

Then similarly we have

$$\mathbb{E}(\widetilde{\Delta}_{k-1}B_{i+\lambda}\Delta_{k-1}B_j) = \int_{\mathscr{D}} (t_2+\lambda-s_2+1)^{2H-2k+2} ds_2\cdots ds_k dt_2\cdots dt_k\,,$$

$$\mathbb{E}(\widetilde{\Delta}_{k-1}B_i\Delta_{k-1}B_{j+1}) = \int_{\mathscr{D}} (t_2-s_2)^{2H-2k+2} ds_2\cdots ds_k dt_2\cdots dt_k\,,$$

$$\mathbb{E}(\widetilde{\Delta}_{k-1}B_i\Delta_{k-1}B_j) = \int_{\mathscr{D}} (t_2-s_2+1)^{2H-2k+2} ds_2\cdots ds_k dt_2\cdots dt_k\,.$$

Taking into account that

$$\begin{aligned}
&((2H-2k+2)(2H-2k+1))^{-1}\int_{t_2}^{t_2+\lambda}\int_{s_2-1}^{s_2}(t_1-s_1)^{2H-2k}ds_1dt_1\\
&= (t_2+\lambda-s_2)^{2H-2k+2}-(t_2+\lambda-s_2+1)^{2H-2k+2}\\
&\qquad -(t_2-s_2)^{2H-2k+2}+(t_2-s_2+1)^{2H-2k+2}\,,
\end{aligned}$$

we finish the proof by plugging the above equations into (3.2). □

Lemma 3.3.3. *Fix $a\in\mathbb{R}$ and $2\le m\in\mathbb{N}$, the series $\sum_{l\in\mathbb{Z}}|\Phi(l,a)|^m$ is absolutely convergent. Moreover, we have the following estimate.*

$$\sum_{l\in\mathbb{Z}\cap[a-\lambda k,\ a+k]^c}|\Phi(l,a)|^m \le C_\theta \lambda^{m(2H-k-\theta)}\,,$$

for any $\theta<-\frac{1}{2}$.

Proof. We consider the case $l>a+k$ first.

$$\begin{aligned}
\Phi(l,a) &= \alpha_k(H)\int_l^{l+\lambda}\int_{t_k}^{t_k+\lambda}\cdots\int_{t_2}^{t_2+\lambda}\\
&\qquad \int_{a+k-1}^{a+k}\int_{s_k-1}^{s_k}\cdots\int_{s_2-1}^{s_2}(t_1-s_1)^{2H-2k}ds_1\cdots ds_k dt_1\cdots dt_k\\
&= \alpha_k(H)\int_l^{l+\lambda}\int_{a+k-1}^{a+k}\int_{[0,\lambda]^{k-1}}\int_{[0,1]^{k-1}} ds_1'\cdots ds_{k-1}' ds_k dt_1'\cdots dt_{k-1}' dt_k
\end{aligned}$$

$$(t_k - s_k + t'_1 + \cdots + t'_{k-1} + s'_1 + \cdots + s'_{k-1})^{2H-2k},$$

where we have used the change of variables, $t_1 - t_2 \to t'_1, \cdots, t_{k-1} - t_k \to t'_{k-1}$ and $s_2 - s_1 \to s'_1, \cdots, s_k - s_{k-1} \to s'_{k-1}$. Note that for negative numbers $\theta, a_1, a_2, \cdots, a_{k-1}$ satisfying $\theta + a_1 \cdots + a_{k-1} = 2H - 2k$, we have the following inequality

$$\begin{aligned} & (t_k - s_k + t'_1 + \cdots + t'_{k-1} + s'_1 + \cdots + s'_{k-1})^{2H-2k} \\ \leq \; & (t_k - s_k)^{\theta} (t'_1)^{a_1} \cdots (t'_{k-1})^{a_{k-1}} \leq (l - a - k)^{\theta} (t'_1)^{a_1} \cdots (t'_{k-1})^{a_{k-1}}. \end{aligned}$$

Therefore,

$$|\Phi(l,a)| \leq (l - a - k)^{\theta} \lambda^{2H-k-\theta}.$$

Similarly, when $l < a - \lambda k$, we have

$$|\Phi(l,a)| \leq (a - l - \lambda k)^{\theta} \lambda^{2H-k-\theta}.$$

□

Denote two intervals $\mathscr{I}_1 = [1, [nt] - k + 1]$ and $\mathscr{I}_2 = [1, [\lambda nt] - k + 1]$. Next we are interested in the convergence of the random vector

$$(Y^1, Y^2) := \Big(\frac{1}{\sqrt{n}} \sum_{i \in \mathscr{I}_1} h(\widetilde{N}^1_{k,i}),\ \frac{1}{\sqrt{\lambda n}} \sum_{i \in \mathscr{I}_2} h(\widetilde{N}^2_{k,i})\Big) \tag{3.3}$$

where the function h is given by (2.2), and

$$\widetilde{N}^1_{k,i} = n^H \frac{\Delta_k B_{\frac{i-1}{n}}}{\sqrt{\rho_{k,H}(0)}}, \quad \widetilde{N}^2_{k,j} = (\lambda n)^H \frac{\Delta_k B_{\frac{j-1}{\lambda n}}}{\sqrt{\rho_{k,H}(0)}},$$

are two standard Gaussian random variables for each $i, j \in \mathbb{Z}$. Then

$$\mathbb{E}(\widetilde{N}^1_{k,i}\widetilde{N}^2_{k,j}) = \rho_{k,H}(0)^{-1}\lambda^{-H}\mathbb{E}(\widetilde{\Delta}_k B_{\lambda(i-1)}\Delta_k B_{j-1}) = \rho_{k,H}(0)^{-1}\lambda^{-H}\Phi(\lambda i - \lambda, j-1)\,. \tag{3.4}$$

Theorem 3.3.4. *Let $k \geq 2$. Let the random vector (Y^1, Y^2) defined by* (3.3)*. Then*

$$(Y^1, Y^2) \to VW_t$$

in law as $n \to \infty$ in the space $\mathscr{D}([0,T])^2$ equipped with the Skorohod topology, where

$$V = v_1 \begin{bmatrix} \sqrt{1 - |v_1^{-2}\rho|^2} & v_1^{-2}\rho \\ 0 & 1 \end{bmatrix},$$

$$\rho = \sum_{m\geq 2} \frac{c_m^2}{m!}\lambda^{-\frac{1}{2}-mH}\rho_{k,H}(0)^{-m}\sum_{l\in\mathbb{Z}}\Phi(l-(k-1)(\lambda-1),0)^m\,,$$

and v_1 is defined by (1.12)*, and $W = \{W_t, t \in [0,T]\}$ is a standard two-dimensional Brownian motion, independent of the fractional Brownian motion B.*

Remark 3.3.5. By Lemma 3.3.3,

$$|\rho| \leq C\sum_{m\geq 2}\frac{c_m^2}{m!}\lambda^{-\frac{1}{2}+mH-mk-m\theta}\,,$$

where $\theta < -\frac{1}{2}$. A sufficient condition for ρ being finite is that $H < k + \theta$ or $H = \frac{1}{2}$.

Proof. Recall that the function $h(N)$ given by (2.2) with N being standard Gaussian has the Wiener Chaos expansion as

$$h(N) = \sum_{m\geq 2} c_m H_m(N)\,.$$

Due to the results of convergence and tightness proved in Theorem 1.13, we need to check the random vector (Y^1, Y^2) converges to the centered Gaussian random vector with covariance $\begin{bmatrix} v_1^2 & \rho \\ \rho & v_1^2 \end{bmatrix}$. Applying Proposition 3.2.1 and its proof, we just need to consider the limit of $\mathbb{E}(Y^1Y^2)$. Using Wiener chaos expansion, we first compute the following quantity.

$$\mathbb{E}(J_mY^1J_mY^2) = \frac{c_m^2}{\sqrt{\lambda}n} \sum_{i\in\mathscr{I}_1} \sum_{j\in\mathscr{I}_2} \mathbb{E}(H_m(\widetilde{N}_{k,i}^1)H_m(\widetilde{N}_{k,j}^2)).$$

By setting $l = j - i\lambda$, we obtain

$$\begin{aligned}
\mathbb{E}(J_mY^1J_mY^2) &= \frac{c_m^2}{\sqrt{\lambda}nm!} \sum_{i\in\mathscr{I}_1} \sum_{j\in\mathscr{I}_2} \mathbb{E}(\widetilde{N}_{k,i}^1\widetilde{N}_{k,j}^2)^m \\
&= \frac{c_m^2}{\lambda^{\frac{1}{2}+mH} nm!\rho_{k,H}(0)^m} \sum_{i=1}^{[nt]-k+1} \sum_{l=1-\lambda i}^{[\lambda nt]-k+1-i\lambda} \Phi(\lambda i - \lambda, l + i\lambda - 1)^m,
\end{aligned}$$

where in the second equality we have used (3.4). Now using Lemma 3.3.1 yields

$$\begin{aligned}
&\mathbb{E}(J_mY^1J_mY^2) \\
= &\frac{c_m^2}{\lambda^{\frac{1}{2}+mH} nm!\rho_{k,H}(0)^m} \sum_{i=1}^{[nt]-k+1} \sum_{l=1-\lambda i}^{[\lambda nt]-k+1-i\lambda} \Phi(l - \lambda k + \lambda + k - 1, 0)^m \\
= &\frac{c_m^2}{\lambda^{\frac{1}{2}+mH} m!\rho_{k,H}(0)^m} \frac{1}{n} \sum_{i=(1-l)\lambda^{-1}}^{([\lambda nt]-k+1-l)\lambda^{-1}} \sum_{l=1-\lambda([nt]-k+1)}^{[\lambda nt]-k+1-\lambda} \Phi(l - \lambda k + \lambda + k - 1, 0)^m.
\end{aligned}$$

From Lemma 3.3.3, we know that the series $\sum_{l\in\mathbb{Z}} \Phi(l,0)^m$ is absolutely convergent. Thus we can compute

$$\lim_{n\to\infty} \mathbb{E}(J_mY^1J_mY^2) = \frac{c_m^2 t}{\lambda^{\frac{1}{2}+mH} m!\rho_{k,H}(0)^m} \sum_{l\in\mathbb{Z}} \Phi(l - \lambda k + \lambda + k - 1, 0)^m.$$

The computation is complete by noting that $\rho = \sum_{m\geq 2} \lim_{n\to\infty} \mathbb{E}(J_m Y^1 J_m Y^2)$. □

3.4 Estimation of the integrated volatility

We consider the stochastic process X_t that satisfies

$$dX_t = f(t, X_t)dt + \sigma_t dB_t^H, \tag{4.1}$$

with initial condition $X_0 \in \mathbb{R}$, where $B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian (fBm) motion of Hurst parameter $H \in (0,1)$, the volatility σ_t is a stochastic process with β-Hölder continuous trajectories, where $\beta > 1 - H$. Under this condition on σ_t, the stochastic integral $\int_0^t \sigma_s dB_s^H$ is well defined as a pathwise Riemann-Stieljes integral (see, for instance, [41]). The drift function $f(t, X_t)$ is Lipschitz continuous and satisifes a boundness conditon, i.e.,

$$|f(t,x) - f(t,y)| \leq C|x-y|, \ \forall x,y \in \mathbb{R}, \forall t \in [0,T]$$

and

$$f(t,x) \leq C|x| + f_0(t), \forall x \in \mathbb{R}, \forall t \in [0,T].$$

Under these assumptions, the above stochastic differential equation has a unique solution (see [32]). As a consequence of Proposition 3.2.5, Theorem 3.1.3, Corollary 3.1.4, and Proposition 3.2.5, we have the following convergence rate for the power variations of the solution X_t to the SDE (4.1).

Theorem 3.4.1. *Let X_t satisfy* (0.1)*, where the sample path of σ_t is Hölder continuous of exponent $a > \max(1-H, \frac{1}{2(p\wedge 1)})$. Assume $k \geq 2$ and $p \in \{2\} \cup [3, \infty)$. We have the following results.*

1. *We have the convergence rate for the power variation of X_t.*

$$n^{-1+pH}V^n_{k,p}(X)_t - c_{k,p}\int_0^t |u_s|^p ds = \mathscr{O}((n/\log\log n)^{-\frac{1}{2}}), \quad \text{a.s..}$$

2. *The following central limit theorem holds true.*

$$\sqrt{n}\left(n^{-1+pH}V^n_{k,p}(X)_t - c_{k,p}\int_0^t |\sigma_s|^p ds\right) \xrightarrow{\mathscr{L}} v_1\int_0^t |\sigma_s|^p dW_s\,, \quad as\ n\to\infty\,,$$

in law in $\mathscr{D}([0,T])$ equipped with the Skorohod topology, where v_1 is defined by (1.12) *and $W = \{W_t, t\in[0,T]\}$ is a Brownian motion, independent of the fractional Brownian motion B.*

Assume we observe one trajectory of X. Now we are interested to estimate the integrated volatility $\int_0^t |\sigma_s|^p ds$. Motivated by Theorem 3.1.3, we construct the k^{th} order power variation estimator $PV_{k,p}(X)_t$ for the integrated volatility $\int_0^t |\sigma_s|^p ds$ as follows

$$PV_{k,p}(X)_t = \frac{n^{-1+pH}V^n_{k,p}(X)_t}{c_{k,p}}, \quad t\in[0,T]\,, \tag{4.2}$$

where the k^{th} order power variation $V^n_{k,p}(X)_t$ is given by (1.2), and the normalizing constant $c_{k,p}$ is given by (1.1). For this estimator we have the following asymptotic consistency and the central limit theorem.

Theorem 3.4.2. *Let X_t satisfy* (4.1)*, where the sample path of σ_t is Hölder continuous of exponent $a > \max(1-H, \frac{1}{2(p\wedge 1)})$. Assume $k\geq 2$ and $p>\frac{1}{2}$. Then the estimator $PV_{k,p}(X)_t$ defined by* (4.2) *converges in probability to $\int_0^t |\sigma_s|^p ds$ uniformly on any compact interval $[0,T]$. Furthermore, the following central limit theorem holds true.*

$$\sqrt{n}\Big(PV_{k,p}(X)_t - \int_0^t |\sigma_s|^p ds\Big) \xrightarrow{\mathscr{L}} \frac{v_1}{c_{k,p}}\int_0^t |\sigma_s|^p dW_s\,, \quad as\ n\to\infty\,,$$

in law in $\mathscr{D}([0,T])$ equipped with the Skorohod topology, where v_1 is defined by (1.12) *and $W = \{W_t, t \in [0,T]\}$ is a Brownian motion, independent of the fractional Brownian motion B^H.*

Proof. By assumption, the stochastic process σ_t has Hölder continuous trajectories of order $a > 1-H$. Then the stochastic process X_t has Hölder continuous trajectories as well. Write $X_t = X_0 + Y_t + \int_0^t \sigma_s dB_s^H$, where $Y_t = -\theta \int_0^t X_s ds$. It is easy to check that $n^{-1/2+pH} V_{k,p}^n(Y)_t \to 0$ almost surely on $[0,T]$. The theorem follows from Theorem 3.1.1, Theorem 3.1.3, and Corollary 3.1.4. □

When $\sigma_t = \sigma$ is time independent, Theorem 3.4.2 gives the following result.

Proposition 3.4.3. Let $k \geq 2$ and $p > \frac{1}{2}$. Then the estimator $PV_{k,p}(X)_t$ converges almost surely to $|\sigma|^p t$ uniformly on any compact interval $[0,T]$. Furthermore,

$$\sqrt{n}(PV_{k,p}(X)_t - |\sigma|^p t) \xrightarrow{\mathscr{L}} \frac{v_1 |\sigma|^p}{c_{k,p}} W_t$$

as $n \to \infty$ in law in $\mathscr{D}([0,T])$ equipped with the Skorohod topology, where v_1 is given by (1.12) and W_t is a Brownian motion independent of the fractional Brownian motion B^H.

This proposition gives another estimator for $|\sigma|$:

$$|\hat{\sigma}_T|^p = \frac{n^{-1+pH} V_{k,p}^n(X)_T}{c_{k,p} T}. \tag{4.3}$$

It is easy to see that Theorem 3.4.2 and Proposition 3.4.3 yield the following result.

Proposition 3.4.4. When $H \in (0, \frac{3}{4})$, set $k \geq 1$. When $H \in [\frac{3}{4}, 1)$, set $k \geq 2$. Assume $p > \frac{1}{2}$. Then, the estimator $|\hat{\sigma}_T|^p$ defined by (4.3) converges almost surely to $|\sigma|^p$.

Furthermore, $\sqrt{n}(|\hat{\sigma}_T|^p - |\sigma|^p) \xrightarrow{\mathscr{L}} N(0, \nu^2)$ as $n \to \infty$, where the asymptotic variance ν^2 is given by

$$\nu^2 = \frac{\Gamma(\frac{1}{2})^2}{2^p \Gamma(\frac{p+1}{2})^2} \sum_{m=2}^{\infty} m! \mathbb{E}^2(H_m(N)|N|^p) \left[1 + 2\sum_{j=1}^{\infty} \left(\frac{\rho_{k,H}(j)}{\rho_{k,H}(0)}\right)^m\right] \frac{|\sigma|^{2p}}{T}. \tag{4.4}$$

Here N is a standard Gaussian random variable.

Usually the variance in (4.4) is complicated to compute. When $p = 2$, we compute the normalized asymptotic variance of $\nu^2 \frac{T}{|\sigma|^{2p}}$ for some H and k in the following Table.

Table 1: Normalized Asymptotic variance $\nu^2 \frac{T}{|\sigma|^{2p}}$ (when $p = 2$)

	k				
H	1	2	3	4	5
0.1	2.7283	3.7127	4.4814	5.1354	5.7147
0.3	2.2504	3.3539	4.1909	4.8855	5.4924
0.5	2.0000	3.0000	3.8889	4.6200	5.2531
0.6	2.1639	2.8308	3.7364	4.4830	5.1282
0.7	3.6088	2.6704	3.5846	4.3443	5.0005
0.8	-	2.5215	3.4348	4.2047	4.8707
0.9	-	2.3872	3.2884	4.0651	4.7393

We see that when H is small (for example when $H \leq 0.6$), it is more efficient to use the first order power variation than the higher order ones. However, when H is large (for example when $H \geq \frac{3}{4}$), the central limit theorem of the first order power variation does not hold, but we always have the central limit theorem for the second order power variation. As long as the central limit theorem of the power variation holds, it is preferable to use the lowest order.

3.5 Estimation for Hurst parameter

In this section, we consider the estimation for Hurst parameter in the SDE (4.1).

Let $\lambda > 1$ be the scaling parameter. Introduce the statistics S_n as

$$S_{\lambda,n,t} := \frac{V_{k,p}^{\lambda n}(X)_t}{V_{k,p}^{n}(X)_t} = \frac{\sum_{i=1}^{[\lambda nt]-k+1} \left|\Delta_k X_{\frac{i-1}{\lambda n}}\right|^p}{\sum_{i=1}^{[nt]-k+1} \left|\Delta_k X_{\frac{i-1}{n}}\right|^p} .$$

Then we propose the estimator for the Hurst parameter H as follows

$$\hat{H}_{\lambda,n,t} = \frac{1}{p}\left(1 - \frac{\log S_{\lambda,n,t}}{\log \lambda}\right) . \tag{5.1}$$

Theorem 3.5.1. *Let $\hat{H}_{\lambda,n,t}$ be defined by* (5.1)*. Then $\hat{H}_{\lambda,n,t} \to H$ almost surely as $n \to \infty$, for all $\lambda > 1$, for any $t \in [0,T]$. Moreover, $|\hat{H}_{\lambda,n,t} - H| = \mathscr{O}((n/\log\log n)^{-\frac{1}{2}})$, and $\sqrt{n}\left(\hat{H}_{\lambda,n,t} - H\right)$ converges in law to the normal distribution with mean* 0 *and variance*

$$\frac{2(v_1^2 - \rho)}{p^2(\log\lambda)^2\left(\int_0^t |\sigma_s|^p ds\right)^2} \int_0^t |\sigma_s|^{2p} ds .$$

Proof. Denote

$$\alpha_n = n^{-1+pH} V_{k,p}^{n}(X)_t, \quad \beta_n = (\lambda n)^{-1+pH} V_{k,p}^{\lambda n}(X)_t, \quad \gamma = c_{k,p}\int_0^t |\sigma_s|^p ds.$$

Since $\alpha_n \to \gamma$ and $\beta_n \to \gamma$ almost surely, $\frac{\beta_n}{\alpha_n} \to 1$ as $n \to \infty$ by Theorem 3.4.1. Note that

$$\frac{\beta_n}{\alpha_n} = \lambda^{-1+pH} S_{\lambda,n,t} .$$

Therefore,

$$\hat{H}_{\lambda,n,t} = \frac{1}{p}\left(pH - \frac{1}{\log\lambda}\log\frac{\beta_n}{\alpha_n}\right) \to H ,$$

almost surely as $n \to \infty$. Note that

$$|\log \beta_n - \log \gamma| = \frac{1}{\gamma^*}|\beta_n - \lambda|,$$

for some γ^* between β_n and γ, so $\log \beta_n - \log \gamma = \mathscr{O}((n/\log\log n)^{-\frac{1}{2}})$. This is valid for $\log \alpha_n - \log \gamma$ as well. Taking into account

$$|\hat{H}_{\lambda,n,t} - H| = \frac{1}{p \log \lambda}|(\log \beta_n - \log \gamma) - (\log \alpha_n - \log \gamma)|,$$

we have $|\hat{H}_{\lambda,n,t} - H| = \mathscr{O}((n/\log\log n)^{-\frac{1}{2}})$. We use $a \sim b$ to denote that a and b have the same asymptotic distribution.

$$\begin{aligned} \sqrt{n} p \log \lambda \left(\hat{H}_{\lambda,n,t} - H\right) &\sim \sqrt{n}(\log \alpha_n - \log \gamma) - \sqrt{n}(\log \beta_n - \log \gamma) \\ &\sim \frac{\sqrt{n}}{\alpha_n}(\alpha_n - \gamma) - \frac{\sqrt{n}}{\beta_n}(\beta_n - \gamma). \end{aligned}$$

From Theorem 3.3.4, we see that the random vector $\sqrt{n}(\alpha_n - \gamma, \beta_n - \gamma)$ converges in law to the centered normal distribution with covariance

$$\int_0^t |\sigma_s|^{2p} ds \begin{bmatrix} v_1^2 & \rho \\ \rho & v_1^2 \end{bmatrix}.$$

Therefore, $\sqrt{n} p \log \lambda \left(\hat{H}_{\lambda,n,t} - H\right)$ converges in law to the normal distribution

$$N(0, \frac{1}{\gamma^2}(2v_1^2 - 2\rho) \int_0^t |\sigma_s|^{2p} ds).$$

□

Chapter 4

Drift parameter estimation for linear stochastic differential equations

In this chapter we consider the fractional Ornstein-Uhlenbeck process defined as the unique pathwise solution to the stochastic differential equation

$$dX_t = -\theta X_t dt + \sigma dB_t^H, \tag{0.1}$$

with initial condition $X_0 \in \mathbb{R}$, where $B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian (fBm) motion of Hurst parameter $H \in (0,1)$, θ is a positive parameter and the volatility $\sigma > 0$ is a constant. The above stochastic differential equation has a unique solution.

Assume that the parameters $\theta > 0$ is unknown and that the process can be observed continuously or at discrete time instants. We want to estimate the drift parameter θ for any $H \in (0,1)$. We assume that the Hurst parameter H and the volatility σ are known and we want to estimate the drift parameter θ. There have been two popular types of estimators for this drift parameter. One is the maximum likelihood estimator and the other one is the least square estimator. In the Brownian motion case, they coincide, but for the fractional Ornstein-Uhlenbeck processes they are different (see [20] and [24]).

A summary of some relevant results are presented below.

(i) In the case of continuous observations, Kleptsyna and Le Breton ([24]) studied the maximum likelihood estimator (MLE) which is defined by

$$\hat{\theta}_{MLE} = -\left\{ \int_0^T Q^2(s) dw_s^H \right\}^{-1} \int_0^T Q(s) dZ_s \,,$$

where

$$Q(t) = \frac{d}{dw_t^H} \int_0^t k_H(t,s) X_s ds, \quad Z_t = \int_0^t k_H(t,s) dX_s,$$

$k_H(t,s) = \kappa_H^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H}$ and $w_t^H = \lambda_H^{-1} t^{2-2H}$ with constants κ_H and λ_H depending on H. They proved the almost sure convergence of $\hat{\theta}_{MLE}$ to θ as T tends to infinity. It is worth noting that Tudor and Viens ([38]) have also obtained the almost sure convergence of both the MLE and a version of the MLE using discrete observations for all $H \in (0,1)$. Bercu, Courtin and Savy proved in [6] the following central limit theorem for the MLE in the case of $H > \frac{1}{2}$:

$$\sqrt{T} (\hat{\theta}_{MLE} - \theta) \xrightarrow[T \to \infty]{\mathscr{L}} N(0, 2\theta) \,.$$

They claimed without proof that the above convergence is also valid for $H \in (0, \frac{1}{2})$.

(ii) Hu and Nualart ([20]) proposed the least square estimator defined by

$$\hat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \theta - \sigma \frac{\int_0^T X_t dB_t^H}{\int_0^T X_t^2 dt} \,, \tag{0.2}$$

where the integral with respect to B^H is interpreted in the Skorohod sense. They also introduced another estimator $\widetilde{\theta}_T$ based on the ergodic theorem given by

$$\widetilde{\theta}_T = \Big(\frac{1}{\sigma^2 H \Gamma(2H) T} \int_0^T X_t^2 dt \Big)^{-\frac{1}{2H}} \,. \tag{0.3}$$

Almost sure convergence and central limit theorems for these two estimators have been proved for $H \in [\frac{1}{2}, \frac{3}{4})$.

However, when $H \in (0, \frac{1}{2}) \cup [\frac{3}{4}, 1)$, the central limit theorems for the least square estimator $\hat{\theta}_T$ have not been known yet. The first objective of this chapter is to prove the asymptotic consistency of $\hat{\theta}_T$ by using a new method, different from that in [20], which is valid for all $H \in (0,1)$. This method involves the relationship between the divergence and Stratonovich integrals and the integration by parts technique and it is based on the pathwise properties of the fractional Ornstein-Uhlenbeck process established in a paper [12] by Cheridito, Kawaguchi and Maejima. The next and the main objective of this chapter is to establish a central limit theorem for the least square estimator $\hat{\theta}_T$ for $H \in (0, \frac{1}{2})$ and a noncentral limit theorem for $H \in [\frac{3}{4}, 1)$. In the later case, we can identify the limit as a Rosenblatt random variable. We will make a comparison of the asymptotic variance for these three estimators and show that the least square estimator performs better than the maximum likelihood estimator when $H \in (0, \frac{1}{2})$. Since the ergodic-type estimator $\widetilde{\theta}_T$ is a function of a pathwise Riemann integral that appears simpler than the other two estimators, we will use $\widetilde{\theta}_T$ to construct a consistent estimator $\bar{\theta}_n$ for high frequency data (if only discrete observations are available). The asymptotic behavior of $\bar{\theta}_n$ in this case is also studied in this paper. The proofs of our results are highly technical and rely on some sophisticated computation, which we shall put in the last section of this chapter.

4.1 Lease squares estimator

We shall focus on the least square estimator as introduced in [20]:

$$\hat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \theta - \sigma \frac{\int_0^T X_t dB_t^H}{\int_0^T X_t^2 dt}\,, \tag{1.1}$$

where dB_t^H denotes the divergence integral. In the paper [20], the almost sure convergence of $\hat{\theta}_T$ to θ is proved for $H \geq \frac{1}{2}$ and the central limit theorem is obtained for $H \in [\frac{1}{2}, \frac{3}{4})$. In this paper, we shall extend these results for a general Hurst parameter $H \in (0,1)$. In addition, we shall also consider a simulation friendly estimator: ergodic type estimator.

To simplify notation, we assume $X_0 = 0$. In this case the solution to (0.1) is given by

$$X_t = \sigma \int_0^t e^{-\theta(t-s)} dB_s^H \,. \tag{1.2}$$

Theorem 4.1.1. *For $H \in (0,1)$, $\hat{\theta}_T \to \theta$ a.s. as $T \to \infty$.*

Proof. Using integration by parts, we can write

$$X_t = \sigma \int_0^t e^{-\theta(t-s)} dB_s^H = \sigma \Big(B_t^H - \theta \int_0^t B_s^H e^{-\theta(t-s)} ds \Big) \,. \tag{1.3}$$

Since X_t is in the first Wiener chaos, we have the relationship between the divergence integral and the Stratonovich integral as

$$\int_0^T X_t dB_t^H = \int_0^T X_t \circ dB_t^H - \ell(T) \,, \tag{1.4}$$

where $\ell(T) = \mathbb{E} \int_0^T X_t \circ dB_t^H$. Using (1.3), $\ell(T)$ can be computed as follows

$$\begin{aligned} \ell(T) &= \sigma \mathbb{E} \int_0^T (B_t^H - \theta \int_0^t B_s^H e^{-\theta(t-s)} ds) \circ dB_t^H \\ &= \sigma \Big[\frac{1}{2} T^{2H} - \theta \int_0^T \int_0^t e^{-\theta(t-s)} \frac{\partial \mathbb{E}(B_s^H B_t^H)}{\partial t} ds dt \Big] \\ &= \frac{\sigma}{2} T^{2H} - m(T) \,, \end{aligned} \tag{1.5}$$

where

$$m(T) := H \theta \sigma \int_0^T \int_0^t e^{-\theta(t-s)} (t^{2H-1} - (t-s)^{2H-1}) ds dt.$$

Making the substitutions $t-s \to u$, $s \to v$ and then integrating first in the variable v yield

$$m(T)=\frac{\sigma}{2}\gamma_{\theta T}^{1}T^{2H}+\sigma\theta^{-2H}\gamma_{\theta T}^{2H+1}(H-\frac{1}{2})-TH\sigma\theta^{1-2H}\gamma_{\theta T}^{2H}\,. \tag{1.6}$$

In the above equation, we use the notation $\gamma_T^{\alpha}=\int_0^T e^{-x}x^{\alpha-1}dx$. Observe that γ_T^{α} converges to $\Gamma(\alpha)$ exponentially fast as $T\to\infty$. Then clearly we have

$$\lim_{T\to\infty}T^{-1}\ell(T)=\lim_{T\to\infty}T^{-1}\big(\frac{\sigma}{2}T^{2H}-m(T)\big)=H\sigma\theta^{1-2H}\Gamma(2H)\,. \tag{1.7}$$

On the other hand, we have

$$\sigma\int_0^T X_t\circ dB_t^H=\int_0^T X_t\circ(dX_t+\theta X_t dt)=\frac{X_T^2}{2}+\theta\int_0^T X_t^2 dt\,. \tag{1.8}$$

Combining (1.4) and (1.8) we obtain

$$\sigma\int_0^T X_t dB_t^H=\frac{X_T^2}{2}+\theta\int_0^T X_t^2 dt-\sigma\ell(T)\,. \tag{1.9}$$

From Lemma 4.5.6, we see $\lim\limits_{T\to\infty}\dfrac{X_T^2}{T}=0$. Therefore, by Lemma 4.5.7, (1.7), and (1.9), we have

$$\lim_{T\to\infty}T^{-1}\sigma\int_0^T X_t dB_t^H=0.$$

As a consequence,

$$\lim_{T\to\infty}\hat{\theta}_T=\lim_{T\to\infty}\left(\theta-\frac{\sigma\int_0^T X_t dB_t^H}{\int_0^T X_t^2 dt}\right)=\theta\,.$$

□

The next theorem shows the asymptotic laws for the least square estimator $\hat{\theta}_T$.

Theorem 4.1.2. *As $T \to \infty$, the following convergence results hold true.*

(i) For $H \in (0, \frac{3}{4})$, $\sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{\mathscr{L}} N(0, \theta\sigma_H^2)$, where

$$\sigma_H^2 = \begin{cases} (4H-1) + \frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1-2H)} & \text{when } H \in (0, \frac{1}{2}) \,, \\ \\ (4H-1)\left[1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)\Gamma(2H)}\right] & \text{when } H \in [\frac{1}{2}, \frac{3}{4}) \,. \end{cases}$$

(ii) For $H = \frac{3}{4}$, $\dfrac{\sqrt{T}}{\sqrt{\log(T)}}(\hat{\theta}_T - \theta) \xrightarrow{\mathscr{L}} N(0, 4\pi^{-1}\theta)$.

(iii) For $H \in (\frac{3}{4}, 1)$, $T^{2-2H}(\hat{\theta}_T - \theta) \xrightarrow{\mathscr{L}} \dfrac{-\theta^{2H-1}}{H\Gamma(2H)} R_1$, where $R_1 = I_2(\delta_{0,1})$ is the Rosenblatt random variable and $\delta_{0,1}$ is the Dirac-type distribution defined in (3.1).

Remark 4.1.3. It is interesting to note that when $H \in (0, \frac{1}{2})$, by the fact $\lim_{z\to 0} z\Gamma(z) = 1$, we have

$$\lim_{H \to \frac{1}{2}^-} \sigma_H^2 = 2$$

which is consistent with $\sigma_H^2 = 2$ if $H = \frac{1}{2}$. Moreover, we also see that $\lim_{H\to 0} \sigma_H^2 = 0$.

Proof. The case $H \in [\frac{1}{2}, \frac{3}{4})$ was proved in [20]. We shall use Malliavin calculus to prove the theorem for $H \in (0, \frac{1}{2}) \cup [\frac{3}{4}, 1)$.

Step 1: We use Theorem 2.3.1 to prove the central limit theorem when $H \in (0, \frac{1}{2})$. By (1.1) and (1.2), we can write our target quantity as

$$\sqrt{T}(\hat{\theta}_T - \theta) = -\frac{\frac{\sigma^2}{\sqrt{T}} \int_0^T (\int_0^t e^{-\theta(t-s)} dB_s^H) dB_t^H}{\int_0^T X_t^2 dt / T} = \frac{-\frac{\sigma^2}{2\sqrt{T}} F_T}{\int_0^T X_t^2 dt / T} \,, \tag{1.10}$$

where

$$F_T = \int_0^T \int_0^T e^{-\theta|t-s|} dB_s^H dB_t^H . \tag{1.11}$$

We introduce the function

$$f(s,t) = \frac{1}{\sqrt{T}} e^{-\theta|s-t|} \mathbf{1}_{[0,T]^2} . \tag{1.12}$$

Then $\frac{1}{\sqrt{T}} F_T = I_2(f)$ is in the second Wiener chaos. Our main objective is to use Theorem 2.3.1 to obtain the central limit theorem for the term $\frac{1}{\sqrt{T}} F_T$ and then we apply Lemma 4.5.7 and Slutsky's theorem for (1.10) to obtain the central limit theorem of $\hat{\theta}_T$. First of all, let us check the variance assumption in Theorem 2.3.1. By the isometry between the Hilbert space $\mathfrak{H}^{\otimes 2}$ and the second chaos $\mathscr{H}_2$, we have

$$\mathbb{E}\left(\frac{1}{T} F_T^2\right) = \frac{2}{T} \langle e^{-\theta|s_1-t_1|}, e^{-\theta|s_2-t_2|} \rangle_{\mathfrak{H}\otimes\mathfrak{H}}.$$

To compute the above norm, we shall use the definition of the tensor product space where the norm in the Hilbert space $\mathfrak{H}$ is defined by (1.3), namely,

$$\mathbb{E}\left(\frac{1}{T} F_T^2\right) = \frac{2}{T} \int_{[0,T]^4} \frac{\partial e^{-\theta|s_1-t_1|}}{\partial t_1} \frac{\partial e^{-\theta|s_2-t_2|}}{\partial s_2} \frac{\partial R_H(s_1,s_2)}{\partial s_1} \frac{\partial R_H(t_1,t_2)}{\partial t_2} ds_1 ds_2 dt_1 dt_2. \tag{1.13}$$

By Equation (5.25) in Lemma 4.5.5, we have

$$\lim_{T\to\infty} \mathbb{E}\left(\frac{1}{T} F_T^2\right) = 4H^2 \theta^{1-4H} \Gamma(2H)^2 \Big((4H-1) + \frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1-2H)}\Big). \tag{1.14}$$

Next, let us check the second condition in Theorem 2.3.1. Without loss of generality we can assume $\theta = 1$. The first contraction of the kernel f is

$$(f \otimes_1 f)(s,t) = \frac{1}{T} \langle e^{-|\cdot - s|} \mathbf{1}_{[0,T]}(\cdot), e^{-|\cdot - t|} \mathbf{1}_{[0,T]}(\cdot) \rangle_{\mathfrak{H}} . \tag{1.15}$$

We want to prove that the norm of the above function in the Hilbert space $\mathfrak{H}^{\otimes 2}$ goes to 0 as $T \to \infty$. Using the identity (1.4), we rewrite

$$\begin{aligned}
(f \otimes_1 f)(s,t) &= \frac{1}{T c_H^2} \int_{\mathbb{R}} \mathscr{F}(e^{-|\cdot - s|} \mathbf{1}_{[0,T]}(\cdot))(\xi) \overline{\mathscr{F}(e^{-|\cdot - t|} \mathbf{1}_{[0,T]}(\cdot))(\xi)} |\xi|^{1-2H} d\xi \\
&= \frac{4}{T c_H^2} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \frac{e^{-is\eta}}{1+\eta^2} \cdot \frac{1 - e^{-iT(\xi - \eta)}}{i(\xi - \eta)} d\eta \Big) \\
&\quad \times \Big(\int_{\mathbb{R}} \frac{e^{it\eta'}}{1+\eta'^2} \cdot \frac{1 - e^{iT(\xi - \eta')}}{-i(\xi - \eta')} d\eta' \Big) |\xi|^{1-2H} d\xi .
\end{aligned}$$

Observe that the function $f \otimes_1 f$ is the inverse Fourier transformation of the following function

$$h(s,t) = \frac{4}{T c_H^2} \int_{\mathbb{R}} \Big(\frac{1}{1+s^2} \cdot \frac{1 - e^{-iT(\xi + s)}}{i(\xi + s)} \Big) \Big(\frac{1}{1+t^2} \cdot \frac{1 - e^{iT(\xi - t)}}{-i(\xi - t)} \Big) |\xi|^{1-2H} d\xi .$$

By the Parseval's identity, the norm of the function $f \otimes_1 f$ in the space $\mathfrak{H}^{\otimes 2}$ can be computed as

$$\begin{aligned}
\|f \otimes_1 f\|_{\mathfrak{H}^{\otimes 2}}^2 &= \frac{1}{c_H^2} \int_{\mathbb{R}^2} |h(\eta, \eta')|^2 |\eta|^{1-2H} |\eta'|^{1-2H} d\eta d\eta' \\
&\leq \frac{C}{T^2} \int_{\mathbb{R}^2} \frac{|\eta|^{1-2H}}{1+\eta^4} \frac{|\eta'|^{1-2H}}{1+\eta'^4} \qquad (1.16) \\
&\quad \times \Big(\int_{\mathbb{R}} \frac{|e^{iT(\xi - \eta)} - 1|}{|\xi - \eta|} \frac{|e^{iT(\xi - \eta')} - 1|}{|\xi - \eta'|} |\xi|^{1-2H} d\xi \Big)^2 d\eta d\eta' .
\end{aligned}$$

(1.17)

Now our task is to show the right-hand side of the above inequality goes to 0 as $T \to \infty$. This can be achieved by studying the asymptotic behavior of the multiple integral in (1.17), which is denoted by I. Making a change of variable $\xi \to x+\eta$ yields

$$\begin{aligned}
I &= \int_{\mathbb{R}^2} \frac{|\eta|^{1-2H}}{1+\eta^4} \frac{|\eta'|^{1-2H}}{1+\eta'^4} \left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|} \frac{|e^{iT(x+\eta-\eta')}-1|}{|x+\eta-\eta'|} |x+\eta|^{1-2H} dx \right)^2 d\eta d\eta' \\
&\leq 2 \int_{\mathbb{R}^2} \frac{|\eta|^{1-2H}}{1+\eta^4} \frac{|\eta'|^{1-2H}}{1+\eta'^4} \left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|^{2H}} \frac{|e^{iT(x+\eta-\eta')}-1|}{|x+\eta-\eta'|} dx \right)^2 d\eta d\eta' \\
&\quad +2 \int_{\mathbb{R}^2} \frac{|\eta|^{3-6H}}{1+\eta^4} \frac{|\eta'|^{1-2H}}{1+\eta'^4} \left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|} \frac{|e^{iT(x+\eta-\eta')}-1|}{|x+\eta-\eta'|} dx \right)^2 d\eta d\eta' .
\end{aligned}$$

Making another change of variable $\eta' \to \eta - y$, we can write

$$\begin{aligned}
I &\leq 2 \int_{\mathbb{R}^2} \frac{|\eta|^{1-2H}}{1+\eta^4} \frac{|\eta-y|^{1-2H}}{1+(\eta-y)^4} \left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|^{2H}} \frac{|e^{iT(x+y)}-1|}{|x+y|} dx \right)^2 d\eta dy \\
&\quad +2 \int_{\mathbb{R}^2} \frac{|\eta|^{3-6H}}{1+\eta^4} \frac{|\eta-y|^{1-2H}}{1+(\eta-y)^4} \left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|} \frac{|e^{iT(x+y)}-1|}{|x+y|} dx \right)^2 d\eta dy \\
&=: 2(I_1+I_2).
\end{aligned}$$

For the term I_2, taking into account that

$$M := \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} \frac{|\eta|^{3-6H}}{1+\eta^4} \frac{|\eta-y|^{1-2H}}{1+(\eta-y)^4} d\eta < \infty,$$

we see

$$I_2 \leq M \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|}{|x|} \frac{|e^{iT(x+y)}-1|}{|x+y|} dx \right)^2 dy = M \|f*f\|^2_{L^2(\mathbb{R})},$$

where $f(x) = \frac{|e^{iTx}-1|}{|x|}$. By Young's inequality

$$I_2 \leq M\|f\|^4_{L^{4/3}(\mathbb{R})} = M\left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|^{4/3}}{|x|^{4/3}}dx\right)^3 = MT\left(\int_{\mathbb{R}} \frac{|e^{ix}-1|^{4/3}}{|x|^{4/3}}dx\right)^3 = CT.$$

Now we consider the term I_1. The measure $\mu(dy) = \int_{\mathbb{R}} \frac{|\eta|^{1-2H}}{1+\eta^4}\frac{|\eta-y|^{1-2H}}{1+(\eta-y)^4}d\eta$ is finite and has a bounded density with respect to the Lebesgue measure. Consider the function $g(x) = \frac{|e^{iTx}-1|}{|x|^{2H}}$. For any $p \geq 2$,

$$I_1 = \|f * g\|^2_{L^2(\mathbb{R},\mu)} \leq C_1\|f * g\|^2_{L^p(\mathbb{R},\mu)} \leq C_2\|f * g\|^2_{L^p(\mathbb{R})}.$$

Let p also satisfy $p > \frac{1}{2H}$ and for such p we can choose α and β such that $\alpha > 1$, $2H\beta > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1 + \frac{1}{p}$. Then by Young's inequality

$$\begin{aligned} I_1 &\leq C_2\|f\|^2_{L^\alpha(\mathbb{R})}\|g\|^2_{L^\beta(\mathbb{R})} \\ &= C_2\left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|^\alpha}{|x|^\alpha}dx\right)^{\frac{2}{\alpha}}\left(\int_{\mathbb{R}} \frac{|e^{iTx}-1|^\beta}{|x|^{2H\beta}}dx\right)^{\frac{2}{\beta}}. \end{aligned}$$

A change of variable $x \to y/T$ tells us that $I_1 \leq CT^{4H-\frac{2}{p}}$. From (1.17), we obtain

$$\|f \otimes_1 f\|^2_{\mathfrak{H}^{\otimes 2}} \leq CT^{(-1)\vee(4H-2-\frac{2}{p})}, \tag{1.18}$$

and this goes to 0 as T tends to infinity. By Theorem 2.3.1, as T goes to infinity, the term $\frac{1}{\sqrt{T}}F_T$ converges in distribution to a centered Gaussian random variable with variance given by (1.14). Applying Slutsky's theorem and Lemma 4.5.7 from Appendix to the equation (1.10), we finish the proof of the theorem for $H \in (0, \frac{1}{2})$.

Step 2: Case $H = \frac{3}{4}$. First note that Lemma 4.5.5 in the Appendix gives the limiting variance of $\frac{F_T}{\sqrt{T\log T}}$. To obtain the central limit theorem, we need to check one of the equivalent conditions in Theorem 2.3.1. This can be dealt with in a similar way as in the proof of Theorem 3.4 in [20] by verifying condition 5 of Theorem 2.3.1. However, it is worth noting that it also suffices to verify the equivalent condition $\|f\otimes_1 f\|^2_{\mathfrak{H}^{\otimes 2}} \to 0$, and the arguments used in the case of $H\in(0,\frac{1}{2})$ can be extended to the case $H\in(0,\frac{3}{4}]$.

Step 3: In this step we will prove the theorem when $H\in(\frac{3}{4},1)$. Recall that the term F_T is given by (1.11). By (1.1) and (1.2), we write

$$T^{2-2H}(\hat{\theta}_T-\theta) = \frac{-\frac{\sigma^2}{2}T^{1-2H}F_T}{\int_0^T X_t^2 dt/T}.$$

Denote

$$\widetilde{F}_T = T^{2H}\int_{[0,1]^2} e^{-\theta T|t-s|}dB_s^H dB_t^H\,. \tag{1.19}$$

By the self-similarity property of the fBm, the process $\{F_T, T>0\}$ has the same law as $\{\widetilde{F}_T, T>0\}$. To prove part (iii) of the theorem, we need to show $T^{1-2H}F_T \xrightarrow{\mathscr{L}} 2\theta^{-1}R_1$. It suffices to prove

$$\lim_{T\to\infty}\mathbb{E}(T^{1-2H}\widetilde{F}_T - 2\theta^{-1}R_1)^2 = 0\,. \tag{1.20}$$

By Equations (5.27) and (5.28), we see immediately that

$$\lim_{T\to\infty}\mathbb{E}\left(T^{2-4H}\widetilde{F}_T^2\right) = \lim_{T\to\infty}\mathbb{E}\left(T^{2-4H}F_T^2\right) = \frac{16\alpha_H^2\theta^{-2}}{(4H-2)(4H-3)}\,,$$

$$\lim_{T\to\infty}\mathbb{E}\left[2\theta^{-1}R_1(T^{1-2H}\widetilde{F}_T)\right] = \frac{16\alpha_H^2\theta^{-2}}{(4H-2)(4H-3)}\,,$$

where $\alpha_H = H(2H-1)$. On the other hand, we have

$$
\begin{aligned}
\mathbb{E}(2\theta^{-1}R_1)^2 &= 8\theta^{-2}\alpha_H^2 \int_{[0,1]^4} \delta_{0,1}(s-t)\delta_{0,1}(s'-t')|s-s'|^{2H-2}|t-t'|^{2H-2} ds dt ds' dt' \\
&= 8\theta^{-2}\alpha_H^2 \int_{[0,1]^2} |t-s|^{4H-4} ds dt = \frac{16\theta^{-2}\alpha_H^2}{(4H-3)(4H-2)} .
\end{aligned}
$$

This shows (1.20) and hence completes the proof of the theorem. □

As an immediate consequence of the proof of the central limit theorem for $\frac{1}{\sqrt{T}} F_T$ when $H \in (0, \frac{3}{4})$, we can derive the total variation distance between $\frac{1}{\sqrt{T}} F_T$ and its limiting distribution. The case $H = \frac{3}{4}$ is similar. This is summarized in the following proposition.

Proposition 4.1.4. Let F_T be given by (1.11) and let $\sigma_T^2 = \mathbb{E}((f_T F_T)^2)$ be its variance, with the normalizing factor $f_T = \frac{1}{\sqrt{T}} \mathbf{1}_{\{H \in (0, \frac{3}{4})\}} + \frac{1}{\sqrt{T \log(T)}} \mathbf{1}_{\{H = \frac{3}{4}\}}$. Let N denote a random variable with the standard normal distribution. Then

$$
d_{\mathrm{TV}}\left(\frac{f_T F_T}{\sigma_T}, N\right) \le \begin{cases} \frac{C}{\sqrt{T}} & \text{when } H \in (0, \frac{1}{2}) \\ \frac{C}{\sqrt{T^{3-4H}}} & \text{when } H \in [\frac{1}{2}, \frac{3}{4}) \\ \frac{C}{\sqrt{\log(T)}} & \text{when } H = \frac{3}{4} . \end{cases}
$$

Proof. It suffices to consider the case $H \in (0, \frac{3}{4})$. The case $H = \frac{3}{4}$ can be treated in a similar way. Recall that $\frac{1}{\sqrt{T}} F_T = I_2(f)$, where the kernel f is given by (1.12). Applying Proposition 2.3.3 yields

$$
\begin{aligned}
d_{\mathrm{TV}}\left(\frac{F_T}{\sqrt{T}\sigma_T}, N\right) &\le 2\sqrt{\mathrm{Var}\left(\frac{1}{2}\left\|\frac{1}{\sigma_T\sqrt{T}} DF_T\right\|_{\mathfrak{H}}^2\right)} \\
&= \frac{1}{\sigma_T^2}\|f \otimes_1 f\|_{\mathfrak{H}^{\otimes 2}} \le C\sqrt{T^{(-1)\vee(4H-2-\frac{2}{p})}} ,
\end{aligned}
$$

for any $p \geq 2$, where for the above identity we used Lemma 5.2.4 from [28], and for the last inequality we used the inequality (1.18). Clearly, when $H \in (0, \frac{1}{2})$, the bound is $C/\sqrt{T}$. When $H \in [\frac{1}{2}, \frac{3}{4})$, $p = 2$ is chosen to derive the bound. $\square$

Remark 4.1.5. We make some comments on the distance between the normalized F_T, $\hat{\theta}_T$, and their limiting distributions.

1. Recall that $\sqrt{T}(\hat{\theta}_T - \theta) = \frac{-\frac{\sigma^2}{2} F_T/\sqrt{T}}{\frac{1}{T}\int_0^T X_t^2 dt}$. We have obtained the asymptotic behavior for the numerator in the preceding Proposition 4.1.4. By Lemma 4.5.7, The denominator converges to a constant almost surely, and the convergence rate is of $\sqrt{T}$ (See [36]). It is challenging to study the total variation distance between $\sqrt{T}(\hat{\theta}_T - \theta)$ with its limiting normal distribution, since it involves the quotient of two dependent random variables. This is left as an open problem.

2. For $H \in (\frac{3}{4}, 1)$, we can get a convergence rate for (1.20) by examining the proof of (5.27) and (5.28) in Lemma 4.5.5. In this way we find that

$$\mathbb{E}\left(T^{1-2H}\tilde{F}_T - 2\theta^{-1}R_1\right)^2 = O(T^{3-4H}).$$

This implies that $T^{1-2H}F_T$ (which has the same law as $T^{1-2H}\tilde{F}_T$ defined by (1.19)) converges to the Rosenblatt random variable in law at the rate of $\sqrt{T^{4H-3}}$.

4.2 Ergodic type estimator

In this section, we shall use the results of the last section to consider a simulation friendly estimator: ergodic type estimator.

Theorem 4.2.1. *Define an ergodic-type estimator for the drift parameter by*

$$\widetilde{\theta}_T = \Big(\frac{1}{\sigma^2 H\Gamma(2H)T}\int_0^T X_t^2 dt\Big)^{-\frac{1}{2H}} . \tag{2.1}$$

Then $\widetilde{\theta}_T \to \theta$ almost surely as $T \to \infty$. Furthermore, we have the following central limit theorem ($H \le 3/4$) and noncentral limit theorem ($H > 3/4$).

(1) When $H \in (0, \frac{3}{4})$, we have $\sqrt{T}(\widetilde{\theta}_T - \theta) \xrightarrow{\mathscr{L}} N(0, \frac{\theta}{(2H)^2}\sigma_H^2)$ as $T \to \infty$, where σ_H^2 is defined in Theorem 4.1.2.

(2) When $H = \frac{3}{4}$, we have $\frac{\sqrt{T}}{\log(T)}(\widetilde{\theta}_T - \theta) \xrightarrow{\mathscr{L}} N(0, \frac{16\theta}{9\pi})$ as $T \to \infty$.

(3) When $H \in (\frac{3}{4}, 1)$, we have $T^{2-2H}(\widetilde{\theta}_T - \theta) \xrightarrow{\mathscr{L}} \frac{-\theta^{2H-1}}{H\Gamma(2H+1)}R_1$, where $R_1 = I_2(\delta_{0,1})$ is the Rosenblatt random variable, and $\delta_{0,1}$ is the Dirac-type function defined in (3.1).

Proof. The paper [20] provides a proof of the theorem when $H \in (\frac{1}{2}, \frac{3}{4})$. Here we present a proof valid for all $H \in (0,1)$. By Lemma 4.5.7, it is easy to see $\widetilde{\theta}_T \to \theta$ almost surely as $T \to \infty$.

We prove the central limit theorem when $H \in (0, \frac{3}{4})$. For $H \in [\frac{3}{4}, 1)$, the proof is similar. By (1.1) and (1.9), we can derive an expression for $\int_0^T X_t^2 dt$, and then express $\widetilde{\theta}_T$ as a function of $\hat{\theta}_T$. In this way, we obtain

$$\sqrt{T}(\widetilde{\theta}_T - \theta) = \sqrt{T}\left[\left(\frac{\sigma^2 H\Gamma(2H)\hat{\theta}_T}{-\frac{X_T^2}{2T} + \sigma T^{-1}\ell(T)}\right)^{\frac{1}{2H}} - \theta\right].$$

By Lemma 4.5.6 and (1.7) we have

$$\begin{aligned}\sqrt{T}(\widetilde{\theta}_T - \theta) &= \sqrt{T}\Big[\Big(\frac{1}{\theta^{1-2H} + o(T^{-1/2})}\Big)^{\frac{1}{2H}}\hat{\theta}_T^{\frac{1}{2H}} - \theta\Big] \\ &= \sqrt{T}\theta^{1-\frac{1}{2H}}(\hat{\theta}_T^{\frac{1}{2H}} - \theta^{\frac{1}{2H}}) + \sqrt{T}\, o(T^{-1/2})\hat{\theta}_T^{\frac{1}{2H}} .\end{aligned}$$

Meanwhile, we can write

$$\sqrt{T}\left[\hat{\theta}_T^{\frac{1}{2H}} - \theta^{\frac{1}{2H}}\right] = \sqrt{T}\left[\frac{1}{2H}\theta^{\frac{1}{2H}-1}(\hat{\theta}_T - \theta) + \frac{1-2H}{8H^2}(\hat{\theta}_T - \theta)^2(\theta_T^*)^{\frac{1}{2H}-2}\right]$$

for some θ_T^* between θ and $\hat{\theta}_T$. Now the theorem follows from Theorem 4.1.2. $\square$

Remark 4.2.2. By the property for gamma function: $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ for $z \notin \mathbb{Z}$, we see $\lim_{H\to 0} \frac{\theta}{(2H)^2}\sigma_H^2 = \frac{\pi^2}{2}\theta$.

Now we have obtained the asymptotic law of the least square estimator (LSE) $\hat{\theta}_T$ and the ergodic type estimator (ETE) $\widetilde{\theta}_T$. Next, we compare these two estimators with the maximum likelihood estimator by computing their asymptotic variance. For convenience, we assume $\theta = 1$. As it can be seen from Figure 1, the asymptotic variance of LSE increases as H increases. When $H \in (0, \frac{1}{2})$, the asymptotic variance of LSE is less than that of MLE, where the converse is true for $H \in (\frac{1}{2}, \frac{3}{4})$. The asymptotic variance of ETE decreases on $H \in (0, \frac{1}{2})$ and then increases on $H \in (\frac{1}{2}, \frac{3}{4})$; however, it does not blow up as fast as LSE does when H is close to $\frac{3}{4}$. If we justify these three estimators only based on asymptotic variance, LSE performs best when $H \in (0, \frac{1}{2})$ and MLE performs best when $H \in (\frac{1}{2}, \frac{3}{4})$. At $H = \frac{1}{2}$, these three estimators have the same asymptotic variance.

4.3 Discrete case

The estimators $\hat{\theta}_T$ and $\widetilde{\theta}_T$ are based on continuous time data. In practice the process can only be observed at discrete time instants. This motivates us to construct an estimator based on discrete observations. We assume that the fractional Ornstein-Uhlenbeck process X given by (1.2) can be observed at discrete time points $\{t_k = kh, k = 0, 1, \ldots, n\}$. We shall use nh instead of T for the time period of the observation. Here h represents

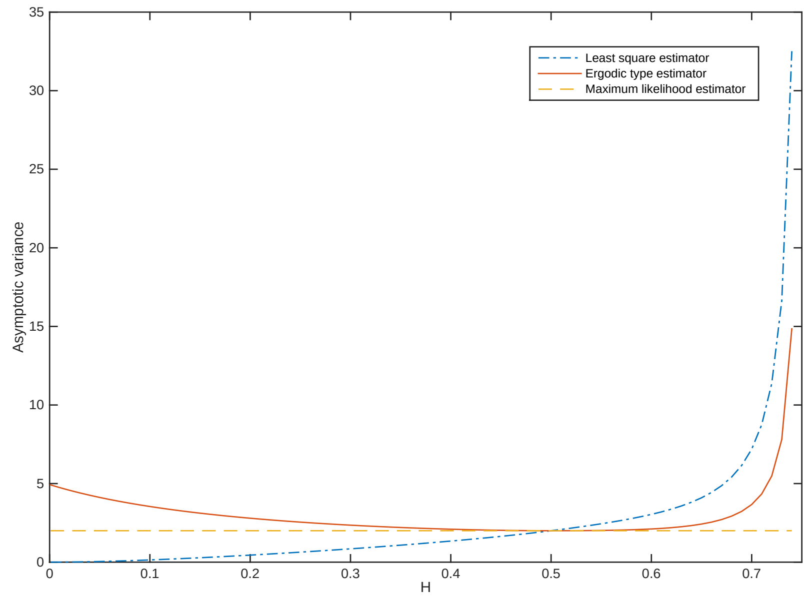

Figure 4.1: Asymptotic Variance of the Three Estimators

the observation frequency and it depends on n. We will only consider the high frequency observation case, namely, we shall assume that $h \to 0$ as $n \to \infty$. We shall use ergodic type estimator since it can be expressed as a pathwise Riemann integral with respect to time. The following Theorem shows its asymptotic consistency and some results on its asymptotic law.

Theorem 4.3.1. *Assume the fractional Ornstein-Uhlenbeck process X given by* (1.2) *is observed at discrete time points $\{t_k = kh, k = 0, 1, ..., n\}$. Suppose that h depends on n and as $n \to \infty$, h goes to 0 and nh converges to ∞. In addition, we make the following assumptions on h and n:*

(1) When $H \in (0, \frac{3}{4})$, $nh^p \to 0$ for some $p \in (1, \frac{3+2H}{1+2H} \wedge (1+2H))$ as $n \to \infty$.

(2) When $H = \frac{3}{4}$, $\frac{nh^p}{\log(nh)} \to 0$ for some $p \in (1, \frac{9}{5})$ as $n \to \infty$.

(3) When $H \in (\frac{3}{4}, 1)$, $nh^p \to 0$ for some $p \in (1, \frac{3-H}{2-H})$ as $n \to \infty$.

Set

$$\bar{\theta}_n = \left(\frac{1}{n\sigma^2 H\Gamma(2H)} \sum_{k=1}^{n} X_{kh}^2 \right)^{-\frac{1}{2H}}. \tag{3.1}$$

Then $\bar{\theta}_n$ converges to θ almost surely as $n \to \infty$. Moreover, as n tends to infinity, we have the following central and noncentral limit theorems.

(1) When $H \in (0, \frac{3}{4})$, $\sqrt{nh}(\bar{\theta}_n - \theta) \xrightarrow{\mathscr{L}} N(0, \frac{\theta}{(2H)^2}\sigma_H^2)$, where σ_H^2 is given in Theorem 4.1.2.

(2) When $H = \frac{3}{4}$, $\frac{\sqrt{nh}}{\log(nh)}(\bar{\theta}_n - \theta) \xrightarrow{\mathscr{L}} N(0, \frac{16\theta}{9\pi})$.

(3) When $H \in (\frac{3}{4}, 1)$, $(nh)^{2-2H}(\bar{\theta}_n - \theta) \xrightarrow{\mathscr{L}} \frac{-\theta^{2H-1}}{H\Gamma(2H+1)}R_1$, where $R_1 = I_2(\delta_{0,1})$ is the Rosenblatt random variable and $\delta_{0,1}$ is the Dirac-type function defined in (3.1).

Before we prove Theorem 4.3.1, we state and prove an auxillary result in the following lemma about the regularity of sample paths of the fractional Ornstein-Uhlenbeck process X.

Lemma 4.3.2. *Let X_t be given by* (1.2). *Then for every interval $[0,T]$ and any $0 < \varepsilon < H$,*

$$|X_t - X_s| \leq V_1 |t-s|^{H-\varepsilon} + V_2 |t-s| \quad \text{a.s.}, \tag{3.2}$$

where the random variables V_i are defined as follows: $V_1 = \sigma\eta_T$ where η_T is given by (1.2) *with $\alpha = H - \varepsilon$, $V_2 = 2\sigma\theta \sup_{u\in[0,T]} |B_u^H|$.*

Proof. Consider the process $Q_t = \sigma\theta \int_0^t B_v^H e^{-\theta(t-v)} dv$. Using (1.3), for any $s,t \in [0,T]$ and $s < t$, we have

$$|X_t - X_s| = \left| \sigma(B_t^H - B_s^H) - (Q_t - Q_s) \right| \leq \sigma \left| B_t^H - B_s^H \right| + |Q_t - Q_s|.$$

Note that

$$
\begin{aligned}
|Q_t - Q_s| &\leq \sigma\theta \left| \int_s^t B_v^H e^{-\theta(t-v)} dv \right| + \sigma\theta \left| \int_0^s B_v^H (e^{-\theta(t-v)} - e^{-\theta(s-v)}) dv \right| \\
&\leq \sigma\theta \sup_{v\in[s,t]} |B_v^H| \int_s^t e^{-\theta(t-v)} dv \\
&\quad + \sigma\theta \sup_{v\in[s,t]} |B_v^H| \left(1 - e^{-\theta(t-s)}\right) \int_0^s e^{-\theta(s-v)} dv \\
&\leq 2\sigma\theta \sup_{v\in[s,t]} |B_v^H| |t-s| .
\end{aligned}
$$

Using the above inequality for $|Q_t - Q_s|$ and Applying (1.2), with $\alpha = H - \varepsilon$, for $B_t^H - B_s^H$ yield

$$
|X_t - X_s| \leq \sigma\eta_T |t-s|^{H-\varepsilon} + 2\sigma\theta \sup_{u\in[s,t]} |B_u^H| |t-s| .
$$

□

Proof of Theorem 4.3.1: Let $T = nh$, $Z_n = \frac{1}{nh}\int_0^{nh} X_t^2 dt$, and $\psi_n = \frac{1}{n}\sum_{k=1}^n X_{kh}^2$. Consider the function

$$
f(x) = \sqrt{x}\mathbf{1}_{\{0<H<3/4\}} + \sqrt{x}/\log(x)\mathbf{1}_{\{H=3/4\}} + x^{2-2H}\mathbf{1}_{\{3/4<H<1\}} .
$$

Step 1: We claim that $f(nh)|Z_n - \psi_n| \to 0$ almost surely as $n \to \infty$. Applying Markov's inequality for $\delta > 0, q > 1$ yields

$$
P(f(nh)|Z_n - \psi_n| > \delta) \leq \delta^{-q} f(nh)^q \mathbb{E}|Z_n - \psi_n|^q . \tag{3.3}
$$

We apply Minkowski's inequality to obtain

$$
\mathbb{E}|Z_n - \psi_n|^q = (nh)^{-q}\mathbb{E}\left| \sum_{j=1}^n \int_{(j-1)h}^{jh} (X_t + X_{jh})(X_t - X_{jh}) dt \right|^q
$$

$$\leq \quad (nh)^{-q}\left(\sum_{j=1}^{n}\int_{(j-1)h}^{jh}\left(\mathbb{E}(|X_t+X_{jh}||X_t-X_{jh}|)^q\right)^{1/q}dt\right)^q .$$

Taking into account of Lemma 4.3.2, we have

$$\begin{aligned}
& \mathbb{E}\,|Z_n-\psi_n|^q \\
\leq \quad & (nh)^{-q}\left(\sum_{j=1}^{n}\int_{(j-1)h}^{jh}\|V_1(X_t+X_{jh})\|_{L^q}|t-jh|^{H-\varepsilon}+\|V_2(X_t+X_{jh})\|_{L^q}|t-jh|dt\right)^q ,
\end{aligned}$$

where the V_i's are defined in Lemma 4.3.2. By Hölder's inequality and the fact $\|X_t\|_{L^q} = (\mathbb{E}|X_t|^q)^{1/q} \leq M_q$ for all $t > 0$, $q > 1$, we can write

$$\|V_i(X_t+X_{jh})\|_{L^q} \leq 2M_{qr_i}\|V_i\|_{qs_i} ,$$

where $1/r_i + 1/s_i = 1$. Therefore,

$$\mathbb{E}\,|Z_n-\psi_n|^q \leq C\left(M_{qr_1}^q\|V_1\|_{qs_1}^q h^{q(H-\varepsilon)} + M_{qr_2}^q\|V_2\|_{qs_2}^q h^q\right) ,$$

where C denotes a generic constant.

By (1.2), $\|V_1\|_{qs_1}^q = CT^{q\varepsilon}$ for $\varepsilon \in (0,H)$. By the self-similarity property of fBm, $\|V_2\|_{qs_2}^q = CT^{qH}$. Using these observations, we obtain

$$\mathbb{E}\,|Z_n-\psi_n|^q \leq C\left((nh)^{q\varepsilon}h^{q(H-\varepsilon)} + (nh)^{qH}h^q\right) ,$$

and plugging this inequality to (3.3), we get

$$P(f(nh)\,|Z_n-\psi_n| > \delta) \leq C\delta^{-q}f(nh)^q\left((nh)^{q\varepsilon}h^{q(H-\varepsilon)} + (nh)^{qH}h^q\right) . \tag{3.4}$$

If the right-hand side of the above inequality is summable with respect to n, then $f(nh)\,|Z_n - \psi_n| \to 0$ almost surely by the Borel-Cantelli Lemma. We show this summability when $H \in (0,1/2)$ and the other cases are similar. The right-hand side of (3.4) can be written as

$$Cn^{-1-\lambda}\left((nh^{\beta_1})^{\gamma_1} + (nh^{\beta_2})^{\gamma_2}\right),$$

where

$$\beta_1 = \frac{q/2 + q\varepsilon + q(H-\varepsilon)}{1+\lambda+q\varepsilon+q/2}, \quad \beta_2 = \frac{3/2q + qH}{1+\lambda+q/2+qH},$$

and γ_i's are the denominator of β_i's. Note that the positive variables ε and λ can be arbitrarily small and q can be arbitrarily large. In this way, we have $\beta_1 \in (1, 1+2H)$ and $\beta_2 \in (1, \frac{3+2H}{1+2H})$. If $nh^p \to 0$ for some $p \in (1, \min(\frac{3+2H}{1+2H}, 1+2H))$, then $nh^{\beta_i} \to 0$ by carefully choosing these free variables.

Step 2: We prove the almost sure convergence of $\bar{\theta}_n$. Denote $\rho = \sigma^2 H\Gamma(2H)$. Recall that $\widetilde{\theta}_T$ is given in Theorem 4.2.1. By the mean value theorem, we can write

$$\bar{\theta}_n - \theta = \left(\frac{\psi_n - Z_n}{\rho} + \widetilde{\theta}_T^{-2H}\right)^{-\frac{1}{2H}} - \theta = \widetilde{\theta}_T - \theta + \int_0^1 g_n(\lambda)d\lambda, \tag{3.5}$$

where $g_n(\lambda) = -\frac{1}{2H}\frac{\psi_n - Z_n}{\rho}\left(\lambda\frac{\psi_n - Z_n}{\rho} + \widetilde{\theta}_T^{-2H}\right)^{-\frac{1}{2H}-1}$.

The result in Step 1 also implies $Z_n - \psi_n \to 0$ almost surely as $n \to \infty$, so $\lim_{n\to\infty} g_n(\lambda) = 0$ a.s. for all $\lambda \in [0,1]$. Meanwhile, for almost all ω, there exists $N := N(\omega) \in \mathbb{N}$ such that for $n > N$,

$$\left|\frac{\psi_n - Z_n}{\rho}\right| < \frac{1}{3}\theta^{-2H}, \quad \left|\widetilde{\theta}_T^{-2H} - \theta^{-2H}\right| < \frac{1}{3}\theta^{-2H}.$$

Then for $n > N$, $|g_n(\lambda)| \le C\theta$. By the dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^1 g_n(\lambda) d\lambda = 0 \quad \text{a.s.}.$$

Then it is clear that $\bar{\theta}_n$ converges to θ almost surely.

Step 3: We prove the asymptotic laws of $\bar{\theta}_n$. Equation (3.5) yields

$$f(nh)(\bar{\theta}_n - \theta) = f(T)(\widetilde{\theta}_T - \theta) + f(nh) \int_0^1 g_n(\lambda) d\lambda \,.$$

Using the result of Step 1 and the similar arguments in step 2, we obtain

$$\lim_{n\to\infty} \int_0^1 f(nh) g_n(\lambda) d\lambda = 0 \quad \text{a.s.}.$$

Then it is clear that $f(nh)(\bar{\theta}_n - \theta)$ converges in law to the same random variable as $f(T)(\widetilde{\theta}_T - \theta)$ when T tends to infinity. By Theorem 4.2.1, we finish the proof.

□

4.4 Monte Carlo simulations

we use the R package Yuima to do some Monte Carlo simulations. The Wood-Chan simulation method is used to generate fractional Gaussian noise, and the Euler-Maruyama scheme is used to produce sample observations of the stochastic differential equation (0.1) (we take $\sigma = 1$).

First we choose $\theta = 1$. For each H value, only one trajectory is generated and $\bar{\theta}_n$ is calculated along this trajectory. The values of $\bar{\theta}_n$ are plotted in Fig. 4.2 as T

increases. As it can be seen, $\bar{\theta}_n$ converges to the true value $\theta = 1$ as sufficient number of observations are obtained.

Next we choose $\theta = 0.5$. For each H value, we perform 5000 Monte Carlo simulations to generate 5000 trajectories. For each trajectory, the quantity $\sqrt{nh}(\bar{\theta}_n - \theta)$ is calculated, and the density plot of these 5000 estimators is obtained, which is displayed in Fig. 4.3. The graphs show that the density plot of the simulation results is close to the kernel of the limiting distribution of $\sqrt{nh}(\bar{\theta}_n - \theta)$ when $H = 0.25, 0.5, 0.6$. For $H > \frac{3}{4}$, the limiting distribution, known as Rosenblatt distribution, is not known to have a closed form. Readers who are intersted in the density plot of Rosenblatt random variable are referred to the paper [39] and the references therein.

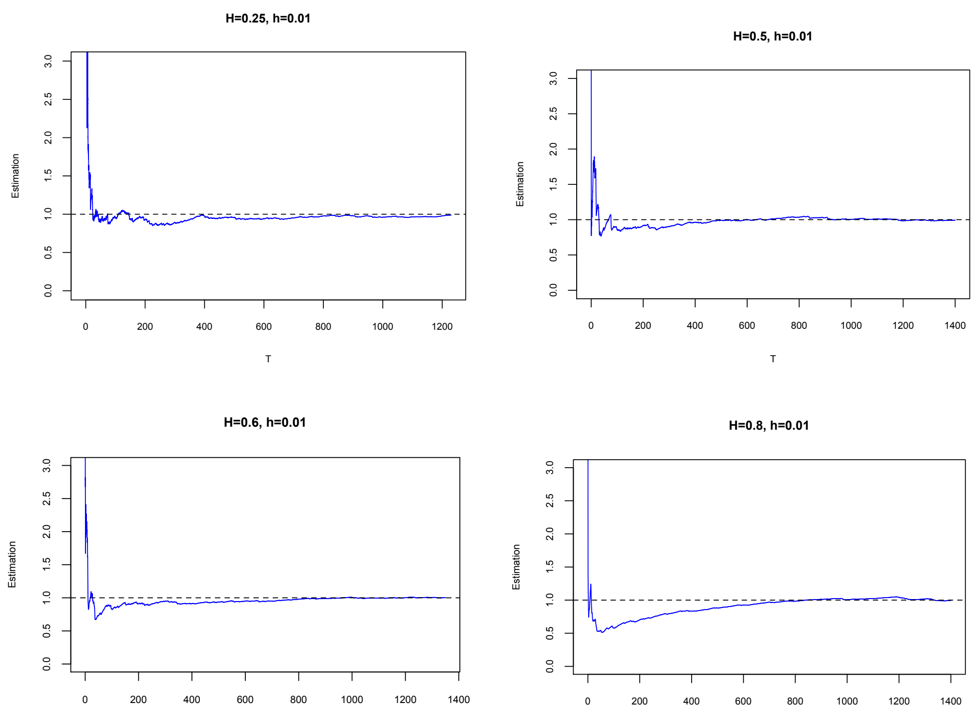

Figure 4.2: The one-trajectory simulation results of $\bar{\theta}_n$ for different H values, with $\theta = 1$ and $h = 0.01$.

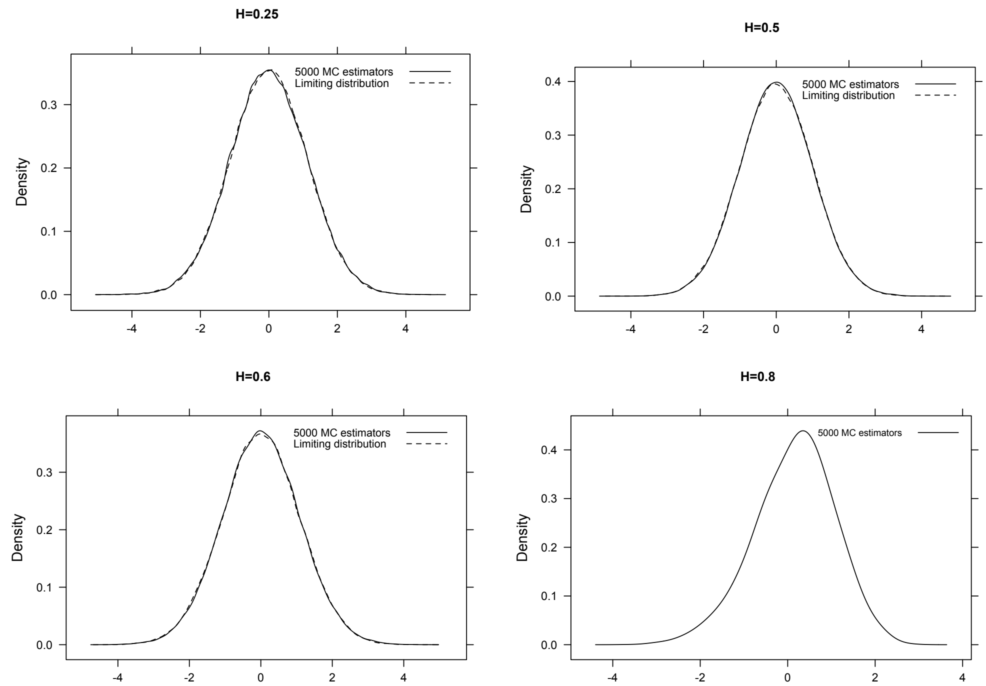

Figure 4.3: Density plots for 5000 simulation results of $\sqrt{nh}(\bar{\theta}_n - \theta)$ and its limiting distribution, with $\theta = 0.5$, $h = 0.01$, $n = 100,000$.

4.5 Some computations

This section contains some technical results needed in the proofs of the main theorems of the paper. First we need to identify the limits of some multiple integrals. Denote

$$\begin{aligned}
\psi(x,u) &:= \psi_T(x,u) = T^{4H+1} e^{-\theta T(u+x)}\,, &(5.1)\\
\varphi_1(x) &:= \int_x^1 [(t-x)^{2H-1} - 1][(1-t)^{2H-1} - (1-t+x)^{2H-1}]dt\,, &(5.2)\\
\varphi_2(x) &:= \int_x^1 [(t-x)^{2H-1} - t^{2H-1}](1-t+x)^{2H-1} dt\,, &(5.3)\\
\varphi_3(x,u) &:= \int_x^1 [\mathrm{sgn}(u-t)|u-t|^{2H-1} - \mathrm{sgn}(x+u-t)|x+u-t|^{2H-1}]dt\,, &(5.4)\\
\varphi_4(x,u) &:= \int_x^1 t^{2H-1}\mathrm{sgn}(x+u-t)|x+u-t|^{2H-1} \\
&\quad -(t-x)^{2H-1}\mathrm{sgn}(u-t)|u-t|^{2H-1} dt\,, &(5.5)
\end{aligned}$$

$$\begin{aligned}\varphi_5(x,u) &:= \int_x^1 \operatorname{sgn}(x+u-t)|x+u-t|^{2H-1}(1-t)^{2H-1} \\ &\quad -\operatorname{sgn}(u-t)|u-t|^{2H-1}(1-t+x)^{2H-1}dt\,.\end{aligned}$$

Fix an $\varepsilon \in (0,\frac{1}{4})$. Denote $[0,1]^2 = \mathscr{I}_1 \cup \mathscr{I}_2$ where $\mathscr{I}_1 = [0,\varepsilon]^2$ and $\mathscr{I}_2 = [0,1]^2 \backslash [0,\varepsilon]^2$.

Lemma 4.5.1. *Let $H \in (0,\frac{1}{2})$. When $(x,u) \in \mathscr{I}_1$, we have the following estimates.*

(i)

$$|\varphi_1(x)| \le x^{2H}\,, \tag{5.6}$$

(ii)

$$|\varphi_3(x,u)| \le C(x^{2H} + u^{2H} + |u-x|^{2H})\,, \tag{5.7}$$

(iii)

$$|\varphi_5(x,u)| \le C(x^{2H} + u^{2H} + |u-x|^{2H})\,, \tag{5.8}$$

where C is a constant independent of x,u.

Proof. First we prove (5.6). Observe that

$$0 \le \varphi_1(x) \le \int_x^1 f(x,t)dt, \tag{5.9}$$

where

$$f(x,t) = (t-x)^{2H-1}[(1-t)^{2H-1} - (1-t+x)^{2H-1}]\,.$$

It is clear that

$$f(x,t) \le \Big(\frac{1-x}{2}\Big)^{2H-1}[(1-t)^{2H-1} - (1-t+x)^{2H-1}]\,,$$

for $\frac{1+x}{2} \le t \le 1$. For $x \le t \le \frac{1+x}{2}$, applying the mean value theorem for the second factor of $f(x,t)$ yields

$$f(x,t) \le (1-2H)(t-x)^{2H-1}\big(\frac{1-x}{2}\big)^{2H-2}x\,.$$

Integrating the right-hand side of the above two inequalities with respect to t, we obtain

$$\begin{aligned}
&\int_x^1 f(x,t)dt \\
\le\ & \frac{1}{2H}\big(\frac{1-x}{2}\big)^{2H-1}\big[\big(\frac{1-x}{2}\big)^{2H}-\big(\frac{1+x}{2}\big)^{2H}+x^{2H}\big]+\frac{1-2H}{2H}\big(\frac{1-x}{2}\big)^{4H-2}x \\
\le\ & \frac{1}{2H}\big(\frac{1-\varepsilon}{2}\big)^{2H-1}x^{2H}+\frac{1-2H}{2H}\big(\frac{1-\varepsilon}{2}\big)^{4H-2}x^{2H}\ ,
\end{aligned}$$

where we have used the inequality $x < x^{2H}$ on $\mathscr{I}_1$ (i.e., $x \in (0,\varepsilon)$). Thus, (5.6) follows from the above inequality and (5.9).

Next we prove (5.7). Note that the antiderivative of the function $\text{sgn}(x)|x|^{2H-1}$ is $(2H)^{-1}|x|^{2H}$, so we can compute $\varphi_3(x,u)$ as follows

$$\varphi_3(x,u) = \frac{1}{2H}\big(|u-x|^{2H}-(1-u)^{2H}+(1-x-u)^{2H}-u^{2H}\big)\,. \tag{5.10}$$

Applying the inequality

$$\big|(1-x-u)^{2H}-(1-u)^{2H}\big| \le 2H(1-x-u)^{2H-1}x \le 2H(1-2\varepsilon)^{2H-1}x^{2H}\ ,$$

and the triangular inequality to (5.10) yields

$$\begin{aligned}
|\varphi_3(x,u)| &\le (2H)^{-1}(|u-x|^{2H}+u^{2H}+2H(1-2\varepsilon)^{2H-1}x^{2H}) \\
&\le C(|u-x|^{2H}+u^{2H}+x^{2H}) \quad \forall\, x,u \in \mathscr{I}_1\,.
\end{aligned}$$

Finally, we prove (5.8). Denote

$$\zeta_{x,u}(t)=\operatorname{sgn}(x+u-t)|x+u-t|^{2H-1}(1-t)^{2H-1}-\operatorname{sgn}(u-t)|u-t|^{2H-1}(1-t+x)^{2H-1}.$$

Let $\delta\in(\frac{1}{2},1)$. Since $\varepsilon\in(0,\frac{1}{4})$ and $(x,u)\in(0,\varepsilon)^2$, the interval $(x,1)$ can be decomposed into the following three intervals, where

$$J_1=(x,\,u+x),\quad J_2=(u+x,\,\delta),\quad J_3=(\delta,\,1)\,.$$

Then $\varphi_5(x,u)=\sum_{k=1}^{3}\int_{J_k}\zeta_{x,u}(t)dt$. We consider the above three integrals separately.

Case 1: When $t\in J_1$, we have

$$(1-t)^{2H-1}\le(1-u-x)^{2H-1}\le(1-2\varepsilon)^{2H-1}\,. \tag{5.11}$$

When t falls in different subintervals of J_1, we bound $(1-t+x)^{2H-1}$ in different ways. Namely, if $t\in(x,u)$ and $u\ge x$,

$$(1-t+x)^{2H-1}\le(1+x-u)^{2H-1}\le(1-\varepsilon)^{2H-1}\,. \tag{5.12}$$

If $t\in(x\vee u,x+u)$,

$$(1-t+x)^{2H-1}\le(1-u)^{2H-1}\le(1-\varepsilon)^{2H-1}\,. \tag{5.13}$$

Applying (5.11) for the first summand in $\zeta_{x,u}(t)$, (5.12) and (5.13) for the second summand, we can bound the integration of $\zeta_{x,u}(t)$ on J_1 as follows

$$\begin{aligned}\Big|\int_{J_1}\zeta_{x,u}(t)dt\Big| &\leq (1-2\varepsilon)^{2H-1}\int_x^{u+x}(x+u-t)^{2H-1}dt\\ &\quad +(1-\varepsilon)^{2H-1}\big(\int_x^u (u-t)^{2H-1}1_{\{u\geq x\}}dt+\int_{x\vee u}^{x+u}(t-u)^{2H-1}dt\big)\,.\end{aligned}$$

Integrating with respect to t yields

$$\Big|\int_{J_1}\zeta_{x,u}(t)dt\Big|\leq C(u^{2H}+(u-x)^{2H}1_{\{u\geq x\}}+x^{2H})\,.$$

Case 2: For $t\in J_2$, we rewrite

$$\begin{aligned}-\int_{J_2}\zeta_{x,u}(t)dt &= \int_{u+x}^{\delta}(1-t)^{2H-1}\big((t-u-x)^{2H-1}-(t-u)^{2H-1}\big)\\ &\quad +(t-u)^{2H-1}\big((1-t)^{2H-1}-(1-t+x)^{2H-1}\big)dt\,,\end{aligned}$$

which is nonnegative. In the above integrand, we bound $(1-t)^{2H-1}$ by $(1-\delta)^{2H-1}$ for the first summand. For the second summand, we apply the mean value theorem for the difference part and bound $(t-u)^{2H-1}$ by x^{2H-1}. Then integrating t yields

$$\begin{aligned}0\leq-\int_{J_2}\zeta_{x,u}(t)dt &\leq \frac{(1-\delta)^{2H-1}}{2H}\big((\delta-u-x)^{2H}-(\delta-u)^{2H}+x^{2H}\big)\\ &\quad +x^{2H}\big((1-\delta)^{2H-1}-(1-u-x)^{2H-1}\big)\\ &\leq \frac{(1-\delta)^{2H-1}}{2H}x^{2H}+(1-\delta)^{2H-1}x^{2H}\leq Cx^{2H}\,.\end{aligned}$$

Case 3: For $t\in J_3$, we rewrite

$$-\int_{J_3}\zeta_{x,u}(t)dt = \int_{\delta}^{1}(t-u-x)^{2H-1}((1-t)^{2H-1}-(1-t+x)^{2H-1})$$

$$+(1-t+x)^{2H-1}((t-u-x)^{2H-1}-(t-u)^{2H-1})dt\,,$$

which is nonnegative. In the above integrand, we bound $(t-u-x)^{2H-1}$ by $(\delta-2\varepsilon)^{2H-1}$ for the first summand. For the second summand, apply the mean value theorem for the difference part and bound $(1-t+x)^{2H-1}$ by x^{2H-1}. Then integrating t yields

$$\begin{aligned} 0\le -\int_{J_3}\zeta_{x,u}(t)dt &\le \frac{(\delta-2\varepsilon)^{2H-1}}{2H}\big((1-\delta)^{2H}-(1-\delta+x)^{2H}+x^{2H}\big)\\ &\quad + x^{2H}\big((\delta-u-x)^{2H-1}-(1-u-x)^{2H-1}\big)\\ &\le \frac{(\delta-2\varepsilon)^{2H-1}}{2H}x^{2H}+x^{2H}(\delta-u-x)^{2H-1}\le Cx^{2H}\,. \end{aligned}$$

In the last step we have applied the inequality $\delta-u-x\ge\delta-2\varepsilon$. □

Lemma 4.5.2. *Suppose $H\in(0,\frac{1}{2})$. Let $\psi(x,u)$ and $\varphi_4(x,u)$ defined by* (5.1) *and* (5.5)*, respectively. Fix $\varepsilon\in(0,1/4)$. Then*

$$\lim_{T\to\infty}\int_{[0,\varepsilon]^2}\psi(x,u)(x^{2H}+u^{2H}+|x-u|^{2H})dxdu=0\,, \tag{5.14}$$

and

$$\begin{aligned} &\lim_{T\to\infty}\int_{[0,1]^2}\psi(x,u)\varphi_4(x,u)dxdu\\ &\qquad=\theta^{-1-4H}\Big(\Gamma(2H)^2(2H-2^{-1})+\frac{\Gamma(2-4H)\Gamma(2H)\Gamma(4H)}{\Gamma(1-2H)}\Big)\,. \end{aligned} \tag{5.15}$$

Proof. We first prove (5.14). For the first summand, making the change of variables $Tx\to x_1$ and $Tu\to x_2$ yields

$$\int_{[0,\varepsilon]^2}T^{4H+1}e^{-\theta T(x+u)}x^{2H}dxdu=T^{2H-1}\int_{[0,T\varepsilon]^2}e^{-\theta(x_1+x_2)}x_1^{2H}dx_1dx_2\,, \tag{5.16}$$

which goes to 0 as $T \to \infty$. A similar argument could be applied to the second summand. For the third summand, by symmetry it suffices to consider the integral on the region $\{u > x\}$. Making the change of variables $T(u-x) \to x_1$, $Tx \to x_2$ yields

$$\int_{[0,\varepsilon]^2} T^{4H+1} e^{-\theta T(x+u)} |u-x|^{2H} dxdu = 2T^{2H-1} \int_{[0,T\varepsilon]^2, x_1+x_2 \le T\varepsilon} e^{-\theta(x_1+2x_2)} x_1^{2H} dx_1 dx_2 \,, \tag{5.17}$$

which goes to 0 as $T \to \infty$.

Next we show (5.15). Set

$$\Theta := \lim_{T\to\infty} \int_{[0,1]^2} \psi(x,u)\varphi_4(x,u)dxdu.$$

Making change of variables, $\theta Tx \to x, \theta Tu \to u, \theta Tt \to t$, we can write

$$\begin{aligned} \Theta &= \theta^{-1-4H} \int_{[0,\infty)^2} e^{-(u+x)} dxdu \\ &\quad \times \int_x^\infty [t^{2H-1}\mathrm{sgn}(x+u-t)|x+u-t|^{2H-1} - (t-x)^{2H-1}\mathrm{sgn}(u-t)|u-t|^{2H-1}]dt \,. \end{aligned}$$

The above integral can be decomposed as follows

$$\Theta = \theta^{-1-4H}(L_1 - L_2 + L_3) \,,$$

where

$$L_1 := \int_{[0,\infty)^2} e^{-(x+u)} dxdu \int_x^{x+u} t^{2H-1}(x+u-t)^{2H-1} dt \,,$$

$$L_2 := \int_{[0,\infty)^2, u>x} e^{-(x+u)} dxdu \int_x^u (t-x)^{2H-1}(u-t)^{2H-1} dt \,,$$

$$L_3 := \int_{[0,\infty)^2} e^{-(x+u)} dxdu \Big(\int_{u\vee x}^\infty (t-x)^{2H-1}(t-u)^{2H-1} dt - \int_{x+u}^\infty t^{2H-1}(t-x-u)^{2H-1} dt \Big) \,.$$

Making the change of variables $t-x \to s$ and integrating u, we obtain

$$L_1 = \Gamma(2H) \int_{[0,\infty)^2} e^{-(x+s)} (x+s)^{2H-1} dxds = \Gamma(2H)^2 2H\,.$$

Denote by $\mathrm{B}(\alpha,\beta)$ the Beta function. Then

$$L_2 = \mathrm{B}(2H,2H) \int_{[0,\infty)^2, u>x} e^{-(x+u)} (u-x)^{4H-1} dxdu\,.$$

By setting $u-x \to v$ and integrating in x first, we deduce $L_2 = \Gamma(2H)^2/2$. To compute L_3, by symmetry it suffices to integrate on the region $\{u<x\}$. For the second integral, we make the change of variables $t-u \to y$. In this way, we obtain

$$L_3 = 2 \int_{0<u<x<y<\infty} e^{-(u+x)} ((y-u)^{2H-1} - (y+u)^{2H-1})(y-x)^{2H-1} dydxdu.$$

The change of variables $x-u \to a, y-x \to b$ yields

$$\begin{aligned} L_3 &= 2 \int_{\mathbb{R}_+^3} e^{-(a+2u)} b^{2H-1} \left[(a+b)^{2H-1} - (a+b+2u)^{2H-1} \right] dudadb \\ &= 2 \int_{\mathbb{R}_+^3} e^{-(a+2u)} b^{2H-1} dudadb \int_a^{2u+a} (1-2H)(b+z)^{2H-2} dz \\ &= 2(1-2H) \int_{\mathbb{R}_+^2} e^{-(a+2u)} duda \int_a^{2u+a} \left(\int_{\mathbb{R}_+} b^{2H-1} (b+z)^{2H-2} db \right) dz\,. \end{aligned}$$

Setting $z/(b+z) \to v$ and integrating v on $[0,1]$, we obtain

$$\begin{aligned} L_3 &= 2(1-2H)\mathrm{B}(2-4H,2H) \int_{\mathbb{R}_+^2} e^{-(a+2u)} duda \int_a^{2u+a} z^{4H-2} dz \\ &= \frac{\Gamma(2-4H)\Gamma(2H)\Gamma(4H)}{\Gamma(1-2H)}\,. \end{aligned}$$

Then, the lemma follows from the above computations of L_1, L_2 and L_3. □

Lemma 4.5.3. *Denote $\mathscr{I}_1 = [0,\varepsilon]^2$ and $\mathscr{I}_2 = [0,1]^2 \backslash [0,\varepsilon]^2$. The functions ψ and φ_i are given by* (5.1) *to* (5.6)*. Suppose $H \in (0, \frac{1}{2})$. For $j = 1,2$ and $i = 1,2,3,5$, we have the following result.*

$$\lim_{T\to\infty} \int_{\mathscr{I}_j} \psi \varphi_i dxdu = 0\,. \tag{5.18}$$

Proof. The proof of (5.18) is divided into the cases $j = 2$ and $j = 1$.

Case $j = 2$: Clearly, for $(x,u) \in \mathscr{I}_2$,

$$\psi(x,u) \le T^{4H+1} e^{-\theta T \varepsilon}, \tag{5.19}$$

which implies

$$\int_{\mathscr{I}_2} \psi \varphi_i dxdu \;\to 0 \quad \text{for } i = 1,2,3,5 \tag{5.20}$$

as $T \to \infty$. Thus, (5.18) holds true for $j = 2$.

Case $j = 1$: For $i = 2$, we evaluate the integral of $\psi\varphi_2$ on $\mathscr{I}_1$ by making change of variables $Tx \to x$, $Tu \to u$ and $Tt \to t$. In this way, we obtain

$$\int_{\mathscr{I}_1} \psi \varphi_2 dxdu = \int_{[0,T\varepsilon]^2} e^{-\theta(u+x)} dxdu \int_x^T [(t-x)^{2H-1} - t^{2H-1}](T-t+x)^{2H-1} dt\,. \tag{5.21}$$

Clearly $(T-t+x)^{2H-1} \le x^{2H-1}$, so the integrand of the above triple integral is bounded by the function $e^{-\theta(u+x)}((t-x)^{2H-1} - t^{2H-1})\mathbf{1}_{\{t\ge x\}} x^{2H-1}$ which is integrable on $[0,\infty)^3$. As $T \to \infty$, $(T-t+x)^{2H-1} \to 0$. Applying the dominated convergence theorem, we have

$$\lim_{T\to\infty} \int_{\mathscr{I}_1} \psi \varphi_2 dxdu = 0. \tag{5.22}$$

The cases $i = 1,3,5$ follows from (5.6), (5.7) and (5.8) and Lemma 4.5.2. $\square$

Lemma 4.5.4. *For $n \geq 0$, and $H \in [\frac{3}{4}, 1)$, set*

$$A_{1,H}(T) = T^{3-4H} \int_0^T \int_0^{T-t} s^n e^{-\theta s} t^{2H-2} (s+t)^{2H-2} ds dt,$$

and

$$A_{2,H}(T) = T^{3-4H} \int_0^T \int_0^T s^n e^{-\theta s} t^{2H-2} (s+t)^{2H-2} ds dt.$$

Then

(i) For $H \in (\frac{3}{4}, 1)$, $\lim_{T\to\infty} A_{1,H}(T) = \lim_{T\to\infty} A_{2,H}(T) = \dfrac{\theta^{-(n+1)}\Gamma(n+1)}{4H-3}$;

(ii) For $H = \frac{3}{4}$, $\lim_{T\to\infty} \dfrac{A_{1,H}(T)}{\log T} = \lim_{T\to\infty} \dfrac{A_{2,H}(T)}{\log T} = \Gamma(n+1)\theta^{-(n+1)}$.

Proof. (i) For $H \in (\frac{3}{4}, 1)$, we have

$$A_{2,H}(T) \leq T^{3-4H} \int_0^T \int_0^T s^n e^{-\theta s} t^{4H-4} ds dt,$$

and

$$A_{1,H}(T) \geq T^{3-4H} \int_0^T \int_0^{T-t} s^n e^{-\theta s} (s+t)^{4H-4} ds dt.$$

For the right-hand sides of the above two inequalities, we integrate first in t to obtain

$$\begin{aligned} \frac{1}{4H-3} \left(\int_0^T s^n e^{-\theta s} ds - T^{3-4H} \int_0^T s^{n+4H-3} e^{-\theta s} ds \right) \\ \leq A_{1,H}(T) \leq A_{2,H}(T) \leq \frac{1}{4H-3} \int_0^T s^n e^{-\theta s} ds. \end{aligned}$$

This yields (i) by letting $T \to \infty$.

(ii) For $H = \frac{3}{4}$, by the L'Hopital rule, we have

$$\lim_{T\to\infty} \frac{A_{2,H}(T)}{\log T} = \lim_{T\to\infty} T\Big[\int_0^T s^n e^{-\theta s} T^{-\frac{1}{2}}(s+T)^{-\frac{1}{2}} ds + \int_0^T T^n e^{-\theta T} t^{-\frac{1}{2}}(T+t)^{-\frac{1}{2}} dt\Big].$$

The second summand on the right-hand side of the above equation goes to 0 as $T \to \infty$, so

$$\lim_{T\to\infty} \frac{A_{2,H}(T)}{\log T} \le \int_0^\infty s^n e^{-\theta s} ds. \tag{5.23}$$

On the other hand, by the inequality $t \le s+t$,

$$\begin{aligned} \frac{A_{1,H}(T)}{\log T} &\ge \frac{1}{\log T} \int_0^T \int_0^{T-t} s^n e^{-\theta s}(s+t)^{-1} ds dt \\ &= \frac{1}{\log T}\Big[\log T \int_0^T s^n e^{-\theta s} ds - \int_0^T s^n e^{-\theta s} \log s ds\Big]. \end{aligned}$$

The function $s^n e^{-\theta s}\log s$ is integrable on $[0,\infty)$. Thus,

$$\lim_{T\to\infty} \frac{A_{1,H}(T)}{\log T} \ge \int_0^\infty s^n e^{-\theta s} ds. \tag{5.24}$$

By (5.23) and (5.24), we conclude the proof of (ii). □

Lemma 4.5.5. *Let F_T, $\widetilde{F}_T$ be defined by* (1.11) *and* (1.19)*, respectively. Moreover, let R_1 be defined in Part (iii) of Theorem 4.1.2. Then we have the following convergence results.*

(i) *When $0 < H < \frac{1}{2}$ we have*

$$\lim_{T\to\infty} \mathbb{E}\left(\frac{1}{T}F_T^2\right) = 4H^2\theta^{1-4H}\Gamma(2H)^2\Big((4H-1) + \frac{2\Gamma(2-4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1-2H)}\Big). \tag{5.25}$$

(ii) *When $H = \frac{3}{4}$, we have*

$$\lim_{T\to\infty} \frac{\mathbb{E}\left(F_T^2\right)}{T\log(T)} = 9/4\theta^{-2} \,. \tag{5.26}$$

(iii) *When $H > \frac{3}{4}$, we have*

$$\lim_{T\to\infty} \mathbb{E}\left(T^{2-4H} F_T^2\right) = \frac{16\alpha_H^2\theta^{-2}}{(4H-2)(4H-3)} \,, \tag{5.27}$$

$$\lim_{T\to\infty} \mathbb{E}[T^{1-2H} R_1 \widetilde{F}_T] = \frac{8\alpha_H^2\theta^{-1}}{(4H-2)(4H-3)} \,, \tag{5.28}$$

where $\alpha_H = H(2H-1)$.

In the above lemma, we do not give a statement when $H \in [\frac{1}{2}, \frac{3}{4})$, because this case has been studied in [20].

Proof. **Part (i):** Assume $H \in (0, \frac{1}{2})$. Applying L'Hopital's rule to (1.13) yields

$$\lim_{T\to\infty} \mathbb{E}\left(\frac{1}{T} F_T^2\right) = \lim_{T\to\infty} 4H^2\theta^2(I_1 + I_2) \,, \tag{5.29}$$

where

$$\begin{aligned} I_1 &= (H\theta)^{-1} \int_{[0,T]^3} e^{-\theta(T-t_1)} \frac{\partial e^{-\theta|s_2-t_2|}}{\partial s_2} [T^{2H-1} - (T-s_2)^{2H-1}] \frac{\partial R_H(t_1,t_2)}{\partial t_2} ds_2 dt_1 dt_2 \,, \\ I_2 &= -(H\theta)^{-1} \int_{[0,T]^3} e^{-\theta(T-s_1)} \frac{\partial e^{-\theta|s_2-t_2|}}{\partial s_2} \frac{\partial R_H(s_1,s_2)}{\partial s_1} \\ &\quad \times [t_2^{2H-1} + (T-t_2)^{2H-1}] ds_1 ds_2 dt_2 \,. \end{aligned} \tag{5.30}$$

To compute the limit of $\mathbb{E}(\frac{1}{T} F_T^2)$ we will consider that of I_1 and I_2.

Computation of $\lim_{T\to\infty} I_1$: We first compute explicitly the partial derivatives in the integrand of I_1. On the region $\{t_2 > s_2\}$, we make change of variables $1-\frac{t_1}{T}\to u$, $\frac{t_2}{T}-\frac{s_2}{T}\to x$ and $1-\frac{s_2}{T}\to t$, and on the region $\{t_2 < s_2\}$, we make change of variables $1-\frac{t_1}{T}\to u$, $\frac{s_2}{T}-\frac{t_2}{T}\to x$ and $1-\frac{t_2}{T}\to t$. In this way, I_1 can be written as

$$
\begin{aligned}
I_1 \;=\; & \int_{[0,1]^3, x\le t} T^{4H+1} e^{-\theta T(u+x)} (1-t^{2H-1}) \\
& \quad \left((1-t+x)^{2H-1} - \operatorname{sgn}(x+u-t)|x+u-t|^{2H-1}\right) dudxdt \\
& - \int_{[0,1]^3, x\le t} T^{4H+1} e^{-\theta T(u+x)} \qquad (5.31) \\
& \left(1-(t-x)^{2H-1}\right)\left((1-t)^{2H-1} - \operatorname{sgn}(u-t)|u-t|^{2H-1}\right) dudxdt\,.
\end{aligned}
\tag{5.32}
$$

Reorganize the terms in the above integrals we have

$$
I_1 = \int_{[0,1]^2} \psi(x,u) \sum_{i=1}^{4} \varphi_i dxdu\,, \tag{5.33}
$$

where the functions ψ, φ_i are given by (5.1) to (5.5).

By (5.18), we see

$$
\lim_{T\to\infty} I_1 = \lim_{T\to\infty} \int_{[0,1]^2} \psi(x,u)\varphi_4(x,u) dxdu\,, \tag{5.34}
$$

whose value is computed in (5.15) of Lemma 4.5.2.

Computation of $\lim_{T\to\infty} I_2$*:* We first compute explicitly the partial derivatives in the integrand of (5.30). On the region $\{s_2 > t_2\}$, we make change of variables $T-s_1\to Tu$, $s_2-t_2\to Tx$ and $T-t_2\to Tt$, and on the region $\{t_2 > s_2\}$, we make change of variables

$T - s_1 \to Tu$, $t_2 - s_2 \to Tx$ and $T - s_2 \to Tt$. In this way,

$$\begin{aligned} I_2 &= \int_{[0,1]^3, t \geq x} T^{4H+1} e^{-\theta T(u+x)} \left((1-u)^{2H-1} + \operatorname{sgn}(x+u-t)|x+u-t|^{2H-1} \right) \\ &\quad \left(t^{2H-1} + (1-t)^{2H-1} \right) dudxdt - \int_{[0,1]^3, t \geq x} T^{4H+1} e^{-\theta T(u+x)} \\ &\quad \left((1-u)^{2H-1} + \operatorname{sgn}(u-t)|u-t|^{2H-1} \right) \left((1-t+x)^{2H-1} + (t-x)^{2H-1} \right) dudxdt \,. \end{aligned}$$

Note that

$$\int_x^1 \left(t^{2H-1} + (1-t)^{2H-1} \right) - \left((1-t+x)^{2H-1} + (t-x)^{2H-1} \right) dt = 0 \,,$$

so I_2 can be simplified and rewritten as

$$I_2 = \int_{[0,1]^2} \psi(x,u) \left(\varphi_4(x,u) + \varphi_5(x,u) \right) dxdu \,, \tag{5.35}$$

where $\psi(x,u)$, $\varphi_4(x,u)$ and $\varphi_5(x,u)$ are given by (5.1), (5.5) and (5.6) respectively.

By (5.34) and the result of (5.18) for $i = 5$, we have

$$\lim_{T \to \infty} I_2 = \lim_{T \to \infty} I_1 \,. \tag{5.36}$$

Then part **(i)** follows from (5.29), (5.34), (5.36) and (5.15).

Part (ii) and (iii): Assume $H \geq 3/4$. Using (1.5), we have

$$\mathbb{E}(F_T^2) = 2\alpha_H^2 I_T \,, \tag{5.37}$$

where $\alpha_H = H(2H-1)$, and

$$I_T = \int_{[0,T]^4} e^{-\theta|s_2-u_2|-\theta|s_1-u_1|}|s_2-s_1|^{2H-2}|u_2-u_1|^{2H-2}du_1du_2ds_1ds_2\,. \qquad (5.38)$$

Applying L'Hopital rule yields

$$\begin{cases} \lim_{T\to\infty} \mathbb{E}(T^{2-4H}F_T^2) = \frac{8\alpha_H^2}{4H-2}\lim_{T\to\infty} T^{3-4H}J_T & \text{when } H\in(\frac{3}{4},1) \quad (5.39) \\ \lim_{T\to\infty} \frac{\mathbb{E}F_T^2}{T\log T} = \frac{9}{8}\lim_{T\to\infty}\frac{J_T}{\log T} & \text{when } H=\frac{3}{4}, \quad (5.40) \end{cases}$$

where

$$J_T = \int_{[0,T]^3} e^{-\theta|T-u_2|-\theta|s_1-u_1|}(T-s_1)^{2H-2}|u_2-u_1|^{2H-2}du_1du_2ds_1\,.$$

Denote

$$h(T) = T^{3-4H}\mathbf{1}_{\{H\in(\frac{3}{4},1)\}} + (\log T)^{-1}\mathbf{1}_{\{H=\frac{3}{4}\}}\,.$$

Then, finding the limits (5.39) and (5.40) is reduced to the computation of $\lim_{T\to\infty} h(T)J_T$.

Making the change of variables $x=T-u_2$, $y=u_1-s_1$ and $z=T-s_1$ in the region $\{u_1>s_1\}$ and the change of variables $x=T-u_2$, $y=s_1-u_1$, $z=T-s_1$ in the region $\{u_1<s_1\}$, we can write J_T as follows

$$\begin{aligned} J_T &= \int_{[0,T]^3,y<z} e^{-\theta(x+y)}z^{2H-2}|x+y-z|^{2H-2}dxdydz \\ &\quad + \int_{[0,T]^3,y+z<T} e^{-\theta(x+y)}z^{2H-2}|y+z-x|^{2H-2}dxdydz. \qquad (5.41) \end{aligned}$$

Consider the functions

$$f_1(x,y,z) = e^{-\theta(x+y)} z^{2H-2} |x+y-z|^{2H-2}, \qquad f_2(x,y,z) = e^{-\theta(x+y)} z^{2H-2} |y+z-x|^{2H-2}.$$

For the first integral of (5.41), we split the integration interval $\{y < z\}$ into $\{x+y < z\} \cup \{x+y \geq z, y < z\}$. For the second integral of (5.41), we write the integration interval as $\{y+z < T\} = \{x+y < T, x \leq y\} \cup \{x+y < T, 0 < x-y < z\} \cup \{x+y < T, x-y \geq z\} \cup \{x+y \geq T\} \backslash \{y+z \geq T\}$. In this way, we can split J_T into seven integrals. It turns out that some of them are bounded by a constant independent of T and they do not contribute to the limit, because $h(T) \to 0$. More precisely, we can derive the following bounds:

$$\begin{aligned} \int_{[0,T]^3, x+y \geq z, y<z} f_1(x,y,z) dxdydz &\leq \int_{[0,T]^3, x+y \geq z} f_1(x,y,z) dxdydz \\ &= C_1 \int_{[0,T]^2} e^{-\theta(x+y)} (x+y)^{4H-3} dxdy \leq C, \end{aligned}$$

where in the second step we integrated in z and the last step follows from the inequality $x+y \geq 2\sqrt{xy}$. It is trivial to show that

$$\int_{[0,T]^3, x+y \geq T} f_2(x,y,z) dxdydz \leq e^{-\theta T} \int_{[0,T]^3} z^{2H-2} |y+z-x|^{2H-2} dxdydz \leq C,$$

and

$$\begin{aligned} & \int_{[0,T]^3, x+y<T, x-y \geq z} f_2(x,y,z) dxdydz \\ \leq & \int_{[0,T]^3, x-y \geq z} e^{-\theta(x+y)} z^{2H-2} (x-y-z)^{2H-2} dxdydz \\ = & C_1 \int_{[0,T]^2} e^{-\theta(x+y)} (x-y)^{4H-3} dxdy \leq C. \end{aligned}$$

The last bounded integral is

$$\begin{aligned}\int_{[0,T]^3, y+z\geq T} f_2(x,y,z)dxdydz &\leq \int_{[0,T]^3, y+z\geq T} e^{-\theta(x+y)} z^{2H-2}(T-x)^{2H-2}dxdydz \\ &\leq \left(\int_0^T e^{-\theta x}(T-x)^{2H-2}dx\right)^2 \leq C\,,\end{aligned}$$

where in the second step we have used the inequality $z^{2H-2} \leq (T-y)^{2H-2}$ and the last step follows from the following inequality

$$\begin{aligned}\int_0^T e^{-\theta x}(T-x)^{2H-2}dx &\leq \int_0^{T/2} e^{-\theta x} x^{2H-2}dx + \int_{T/2}^T e^{-\theta(T-x)}(T-x)^{2H-2}dx \\ &\leq 2\int_0^\infty e^{-\theta x} x^{2H-2}dx\,.\end{aligned}$$

With these observations,

$$\begin{aligned}\lim_{T\to\infty} h(T)J_T &= \lim_{T\to\infty} h(T)\int_{x+y<z} f_1(x,y,z)dxdydz \\ &\quad + \lim_{T\to\infty} h(T)\int_{\{x+y<T, x\leq y\}} f_2(x,y,z)dxdydz \\ &\quad + \lim_{T\to\infty} h(T)\int_{\{x+y<T, 0<x-y<z\}} f_2(x,y,z)dxdydz\,.\end{aligned}$$

We make change of variables $z-(x+y)\to u, x+y\to v, y\to y$ for the first term, $y-x\to u, z\to v, y\to y$ for the second term, and $x-y\to u, z-x+y\to v, y\to y$ for the third term. In this way, we obtain

$$\begin{aligned}\lim_{T\to\infty} h(T)J_T &= \lim_{T\to\infty} h(T)\int_{[0,T]^3, u+v<T, y<v} e^{-\theta v}(u+v)^{2H-2}u^{2H-2}dydudv \\ &\quad + \lim_{T\to\infty} h(T)\int_{[0,T]^3, u<y<(T+u)/2} e^{-\theta(-u+2y)}v^{2H-2}(u+v)^{2H-2}dydudv \\ &\quad + \lim_{T\to\infty} h(T)\int_{[0,T]^3, u+v<T, y<(T-u)/2} e^{-\theta(u+2y)}(u+v)^{2H-2}v^{2H-2}dydudv\,.\end{aligned}$$

Finally, the limits (5.26) and (5.27) follow from integrating in the variable y and an application of Lemma 4.5.4.

We proceed now to the proof of (5.28). Assume $H > 3/4$. Recall that $R_1 = I_2(\delta_{0,1})$ is given in Theorem 4.1.2 and $\widetilde{F}_T$ is given by (1.19). By (3.2), we can write

$$\mathbb{E}(R_1(T^{1-2H}\widetilde{F}_T)) = 2\alpha_H^2 T \int_{[0,1]^3} e^{-\theta T|t-s|}|t-t'|^{2H-2}|s-t'|^{2H-2}dsdtdt'.$$

We make the change of variables $Tt \to x, Ts \to y, Tt' \to z$ to rewrite the above equation as

$$\mathbb{E}(T^{1-2H}R_1\widetilde{F}_T) = \frac{2\alpha_H^2}{T^{4H-2}} \int_{[0,T]^3} e^{-\theta|x-y|}|x-z|^{2H-2}|y-z|^{2H-2}dxdydz.$$

By the symmetry of x, y in the above equation, applying L'Hopital's rule yields

$$\lim_{T\to\infty} \mathbb{E}(T^{1-2H}R_1\widetilde{F}_T) \tag{5.42}$$

$$= \frac{\alpha_H^2}{2H-1} \lim_{T\to\infty} T^{3-4H}\Big(2\int_{[0,T]^2} e^{-\theta(T-y)}(T-z)^{2H-2}|y-z|^{2H-2}dydz$$
$$+\int_{[0,T]^2} e^{-\theta|x-y|}(T-x)^{2H-2}(T-y)^{2H-2}dxdy\Big) \tag{5.43}$$

$$=: \frac{\alpha_H^2}{2H-1} \lim_{T\to\infty} T^{3-4H}(2L_1+L_2). \tag{5.44}$$

To compute L_1, on the region $\{y > z\}$ we make the change of variables $y - z \to t$, $T - y \to s$ and on the region $\{y < z\}$ we make the change of variables $z - y \to s$, $T - z \to t$. In this way we obtain

$$L_1 = \int_{[0,T]^2, s+t<T} e^{-\theta s}(s+t)^{2H-2}t^{2H-2}dsdt + \int_{[0,T]^2, s+t<T} e^{-\theta(s+t)}t^{2H-2}s^{2H-2}dsdt$$

For the term L_2, by symmetry it is sufficient to consider the region $\{x > y\}$ and making the change of variables $T - x \to t,\ x - y \to s$, we obtain

$$L_2 = 2 \int_{[0,T]^2, s+t<T} e^{-\theta s} t^{2H-2} (s+t)^{2H-2} ds dt \,.$$

Notice that the second summand of L_1 is bounded by $\int_{[0,\infty)^2} e^{-\theta(s+t)} t^{2H-2} s^{2H-2} ds dt$. Therefore,

$$\begin{aligned} \lim_{T\to\infty} \mathbb{E}(T^{1-2H} R_1 \widetilde{F}_T) &= \frac{4\alpha_H^2}{2H-1} \lim_{T\to\infty} T^{3-4H} \int_{[0,T]^2, s+t<T} e^{-\theta s} (s+t)^{2H-2} t^{2H-2} ds dt \\ &= \frac{4\alpha_H^2 \theta^{-1}}{(2H-1)(4H-3)}, \end{aligned}$$

where the last step is due to Lemma 4.5.4. This finishes the proof of Lemma 4.5.5. □

Lemma 4.5.6. *Let Y_T be defined by*

$$Y_t = \sigma \int_{-\infty}^{t} e^{-\theta(t-s)} dB_s^H = X_t + e^{-\theta t} \xi \,, \tag{5.45}$$

where

$$\xi = \sigma \int_{-\infty}^{0} e^{\theta s} dB_s^H \,. \tag{5.46}$$

For any $\alpha > 0$, $\frac{Y_T}{T^\alpha}$ converges almost surely to zero as T tends to infinity.

Proof. The case $H \geq \frac{1}{2}$ was proved in [20]. Here, we present a different proof valid for all $H \in (0,1)$. We denote $\beta := \mathbb{E}\xi^2 = \sigma^2 \theta^{-2H} H \Gamma(2H)$, which is computed in Lemma 4.5.7. Notice that the covariance of the process Y_t for $t > 0$ is computed as

$$\begin{aligned} \mathrm{Cov}(Y_0, Y_t) &= e^{-\theta t} \mathbb{E}\Big(\xi \Big[\xi + \sigma \int_0^t e^{\theta u} dB_u^H \Big] \Big) \\ &= e^{-\theta t} \beta + e^{-\theta t} \sigma^2 \mathbb{E}\Big(\int_{-\infty}^{0} e^{\theta s} dB_s^H \int_0^t e^{\theta u} dB_u^H \Big). \end{aligned}$$

We use integration by parts for both integrals in the above equation to rewrite

$$\mathrm{Cov}(Y_0, Y_t) = e^{-\theta t}\beta + g_1(t) - g_2(t)\,,$$

where

$$g_1(t) = e^{-\theta t}\sigma^2\theta^2\mathbb{E}\Big(\int_{-\infty}^{0}\int_0^t B_s^H B_u^H e^{\theta(u+s)}duds\Big), \quad g_2(t) = \sigma^2\theta\mathbb{E}\Big(\int_{-\infty}^{0} B_s^H B_t^H e^{\theta s}ds\Big)\,.$$

By Fubini theorem and the explicit form of the covariance of fBm,

$$\begin{aligned} g_1(t) &= \frac{1}{2}e^{-\theta t}\sigma^2\theta^2\int_{-\infty}^{0}\int_0^t (|s|^{2H} + u^{2H} - (u-s)^{2H})e^{\theta(u+s)}duds \\ &= \beta(1 - e^{-\theta t}) + \frac{1}{2}e^{-\theta t}\sigma^2\theta\int_0^t e^{\theta u}u^{2H}du - \frac{\beta}{2}(e^{\theta t} - e^{-\theta t})\,. \end{aligned}$$

When we compute the above double integral, we write the integrand as three items by distributing $e^{\theta(u+s)}$ and then integrate the terms one by one. For the term involving $(u-s)^{2H}$, we make the change of variables $u - s \to x, s \to y$ and integrate in the variable y first. Similarly,

$$\begin{aligned} g_2(t) &= \frac{1}{2}\sigma^2\theta\int_{-\infty}^{0}(|s|^{2H} + t^{2H} - (t-s)^{2H})e^{\theta s}ds \\ &= \beta + \frac{1}{2}\sigma^2 t^{2H} - \beta e^{\theta t} + \frac{1}{2}\sigma^2\theta e^{\theta t}\int_0^t e^{-\theta s}s^{2H}ds\,. \end{aligned}$$

Denote $a_t = o(b_t)$ if $\lim_{t\to 0}\frac{a_t}{b_t} = 0$. Notice that

$$\int_0^t e^{\theta(u-t)}u^{2H}du - \int_0^t e^{\theta(t-s)}s^{2H}ds = o(t^{2H}).$$

Based on the above computations, for t small, we have

$$\mathrm{Cov}(Y_0, Y_t) = \beta\left[1 - \frac{\theta^{2H}}{\Gamma(2H+1)} t^{2H} + o(t^{2H})\right].$$

The lemma now follows from Theorem 3.1 of [34].

□

Lemma 4.5.7. *Let the stochastic process X_t satisfy* (0.1) *(with $\sigma_t = \sigma$). Then*

$$\frac{1}{T}\int_0^T X_t^2 dt \to \sigma^2 \theta^{-2H} H\Gamma(2H)$$

a.s. and in L^2, as $T \to \infty$.

Proof. When $H \geq \frac{1}{2}$, the Lemma is proved in [20]. We shall handle the case of general Hurst parameter in a similar way. The process $\{Y_t, t \geq 0\}$ defined by (5.45) is Gaussian, stationary and ergodic for all $H \in (0,1)$. By the ergodic theorem,

$$\frac{1}{T}\int_0^T Y_t^2 dt \to \mathbb{E}(Y_0^2), \quad \text{as } T \text{ goes to infinity,}$$

almost surely and in L^2. This implies

$$\frac{1}{T}\int_0^T X_t^2 dt \to \mathbb{E}(Y_0^2),$$

as T goes to infinity, almost surely and in L^2. Moreover, integrating by parts yields

$$\begin{aligned}\mathbb{E}(Y_0^2) = \mathbb{E}(\xi^2) &= \sigma^2 \mathbb{E}\Big(\int_{-\infty}^0 e^{\theta s} dB_s^H\Big)^2 = \theta^2\sigma^2 \mathbb{E}\int_{-\infty}^0 \int_{-\infty}^0 B_s^H B_r^H e^{\theta(s+r)} dsdr \\ &= \theta^2 \sigma^2 \int_0^\infty \int_0^\infty e^{-\theta(s+r)} R_H(s,r) dsdr = \sigma^2 \theta^{-2H} H\Gamma(2H).\end{aligned}$$

In the last step of the above computation, we use the same idea as near the end of the proof for Lemma 4.5.6. Namely, one writes out the explicit form of $R_H(s,r)$, split the integrand into three items by distributing $e^{-\theta(s+r)}$ to the summands of $R_H(s,r)$, and then integrate the three items one by one. For the item involving $|s-r|^{2H}$, noticing the symmetry of s,r, one can make change of variables $s-r \to u, r \to v$, and then integrate in the variable v first. $\square$

Chapter 5

Drift parameter estimation for nonlinear stochastic differential equations

5.1 Main results

In this chapter, we study a parameter estimation problem for the following stochastic differential equation (SDE) driven by a fractional Brownian motion (fBm)

$$dX_t = -f(X_t)\theta dt + \sigma dB_t\,, \quad t \geq 0\,, \tag{1.1}$$

where $X_0 = x_0 \in \mathbb{R}^m$ is a given initial condition. The notations appearing in the above equation are explained as follows. For the diffusion part, $B = (B^1, \dots, B^d)$ is a d-dimensional fBm of Hurst parameter $H \in (0,1)$. The diffusion coefficient $\sigma = (\sigma_1, \dots, \sigma_d)$ is an $m \times d$ matrix, with σ_j, $j = 1, \dots, d$ being given vectors in $\mathbb{R}^m$. For the drift part, the function $f : \mathbb{R}^m \to \mathbb{R}^{m\times l}$ satisfies some regularity and growth conditions that we shall specify below. We write $f(x) = (f_1(x), \dots, f_l(x))$, with $f_j(x)$, $j = 1, \dots, l$, being vectors in $\mathbb{R}^m$. We assume that $\theta = (\theta_1, \dots, \theta_l) \in \mathbb{R}^l$ is an unknown constant parameter. In equation (1.1) we have used matrix notation, where the vectors are understood as column vectors. With above notations, we may write (1.1) as

$$dX_t = -\sum_{j=1}^{l} \theta_j f_j(X_t)dt + \sum_{j=1}^{d} \sigma_j dB_t^j\,.$$

Our objective is to estimate the parameter vector θ, from the continuous observations of the process $X = \{X_t, t \geq 0\}$ in a finite interval $[0,T]$. We consider a least squares type estimator, which consists of minimizing formally the quantity $\int_0^T |\dot{X}_t + f(X_t)\theta|^2 dt$, where and in what follows we use $|\cdot|$ to denote the Euclidean norm of a vector or the Hilbert-Schmidt norm of a matrix. From this procedure, the least squares estimator (LSE) is given explicitly by

$$\hat{\theta}_T = -\left(\int_0^T (f^{tr} f)(X_t)dt\right)^{-1} \int_0^T f^{tr}(X_t)dX_t\,, \tag{1.2}$$

where f^{tr} denotes the transpose of the matrix f. Substituting (1.1) into the above expression we have

$$\hat{\theta}_T = \theta - \left(\int_0^T (f^{tr} f)(X_t)dt\right)^{-1} \int_0^T f^{tr}(X_t)\sigma dB_t\,. \tag{1.3}$$

In the above equation, the stochastic integral with respect to the fBm is understood as a divergence integral (or Skorohod integral). See Section 2 for its definition.

In order to state the main result of the paper, we introduce the following hypothesis.

Hypothesis 5.1.1. The functions f_j, $1 \leq j \leq m$ are continuously differentiable and there is a positive constant L_1 such that the Jacobian matrices $\nabla f_j(x) \in \mathbb{R}^{m\times m}$ satisfy $\sum_{j=1}^{l} \theta_j \nabla f_j(x) \geq L_1 I_m$ for all $x \in \mathbb{R}^m$, where I_m is the $m \times m$ identity matrix.

In the above hypothesis and in what follows we use the notation $A \geq B$ to denote the fact that $A - B$ is a non-negative definite matrix.

We denote by $\mathscr{C}_p^1(\mathbb{R}^m)$ the class of functions $g \in \mathscr{C}^1(\mathbb{R}^m)$ such that there are two positive constants L_2 and γ with

$$|g(x)| + |\nabla g(x)| \le L_2(1 + |x|^\gamma), \tag{1.4}$$

for all $x \in \mathbb{R}^m$. We denote by $\mathscr{C}_p^2(\mathbb{R}^m)$ the class of functions $g \in \mathscr{C}^2(\mathbb{R}^m)$ such that there are two positive constants L_2 and γ with

$$|g(x)| + |\nabla g(x)| + |\mathbb{H}(g)(x)| \le L_2(1 + |x|^\gamma), \tag{1.5}$$

for all $x \in \mathbb{R}^m$, where $\mathbb{H}(g) = \left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)_{1 \le i,j \le m}$ denotes Hessian matrix of g.

It is easy to see that under Hypothesis 5.1.1, f satisfies the one-sided dissipative Lipschitz condition:

$$\langle x - y, (f(x) - f(y))\theta \rangle \ge L_1 |x - y|^2, \quad \forall\, x, y \in \mathbb{R}^m. \tag{1.6}$$

According to the papers [16, 17, 2] and the references therein, under Hypothesis 5.1.1 and assuming $f_{ij} \in \mathscr{C}_p^1(\mathbb{R}^m)$, for all $1 \le i \le m$, $1 \le j \le l$, the SDE (1.1) admits a unique solution X_t in $\mathscr{C}^\alpha(\mathbb{R}_+; \mathbb{R}^m)$ for all $\alpha < H$. Now we state the main result of this paper.

Theorem 5.1.2. *Assume Hypothesis 5.1.1 and that the components of f belong to $\mathscr{C}_p^1(\mathbb{R}^m)$ when $H \in [\frac{1}{2}, 1)$, and they belong to $\mathscr{C}_p^2(\mathbb{R}^m)$ when $H \in (\frac{1}{4}, \frac{1}{2})$. Suppose that $\mathbb{P}\left(\det(f^{tr} f)(\overline{X}) > 0\right) > 0$, where $\overline{X}$ is the random variable appearing in Theorem 5.2.1. Then the least squares estimator $\hat{\theta}_T$ of the parameter θ is strongly consistent in the sense that $\lim_{T \to \infty} |\hat{\theta}_T - \theta| = 0$ almost surely.*

Remark 5.1.3. Condition $\mathbb{P}\left(\det(f^{tr}f)(\overline{X})>0\right)>0$ means that $\nu(\det(f^{tr}f)>0)>0$, where ν is the invariant measure of the SDE (1.1). A sufficient condition for this to hold is $\det(f^{tr}f)(x)>0$ for all $x\in\mathbb{R}^m$.

Remark 5.1.4. When $f(x)=x$ is linear, this inference problem of θ has been extensively studied in the previous chapter.

5.2 Ergodicity of the stochastic differential equations

First, let us recall an ergodic theorem for the solution to equation (1.1) that is crucial for our arguments. Recall that the d-dimensional fBm $B=\{(B_t^1,\dots,B_t^d),t\geq 0\}$ with Hurst parameter $H\in(0,1)$, is a zero mean Gaussian process whose components are independent and have the covariance function

$$\mathbb{E}(B_t^iB_s^i)=R_H(t,s):=\frac{1}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H}), \tag{2.1}$$

for $i=1,\dots,d$. The probability space $(\Omega,\mathscr{F},\mathbb{P})$ we are taking is the canonical probability space of the fractional Brownian motion. Namely, $\Omega=C_0(\mathbb{R}_+;\mathbb{R}^d)$ is the set of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$ equipped with the uniform topology on any compact interval; $\mathscr{F}$ is the Borel σ-algebra, and $\mathbb{P}$ is the probability measure on $(\Omega,\mathscr{F})$ such that the coordinate process $B_t(\omega)=\omega(t)$ is a fractional Brownian motion with Hurst parameter $H\in(0,1)$.

We define the shift operators $\mu_t:\Omega\to\Omega$ as

$$\mu_t\omega(\cdot)=\omega(\cdot+t)-\omega(t),\ t\in\mathbb{R},\omega\in\Omega.$$

The probability measure $\mathbb{P}$ is invariant with respect to the shift operators μ_t. The ergodic property of the SDE (1.1) is summarized in the following theorem (see [17, 2]).

Theorem 5.2.1. *Assume the drift function f satisfies Hypothesis 5.1.1 and its components belong to $\mathscr{C}_p^1(\mathbb{R}^m)$. Then, the following results hold:*

(i) There exists a random variable $\overline{X}:\Omega\to\mathbb{R}^m$ with $\mathbb{E}|\overline{X}|^p<\infty$ for all $p\geq 1$ such that

$$\lim_{t\to\infty}|X_t(\omega)-\overline{X}(\mu_t\omega)|=0 \tag{2.2}$$

for $\mathbb{P}$-almost all $\omega\in\Omega$.

(ii) For any function $g\in\mathscr{C}_p^1(\mathbb{R}^m)$, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g(X_t)dt=\mathbb{E}[g(\overline{X})] \qquad \text{P-}a.s. \tag{2.3}$$

5.3 Moment estimates and maximal inequality for divergence integrals with respect to fBm

When $H>\frac{1}{2}$, thanks to (1.7) and (2.1), the following lemma provides a useful estimate for the p-norm of the divergence integral with respect to fBm.

Lemma 5.3.1. *Let $H\in(\frac{1}{2},1)$ and let u be an element of $\mathbb{D}^{1,p}(\mathfrak{H}^d)$, $p>1$. Then u belongs to the domain of the divergence operator δ in $L^p(\Omega)$. Moreover, we have*

$$\mathbb{E}(|\delta(u)|^p)\leq C_{p,H}\left(\|\mathbb{E}(u)\|^p_{L^{1/H}([0,\infty);\mathbb{R}^d)}+\mathbb{E}\left(\|Du\|^p_{L^{1/H}([0,\infty)^2;\mathbb{R}^{d\times d})}\right)\right).$$

Now we consider the case of $H \in (0, \frac{1}{2})$. First we will derive an estimate for the p-norm of $\|u\mathbb{1}_{[a,b]}\|_{\mathfrak{H}\otimes\mathbb{W}}$, where u is a stochastic process with values in a Hilbert space $\mathbb{W}$.

Consider the functions L^t and $L^{t,s}$ defined for $0 < s < t < b$ by

$$L^t(\lambda_0, \lambda_1) := (b-t)^{\lambda_0} t^{\lambda_1}\,, \tag{3.1}$$

$$L^{t,s}(\lambda_2, \lambda_3, \lambda_4) := (b-t)^{\lambda_2}(t-s)^{\lambda_3} s^{\lambda_4}\,. \tag{3.2}$$

where the λ_i's are parameters. We denote by C a generic constant that depends only on the coefficients of the SDE (1.1), the Hurst parameter H and the parameters introduced along the paper.

Proposition 5.3.2. Let $p \geq 2$ and $H \in (0, \frac{1}{2})$. Fix $b \geq 0$. Let $\mathbb{W}$ be a Hilbert space and consider a $\mathbb{W}$-valued stochastic process $u = \{u_t, t \geq 0\}$ satisfying the following conditions:

(i) $\|u_t\|_{L^p(\Omega;\mathbb{W})} \leq K_1 L^t(\lambda_0, \lambda_1)$, for all $t \geq 0$;

(ii) $\|u_t - u_s\|_{L^p(\Omega;\mathbb{W})} \leq K_2 L^{t,s}(\lambda_2, \lambda_3, \lambda_4)$, for all $s < t \leq b$,

where the parameters λ_i satisfy $\lambda_0 > -H$, $\lambda_1, \lambda_4 \geq 0$, $\lambda_2 > -\frac{1}{2}$, and $\lambda_3 > \frac{1}{2} - H$. Then for all $0 \leq a \leq b$,

$$\begin{aligned} & \mathbb{E}(\|u\mathbb{1}_{[a,b]}\|^p_{\mathfrak{H}\otimes\mathbb{W}}) && (3.3)\\ \leq\ & CK_2^p b^{p\lambda_4}(b-a)^{pH+p\lambda_2+p\lambda_3} + CK_1^p b^{p\lambda_1}(b-a)^{pH+p\lambda_0}\,. && (3.4) \end{aligned}$$

Proof. To simplify we assume $\mathbb{W} = \mathbb{R}$. Using the isometry of the operator K_H, we can write

$$\mathbb{E}(\|u\mathbb{1}_{[a,b]}\|^p_{\mathfrak{H}}) = \mathbb{E}\left(\|K_H(u\mathbb{1}_{[a,b]})\|^p_{L^2([0,b])}\right)\,.$$

We decompose the integral appearing in (1.10) into sum of three terms according to the cases where one of s,t is in the interval (a,b) or both. In this way, we obtain

$$\begin{aligned} K_H(u\mathbb{1}_{[a,b]}) &= K_H(b,s)u_s\mathbb{1}_{[a,b]}(s)+\left(\int_s^b (u_t-u_s)\frac{\partial K_H}{\partial t}(t,s)dt\right)\mathbb{1}_{[a,b]}(s) \\ &\quad +\left(\int_a^b u_t\frac{\partial K_H}{\partial t}(t,s)dt\right)\mathbb{1}_{[0,a]}(s) \\ &=: I_1+I_2+I_3\,. \end{aligned}$$

Thus,

$$\mathbb{E}(\|u\mathbb{1}_{[a,b]}\|_{\mathfrak{H}}^p)\leq C\sum_{i=1}^3 A_i\,, \tag{3.5}$$

where $A_i=\mathbb{E}\left(\|I_i\|^p_{L^2([0,b])}\right)$. Now we estimate each term A_i in (3.5). For A_1, applying Minkowski inequality and condition (i), we obtain

$$\begin{aligned} A_1 &\leq C\left(\int_a^b \left((b-s)^{2H-1}+s^{2H-1}\right)\|u_s\|^2_{L^p(\Omega)}ds\right)^{\frac{p}{2}} \\ &\leq CK_1^p\left(\int_a^b ((b-s)^{2H-1}+s^{2H-1})(b-s)^{2\lambda_0}s^{2\lambda_1}ds\right)^{\frac{p}{2}} \\ &\leq CK_1^p\left(\int_a^b \left((b-s)^{2H-1}+(s-a)^{2H-1}\right)(b-s)^{2\lambda_0}s^{2\lambda_1}ds\right)^{\frac{p}{2}} \\ &= CK_1^p b^{p\lambda_1}(b-a)^{pH+p\lambda_0}\,. \end{aligned}$$

For the term A_3, applying again Minkowski inequality and condition (i), we can write

$$\begin{aligned} A_3 &\leq C\left(\int_0^a\left(\int_a^b \|u_t\|_{L^p(\Omega)}(t-s)^{H-\frac{3}{2}}dt\right)^2ds\right)^{\frac{p}{2}} \\ &\leq CK_1^p\left(\int_0^a\left(\int_a^b (b-t)^{\lambda_0}t^{\lambda_1}(t-s)^{H-\frac{3}{2}}dt\right)^2ds\right)^{\frac{p}{2}}\,. \end{aligned}$$

Denote $g(t) = (b-t)^{\lambda_0} t^{\lambda_1}$ which is positive. Then

$$A_3 \le C K_1^p \left(\int_{[a,b]^2} g(t_1) g(t_2) dt_1 dt_2 \int_0^a (t_1 - s)^{H-\frac{3}{2}} (t_2 - s)^{H-\frac{3}{2}} ds \right)^{\frac{p}{2}} .$$

Now

$$\int_0^a (t_1 - s)^{H-\frac{3}{2}} (t_2 - s)^{H-\frac{3}{2}} ds \le \int_0^a (t_1 - a)^{H-\frac{3}{2}} (t_2 - s)^{H-\frac{3}{2}} ds \le C (t_1 - a)^{H-\frac{3}{2}} (t_2 - a)^{H-\frac{1}{2}} .$$

In the same way we have

$$\int_0^a (t_1 - s)^{H-\frac{3}{2}} (t_2 - s)^{H-\frac{3}{2}} ds \le C (t_2 - a)^{H-\frac{3}{2}} (t_1 - a)^{H-\frac{1}{2}} .$$

Using the fact that if $u \le a_1$ and $u \le a_2$, then $u \le \sqrt{a_1 a_2}$, we see that

$$\int_0^a (t_1 - s)^{H-\frac{3}{2}} (t_2 - s)^{H-\frac{3}{2}} ds \le (t_1 - a)^{H-1} (t_2 - a)^{H-1} .$$

Therefore, we have

$$A_3 \ \le \ C K_1^p \left(\int_a^b (b-t)^{\lambda_0} (t-a)^{H-1} t^{\lambda_1} dt \right)^p \le C K_1^p b^{p\lambda_1} (b-a)^{pH + p\lambda_0} .$$

For A_2, applying Minkowski inequality and condition (ii), yields

$$\begin{aligned}
A_2 \ &\le \ C \left(\int_a^b \left(\int_s^b \|u_t - u_s\|_{L^p(\Omega)} (t-s)^{H-\frac{3}{2}} dt \right)^2 ds \right)^{\frac{p}{2}} \\
&\le \ C K_2^p \left(\int_a^b \left(\int_s^b (b-t)^{\lambda_2} (t-s)^{\lambda_3} s^{\lambda_4} (t-s)^{H-\frac{3}{2}} dt \right)^2 ds \right)^{\frac{p}{2}} \\
&\le \ C K_2^p \left(\int_a^b (b-s)^{2\lambda_2 + 2\lambda_3 + 2H - 1} s^{2\lambda_4} ds \right)^{\frac{p}{2}} \\
&= \ C K_2^p b^{p\lambda_4} (b-a)^{pH + p\lambda_2 + p\lambda_3} .
\end{aligned}$$

This completes the proof. $\square$

Suppose now that u is a d-dimensional stochastic process. We will make use of the notation $\|u\|_{p,a,b} := \sup_{a\le t\le b} \|u_t\|_{L^p(\Omega;\mathbb{R}^d)}$. Consider the following regularity conditions on u:

Hypothesis 5.3.3. Assume that there are constants $K > 0$, $\beta > \frac{1}{2} - H$ and $\lambda \in (0,H]$, such that the $\mathbb{R}^d$-valued process $u = \{u_t, t \ge 0\}$ and its derivative $\{Du_t, t \ge 0\}$ satisfy the following conditions:

(i) $\|u\|_{p,0,\infty} = \sup_{t\ge 0} \|u_t\|_{L^p(\Omega;\mathbb{R}^d)} < \infty$,

(ii) $\|u_t - u_s\|_{L^p(\Omega;\mathbb{R}^d)} \le K(t-s)^{\beta}$,

(iii) $\|Du_t\|_{L^p(\Omega;\mathbb{H}^d\otimes\mathbb{R}^d)} \le Kt^{\lambda}$,

(iv) $\|Du_t - Du_s\|_{L^p(\Omega;\mathbb{H}^d\otimes\mathbb{R}^d)} \le K(t-s)^{\beta} s^{\lambda}$,

for all $0 \le s < t$.

As an application of (2.1) and Proposition 5.3.2, we give the following estimate for the p-th moment of the divergence integral $\delta(u\mathbb{1}_{[0,T]})$.

Proposition 5.3.4. Let $H \in (0, \frac{1}{2})$ and $p \ge 2$. Assume that the $\mathbb{R}^d$-valued stochastic process $\{u_t, t \ge 0\}$ satisfies Hypothesis 5.3.3. Then for any $T > 0$, the divergence integral $\delta(u\mathbb{1}_{[0,T]})$ is in $L^p(\Omega)$, and

$$\mathbb{E}(|\delta(u\mathbb{1}_{[0,T]})|^p) \le CT^{pH}(1+T^{p\lambda})(1+T^{p\beta}),$$

where the constant C is independent of T.

Proof. We will use inequality (2.1) to prove the proposition and it suffices to compute the right-hand side of (2.1). Applying Proposition 5.3.2 to $\mathbb{W}=\mathbb{R}^d$, $\lambda_3=\beta$ and $\lambda_i=0, i\neq 3$, we obtain

$$\mathbb{E}(\|u\mathbb{1}_{[0,T]}\|^p_{\mathfrak{H}^d}) \leq C\left(\|u\|^p_{p,0,\infty}T^{pH}+K^pT^{p\beta+pH}\right).$$

To compute the p-th moment of the derivative of u, we use the functions L^t and $L^{t,s}$ introduced in (3.1) and (3.2), respectively, to write the conditions (iii) and (iv) of Hypothesis 5.3.3 as

$$\|Du_t\|_{L^p(\Omega;\mathbb{H}^d\otimes\mathbb{R}^d)} \leq KL^t(0,\lambda),$$

and

$$\|Du_t-Du_s\|_{L^p(\Omega;\mathbb{H}^d\otimes\mathbb{R}^d)} \leq KL^{t,s}(0,\beta,\lambda).$$

Then we use Proposition 5.3.2 for $\mathbb{W}=\mathbb{H}^d\otimes\mathbb{R}^d$ and take into account the isomorphism $\mathfrak{H}\otimes(\mathfrak{H}^d\otimes\mathbb{R}^d)\cong\mathfrak{H}^d\otimes\mathfrak{H}^d$ to obtain

$$\mathbb{E}(\|Du\mathbb{1}_{[0,T]}\|^p_{\mathbb{H}^d\otimes\mathbb{H}^d}) \leq CK^pT^{pH+p\lambda}(1+T^{p\beta}).$$

This completes the proof of the proposition. □

When $H\neq\frac{1}{2}$, the divergence integral $\left\{\int_0^t u_s dB_s, t\geq 0\right\}$ is not a martingale, so we cannot apply Burkholder inequality to bound the maximum of the integral. However, if the process u satisfies some regularity conditions in Hypothesis 5.3.3, we can use a factorization method to estimate the maximum, as it has been done in [1]. This result is given in the following theorem.

Theorem 5.3.5. *Let $\{u_t, t\geq 0\}$ be an $\mathbb{R}^d$-valued stochastic process. For the divergence integral $\int_0^t u_s dB_s$, $t\geq 0$, we have the following statements:*

1. *Let $H \in (\frac{1}{4}, \frac{1}{2})$ and $p > \frac{1}{H}$. Assume that the stochastic process u satisfies Hypothesis 5.3.3. Then the divergence integral $\int_0^t u_s dB_s$ is in $L^p(\Omega)$ for all $t \geq 0$ and for any $0 \leq a < b$ we have the estimate*

$$\mathbb{E}\left(\sup_{t\in[a,b]}\left|\int_a^t u_s dB_s\right|^p\right) \leq C(b-a)^{pH}(1+(b-a)^{p\beta})(1+b^{p\lambda}),$$

where C is a generic constant that does not depend on a, b.

2. *Let $H \in (\frac{1}{2}, 1)$ and $\frac{1}{p} + \frac{1}{q} = H$ with $p > q$. Suppose that for all $T > 0$*

   (i) $\int_0^T \mathbb{E}(|u_s|^p) ds < \infty$,

   (ii) $\int_0^T \int_0^s \mathbb{E}(|D_t u_s|^p) dt ds < \infty$.

   Then the divergence integral $\int_0^t u_s dB_s$ is in $L^p(\Omega)$ for all $t \geq 0$ and for any interval $[a, b]$, we have

$$\begin{aligned} & \mathbb{E}\left(\sup_{t\in[a,b]}\left|\int_a^t u_s dB_s\right|^p\right) \\ \leq\ & C\left((b-a)^{\frac{p}{q}}\int_a^b \mathbb{E}(|u_s|^p)ds + (b-a)^{\frac{2p}{q}}\int_a^b\int_a^s \mathbb{E}(|D_t u_s|^p)dtds\right), \end{aligned}$$

   where the constant C does not depend on a, b.

Proof. We may assume that u is a smooth function. The general case follows from a limiting argument. We will use the elementary integral $\int_s^t (t-r)^{\alpha-1}(r-s)^{-\alpha} dr = \frac{\pi}{\sin(\alpha\pi)}$ for any $\alpha \in (0,1)$, and a stochastic Fubini's theorem. For any $\alpha \in (\frac{1}{p}, 1)$, we have

$$\begin{aligned} & \mathbb{E}\left(\sup_{t\in[a,b]}\left|\int_a^t u_s dB_s\right|^p\right) \\ & = \left(\frac{\sin(\alpha\pi)}{\pi}\right)^p \mathbb{E}\left(\sup_{t\in[a,b]}\left|\int_a^t\left(\int_s^t (t-r)^{\alpha-1}(r-s)^{-\alpha}dr\right)u_s dB_s\right|^p\right) \end{aligned}$$

$$
\begin{aligned}
&\left(\frac{\sin(\alpha\pi)}{\pi}\right)^p \mathbb{E}\left(\sup_{t\in[a,b]}\left|\int_a^t \left(\int_a^r (r-s)^{-\alpha} u_s dB_s\right)(t-r)^{\alpha-1} dr\right|^p\right) \\
&\leq \left(\frac{\sin(\alpha\pi)}{\pi}\right)^p \mathbb{E}\left(\sup_{t\in[a,b]}\left(\int_a^t \left|\int_a^r (r-s)^{-\alpha} u_s dB_s\right|^p dr\right)\left|\int_a^t (t-r)^{\frac{p(\alpha-1)}{p-1}} dr\right|^{p-1}\right) \\
&\leq C_{\alpha,p}(b-a)^{p\alpha-1}\int_a^b \mathbb{E}(|G_r|^p) dr, \qquad (3.6)
\end{aligned}
$$

where

$$
G_r := \int_a^r (r-s)^{-\alpha} u_s dB_s, \qquad r\in[a,b].
$$

Case $H\in(\frac{1}{2},1)$: Using Lemma 5.3.1 for $\alpha\in(\frac{1}{p},\frac{1}{q})$ and $\frac{1}{p}+\frac{1}{q}=H$, we get

$$
\begin{aligned}
\mathbb{E}(|G_r|^p) &\leq C_{p,H}\Bigg(\left(\int_a^r (r-s)^{-\frac{\alpha}{H}}|\mathbb{E}(u_s)|^{\frac{1}{H}} ds\right)^{pH} \\
&\quad + \mathbb{E}\left(\int_a^r\int_a^s (r-s)^{-\frac{\alpha}{H}}|D_\mu u_s|^{\frac{1}{H}} d\mu ds\right)^{pH}\Bigg) \\
&\leq C_{p,H}\left(\int_a^r (r-s)^{-\alpha q} ds\right)^{\frac{p}{q}}\left(\int_a^r |\mathbb{E}(u_s)|^p ds\right) \\
&\quad + C_{p,H}\left(\int_a^r\int_a^s (r-s)^{-\alpha q} d\mu ds\right)^{\frac{p}{q}}\left(\int_a^r\int_a^s \mathbb{E}(|D_\mu u_s|)^p d\mu ds\right) \\
&\leq C_{\alpha,p,q,H}\Bigg((r-a)^{\frac{p}{q}-\alpha p}\int_a^r \mathbb{E}(|u_s|^p) ds \\
&\quad + (r-a)^{\frac{2p}{q}-\alpha p}\int_a^r\int_a^s \mathbb{E}(|D_\mu u_s|^p) d\mu ds\Bigg).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\left(\sup_{t\in[a,b]}\left|\int_a^t u_s dB_s\right|^p\right) &\leq C\Bigg((b-a)^{\frac{p}{q}}\int_a^b \mathbb{E}(|u_s|^p) ds \\
&\quad + (b-a)^{\frac{2p}{q}}\int_a^b\int_a^s \mathbb{E}(|D_\mu u_s|^p) d\mu ds\Bigg).
\end{aligned}
$$

Case $H \in (0, \frac{1}{2})$: Denote $\psi(t) = (r-t)^{-\alpha} u_t$ for $t \in [a, r)$. Then by (2.1),

$$\mathbb{E}(|G_r|^p) \le \mathbb{E}(\|\psi \mathbb{1}_{[a,r)}\|^p_{\mathfrak{H}^d}) + \mathbb{E}(\|D(\psi \mathbb{1}_{[a,r)})\|^p_{\mathfrak{H}^d \otimes \mathbb{H}^d}) . \tag{3.7}$$

We will estimate the above two items on the right-hand side one by one. For $a \le s < t < r$,

$$\begin{aligned} |\psi(t) - \psi(s)| &= |(r-t)^{-\alpha}(u_t - u_s) + \left((r-t)^{-\alpha} - (r-s)^{-\alpha}\right) u_s| \\ &\le (r-t)^{-\alpha}|u_t - u_s| + (r-t)^{-2\alpha}(t-s)^{\alpha}|u_s| , \end{aligned}$$

where we have used the inequality $1 - (r-t)^{\alpha}(r-s)^{-\alpha} \le (r-s)^{-\alpha}(t-s)^{\alpha}$. Thus, using Hypothesis 3.3 (ii), we can write

$$\begin{aligned} \|\psi(t) - \psi(s)\|_{L^p(\Omega;\mathbb{R}^d)} &\le (r-t)^{-\alpha}\|u_t - u_s\|_{L^p(\Omega;\mathbb{R}^d)} + (r-t)^{-2\alpha}(t-s)^{\alpha}\|u_s\|_{L^p(\Omega;\mathbb{R}^d)} \\ &\le K(r-t)^{-\alpha}(t-s)^{\beta} + \|u\|_{p,a,b}(r-t)^{-2\alpha}(t-s)^{\alpha}, \end{aligned} \tag{3.8}$$

and

$$\|\psi(t)\|_{L^p(\Omega;\mathbb{R}^d)} = (r-t)^{-\alpha}\|u_t \mathbb{1}_{[a,r)}\|_{L^p(\Omega;\mathbb{R}^d)} \le (r-t)^{-\alpha}\|u\|_{p,a,b} , \tag{3.9}$$

This means that ψ satisfies the assumptions of Proposition 5.3.2 with $W = \mathbb{R}^d$ with the functions $L^t(-\alpha, 0)$ and $L^{t,s}(-\alpha, \beta, 0) + L^{t,s}(-2\alpha, \alpha, 0)$ if we choose $\alpha \in (\max(\frac{1}{p}, \frac{1}{2} - H), H)$, which requires $H \in (\frac{1}{4}, \frac{1}{2})$. In this way, we obtain

$$\mathbb{E}(\|\psi \mathbb{1}_{[a,r]}\|^p_{\mathfrak{H}^d}) \le C(r-a)^{pH - p\alpha}(1 + (r-a)^{p\beta}) . \tag{3.10}$$

Similarly, using Hypotheses 3.3 (iii) and (iv), we have

$$\begin{aligned}
&\|D\psi(t)-D\psi(s)\|_{L^p(\Omega;\mathfrak{H}^d\otimes\mathbb{R}^d)}\\
\leq\ & (r-t)^{-\alpha}\|Du_t-Du_s\|_{L^p(\Omega;\mathfrak{H}^d\otimes\mathbb{R}^d)}+(r-t)^{-2\alpha}(t-s)^{\alpha}\|Du_s\|_{L^p(\Omega;\mathfrak{H}^d\otimes\mathbb{R}^d)}\\
\leq\ & K(r-t)^{-\alpha}(t-s)^{\beta}s^{\lambda}+\ K(r-t)^{-2\alpha}(t-s)^{\alpha}s^{\lambda} \qquad (3.11)
\end{aligned}$$

and

$$\|D\psi(t)\|_{L^p(\Omega;\mathfrak{H}^d\otimes\mathbb{R}^d)} \;=\; (r-t)^{-\alpha}\|Du_t\|_{L^p(\Omega;\mathfrak{H}^d\otimes\mathbb{R}^d)}\leq K(r-t)^{-\alpha}t^{\lambda}. \qquad (3.12)$$

This means that $D\psi$ satisfies the assumptions of Proposition 5.3.2 with $\mathbb{W}=\mathbb{H}^d\otimes\mathbb{R}^d$ with the functions $L^t(-\alpha,\lambda)$ and $L^{t,s}(-\alpha,\beta,\lambda)+L^{t,s}(-2\alpha,\alpha,\lambda)$. Using Proposition 5.3.2 for $D\psi$ with $W=\mathfrak{H}^d\otimes\mathbb{R}^d$, we have

$$\mathbb{E}(\|D(\psi\mathbb{1}_{[a,r]})\|^p_{\mathbb{H}^d\otimes\mathbb{H}^d}) \;\leq\; C(r-a)^{pH-p\alpha}(1+(r-a)^{p\beta})b^{p\lambda}. \qquad (3.13)$$

Substituting the bounds of (3.10) and (3.13) into (3.7), we have

$$\mathbb{E}(|G_r|^p) \;\leq\; C(r-a)^{pH-p\alpha}(1+(r-a)^{p\beta})(1+b^{p\lambda}). \qquad (3.14)$$

Finally, putting this estimate into (3.6), we complete the proof. □

5.4 Estimates of the solution of stochastic differential equations

Before we present the proof of the main theorem, we need some auxiliary results. First, we prove some estimates for the p-th moment of the solution of the SDE (1.1).

Proposition 5.4.1. Let $H \in (0,1)$ and $p \geq 1$. Assume the drift function f of the SDE (1.1) satisfies Hypotheses 5.1.1 and its components belong to $\mathscr{C}_p^1(\mathbb{R}^m)$. Let X be the unique solution to (1.1). Then we have the following statements:

(1) There exists a constant $C_p > 0$ such that $\|X_t\|_{L^p(\Omega;\mathbb{R}^m)} \leq C_p$, and

$$\|X_t - X_s\|_{L^p(\Omega;\mathbb{R}^m)} \leq C_p |t-s|^H$$

for all $t \geq s \geq 0$.

(2) The Malliavin derivative of the solution X_t satisfies for all $0 \leq s \leq t$

$$|D_s X_t| \leq |\sigma| e^{-L_1(t-s)}, \text{ a.s.} \tag{4.1}$$

Moreover, if $v \leq u \leq s \leq t$, we have

$$\|D_u X_t - D_v X_t\|_{L^p(\Omega;\mathbb{R}^{m\times d})} \leq C e^{-L_1(t-u)} (1 \wedge |u-v|), \tag{4.2}$$

$$\|D_u X_t - D_u X_s\|_{L^p(\Omega;\mathbb{R}^{m\times d})} \leq C e^{-L_1(s-u)} (1 \wedge |t-s|), \tag{4.3}$$

and

$$\|D_u X_t - D_v X_t - (D_u X_s - D_v X_s)\|_{L^p(\Omega;\mathbb{R}^{m\times d})} \leq C e^{-L_1(s-u)} (1 \wedge |u-v|)(1 \wedge |t-s|), \tag{4.4}$$

where C is a generic constant.

Proof. For the proof of the first result we refer to [16], [17], and [2].

To show the second part of this proposition, taking the Malliavin derivative for $s \leq t$ on both sides of equation (1.1) yields

$$D_s X_t = -\int_s^t \sum_{j=1}^{l} \theta_j \nabla f_j(X_r) D_s X_r dr + \sigma\,, \tag{4.5}$$

where $\sigma = (\sigma_1, \ldots, \sigma_d) \in \mathbb{R}^{m \times d}$. Denote $Z_t = D_s X_t$ for $t \geq s$. We can write the above equation as the following ordinary differential equation for $t \geq s$:

$$\begin{cases} dZ_t = -\sum_{j=1}^{l} \theta_j \nabla f_j(X_t) Z_t dt, \\ Z_s = \sigma. \end{cases}$$

Differentiating $|Z_t|^2$ with respect to t, and using (1.6), we get

$$\frac{d|Z_t|^2}{dt} = 2\langle Z_t, -\sum_{j=1}^{l} \theta_j \nabla f_j(X_t) Z_t \rangle \leq -2L_1 |Z_t|^2\,.$$

By Gronwall's lemma, we obtain

$$|Z_t|^2 \leq e^{-2L_1(t-s)} |\sigma|^2\,,$$

and this implies (4.1).

We now proceed to the proof of (4.2). For $v \leq u \leq t$, equation (4.5) implies

$$D_u X_t - D_v X_t = -\int_u^t \sum_{j=1}^{l} \theta_j \nabla f_j(X_r)(D_u X_r - D_v X_r) dr + \int_v^u \sum_{j=1}^{l} \theta_j \nabla f_j(X_r) D_v X_r dr\,. \tag{4.6}$$

Repeating the above arguments for $D_uX_t - D_vX_t$, $t \geq u$, we can write

$$|D_uX_t - D_vX_t| \leq e^{-L_1(t-u)}\left|\int_v^u \sum_{j=1}^l \theta_j \nabla f_j(X_r) D_vX_r dr\right|.$$

Applying Minkowski inequality and (4.1) to D_vX_r, and then using the fact that the L^p-norm of $|\nabla f_j(X_r)|$ is bounded due to condition (1.4), we obtain

$$\begin{aligned}
\|D_uX_t - D_vX_t\|_{L^p(\Omega;\mathbb{R}^{m\times d})} &\leq e^{-L_1(t-u)}\int_v^u \|\sum_{j=1}^l \theta_j \nabla f_j(X_r) D_vX_r\|_{L^p(\Omega;\mathbb{R}^{m\times d})} dr \\
&\leq Ce^{-L_1(t-u)}\int_v^u e^{-L_1(r-v)} dr \leq Ce^{-L_1(t-u)}(1 \wedge |u-v|).
\end{aligned}$$

This proves (4.2). To prove (4.3), we use equation (4.5) to obtain

$$\mathbb{E}(|D_uX_t - D_uX_s|^p) = \mathbb{E}\left(\left|\int_s^t \sum_{j=1}^l \theta_j \nabla f_j(X_r) D_uX_r dr\right|^p\right).$$

Applying Minkowski inequality and using (4.1) for D_uX_r, and the fact that the L^p-norm of $|\nabla f_j(X_r)|$ is bounded, we obtain

$$\begin{aligned}
\|D_uX_t - D_uX_s\|_{L^p(\Omega;\mathbb{R}^{m\times d})} &\leq \int_s^t \left\|\sum_{j=1}^l \theta_j \nabla f_j(X_r) D_uX_r\right\|_{L^p(\Omega;\mathbb{R}^{m\times d})} dr \\
&\leq C\int_s^t e^{-L_1(r-u)} dr \leq Ce^{-L_1(s-u)}(1 \wedge |t-s|).
\end{aligned}$$

Finally we prove (4.4). Using (4.6), we have the following estimate

$$\begin{aligned}
&\|D_uX_t - D_vX_t - (D_uX_s - D_vX_s)\|_{L^p(\Omega;\mathbb{R}^{m\times d})} \\
= &\left\|\int_s^t \sum_{j=1}^l \theta_j \nabla f_j(X_r)(D_uX_r - D_vX_r) dr\right\|_{L^p(\Omega;\mathbb{R}^{m\times d})}.
\end{aligned}$$

Applying Minkowski inequality and Cauchy-Schwartz inequality yields

$$\begin{aligned}
&\|D_uX_t - D_vX_t - (D_uX_s - D_vX_s)\|_{L^p(\Omega;\mathbb{R}^{m\times d})} \\
\leq\ & C\int_s^t \|\nabla f_j(X_r)\|_{L^{2p}(\Omega;\mathbb{R}^{m\times m})}\|D_uX_r - D_vX_r\|_{L^{2p}(\Omega;\mathbb{R}^{m\times d})}dr \\
\leq\ & C(1\wedge|u-v|)\int_s^t e^{-L_1(r-u)}dr \leq Ce^{-L_1(s-u)}(1\wedge|u-v|)(1\wedge|t-s|)\,.
\end{aligned}$$

This proves (4.4) and proof of the proposition is complete. □

Remark 5.4.2. It is worth pointing out that the solution of the SDE (1.1) is Hölder continuous in L^p for all $p\geq 1$ with exponent H, i.e., $\|X_t - X_s\|_{L^p(\Omega;\mathbb{R}^m)} \leq C|t-s|^H$. However, the Malliavin derivative of X_t is more regular, i.e.,

$$\|D_uX_t - D_uX_s\|_{L^p(\Omega;\mathbb{R}^{m\times d})} \leq C|t-s|.$$

That is, the Hölder continuity exponent is improved from H to 1. This is because the noise in the SDE is additive.

The next lemma provides bounds for the norm of the derivative of a function of the solution to equation (1.1).

Lemma 5.4.3. *Let $H\in(0,\frac{1}{2})$ and $p\geq 2$. Consider a function $g=(g^1,\dots,g^d):\mathbb{R}^m\to\mathbb{R}^d$ whose components belong to $\mathscr{C}_p^2(\mathbb{R}^m)$. Then for all $0\leq s\leq t$, we have*

$$\|Dg(X_t) - Dg(X_s)\|_{L^p(\Omega;\mathfrak{H}^d\otimes\mathbb{R}^d)} \leq K(t-s)^H s^\lambda\,, \tag{4.7}$$

and

$$\|Dg(X_s)\|_{L^p(\Omega;\mathfrak{H}^d\otimes\mathbb{R}^d)} \leq Ks^\lambda\,, \tag{4.8}$$

for any $\lambda\in(0,H]$, where K is a constant that may depend on λ.

Proof. Consider the $\mathfrak{H}^d \otimes \mathbb{R}^d$-valued function $\phi := Dg(X_t) - Dg(X_s)$. We can writre

$$\begin{aligned}\|\phi\|^2_{\mathfrak{H}^d\otimes\mathbb{R}^d} \le C\|\phi\|^2_{K_t^d\otimes\mathbb{R}^d} \quad &\le \quad C\int_0^t |\phi(u)|^2\left((t-u)^{2H-1}+u^{2H-1}\right)du \\ &\quad +C\int_0^t\left(\int_v^t |\phi(u)-\phi(v)|(u-v)^{H-\frac{3}{2}}du\right)^2 dv \\ &\quad =: C(A_1+A_2)\,.\end{aligned}$$

Therefore,

$$\|\phi\|_{L^p(\Omega;\mathbb{H}^d\otimes\mathbb{R}^d)} \le C\sum_{i=1}^2 \|A_i\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)}\,.$$

It remains to estimate $\|A_i\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)}$ for $i=1,2$. First, we write $\phi(u)$ as

$$\phi(u) = \nabla g(X_t)\cdot(D_uX_t - D_uX_s) + (\nabla g(X_t) - \nabla g(X_s))\cdot D_uX_s\,. \tag{4.9}$$

Thus, by the submultiplicativity of Hilbert-Schmidt norm, i.e., $|AB|\le|A||B|$, we have

$$|\phi(u)| \le \begin{cases} |\nabla g(X_t)||D_uX_t - D_uX_s| + |X_t - X_s||D_uX_s| & \\ \qquad\qquad \times \int_0^1 \|\mathbb{H}(g(X_s+r(X_t-X_s)))\|dr & \text{when } u\le s\le t\,; \\ |\nabla g(X_t)||D_uX_t| & \text{when } s\le u\le t\,. \end{cases}$$

Here $\mathbb{H}(g) = (\mathbb{H}(g^1),\dots,\mathbb{H}(g^d))$ is understood as the third order tensor, and $\|\mathbb{H}(g)\|^2 = \sum_i |\mathbb{H}(g^i)|^2$. Since the components of g belong to $\mathscr{C}_p^2(\mathbb{R}^m)$, Proposition 5.4.1 says that the L^p norm of $|\nabla g(X_t)|$ and $\|\mathbb{H}(g(X_t))\|$ are both bounded for any $t\ge 0, p\ge 1$. Due to these facts and the inequalities (4.1) and (4.3), we have

$$
\begin{aligned}
\left(\mathbb{E}(|\phi(u)|^p)\right)^{\frac{1}{p}} &\leq C\begin{cases}
\left(\mathbb{E}(|\nabla g(X_t)|^{2p})\right)^{\frac{1}{2p}}\left(\mathbb{E}(|D_uX_t-D_uX_s|^{2p})\right)^{\frac{1}{2p}} \\
\qquad +e^{-L_1(s-u)}\int_0^1\left(\mathbb{E}(\|\mathbb{H}(g(X_s+r(X_t-X_s)))\|^{2p})\right)^{\frac{1}{2p}}dr \\
\qquad \times\left(\mathbb{E}(|X_t-X_s|^{2p})\right)^{\frac{1}{2p}} \qquad\qquad \text{when } u\leq s\leq t; \\
\left(\mathbb{E}(|\nabla g(X_t)|^{2p}\right)^{\frac{1}{2p}}\left(\mathbb{E}(|D_uX_t|^{2p})\right)^{\frac{1}{2p}} \qquad \text{when } s\leq u\leq t
\end{cases} \\
&\leq Ce^{-L_1(s-u)}(t-s)^H\mathbb{1}_{\{u<s\}}+Ce^{-L_1(t-u)}\mathbb{1}_{\{u>s\}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|A_1\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)} &\leq \left(\int_0^t\left(\mathbb{E}(|\phi|^p)\right)^{\frac{2}{p}}\left((t-u)^{2H-1}+u^{2H-1}\right)du\right)^{\frac{1}{2}} \\
&\leq C(t-s)^H\left(\int_0^s e^{-2L_1(s-u)}\left((t-u)^{2H-1}+u^{2H-1}\right)du\right)^{\frac{1}{2}} \\
&\quad +C\left(\int_s^t e^{-2L_1(t-u)}\left((t-u)^{2H-1}+u^{2H-1}\right)du\right)^{\frac{1}{2}} \\
&\leq C(t-s)^H,
\end{aligned}
$$

where in the last inequality we have used the following arguments. For the second summand, we have bounded $e^{-2L_1(t-u)}$ by 1 and applied the inequality $t^{2H}-s^{2H}\leq(t-s)^{2H}$. For the first summand, we bound $(t-u)^{2H-1}$ by $(s-u)^{2H-1}$ and decompose the integral in the intervals $[0,1]$ and $[1,s]$ (if $s\geq 1$).

Now we discuss A_2. For $v<u$, we decompose

$$\phi(u)-\phi(v) = \begin{cases} (\nabla g(X_t)-\nabla g(X_s))\cdot(D_uX_t-D_vX_t) \\ \qquad +\nabla g(X_s)\cdot(D_uX_t-D_vX_t-(D_uX_s-D_vX_s)) \\ \qquad\qquad \text{when } v<u<s<t\,; \\ (\nabla g(X_t)-\nabla g(X_s))\cdot(D_uX_t-D_vX_t) \\ \qquad +\nabla g(X_s)\cdot(D_uX_t-D_vX_t+D_vX_s) \quad \text{when } v<s<u<t\,; \\ \nabla g(X_t)\cdot(D_uX_t-D_vX_t) \quad \text{when } s<v<u<t\,. \end{cases}$$

We shall consider the above three cases separately.

Case 1): $v<u<s<t$. In this case we have

$$\begin{aligned} & (\mathbb{E}(|\phi(u)-\phi(v)|^p))^{\frac{1}{p}} \\ \leq\ & \int_0^1 \left(\mathbb{E}(\|\mathbb{H}(g(X_s+r(X_t-X_s)))\|^{4p})\right)^{\frac{1}{4p}} dr \\ & \times \left(\mathbb{E}(|X_t-X_s|^{4p})\right)^{\frac{1}{4p}} \left(\mathbb{E}(|D_uX_t-D_vX_t|^{2p})\right)^{\frac{1}{2p}} \\ & + \left(\mathbb{E}(|\nabla g(X_s)|^{2p})\right)^{\frac{1}{2p}} \left(\mathbb{E}(|D_uX_t-D_vX_t-(D_uX_s-D_vX_s)|^{2p})\right)^{\frac{1}{2p}}\ . \end{aligned}$$

Case 2): $s<v<u<t$. We have

$$\begin{aligned} (\mathbb{E}(|\phi(u)-\phi(v)|^p))^{\frac{1}{p}} &= (\mathbb{E}(|\nabla g(X_t)\cdot(D_uX_t-D_vX_t)|^p))^{\frac{1}{p}} \\ &\leq \left(\mathbb{E}(|\nabla g(X_t)|^{2p})\right)^{\frac{1}{2p}} \left(\mathbb{E}(|D_uX_t-D_vX_t|^{2p})\right)^{\frac{1}{2p}}\ . \end{aligned}$$

Case 3): $v<s<u<t$. We have

$$\phi(u)-\phi(v) = \nabla g(X_t)\cdot D_uX_t - \nabla g(X_t)\cdot(D_vX_t-D_vX_s) - (\nabla g(X_t)-\nabla g(X_s))\cdot D_vX_s\,,$$

so

$$
\begin{aligned}
& \left(\mathbb{E}(|\phi(u)-\phi(v)|^{p})\right)^{\frac{1}{p}} \\
\leq \; & \left(\mathbb{E}(|\nabla g(X_t)|^{2p})\right)^{\frac{1}{2p}} \left(\left(\mathbb{E}(|D_u X_t|^{2p})\right)^{\frac{1}{2p}} + \left(\mathbb{E}(|D_v X_t - D_v X_s|^{2p})\right)^{\frac{1}{2p}} \right) \\
& + \int_0^1 \left(\mathbb{E}(\|\mathbb{H}(g(X_s + r(X_t - X_s)))\|^{4p})\right)^{\frac{1}{4p}} dr \\
& \times \left(\mathbb{E}(|X_t - X_s|^{4p})\right)^{\frac{1}{4p}} \left(\mathbb{E}(|D_v X_s|^{2p})\right)^{\frac{1}{2p}} .
\end{aligned}
$$

Combining the above cases, and using the inequalities (4.1) to (4.4) in Proposition 5.4.1, we obtain

$$
\begin{aligned}
& \left(\mathbb{E}(|\phi(u)-\phi(v)|^{p})\right)^{\frac{1}{p}} \qquad (4.10)\\
\leq \; & C|t-s|^{H} e^{-L_1(s-u)}|u-v|^{\varepsilon}\mathbb{1}_{\{v<u<s<t\}} + Ce^{-L_1(t-u)}|u-v|^{\varepsilon}\mathbb{1}_{\{v>s\}} \\
& + C\left(e^{-L_1(t-u)} + e^{-L_1(s-v)}|t-s|^{H}\right)\mathbb{1}_{\{v<s<u<t\}} \\
& := \sum_{i=1}^{4} A_{2i}\,, \qquad (4.11)
\end{aligned}
$$

where we have used $1 \wedge |u-v| \leq C|u-v|^{\varepsilon}$ for any $\varepsilon \in [0,1]$ and $1 \wedge |t-s| \leq C|t-s|^{H}$. Now we apply Minkowski's inequality to $\|A_2\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)}$ and then an application of (4.11) yields

$$
\|A_2\|^{\frac{1}{2}}_{L^{\frac{p}{2}}(\Omega)} \leq \left(\int_0^t \left(\int_v^t \left(\mathbb{E}|\phi(u)-\phi(v)|^{p}\right)^{\frac{1}{p}} (u-v)^{H-\frac{3}{2}} du\right)^2 dv\right)^{\frac{1}{2}} \leq \sum_{i=1}^{4} A_2^{(i)}\,,
$$

where

$$
A_2^{(i)} = \left(\int_0^t \left(\int_v^t A_{2i}(u-v)^{H-\frac{3}{2}} du\right)^2 dv\right)^{\frac{1}{2}} .
$$

For $i=1$, fix $\lambda \in (0,H]$ and set $\varepsilon = 1-H+\lambda$ for A_{21} in (4.11). In this way, we obtain

$$\begin{aligned} A_2^{(1)} \quad &\leq \quad C(t-s)^H \left(\int_0^s \left(\int_v^s e^{-L_1(s-u)} (u-v)^{\lambda - \frac{1}{2}} du \right)^2 dv \right)^{\frac{1}{2}} \\ &\leq C(t-s)^H \left(\int_0^s (s-v)^{2\lambda - 1} dv \right)^{\frac{1}{2}} \leq C(t-s)^H s^{\lambda}, \end{aligned}$$

where the second inequality follows from the following estimate. For any $\alpha \in (-1,0)$,

$$\begin{aligned} \int_v^s e^{-L_1(s-u)} (u-v)^{\alpha} du \quad &\leq \quad \int_0^{s-v} e^{-L_1(s-v-x)} x^{\alpha} dx \\ &\leq \quad \int_0^{\frac{s-v}{2}} e^{-L_1(\frac{s-v}{2})} x^{\alpha} dx + \int_{\frac{s-v}{2}}^{s-v} (\frac{s-v}{2})^{\alpha} e^{-L_1(s-v-x)} dx \\ &\leq \quad C \left(e^{-L_1(\frac{s-v}{2})} (\frac{s-v}{2})^{\alpha+1} + (\frac{s-v}{2})^{\alpha} \right) \qquad (4.12) \\ &\leq \quad C(s-v)^{\alpha}, \qquad (4.13) \end{aligned}$$

taking into account the fact that the function $xe^{-L_1 x}$ is bounded on $[0,\infty)$.

For $i=2$, choosing $\varepsilon = 1$, we can write

$$A_2^{(2)} \leq C \left(\int_s^t \left(\int_v^t e^{-L_1(t-u)} (u-v)^{H-\frac{1}{2}} du \right)^2 dv \right)^{\frac{1}{2}}$$

Using (4.13) by setting $\lambda = H - \frac{1}{2}$, we have

$$A_2^{(2)} \leq C \left(\int_s^t (t-v)^{2H-1} dv \right)^{\frac{1}{2}} \leq C(t-s)^H .$$

For $i=3$,

$$A_2^{(3)} \quad \leq \quad C \left(\int_0^s \left(\int_s^t e^{-L_1(t-u)} (u-v)^{H-\frac{3}{2}} du \right)^2 dv \right)^{\frac{1}{2}}$$

$$\leq C\int_s^t \left(\int_0^s e^{-2L_1(t-u)}(u-v)^{2H-3}dv\right)^{\frac{1}{2}} du \leq C\int_s^t (u-s)^{H-1}du \leq C(t-s)^H .$$

For $i=4$,

$$\begin{aligned} A_2^{(4)} &\leq C(t-s)^H \left(\int_0^s \left(\int_s^t (u-v)^{H-\frac{3}{2}}du\right)^2 e^{-L_1(s-v)}dv\right)^{\frac{1}{2}} \\ &\leq C(t-s)^H \left(\int_0^s (s-v)^{2H-1}e^{-L_1(s-v)}dv\right)^{\frac{1}{2}} \leq C(t-s)^H . \end{aligned}$$

This finishes the proof of (4.7). The proof of (4.8) is similar. □

We next apply Proposition 5.4.1 and Lemma 5.4.3 to deduce the estimate for the p-th moment of the divergence integral $Z_{g,t}$ which is defined as

$$Z_{g,t} := \int_0^t g(X_s)dB_s\,, \tag{4.14}$$

where $\{X_t, t \geq 0\}$ is the solution of the SDE (1.1), and the function $g:\mathbb{R}^m \to \mathbb{R}^d$ satisfies some regularity and growth conditions.

Proposition 5.4.4. Let the divergence integral $Z_{g,T}$ be defined by (4.14).

1. If $H \in (\frac{1}{4}, \frac{1}{2})$ and $p \geq 2$, assume that the components of the function $g:\mathbb{R}^m \to \mathbb{R}^d$ belong to the space $\mathscr{C}_p^2(\mathbb{R}^m)$. Then we have

$$\mathbb{E}(|Z_{g,T}|^p) \leq CT^{pH}(1+T^{p\lambda})(1+T^{pH})\,,$$

for any $\lambda \in (0,H]$, where $C>0$ is a constant independent of T.

2. If $H \in (\frac{1}{2}, 1)$, assume that the components of the function $g : \mathbb{R}^m \to \mathbb{R}^d$ belong to the space $\mathscr{C}_p^1(\mathbb{R}^m)$. Then for $p > \frac{1}{H}$, we have

$$\mathbb{E}(|Z_{g,T}|^p) \le CT^{pH},$$

for all $T > 0$, where $C > 0$ is independent of T.

Proof. First, for $H \in (\frac{1}{4}, \frac{1}{2})$, by Proposition 5.4.1, the process $\{g(X_t), t \ge 0\}$ satisfies conditions (i) and (ii) of Hypothesis 5.3.3 with $\beta = H$, which requires $H > \frac{1}{2} - H$, i.e., $H > \frac{1}{4}$. By (4.7) and (4.8) of Lemma 5.4.3, $Dg(X_t)$ satisfies conditions (iii) and (iv) of Hypothesis 5.3.3 with $\beta = H$ and $\lambda \in (0, H]$. By Proposition 5.3.4, we obtain the result.

Second, for $H \in (\frac{1}{2}, 1)$, applying the results in the preceding Proposition 5.4.1, we get that $g(X_t)$ and $\nabla g(X_t)$ are bounded in $L^p(\Omega)$, so clearly $g(X_t)$ is in the space $\mathbb{D}^{1,p}(\mathfrak{H}^d)$. Applying Lemma 5.3.1 to $Z_{g,T}$ yields

$$\mathbb{E}(|Z_{g,T}|^p) \le C_{p,H} \left(\left(\int_0^T \mathbb{E}(|g(X_t)|^{\frac{1}{H}}) dt \right)^{pH} + \mathbb{E} \left(\int_0^T \int_0^t |D_s g(X_t)|^{\frac{1}{H}} ds dt \right)^{pH} \right).$$

Then we use (4.1) and integrate s to obtain

$$\begin{aligned}
&\mathbb{E}(|Z_{g,T}|^p) \\
\le\ & C_{p,H} \left(\left(\int_0^T \mathbb{E}(|g(X_t)|^{\frac{1}{H}}) dt \right)^{pH} + \frac{|\sigma|^p H^{pH}}{L_1^{pH}} \mathbb{E} \left(\int_0^T |\nabla g(X_t)|^{\frac{1}{H}} (1 - e^{-\frac{L_1}{H} t}) dt \right)^{pH} \right) \\
\le\ & C_{p,H} \left(\int_0^T \mathbb{E}(|g(X_t)|^{\frac{1}{H}}) dt \right)^{pH} + C_{p,H,L_1,\sigma} \left(\int_0^T (\mathbb{E} |\nabla g(X_t)|^p)^{\frac{1}{pH}} dt \right)^{pH} \le CT^{pH}.
\end{aligned}$$

This concludes the proof. □

5.5 Proof of the strong consistency of the least squares estimator

The following lemma is an important ingredient of the proof of Theorem 5.1.2.

Lemma 5.5.1. *Suppose* f *satisfies* $\mathbb{P}\left(\det(f^{tr}f)(\overline{X})>0\right)>0$*, then* $\mathbb{E}\left((f^{tr}f)(\overline{X})\right)$ *is invertible.*

Proof. Let ν be the law of $\overline{X}$. Applying Minkowski determinantal inequality and Jensen's inequality yields

$$\det\left(\int_{\mathbb{R}^m}(f^{tr}f)(x)\nu(dx)\right)^{\frac{1}{l}} \geq \int_{\mathbb{R}^m}\det\left((f^{tr}f)(x)\right)^{\frac{1}{l}}\nu(dx)\,,$$

which is positive under our hypothesis. □

Next we proceed to prove Theorem 5.1.2. Recall that the estimator $\hat{\theta}_T$ is given by (1.3). By Theorem 5.2.1, we have

$$\frac{1}{T}\int_0^T(f^{tr}f)(X_t)dt \to \mathbb{E}\left((f^{tr}f)(\overline{X})\right) \quad \text{a.s.}\,,$$

which is invertible. Therefore,

$$\left(\frac{1}{T}\int_0^T(f^{tr}f)(X_t)dt\right)^{-1} \to \left(\mathbb{E}\left((f^{tr}f)(\overline{X})\right)\right)^{-1} \quad \text{a.s.}\,. \tag{5.1}$$

Fix $j=1,\dots,l$ and consider the function $g_j(x)=f_j^{tr}(x)\sigma:\mathbb{R}^m\to\mathbb{R}^d$. Denote

$$Z_{j,t}=\int_0^t g_j(X_s)dB_s=\int_0^t f_j^{tr}(X_s)\sigma dB_s.$$

for $j = 1, \ldots, l$. Taking into account (5.1), to show $\lim_{T\to\infty} \frac{1}{T}|\hat{\theta}_T - \theta| = 0$ it suffices to show

$$\lim_{T\to\infty} \frac{1}{T} Z_{j,T} = 0 \tag{5.2}$$

for each $j = 1, \ldots, l$. The proof of (5.2) will be done in two steps.

Step 1: Fix $j = 1, \ldots, l$. We first show that

$$\lim_{n\to\infty} n^{-1} Z_{j,n} = 0.$$

Since the components of f belong to the space $\mathscr{C}_p^i(\mathbb{R}^m)$ with $i = 1, 2$, depending on $H > \frac{1}{2}$ or $H < \frac{1}{2}$, respectively, clearly the function $g_j(x)$ satisfies the conditions in Proposition 5.4.4. Applying Proposition 5.4.4,

$$\mathbb{E}(|Z_{j,n}|^p) \le \begin{cases} Cn^{pH} & \text{when } H \in (\frac{1}{2}, 1) \\ Cn^{p(2H+\lambda)} & \text{when } H \in (\frac{1}{4}, \frac{1}{2}) \end{cases} \tag{5.3}$$

for any $\lambda \in (0, H]$. We will choose p and λ in such a way that $p > \frac{1}{1-H}$ if $H \in (\frac{1}{2}, 1)$ and $0 < \lambda < 1 - 2H$ and $p > \frac{1}{1-2H-\lambda}$ if $H \in (0, \frac{1}{2})$.

On the other hand, for any $\varepsilon > 0$, by Chebyshev inequality and the above estimates we have

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}(|n^{-1} Z_{j,n}| > \varepsilon) &\le \sum_{n=1}^{\infty} \varepsilon^{-p} \mathbb{E}\left(|n^{-1} Z_{j,n}|^p\right) \\ &\le \begin{cases} C\sum_{n=1}^{\infty} \varepsilon^{-p} n^{(H-1)p} & \text{when } H \in (\frac{1}{2}, 1) \\ C\sum_{n=1}^{\infty} \varepsilon^{-p} n^{(2H+\lambda-1)p} & \text{when } H \in (0, \frac{1}{2}) \end{cases} \\ &< \infty. \end{aligned}$$

By Borel-Cantelli lemma, $n^{-1}Z_{j,n} \to 0$ a.s. as $n \to \infty$.

Step 2: For any $T > 0$ we define the integer k_T by $k_T \le T < k_T + 1$. We write

$$\frac{1}{T}Z_{j,T} = \frac{k_T}{T}\frac{1}{k_T}\int_0^{k_T} g_j(X_t)dB_t + \frac{1}{T}\int_{k_T}^{T} g_j(X_t)dB_t \,.$$

Thus,

$$\frac{1}{T}\left|Z_{j,T}\right| \le \frac{1}{k_T}\left|\int_0^{k_T} g_j(X_t)dB_t\right| + \frac{1}{T}\left|\int_{k_T}^{T} g_j(X_t)dB_t\right| .$$

Clearly from Step 1 the first summand converges to 0 almost surely as $T \to \infty$. For the second summand, observe that

$$\frac{1}{T}\left|\int_{k_T}^{T} g_j(X_t)dB_t\right| \le \frac{1}{k_T}\sup_{t\in[k_T,k_T+1]}\left|\int_{k_T}^{t} g_j(X_s)dB_s\right| . \tag{5.4}$$

Now we apply Theorem 5.3.5 to the p-th moment of $\sup_{t\in[k_T,k_T+1]}\left|\int_{k_T}^{t} g_j(X_s)dB_s\right|$. When $H \in (\frac{1}{2},1)$, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[k_T,k_T+1]}\left|\int_{k_T}^{t} g_j(X_s)dB_s\right|^p\right] \\
\le\ & C\left(\int_{k_T}^{k_T+1}\mathbb{E}(|g_j(X_s)|^p)ds + \int_{k_T}^{k_T+1}\int_{k_T}^{s}\mathbb{E}(|D_\mu g_j(X_s)|^p)d\mu ds\right) \\
\le\ & C\int_{k_T}^{k_T+1}\mathbb{E}\left(|g_j(X_s)|^p + |\nabla g_j(X_s)|^p\right)ds \le C\,.
\end{aligned}$$

Similarly, for $H \in (\frac{1}{4},\frac{1}{2})$, g_j belongs to $\mathscr{C}_p^2(\mathbb{R}^m)$, so by Lemma 5.4.3 it satisfies Hypothesis 5.3.3. Then applying Theorem 5.3.5 yields

$$\mathbb{E}\left[\sup_{t\in[k_T,k_T+1]}\left|\int_{k_T}^{t} g_j(X_s)dB_s\right|^p\right] \le C(k_T+1)^{p\lambda}$$

for all $p > \frac{1}{H}$, and any $\lambda \in (0,H]$. By Chebyshev inequality,

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{k_T}\sup_{t\in[k_T,k_T+1]}\left|\int_{k_T}^t g_j(X_s)dB_s\right| > \varepsilon\right)\\
&\leq \varepsilon^{-p}\mathbb{E}\left(\frac{1}{{k_T}^p}\sup_{t\in[k_T,k_T+1]}\left|\int_{k_T}^t g_j(X_s)dB_s\right|^p\right) \leq C\varepsilon^{-p}k_T^{p\lambda-p}.
\end{aligned}$$

Choosing p large enough, the above right-hand side is summable with respect to k_T and the desired result just follows from Borel-Cantelli Lemma. This completes the proof of Theorem 5.1.2.

Remark 5.5.2. From the proof of Theorem 5.1.2 we can see that the random variables $\xi_t = t^{-1}Z_{j,t}$ converge to 0 as t tends to infinity for every $j = 1,\dots,l$ in the following sense. For any $\varepsilon > 0$,

$$\lim_{n\to\infty}\sum_{k=n}^{\infty}\mathbb{P}(\sup_{k\leq t\leq k+1}|\xi_t| > \varepsilon) = 0.$$

This type of convergence is analogous to the complete convergence of a sequence of random variables (see [18]), which implies the almost sure convergence.

Remark 5.5.3. If we assume that the parameter vector θ belongs to a compact set $\Theta \subset \mathbb{R}^l$, the upper bound of the p-th moment of X_t would be independent of θ, and, correspondingly, the constants C and K that appear in Proposition 5.4.1, Lemma 5.4.3 and Proposition 5.4.4 would be independent of θ as well. As a consequence, we get the uniform strong convergence of the random variables $\xi_t = t^{-1}Z_{j,t}$ to 0 as t tends to infinity for every $j = 1,\dots,l$, in the sense of

$$\lim_{T\to\infty}\sup_{\theta\in\Theta}\mathbb{P}(\sup_{t\geq T}|\xi_t| > \varepsilon) = 0$$

for any $\varepsilon > 0$. Furthermore, if the function f satisfies $(f^{tr} f)^{-1} \leq L_3 I_l$ where $L_3 > 0$ is a constant independent of θ and I_l is an $l \times l$ identity matrix, the uniform strong consistency of $\hat{\theta}_T$ can be established by observing that $\left(\frac{1}{T}\int_0^T (f^{tr} f)(X_t)dt\right)^{-1} \leq L_3 I_l$.

Chapter 6

Summary and future research work

In this chapter, we summarize the obtained results in this dissertation and discuss some other research work that could be completed in the near future.

6.1 Summary

In this study, we consider the parameter estimation for stochastic differential equations driven by fractional Brownian motion.

In Chapter 1, we recall the background on Malliavin calculus and Gaussian analysis elements that play important roles in this research.

In Chapter 2, we have investigated the asymptotics of iterated power variations. The law of iterated logarithm of fBm has been obtained and correspondingly the convergence rate of the iterated power variations is discovered for the first time. We have also obtained the joint convergence along different subsequences of power variations. As a consequence, we have applied these results to construct the estimators for integrated volatility, volatility and Hurst parameter in the SDEs. These estimators are strongly consistent and admit central limit theorems.

In Chapter 3, we have studied the drift estimation for the fractional Ornstein-Uhlenbeck process that is the solution to the linear SDE. Through minimizing the L^2 norm of the noise part, we have derived the least squares estimator and prove the strong consistency and limit theorems. We have also studied the discrete case and obtained a strongly consistent estimator. Monte Carlo simulations have been carried out to valid our results.

In Chapter 4, we have considered least squares estimation for the drift parameter vector in the nonlinear SDEs. To prove the consistency of the estimator, we have used the ergodicity of the SDE and investigated the regularity of the SDE's solution. The maximum inequality of Skorohod integrals has been developed as well.

6.2 Future research work

Besides the asymptotics and convergence rate for iterated power variations that have been obtained in this dissertation, there are several other things that we can contribute to the well estabilished limiting theory using Stein's method and Malliavin calculus. As one of my ongoing projects, the convergence rate of a general smooth functional of a stationary Gaussian sequence is under investigation. Later on, this research work could be extended for the non-stationary Gaussian case and even other non-Gaussian distributions.

Moreover, as an important application of limiting theorems, the inference problems of stochastic processes are actively studied in the recent decades along with the development of Gaussian analysis. There are several things that could be completed in addition to the results in this dissertation. Firstly, we could consider the consistency of the least squares estimator in the discrete case for a general nonlinear SDE, especially when $H < \frac{1}{2}$. Secondly, it is unknown whether the least squares estimator for the nonlinear SDE admits the central limit theorem. This is a challenging problem which relies on the research of the limiting theorem of Skorohod integrals. Thirdly, in addition to least squares estimation and maximum likelihood estimation, many other estimation methods can be considered including moment estimation and Bayesian method. Moreover, We can extend these estimation methods to other stochastic models driven by a general Gaussian noise and some reflected SDEs.

Finally, it is worth mentioning that it would be interesting to apply these estimation methods to deal with real world data.

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